# Mathematical realism and conceptual semantics

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**Abstract:** The dominant approach to analyzing the meaning of natural language sentences that express mathematical knowledge relies on a referential, formal semantics. Below, I discuss an argument against this approach and in favour of an internalist, conceptual, intensional alternative. The proposed shift in analytic method offers several benefits, including a novel perspective on what is required to track mathematical content, and hence on the Benacerraf dilemma. The new perspective also promises to facilitate discussion between philosophers of mathematics and cognitive scientists working on topics of common interest.

**Keywords**: semantics, epistemology, cognition, naturalism, Chomsky, Benacerraf, Jackendoff.

## Introduction

In his celebrated paper, Paul Benacerraf (1973) challenges us simultaneously to provide a plausible semantic account of mathematical statements and an explanation of our epistemic access to their contents. This well-known, two-tiered problem is interesting insofar as it brings into sharp focus a cluster of unresolved issues in the philosophy of mathematics, epistemology, and the philosophy of language. Advances in linguistics bear on the topic in important ways. Benacerraf himself, and indeed the bulk of the philosophical literature, makes some apparently benign assumptions concerning the semantics of natural languages. Those commitments have now been shown more tendentious than first realized. The extensional, formal semantics inherited from *inter alia* Alfred Tarski, Richard Montague, and Donald Davidson, has come under increasing pressure by mainstream linguists, particularly those working in the Chomskyan tradition, for reasons quite independent of any controversies within philosophy of mathematics. The resultant shift in semantic perspective forces a reconceptualization of what has come to be known as one of the defining problems in the philosophy of mathematics.

## 1. The received view

Benacerraf makes two important assumptions concerning the semantics of natural languages. He supposes that any acceptable semantic theory must treat sentences that (according to realists) express mathematical truths no differently from more pedestrian factual statements.[1] And he accepts that there is but one promising such theory: extensional, formal semantics (FS). Here is the relevant passage:

> The semantical apparatus of mathematics [should] be seen as part and parcel of that of the natural language in which it is done, and thus whatever *semantical* account we are inclined to give of names or, more generally, of singular terms, predicates, and quantifiers in the mother tongue [ought to] include those parts of the mother tongue we classify as mathematese. . . I take it that we have only one such account: Tarski's. (Benacerraf, 1973)

Modern formal semantics can be traced to Mongague's (1974) seminal development of Tarski's work.[2] In brief, natural human languages can be viewed as sets of legal strings defined over an alphabet. Viewed thus, they can be interpreted using the same model-theoretic tools used to

[1]The dispute between realists and antirealists is not at issue here; both are required to make use of an acceptable semantic framework. I defend an ontological realism about mathematics in my (2009). For a helpful discussion of fictionalism and semantics, see Stanley (2001).

[2]For a textbook treatment see Heim & Kratzer (1998).

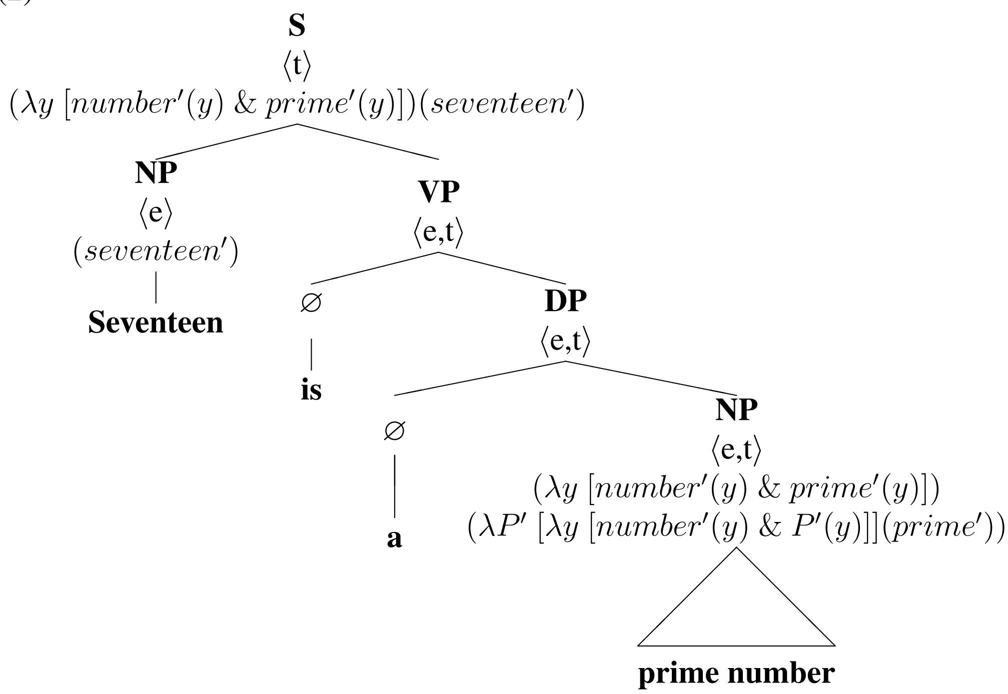
interpret formal calculi. We take the core meaning of a declarative sentence to be that aspect of its interpretation which does not vary with context: its truth-conditions. The meanings of sentence-parts are the contributions those parts make to the truth-conditions of the whole. The meaning of a sentence is thus a monotonic function of the meaning of its components and how these are assembled. On this view, noun phrases are mapped to elements of what Quine calls ‘the passing show’: the things, in the broadest sense, that we refer to in our discourse. Other parts of speech are mapped to functions that reflect the role they play. The result is a flexible, systematic, and infinite hierarchy of semantic types—one that is capable of classifying a rich portion of any natural language.[3]

Consider, as an illustration, the following example:

(1) [$_{S}$ [$_{NP}$ **Seventeen**] [$_{VP}$ **is** [$_{DP}$ **a** [$_{NP}$ [$_{AP}$ **prime**] [$_{NP}$ **number.**] ] ] ] ]

Notice that it has been analyzed into its syntactic components. Its standard semantic interpretation is this:

(2)




For the sentence to be of semantic type ⟨t⟩, the initial NP must be of type ⟨e⟩ and to designate an entity. And “seventeen” is indeed taken to refer to seventeen, the number itself. The final NP is of semantic type ⟨e,t⟩, and designates a function from entities to truth values. Unsurprisingly, “prime number” refers to the property of being a prime number.[4] The VP constitutes an unsaturated function from entities to truth values. Both the copula and the indefinite article are semantically inert, contributing nothing to the overall interpretation. The whole is true if and only if there in fact is an entity, picked out by ‘seventeen’ which has the property of being a prime number. So far, all seems well.

[3] That portion of NL which involves tense, referentially opaque contexts, or other varieties of context-sensitivity require us to move beyond a purely referential semantics to one that includes *intensions*. I’m bypassing those issues here.

[4] I am treating ‘prime’ as an intersective adjective since not everything that is prime in the relevant sense is *ipso facto* a number. Primeness has been generalized to other branches of mathematics. This helps simplify the analysis somewhat.

## 2. Limitations of FS

The troubles start when we consider more closely the nature of the word-world relation on which FS hinges. A number of linguists and philosophers working in a broadly Chomskyan tradition have expressed worries on this score.[5] It's important to tread carefully: there is little doubt that FS offers a formally precise, finitely specified, and maximally nuanced taxonomy of semantic types. However, even so, we expect scientific theories to do more than *classify* the phenomena they cover; we expect them also to *explain*. What is at issue is whether FS offers an account of the meaningfulness of natural language (NL) expressions or whether it piggy-backs on our antecedent personal understanding. Below, I take up the two relata of the word-world relation in turn.

*Referents.* Consider first the right-hand side of the relation. In order for regularities that appear in 'the passing show' to figure as explanans in deductive-nomological explanations, they must have stable, objective identity conditions—i.e., ones specifiable without reference to subjective human viewpoints or concerns. This is evidently true of pH, momentum, capacitance and other well-behaved posits of the mature sciences. In contrast, consider the following sentence:

(3) The thin, blue book, weighing four ounces, standing second from the left, was published in 1759 and caused such a scandal in Europe that it was publicly burned.

We have no trouble understanding it or even supposing that it's true. But what sort of object is in question in the leading NP? Apparently, it's thin, light, resides on the shelf and was burned some time ago! In the face of this absurdity it's tempting to reach for the type/token distinction. But that won't help here. Perhaps the type was published but it cannot weigh four ounces. Nor can it sit on the shelf. Moreover, the same interpretive difficulties accrue to proper nouns, as in Chomsky's (2000) example: "London is so unhappy, ugly, and polluted that it should be destroyed and rebuilt 100 miles away." They are, if anything, all the more glaring with respect to properties, such as that of being red, interesting, or dishonest. And trying to establish crisp identity conditions for verbs—such as smiling or hurrying—that make no mention of human recognitional capacities seems entirely hopeless.

The lesson to derive is not that *Candide*, a patch of red, a smile, or an aspect of the city of London are somehow non-existent or fictional. Rather, it's that in order recognize them as such one must necessarily invoke human recognitional capacities. These cannot be abstracted from semantic analysis; they must form an integral part of it. Far from scientifically explaining how human beings understand their languages, FS presupposes this understanding.

*Words.* It's worth briefly mentioning that parallel problems arise to the left-hand side of the word-world relation. Consider what counts as 'the same' word. Let's agree to leave aside complexities due to medium (clay tablet, telegraph, email) and focus solely on the spoken word. It's tempting to look to the physical characteristics of the acoustic signal for the identity conditions of individual words. This is a vain hope. We can see by looking at a spectrogram energy pattern that the acoustic signal of human speech is continuously variable, and not, as one might perhaps expect, conveniently segmented at word boundaries. Indeed, one reason why spectrograms have not delivered the anticipated quick strides in language recognition is that in order to identify lexical items in the signal, finer-grained elements within the energy patterns need to be matched with appropriate phonological representations—those phonological representations, in turn, are used to access appropriate lexical items. These are nontrivial tasks. Matching acoustic signal to phonological target is complicated by a number of factors. Most obviously, there

[5] See especially the work of James McGilvray (1998), Paul Pietroski (2005), and Robert Stainton (2006), as well as some (2000) comments by Noam Chomsky himself.

are individual differences between speakers due to differences in their vocal tracts. As well, the local speaking environment often changes how people use their voices and hence what the spectrogram registers. We often alter how we speak depending on social context—our sociolect can be more or less crisp depending on whether we are aiming to sound formal or casual. Finally, there are more technical reasons for variation. The pronunciation of individual sounds is often affected in subtle ways by what precedes and follows them. These coarticulation effects stymy any simple mapping between features of the physical signal and the phonological targets. The upshot from the foregoing is that what common sense identifies as 'words' are not, in fact, freestanding entities (or even event tokens) discernible solely in terms of their intrinsic physical characteristics.[6]

A scientific theory faces a number of simultaneous challenges: among them, descriptive and explanatory adequacy. The explanation of the meanings of natural language expressions offered by traditional formal semantics relies crucially on mappings between freestanding linguistic items and the elements of a non-lingusitic domain—that is, on referential mappings between words and world. Available evidence strongly suggests that this picture is overly simple. If that's the case then FS meaning postulates cannot play the same role in deductive-nomological explanations as the objective regularities delivered by mature sciences. FS meaning postulates may, of course, still appear in scientific explanations, but only as a sort of short-hand that itself stands in further need of elucidation by a more basic theory. Arguably, it's that more basic theory which deserves to be called natural language semantics.

## 3. Alternative approach

A more promising approach to the study language—one that explicitly treats what FS takes for granted—lies within the Chomskyan tradition. This is an oft-told tale, so I'll be brief. The Chomskyan abandons the view that natural languages can profitably be understood as abstract sets of well-formed formulae. The familiar appearance that there exist publicly shared vernaculars (French, Russian, etc.) is an effect of the interaction between the functionally specialized regions of the minds/brains of individual speakers. It's useful to distinguish between E-language—the external, public phenomenon—and I-language, the internal linguistic competence encoded in the mind/brain of the individual speaker. The Chomskyan takes the latter to be the linguists' proper focus of research. The work proceeds on several fronts. Every neurotypical human child is born with a 'language acquisition device': an innate mind/brain structure geared toward acquiring a competence in the local dialect.[7] Language acquisition involves the unconscious setting or ranking of a number of innate parameters, each of which encodes some aspect of the local language (for instance, head-initial for Spanish phrases or head-final for Japanese ones). The task for the linguist interested in NL syntax is to map the competence of the mature I-language user but also ultimately to arrive at a theory of the universal linguistic endowment shared by all humans.[8]

The shift from E-language to I-language in the study of NL syntax has a natural corollary in semantics. Ray Jackendoff (*passim*) has proposed an internalist theory that views NL semantics as an interface between the language organ and other mental modules—in particular, the sensory modalities, motor regions, spatial cognition, and social cognition. The aim of semantic theory, according to the conceptual semanticist, is to offer an account of the psychologically real process whereby the syntactic structures generated by the language organ come to be matched with appropriate information-bearing structures elsewhere in the mind/brain. To do this, conceptual

[6]See Ingram (2007), Pulvermuller (2002) and Jackendoff (2002) for an extended treatment.

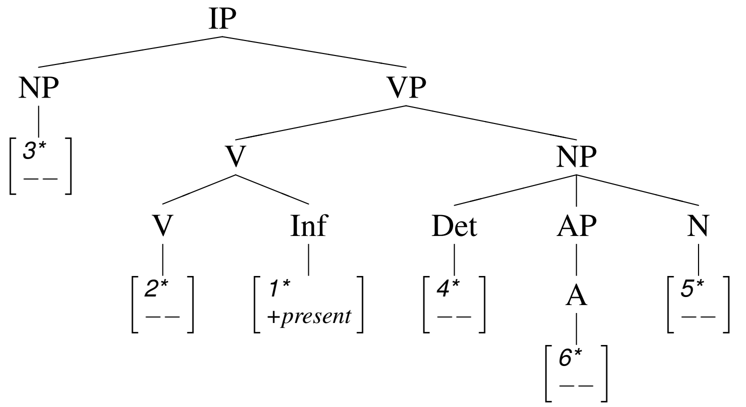
[7]For a recent review of some of the evidence, see Pietroski & Crain (2002).

[8]For useful overviews of the Chomskyan project, see Cook & Newson (2007) and Smith (1999).

semantics (CS) employs a rich range of semantic types. These include ⟨PERSON⟩, ⟨THING⟩, ⟨EVENT⟩, ⟨AMOUNT⟩, ⟨PROPERTY⟩, ⟨PATH⟩, ⟨MANNER⟩, ⟨LOCATION⟩. Which list of types will figure in the finished theory is an empirical question for cognitive scientists. Whatever the outcome, a CS analysis of a sentence is tantamount to its interpretation in a many-sorted logic where each type is intended to correspond to a psychologically real process-type in the mind/brain of the interpreter. Since this is an internalist account, there can be no guarantee that the conceptual types humans employ will find straightforward correlates in our physical surroundings. Conceptual semantics is not a form of idealism or solipsism however. It's important to realize that while on the CS view linguistic interpretation is a fairly high-level cognitive phenomenon, it is well understood that mental processes ultimately bottom out on our sensory surfaces and the contact these make with external reality. This simply occurs quite far downstream of our concepts, the language module, or its semantic interface.[9]

*Reanalysis.* It may be helpful to reanalyze our previous example from a CS standpoint. The first task, once again, is to provide an appropriate syntactic decomposition. The key difference is that words can no longer figure in our analysis; they are not *bona fide* I-linguistic posits. Instead, we must content ourselves with pointers to semantic and phonological structures.[10]

(4)




This is the aspect of (1) that the syntactic portion of the mind/brain of the individual speaker encodes. I have labelled each pointer with a starred number for the sake of clarity. The order of the pointers corresponds to the structural order of the constituents. The VP is taken to govern the sentence as a whole. The maximal projection of the first NP acts as the IP's specifier. The second NP plays the role of a complement to the verb phrase. It is taken here to project three elements which facilitate the explanation of the syntax-semantics interface. As a general rule, we can expect every major phrasal constituent to map to a major semantic category. We can expect the leaf nodes to map to functions (though these may be zero-place functions). Since the syntactic structure we're working with contains six phrasal constituents, we can expect the corresponding semantic structure to contain the same number of fundamental components. Here is the matching conceptual structure:

(5)
$$\left[ PRESENT \left[ BE_{\text{ident}}\left( \left[ TOKEN : Seventeen \right]_{\text{Thing } 3}, \left[ \begin{array}{l} TYPE : number \\ \left[ prime \right]_{\text{Property } 6} \end{array} \right]_{\text{Indef4 Thing5}} \right) \right]_{\text{State } 2} \right]_{\text{Situation } 1}$$

[9]The CS way of approaching semantics has interesting ties to Kant, particularly as he is read by Patricia Kitcher (1990).

[10]This version is in line with (Culicover & Jackendoff, 2005). Whether it is correct is, in the long run, an empirical question. I am using it here to make the fit between syntactic structure and conceptual semantics as perspicuous as possible.

The sentence as a whole expresses a situation which obtains in the present tense. This situation is represented in conceptual structure as involving the existence of a state between a token item and a type. The token is a ⟨thing⟩: the number seventeen. The type is complex: it involves being a number, also a species of ⟨thing⟩. But it is further delimited by the ⟨property⟩ of being prime. So the sentence expresses that the token entity named by 'seventeen' is found among the entities designated by 'number'; more precisely, it is among those entities of that type which are also prime. None of this is terribly surprising, of course. The usefulness of this analysis, again, hinges on the correspondence of the semantic types to biologically real structures and processes. Among cognitive scientists' research goals is to characterize these semantic constituents in the terms of computational neuroscience.

## 4. Realism reconsidered

A standard argument for ontological realism in mathematics runs as follows: Mathematical statements are true, false, or lack content altogether (the nominalist option). It's radically implausible that accepted mathematical statements are uniformly false. For its part, the no-content option fails to explain the surprising (indeed shocking) indispensability of apparently useless maths to theory construction in natural science.[11] So some mathematical statements must be true. But given standard formal semantics, the truth of mathematical statements entails the existence of at least some mathematical entities. And so we should be ontological realists about maths' posits. Now, unfortunately, this argument is invalidated by the acceptance of a conceptual semantics. Ironically, by moving to a cognitively responsible and scientifically realistic view of language, we seem to undermine realism about maths. I suspect, in fact, that the unwillingness to bite this bullet lies at the root of some theorists reluctance to accept a cognitive conception of NL syntax and semantics.

I think the worries are exaggerated. In a somewhat different context, Tarski (1944) admonishes us that semantics is not a cure for the ills of the world, nor is it a means of showing that everyone except the speaker and her friends are speaking nonsense. The observation is still apt. Our theory of meaning does not go deep enough to somehow determine by fiat the outcome of controversies in the ontology of mathematics. It is no more and no less than an account of how a particular sort of creature comes to interpret its natural language and perhaps other, related symbol systems. As you might expect then, there is a way of articulating a robustly ontologically realist conception of mathematics even if we accept cognitivism.

The key, I think, is take one's initial cue from theories of vision and other sensory modalities (as indeed Gödel (1947) suggests). Some cognitive processes are relatively unconstrained by independently existing realities. Conceptual free association is one example. Human beings are notoriously bad at generating truly random responses. Nonetheless, free association is vastly less constrained than, say, vision or hearing—so long as we are awake, these modalities are constrained by our interactions with our surroundings. Not all constraints on cognitive processing come from without. We cannot help but to recognize certain sentences in our native tongue as ungrammatical. There, we are constrained by our idiolect's parameter settings. Our judgement, as in the case of perceptual experience, is evidently not foolproof; garden path sentences can make us *think* that a constraint has been violated when it has not.[12] This is no more troubling to the linguist however than visual illusions are to the vision-scientist. The correct syntactic theory accounts both for our intuitions of grammaticality and for why we find garden-path sentences so difficult to parse. Indeed, in each of the above cases, the fact that we can talk about

[11] For a recent discussion to which I am sympathetic, see Maddy (2011). For an older, anti-naturalist defence, see Steiner (1998).

[12] A typical, grammatical garden path sentence would be: "The horse ran past the barn fell."

making *errors* means also that we can talk about getting things right. And to claim that some representation is correct is (at the very least) to maintain that it does not violate any of the relevant, local, operative constraints. Evidently, 'correctness' is a very general notion; it applies to statements, procedures, choices, and so on. But when we limit our purview to contentful, affirmative judgements, to judge correctly is to judge *truly*.

Bearing this in mind, here is a revised argument for mathematical realism: Mathematical statements are true, false or contentless. The latter two options are (to a realist) implausible. Assuming a conceptual semantics, the truth of (some) mathematical statements entails that objective constraints on mathematical judgements obtain. These constraints are not due to our culture, opinions, conventions, or (crucially) even our neurophysiology. *That* would be psychologism! For our mathematical judgements to be universal, necessary, and objective it suffices for the constraints on the relevant judgement to be mind-independent, inescapable, and applicable under all circumstances.[13]

If we accept this picture then the epistemic issue concerning mathematical knowledge henceforth shifts away from how we 'make contact' with the truth-makers of mathematical expressions; instead, we need to ask how the semantic structures employed in the course of mathematical reasoning are forced to follow their rigidly constrained course. What, in effect, makes the geography of peaks, valleys and hidden trails in Allain Connes' (1995) mathematical landscape so implacably resistant to our ambitions, desires, whims, and wishful thinking? This is to trade a metaphysical mystery for a research problem; surely a trade worth making.

## 5. Conclusion

Here's what I have argued: the Benacerraf problem, as it's traditionally articulated, builds on a view of language at odds with the best available work in linguistics and cognitive science. By opting instead for an internalist view of NL syntax and a conceptual theory of semantics, we gain a better understanding of the nature of linguistic meaning—and *ipso facto* of sentences that express mathematical truths. The shift to a cognitively responsible theory of language need not entail psychologism about the truths of mathematics. Within the new framework, Benacerraf's worry concerning mathematical truth reasserts itself, albeit in modified form: viz., we ask whether it is possible to offer a theory of the objective constraints on the information-bearing states of the mind/brain that constitute the meanings of mathematical expressions. The realist supposes that such constraints do indeed exist and sets about building the relevant theory; an anti-realist doubts that anything beyond our conventions, fictions, or perhaps our neurophysiology limits our mathematical research.[14] I admit the shift in theoretical perspective may be jarring for some. But, I submit, it offers a more promising path toward a detailed explanation of human mathematical competence than the hunt for extrasensory abstracta.

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[13]That said, there may be alternative ways of ranking the relative weight of the transcendental constraints, as for instance when one decides between alternative axioms in set theory. The details of the applicable theory is a topic for another time.

[14]The literature in cognitive psychology and cognitive neuroscience on mathematical reasoning is vast—most of it, unfortunately, anti-realist and focused on arithmetic. See Dehaene (2011) for a review of some of the results.

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