The Case Against Infinity

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INTRODUCTION

Infinity is a profound notion, holding a powerful emotional appeal throughout mathematics, science, cosmology, and even everyday discourse. But while infinity is certainly a popular notion, it is simply taken for granted that infinity makes sense, merely assumed that infinity is a logically coherent concept. I argue that such an assumption is unwarranted; as it is traditionally defined in mathematics, infinity does not make sense—it's a logical absurdity and should be rejected as such.[1]

The reason infinity is often assumed to make sense is because both the noun "infinity" and its adjective "infinite" can be used to make grammatically correct sentences, and notions of infinity and infinite amounts can also be symbolized for making structurally correct equations. However, not all grammatically correct sentences make logical sense, and not all structurally correct equations do either. For example, consider this sentence: "The sound of one hand clapping is orange." That's a grammatically correct sentence, but it is meaningless if taken literally. Then too, consider $0/0 - 1 = 0/0 - 2$, a structurally correct equation which is also meaningless since zero divided by zero is mathematically undefined. Not all grammatically correct sentences or structurally correct equations are logically meaningful. Even worse, there are some formally correct sentences and equations that only *seem* to make sense because it is taken for granted that their meanings are coherent when, in reality, they don't actually make sense at all. I argue that expressions about "infinity" or "the infinite" are like these nonsensical sentences and equations. The traditional notion of infinity is pseudo-rational—it doesn't really make sense at all; rather, it only seems to make sense when its users assume that it's a meaningful notion.

If, to the contrary, infinity is self-contradictory, like postulating the existence of a black cat that is entirely white, then infinity (at least, in its traditional understanding) is logically absurd. I propose that such is the case.

I realize this view demands a lot of proof. After surveying the opinions of mathematicians, it seems that most of them dismiss skepticism of infinity, and some even assert that no concept in mathematics could be any clearer or more consistent.[2] I aim to show the opposite to be true.

I'll first argue against the traditional notion of infinity as a mathematical value, proving that infinity's standard definitions are logically self-contradictory when they aren't too vague to be meaningful. This forces the realization that no mathematical system is logically consistent when it uses infinity as a mathematical value. Because the mathematical definitions for infinity are self-contradictory, the use of infinity as a value in calculation should be abandoned as irrational in favor of a more logically coherent alternative.

Since no mathematical system would be complete unless it replaced infinity as a value with a more coherent concept, I propose that the traditional notion of infinity should be replaced with a concept called "indefiniteness" or "the indefinite."[3] I will further show that indefiniteness is not simply conceptual sleight-of-hand; it really is different from infinity in that it avoids the logical contradictions inherent in the notion of infinity but while still retaining the mathematical utility that infinity has. Indeed, to throw out the notion of infinity as a mathematical value in favor of indefiniteness would not lose anything significant to the field of mathematics; to the contrary, it would actually make mathematics even more logically consistent than it is now with the notion of infinity still in use. And so I will offer indefiniteness as the new mathematical value that ought to be used in place of infinity.

Now the logical failure of the traditional notion of infinity, and the necessity of replacing infinity as a mathematical value with indefiniteness, carries some serious implications, especially for physics and cosmology: If infinity as a mathematical value is logically absurd, and if logical absurdities refer to nothing that can actually exist, then infinity as a mathematical value refers to nothing that really exists—at least, not according to the traditional definition of infinity. As a result, measures of space and time cannot really be "infinite" in the usual sense of the term. I conclude that this is exactly the case—no matter how *indefinite* the vastness of space or time may be, the Universe as a whole must still be finite.

FINITE

Beginning with the Finite

If a logically coherent and meaningful definition for the traditional understanding of infinity can be found, then my case against infinity won't be compelling; but if it can't be found, my case will stand. To find out which is so, let's start by examining some definitions and seeing where they lead. We will begin this inquiry into infinity by exploring definitions for its related terms, such as "finite" and "finitude."

The word "finite" comes from the Latin words "finitus" (which means to end, bound, or limit) and "finis" (end or limit). In short, if something was regarded as finite, it was thought to have a bound, end, or limit to its measure.[4] The corresponding term "finitude" means simply the quality or condition of being finite. There is nothing illogical about these etymological definitions.

However, there is one caveat: while that which has a boundary or end is indeed finite, something can be finite and yet not have an end or boundary. All bounded things are finite, but not all finite things are bounded. Though that might sound paradoxical at first, a simple example should clear it up: consider a circle. In fact, take any figure, set, series, process, or system in which the sequence of elements within it curves back on itself—such as the circumference of a circle or sphere. The circumference has no end points or boundaries and yet is still finite in the sense of being limited in size or scope. A circumference is said to be limited but unbounded. So, I take the ordinary meaning of finite as referring to that which is *limited*, but not necessarily having an end or boundary, since there are extents like the circumference of a circle that have no end points or boundaries (edges) and yet are still finite in the sense of having limits in size or scope. In ordinary language then, to be finite is to be limited.

Now let's get a little more technical.

Finite—Complete and Incomplete

In mathematics, to be finite is to be *limited in quantity*. By "quantity" I mean to have a positive or negative numerical value. If something is mathematically finite, it is measurable by a positive or negative numerical value that is limited. This is a rather broad way to define finitude, but as we'll see, it is the right definition for mathematics.

Unfortunately, it isn't always the definition used in mathematics. Mathematicians sometimes say that to be finite is to have a limit in reference to the scale of *natural numbers*. Natural numbers are usually identified with positive integers such as 1, 2, 3, 4, 5, etc. Sometimes mathematicians include 0 in the set of natural numbers, in which case the natural numbers refer to all numbers that are non-negative integers. Other mathematicians refer to only the set of positive integers as the natural numbers, while the set of natural numbers together with zero constitutes the set of *whole numbers*. For the sake of clarity, I will take the latter position, identifying natural numbers as the set of positive integers and whole numbers as the set of natural numbers with zero included. With these distinctions in place, one of the usual mathematical definitions of the word “finite” is to have a number of elements (for example, objects) capable of being put into a one-to-one correspondence with a segment of natural numbers.[5]

A “one-to-one correspondence” refers to the ordered matching of members between two sets. The members of one set (which we’ll call A) are evenly matched with the members of a second set (we’ll call it B). As mathematician J.R. Maddocks explains,

> Evenly matched means that each member of A is paired with one and only one member of B, each member of B is paired with one and only one member of A, and none of the members from either set are left unpaired. The result is that every member of A is paired with exactly one member of B and every member of B is paired with exactly one member of A. In terms of ordered pairs (*a*, *b*), where *a* is a member of A and *b* is a member of B, no two ordered pairs created by this matching process have the same first element and no two have the same second element. When this type of matching can be shown to exist, mathematicians say that "a one-to-one correspondence exists between the sets A and B.” [6]

As a simple example, take a set of five apples that can be numbered completely with the segment of natural numbers 1 through 5. By counting the apples—one, two, three, four, five apples—we put the apples and the positive integers (which, in this case, represent a segment of natural numbers) into a one-to-one correspondence. To be “finite” in this sense, then, is to be able to be put into just such a correspondence. In this sense of the term, to be finite is for a quantity or set to be either (a) *countable* using a terminating sequence of natural numbers or (b) accurately *designated* as belonging to a terminating sequence of natural numbers.[7] The example of counting the apples is a case of the former and just labeling the apples one through five would be a case of the latter.

These mathematical definitions of what it means to be finite imply that the scale of natural numbers *contains* sequences of numbers that terminate when they are selected as subsets taken from the entire scale, but the definitions leave vague whether or not the scale itself has an end and so is limited. If the scale itself has an end, then it is finite. If the scale as such is somehow not terminating, then it may not be finite—it could be *infinite*.

Let’s consider the first option—that the scale of natural numbers has an end. Such cannot be true. After all, for any number in the scale, you can always calculate one number higher. If you can always go higher, then you can go higher *without end*. The scale of natural numbers has no end. But does that mean it is infinite?

That brings us to the second option: it might be assumed that since the scale of natural numbers is without end, it surely has no limit at all and so *must* be infinite. Ah, but not so fast! Perhaps the scale of natural numbers has no end, but that doesn’t mean the scale of natural number has *no limit*.

There is a difference between being endless and being limitless. As we’ve seen, a circle’s circumference is endless (there are no end points) but it is also limited, not limitless—one circle can fit inside another like the rings on a target, and the circumference of a circle can be expressed by a rational number. Hence, to be unending is not necessarily the same thing as to be without limit, and so to be unending is not necessarily the same as being *infinite*. As a circle’s circumference is endless but not limitless, so too the natural number scale may be endless but not limitless. Therefore, the scale of natural numbers may be endless, but that doesn't necessarily mean it is "endless" in the sense of being *infinite*.

On the other hand, while the natural number scale could be endless without also being limitless or "infinite," it also does not need to be endless by being unbounded like a circle. Instead, perhaps the natural number scale should be thought of as "endless" merely in the sense of being always *incomplete*.

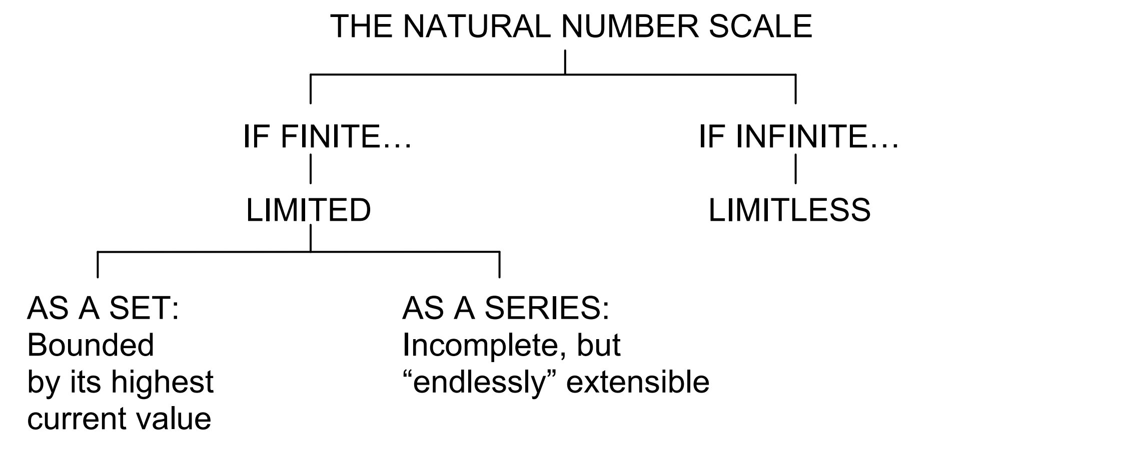

FIGURE 1

The best way to grasp this view of an endless-but-limited natural number scale is by an analogy that distinguishes between *sets* and *series*. A set is a collection of distinct things that exist all at the same time (like the set of inner planets: Mercury, Venus, Earth, and Mars) while a series is the sum of a sequence of things that have existed over time (such as a series denoting the sequence of all species that have ever lived on Earth). With this distinction in mind (also see **Figure 1**), we're ready for an analogy that expresses why the natural number scale can be "endless" but also limited or finite:

Imagine a round tower built of bricks. The bricks spiral upward from the base to the top of the tower. If the top of the tower is capped by a rooftop, we'll call the tower *complete*—it cannot be made any taller. The capped tower then has an end (and so a limit) to its height; it is finite. But suppose the tower has no roof and instead has merely an unfinished sequence of brickwork at the top. If construction has ceased, the tower stands as an *incomplete* set of bricks but is finite nonetheless. Considered just as a set of bricks, the tower can be complete or incomplete—limited in either case.

Now let's consider the tower as a series rather than as just a set. Suppose that construction has not stopped and will never stop. For as long as there is time the tower will always be made ever taller. In that case, because it is perpetually under construction, being made ever taller, the tower always remains incomplete as well—it is never finished. It is an ongoing process, a series of bricks being laid "without end." But even though the top of the tower is a sequence of brickwork that continually grows higher, the tower always remains finite because it always has a limited number of bricks from its base to its top at any time. No matter how many bricks are added, no matter how high the tower becomes, there is always a countable number of steps to the top.

By analogy, the sequence of numbers in the scale of natural numbers is like the increasing brickwork in the perpetually heightening tower. Since there is no “roof” to the scale of natural numbers—no highest number beyond which we cannot continue to extrapolate—the scale of natural numbers is never complete. Instead, the scale of natural numbers is like the tower that is never finished—there is always one more brick that could be laid, always one more number that could be calculated, but never a last one. Just as with the growing tower that is always incomplete, so too the natural number scale can be continually extended but will always be incomplete. And because both the fictional tower and natural number scale are inherently incomplete, they are both “endless” processes under construction.

Moreover, although they are both endless as processes with series of steps, the tower and the scale of natural numbers are always finite at any given time. There are a limited number of steps to the highest brick in the tower, and there are a limited number of values to the highest number in the scale of natural numbers at any given moment. Whatever the distance the highest brick is to the base of the tower or highest number in the scale of natural numbers is to the number 1, there are a finite number of steps to the top at any time; no matter how long we go on building the tower or inventing new numerical values, the steps are always countable and there is a highest definite number. Both are processes containing series of steps “without end,” but both are also limited as sets of elements, and so finite. As a perpetually constructing tower remains always unfinished but nevertheless finite, so the scale of natural numbers remains always incomplete but finite.

Put another way, the natural number scale could be construed as “terminating” wherever the sequence of natural numbers currently leaves off—it would be finite in that regard since there is always a highest number in the set that has been or that can be calculated for it at any given time. And the scale would also be “endless” as a series because its limit can always be *extended* as long as there is time to do so and no final values have been or will be calculated. Thus, having a limit that is “endlessly” extensible in this way merely means that the natural number scale is inherently and always *incomplete*, not that it is infinite.

If this analysis is correct, then finitude could be a property describing the entire scale of natural numbers. To be finite would not, as traditional mathematics has it, to be able to be put into a one-to-one correspondence with just a *segment* of natural numbers. No, finitude would also be a property of the scale of natural numbers taken as a set at any given time and as an incomplete series. That is not to say, of course, that segments of the natural number scale are not finite—far from it! A segment of five numbers is definitely finite. It is merely to say that the term “finite” should not be defined in too narrow of a sense as corresponding *only* to segments of natural numbers; it can also be applied to the scale of natural numbers itself.

Finite—Definite and Indefinite

So far we’ve seen finite sets and series. We’ve also seen how they can each be either complete or incomplete. Since there are real finite towers that are roofed and finished while others are left unfinished or in ruins, we can say that the property of finitude contains both complete and incomplete sets. And since scales of numbers are extended by processes of calculation over time in which new values arise in a series, we can also say finitude encompasses complete and incomplete series.

But now completeness and incompleteness also need to be distinguished from a couple of related categories—*definiteness* and *indefiniteness*. Along with being complete or incomplete then, a finite set or series can also be “definite” or “indefinite.”

At this point I should acknowledge that some philosophers distinguish three categories of numeric values or collections: the finite, the infinite, and the indefinite.[8] I am taking a different approach. The indefinite is not a separate category that stands *alongside* the quantitative categories of the finite and the infinite as something different and outside those two categories together. Instead, the indefinite is a subcategory of the finite along with another subcategory—the definite.

> The *definite* is that which the form and limits are determined and are apprehended by us. That of which we know not the limits, [fallaciously] comes to be regarded as having none; and hence *indefinite* has been confounded with the *infinite*, though these two must be carefully distinguished. The *infinite* is absolute; it is that which has and can have no limit; the *indefinite* is that of which the limits are not known to us. You can suppose it enlarged or diminished, but still it is finite.[9]

Along with the complete/incomplete distinctions then, the finite also contains two additional subcategories: the definite and the indefinite (**Figure 2**). Since the infinite is the category opposite to the finite, as a quantitative category it is not yet clear in our analysis if categories such complete/ incomplete or definite/indefinite even apply to it, but this will be become clearer as we proceed.

These distinctions become important when we analyze the scale of natural numbers. Any known segment of the natural number scale is composed of finite values that are definite (defined or determined). But suppose the natural number scale also contains unknown finite values that are indefinite (undefined or undetermined).

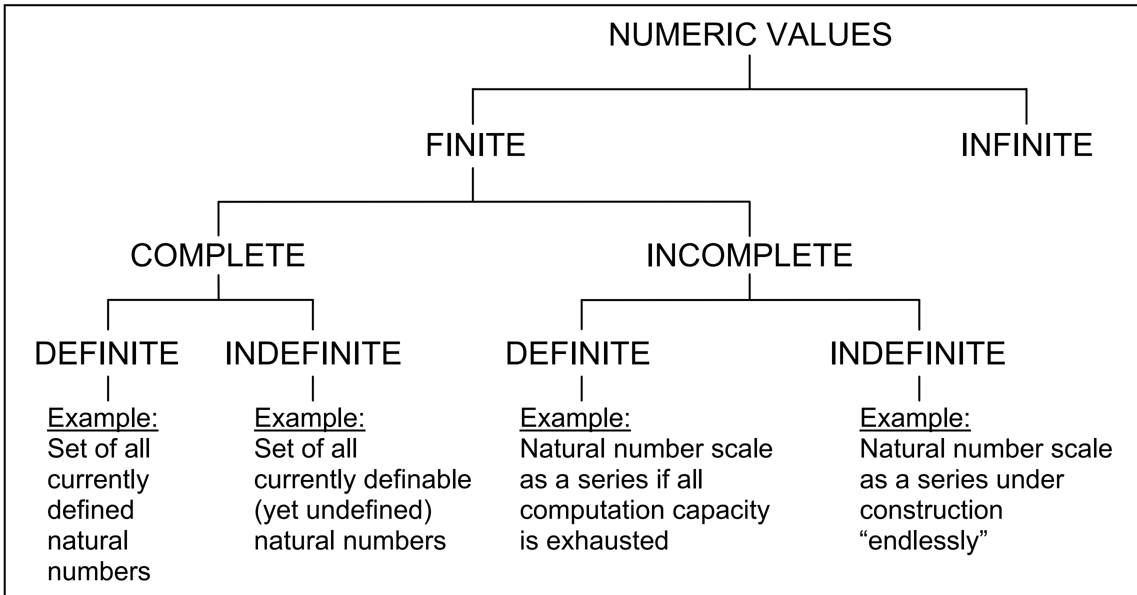

FIGURE 2

There are finite numbers no one has ever calculated and that cannot currently be calculated because their values are off any known scale of natural numbers. That is, there are some finite numbers that have not been invented since they either (1) have values vast beyond the largest finite numbers *so far* conceived or (2) also have values beyond the largest finite numbers that can *in practice* be defined, and yet have finite values that are nevertheless definable *in principle*.

Indefinite numeric values are those values off the defined and known scale of natural numbers, but such numbers while not necessarily definable in actual practice are still definable "in principle"—if only we just had the means (say, a more powerful computer), we could have defined these higher finite numbers in the set until if/when we have exhausted all our current computing ability to go higher.[10] Such numbers are still finite rather than "infinite" because their values would have limits if they could be calculated; it's simply that those limits would be higher than any value anyone can now *actually* calculate. You can think of these unknown values as potentials latent in the scale of natural numbers—they are reachable in principle even if they cannot be reached in practice. Because these numbers are still finite, it isn't the term "infinite" that describes them; instead, they are best referred to as simply *indefinite*.

To get a clearer idea of what a finite-but-indefinite value would be, consider an example from the classic novel *Watership Down*, by Richard Adams. In the novel, rabbits have their own culture and language. They also have a very primitive mathematics; the rabbits can only count up to the number four. In a footnote, the author explains that to the rabbits any number above four is *hrair*—"a lot."[11] In contrast to Adams's rabbits, we know that four is followed by five and a whole series of higher, but nevertheless finite, numbers. And yet, even our scale of natural numbers, no matter how high the numbers in the scale are calculated, has a *highest* number ever defined up to the present. Perhaps somewhere in the Universe there is an extraterrestrial species with a computing system that can count so fantastically higher than we that our number scales would seem as puny to theirs as the number scale of Adams's rabbits do to ours. Even so, whatever number that anyone, or any being in the Universe, shall ever define as the highest number calculated, such a number and the scale to which it belongs would still be finite wherever the scale leaves off. All the numbers that would lie beyond what anyone knows are simply "a lot"—they go undefined even though they too would be finite if they could be counted.

Now it is true that despite what anyone will ever come up with by calculation or extrapolation (mentally or mechanically) as the highest number conceived, there is nevertheless a logical possibility that, if one only had more computing power, yet another finite number could be defined. But this means only that it is logically possible for there to be numbers that, if they *could* be invented, *would* nevertheless have terminating values and so still be finite. And because this implies limits, albeit extended limits, the scale of natural numbers would remain finite at any step of the way. The natural number scale would have only a finite number of values actually *defined* at any time no matter how far we calculate or extrapolate it.

Looking at the scale of natural numbers this way, we see that there is more than one way for a set of anything to be finite: One way is for the set to be limited because it is both complete and its elements are entirely defined, although that does not describe the natural number scale, which contains defined elements but not all of its elements are defined. Another way is to be limited because the set contains a subset of undefined elements as well as a subset of defined ones, and those undefined elements are identical to the subset of *elements currently definable* within the larger set. However, since those undefined elements are in principle definable, that also means the subset of defined elements could be increased, which in turn means the superset as a whole (containing both defined and undefined elements) must *also* be considered as a *series*. Suppose the currently undefined elements in that series later become defined, but as a result new undefined elements are generated with the potential to be defined. That would make the series inherently incomplete—a work in ongoing progress in which there are always new definable values that later become defined. And being incomplete, the series would "terminate" or stop only where it leaves off at any time according to its defined and definable values. So, the series as such would be "indefinite." This describes the scale of natural numbers.

The natural number scale as a set, then, contains defined values and undefined ("indefinite") values at any given time, while the scale itself is also "indefinite" as a series. When the currently undefined values become defined later on, yet more undefined values are generated as new potentials to be defined. And so the undefined values currently in the natural number scale that lie latent by virtue of being logically possible (but either not invented yet or not conceivable in actual practice) are only undefined or "indefinite" temporarily, and potentially new ones are always created as the defined subset of numbers in the scale increases in length. (This ongoing process is simplified in **Figure 3**, as I have no idea what the highest value calculated to date is on the natural number scale.)

Since this view fits our experience of working with natural numbers, I am proposing that finitude should be regarded as characterizing the scale of natural numbers in this way—as a set and as a series.

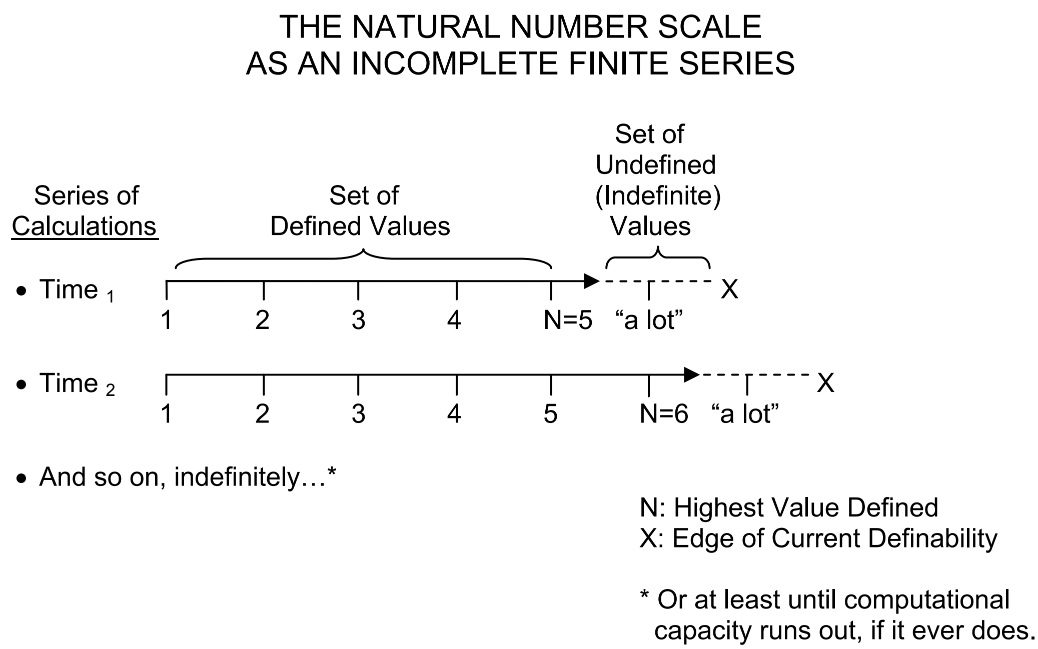

FIGURE 3

Indefinite but still Limited

To briefly sum up on the subject of indefinite finitudes in sets and series:

The indefinite applies to finite sets, but only to those portions of which are currently undefined. As a subset of the finite natural number scale, the indefinite contains those higher values that are undefined in practice, but are potentially definable in principle. That is, on the scale there are always potential values that are undefined or "indefinite" since they are never reached in actual practice. The scale of natural numbers also contains a *highest* known and defined value at the present time as well as a highest unknown but potentially definable value at the present time (hence these current limits also make the scale finite).

So, while at any given time a set of numbers will have a highest value, some sets can be extended in magnitude. For example, the set of all natural numbers may in principle be definable up to a highest value possible right now, but that value could be surpassed when computing power increases and a new possible highest value becomes a potential. Just as Olympic records were meant to be broken, so too the highest values possible now are intended to be overcome by higher possible values later. As computation proceeds from one occasion to the next, topping the last highest number able to be calculated, we see that the indefinite must apply to finite *series*. As already suggested, the natural number scale may be one of these series. If so, defining finite numbers and values for the natural number scale is an ongoing process that is worked out in a series of steps that is never complete, making the series itself indefinite. As a series, the indefinite is both non-zero and has "no end" in the sense that the scale, though finite in sequence at any time, can at least in principle be perpetually added onto with more, limited, values that have not yet been defined.

This assumes, of course, that we never run out of time and computing power, which is disputable. Some computer scientists believe that it may not be physically possible to compute the information content for real numbers beyond 2^{512} bytes of computer memory storage. To get an idea of how large a value that is, 2^{32} bytes may only be 4 gigabytes but 2^{1000} is larger than the number of electrons in the perceivable universe. So, rational numbers having integers composed of 2^{512} digits are outside the range of computable numbers. Even computing the value of a real number equal to 2^{50} bytes worth of storage may not be feasible. If true, these limits would make the scale of natural numbers also explicitly bounded in terms of the how many digits a value can have.[12] Nevertheless, this still leaves a lot of room for defining higher values of natural numbers for the foreseeable future.

Regardless, the indefinite can be either a set or a series—a set that contains undefined numbers and values or a series that is itself incomplete—or it may be considered as both a set in one sense and a series in another. Recall that the tower analogy distinguished between sets and series, but the perpetually unfinished tower of natural numbers was described as indefinite in both senses. The natural number scale could be limited as a set containing indefinite values, while simultaneously able to have more values defined for it without any foreseeable end—hence growing "indefinitely" as a series.

Once again we see this take on the natural number scale implies that there is no need for the natural number scale to be necessarily *infinite*.

DEFINING INFINITUDES

The Infinite and Zero

Now what of infinity? Infinity is the quality or condition of being infinite. The word "infinite" literally means "not finite" or to have no limit. But if that is all there is to the infinite, there would not be much need for discussion; there are some things that have a limit and others that don't. It gets more complicated though, when we stop to ponder what it means for something to be *limitless*.

If something has a limit, it may be quantifiable (at least in principle) depending on what it is. Conversely, if something doesn't have a limit, it could be because that thing has absolutely no quantity to it—being without scale, magnitude, degree, extent, etc. or simply unable to be measured at all with integers from the natural number scale. If there is no quantity to measure, there is certainly no limit to consider. But to have no quantity is to equal zero. Thus, to have *no limit* by virtue of having no quantity is to equal zero.

Zero has "no limit," in the sense of having no quantity and so no positive or negative limits. On the other hand, though zero *has* no limit, it can nevertheless *be* a limit, as when we say that fractions can be divided ever closer to zero *as* a limit. Even so, it is the former aspect of zero—its lack of limit by virtue of lacking quantity—that concerns us here. We saw earlier that we must be careful to distinguish the infinite from the indefinite. As it turns out, if we are to preserve the traditional notion of the infinite, we must also be careful to distinguish the infinite from something else that is "limitless"—zero. To define the infinite as that which is "not finite," only in the sense of having no quantity at all, makes the infinite identical to zero—to be infinite in quantity and zero in number would mean one and the same thing; the infinite would ceases to be distinct from zero as a *mathematical value*.

While zero may be "infinite" in some sense, the traditional notion of *infinity* is supposed to be *other* than zero. Besides, although "infinite" may be a useful term for designating zero's status as having no quantity with a limit, infinity would become redundant as a mathematical value since zero suffices on its own. So, to define the infinite as simply "not finite" in the sense of having no quantity at all would dissolve the usefulness of infinity as a mathematical value for making calculations. Perhaps there is another option for defining the infinite as "not finite."

Mathematically speaking, the infinite has been defined as having a number of elements not capable of being put into a one-to-one correspondence with a segment of natural numbers.[13] Although this definition does not *intend* to identify infinity with zero, it too fails to distinguish itself from zero.

Let's assume we have two sets of numbers: Set A is the set of all natural numbers. Set B is another set of numbers. If Set B cannot be matched with a finite segment of Set A, then Set B is supposed to be "infinite." However, suppose the number of elements in Set B is zero. If so, then no such matching between the two sets can take place; if the elements in the set are zero in number, then there are no elements in the set that are capable of being put into a one-to-one correspondence with a segment of natural numbers. Thus, having elements or objects that are zero in number satisfies this definition of infinite—Set B, the zero set, can't be put into a one-to-one correspondence with a segment of Set A, the set of natural numbers. So, Set B must be infinite, which is absurd because Set B is simply zero and the infinite is supposed to stand *in contrast* to zero either as an incalculably *greater* value or as a value that *tends toward* zero but never reaches zero.

Defining the infinite as simply "not finite" either by lack of quantity or by lack of one-to-one correspondence with natural numbers fails to capture what is traditionally meant by the infinite. The reason it fails is because it is a purely negative definition—it tells us what infinity is not, rather than what it *is* or what it is supposed to be. Yes, to be infinite is to be not finite, but in what way? If the infinite is simply limitless because it lacks quantity or because it can't be put into a one-to-one correspondence with a segment of natural numbers, then it is indistinguishable from zero—a problem for the traditional notion of infinity.

Infinity and the Infinite

Perhaps to be infinite is to be limitless, but not by lacking quantity. Instead, suppose that to be infinite is to have a quantity (positive or negative) but a quantity that has no limit—the infinite is *limitless quantity*.

To understand what is meant by "limitless quantity" requires yet another distinction to be made, a distinction between two kinds of infinitude: *infinity* and *the infinite*. This is a distinction that I cannot put off making from here on. Previously I had noted a definition of infinity as “the condition of being infinite,” which while accurate, is not the full story. That simple definition of infinity doesn’t really reveal the traditional distinction between infinity and the infinite that mathematicians usually make.

Both the infinite and infinity indicate limitlessness in quantity, but each in a different fashion. Mathematicians represent the infinite by the symbol $\aleph_0$ known as aleph-naught, which designates a *limitless set*, while they represent infinity by a “lazy eight” symbol called the *lemniscate* or *Mobius strip*, denoted as ∞. Infinity is not a limitless set—a collection of things that has no limit to its members. Instead, mathematicians sometimes define infinity as “the limit that a function *ƒ* is said to approach at $x = a$ when *ƒ(x)* is larger than any preassigned number for all *x* sufficiently near *a*.”[14] But there is a problem with this definition: the term “infinity” literally means to *be* limitless and yet this definition calls it a limit.[15] So, we have a limitless limit, which is self-contradictory. Worse still, there is really no sense in which you can “approach” or draw “near” infinity since infinity is supposed to be just as far away no matter how long you make progress. The whole idea of infinity is supposed to be that you cannot come close to it, let alone reach it, by going through a series of steps like counting. Consequently, it would be more precise to say that infinity is supposed to be a *limitless sequence*—a sequence that cannot be enumerated no matter how long one counts. So, both the infinite and infinity are “limitless quantity,” but the former as an entire set of things with no limit and the latter as a sequence that can’t be enumerated.

By arguing against the traditional notion of infinity, I am actually arguing against both infinity *and* the infinite—that is, against both infinity as a limitless sequence and against the infinite as a limitless quantitative set. It’s my contention that neither notion of infinitude really makes logical sense.

I will also argue that between our two options for defining infinity or the infinite (first as “not finite” because it lacks quantity and second as “not finite” because it is a limitless quantity), the first is actually the only logical choice, contrary to tradition though it may be. Unless the terms “infinity” and “infinite” refer simply to that which is not finite because it lacks quantity (and so is equivalent to zero), the terms must be taken as limitless quantity, and so taken they become either too vague or lack any logically coherent meaning at all.

Infinity versus the Indefinite

So far we’ve had a hard time distinguishing the “limitlessness” of the traditional notions of infinitude from the limitless aspects of zero, and we’ve also encountered trouble in distinguishing the infinite from the indefinite. We are not yet through these troubles; there is yet another difficulty with distinguishing the infinite from the indefinite.

The infinite has been defined by mathematicians as a quantity that is limitless because its value is *greater* than any terminating sequence of natural numbers.[16] And this mathematical definition in turn implies that infinity is a value that cannot be counted or “a value greater than any computable value.”[17] Though these definitions are meant to distinguish the infinite (and infinity) from indefinite, they still have a few problems making that distinction.

An “indefinite value” could be taken to designate a value in a set that would, in principle, be defined if anyone had the means to reach it even when in actuality they don’t. It *would* fall in any known natural number scale if it could be counted, but because it cannot be counted its finite value is off the scale. A value can also be “indefinite” in a series if it corresponds to the “next number” off the (incomplete) scale of natural numbers—a value too big for anyone to think up or any machine to calculate because you cannot reach the “next number" beyond what can be actually computed in the present. In either set or series then, "the indefinite" designates numeric value that is also "greater than any terminating sequence of natural numbers" in the sense that, though finite, the value would lie just beyond where the known set of natural numbers actually does leave off ("terminate") at any time—so, such value is indefinite.

As we saw earlier, a good example is to suppose no one could ever count higher than 4. In that case, the sequence of natural numbers "terminates" at 4 since no one could count higher. The next number higher than 4 that could be counted if anyone had the ability to count higher (and in reality we know it is a finite number) would have to be labeled as "indefinite" by everyone since they wouldn't be able to count higher. Whatever sequence of natural numbers that could ever in practice be invented, there is always in principle a number or value that is finite (just a little greater), a number that could be invented if only the time or resources existed—that value is not "infinite" since it would have a limit; it is simply undefined or "indefinite" since no one is able to invent it.

Another way to designate the indefinite is to represent the set of natural numbers as 1, 2, 3, 4…*n* where *n* is equal to a finite number that is indefinite (not defined) because it is above the highest number actually defined thus far. Hence, *n* is "greater" than a set of "terminating natural numbers"—numbers that anyone could define or compute, but whatever value *n* could have, it is still finite even if its limit is not defined (given a specified value) as the limits of values calculated from the former numbers in the scale are defined.

Notice that the indefinite fits the very definition of what is supposed to be infinity. And this in turn implies that the indefinite, like infinity, is also "greater than any computable value" because indefinite values can also be too vast to count in actual practice. If indefinite values can be defined as greater than any computable value, then this makes what was supposed to be infinite indistinguishable from what is actually indefinite. Indefiniteness and infinity become conflated.

This conflation is a problem because infinity is not supposed to be the same thing as the indefinite: to be infinite is to be *not finite*; to be indefinite is to be *undefined* in value but still finite in principle. Unlike the indefinite, infinity doesn't have a limit that is beyond what is actually calculable; rather, infinity has *no limit at all*. That's supposed to be the difference. And this difference indicates that we cannot mathematically define infinity as we did in terms of being simply incomputable or as being greater than any terminating sequence of defined natural numbers. If we do, we have no way of distinguishing infinity and the indefinite since both have values that are off any actually defined scale.

This conflation of mathematical definitions is also what I meant when I said that infinity, as limitless quantity, becomes too vague of a notion unless it is more strictly defined. There are, however, other mathematical features that infinitude is supposed to have that better distinguish it from the concept of indefiniteness. What remains to be seen is whether or not those features make logical sense.

The Infinite—Complete or Incomplete?

If something has a quantity, it is either a set containing some number of elements or a series containing some number of steps in sequence. The infinite is supposed to be quantitative, and so it is supposedly a set or series of elements.

Sets and series can also be either complete or incomplete, so we can also inquire whether or not something allegedly infinite is complete or incomplete. If the infinite is a set, it can be a complete set or an incomplete set. If it is a series, it could be a complete series by reaching an end (and so have a complete quantity) or it could remain incomplete by never being brought to an end (and so be perpetually incomplete in quantity). Those are the possibilities.

As illustrated by the earlier tower analogy, these categories are certainly true of finite things—the tower could be complete or incomplete as a set or as a series of arranged bricks. If complete as a set, the tower is finished and is made no higher. There are many complete towers in the world. Alternatively, if the tower is incomplete, it could be incomplete as a set of bricks and just remain so without further construction. Either way, the tower would be finite as a set.

The tower could also be considered as a series of bricks being laid one after another. If it is a complete series, the bricks are laid in sequence until the tower is finished. If it is incomplete as a series, the tower is perhaps in the process of being constructed and so has yet to be completed. But we could also suppose that the tower could, in a logically possible (yet still fictional) world, remain incomplete as an *ongoing* series—a tower undergoing a further, and in the analogy perpetual, process of construction in which bricks are laid in sequence without end. And yet, even though construction may never end, the tower would always remain finite given its starting point.

That last possibility fits the concept of a finite, but indefinitely tall, tower being built. But consider a tower that is supposed to be *infinite*. If the tower is infinite, we have the same four options: it could be infinite as a complete set or a complete series, or as an incomplete set or an incomplete series. However, the infinite as either an incomplete set or an incomplete series would seem to be the same thing as being a finite but *indefinite* set or series, and so a kind of misnomer. Instead, we will consider only the traditional notions of the infinite as a complete set or series.

Whether set or series, an infinite tower must have a *limitless quantity* of bricks to it. We will see whether such a condition is logically possible. First, let's turn to the notion of the infinite as a complete-limitless set. Later, we'll consider whether or not the infinite could be a complete-limitless series.

The Infinite as Complete and Limitless

A traditional notion of the infinite is that it is supposed to be a non-zero set that has no limit in the sense of having no highest definable value *even in principle*. So, if the scale of natural numbers were infinite, rather than indefinite, then even if you had the means to calculate higher values on the number scale than those that can actually be defined now or ever, you still would never reach a value you could call the highest value of all. Infinity, as a limitless sequence within the set, cannot be reached because the infinite (the set as a whole) is inexhaustible.

But the infinite is not just limitless; the term "infinite" is also meant to describe any set that is *complete* as well as limitless, while in contrast the indefinite is that which is limited where it is left *incomplete* or simply *undefined*. Unlike the indefinite, the infinite is often stated to be complete as a set of definite things, while those things are also supposed to be limitless in quantity.

As we shall see, the very notion of the infinite as both complete and limitless makes the infinite an inherently self-contradictory notion. A complete-limitless thing is an oxymoron. I'll show that this is exactly the case with a demonstration of how assuming that the infinite is both a complete set and simultaneously also limitless set creates contradictions.

TRANSFINITE CALCULATIONS

Computing Incomputable Infinities

If the infinite is a complete-limitless set, then one should be able to conceive of the possible existence of various complete sets of things that are limitless in quantity. For example, if we can conceive of an infinite number of things, then we should be able to conceive of an infinite number of integers (numbers that are not fractions) and there should be conceivable various infinite sets of integers or numbers. One could have an infinite set of natural numbers (1, 2, 3, 4, 5, 6, 7, 8, 9…) and also an infinite set of even natural numbers (2, 4, 6, 8…).

But here a contradiction creeps in. Because in the first segment of all natural numbers there are more integers between 1 and 9 (namely, the integers 2, 3, 4, 5, 6, 7, 8) than in the second segment of even natural numbers (which contains just 2, 4, 6, 8 between 1 and 9), then it would seem that the set of all natural numbers must contain more integers than the set of all *even* natural numbers—no matter how large the two sets are calculated. Taking a look at these two sets of numbers, the entire set of natural numbers appears to be bigger than the entire set of even natural numbers because the set of even natural numbers is just a subset of the natural numbers. It appears that if the two sets are infinite, one infinite set should be bigger than the other. This suggests that two infinite sets can come in different "sizes." However, if infinity is not computable because it is *greater than any terminating number*, then one infinite amount cannot be said to be "bigger" or "smaller" than any other infinite amount. Different in size, yet the same in size—a contradiction.

Worse still, if the two sets are different sizes, we then ought to be able to compare the relative sizes of the infinite sets. And in making a comparison of their sizes, seeing that one set should be larger, we should then be able to measure and compute their size differences. That is, because the two infinite sets seem to have relative sizes, we should be able to add them together, employing an arithmetic of infinities (infinite sets). But that is something we should not be able to do; these sets of natural numbers, when carried to infinity, are supposed to have no *computable value*. An inability to make computations is what is supposed to define the infinite; to be computable is only a feature of finitude. Because they seem to be computable, they cannot be infinite, but that is what they are supposed to be. So, we now have sets of infinite (incomputable) amounts that must nevertheless be computable, making infinity a self-contradictory notion.

But suppose this is merely a paradox (the illusion of a contradiction) and not an actual contradiction. There are two ways we might resolve the paradox.

One way to solve the paradox is to suggest that infinities can be computed but they don't really have "values" since mathematical values are numbers and no number is equal to infinity. So, it's still true that the infinite is greater than any computable *value*. While that answer follows the previous definitions to the letter, it is actually a disingenuous dodge since the term "value" is defined in more than one way. Another mathematical definition of value is "an assignment of a significance to a variable"[18] as when the variable x is equal to ∞. So, in traditional mathematics, infinity can indeed be considered a value. This in turn means that for infinite sets of numbers, there must be an infinite number of values—which compels us to conclude that there must be "infinite values" after all. So, this way out of the paradox won't work.

The second suggestion for resolving the paradox is more common: it might be proposed that a given infinite set cannot be said to be larger than another infinite set; it is merely that the two sets have the *illusion* of being different in size. Such an argument has already been offered in the field of mathematics.

Mathematicians realized that the set of natural numbers and the set of even natural numbers can be brought into a one-to-one correspondence with each other through a mapping that matches 1 to 2, 2 to 4, 3 to 6, 4 to 8, and so forth:

1	2	3	4…
↕	↕	↕	↕
2	4	6	8…

In the 1870s, the mathematician George Cantor proposed that this one-to-one correspondence means that two paired-off sets, such as the set of natural numbers and the set of even natural numbers, when extended *infinitely* are really to be defined as "the same size" even though comparing finite segments taken from the two sets gives the illusion that the whole of one infinite set is larger than the whole of the other infinite set.[19]

Now suppose we concede the argument that it is merely an illusion that one countable infinite set can be larger than another and that both sets are really the same size. Two problems nevertheless result.

First, even if we concede that it is merely an illusion that the two sets must be of different sizes (given a finite segment of each having a range between two equal values) and that both are therefore actually the same size when paired up, we still need not conclude that the two sets *must* be "the same size" because they go on infinitely. Instead, we could easily suppose that the two sets of numbers are actually calculated as a pair of *series*, and that as a pair of series, they could be made to go on *indefinitely* rather than exist already complete all at once in an infinite set. Calculating novel values is, after all, a process of extrapolating, inventing, or generating a series, not of discovering something that is already there. The two series are merely "the same size" as long as one goes on extrapolating new values for both, tit-for-tat, making them match up in one-to-one correspondence. So, the one-to-one correspondence of two sets that can be extended by a process of pairing off new values does not in itself show that the sets are the same size by being "infinite" (complete and yet limitless).

Second, infinity is supposed to be incomputable, and yet Cantor's work still attempts to make the infinite computable anyway.

Numbers that can be put into a one-to-one correspondence with natural numbers are, in Cantor's view, "countable" and numbers that can't be put into such a correspondence (like irrational numbers) he called "uncountable." Since he believed that the natural numbers extend to infinity, any set of numbers that can be put into a one-to-one correspondence with them are "countably infinite." All countably infinite sets are really the "same size" and merely have the illusion of being different in size. Granted. But while sets able to be put into an infinite one-to-one correspondence are allegedly countable infinities, Cantor also claimed that they are only one kind of infinite set. Another kind of infinite set, he proposed, is an "uncountable infinite." A set of irrational numbers, assumed to extend to infinity, represents an example of an uncountable infinite.

Cantor further showed that if countable and uncountable infinite sets are assumed to be somehow complete—all "there" existing in some conceptual world awaiting discovery—then some infinite sets must nevertheless still be bigger than others. The "smallest" sort of infinite is a countably infinite set. It is represented by aleph-naught or $\aleph_0$. The natural number scale is supposed to be an example of that. From there "larger" or "stronger" uncountable infinities can be proposed. Decimal scales and irrational numbers are purportedly uncountable infinities. Cantor came up with a system of *infinite cardinals* (all symbolized by the aleph or $\aleph$ as $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, etc.) which are used to measure the "apparent size" or "strength" of infinite sets from the countable infinite sets through all uncountable infinite sets.

If the idea of infinite sets of various sizes or strengths is coherent, then one could also conceive of adding up different infinite sets, and that would make the infinite computable once more. Cantor aimed to do just that. He invented a *transfinite mathematics* which is used to compute these infinite sets. However, the infinite was supposed to be incomputable, and since a computable-incomputable thing is a contradiction, so must be this notion of the infinite.

Even so, let's throw out the requirement that the infinite must be incomputable and instead allow Cantor his computable infinities for a bit. According to transfinite mathematics, infinities can be measured or calculated based on their relative sizes, and they can also be added, multiplied, and raised in power.[20] But even if Cantor is correct that infinite sets can be represented in such a way as to make them computable, the computations of infinite sets still result in contradictions that can only be avoided with fallacious maneuvers, as we shall see.

Contradictions in Infinite Sets

Philosopher William Lane Craig used an analogy to show the mathematical contradictions involved with making certain kinds of calculations with infinite sets. Craig asks us to imagine that he has an infinite number of marbles in his possession, and that he wants to give you some of them. Suppose, in fact, that he wants to give you an infinite number of marbles.

One way he could do that would be to give you the entire set of marbles. In that case he would have zero marbles left for himself.[21] However, another way he could do it would be to give you all the odd numbered marbles. Then he would still have an infinite number left over for himself, and you would have an infinite set too. You'd have just as many as he would—in fact, each of you would have just as many as Craig originally had before the marbles were divided into odd and even.[22]

These illustrations demonstrate that performing simple calculations involving an infinite number of things leads to contradictions. In the first case in which Craig handed out all the marbles, an infinite minus an infinite is zero ($\aleph_0 - \aleph_0 = 0$); in the second case in which he gave you all the odd-numbered marbles, an infinite minus an infinite is still infinite ($\aleph_0 - \aleph_0 = \aleph_0$). In each case, the identical value was subtracted from the identical value, but with contradictory results. "For that reason," says Craig, "mathematicians are forbidden from doing subtraction and division in transfinite arithmetic, because this would lead to contradictions."[23]

Note that one must forbid in transfinite mathematics what is easily done in ordinary, finite math: in transfinite math, you can add but not subtract, multiply but not divide. With these rules in place, the logical contradictions resulting from subtracting or dividing infinities are avoided. Framing the rules of transfinite mathematics in this manner, calculating with infinities does become coherent, but at the price of being ad hoc. That is to say, infinities are used in a mathematically consistent manner, but only by preserving the traditional notion of the infinite from refutation for its own sake. As a result, the rules of transfinite mathematics end up telling us nothing about infinity as a quantitative condition, but merely create a kind of self-consistent mathematical game that has no bearing in reality.

Claiming that infinities can be calculated as long as you don't allow subtraction and division is like saying square circles exist because we could create a formula that allows them to be used, provided some qualifications are put in place that don't allow us to expose the contradictions involved with having a round square or a circle with corners. What would a square circle formula look like? In principle we could create a formula using hyperbolic spaces that will allow us to combine squares and circles into "square circles," or at least transform the transcendental numbers used to calculate the area of a circle into the algebraic numbers used to calculate the area of a square, thereby converting the area of a circle into the area of a square. Suppose we could get such a formula to work consistently but we also have to stipulate that it only works with square circles and that certain other operations in algebra and transcendental math break down in attempts to use them with square circles.

The result would be that square circles are not any less self-contradictory; all that we would end up proving in the construction of such a formula is that a coherent game of square circle calculation can be made as long as the rules are limited in an ad hoc fashion so that the illogic of square circles is not allowed to be exposed by taking the concept to its, well, logical end. Transfinite mathematics is in the same boat—the game is coherent only because we won't allow logic to proceed down its natural path so that the absurdity of its subject would be exposed. This allows the idea of computable infinities to retain the illusion of being a logically coherent notion.

In response to this position, it has been objected that infinite sets are qualitatively and not simply quantitatively different from finite sets, and so transfinite numbers can't be treated like finite numbers. One philosopher, Blake T. Ostler, states that "transfinite mathematics does not simply prohibit subtraction and division by fiat; rather, transfinite mathematics shows why operations that can be done with finite numbers sometimes give indefinite results for transfinite numbers."[24] In other words, the infinite simply doesn't work like the finite.

There are two problems with this response. First, it seems to assume that infinite sets already exist either in nature or in some conceptual world awaiting discovery. But that is a big assumption, and one which we are not intellectually compelled to make. It could equally be assumed that infinite sets are mathematical inventions, in which case the rules of manipulating infinite sets do not indicate "how infinity works" as if those rules are merely descriptions of an infinite set's behavior that mathematicians discover like the principles of atomic motion in a condensed gas are discovered by physicists. Instead, it could be that the rules for calculating infinite sets are only "discovered" in the same way that a new strategy in chess is discovered. This kind of "discovery" is actually a form of invention; it is the invention of new rules of inference for manipulating concepts in a pattern coherent with the rules previously established for those concepts. If this is correct, then the rules of transfinite mathematics are not really the discovery of some phenomenon independent of human activity, but are rather the creation of intellects concocting rules for a mathematical game.

Second, the response also misses the point: no one is denying that transfinite mathematics plays by different rules than finite math. That's not the issue. The issue is why. Why does transfinite mathematics have to avoid straightforward functions that should be able to occur if the traditional notion of the infinite is really coherent? If you can imagine a scenario in which an infinite set of marbles exists, you should be able to imagine subtracting or dividing that set. If you would get nonsensical results, then reiterating that transfinite math just doesn't allow one to subtract and divide infinite sets is not an answer, it's a dodge.

To get around this problem Ostler reports, "Graham Oppy has pointed out that there are a number of other theories of transfinite numbers that have been developed that deal with inverse operations with transfinite numbers without contradiction."[25] Oppy pointed specifically to three such theories. However, two of Oppy's examples are not, strictly speaking, transfinite—they deal with what might be called *indefinite* numbers that are still finite. His third example, on the other hand, does show that a system can be constructed to carry out inverse operations with infinite sets, but not without heavily modifying the assumptions of Cantorian transfinite mathematics.[26] And it isn't clear that such a modification is not really just a rationalization, for it is still dubious such modifications succeed in making transfinite mathematics applicable to logically possible scenarios in the actual world. Besides, it isn't the possibility of making a coherent system that matters (we might even manage it with square circles), but whether or not such a system is made to cover up a contradiction in the very meaning of "infinite."

Symbol Games

Some might argue that because we can use the lemniscate (∞) or the aleph-naught ($\aleph_0$) consistently in mathematical equations, then the concept that the symbols refers to—infinity or the infinite—must be logically coherent in meaning. This is mistaken. Infinities don't need to be conceptually or logically coherent in meaning to be used consistently in mathematics.

For one thing, we could use the infinite symbols to represent some other concept and, as long as we use them consistently, they will seem to make mathematical sense even though they are logically absurd. One could just as well say that a set of things gets so big, or a series of events goes on for so long, that the set or series is no longer itself—a property we'll call "identical non-identity." Surely this would be grammatically correct and we could even use a Mobius strip to symbolize identical non-identity. On top of that, the Mobius strip could be used in mathematical equations to show how a series or set as an identical non-identity can be computed, but that doesn't make identical non-identity any more intelligible, and neither is using the lemniscate and aleph-naught as symbols for infinities in mathematics sufficient to make the traditional idea of infinitude coherent in conception.

In terms of purely formal considerations, infinities need refer to nothing at all. An infinite set, for instance, can be symbolized not just by an aleph-naught but also by a voltage level in a computer. And a machine without intellect can use symbols of the infinite to give consistent outputs with even more ease than a human can. Because the computer can use the symbol of infinity consistently without comprehending a thing about its alleged meaning, this indicates that the ability to use the symbol of infinity consistently in equations does not show that infinity is necessarily a conceptually meaningful notion. It may indicate only that the symbol for infinity is a convenient placeholder governed by the rules of an invented game.

Then too, while symbols for infinity like the lemniscate can be shuffled about in equations consistently if the rules for doing so are written carefully enough, this is no guarantee that infinity is *logically* meaningful either. One could use the expression 0/0 consistently in an equation like $0/0 \times (3+5) = 0/0 \times (4+4)$ just as well, though 0/0 has no defined meaning. Further, if the mathematical rules for using 0/0 are not framed carefully enough, this expression too is revealed to be logically incoherent. The same is true with infinity, as is shown when we use transfinite mathematics to compute infinite sets.

Is Transfinite Mathematics Necessary?

The reason that contradictions would result in transfinite mathematics unless avoided in an ad hoc manner could be that the infinite as a "complete-limitless" set is a genuine contradiction in terms. To be *complete* is to be a whole that lacks no elements or members. To be complete assumes that there are more than zero members in the set or collection that is complete. And to have elements or members in completion implies that the set of those members is *divisible*, at least in principle. Whether I have a complete set of chess pieces or a complete set of 1-100 integers, the complete sets can be divided—black and white pieces, odd and even integers—and made incomplete by the act of division. Now I don't mean that to be complete is actually to be divided; rather, I simply mean that completeness implies the set is *able* to be divided. Take limitlessness; for a set or series to be "limitless" implies that it is *indivisible*—it can't be divided. Either you can't take away from the limitless or no matter how much you do take away you never end up with less than you had. That which is limitless is that which is also indivisible. The infinite is therefore something that is complete, thus divisible, and yet limitless, therefore indivisible. Since being a divisible-yet-indivisible thing is a contradiction in terms, so too the traditional notion of the infinite is self-contradictory. Little wonder then, that subtracting or dividing infinite amounts results in contradictions.

To get beyond the self-contradiction requires throwing out the traditional understanding of infinity. I indicated earlier that taking infinity, or the infinite, as a condition that zero can be said to have in some contexts might be the best way to make the term "infinite" intelligible. The contradictions involved with defining "the infinite" as both *complete* and *limitless* support that position. While there is no sense in which zero could be said to be "complete" or "incomplete," as such categories simply do not apply to it, to be limited or limitless are categories that do apply to zero. As noted earlier, zero can *be* a limit on a scale, but in another sense zero itself is "limitless," or *has* no limits, because it is without quantity and so has no positive or negative amount of members with defined limits. Which in turn means that zero cannot be divided and so is "limitless" in that sense as well. This is certainly not how infinitude is traditionally understood, but it may be the most logically consistent given the breakdown of the traditional notion.

If this analysis is correct, then transfinite mathematics need not be about computing "transfinite" or infinite objects after all. If a set or series with members can be computed, it must be finite rather than infinite. (That's not to say that all finite amounts can be computed, for we ran into indefinite, but nevertheless finite, amounts earlier that can't be computed with precise values; it is only to say that if it can be computed, it is finite.) In this view, whatever the cardinals of transfinite mathematics can be used to measure, they do not measure so-called infinite sets.

Moreover, if infinity is not really supposed to be computable after all, then it makes sense to suppose that it is meaningless to regard infinite sets as "bigger" or "smaller" or even "the same size" as other infinite sets, and equally meaningless to say that infinite sets can be added, multiplied, or raised in power as Cantor's transfinite mathematics attempts. To assume otherwise, that there really is a mathematics of complete-limitless sets, brings us right back to the need for those ad hoc maneuvers in transfinite math.

Instead, if transfinite math has any application at all, it would have to have application in regard to something finite. Perhaps the same mathematical rules found in transfinite mathematics could be used with reference to *indefinite sets* that are not infinite.

As shown earlier, the category of all things finite can be divided into two subcategories: the definite and the indefinite. The definite makes up all numeric values that are defined—any number that has ever been calculated or computed in ordinary, finite mathematics. In contrast is the indefinite. We saw that one definition for indefinite values makes them uncountable or incomputable in practice even though the values are finite (limited somewhere above where any calculated scale leaves off). In attempting to compute infinities perhaps we are not really dealing with well-defined sets or complete series that are infinite at all; perhaps we are really dealing with finite, yet *indefinite*, sets or series.

Because indefinite values are not defined, we could regard the indefinite sets to which they belong to have *relative sizes to one another*. When two sets of natural numbers are both extended toward *no clearly defined end* the sets may be in a sense "the same size" or "equivalent" in length as one another, not because their numbers are somehow "infinite" but simply because the two sets are really finite but incomplete, and so without highest *defined* values. That is, the two scales of numbers have a one-to-one correspondence and are in this sense "the same size," but only because there is always some further number in each series that can be matched with the other when that further number is conceived of or calculated. One more way to put it would be to say that both sets of numbers are "the same size" in the sense that neither set is ever complete but both have the same number of integers whenever the one-to-one correspondence procedure is carried out. Thus, the *incompleteness* of the sets also means that they each are "equal" since both of them have no *definite* end, while neither needs to be "infinite."

If this is right, then what Cantor really may have proved by the one-to-one correspondence of the segment of natural numbers with the segment of even or odd natural numbers is that there simply is never a complete *series* of natural numbers. Scales of natural numbers are simply incomplete or indefinite as series. By attempting to label so-called "infinite" sets using his cardinals then, perhaps Cantor actually succeeded in designating *indefinites*—amounts with no clearly defined values in the series of natural numbers—as having a set of relative "values" nonetheless. So, his mathematics may still be useful for calculating the relative sizes of sets with undefined, yet finite values that are continually being created in series.

Although indefinite numbers could not be computed with exact values, we could just as well use Cantor's "transfinite mathematics" to label the sizes of finite sets containing no clearly defined values and then compute the relative extent of those sets. That means that if there are sets of things infinite in size, they are not necessarily computed by Cantor's mathematics. If so, then Cantor's aleph cardinals do not need to represent actual infinities but rather could be redefined as referring only to finite sets of indefinite (undefined) size or finite series of indefinite (incomplete) succession. His "transfinite" mathematics would in that case become a mathematics of the indefinite rather than the infinite, which also explains why its operations must be limited in an ad hoc fashion to increasing and never decreasing values—the indefinite is always incomplete and inexact while the scale considered continues to grow.

It might be objected that Cantor's idea of infinite sets has a feature not found in finite sets, however indefinite those sets may be. A finite set is said to be one in which no subset can exhaust all the members of the original set, "that is to say, at least one member of the original set is not also a member of the subset."[27] In contrast, an infinite set "is one that is such that the whole set has the same number of members as the proper subset."[28] In other words, the whole of an infinite set is equal to an infinite subset of the whole. In reply, we could instead say that the first condition only applies to finite *definite* sets, while the second applies only to finite *indefinite* sets. If the number of members in an indefinite set is not defined (especially if the set belongs to an undefined series that is incomplete), then one could just as well suppose that the whole of the indefinite set is equal to some indefinite subset in the whole by virtue of being equally incomplete, and so indefinite, as a series under construction.

I would also point out that because the very parameters of transfinite mathematics were framed to preserve the traditional notion of infinity from refutation as inherently illogical, it may be better to reinterpret and redefine Cantor's transfinite mathematics as a mathematics of the indefinite. (Even then, though, we don't really *need* Cantor's math. It is not used in the physical or behavioral sciences; it's really just a form of professional entertainment for mathematicians.)

What about Infinite Objects?

If the numbers on any conceivable number scale, when extended toward no clearly defined end (or "indefinitely") are really finite in the sense that the scale terminates where it is left *incomplete*, then this also means that any set of objects designated with values from this scale must also be finite—no matter how vast or extended how far beyond our ability to compute. When we try to conceive of an "infinite" number of stars and planets, we are really conceiving these not to be infinite (non-finite), but to be finite though having a number vast beyond any scale of computation that could be completed or defined (i.e., indefinite). So far then, it looks like sets of things, even sets of numbers, are not necessarily infinite in size.

Some would argue that the tradition notion of infinite sets *must* make sense since we use infinite sets and infinity in mathematics all the time. For example, what about "infinite objects" (numbers or values used in mathematics that supposedly are infinite in size)? Earlier, irrational and transcendental numbers were mentioned. The values of such numbers are supposedly "infinite." For example, take the value of *pi* (the ratio of a circle's circumference to its diameter, denoted by the sixteenth letter of the Greek alphabet, **π**). Pi, which equals approximately 3.14159, never has a final value; pi is an "irrational number"—a number that cannot be written as a simple fraction and that has no final value when written in decimal form. Pi is equal to 3.14159...where further decimal places continue to give an ever more "precise" or explicit value, but a value that has no final or "definite" value. So far pi has been calculated to billions of decimal places. But, even with the use of supercomputers, there will never be a terminating value for pi because we can always calculate further, and so invent new "more precise" values for pi. The future values for pi are not there waiting to be discovered as one might discover a new island; rather, they are "discovered" only in the sense that they are *calculated, extrapolated,* or *invented* using the given mathematical rules. And since new calculations can always be carried on, provided there is time to do so, new decimal places for pi can always be computed so long as a human or machine is around to do the computing; thus, pi will remain indefinite for all time. Anytime your calculator or computer runs out of computing power for an irrational number, it has indicated that the next value that would have been computed is off the charts and so is undefined or "indefinite." The limit of the computable can always be pushed back by increasing computing power, but there will always be undefined values left over so long as there is time to do calculations.

A similar example is the square root of two (√2), sometimes considered to be an "infinite object" because its value is equal to a number having an unending series of decimal places (1.4142136…). The key word here is "series" in designating the decimal places of the expression: like pi, the square root of two does not necessarily have to exist as a set of decimal places already complete in some Platonic world of forms awaiting discovery by mathematicians. Instead, it is better to consider the square root of two as having a *series* of decimal places, rather than a complete set of decimal places, that can "endlessly" be extended because the number of places in the series' decimal value is always *incomplete*. Since no one has calculated a terminating decimal place for √2, we could simply consider its value to be just *the highest value that anyone has calculated so far* for it and so further consider this irrational number to be actually finite but simply *indefinite* in extent since it is always incomplete.

Supposedly "infinite objects," then, are better regarded as *indefinite objects* or incomplete objects. Whether we are talking about the value of pi, the square root of two, or some other irrational or transcendental value, such values are indefinite but not necessarily infinite. Thus nowhere, not even if only in the world of mathematical conception, does an infinite value or set need to exist, nor does the idea of infinity necessarily make sense—it seems to generate logical contradictions that can only be avoided in an arbitrary or ad hoc manner as we've seen.

INFINITY LOST

Boundless Space and Time

That there is either no clear or no non-contradictory notion of infinity when applied to abstract sets of numbers and objects, or even mathematical equations, is one thing, but what about space and time? Many cosmologists assume that the Universe must be infinite in space and time; the Universe is often regarded as limitless in space and most believe it will be around for an infinite amount of time. Do these ideas make sense?

Time and space are sometimes said to be infinite because they are "endless" or "boundless." But as we saw previously, we must be careful with such terms. To be finite means to have a limit, but not all limits have ends or boundaries. A line segment is a bounded thing because it has ends, but circles have no ends and so are "boundless" along their circumference while their "closed" geometry still proves them to be finite. A two-dimensional plane can also curve back upon itself forming a three-dimensional sphere; the surface of the sphere has no end around its circumference either—it is also endless and unbounded while being finite nonetheless. Circles and spheres are both examples of things unbounded, then, but they are both finite and so being "unbounded" in this sense is not the same as being infinite. An infinite extent, as for example the measure of a supposedly infinite line, cannot be "boundless" by simply being a finite extent that curves back upon itself. No, instead the infinite is supposed to be that which is endless or unbounded by having no limit at all.

So, perhaps the infinite is the condition of being *unbounded* in spatial extent, but it must be so in the sense of being *unlimited* in extent. And yet, to be unlimited and also non-recursive is just what is incomprehensible. We cannot actually picture say, a line extending infinitely, nor can we experience infinity in terms of space. Even if we use an arrow that points off into the distance in order to represent a space that "keeps going without end," this doesn't help much, for, phrases like "without end" and "keeps going" apply just as well to vaster *finite* extents of space that are indefinite, rather than infinite, in length. Perhaps such extents just can't be reached since, given all the time that will exist, we'll never have enough time to reach the indefinite end. Or, perhaps space is curved back on itself as a closed hypersphere of finite but indefinite proportions. Or, maybe space is continually growing indefinitely larger without end. In all of these logical possibilities one could "keep going without end" or travel through an "unbounded" space in the sense of travelling through space as a finite but indefinitely vast extent. But that is not what an *infinite* extent of space is supposed to mean.

Like space, time also is often assumed to be infinite. Maybe we can use "boundless time" to get a handle on the concept of the infinite. Infinity is sometimes thought to be "unbounded" in the sense of having a value such that, in attempting to calculate it, the calculation would take one "forever" to complete. Such temporal terms as "forever", "eternity", "boundless time", "endless time", etc. are supposed to be understood as "unending succession."

But what can it mean for a succession of moments not to end? The only successions we *do* experience are successions that terminate at some point, as when the sand in an hourglass runs out. Beyond that, there is only one clear kind of unending succession so far and that is what mathematicians call "finite unbounded" succession.

Finite unbounded succession, or what I like to term simply the *finite-unbounded*, is any succession of steps or events that is finite in one sense and unbounded in another sense. A finite-unbounded succession is one with a limited number of steps that returns to its starting point in a circuit. It is best understood by analogy. Consider the example of a series of cartoon panels in strip. One can imagine the first panel as depicting a small rubber ball flying through the air. The next panel of the cartoon shows a man being hit in the back of his head with the ball. The next series of panels depicts the man stooping down, picking up the ball, and throwing the ball ahead of him. Now imagine taking the cartoon strip and looping it back upon itself so that the end panel is attached to the beginning panel and the content of what was the beginning panel follows as a consequence of the content in what was the end panel. Now the complete cartoon loop represents a kind of "unending succession"—in fact, the completed loop has no end or beginning even though the number of panels is finite. The last panel of the cartoon strip shows the ball in flight and that panel is looped around and attached to the beginning panel so that the thrown ball appears to strike the man who had thrown the ball in the back of the head as a result of *his own* throw. Here we have a cartoon depicting a succession of events in which each event only occurs once and no event is repeated, but the events also occur in succession without beginning or end. When the loop is made, there ceases to be a "beginning" panel or an "end" panel, yet there are a finite number of panels in the loop. That is an example of a finite unbounded succession.

The rock band Pink Floyd also played with the finite-unbounded concept in their album *The Wall* in which the end of the album asks, "Isn't this where…" and the beginning of the album finishes the question with, "…we came in?" This also represents what an unending succession is supposed to be.

These examples show, at least in analogy, the finite-unbounded—a series of moments that has no beginning and no end, but which is finite. And yet, a finite unbounded succession is not supposed to be what the "unending succession" of *infinite* time means. Infinite time is supposed to be unending time, but not in such a manner that time is looped back upon itself in a circuit. Again we are left to wonder what an example of unending non-finite succession, an example of infinite time, would be.

There are two remaining possibilities for what "boundless time" might mean. The first possibility is that boundless time means a *cyclic succession of moments*. In contrast to the circuit of the finite-unbounded given in the cartoon loop analogy, a cyclic succession has events that repeat over and over. It is the kind of succession that repeats in cycles as, for instance, with the seasonal changes.

But all the cycles we know of have a beginning and an end. It does no good to imagine a particular cycle, like the revolution of the Earth around the Sun, continuing after human beings are extinct as an instance of an "unending" cycle. One reason is because we know that even the Earth's cycle of solar revolution must come to an end according to the laws of physics. The other reason is simply that at some point, one stops imagining the cycle's continued repetition anyway. So, one can only imagine a cycle continuing a finite number of times, however vast that number. And even if we throw out trying to *imagine* infinite time and instead simply try to *conceive* of infinite time in the abstract, it is of no help for again to have a series "without end" is not to have a clear idea of an infinite series. An "unending succession" merely indicates a series or cycle of events that is left *incomplete* and so without defined end. To have no defined end is just to be *indefinite* rather than infinite. It's possible that once a cycle starts, it continues indefinitely, but that only means that, no matter how long it goes, it remains finite during every revolution even if it continues on "without end." So we still have no clear idea about what infinite time would be.

Infinity is supposed to be different from the merely indefinite, which always remains finite, but how is it different? Until we know that, we won't really know how infinite time differs from time being always finite though going on indefinitely, or how infinite space differs from space being finite though indefinitely vast. The infinite still needs defined as "unending" but in *non-finite* terms rather than merely in terms of being indefinite. So far in defining infinities as being unbounded or unending we have not yet adequately distinguished the infinite from the indefinite.

And that brings us to the second remaining possibility for what "boundless time" (or "boundless space" for that matter) might mean. To be "boundless" in the sense of having no limit, in the sense of infinity, is not to be "endless" in the sense of being a closed, recursive extent or process like a circle or circuit. Instead, the infinite is supposed to be "boundless" in the sense of being both *open and endless*. If time (or space) is infinite by being boundless then, it must be boundless in this sense.

Yet here too we must take care. We just considered cycles which can also be "open and endless" in the sense of being incomplete series, but infinity is not supposed to be merely incomplete, for that implies finitude. So we have another problem: To be an open and unending extent or process—whether linear or cyclic—does not necessarily differ from simply being an incomplete, finite, extent or process. You could always add on to an incomplete finite set that is open, making it an indefinite series, and infinity is supposed to "keep going" as well, but not because it is incomplete or indefinite. Rather, infinity is supposed to be *complete* while also being limitless in the sense of being both open and endless. Infinity is that which is *limitless* (open and endless) *and complete*. And that is the sense in which infinity, in terms of either time or space, is supposed to be "boundless."

However, as we saw before, if infinity is supposed to be both limitless (which implies indivisibility) and also complete (which implies divisibility), then we have a contradiction in terms: a divisible/indivisible thing. Hence, being "boundless" in space or time as a complete-yet-limitless extent or process also implies a logical absurdity.

So far we've tried to nail down infinity with examples of "boundless" space or time but came away with only a logical contradiction. Assuming the traditional notion of infinity still makes sense in relation to space and time, perhaps we can find out what infinity means by supposing that infinite space and infinite time can describe *each other*. After all, we know that a finite amount of space can be used to describe a finite amount of time and a finite amount of time can be used to reference a finite amount of space. For example, astronomers use the light-year to describe how far light travels through the vacuum of space in one Earth year (about 9.46 trillion—9.46×10^{12}—kilometers or 5.88 trillion—5.88×10^{12}—miles. In this case, we have extents of space defined by time. Then too, any timeline is really a spatial image—a line or linear figure of some sort—used to represent periods of time: a biography, a history, geologic epochs, astronomical ages, etc. All of these periods of time can be represented by reference to space. Let's try to get a handle on infinity by using analogies involving infinite space and infinite time.

We could, for instance, attempt to use infinite time to describe what an infinite amount of space means—"Travel out into space and no matter how long you live, even if you never die, you could never reach the end of space: space is infinitely large." And we could use infinite space to describe what infinite time is supposed to be—"Think of a 1 followed by a string of 0s with each 0 representing ten years: no matter how long that string of zeros is, it can't express the infinitude of time."

Unfortunately, using spatial representations such as timelines does not distinguish infinite time from *indefinite* time. Every timeline we use simply ends in an arrow pointing off into an indefinite distance. Even the so-called "infinite" string of zeros representing years doesn't necessarily clear up infinity for we can't actually generate a complete but non-finite string of zeros; instead, we have to fall back on a placeholder like an ellipsis (1000000…) to indicate that the series keeps going. But that just indicates the zeros continue *indefinitely* and not necessarily infinitely. Infinite space is hard to define: it's supposed to take "forever" to reach an infinitude of space, but even if one is immortal, one will not necessarily live for an infinite amount of time, just an indefinite amount of time. Start counting forward from your birth and at any point you will only have a finite number of years since your birth—even if you never die. That's indefinite time, but it only reveals an indefinite amount of space, not an infinite amount of space; it only shows only that space is indefinitely vast from our perspective and not necessarily "infinitely" vast.

Using infinite space to illuminate infinite time or infinite time to illustrate infinite space doesn't seem to be shedding any light on what the infinite is in itself. Circularity is not a bad thing in itself, but being *viciously* circular—circular in a *logical* sense—needs to be avoided if a concept is to be meaningful. Unfortunately, the circularity we run into with trying to describe infinity in terms of space and time is indeed turning out to be vicious. In terms of using infinite space to describe infinite time, an infinite amount of space turns out to be indistinguishable from an indefinitely vast, but finite, amount of space. And if infinite space as something distinct from the indefinite is incomprehensible, then attempts to use infinite space as a metaphor to describe infinite time renders infinite time incomprehensible as well. Working the other way around—trying to understand an infinite amount of space in terms of infinite time—achieves the same result: infinite time becomes indistinguishable from indefinite time. We still haven't found out what infinite time really means as something distinct from the finite but indefinite, and so we get no closer to understanding what infinity is supposed to refer to.

So far, then, it seems that the only way to comprehend the infinitude of time is with reference to infinite space, and the only way to comprehend the infinitude of space is with infinite time—one kind of infinity is needed to comprehend the other kind of infinity, and neither can be comprehended alone. We are trapped in a vicious circularity. The very concept of infinity itself becomes circular, making the whole notion logically unintelligible.

And falling back on symbols doesn't seem to help. The lemniscate or Mobius strip (∞) is certainly an unbounded mathematical symbol. But note that while it is true the lemniscate is "unbounded" or "without end" since it is a line curved back upon itself, this is just to be boundless in the same finite way that a circle or a sphere is boundless. Furthermore, the shape of the lemniscate itself does not reveal what infinite space or infinite time is supposed to be. The symbol is just a shorthand way of *designating* infinity, whatever it is.

We are also told that if one were to trace the path of the lemniscate round and round without ever stopping, one could continue "without end"—for an infinite amount of time. But using the unbounded symbol of infinity to represent infinite time really does nothing to clarify the notion of infinite time, for again one is tracing the path only indefinitely, no matter how high one counts. Further, one must first understand what is meant by "time without end" to understand what it means to trace the path of the infinity symbol "endlessly." And time "without end" is again indistinguishable from either the finite unbounded succession already illustrated or indefinite time, which is also always finite. Neither does the lemniscate shed light on the supposed infinitude of space; the symbol itself is finite in length and is not useful as a "picture" for infinity.

Once again a clear notion of infinite space has not been found. So, neither infinite space nor infinite time seems to be a clear notion and that does not bode well for the traditional notion of infinity. Worse still, when notions of infinite space and time are made less vague, they result in logical contradictions, as we shall see.

Not-So-Infinite Space

There are yet more problems with the idea of real spatial infinities. The idea of infinities in space again suggests that any given space can be measured in terms of sets, as in sets of infinite places or sets of infinitesimal points in space. And this is a very problematic notion.

Any line is supposedly made up of an infinite set of *infinitesimal points*. That is, space is supposedly composed or made up of points that are “infinitely small” or “approaching” zero in size or increment but not equal to zero. However, assuming points to be real extents of space that are infinitesimal brings back the same logical problems we faced with infinite sets: if any given amount of space is composed of a set of infinitesimal points, then this implies an *infinite set* of infinitesimal points making up the given space. That means we can once again add an infinite set of points in one space (like the infinite number of points making up the length of one book shelf) to infinite sets of points in other spaces (such as the infinite number of points running the lengths of the other shelves in the bookcase) and so we are back to transfinite mathematics and the same logical problems we’ve already seen with calculating infinities: If I can *add* the infinite number of points in one space to the infinite number of points in another space (as with adding the infinite sets of infinitesimal points extending along two bookshelves), there is no plausible reason why I can’t *subtract* one infinite set of points from another or *divide* an infinite set of points (such as when I cut the length of a bookshelf in half and end up with two sets of infinitesimal points), but subtracting and dividing infinities is denied by transfinite mathematics to avoid the contradictions we saw before. So once again, the arbitrariness inherent in the notion of infinity is shown—the operations are forbidden so that the illusion of logic can be preserved in relation to the infinite. The reason such logical problems occur and ad hoc machinations are needed to overcome them is because the very notion of an infinite set of points or an infinite anything does not make coherent sense.

Still, there are some further objections that have to be considered: it could be countered, for instance, that dividing an infinite extent *does* make sense because we can always divide a given space into smaller spaces, on down to the infinitesimal, so infinitesimal points *must* exist and therefore infinity is coherent and must exist too. In response, we cannot in actual *practice* divide anything up into infinitely small parts. And even if the process of division is carried out in mere conception alone, we still run into the above problems with infinity unless we suppose that points are neither infinitesimal nor do they have any extension at all. A given space is not a set of an infinite number of points; instead, a mathematical point is simply a geometrical unit with *no size*—no extension—at all and space is not “composed” of points in the way that necklace is made of pearls. Points do not “make up” a given line, area, or volume like atoms making up a material body. Instead, the line, area, or volume is what is basic and points are arbitrary divisions of these fundamental wholes.

Space is a continuum that is only *measured* geometrically by given points where continuous lines, planes, or volumes intersect. Points do not make up space like a set of bricks forming a wall or a series of dots added together to compose a line. Instead, imagine that the line is what is taken as basic, and while a person measuring the line can artificially divide the line up into a set of dots; the line prior to any such division is fundamentally a seamless continuum made of no discrete parts. Of course, in reality if the mark of a line were drawn with some material medium like a fountain pen, the material line would be composed of atoms in the ink that are themselves discrete—a row of dots forming a line—but the material mark of the line is not important; an imaginary line could also be used to designate just a region of the immaterial continuum of space itself. So too, points are not entities in and of themselves, combined contiguously to form the whole of space. To the contrary, space is itself a continuum—a whole without independent parts. It is only when we use geometry to measure space by dimensions that we divide space up into given sets of points. Points are then just imaginary units of division employed by geometry to *describe* space; points are not discrete entities with their own existence. As P.O. Johnson put it:

> A line might be divisible at any arbitrarily chosen point, but this does not mean it is constituted by discrete points…The line is not a complex of which points are simple constituents. Moreover, if it was such a complex, and did contain or consist of points, then these could not be infinitely numerous, for this would imply that the series of points was unending, and therefore that they could not form a complex or constitute a whole.[29]

Rather, space is a "fluid" or continuous entity, and we only measure space according to dimensions that we in turn divide up into some set of points that are arbitrary in division. So, there is no problem with the space of a line having dimension while points do not, for the points of the line represent only artificial divisions of what is really a continuum.

This is not to belittle the idea of points: the idea of spatial points is of indispensable aid in making measurements. Even if there are no actual physical points composing space like building blocks, we can still consider points to be handy concepts for dividing up the continuum of real, objective space into quantities in order to make measurements of distance. Points are useful heuristic devises or geometrical tools that are superimposed by imagination on portions of the real continuum of space for making accurate references to *portions* of the continuum of space. So, even if the continuum of space can only be divided up artificially into points, points nevertheless can refer to definite regions of space where we choose to map space with mathematical lines and planes that intersect. It is simply that, as heuristic devices, points need not be "infinite" in number, whatever that means.

If space is not composed of infinitely small points, we again come to the conclusion that actual infinite collections need not exist, not even in empty space. Indeed, from the former arguments above, infinitesimal points make no sense at all because the concept of the infinite is self-contradictory.

And yet, even if one concedes these arguments, one might want to object again by stating that it has only been proven that infinitesimals do not exist—it has not been shown that infinite spans of space altogether do not exist. It might be argued that there exist infinitely many sets of *finite* extents of space. So, space could still be actually infinite in extension. This objection won't hold, however. The reason that the objection fails is simply that such an objection does not withstand the logical difficulties already raised with actual infinite *sets* of things, be those sets of numbers, objects, points of space or other increments of space; an infinite set of finite things still implies the same logical contradictions already encountered when sets are rendered non-finite. We must conclude that neither in abstract sets, nor in actual things, nor in numbers, nor in space are actual infinities to be found.

Infinite Time and a Series of Troubles

Space can be measured as a *set* of places that exist all at once in any finite amount that can be computed, but an infinite amount of space is problematic since infinite sets of spatial coordinates are logically incoherent. Recall that a set is a collection the members of which exist *all at once*, while a series is a collection the members of which come into existence *in* *succession*. Perhaps the problem with attempting to measure infinite space is that the infinite doesn't really exist as a *set*; maybe instead the infinite exists as a *series*.

While space can't be measured as a series, *time* can be. Suppose we can take some period of time and see if an infinite series can be found within it; maybe time has an infinite series of moments or events making it up—an infinite past and/or an infinite future. And maybe any finite length of time is made up of an infinite series of temporal moments or instants.

The question of infinity then becomes, can there actually be an infinite series—a *complete-limitless* series—of things such as instants, moments, or events in time? Some philosophers have argued that the notion of an infinite series or an infinite amount of time could exist. I'll start by showing that time cannot be an infinite series.

For a series to be infinite means that the members of the series each come into existence, one *after* another, in *succession* until an infinite number exists or has existed. In other words, an infinite series is not a series that never reaches completion, for that would just be an *indefinite* though finite series. No, an infinite series is a series that has or will *complete* an infinite number of steps. The question is this: can an infinite number of steps ever be completed and does such an idea even make sense?

One attempt to argue for an infinite series or number of steps being completed is to suppose that any finite period of time contains an infinite number of infinitely short moments or "instants" of time that are completed from one period of time to the next. Any period of time is made up of smaller periods of time. An hour is made up of minutes. A minute is made up of seconds. A second is divided further into milliseconds (each millisecond is one thousandth of a second), into microseconds (each being a millionth of a second), into nanoseconds (each a billionth of a second), into picoseconds (each a trillionth of a second), into femtoseconds (each a quadrillionth of a second), and so on. Since a single period of time can always be divided into smaller units of time, it could still be objected that an "instant" of time must exist as a temporal period so short in duration that it is *infinitely* below the scale of femtoseconds. An "instant" of time in this view is the temporal equivalent of an "infinitesimal" point of space. Since no one wants to deny that instants of time exist, and time continues to pass for any given duration, a collection of temporal instants can exist as an actual, infinite set of temporal periods strung together into whatever duration you want to consider, be it a second, a minute, an hour, etc. Any finite period or duration of time then, contains an actual infinite set of instants—infinitely short moments of time. And since we move from one second, one minute, etc., to the next, then we must move from one complete infinite series of instants in time to the next. So, infinite amounts of time do exist.

But this argument is flawed because the same problems with spatial infinitesimals also apply to temporal infinitesimals: an infinitely small number of instants cannot add up like temporal atoms to make a given duration of time since infinities cannot be calculated or completed by addition according to the very definition of infinity as being incomputable. And even if we make infinities computable with transfinite mathematics, we still run into the same arbitrariness that we had before with transfinite mathematics allowing some operations but not others to conceal contradictions.

The proper conclusion is that instants of time are not "infinitely short" periods of time that together make up seconds and minutes like links make up a chain. Nor are instants of time periods so short in duration as to be finite but nevertheless indefinite in principle as well as in practice. Instead, instants have *no duration at all*; instants of time are just moments of time *without duration*. They are like points of space—instants have no temporal extension and so are not building blocks of time. Rather, instants of time are only our own arbitrary or artificial *divisions* or "slices" taken from what is fundamentally an unbroken, fluid, and continuous process of change in space. It is the process that is basic, not the instants of it. Thus, the whole of a given period of time can be arbitrarily divided into instants for heuristic purposes of measurement, but time itself has no "infinitesimal" periods that make it up like a string of pearls. Moreover, instants of time do not constitute an actual *set* of temporal moments that somehow exist "all at once," side-by-side, in a block of "space-time" either. Rather, instants of time exist only as a sequence or *series* of our own artificial divisions of a seamless process of change and this series is simply a convenient mathematical mapping of moments used to measure the order of events that take place in the continuum of change we call "time." If instants of time, like spatial points, are just heuristic tools of measurement used for conceptually dividing up what is in reality a continuous process of change, then there are again no infinitesimals and there need be no actual infinities making up any finite length of time.

In an attempt to counter that infinite series and infinite amounts of time still make sense, some philosophers have argued that an infinite number of steps can be completed by going through the sequence of steps really fast—infinitely fast.[30] So, it is argued, while it is true that if any *finite* number is "taken from" an infinite series other members will remain, there is no reason that an *infinite* number of steps cannot be taken from an infinite sequence by means such as infinite acceleration.[31] If so, then an infinite sequence can exist as a complete infinite set. Therefore, it is argued, if a complete infinite sequence of steps can exist, then an infinite amount of time could also exist as a completed series of instants, moments, or events (at least, relative to some infinitely accelerated frame of reference).

Such an argument is fallacious for trying to prove that an actual infinite amount of time or temporal, infinitesimal "instants" can exist, though, for there is still a mathematical problem with trying to "take an infinite number from" an infinite set of steps by infinite acceleration. Either taking an infinite number away from the infinite sequence would still leave an infinite amount of steps left to complete (infinity minus infinity is infinity) or it would complete the sequence all in one jump (infinity minus infinity is zero). Those proposing infinite acceleration deny the former and affirm the latter. But even if we allow this denial to get them out of mathematical contradiction, completing an infinite number of things all at once by infinite acceleration simply amounts to confusing infinite acceleration with *instantaneity*.

Moving instantaneously from here to there takes *no* time, not an "infinitely short" period of time; even infinite acceleration is not really an instantaneous "all at once" jump. That is, it still takes time, even if it is an infinitely short amount of time, to get up to an infinite amount of speed. So, an infinite sequence can't be crossed "all at once" by infinite acceleration. That means infinity minus infinity, at least in this case, is still infinity: an actual infinite has not been *completed*, for an infinite number of steps yet remain.

Further, infinite acceleration means that there is another infinite set to cross: because we have to start at speed zero and then work up to an infinite speed by accelerating, that means another infinite number of steps have to be completed—namely the infinite number of *speeds* that must be crossed to reach *infinite speed*. But then wasn't the whole point of proposing infinite acceleration in the first place to find out if an infinite series could be completed? Proposing infinite acceleration begs the question because infinite acceleration (going faster and faster) involves traversing an infinite series of steps from one speed to another in an infinite sequence of speeds, and *completing* an infinite sequence is what we were questioning to begin with.

Consequently, it simply isn't clear that an actual infinite series could be completed. And if that is not clear, then it isn't clear that an infinite amount of temporal instants could ever add up to make a complete infinite set of instants within any finite period of time. So the question still remains if infinite time is even a coherent notion at all.

Moments Don't Add Up into Infinite Sets

We've seen the problem with trying to complete an infinite series. This entails that the future is not infinite in the sense that there will ever be an actual infinite amount of time that has taken place. To see why the future is not actually infinite, consider the following logical problem:

Tristram Shandy lives forever and writes one day of his autobiography each year. Though he falls further behind every year, he can still write about everyday of his life if he lives forever. How can this be; how can he fall further and further behind and yet write about every day he will ever live? This is really only a conundrum if one thinks both that Shandy must forever be falling behind in his task and yet at some point of time infinitely in the future will *complete* his autobiography. There is no such point of time in the future, however. Shandy will never complete his autobiography, but he will write about everyday of his life.[32] This means only that though the whole of the autobiography will never be written, no part of the autobiography will remain unwritten forever. In other words, for any day of Shandy's life, Shandy will write about that day at some point in the future.

But now notice that we don't have an infinite future that is ever complete or realized. This "infinite" future is not actually infinite since the whole autobiography will never be written; at most you could consider the future to be a sequence or series of *potential* moments that do not yet actually exist in relation to the present moment. This idea is itself dubious, but if it were consistent, then the future could be "potentially" infinite even if it can't be *actually* infinite.

What about the past? Has an infinite amount of time already taken place? One could conceivably represent the infinite past with a straight timeline arrow pointing indefinitely far away, into an endless past. Since the past contains only events that have already, actually, occurred and our representation of the past as infinite entails that the Universe had no beginning, then have we found an actual, complete infinity? That all depends on what is meant by an "infinite" past.

In conceiving the past to be infinite, we must not assert that the Universe has in its past a beginning infinitely far removed from the present. To have an infinite past does not mean that the Universe did have a beginning or "first moment," which can be represented as time *t*, but that the beginning *t* is simply a moment infinitely removed from the present moment. In other words, to say the past is infinite is not to assert that the Universe began at some infinitely distant moment of time *t* in the past such that the Universe had to pass through infinite time to get from *t* to the present moment. So, for example, we must not try to imagine that there is a beginning of time infinitely in the past from which one could have begun counting until one had reached the present. Rather, to have an infinite past implies that the Universe had *no beginning at all*, not a beginning infinitely in the past.

So far, so good. It is true that you can't say time really had a beginning but that beginning is infinitely in the past, for an infinite past implies *no beginning*. In support of this point, suppose, similar to Tristram Shandy's case, that one had an infinite life span and that for each day of one's life one could think back to the events of an entire year before the last year one had contemplated the day previous. One would eventually think back to the year in which one was born. Let's say one then begins pondering the years prior to one's birth, again contemplating each day one year previous to the year contemplated the day before. One would then be thinking back through historical events in the past before one's own birth, back through the centuries, millennia, and beyond. If in sequentially contemplating prior moments without end, one were never able, even in principle, to arrive at a beginning, the past would be infinite. As the events of the past are thought of, the thinker could not even in principle exhaust the events of the infinite past. Hence, it is impossible to traverse an infinite past by going successively backward through time, even if such traversal is attempted only in imagination.[33] Nor is the number of events traversed in time a function of the direction of movement: one cannot cross an actual infinite by counting *forward*, as shown before, anymore than one can cross an infinite by going backward in time. This much is consistent enough.

But this qualification of what an infinite past means is also misleading, for it is here that another problem with time creeps in: In order to arrive at the present, all events in the past had to occur *in succession*. If any sequence of prior events did not actually occur, then the events succeeding those events could not have occurred. For example, if any sequence of events comprising the year 1492 CE did not occur, then neither did the years 1493, 1494, etc. occur. Thus, the events in years succeeding those years could not have occurred to bring us to the present and so the present moment would not exist, which it must. There must have been events in the past that were succeeded by a sequence of events to bring us to the present. But now consider an infinite past. If events stretch infinitely into the past, then there must be an event or a sequence of events that lie *infinitely* removed from the present. Therefore, even if time is "without beginning" in the sense of having an infinite past, there must still be events that are *infinitely* in the past, that have occurred, and that have been succeeded by other events to get us to the present moment.

So, while one couldn't start counting from a supposed "beginning of time" that lies infinitely in the past and come to the present, one should nevertheless have been able, given an immortal life span, to count from *some* past event *infinitely removed* from the present and yet still complete the infinite sequence to reach the present moment. And that is precisely the problem, because just as Tristram Shandy never reaches an infinite future event, so too one in the infinite past cannot reach the present, which for that individual always lies infinitely far ahead.

Therefore, the contradiction remains: if the past is infinite, then there must be an event or events infinitely in the past, but there can't be because the present could never be reached from it and so we wouldn't exist now. The only way out of this contradiction is to assert that the past must therefore be finite.

Suppose, though, that we try to resolve this contradiction by changing our assumptions. If might be first conceded that because infinity is not an actual number, but only a mathematical property, you can't start counting 1, 2, 3, etc. and then—poof—suddenly arrive at ∞ somewhere down the road, reaching infinity as if it is the endpoint of a process. And it then might be supposed that if there is no endpoint that is infinite, there can be no starting point that is infinite either. So, perhaps it is meaningless to speak of "starting" or "arriving" at a moment or event that is infinitely removed from the present. We may then dismiss the idea that an infinite future implies some event "infinitely" in the future and we may also dismiss the notion that an infinite past implies some moment or event "infinitely" in the past. In this view, there is no event or moment lying infinitely in the past that one could use as a starting point to begin counting forward in time.[34] Instead, there are no particular moments or events that are "infinitely" in the past or future at all; every event lays only a finite amount of time away from every other event, no matter how long the time is between them. Hence, although you can never get to infinity by successive addition, the past or future—indeed, time itself—could nevertheless be "infinite" as a whole.

Does that solve the problem? Actually, it doesn't, for two reasons: First, if the past is infinite, then it is wrong to say that *no* event exists *infinitely* far in the past. If there is no event infinitely in the past, then the past is not really infinite. If any and every event *e* in the past is only finitely removed from the present, then the past is finite no matter how far back we go to reach *e*. For every event in the past to be finite and yet for time to have no beginning only implies that time is a finite unbounded circuit or loop as described earlier; not an infinite sequence.

Second, even if it is conceded that time is somehow infinite "as a whole" but that no event *in* time is "infinitely" removed from the present, another problem persists. An infinite past still implies an infinite *number of events* in the past. Now take the infinite number of events from the present to the endless past as a whole—just as you can't start counting by regress and reach an event lying infinitely in the past, so too you can't invert the direction of movement along *that same number of events* and have the present be reachable either. As philosopher J.P. Moreland said,

> [If the past were infinite], then reaching the present moment would be like counting to zero from negative infinity (*n*,…, -3, -2, -1). Counting to positive infinity from zero (0, 1, 2, 3,…, *n*) involves the same number of events as does counting to zero (the present moment) from negative infinity. The number of events traversed is not a function of direction, and the latter task is as problematic as the former—not because both allegedly involve starting from some point (the former has such a point, the latter does not), but because of the impossibility of traversing an actual infinite by successive addition.[35]

The problems with trying to get to infinity by successive addition have already been shown. But those problems are compounded when time is brought into the picture, for events in the past had to have occurred, *in succession*, in order for time to proceed to the present. That means if the past is infinite, then the present couldn't be reached and so it doesn't exist. And yet, here we are. Moreland elucidates:

> Further, coming to the present moment by traversing an infinite past is worse than counting to positive infinity from zero, because the former cannot even get started. It is like trying to jump out of a bottomless pit. The whole idea of getting a foothold in the series in order to make progress is unintelligible. Take any specifiable event in the past. In order to reach that event, one would already have to traverse an actual infinite, and the problem is perfectly iterative—it applies to each point in the past.
>
> By the way, this was a major point of several of Zeno's paradoxes. He was not merely trying to show that one could not finish crossing an actual infinite, but that one could not even begin moving, because the problem of crossing was perfectly general and iterative.[36]

Moreland has correctly concluded that if the past were infinite, then the Universe would have had to have undergone an actual infinite *series* of steps to get to the present, which is impossible. And since one cannot cross an actual infinite, then the past must have been finite.

Craig used the analogy of dominoes to make a similar point: "If there were a beginningless series of falling dominoes, would not the number of dominoes fallen prior to today be actually infinite?"[37] Indeed it would. In fact, if an infinite number of dominoes could fall in a series so that an infinite number of falls have occurred before the present, then that means an infinite number of dominoes have already *completed* falling. But then, if the dominoes could complete a sequence of falls, they could not have been falling *without limit*—infinitely. A contradiction. Hence, an infinite past is impossible.

There are some academics who don't agree, of course. Mathematician and science writer Martin Gardner is one of them. Gardner states,

> An endless regress is absurd only to someone who finds it ugly or disturbing. There is nothing logically absurd about an infinite regress. We may feel uncomfortable with the infinite set of integers, but who wants to deny that the sequence goes to infinity in both positive and negative directions? Fractions in the sequence 1/2, 1/3, 1/4, 1/5…get smaller and smaller but the sequence never ends with a smallest fraction.[38]

Who would want to deny an infinite set of integers or fractions? I do for one. Since the traditional notion of infinite sets tends to generate so many apparent contradictions, it is my contention that it should be replaced by a more logically coherent notion, such as the concept of an *indefinite set* or an *indefinite series*. Instead of supposing that integers and fractions belong to limitless sets that exist already complete in a Platonic world awaiting discovery, we could just as well consider integers and fractions as belonging to *indefinite* sets or series. An indefinite series may contain both a set of defined values (like 1/2, 1/3, 1/4, 1/5 in Garner's example) and also a set of undefined, or "indefinite" values (indicated by the ellipses…or higher than any fraction we can actually calculate). As higher finite values are calculated those newly defined values represent the definition of values that were previously undefined, values that were but potentials in the series. As those values are defined, yet more undefined values lying beyond those invented or calculated for the series come to exist as new potentials for definition. At any given time, though, both the number of defined and the number of undefined values in the series are finite. So integers, fractions, and the like need not be somehow already there in a supposedly infinite set, but could simply constitute a finite set at any instant in the calculation process while new values are produced *indefinitely* in the sequence of values as a series.

Moreover, because actual infinite sets remain logically absurd for all the reasons given earlier, that means infinite *regresses* remain absurd since they imply infinite sets; a regress of steps can't be infinite because if it were, then the number of steps would equal an infinite *set* of steps and we'd run into the same old problems involved with trying to cross an infinite set of things, just as in the Tristram Shandy illustration. Regresses can be carried on indefinitely or "irrationally," but they can't be carried on infinitely. Thus, it isn't that infinite regresses are seen as absurd because they are supposedly ugly or disturbing; rather, they must be rejected due to being logically incoherent.

So far it appears that time, like space, must be finite. But this isn't quite the end of the matter. It's only been concluded that the concept of infinity refers to nothing that can *actually* exist. Even if infinities don't refer to *actually* endless sets or sequences, perhaps infinities could still exist as *potentially* unlimited sets or series. In that case, maybe time is "infinite" in the sense that it has the "potential" to unfold without limit.

Potential Infinities

Since infinite sets don't actually exist and infinite series can't actually be completed, perhaps infinities nevertheless exist as series that merely have the "potential" to be unending. In his book, *Achilles in the Quantum Universe, The Definitive History of Infinity*, Richard Morris pointed out that before Cantor's transfinite mathematics became popular, "mathematicians generally denied that it was meaningful to speak of infinite numbers." Instead,

> They did make use of unending "infinite series" of numbers. A simple example of such a series would be the set 1, 2, 3,…that we have already encountered. Another would be the series of fractions 1/2, 1/4, 1/8…But here the basic idea was that such a series, like the Energizer bunny, kept going, and going, and going. Even in a steadily increasing series such as 1, 2, 3,…you never got to infinity, only to progressively larger numbers…[39]

These earlier mathematicians were on the right track, and academics have since lost sight of that when they became awestruck by the profoundness of Cantor's work, overlooking the logical limitations and arbitrary parameters of transfinite mathematics in their awe of the idea that the infinite could be made mathematically rigorous. But since the infinite is a self-contradictory notion, no infinite sets of numbers need exist; there are only *indefinite* sets of numbers beyond those defined, and no *series* can exist summed up into an actually complete infinite set.

This position agrees more or less with that of Aristotle, who seems to have also believed that an actual infinite is supposed to be a complete collection—a set of things that exist all together at once—and no such collection actually exists because infinite collections or sets imply logical contradictions. In contrast, he asserted that a "potential infinite" is not a set but a series that is finite at every stage but can never actually be completed.[40]

Aristotle's position that actual infinities are absurd entails that periods of time, whether instants in a second or events stretching from past to future, can also not be totaled up as actual, infinite sets. An actual infinite amount of time has never existed and will never exist. But even if time cannot exist as a series of events that can be summed into an actual, complete infinite set of events, perhaps time nevertheless exists as a series of events that can be *potentially* carried out in an "infinite" way.

A potential infinity of time, in contrast to an actual infinity of time, Aristotle construed as a series of moments that can increase in number "forever" but is always finite at any given interval, and at no moment in time is a subset of completed tasks in the series equal to the completion of the whole series. For instance, if one were to start counting toward infinity beginning with 1, 2, 3, etc., at no moment in time would one's counting reach infinity—no matter how high a number you reach, the value is always finite even though you can potentially carry the series on without end. At each step in counting the series maintains only a *potential infinite*. In this view, time "never ends" because it is a process that is left always incomplete. Since time is never complete, time remains finite at any given interval and so avoids the problems involved with "actually infinite" sets of moments.

If time is in some regard infinite then, it is only potentially and not actually infinite. But now we must ask if this is really the case—is time potentially infinite and, if so, does this potential infinity lay in the past or in the future, or are both past and future "potentially infinite"?

In terms of the past, moments have already occurred in succession and as moments occur they move from being potential to actual. The past therefore contains all the moments that, relative to the present, have already been realized or *actualized*—made "actual." This in turn implies that the series of moments that have already occurred can be summed up into the set of all actualized moments. So, if the past does stretch back infinitely, it must contain an infinite set of actualized moments—an actual infinite—and not just a *potential* infinite set of moments. But the past can't be an actual infinite, for we've already seen the logical contradictions that result with actual infinite sets. Our only conclusion can be that while the past may contain an actual set of moments, it does not contain an actual *infinite* set of moments, and since its moments have all been actualized, neither can the past be *potentially* infinite. Whether actual or potential, the past must be finite.

So that at leaves the future; is the future potentially infinite? The only way that the future could be infinite is if time has the potential to continue on infinitely. But is even this the case? It's possible, but only if infinity is a logically coherent concept. Only logically coherent concepts—for example, concepts like the concept of a square or the concept of a circle—refer to things that can exist. As pointed out earlier, there are some concepts that appear at first glance to be logical because we are able to use them in grammatically correct ways, but which are nonetheless logically meaningless. Concepts such as "square circles" may be used in grammatically correct sentences, but these concepts have no coherent meaning themselves. Their internal logical incoherence indicates that they can refer to nothing that exists either actually *or* potentially: there cannot even potentially be such a thing as a square circle. Now I have argued that actual infinities are logically incoherent in this manner. If my argument is sound, then time does not even have the possibility of being actually infinite. So it becomes vacuous to say time can be "potentially" infinite because that which has no possibility of being realized—of being made actual—has no real potential at all. Because the future has no *possibility* of being actually infinite then, it has no *potential* to be infinite—no potential to be a complete-limitless set or series. In terms of potentiality, the future as a series of events must therefore be finite, no matter how long that series goes on.

It must be conceded, however, that although time is finite, it could still go on "without end" in some sense. On the one hand, time could still be a process that continues toward an end that is indefinitely far removed from the present according to any method of calculation. But because it would still have an end, Aristotle's "potential infinite" would not apply in such a circumstance. Alternatively, time could loop back on itself in a closed circuit, and so be "endless" in that sense. Or perhaps time is simply be an open, linear process that unfolds indefinitely in the sense of continuing "without end" while remaining actually finite at any step in its progress—forever an incomplete process.

The third interpretation of "endless" time better fits Aristotle's idea of a potential infinite insofar as time can continue indefinitely, but in stating time is "potentially infinite" Aristotle misunderstood and mislabeled what I've been calling the *indefinite*. Time has no potential to be infinite because actual infinities cannot be realized, but it does not violate logic to hold that the future is always finite while unfolding "indefinitely." Aristotle's "potential infinite" is just a misnomer for the indefinite.

Either version of indefinite time stands as a more coherent notion than infinite time, but whether or not time really is indefinite is a topic for another paper. For now the important issue is whether or not the *infinite* makes any sense, whether we're talking about infinite sets, infinite space, infinite series, or infinite time. Since it's been shown that infinities can be found neither in sets nor series, neither space nor time, neither in the actual nor in the potential, then I submit the infinite could very well refer to nothing that can exist and its logical coherence is highly suspect.

FINITISM

From Plato to Aristotle

We saw that a traditional definition for infinity is to have "a quantity unable to be counted as its value is greater than any terminating sequence of natural numbers" and another is to be "a value greater than any computable value." These definitions were found to be logically problematic. One problem with these definitions is their negative character—telling us what infinity isn't instead of what it is—and another problem is that these definitions are too vague, not distinguishing between the infinite and zero (which is also non-finite as it is unable to be counted with any segment of natural numbers) or between the infinite and the indefinite (finite amounts that are too large to be computed in practice because they are "off the scale" as undefined or incomplete quantities). The words "infinity" and "infinite" seem to share only two clear meanings: either "indefinite" or simply "not finite" (zero).

In the first case, infinity is synonymous with any finite but indefinite value and so is not really "infinite" at all—instead it is a kind of misnomer for indefinite finite amounts. In the second case infinity is simply that which is equivalent to zero. Since the second case is coherent and doesn't reduce to a misnomer, it would be best for both the lay individual and scholars alike to use the word "infinity" according to the second—infinity's only relevant—definition. It may make the categories "finite" and "infinite" mathematically trivial since all mathematics would be about what is finite, but changing the definition of infinity would be at least logically coherent. Because of its long tradition, however, I doubt folks will be using "infinity" in this way anytime soon. Even so, unless infinity means simply "not finite" by being no different from "zero," it is either a misnomer or simply a logically incoherent notion.

Without such a redefinition, infinity and the infinite lead to logical contradictions. We have seen that sets, spatial extents, and amounts of time must have limits and so be finite in order to avoid the problems that infinities run into. That means that when we try to conceive of sets, extents, or amounts as being complete and yet having no limits, we end up with limited things that are supposedly limitless. We have lines, all of which must have ends, claimed nevertheless not to end. We have a series of moments, that must be finite and only indefinite at best, but the series is claimed to be complete and limitless. Endless ends and uncompleted completions. These are no paradoxes; these are contradictions. It must be concluded that the notion of infinity, as traditionally defined in mathematics, must refer to a logical impossibility by virtue of being self-contradictory. It is no wonder then, that infinity has no clear meaning. As a logical absurdity, it loses *any* coherent meaning.

That's my conclusion, but I should now admit that my case hinges on taking a certain metaphysical position with relation to the question of infinity. Historically, the two most influential philosophers of mathematics have been Plato and Aristotle. Whether mathematicians realize it or not, all mathematical thought on infinity has been influenced by those two philosophers. For Plato, mathematical objects exist independently of human conception—they are not invented but discovered, as one would discover a new property of stellar objects. Aristotle, on the other hand, regarded mathematical objects as abstractions that are invented during the process of calculation. While Plato regarded mathematical objects—numbers, values, geometric figures, etc.—as existing independently in a metaphysical "world of forms," Aristotle believed that mathematical objects have no metaphysical existence independent of human conception; they exist only as descriptions or deductions. I have taken the Aristotelian approach over the Platonic.

Whether you retain the infinite in your mathematics or replace it with the indefinite will depend on whether you are of a more Platonic bent of mind or have a more Aristotelian outlook. These days most mathematicians and theoretical physicists are Platonists: they believe that nature is inherently mathematical or quantitative in structure and so numbers have a metaphysical reality of their own. As Cambridge Professor of Mathematical Sciences Dr. John D. Barrow states, "…mathematical properties of things are real and intrinsic to them. They are more than mere labels. We discover them; we do not merely invent them."[41] From that point of view, if a numeric property seems paradoxical, then that indicates only our limitations in understanding the nature of numbers. Infinity is supposedly such a property. So, even if infinity is paradoxical, it is nevertheless a genuine feature of certain sets or series. The infinite exists already out there in nature, awaiting "discovery" by mathematicians or physicists when they calculate the world. From this perspective, when infinities arise in calculating acceleration toward light speed, in determining black hole masses, or in extrapolating the number of supposed parallel universes, physicists are finding out something profound about the Universe that our finite minds don't fully comprehend.

In contrast, most philosophers lean toward the Aristotelian view: mathematics is a useful tool for describing certain features of the natural world, as when the Celsius and Fahrenheit scales are used to measure heat, but numbers certainly have no independent existence in a Platonic world of forms. Rather, new numbers are invented or "extrapolated" during the act of calculation and have no prior existence all their own. If they have any other existence, they exist only as logical possibilities of calculation or extrapolation. Being calculable, at least in a logically possible sense, is the defining feature of a number. In this view, mathematics isn't a world that exists apart from the possibilities of calculation or invention. Mathematics isn't an intrinsic part of nature; instead, it's like a strategic game with rules that are invented by the game makers—the mathematicians. And when the rules break down, it's time to revise the rules.

In a modern Aristotelian view, if a numeric property is "paradoxical," then that paradox results from the artificial rules of an invented mathematics. And if the solution of the paradox can't be derived without ad hoc maneuvers, it's a good bet the supposed paradoxical property is pointing to an outright contradiction somewhere in the mathematical system. In other words, there's likely a fallacious assumption somewhere in the system causing the contradiction. If so, then the assumption needs altered or thrown out in favor of a more consistent idea so that the contradiction can be avoided. One such "paradoxical" property is infinity. When the paradoxes of infinity are exposed as genuine contradictions, it can be clearly seen that infinities don't reveal anything about the Universe—instead, they point to bad assumptions about the nature of mathematics and its place in describing the Universe.

Now Aristotle himself thought that only an "actual infinity" was self-contradictory while a "potential infinity" was still a coherent idea, but I go a step further and claim that the traditional understanding of infinity itself is the problem. In arguing that infinity, as a device for calculation, is a logical absurdity, I have taken a reformed *neo*-Aristotelian approach over the Platonic approach to mathematics. From this perspective, the notion of an actual infinitude generates self-contradictions and potential infinity seems to be a misnomer for the indefinite, so it is best to abandon the idea of using infinities to make mathematical calculations altogether.

Beyond Infinity

Most of the time, when folks use a word like "infinite" to describe something they simply mean that it is "uncountable" or "vast beyond comprehension." Either of these meanings though, could just as easily be contributed to quantities that are finite, but indefinitely large. The lexical definition of infinity employed by the lay person is even vaguer than the technical definitions used by logicians and mathematicians. It would be better for us all to watch our language and use other words like "vast" and "indefinite" (or even "endless" and "boundless") to get our meaning across than to inaccurately use words like "infinite."

We also need something more than just a change in vocabulary. If infinity is abandoned as a complete-limitless sequence, mathematicians need a more logical concept to replace it as a magnitude, and they also need a more logical concept to replace the notion of the infinite as a complete-limitless set. What are we to use in place of the concept when we want to refer to finite quantities or values that are nevertheless “off the scale” of known or computable natural numbers? We must still have a symbol to replace the lemniscate or Mobius strip in equations leading to off-the-scale values and a new corresponding concept to replace the old incoherent concept of infinity. We also need a new concept, and possible symbol set, to replace the idea of the infinite in set theory. In both cases, I propose that *indefiniteness* and *the indefinite* are the terms referring to the alternative concept we need.

If a value is “indefinite,” then that means either (A) it would be found to terminate the series if it could be measured or counted, but is currently either too minute or too vast to be measured or counted in actual practice or (B) it is finite in the sense of being currently undefined as the *next* value, or sequence of values, beyond the highest or lowest value that can be actually computed. In A, it is merely our practice of counting or computing that limits us from finding the end and eliminating the indefinite. In B our number system just trails off, implying more to come (1, 2, 3…), but simply doesn’t specify how much more. If the highest natural number that could even *in principle* be invented were ten, then the next number after that would be “the indefinite.” In either case, a higher finite value is indicated but not defined, so the value is “indefinite.” Values of irrational and transcendental numbers, such as pi or the square root of two, are also indefinite for the same reason. As mentioned earlier, new values for these ratios could always be calculated or invented, given whatever time remains for us to do so.

With these meanings for *the indefinite* in place, the infinite could be dropped from its traditional use in mathematics and ordinary discourse, and replaced with the concept of the indefinite. Any value that was previously labeled “infinite” can now be labeled as “indefinite” since we can always, given the time, calculate new values, with the next value in line remaining undefined until the limit is pushed back further. Functionally speaking, nothing would be lost—values that were thought to be “infinite” could just as well be thought of as “indefinite.” In terms of ordinary discourse, on those occasions in which one would use the adjective “infinite” one could just as well say “indefinite,” and the traditional use of the adverb “infinitely” could be replaced with the adverb “indefinitely,” as more accurate in reference.

The old concept of infinity can be replaced by the concept of indefiniteness. And indefiniteness can be given its own symbol to use in mathematical equations. Since we can replace the concept of infinity with indefiniteness, then we can also replace the symbol for infinity (∞) with a new symbol for indefiniteness.

There are many ways we might come up with a symbol for indefiniteness and I am open to suggestions. For now, here is one solution we might take: Because any indefinite positive value is a value greater than any definite positive value and any indefinite negative value is less than any definite negative value, we can represent indefiniteness by a symbol that captures this vagueness in value. A symbol for indefiniteness could be formed by combining angle brackets (“less than” and “greater than”) with a tilde (sometimes used to represent approximation) into an expression:

<~>

Such an expression could then be used as a single symbol for indefiniteness.

This *indefiniteness symbol* could easily be used in place of the lemniscate in equations without loss of mathematical function (or perhaps I should say “malfunction” since infinities really indicate we’ve run off the scale anyway). So, for example, take an equation that, traditionally, would use the lemniscate:

$$\mathit{int}\ \{a\}\ \hat{}\ \{b\}\ f(t)\ dt = \infty$$

This equation is taken to mean that *f*(*t*) does not limit a nonzero area from *a* to *b* as finite. But such an equation can be rewritten as:

$$\mathit{int}\ \{a\}\ \hat{}\ \{b\}\ f(t)\ dt = \mathbf{<\sim>}$$

In the second version of the equation, although *f*(*t*) means there is no *definite* finite area from *a* to *b*, this only means that we have an instance of a finite area that is *indefinitely* large (off the defined scale) rather than infinite (complete but limitless). Any attempt to calculate the area goes on indefinitely rather than infinitely. The mathematical function of the indefiniteness symbol thus remains the same as that of the lemniscate, but the *meaning* is different. Now we have a logically meaningful expression without loss of mathematical function.

In a similar manner, we could devise substitute symbols for Cantor's aleph cardinals to designate various degrees of *indefinite* sets instead of infinite sets (if such abstractions are ever really needed). For example, instead of the Hebrew aleph, we could use a different Hebrew letter, such as qof:

ק

We could then assign subscript numerals to the qof for designating the various powers of indefinite sets. So, like Cantor's aleph-naught, we could have qof-naught and various higher powers of indefinite sets: ק$_0$, ק$_1$, ק$_2$, ק$_3$, etc. And, as with the indefiniteness symbol replacing the lemniscate, we could use the qof cardinals in place of Cantor's aleph cardinals.

In fact, if we want to salvage the system of transfinite mathematics for the sake of enriching mathematical entertainment, it would make more sense to re-frame Cantor's "transfinite mathematics" as an *indefinite mathematics*, or mathematics of the indefinite, for then we would no longer think of the infinite as being a paradoxical complete-limitless property that supposedly describes things out there in the real world; instead we would use the indefinite as indicating merely our inability to describe reality completely—a symbol for the limits of our ability to calculate rather than for a supposed definite but unquantifiable property of nature.

Whether the professional field of mathematics will ever replace infinity (∞) with *indefiniteness* (<~>), and the infinite ($\aleph_0$) with *the indefinite* (ק$_0$), or not remains to be seen. I pass the torch to anyone with more influence to fight that battle. Certainly replacing the old concepts and symbols with these would be the correct course to take and would greatly impact other fields of inquiry, in particular physics and cosmology.

The Finite Universe

Throughout science infinities are asserted, especially in physics. The faster a body is accelerated, the harder it is to get it up to the speed of light. Einstein's Theory of Relativity proposes that nothing can go as fast as light because the mass of a body keeps increasing as it approaches the speed of light. The heavier it gets, the harder it is to push it faster. According to Einstein, to get a body up to the speed of light would take an *infinite* amount of energy to overcome the *infinite* mass that the body would acquire at light speed.

Infinities are also believed to exist in other areas of physics. For example, many theories in quantum mechanics assume infinities. Some quantum theories propose an infinite number of Feynman paths when calculating a set of interactions between particles like electrons and neutrinos. In addition, the Standard Model of particle physics conjectures that it is impossible to remove a quark or gluon from a hadron because it would take an infinite amount of energy to overcome the short distance interactions between the infinitely many degrees of freedom these particles allegedly have in their bound states.

Then too, cosmology has been a playground for the notion of infinity. The popular Big Bang Theory holds that space and time sprang into existence out of an infinitesimal spec of space that existed for an infinitesimally short amount of time in which the Universe was infinitely hot and dense due to having an infinite amount of energy and mass compressed in that spec. Such regions of "spacetime" are called "singularities." The core of each so-called "black hole" is supposedly a singularity as well.

Mainstream cosmology asserts that another kind of infinite characterizes the opposite scale of space and time. Today, many cosmologists believe our observable universe is just one in an infinite number of "parallel universes" making up "the Multiverse." In addition, some cosmologies even propose an infinite number of other spatial *dimensions* beyond the usual three of height, width, and length.

These are just a few examples of the many instances in which infinities are assumed in physics and cosmology. But if the case against infinity holds, then the infinities that are being conjectured in science are absurd; they signal either that a theory has taken a wrong turn or at least that the theory must be reinterpreted in terms of indefinite finitudes. Just as rejecting the infinite for the indefinite would give us a more coherent mathematics, so too it can give us a more accurate science. All that remains is for the theorists to reevaluate the notion of infinity.

And reevaluate it they must, for we have seen that the traditional notion of infinity is self-contradictory. There are no infinite magnitudes. However vast the Universe, however indefinite our calculations of space and time may be, all that is, is finite.

NOTES

1. Dr. Peter Fletcher of Keele University asserts, "if you believe that all talk of actual infinity is meaningless then you cannot even pose the question of whether actual infinity exists: if you believe that no actual infinities exist then you evidently believe the question is meaningful and answerable." (Fletcher, Peter. (2007) "Infinity," Chapter in *Handbook of the Philosophy of Science*, Amsterdam: Elsevier, p.548.) Fletcher's statement is true enough, but it does not apply to this paper. I am not asserting that "all talk" of infinity is meaningless. I am simply asserting that the traditional mathematical meaning of infinity is logically incoherent, just as the idea of a four-sided triangle is incoherent. And just as one logical absurdity, a four-sided triangle, cannot exist, so too another logical absurdity, (the traditional notion of) infinity, cannot refer to something that exists.

2. There is also opposition to skepticism of infinity (or the infinite) from outside the field of professional mathematicians. For example, one philosopher on the Web stated that skepticism of infinity constitutes the fallacy of *argument from personal incredulity* since those arguing against infinity allegedly assert that the infinite must not make sense merely because they have a hard time imagining infinite sets and series. He goes on to say that, "our limited human imaginations are poor guides to what properties the universe can have." (See Ash, Thomas. (2001) "The Case Against the Cosmological Argument." Article retrieved August 2, 2009 from http://www.bigissueground.com/atheistground/ash-againstcosmological.shtml.)

There are at least two problems with making that charge of fallacy: First, the charge of committing the argument from personal incredulity may apply to some arguments that have been made against the infinite, but it does not apply to arguments such as mine (or even to the particular argument opposed by the quoted philosopher making the charge). My arguments make the case that there are inherent *logical contradictions* in the traditional notion of infinity. That issue has nothing to do with the limits of human imagination. If something is self-contradictory, it isn't a rational concept—in which case it is little wonder one can't imagine it! Second, to dismiss skepticism of the infinite by claiming that it only looks contradictory because of "our limited human imaginations" is not really evidence; rather, such a claim is an instance of the fallacy of *the appeal to mystery*.

So, to disprove my case against infinity, one would need more than flip charges or appeals to mystery: it would have to be shown where my reasoning is inconsistent and why the contradictions I show infinity to have are not really contradictions after all. If such can be shown, I will recant.

3. In some passages I use the terms "indefiniteness" and "the indefinite" interchangeably. Other passages draw a distinction between these terms, in which "indefiniteness" indicates an undefined value or sequence of values beyond the defined values in a *series* while "the indefinite" refers to a *set* of undefined values.

4. See entry for "finite, a. and n." in *The Oxford English Dictionary* (2nd Edition, 1989) accessible online at http://dictionary.oed.com.

5. See entry for "finite, 1" in Borowski, E.J. and Borwein, J.M. (1991) *The HarperCollins Dictionary of Mathematics*. New York: HarperPerennial division of HarperCollins, p. 221.

6. Maddocks, J.R. (2010) "One-to-One Correspondence." Article retrieved February 13, 2010 from http://science.jrank.org/pages/4861/One-One-Correspondence.html

7. See entry for "countable" in Borowski and Borwein, 1991, p. 130.

8. Craig, William Lane and James D. Sinclair (2009) "The Kalam Cosmological Argument," Chapter in *The Blackwell Companion to Natural Theology*. (Craig, William Lane and J.P. Moreland (eds.). Oxford: Blackwell Publishing, p. 105.

9. Fleming, William. (1890) *Vocabulary of Philosophy: Psychological, Ethical, Metaphysical*. 4th ed. NY: Scribner & Welford, p. 211.

10. See Endnote 12.

11. Adams, Richard. (2005 ed.) *Watership Down*. New York: Simon & Schuster, Footnote p. 5.

12. Kornai, Andras. (February, 2003) "Explicit Finitism." *International Journal of Theoretical Physics*. Vol. 42, No. 2, pp. 301-307. Retrieved January 12, 2010 from http://metacarta.com/Collateral/Documents/English-US/Explicit-finitism-Kornai.pdf. See also Sazonov, Vladimir. (1995) *On Feasible Numbers* in Leivant D., ed., "Logic and Computational Complexity." *Lecture Notes in Computer Science*. Vol. 960, pp.30-51. Slides retrieved January 12, 2010 from http://www.csc.liv.ac.uk/~sazonov/papers.html.

13. Compare entry for "infinite, 2" in Borowski and Borwein, 1991, p. 292.

14. See entry for "infinity" in *The American Heritage Dictionary of the English Language, Fourth Edition*. Retrieved March 16, 2010, from Dictionary.com website: http://dictionary.reference.com/browse/infinity.

Notice that the mathematical function could just as well indicate a series of steps carried on *indefinitely* rather than infinitely. The function simply indicates a series of ongoing refinements: it's rather like trying to get your grade point average back up to 4.0 after getting a 3.0 on one assignment—an impossible task no matter how many assignments you ace, unless the original 3.0 is waived. So, the "limit" approached need not be designated as "infinity" since zero or any definite number would do just as well.

15. I said earlier that although zero *has* no limit, and is therefore "infinite" (limitless) in the sense of having no quantity or extension at all that can be limited, zero can nevertheless *be* a limit as when we say that fractions can be divided ever closer to zero *as* a limit. I suppose it might be countered that if *infinity* cannot be a limit, then it may not make sense to say zero can be "infinite." Indeed it would not if I made zero and infinity as such *synonymous*. However, I am doing no such thing. Rather, I am simply suggesting that "infinity" is a *condition* that zero *has* in the context of being equal to a set that has no quantity to it; infinity is "the condition of being infinite" (limitless) and zero has that condition by lying outside the natural number scale. It would *not* be correct to say zero is defined as limitless in all respects, in all contexts. It would therefore also not be correct to say that zero could function mathematically as a value identically to all the ways in which infinity is supposed to function as a value.

16. Compare entries for "infinite, 1 and 2" in Borowski and Borwein, 1991, p. 292.

17. See entry for "infinity, 1" in Borowski and Borwein, 1991, p. 294.

18. See entry for "value, 2a" in Borowski and Borwein, 1991, p. 622.

19. Cantor, George. (1952 ed.) *Contributions to the Founding of the Theory of Transfinite Numbers*, translation by Philip E. B. Jourdain, Illinois: Open Court Publishing Co.

20. Cantor, *Contributions*.

21. Craig, William Lane, quoted in Strobel, Lee. (2004) *The Case for a Creator*. Grand Rapids, Michigan: Zondervan, p. 103.

22. Ibid., p. 103.

23. Ibid., p. 103.

24. Ostler, Blake T. (2006) "Do Kalam Infinity Arguments Apply to the Infinite Past?" *FAIR Journal*. Retrieved July 4, 2009 from http://www.fairlds.org/New_Mormon_Challenge/ TNMC01.html.

25. See Oppy, Graham. (1995) "Reply to Craig: Inverse Operations with Transfinite Numbers and the Kalam Cosmological Argument." *International Philosophical Quarterly*. Vol. 35, No. 2, pp. 219-221, cited in Ostler (2006).

26. For another example of such modification, see Guminski, Arnold T. (Fall/Winter 2002) "The Kalam Cosmological Argument: The Question of the Metaphysical Possibility of an Infinite Set of Real Entities." *Philo*. Vol. 5, No. 2. Retrieved July 4, 2009 from http://www.philoonline.org/ library/guminski_5_2.htm.

27. Craig & Sinclair, 2009, p. 104.

28. Ibid., p. 104.

29. Johnson, P.O. (July 1992) "Wholes, Parts, and Infinite Collections." *Philosophy*, Vol. 67, No. 261, p. 373.

30. See, for example, both Bertrand Russell and Rudy Rucker's proposals of the ability to cross an infinite provided infinite acceleration is assumed. Russell's illustration is mentioned in James Thomson's article "Infinity in Mathematics and Logic" in *The Encyclopedia of Philosophy*, vol. 4, Paul Edwards, Ed. (New York: Macmillan, 1967), 188. Rucker's illustration is in his book *Infinity and the Mind* (New York: Bantam Books, 1983), 69). However, both of their illustrations end up representing not actual infinities, but potential infinities. Hence, even the examples of Russell and Rucker do not show that actual infinities can be crossed.

31. See the refutation of this claim offered by Black, Max. (April, 1951) "Achilles and the Tortoise." *Analysis*, Vol. 11, pp. 91-101.

32. Johnson, 1992, pp. 370-372. Johnson cites the Tristram Shandy example from Bertrand Russell's book *The Principles of Mathematics* (New York: W. W. Norton & Company, 1980), Chapter XLIII, sections 340-341. Be sure to see also Johnson's rebuttals to Raymond Godfrey's remarks on his article in "More About Infinite Numbers", *Philosophy*, vol. 69, no. 269 (July 1994), 369-370.

33. J.P. Moreland, "Yes! A Defense of Christianity", chapter in Kai Nielsen, *Does God Exist? The Debate between Theists and Atheists*, (New York: Prometheus Books, 1993), 37.

34. Mackie, J.L. (1982) *The Miracle of Theism*. Oxford: Clarendon Press, p. 93.

35. Moreland, "Atheism & Leaky Buckets: The Christian Rope Pulls Tighter", Chapter in *Does God Exist?*, p. 230.

36. Ibid., p. 230.

37. Craig, William Lane. (1999) "A Swift and Simple Refutation of the Kalam Cosmological Argument?" *Religious Studies*. Vol 35. Retrieved July 2, 2009 from http://www.leaderu.com/offices/billcraig/docs/kalam-oppy.html.

38. Martin Gardner, *The Whys of a Philosophical Scrivener* (New York: St. Martin's Griffin, 2nd Ed., 1999), 194.

39. Morris, Richard. (2003) *Achilles in the Quantum Universe, The Definitive History of Infinity,* London: Souvenir Press Limited.

40. Aristotle, *Physics*, Book 3, Chapter 6.

41. Barrow, John D. (2007) *New Theories of Everything*. Oxford University Press, p. 204.