McKinsey Algebras and Topological Models of S4.1

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Abstract. The aim of this paper is to show that every topological space gives rise to a wealth of topological models of the modal logic S4.1. The construction of these models is based on the fact that every topological space defines a Boolean closure algebra (to be called a McKinsey algebra) that neatly reflects the structure of the modal system S4.1 in that all its elements satisfy the topological interpretation of the McKinsey axiom. It is shown that the class of topological models based on McKinsey algebras contains a canonical model that can be used to prove a completeness theorem for S4.1. Finally, it is proved that the McKinsey algebra MKX of a space X endowed with an α-topology satisfies Esakia's GRZ axiom.

Keywords. McKinsey Axiom Schema, Modal Logic, Topological Models of S4, S4.1, Boundary, Closure Algebra, Completeness.

1. Introduction. Let S4.1 be the propositional modal logic over S4 defined by adding the so called McKinsey axiom (schema)

(1.1) $$\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$$

to the axioms and rules of S4. Let (X, τ) be a topological space with closure operator cl and interior operator int (cf. Kuratowski and Mostowski (1976), Willard (2004)). Interpreting the modal operators $\Box$ and $\Diamond$ by the topological operators int and cl in the familiar way, the McKinsey axiom defines the following subset MKX of the power set PX of X:

(1.2) $$MKX := \{a;\ intcl(a) \subseteq clint(a),\ a \subseteq X\}$$

In view of (1.1) I propose to call the elements of MKX as McKinsey sets. The power set PX of a topological space X has the structure of a complete interior Boolean algebra (PX, $\subseteq$, Ø, X, int) or, equivalently, the structure of a complete closure algebra (PX, $\subseteq$, Ø, X, cl). Throughout this paper PX is assumed to carry this structure of a modal Boolean algebra. It is natural to ask then how the subset MKX of PX behaves with respect to this structure. The main technical result of this paper is the following theorem:

(1.3) Theorem. Let (X, τ) be a topological space. The set MKX is a Boolean subalgebra of the Boolean algebra PX. More precisely, MKX is a modal subalgebra of PX. The modal algebra MKX is to be called the McKinsey algebra of the topological space X. The elements of MKX are referred to as McKinsey sets.♦[1]

The following remarks on (1.3) may be in order: (i) Let OX and CX denote the sets of open and of closed sets of a topological space (X, τ), respectively. Then OX ∪ CX ⊆ MKX: For a ∈ OX this is proved as follows. From a = int(a) one obtains and intcl(a) = intclint(a) ⊆ clint(a) by the fundamental properties of the interior operator int. Hence a ∈ MKX. The assertion for CX is proved similarly. As will be shown later MKX not only contains open and closed sets but also all kinds of „mixed“ sets, in particular finite intersections and unions of open and closed sets and nowhere dense sets.
In a sense MKX may be considered as a „finite“ counterpart or analogue of the σ-complete Boolean algebra BOX of Borel sets of X. This is nicely evidenced by Theorem (2.4), to be proved in section 4 according to which certain quotients of MKX and BOX, respectively, are isomorphic Boolean algebras. Nevertheless MKX and BOX are clearly different: Neither MKX is a subset of BOX, nor is BOX a subset of MKX.
(ii) A particularly important role for the structure of MKX is played by the ideal NDX of nowhere dense sets. Recall that a subset a of a topological space is called nowhere dense if and only if intcl(a) = Ø. One immediately observes that NDX ⊆ MKX since by definition one has Ø = intcl a ⊆ clint a = Ø. As will be shown later, OX and NDX generate MKX as a Boolean algebra in a precise sense.

[1] Stone (1937) seems to have been the first paper where MKX showed up (not under this name of course). Stone called it the „complete basic ring of the topological space X“ and proved some of ist the basic properties of McKinsey sets (cf. Theorem 24). Much later, Esakia (2004) mentions as a „key observation“ that the subset {a ∈ B; intcla ≤ clinta} of a Boolean closure algebra B is also a Boolean closure algebra (cf. Esakia (2004, section 8.4). He explicitly mentioned the McKinsey axiom but seems to be unaware of the fact that already Stone made the just mentioned „key observation“. Independently of Stone, McKinsey, and Esakia, the set MKX was mentioned by several topologists who dubbed the „McKinsey sets“ of this paper „delta sets“, „NDB sets (= sets with nowhere dense boundary), or „α-open sets“. These authors, however, never noticed that these sets played a role in modal logic, nor realized that MKX is a Boolean algebra.

(iii) In general MKX is a proper subalgebra of PX as is shown by the following canonical example: Consider the real line **R** endowed with its standard Euclidean topology. Then one obtains for the set **Q** of rational numbers

(1.4) $$\text{intcl}(\mathbf{Q}) = \mathbf{R} \text{ and } \text{clint}(\mathbf{Q}) = \emptyset.$$

Hence **Q** is not an element of MK**R**. This proves that in general MKX ≠ PX. Since every singleton {r}, r ∈ **R**, is clearly a McKinsey set, the set **Q** as a countable union of rational numbers evidences that MKX in general is not a σ-complete Boolean algebra. On the other hand it is well known that there are topological spaces X for which PX = MKX, i.e., every subset a of X is a McKinsey set (cf. Bezhanishvili, Esakia, Gabelaia (2005), van Benthem, Bezhanishvili (2007, Chapter 5, p. 253).[2]

The main aim of this paper is to use the modal algebra MKX to define a profusion of sound topological models of S4.1 and related logics. Among these models there is also a canonical complete model that allows us to prove a completeness theorem for S4.1.

The outline of this paper is as follows. In the next section we explore the lattice-theoretical context in which MKX is embedded. That is to say, we study in some detail the various topologically defined lattices to which MKX is related. This prepares the ground for proving (1.2) in the following section, exploiting a formula of A.H. Stone mentioned. Then we show that McKinsey algebras may be used to construct sound topological models of S4.1. Among these topological models for S4.1 there is a canonical model endowed with a canonical interpretation. This enables us to prove a completeness for S4.1 following the lines of the elegant completeness proof for S4 presented by Aiello, van Benthem, Bezhanishvili (2003)).

2. McKinsey algebras and other lattices of topological spaces. The algebra MKX is in no way the only lattice that is related with a topological space (X, τ). On the contrary, MKX may be considered as an almost completely neglected member of an extended family of lattice structures that can be associated to any topological space X. For later use, some of them are collected in the following list:

[2] Bezhanishvili, Esakia, Gabelaia (2005) proposed to call a topological space X a McKinsey space if the set of dense subsets of X forms a filter (ibid., p. 326). They showed that for McKinsey spaces the algebras MKX and PX coincide, i.e., all subsets of a X satisfy the requirement intcl(a) ⊆ clint(a). Hence calling MKX the McKinsey algebra of X is quite in line with this usage.

(2.1) Definition. Let (X, τ) be a topological space. The topological structure on X gives rise to (at least) the following lattices:

(1)	PX	:=	modal Boolean algebra of subsets of X.
(2)	OX	=	Heyting algebra of open sets.
(3)	CX	=	Co-Heyting algebra of closed sets.
(4)	ROX	=	Boolean algebra of regular open sets.
(5)	RCX	=	Boolean algebra of regular closed sets.
(6)	CLX	=	Boolean algebra of clopen sets.
(7)	BOX	=	Boolean algebra of Borel sets.
(8)	MKX	=	modal Boolean algebra of McKinsey sets (with closure operator cl).♦

The general aim of this paper is to show that for virtually all considerations of modal logic the modal algebra (PX, $\subseteq$, cl) can be replaced by the smaller modal algebra (MKX, $\subseteq$, cl). As an extra bonus of this replacement one obtains an easy completeness theorem for S4.1 instead of S4.

The following inclusion relations between these lattices listed in (2.1) are well-known:

(2.2) Proposition: Let (X, τ) be a topological space. Then the following inclusion relations (as inclusions of order structures) between the lattices defined above hold.

$$\mathrm{CLOX} \subseteq \begin{array}{ccc} \mathrm{ROX} & \subseteq & \mathrm{OX} \\ & & \\ \mathrm{RCX} & \subseteq & \mathrm{CX} \end{array} \subseteq \begin{array}{c} \mathrm{MKX} \\ \\ \mathrm{BOX} \end{array} \subseteq \mathrm{PX}.♦$$

(BOX[3])

Recall that a subset a of a topological space X is meager if and only if a is the countable union of nowhere dense sets. The set of meager sets is denoted by MGX. MGX is well-known to be an ideal of BOX. In Halmos (1963) the following somewhat surprising relation between the Boolean algebras ROX, and BOX is proved (cf. Halmos 1963, § 13, Theorem 4, p. 58):

[3] In general, MKX is not a subset of BOX, nor is BOX a subset of MKX. Already in (1.3) it was pointed out that the set **Q** of rational numbers is a Borel set but not a McKinsey set. In order to show that in general MKX is not a subset of BOX one may argue as follows. As is well-known the Cantor set C is a closed nowhere dense set of **R**. Hence it is an element of MK**R**. This entails that all subsets of C are also nowhere dense and therefore are elements of MK**R**. Among them there are all kinds of non-Borel sets. Hence MK**R** is not a subset of BO**R**. If one wants to avoid non-Borel sets one can replace MKX by the algebra MKX $\cap$ BOX of Borel-McKinsey sets, or, as will be shown in (4.9), one may restrict one's attention to special classes of topological spaces such as α-topological spaces.

(2.3) BOX/MGX = ROX.♦

An analogous result involving the McKinsey algebra MKX and the ideal NDX of nowhere dense sets instead of BOX and MGX is the following one:

(2.4) Theorem. The Boolean algebra ROX of regular open sets of X is the quotient of the McKinsey algebra MKX and the ideal NDX of nowhere dense sets: MKX/NDX = ROX.

This proposition will be proved in section 4. Informally it may be formulated as the assertion that MKX can be interpreted as a finite counterpart of BOX.

In order to prove that MKX is indeed a Boolean subalgebra of PX, we have to recall some elementary facts about the topological operators int and cl (cf. for example Kuratowski and Mostowski 1976, chapter 1, §8, Willard (2004, chapter 2):

(2.5) Lemma. Let (X, τ) be a topological space. Then the topological operators int(interior kernel) and cl(closure) enjoy the following properties:

$(i)_1$ int(a) = **C**cl(**C**a), $(i)_2$ cl(a) = **C**int(**C**b).

$(ii)_1$ int(a ∩ b) = int(a) ∩ int(b) $(ii)_2$ cl(a ∪ b) = cl(a) ∪ cl(b).

$(iii)_1$ intcl intcl(a) = intcl(a) $(iil)_2$ clintclint(a) = clint(a).

$(iv)_1$ If a ∈ OX or b ∈ OX then intcl(a ∩ b) = intcl(a) ∩ intcl(b).

$(iv)_2$ If a ∈ CX or b ∈ CX then clint(a ∪ b) = clint(a) ∪ clint(b).♦

In our calculations we heavily rely on the topological boundary operator bd. The following lemma recalls the definition and the basic properties of this operator:

(2.6) Lemma. Let (X, τ) be a topological space. The boundary operator bd is defined as bd(a) := cl(a) ∩ cl(**C**a). It has the following properties:

(1) bd(Ø) = bd(X) = Ø.

(2) bd(a) = bd(**C**a).

(3) bd(bd(a)) ⊆ bd(a).

(4) a ⊆ b ⇒ bd(a) ⊆ b ∪ bd(b).♦

It is possible to take the notion of boundary bd as basic and define all other topological concepts such as the closure operator cl and the interior operator int in terms of boundary (see Zaritsky 1927, Mormann 2012).

Not so well known is the following boundary formula that describes a relation between the boundaries of sets a, b, $a \cap b$, and $a \cup b$: (cf. Kuratowski and Mostowski 1976, Mormann 2012):

(2.7) Proposition. If a and b are subsets of a topological space X, then

$$bd(a) \cup bd(b) = bd(a \cap b) \cup bd(a \cup b) \cup (bd(a) \cap bd(b))$$

Proof: Kuratowski and Mostowski(1976, p. 32, Exercise 2(a)), Mormann (2012)[4].♦

The following lemma relates the boundary operator bd and the McKinsey sets in the sense of (1.1):

(2.8) Lemma. Let X be a topological space. Then $a \in MKX$ if and only if $intbd(a) = \emptyset$.

Proof. Assume $a \in MKX$, $intcl(a) \subseteq clint(a)$. Then the following equivalences hold

$$\begin{aligned} intcl(a) \subseteq clint(a) &\Leftrightarrow intcl(a) \cap \mathbf{C}clint(a) = \emptyset \\ &\Leftrightarrow intcl(a) \cap intcl(\mathbf{C}a) = \emptyset \\ &\Leftrightarrow int(cl(a) \cap cl(\mathbf{C}a)) = \emptyset \\ &\Leftrightarrow int(bd(a)) = \emptyset. \end{aligned}$$ ♦

Now we can prove that MKX is a Boolean subalgebra of PX. For this task one has to show that MKX is closed with respect to set-theoretical complement **C**, union $\cup$ and intersection $\cap$. By definition of the boundary bd one has $bd(A) = bd(\mathbf{C}A)$, and therefore $intbd(A) = intbd(\mathbf{C}A)$, i.e., MKX is closed with respect to set-theoretical complement **C**. In particular one has $intbd(X) = intbd(\emptyset) = \emptyset$. Hence $\emptyset$ and X are elements of MKX.

Thus the only thing that remains to show is that $a, b \in MKX$ entails that $a \cap b$, $a \cup b \in MKX$. For any subset c of X one has $int(c) = \emptyset$ if and only if $cl(int(c)) = \emptyset$. Now assume $intbd(a) = intbd(b) = \emptyset$. Then we obtain by (2.5):

$$\emptyset = cl(intbd(a) \cup intbd(b))$$

[4] Kuratowski and Mostowski ascribe this result to A.H. Stone but without giving precise source. For another proof one may use the results of Zaritsky (1927);

$=\quad$ cl int (bd(a) $\cup$ (bd(b))

$=\quad$ cl int[bd(a $\cap$ b) $\cup$ bd(a $\cup$ b) $\cup$ (bd(a) $\cap$ bd(b))]

$=\quad$ cl int(bd(a $\cap$ b)) $\cup$ clint(bd(a $\cup$ b)) $\cup$ clint(bd(a) $\cap$ bd(b)).

Hence cl(int(bd(a $\cap$ b)) = cl(int(bd(a $\cup$ b)) = Ø. Thus MKX is a Boolean subalgebra of PX. In order to show that MKX is modal subalgebra of PX we need only to recall that OX is a subset of MKX, since the boundary of a closed (open) set is well-known to have empty interior.♦

3. McKinsey Algebras and Topological Models of S4.1. After these preparations we are ready to prove that S4.1 is sound and complete with respect to topological McKinsey models. First let us recall the standard definition of a topological model of a modal logic.
Let L be the standard language of propositional modal logic. The formulas of L are defined recursively starting from a countable sets of propositional constants p, q, ... With the help of the standard Boolean operators $\neg$ (negation), $\wedge$ (conjunction), and $\vee$ (disjunction), and possibly others (definable in terms of them) the well-formed formulas of propositional logic are defined in the familiar way. If φ is a well-defined formula, $\Box\varphi$ (necessarily φ) and $\Diamond\varphi$ (possibly φ) are also well-defined formulas. The modal operators $\Box$ and $\Diamond$ are interdefinable in the usual way $\Diamond\varphi = \neg\Box\neg\varphi$. No other formulas are well-defined formulas of L. Modal formulas are denoted by ϕ, ψ.
From the various equivalent axiomatizations of S4 we choose one that seems to be best adapted to topological considerations (cf. van Benthem and Bezhanishivili (2007, p. 297)):

(3.1) Axiomatization of propositional modal logic S4:

(N) $\Box$T

(R) $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$

(T) $\Box\varphi \rightarrow \varphi$

(4) $\Box\varphi \rightarrow \Box\Box\varphi$

Modus Ponens (MP) and monotonicity (M) are the rules of inference:

(MP) $(\varphi \rightarrow \psi)$, φ // ψ

(M) $\varphi \rightarrow \psi$ // $\Box\varphi \rightarrow \Box\psi$.♦

(3.2) Definition. Let X be a topological space with interior operator int. A topological model of S4 is a pair (X, ν) with ν being a map that maps the formulas of S4 onto elements of PX in the following way:

(1) ν(p) ∈ PX, p a propositional letter of the language L.

(2) ν(¬φ) := **C**ν(φ).

(3) ν(φ ∧ ψ) := ν(φ) ∩ ν(ψ).

(4) ν(φ ∨ ψ) := ν(φ) ∪ ν(ψ).

(5) ν(□φ) := int(ν(φ)).♦

Due to the well-known properties of the topological operator int topological models (X, ν) are sound S4-models.

Now we are going to show that the modal logic S4.1 is sound and complete with respect to a special class of topological models, to be called MK-models. Moreover we will show that the class of models contains a canonical topological model that can be used to prove a completeness theorem.

(3.3) Definition. Let X be a topological space with interior operator int. A topological model of S4.1 is a pair (X, ν) with ν an interpretation that maps the formulas of S4.1 onto elements MKX in the following way:

(1) ν(p) ∈ MKX, for p a propositional letter of the modal language.

(2) ν(¬φ) := **C**ν(φ).

(3) ν(φ ∧ ψ) := ν(φ) ∩ ν(ψ).

(4) ν(φ ∨ ψ) := ν(φ) ∪ ν(ψ).

(5) ν(□φ) := int(ν(φ)).♦

(3.4) Proposition. A topological model (X, ν) of S4.1 as defined in (3.2) is a sound model of S4.1.

Proof. By (1.3) the algebra MKX of X is a modal Boolean algebra. Hence the topological models (3.3) satisfy the theorems of S4. By the very definition of MKX the S4.1 axiom schema (1.1) □◇φ→◇□φ is satisfied since ν maps φ onto the set ν(φ) ∈ MKX and therefore intclν(φ) ⊆ clintν(φ). Hence the topological models of S4.1 as defined by (3.3) are sound models of S4.1.♦

Before we go on to prove a completeness theorem for S4.1 let us briefly recall the basic notions of truth a validity for propositional modal logic with respect to topological models.

<u>(3.5) Definition.</u> Let $((X, \tau), \nu)$ be a given topological model of S4 and φ a formula of S4:

(i) φ is true at $x \in X$ if and only if $x \in \nu(\varphi)$;

(ii) φ is true in a topological model (X, ν) if it is true at every point $x \in X$;

(iii) φ is said to be valid in X if φ is true in every topological model $((X, \tau_\lambda), \mu_\lambda)$ based on the set X;

(iv) φ is said to be true in a class of topological spaces if φ is valid in every member of the class;

(v) A logic is sound and complete with respect to a class C of topological models (X, ν) if and only if it is sound and complete in the usual sense.♦

Now we are going to show that the class of topological models with target MKX is sound and complete with respect to S4.1. It has already been shown that this class of models is sound with respect to S4.1. The only thing left is to show completeness by constructing a canonical model (X, ν) with a canonical interpretation ν.

Since S4.1 is a normal logic the Lindenbaum lemma is valid for S4.1, and maximal consistent sets of formulas for exist ((cf. Blackburn, de Rijke, and Venema 2001, Lemma 4.17, p. 197). Hence the set W of maximally S4.1 consistent sets of formulas (or „theories“) is well defined and not empty. Since theories T are maximal consistent sets of formulas, for any formula of S4.1 either ϕ or $\neg\phi$ belongs to a theory T. In particular, all theories T of S4.1 contain all instantiations of the McKinsey schema $\Box\Diamond\phi \rightarrow \Diamond\Box\phi$.

In order to construct a canonical topological model for S4.1 we have to define a topological structure on W.

<u>(3.6) Proposition</u>. Let W be the set of all maximal S4.1 consistent sets T of formulas. A topology τ_W on W is defined by the basis $B := \{\{T; \Box\phi \in T, \varphi \in L\}, \phi$ running through all formulas of $L\}$.

<u>Proof</u>. This assertion is proved exactly in the same way as Aiello, van Benthem, and Bezhanishvili (2003) proved the corresponding assertion for the theories of S4. First, one observes that $X \in B$ since all tautologies of S4.1 are elements in all maximal consistent

sets T ∈ W. Secondly, finite intersections of elements of B are again elements of B, since S4.1 is a normal modal logic. Hence we get

$$\{T; \Box\phi \in T\} \cap \{T; \Box\psi \in T\} = \{T; \Box(\phi \wedge \psi) \in T\}$$

In other words, B is a basis for a topology τ_W on W.♦

<u>(3.7) Proposition. (A Canonical Topological Model of S4.1)</u>. Let (W, τ_W) be the canonical topological space of theories of S4.1. Then the canonical S4.1 interpretation ν is the map inductively defined for the formulas of S4 as follows: For propositional constants p of L the map ν is defined as

$$\nu(p) := \{T; T \in W \text{ and } p \in T\}$$

This definition can be inductively extended from atomic sentences p to all formulas of L in the usual way (see e. g. (Aiello, van Benthem, and Bezhanishvili 2003, 895) or (Parikh, Moss, and Steinsvold 2007, 302). The resulting map ν renders (W, ν) a sound topological MK-model of S4.1.

<u>Proof</u>. The proof is by induction on φ. The induction for molecular sentences using only Boolean operators ¬, ∧, and ∨ is trivial. Hence it sees sufficient to consider only the induction step for the modal operator □ in some more detail. In order to show that ν is an interpretation one has to show that $\nu(\Box\varphi) := \{T; \Box\varphi \in T\} = \text{int}\{T; \varphi \in T\}$.

First we show that $\{T; \Box\varphi \in T\} \subseteq \text{int}\{T; \varphi \in T\}$. Suppose that $T \in \nu(\Box\varphi)$. Then $\Box\varphi \in T$. The set $\nu(\Box\varphi)$ is an open set containing T. Since $\nu(\Box\varphi) \subseteq \nu(\varphi)$ one obtains $T \in \nu(\Box\varphi) \subseteq \nu(\varphi) = \{T; \varphi \in T\}$ and therefore $T \in \text{int}\{T; \varphi \in T\}$ by the definition of the interior operator int.

Now let us show that $\text{int}\{T; p \in T\} \subseteq \{T; \Box p \in T\}$. The proof is carried out by reductio. Assume that there is an open base set $\{T; \Box\varphi \in T\}$ contained in $\text{int}\{T; p \in T\}$ that is not contained $\{T; \Box p \in T\}$.

That is to say that there is a φ in L with (i) $\{T; \Box\varphi \in T\} \subseteq \{T; p \in T\}$, and (ii) $\{T; \Box\varphi \in T\} \not\subset \{T; \Box p \in T\}$ for some T. From (i) one obtains that $\Box\varphi \rightarrow p$ for all T. The necessitation rule (N) entails $\Box(\Box\varphi \rightarrow p)$ is an element of every T. Normality (K) of S4.1 yields $(\Box\Box\varphi \rightarrow \Box p)$, from reflexivity one obtains $(\Box\varphi \rightarrow \Box p) \in T$, and eventually modus ponens (MP) yields $\Box p \in T$. This is a contradiction to (ii). Hence $\{T; \Box p \in T\} = \text{int}\{T; p \in T\}$.♦

Let us pause for a moment and take stock. We have shown that (W, ν) is a topological S4 model. The essential task remains still unsolved, namely, to show that (W, ν) is even an

S4.1-model. For this claim one has to show $\nu(\varphi) \in$ MKW. Since MKW is a modal Boolean algebra, it suffices to prove this for $\nu(p)$, p propositional letter of L. In other words one has to prove that

(3.8) $$\mathrm{int}(\mathrm{bd}(\nu(p)) = \mathrm{int}(\mathrm{cl}(\nu(p)) \cap \mathrm{cl}(\mathbf{C}(\nu(p)) = \emptyset$$

By definition one has $\nu(p) = \{T;\, p \in T\}$. The theories T are maximally S4.1-consistent sets of formulas, hence $\mathbf{C}\{T;\, p \in T\} = \{T;\, \neg p \in T\}$. Since ν is an S4-interpretation one has $\mathrm{int}\{T;\, p \in T\} = \{T;\, \Box p \in T\}$. From $\mathrm{cl} = \mathbf{C}\mathrm{int}\mathbf{C}$ and the distributivity of int over $\cap$ (cf. (2.5)) one obtains

$$\begin{aligned} \mathrm{int}(\mathrm{bd}(\nu(p)) &= \mathrm{int\ cl}\{T;\, p \in T\} \cap \mathrm{int\ cl}\{T;\, \neg p \in T\} \\ &= \{T;\, \Box\Diamond p \in T\} \cap \{T;\, \Box\Diamond\neg p \in T\}. \blacklozenge \end{aligned}$$

<u>(3.9) Lemma</u>. Every maximal S4.1-consistent set T of formulas contains the formulas

$$\Box\Diamond p \rightarrow \Box\Diamond\Box\, p \quad \text{for all } p \in L.$$

<u>Proof</u>. By the McKinsey axiom the formula $\Box\Diamond p \rightarrow \Diamond\Box p$ is a S4.1 tautology for all p. By necessitation (N) one obtains $\Box(\Box\Diamond p \rightarrow \Diamond\Box p) \in T$. The K-rule gives us $(\Box\Box\Diamond p \rightarrow \Box\Diamond\Box\, p) \in T$, and reflexivity of $\Box$ eventually yields:

$$\Box\Diamond p \rightarrow \Box\Diamond\Box p \in T. \blacklozenge$$

<u>(3.10) Corollary</u>. If T is an S4.1 theory with $T \in \mathrm{intbd}\nu(p)$ then T contains the formula $\Box\Diamond\Box p \wedge \Box\Diamond\Box\neg p$.♦

Applying the modal-logical counterpart of $(2.5)(\mathrm{iii})_1$ one obtains

(3.11) $$\Box\Diamond\Box p \wedge \Box\Diamond\Box\neg p \Leftrightarrow \Box\Diamond(\Box p \,\&\, \Box\neg p) \Leftrightarrow \Box\Diamond(\bot) = \bot.$$

Hence there is no S4.1 consistent theory $T \in \mathrm{intbd}\nu(p)$, since this set is empty. In other words, the interpretation ν maps atomic formulas p onto $\nu(p) \in$ MKW. This entails that ν maps <u>all</u> formulas φ onto elements $\nu(\varphi) \in$ MKW. Hence ν is sound with respect to S4.1. Hence ν is not only an S4 but indeed an S4.1 interpretation.♦

Now we can prove a completeness theorem for S4.1 with respect to topological models in exactly the same way as the completeness theorem for S4 was proved in (Aiello, van Benthem, and Bezhanishvili (2003)).

(3.12) Theorem. For any set of formulas S

$$\text{IF } S \models \varphi \quad \text{THEN } S \vdash_{S4.1} \varphi$$

Proof. The proof is by reductio. Suppose that $S \models \varphi$ and not $S \vdash_{S4.1} \varphi$. Then $S \cup \{\neg\varphi\}$ is consistent. By the Lindenbaum lemma there is a maximal S4.1 consistent set T that contains $S \cup \{\neg\varphi\}$. The theory T is an element of W for which $\neg\varphi$ is true under the canonical interpretation ν. Hence (W, ν) is a model of S4.1 for which φ is not true. This is a contradiction. Hence $S \vdash_{S4.1} \varphi$.♦

4. ROX as a quotient of MKX. In this section we prove Theorem (2.4), namely, that MKX/NDX = ROX. For this purpose we show that the concatenation of maps

(4.1) $$\text{ROX} \xrightarrow{\ i\ } \text{MKX} \xrightarrow{\ p\ } \text{MKX/NDX}$$

is an order-isomorphismus, and therefore a Boolean lattice isomophism. Here i is the embedding of ROX into MKX, and p is the projection of the Boolean lattice MKX onto the Boolean lattice of equivalence classes of elements of MKX with respect to the ideal NDX of nowhere dense sets. It may be noted that the quotient of MKX by the NDX - as being isomorphic to ROX – is a complete Boolean lattice although neither MKX nor NDX are complete. This may be compared with the analogous case that ROX comes out also as the quotient BOX/MGX of the Borel σ-complete algebra BOX of Borel sets of X and the σ-complete ideal of meager sets MGX.

Since $\text{ROX} \xrightarrow{\ i\ } \text{MKX}$ and $\text{MKX} \xrightarrow{\ p\ } \text{MKX/NDX}$ are order preserving, the resulting map $p \bullet i$ is clearly order preserving. Hence we only need to show that $p \bullet i$ is an epimorphism and a monomorphism. For this purpose we first show that for every $a \in$ MKX the equivalence class $[a] \in$ MKX/NDX contains at least one element of ROX, namely a** with a* = int**C**a, and therefore a** = int**C**int**C**a = intcl(a). This is proved by calculating that p(a) = p(a**) for all $a \in$ MKX. In other words, we have to show that the symmetric difference

$$a \Delta a^{**} := (\mathbf{C}a \cap a^{**}) \cup (a \cap \mathbf{C}a^{**})$$

is an element of NDX, i.e., $intcl((\mathbf{C}a \cap a^{**}) \cup (a \cap \mathbf{C}a^{**}) = 0$. For this it is sufficient to prove that

$$cl(intcl((\mathbf{C}a \cap a^{**}) \cup (a \cap \mathbf{C}a^{**})) = 0$$

and

$$cl(int(cl((\mathbf{C}a \cap a^{**}) \cup cl((a \cap \mathbf{C}a^{**})) = 0$$

Hence is equivalent with proving that $cl(int(cl((\mathbf{C}a \cap a^{**}) = 0$ and $cl(intcl(a \cap \mathbf{C}a^{**})) = 0$. Let us deal with the first factor. Evidently it suffices to prove that $int(cl((\mathbf{C}a \cap a^{**}) = 0$. Since $a^{**} \in ROX$ (2.5) can be applied and one obtains

$$int(cl((\mathbf{C}a \cap a^{**}) = intcl\mathbf{C}a \cap a^{**} = (\mathbf{C}a)^{**} \cap a^{**} = \mathbf{C}clinta \cap intcla$$

Since $a \in MKX$ one one has $intcla \subseteq clinta \Leftrightarrow \mathbf{C}clinta \subseteq \mathbf{C}intcla$. This entails

$$\mathbf{C}clinta \cap intcla \subseteq \mathbf{C}intcla \cap intcla = 0.$$

The second factor $cl(intcl(a \cap \mathbf{C}a^{**}))$ is treated as follows. Since $a \in MKX$ one has $a \cap \mathbf{C}a^{**} \in MKX$. Thus we obtain $cl(intcl(a \cap \mathbf{C}a^{**})) \subseteq cl(clint(a \cap \mathbf{C}a^{**})) = clint(a \cap \mathbf{C}a^{**})) = cl(int(a) \cap int(\mathbf{C}a^{**}))$. Now

$$
\begin{aligned}
int(a) \cap int(\mathbf{C}a^{**})) = 0 \quad &\Leftrightarrow \quad int(a) \cap \mathbf{C}cl\mathbf{C}\mathbf{C}intcl(a) = 0 \\
&\Leftrightarrow \quad int(a) \cap \mathbf{C}clintcl(a) = 0 \\
&\Leftrightarrow \quad int(a) \subseteq clintcl(a).
\end{aligned}
$$

This last inclusion, however, is obviously valid. Hence every equivalence class of MKX/NDX contains at least one regular open element a^{**}. In other words, the map $ROX \xrightarrow{p \bullet i} MKX/NDX$ is an epimorphism.

Now let us show that this map is a monomorphism by proving that every equivalence class of MKX/NDX contains exactly one element of ROX. Suppose that is not the case. Then there are two different $a, b \in ROX$ with symmetric difference $a \Delta b$ that is nowhere dense, i.e. $intcl(a \Delta b) = 0$. Since a and b are different elements of ROX, either $a \wedge b^* \neq 0$ or $b \wedge a^* \neq 0$. Assume $a \wedge b^* \neq 0$. Then $0 \neq a \wedge b^* = a \cap int\mathbf{C}b \subseteq a \cap \mathbf{C}b \subseteq a \Delta b = 0$. This is a contradiction. Hence there is exactly one $a^{**} \in ROX$ in every equivalence class of MKX/NDX, i.e., we get a 1-1 order-preserving isomorphism $MKX/NDX \xrightarrow{\rho} ROX$ with $\rho([a]) := a^{**} = intcl(a)$. Since order-preserving isomorphisms of lattices are lattice isomorphisms (Davey and Priestley, Lemma 5.9) we are done.♦

Theorem (2.4) relates the Boolean modal algebra MKX in a global way to other, better-known „topological“ lattice structures such as ROX, NDX, BOX, and MGX. This does not mean, however, that thereby all questions concerning MKX can already be answered by looking at these structures alone. An interesting example is provided by the problem of of the resolvability of McKinsey algebras (and McKinsey spaces (cf. Bezhanishvili, Esakia, and Gabelaia (2005, p. 326). First, let us recall the pertinent definition (cf. Esakia (2011, p.50)):

(4.2) Definition. Let (B, ≤, cl) be a Boolean closure algebra. An element $0 \neq e \in B$ is resolvable if and only if there are non-zero elements a, b ∈ B with $a \wedge b = 0$, and a, b ≤ e and e ≤ cl(a), cl(b). If no element of B is resoluble, the algebra B is also said to have the Hewett property (cf. Esakia 2011, p. 50).♦

A corresponding definition of resolubility may be given for topological spaces. By definition Polish topological spaces such as the real line **R** and the Cantor space C are resolvable. Probably the best-known couple that demonstrates the resolvability of **R** are the sets **Q** and **CQ** of rational and irrational numbers, respectively. Already in the introduction of this paper it was pointed out that **Q** is not a member of MK**R**. Hence one may ask, whether the McKinsey algebras MKX of Polish spaces X contain resolulble elements at all. The following lemma gives a partial answer:

(4.3) Lemma. Resoluble elements of a McKinsey algebra MKX are nowhere dense.

Proof. Assume e ∈ MKX to be resoluble, i.e. there are a, b ∈ MKX with $a \cap b = 0$, $a, b \subseteq e$ and $e \subseteq cl(a)$, $e \subseteq cl(b)$. Since cl is a topological closure operator one obtains $cl(e) \subseteq cl(a) \cap cl(b)$. But $a \cap b = 0$ entails that $cl(e) \subseteq cl(a) \wedge cl(b) \subseteq cl(a) \wedge cl(\mathbf{C}a) = bd(a)$ and therefore $intcl(e) \leq int(cl(a)) = 0$, since a ∈ MKX. Hence, all resoluble elements of MKX, if there are any, are nowhere dense.♦

The following example shows that many MKX algebras indeed contain resoluable elements:

(4.4). Examples. (a) Let $\mathbf{R}^2$ be the Euclidean plane endowed with the standard topology. Then MK$\mathbf{R}^2$ contains a profusion of nowhere dense resoluble elements. Embed the real line **R** into $\mathbf{R}^2$ by the map **R** ——i——>$\mathbf{R}^2$ defined as r———>(r, 0). Then the sets i(**Q**) and i(**CQ**) are a resolution of (**R**). Even for the real line **R** itself one may find resoluble elments by observing that there is a partition of the Cantor set C into two disjoint sets a and b the closures of which cover C. As is well-known, the Cantor set C is nowhere dense in **R**.♦

In order to show that there are at least some McKinsey algebras MKX that do not contain any resolvable we need two further definitions:

<u>(4.5) Definition (Njastad 1965)</u>. Let (X, τ) be a topological space and NDX its ideal of nowhere dense elements. Then a new topology $\tau(\alpha)$ on X, the so-called the α-topology, is defined by the base:

$$B(\alpha) := \{a - n;\ a \in OX \text{ and } n \in NDX).♦$$

Clearly, the topology $\tau(\alpha)$ defined by $B(\alpha)$ on X is finer than the original topology τ. Further, all nowhere dense sets of (X, $\tau(\alpha)$) are closed, i.e., $NDX(\alpha) \subseteq CX(\alpha)$. Moreover, the Boolean lattice $ROX(\alpha)$ of regular open sets of (X, $\tau(\alpha)$) coincides with the Boolean lattice ROX of regular open sets of (X, α).♦[5]

<u>(4.6). Definition (GRZ Algebras (Esakia 2011, p. 49)</u>. Let (B, cl) be a Boolean closure algebra. For $a \in B$ define

$$p(a) := a \cap \mathbf{C}cl(cl(a) \cap \mathbf{C}a).$$

B satisfies the GRZ-axiom if and only if for all $a \in B$ $cl(p(a)) = cl(a)$.♦

<u>(4.7) Lemma</u>. Let MKX be the McKinsey of a topological space endowed with an α-topology. Then MKX satisfies the GRZ-axiom.

<u>Proof</u>. Assume $a \in MKX$ and observe that $cl(cl(a) \cap \mathbf{C}a) \subseteq cl(cl(a) \cap cl(\mathbf{C}a)) = bd(a)$. Since $a \in MKX$ bd(a) is nowhere dense and therefore $(cl(a) \cap \mathbf{C}a)$ as well. The topology of X is an α-topology, hence by (4.3) $(cl(a) \cap \mathbf{C}a)$ is closed, i.e., $cl(cl(a) \cap \mathbf{C}a) = cl(a) \cap \mathbf{C}a)$. Since $cl(a) \cap \mathbf{C}a) \subseteq \mathbf{C}a)$ and therefore $a = \mathbf{CC}a \subseteq \mathbf{C}(cl(a) \cap \mathbf{C}a)$. Thus one calculates $p(a) = a$ and therefore $cl(p(a)) = cl(a)$, i.e., MKX satisfies the GRZ-axiom.♦

<u>(4.8) Corollary</u>. Let $MKX(\alpha)$ the McKinsey algebra of the α-topological space (X, $\tau(\alpha)$). Then $MKX(\alpha)$ has the Hewett property, i.e., no element of $MKX(\alpha)$ is resoluble.

[5] In (Bezhanishvili, Esakia, Gabelaia, 2005) spaces with an α-topology are called „nodec spaces“.

Proof. Esakia proved that for a closure algebra B the validity of the GRZ axiom and the Hewett property are equivalent (cf. Esakia (2011, Theorem 1, p. 50)).♦

(4.9) Corollary. Let X be an α-topological space (X, $\tau(\alpha)$). Then MKX is a subalgebra of the Borel algebra BOX.

Proof. By (2.4) every element of MKX is a symmetric difference a Δ n, a $\in$ ROX and n $\in$ NDX. Since the topology of X is an α-topology this entails that n as a nowhere dense set is closed and therefore an element of BOX. Hence a $\in$ BOX.♦

5. Concluding Remarks. Since the trail-blazing work of Tarski and McKinsey topological considerations have obtained an ever-growing importance for modal logics. Some years ago Esakia (2004) distinguished between three different frameworks based on the different algebraic structures (lattices) to which a topological space X gives rise (cf. Esakia (2004, 156), in our notation):

(1)	OX	=	The algebra of open sets.
(2)	(PX, cl)	=	Closure algebra.
(3)	(PX, d)	=	Derivative algebra.

(1) and (2) have been treated in sufficient detail in this paper. The derivative algebra (PX, d) however, has not been mentiioned at all. Hence the following very succinct recapitulation of this concept may be in order. Recall that for A $\in$ PX the derivative d(A) of A is the set of limit points of A, x $\in$ X being a limit point of A if for each open neighborhood U(x) of x the set A $\cap$ U(x) – {x} is non-empty. As is well-known cl(A) = A $\cup$ d(A). The operator d satisfies the requirements of a derivative

(5.1) (i) $d(0) = 0$, (ii) $d(a \vee b) = d(a) \vee d(b)$, (iii) $dda \leq a \vee da$.

These three kinds of algebraic systems gives rise to three logical systems:

(1)*	HC	=	The Heyting calculus of intuitionist logic.
(2)*	S4	=	The Lewis modal system.
(3)*	wK4	=	A weak version of the modal system K4.

In an obvious sense the system (PX, cl) may be considered as comprehending the other two systems, i.e., the Heyting calculus HC and the derivative algebra (PX, d) can be defined as specific subsysystems: the Heyting algebra OX of intuitionist logic is defined as the set of open sets of the interior algebra (PX, int), and the Heyting negation a* (a* := int **C**a) and the Heyting implication ⇒ are also defined in terms of the topological operator int and the Boolean connectives of the Boolean algebra PX. Analogous remarks hold for the derivative algebra (PX, d).

Th modal algebra (PX, cl) is not the only possible frame for these systems. Indeed, (PX, cl) is unnececessarily large for this task – everything could have been carried out in the much smaller Boolean subalgebra (MKX, cl) of McKinsey sets (under a very mild restriction). For the Heyting algebra OX this contention is rather obvious and has been explicitly proved in the course of this paper by pointing out that OX ⊆ MX. Proving that the Heyting implication ⇒ can be defined in terms of the topological operator int is routine.

An analogous result can be proved for the derivative algebra (PX, d) by showing that the subalgebra MKX is closed with respect to d. Actually, this can be done only under some mild restriction: Recall that a topological space X is a T_d-space if for each x ∈ X there exists an open neighborhood U(x) of x such that {x} is closed in U(x). Equivalently, X is a T_d-space iff dd(A) ⊆ d(A).

<u>(5.1) Lemma</u>. Let X be a T_d-space. Then for all A ∈ PX the derivative set d(A) ∈ PX.

<u>Proof</u>. We prove that the complement **C**d(A) is open. Suppose x ∉ d(A). By definition this is the case if there is an open neighborhood U(x) of x such that U(x) – {x} ∩ A = Ø. Since X is a T_d-space there is a neighborhood V(x) such that {x} is closed in V(x), i.e. V(x) – {x} is open. Hence {x} is closed in U(x) ∩ V(x) := W(x). Clearly, W(x) is an open neighborhood of x such that W(x) – {x} ∩ A = Ø. Hence the set-theoretical complement of d(A) is open, and therefore d(A) is closed.♦

<u>(5.2) Corollary</u>. (MKX, d) is a derivative subalgebra of (PX, d).♦

The Cantor derivative d is not, however, the only derivative operator defined on MKX. It is easy to show that also the boundary bd operates on MKX as a kind of a rather special derivative:

<u>(5.3) Proposition</u>. (MKX, bd) is a kind of derivative algebra with respect to the boundary operator bd. On MKX, bd satisfies the following conditions:

(i) bd(0) = 0.

(ii) bd(a) = bd(**C**a).

(iii) bd(a) ∪ bd(b) = bd(a ∪ b) ∪ bd(a ∩ b) ∪ bd(a) ∩ bd(b).

(iv) a ⊆ b ⇒ bd(a) ⊆ bd(b) ∪ b

(v) bd(bd(a)) = bd(a).

Proof. By definition bd(a) is closed for all a ∈ PX. Hence bd operates on MKX. In order to show that (iv) holds it is sufficient to observe that bd(bd(a)) = cl(bd(a)) ∩ cl(**C**bd(a). But cl(**C**bd(a)) = **C**int**C** **C**bd(a)) = **C** intbd(a) = 1, since a ∈ MKX.♦

In sum, the McKinsey algebras MKX have a sufficiently rich structure to serve as a convenient algebraic framework for a variety of topological concepts dealing with modal logics such as S4 and S4.1 and similar systems.

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