A general framework for a Second Philosophy analysis of set-theoretic methodology

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Abstract

Penelope Maddy's Second Philosophy is one of the most well-known approaches in recent philosophy of mathematics. She applies her second-philosophical method to analyze mathematical methodology by reconstructing historical cases in a setting of means-ends relations. However, outside of Maddy's own work, this kind of methodological analysis has not yet been extensively used and analyzed. In the present work, we will make a first step in this direction. We develop a general framework that allows us to clarify the procedure and aims of the Second Philosopher's investigation into set-theoretic methodology; provides a platform to analyze the Second Philosopher's methods themselves; and can be applied to further questions in the philosophy of set theory.

1 Introduction

In *Defending the Axioms*, Maddy [2011] applies her position of Second Philosophy to set theory. In *Second Philosophy*, Maddy [2007] elaborates this position by describing the work of a certain idealized enquirer, the Second Philosopher persona:

> [The Second Philosopher has] good reason to pursue mathematics herself, as part of her investigation of the world, but she also recognizes that it is developed using methods that appear quite different from the sort of observation, experimentation, and theory formation that guide the rest of her research. [Maddy, 2011, 39]

Maddy notes that there are two types of questions for the Second Philosopher, one about set-theoretic methodology and the other about the nature of set-theoretic activity itself. The first question already encompasses quite a wide array of topics: The Second Philosopher wants to learn about set-theoretic methodology from a broad point of view and therefore considers not only localized methods (like specific theorem proving techniques), but also more general approaches (like adopting axioms). The second question, about the nature of set-theoretic activity itself, is based upon the results acquired by answering the first question. Maddy here develops the positions of Thin Realism and Arealism, which, she argues, are interchangeable and equally correct.

Although the questions of the second type seem to be the philosophically "richer" option, in this article we will concentrate on the first group of questions. One reason for this focus is that little has been done on this beside Maddy's own work.[1] More to the point, we know of no works that use Maddy's account of Second Philosophy to further investigations into the philosophy of set theory.[2] We feel that this is due to the underdeveloped status of central notions and motivations of the Second Philosopher's approach to set theory and we would like to contribute clarification to increase the usability of Maddy's approach. As we will show in section 5, there is potential for further work in this direction.

Another reason for focusing on the first group of questions is that they are a crucial prerequisite when considering subsequent discussions, like Maddy's philosophical positions developed in the second type of questions, as well as her work on the universe/multiverse debate [as e.g. in Maddy, 2017]. Here a meta-analysis of the Second Philosopher's methods will be a useful tool to get a clear picture on how moves, such as excluding heuristic reasoning, influence the Second Philosopher's analysis, as well as on what the limitations of her methodology are and how it can be changed.

In this article, we propose a *general procedural framework* for the Second Philoso-

[1]The articles that refer to Maddy [2011] are mostly concerned with questions of the second type or more general considerations into the naturalistic nature of Second Philosophy.

[2]One exception may be Ternullo [2019] on the evaluation on multiverse accounts according to Maddy's philosophical approach. However, he does not focus on framing the philosophical methodology itself, but rather on Maddy's arguments in the so-called universe/multiverse debate.

pher's investigation into set-theoretic methodology, i.e. a framework for her own methodology. We do so by developing Maddy's exemplifying descriptions in Maddy [2011] and [Maddy, 1997] into a general framework. This exercise has three aims. We want to

A) provide *clarification* on the use of central notions and aims in Maddy's work;

B) facilitate a general *analysis* of the second-philosophical investigation into set-theoretic methodology; and

C) increase the *applicability* of this investigation for further work in the philosophy of set theory.

We will show that providing such a framework is not at variance with the Second Philosopher's stance and we will give examples on how the framework can be used to further investigate open questions in the philosophy of set theory. Also note that we will work on two different levels of analysis: Aims (A) and (B) are concerned with studying the very methods of the Second Philosopher (with respects to her inquiry into mathematics), whereas goal (C) is concerned with the Second Philosophy's actual application to set theory.

We address aim (A) in section 2 by clarifying how we can provide a general procedural framework for Second Philosophy while at the same time clarifying central relevant notions. In section 3 we examine in detail Maddy's case study of Cantor's introduction of sets as given in Maddy [2011], contrast the latter with her case study of Zermelo's defense of the Axiom of Choice and, finally, propose a first schema for her procedure. Here we are interested in the specific details of a second-philosophical analysis. We then use this first rundown to formulate the general procedural framework in section 4. We show that it is indeed a framework for Second Philosophy and that it can accommodate possible changes in the Second Philosopher's methodology itself. We also address the possibility of separating Second Philosophy from Maddy's description of it. We therefore provide a platform for a general analysis, thereby addressing aim (B). Section 5 shows how we can use the framework to further applications of a second-philosophical analysis of set-theoretic methodology and to outline how this can impact discussions in the philosophy of set theory. This will a (first) answer to how we can achieve aim (C).

2 Some comments on meta-philosophy

In this section we address aim (A) from above, namely, the clarification of central notions and issues in Maddy's work on Second Philosophy. This includes the general question of whether we should give a general procedure at all, as well as clarifying Maddy's terminology and introducing the central method of means-ends relations.

2.1 Is a procedural framework desirable?

Here we want to address the question if it is at all in the interest of Second Philosophy to provide a generalized procedure as propose in this article. Maddy [2007, 1-2] herself points out that Second Philosophy not only "has no theory", but also states that its methods will remain "without any definitive way of characterizing exactly what that term entails". Instead, she goes on to describe through examples how an idealized inquirer, the Second Philosopher persona, is supposed to work. She then concludes:

> Though 'Second Philosophy' is never explicitly defined in all this, I hope that Parts I and II [of [Maddy, 2007, I, II]] provide enough guidance for at least some sympathetic readers to get the hang of how to carry on. [Maddy, 2007, 4]

Maddy [2011, 39] makes this even more explicit: "Indeed any attempt at a once-and-for-all characterization of our inquirer's methods would run counter to the ever-improving, open-ended nature of her project." But, Maddy also states that the Second Philosopher "looks into the matter of how and why the methods she and others use in their inquiries work when they do and don't work when they don't" [2011, 38]. The possibility of evaluating the Second Philosopher's methods presupposes the possibility of making them more explicit.

One main aim in the paper is exactly this: making the Second Philosopher's methodology, as currently stands, more explicit and make it amenable to a general analysis. We will show in section 4.3 that our framework provides a more explicit presentation of the second-philosophical methodology while at the same time being flexible enough not to impede the ever-improving, open-ended nature of the Second Philosopher's project. It might even be possible to use the framework to distinguish between Second Philosophy and Maddy's interpretation of it. Maddy [2007, 3] herself hints at such a possibility when writing that "one might sign on as a Second Philosopher while thinking I've gone astray in my pursuit of the particulars".[3]

2.2 Methodology and practice

First, it is crucial to note that Maddy's understanding of the notion 'set-theoretic methodology' differs from other such understandings in the philosophical literature. Maddy includes the whole setup of set theory in it, going back as far as set theory's fundamental assumptions. While many set theorists would not talk about 'adopting axioms' as a method they use to pursue set-theoretic research, this is one of the most important parts of set-theoretic methodology in Maddy's view, because the adopted axioms delineate what is allowed in a proof and what is not. Typical questions about set-theoretic methodology are: "[W]hat are the proper grounds on which to introduce sets, to justify set-theoretic practices, to adopt set-theoretic axioms?" [Maddy, 2011, 41] The reason for this very broad interpretation of set-theoretic methodology also lies in the goal of Maddy's investigation, namely, to answer philosophical questions of the second type, as pointed out in section 1. In the following, we will use the term 'set-theoretic methodology' to include mathematical methods of set theory (like forcing, transfinite induction, etc.) as well as the more foundational ones included by Maddy.

Second, it is also helpful to note that Maddy uses the term "set-theoretic practice" in a different way than the term "mathematical practice" is used for example in the

[3] See also Rittberg [2016, Chapter 6], where he discusses the possibility of "stripping away the idiosyncrasies of Maddy's investigations" [2016, 138].

context of the philosophy of mathematical practice. For one, when discussing case studies, she mainly refers to published material, therefore concentrating on what could be called "the front" of mathematical practice.[4] This raises the question of the Second Philosopher's adequate description of practice. For example, Rittberg [2016, 137] argues "that the Second Philosophical argument for UNIFY does not lead to a faithful description of contemporary set-theoretic practice".

The difference does not lie only in the source material used, but also in how this material is used for (second-)philosophical considerations. The most obvious example here is probably the use of heuristic reasoning. As we will see in section 4.2, Maddy excludes heuristic reasoning from set-theoretic methodology—not in the sense that set theorists should not use heuristic reasoning, but on the ground that such reasoning should not inform our philosophical investigations into the nature of set-theoretic activity.[5] This assessment would not be shared by authors such as Francois and Bendegem [2010].[6]

However, what she does share with many of the proponents of the philosophy of mathematical practice is the reliance on historical material and analysis. She is, for example, seen as furthering the practical turn in the philosophy of mathematics: "an exemplary case of the practical turn is the philosophy of Penelope Maddy" [Ferreirós, 2016, 23], which Ferreirós bases on the following principle: "Faithfulness to actual practice, both presently and historically, is a key requirement for this kind of work" [Ferreirós, 2016, 24]. This, of course, stands in stark contrast to the aforementioned critique by Rittberg [2016].

[4]In contrast, many studies from the philosophy of mathematical practice focus on the so-called "back" of the practice, as for example here in relation to explanation in mathematics: "The mathematical practice which we consider is drawn from the 'back' of mathematics. This contrasts with the 'front' mathematical practice" [Pease et al., 2019, 17]. The distinction goes back to Hersh [1991].

[5]For further discussion on the role of heuristic reasoning see section 4.2.

[6]Cf. Francois and Bendegem [2010, 118]:

> From the proof viewpoint, it seems obvious that different layers are required: the search for a proof and related concepts involves heuristics, proof search methods. As in many cases such techniques become interiorized [...]. Thus, this proof practice "feeds" a particular metaphysical view that supports and stimulates the mathematician's search and that definitely should be taken into account.

We think that one reason for this divergence of opinions lies in the ambiguity of the relation between set-theoretic practice and set-theoretic methodology. From the outset it could seem that practice is simply the more general term, with methodology being a part of the practice. Then we could simply describe set-theoretic methodology by collecting together all the instances of methods that come up in set-theoretic practice and perhaps augmenting this with an ordering according to importance, simplicity or whatever other feature we are interested in. However, this is not the route Maddy takes. She still takes practice as the basis for her investigation, but instead of just amassing together case studies of methods, she evaluates them. Only methods that pass certain criteria are allowed to be considered part of set-theoretic methodology and then, later on, be used for further philosophical consideration.

2.3 Means-ends relations

According to Maddy, she succeeds in isolating the proper set-theoretic methods, by finding "an array of new methods for justifying claims, methods that appear to be both rational and autonomous" [2011, 54]. We want to know *how* she manages to identify those methods. Here Maddy gives advice:

> In the cases we've surveyed, the community eventually reached a consensus that the controversial method was admissible because it led to certain varieties of mathematics, that is, because it was an effective means to particular desirable ends. Thus the positive counsel of history is to frame a defence or critique of a given method in two parts: first, identify a goal (or goals) of the relevant practice, and second, argue that the method in question either is or isn't an effective means toward that goal. [Maddy, 1997, 194]

Here Maddy observes how the community accepts a certain method (because it is an effective means towards a desirable end) and she proposes to take this justificatory structure as a guideline on how we should advance our own inquiries into set-theoretic methodology. This guideline tells us to frame the investigations into methodology in terms of a means-ends relation and test the means in the relation against its effectiveness towards the goal. The formulation of the means-ends relation and the evaluation of the effectiveness of the means takes place through historical analysis and consideration of contemporary practice; for example. through a consensus of the community.

Regarding the role of consensus: In the above quote it is presented as the outcome of some concerted reasoning in the community. However, in our general framework it can play two further roles: First, a consensus in the community can suggest a method that should be incorporated into set-theoretic methodology (for example the adoption of a certain axiom, as Maddy discusses it in regard to the Axiom of Choice). Such a suggestion does not imply that a method can be included directly into the methodology; we still have to frame it as part of a means-ends relation that we will then evaluate by some criteria. This also means that it is unproblematic if the method is only suggested by part of the community. Second, consensus can be used as a criterion in the evaluation of the means-ends relation. In contrast to the previous case, it is indeed important to figure out if the consensus is only partial and what it is comprised of. Further, consensus cannot be the only criterion for evaluation, for that would make the investigations trivial (a method is suggested for investigation by consensus and then passes the threshold of acceptance because of consensus). Thus it is important to keep these two roles distinct. In the following, we will call the former the *proposing role* and the latter the *evaluative role* of consensus.

A note on terminology: As far as we can see, the means in the means-ends relation is always a method, whereas the end in the relation is always a goal. We pointed out at the beginning of section 2.2 the broad notion of methods at play; the same will hold for goals, meaning they can "range from relatively local problem-solving, to providing foundations, to more open-ended pursuit of promising mathematical avenues" [Maddy, 2011, 52]. Moreover, being a method or a goal is not mutually exclusive; a specific part of practice can be either in different mean-ends relations: "we should expect that some goals will take the shape of means toward higher goals, and that goals at various levels may conflict, requiring a subtle assessment of weights and balances. But the simple counsel remains: identify the goals and evaluate the methods by their relations to those goals" [Maddy, 1997, 194]. Whether we want to consider something as a goal or as a method is then determined by the context and the role something plays in the means-ends relation under consideration.

At the end of this process, a method is accepted as part of the methodology if we can show that it is an effective means towards a goal. Maddy also calls such methods "rational" or claims that it is rational to include them into the methodology. As we want to avoid the discussion about what role standards of rationality play in this context,[7] we will use the effectiveness criterion for a specific mean in a means-ends relation to pass the Second Philosopher's test.

3 Two case studies from Maddy

Our strategy in this section is the following: Building on Maddy's own account of the second-philosophical procedure, we analyze what she does in the case study on *Cantor's introduction of sets* (3.1); we then propose a detailed schema for her methodology in this case study, from which we extract three key features. Subsequently, we test the schema in a further case study, *Zermelo's defense of his axiomatization* (3.2).

3.1 Cantor's introduction of sets

The analysis Maddy [2011, II.2.(i)] gives of Cantor's introduction of sets is typical for a second-philosophical analysis. The guiding question of the present section is how Maddy interprets the historical and mathematical material in order to finally identify a specific means-ends relation and then can conclude that specific methods and goals found in set-theoretic practice are part of the justificatory structure of set theory.

In her reconstruction of the historical case study, Maddy identifies the method with the introduction of sets as new entities. She draws this from historical analysis,

[7] For elaboration, see Maddy [1997, 197].

i.e. Ferreirós interpretation of Cantors work. Later on she implicitly uses the proposing role of consensus, for instance when she assumes that Cantor's introduction of sets is a *typical* example for set-theoretic methodology [Maddy, 2011, 52]. She is justified in the assumption insofar as the point can be made that today there is a consensus on sets as accepted entities in the mathematical community.[8]

In investigating this method, she describes how Cantor was led to the introduction of his definition of derived point set [Cantor, 1872] and explains why this can be considered to be the moment of the introduction of sets as new entities. In the setting of this case study, she identifies a specific research goal of Cantor, based on her observation that "Cantor was engaged in a straightforward project in analysis: generalizing a theorem on representing functions by trigonometric series" [Maddy, 2011, 41]. This suggests that, in general, the research goal of the involved agent(s) can be interpreted as a goal of the relevant practice.

She establishes a means-ends relation in the following way: "a new entity—a set—has been introduced as an effective means toward an explicit and concrete mathematical goal: extending our understanding of trigonometric representations" [Maddy, 2011, 42]. Thus, when establishing the means-ends relation, she already evaluates the method as effective, based on her prior analysis in which she points out that Cantor could prove an important theorem about trigonometric series by using point sets in this specific way. Finally, the method of introducing sets as new entities is included into set-theoretic methodology.

We can now give a schema for Maddy's procedure which is extracted from our analysis of the Cantor example and her description of the second-philosophical method; it is to be understood as a coherent interpretation of Maddy's methodology. In the following, M denotes the means and X denotes a goal:

We start with the observation that some M, often suggested by consensus, is an established part of mathematical methodology. Two questions present themselves:

[8] Though there is an ongoing debate about the foundational role of set theory with respect to other candidates such as category theory and homotopy type theory, the use of sets is widespread over various (if not all) mathematical research fields, and the argument here is not about the importance of set theory as a foundation, but on the mathematical notion of set and its introduction into mathematics.

What is the justification for M? (This is the research question.) When was M introduced and by whom? (This is the methodological question.) As a next step, we have to choose the material from which to draw conclusions: we consider historical reconstructions about and source material of (one of) the event(s) surrounding the introduction of M. Then we reconstruct the event and try to identify a goal X. The central evaluation is carried out by assessing whether there is evidence that M is an effective means for X. If not, we can try to identify a different goal and repeat the assessment. If we succeed, we have found an argument in favour of M, if we fail we should not consider M to be a justified part of set-theoretic methodology.

There are three key components of Maddy's methodology that we extract from this schema: First, the focus on a particular philosophical method, the analysis via means-ends relations. Second, an evaluative component, which is used when assessing the means-ends relation at hand and when identifying the goals. And, third, the use of data both from contemporary practice as well as historical analysis to argue for or against the choice of methods and goals and to investigate the overall effectiveness of the means towards the end.

Let us now turn to one of the other case studies Maddy gives in Maddy [2011] and see if we can find these three key components at work.

3.2 Zermelo's defense of his axiomatization

Maddy [2011, 45-47] offers a case study of Zermelo's defense of the adoption of the Axiom of Choice, building on the evidence found in Zermelo [1908a] and Zermelo [1908b]. Here we will not reconstruct the entirety of Maddy's procedure, as we did in the Cantor case, but only discuss the main differences between the two case studies.

The Zermelo case study adds two aspects to the conclusion drawn from the Cantor case: it introduces a new method, namely the adoption of axioms and it discusses criteria for choosing goals.

As to the first aspect, it may not be clear from the outset how the adoption of an axiom is to be framed as part of a means-ends relation. For one, it is not clear why the adoption of axioms is a method and not a goal that one aims to justify. To clarify this, let us consider how axiom acceptance is used in Maddy's case study. In the Zermelo case, the most convincing arguments according to Maddy are the mathematical outcomes the adoption of an axiom brings about. These can be framed in terms of clear mathematical goals: we aim to prove a certain theorem, we stipulate that a certain mathematical feature should hold or not, etc. How can we reach these goals? By adopting a certain axiom that allows us to prove the theorem, by getting the mathematical feature to hold, etc. In this sense, the adoption of an axiom can be framed in a means-ends relation: adopting an axiom is the means that allows us to reach a certain goal. As we mentioned in section 2.3, this might not be the only way to consider the adoption of an axiom, as means in one case can be ends in another. But, if we want to assess whether they should be included in set-theoretic methodology, we have to consider them as means, because the final evaluation of the means-ends relation judges the effectiveness of means.[9] Note that when presenting the Zermelo case study, Maddy does not outright frame it via means-ends relations. However, she does so in comparable examples like [Maddy, 2017, 303].

Secondly, Maddy discusses criteria for choosing goals by introducing the well-known distinction between intrinsic and extrinsic justification. This is based on the observation that Zermelo proposed both kinds of arguments in order to defend his Axiom of Choice. Maddy assesses the two types of justification differently. While extrinsic justifications are proper justificatory devices, intrinsic reasons are not and may be devoid of justificatory power. [10]

[9]Note that we do not claim that this framework of methods and goals follows the actual developments of the practice. The mathematical work (proving theorems, etc) might come first and only later an axiom is put forward that will entail desirable mathematics. Nonetheless, we would then say that the axiom is justified because it is a means towards this desirable mathematics and then we consider again the same means-ends relation.

[10]In [Maddy, 2011], this can also be detected in her comment to the effect that

> [Zermelo's] claim is that Choice is [intrinsically justified] ...; we might now express this by saying it is part of the informal 'concept of set'. But, as we've seen, Zermelo despairs of defining [the concept of set] with a precision adequate to the development of set theory. Instead he appeals to a second standard of evidence [i.e. extrinsic justification]. [Maddy, 2011, 46]

Here we have an example on how goals are evaluated, meaning that even before the final evaluation of the means-ends relation takes place, goals have to satisfy certain criteria. In this instance, they

For the method of adopting the Axiom of Choice in the Zermelo case, Maddy mentions several goals, that this leaves us with two types of extrinsic goals: A mathematical one, as we need the Axiom of Choice in order to prove certain theorems that mathematical practice (not just set theory!) wants to hold. The second type of goal is foundational, i.e. to create more "productive science" and choose an axiomatization that is "sufficiently wide to retain all that is valuable in this theory" (Zermelo, cited in Maddy [2011, 46-47], cf. Maddy [2011, 52]). In section 4.2 we will include criteria connected to extrinsic justification under the content-based criteria.

After studying Maddy's representation of the Second Philosopher's work in this section, we will proceed to formulate a general framework in the next section.

4 A general framework for a Second Philosophy analysis of set-theoretic methodology

From the work surveyed in the last two sections, let us propose the following procedural framework for a second-philosophical analysis of set-theoretic methodology: The *core procedure* is given by the method of analysing set-theoretic methodology via historically informed and contemporary means-ends relations. This includes the formulation of the method qua means, the identification of the goal(s), as well as the final evaluation of the effectiveness of the method as means to realize the goal. This core part is supported by an *evaluative apparatus* that contains the criteria with which the Second Philosopher evaluates the choice of the methods, goals and case studies. The evaluation is carried out with the help of historical analysis and consideration of contemporary practice. The framework is procedural in the sense that usually the Second Philosopher will formulate a certain means-ends relation and then test the methods and goals according to the criteria of the evaluative apparatus. If they turn out to be unsatisfactory, she may re-formulate them, resulting in a different means-ends relation that again is evaluated—and so on. In the end, she either reach a satisfactory formulation of the methods and goals, in which case

have to be extrinsically, not intrinsically, defined goals.

we can proceed to evaluate the effectiveness of the means; or we won't find such a relation. According to the outcome, she can add the method under investigation to set-theoretic methodology or not.

In the following, we will elaborate on the two parts of the framework. We are especially interested in providing a way to analyze the Second Philosopher's methodology itself, therefore addressing goal (B) from section 1. We will show that one major feature of the procedural framework is to provide a background in which we can analyze the methodology of the Second Philosopher herself on different levels, while keeping track of how changes in different parts of the framework will alter and influence the overall methodology. This will also show that the framework allows for enough flexibility to accommodate the "ever-improving, open-ended nature of [the Second Philosopher's] project" [Maddy, 2011, 38] and therefore address the worry voiced in section 2.1.

4.1 The core procedure

The core procedure contains the methods used by the Second Philosopher in her investigations into set theory. Building on Maddy [2011], we identified the analysis using means-ends relations as the method used to investigate set-theoretic methodology. We discussed in section 2.3 what the method and goals consist in. For the final evaluation, the method has to be effective to reach the goal in question. Providing that the means and ends satisfy the criteria of the evaluative apparatus, this can be judged through the analysis of the historical case study and of contemporary practice.

It is important to note that, by describing the core procedure in this way, we do not claim that an analysis via means-ends relation is the only method the Second Philosopher should use. Instead, as the Second Philosopher constantly reflects on her own methodology, she is free to add other methods to the core procedure; change details of how the means-ends analysis works; or even discard this method entirely. Of course, a change in the core procedure would require a thorough analysis. In section 4.3 we will develop this further and give some hints as to why some changes concerning the method of employing the analysis of means-ends relations might be necessary to investigate *contemporary* set theory.

4.2 The evaluative apparatus

As we mentioned before, the Second Philosopher's ultimate goal of answering questions about the very nature of set-theoretic activity requires a careful choice of means-ends relations. While the choice of the the relevant setting by fixing a method and goal could be attained by simply picking a method and goal from practice (perhaps using the proposing role of consensus), here an evaluative component comes in. For example, the Second Philosopher wants to avoid including tangential methods or methods that lead practice astray in some sense.[11] This refers back to the evaluative role of consensus given in section 2.3. Additionally, we have seen that Maddy favours extrinsic over intrinsic justification. This gives rise to criteria that say something about mathematical productiveness or relevance. But it also hints at the possibility of a negative criterion on what methods and goals should not be.

Maddy provides one such negative criterion under the label of *heuristic aid* or *heuristic device*. Let us give two examples to illustrate what this criterion is meant to subsume:

Example 1

> Dedekind's paper on the natural numbers includes his belief that they are 'free creations of the human mind' (Dedekind [1888], p. 791). Given the wide range of views mathematicians tend to hold on these matters, it seems unlikely that [they] would all agree on any single conception of the nature of mathematical objects in general, or of sets in particular; the Second Philosopher concludes that such remarks should be treated as colorful asides or *heuristic aides*, but not as part of the evidential structure of the subject. What matters for her methodological purposes

[11] This also allows the Second Philosopher to reject examples of individual practices or the practice of small sub-communities as template for general set-theoretic methodology.

> is that all concerned do feel the force of the kinds of considerations we've been focusing on here; these are the shared convictions that actually drive the practice. [Maddy, 2011, 52–53, our emphasis].

In this example Maddy uses the evaluative role of consensus in the community to reject a particular group of beliefs as relevant for set-theoretic methodology. Not only does she claim that there was no consensus about such beliefs, but even more, that such a consensus would likely have never come about. She then uses consensus to identify such heuristic aides, aiming at excluding such beliefs pertaining to individuals or subgroups of practitioners from the justificatory structure of set theory. In the next example, we can see that consensus is not the only aspect of the "heuristic aid" argument.

Example 2

> [...] Linnebo and Pettigrew propose that the iterative conception justifies ZFC [...]. My own view is that the iterative conception is a brilliant *heuristic device*, but that the justification for the axioms it suggests [...] rests on their power to further various mathematical goals of set theory, including its foundational goals. [Maddy, 2017, 303, our emphasis]

Embedding this example in a means-ends relation, Maddy claims that choosing the goal of "compliance with the iterative conception" for the method "adopting ZFC as axioms" will not give rise to a usable means-ends-relation because the ends are not of the right type; instead they should be mathematical and foundational goals. Here, Maddy contrasts heuristics with subject-based considerations, i.e. mathematical or foundational considerations in mathematics.

These two examples show that there are different flavours of the "no heuristics" criterion and we argue that it is not *one* criterion after all. Instead, different aspects of it can be subsumed under different criteria, depending on which flavour we want to address. In Example 1, we claim that the "no heuristics" argument is actually part of a consensus-based criterion. As we discussed the different roles of consensus before, let us turn to Example 2. Here, we claim that the argument is part of what we call, for lack of a better term, a subject-based criterion. Let us give a broader background for this criterion:

Judging from Example 2, one could think that the difference between heuristic and non-heuristic arguments lies in philosophical vs mathematical considerations. Maddy [1997] describes the role of historical analysis as including negative counsel that "advises us that certain typically philosophical issues are ultimately irrelevant to the defence or criticism of mathematical methods" [1997, 194]. At the same time, she concedes that "[d]isappointing as this may be, I think there is no principled distinction to draw between mathematics and philosophy [...]. At least, this isn't the course I take" [Maddy, 1997, 193]. Instead there are instances of philosophical considerations that can and should be included into our analysis. One example is her argument that questions, such as whether CH has a definite truth value, are still legitimate even if they are independent from ZFC. They are legitimate because they can be framed and answered in mathematical terms [Maddy, 1997, 194-195]. Also, Maddy [2011] views foundational goals, like providing a unified basis for mathematics, as legitimate goals and does not characterize them as part of heuristics.[12] In the end, it seems to come down to a fine-grained analysis of the considerations in question, with the guideline that considerations are the more allowable the more they are mathematically informed, i.e. tied to mathematical content or mathematical ways in which they can be phrased instead of only concerning philosophical thinking. Coming back to Example 2, this raises the question why the iterative conception should be so decisively disregarded. There are ways in framing the iterative conception more mathematically, for example as done in Boolos [1971].[13] As we hinted at in section 2.2, there are also arguments to be made that heuristics indeed is an important part of mathematical methodology and should inform further philosophical debate.[14] However, in the following we will include the "no heuristics" argument, as it plays an

[12]For a comprehensive discussion of what foundational goals are, see Maddy [2017].

[13]Many thanks to Neil Barton for pointing this out to us.

[14]Notably in her very early work, Maddy [1988] does not use a "no-heuristics"-criterion. To the contrary, she relates heuristics to extrinsic reasoning when she emphasises that also ZFC axioms are supported by "'extrinsic' (pragmatic, heuristic) justifications, stated in terms of their consequences, or intertheoretic connections, or explanatory power, for example" [1988, 483].

important part in Maddy's work and the goal of the framework is to structure this work.

Finally, let us list three criteria which we identified in Maddy's work and which are at play in the evaluative apparatus:

Consensus-based criteria[15]: "typical" [Maddy, 2011, 56], "natural" [Maddy, 2011, 85] shared beliefs that actually "drive the practice" [Maddy, 2011, 53].

Content-based criteria: part of the "evidential structure of the subject "[Maddy, 2011, 53], "productive" [Maddy, 2011, 85], "methodologically relevant" [Maddy, 1997, 193].

Subject-based criteria: no heuristics[16], no "philosophical" but mathematically informed content (in the way detailed above).

Three remarks are in order here: First, the dividing line between these criteria are not sharp. The idea behind the distinction between the first two criteria is that consensus-based criteria rely on judgments of the practitioners, whereas content-based criteria rely on the content being produced by the practitioners, like theorems, proof methods etc. Of course, it could be argued that 'relevance' is also a consensus-based criterion, however such ambiguities will not damage the general structure of the framework.

Second, it would be interesting to find and analyse a method or goal on which the criteria disagree. For example, what if a strong consensus emerges in the community about a goal or method that the Second Philosopher dismissed as heuristic device? Without being able to elaborate here, we think that the previously discussed case of the iterative conception could be construed in this way. This goes to show that we do not believe that the outcome of the procedural schema is guaranteed to be conclusive. More than that, we perceive the possibility of an inconclusive outcome as a desirable property of our framework as scientific practice is in flux and sometimes inconclusive itself. It should therefore be reflected by an analysis into its methodology.

Finally, we can see all of the criteria at work in Maddy's case study of Cantor's introduction of sets. As we pointed out in section 3.1, the consensus-based criteria

[15] Note that this refers to consensus in its evaluative role, not its proposing role

[16] [Maddy, 2017, 303]

are met because sets as entities are regarded as a standard method in set theory. The content-based criteria are also met: the whole mathematical system of set theory is evidence for this. And, finally, the subject-based criteria are met, as evidenced by the following observation: Maddy uses terms like "exists" or even "ontology" not in "any philosophically loaded way: I just mean what the practice asserts to exist, leaving the semantic or metaphysical issues open" [Maddy, 2017, 296, footnote 13]. So, when analyzing the method of introducing sets as new entities, we do not attach a metaphysical claim to this method, in the sense that sets are introduced as objects that "really exist", a belief which seems not to be a potential candidate for overall consensus. In this sense, no heuristics is involved in the method.

The procedural framework we have developed in the last two sections is modeled after Maddy's presentation of Second Philosophy. Our choice of criteria is guided by Maddy's analysis. However, we have also seen that some of the criteria can be called into question. In the following section, we will show that the procedural framework does allow for changes, for example in the choice of criteria. This opens up the possibility to be a Second Philosopher without having to agree with the entirety of Maddy's interpretation of it.

4.3 Analysing the Second Philosopher's methodology

In Section 2, we raised the question whether it is in the interest of the Second Philosopher to provide a generalized framework in light of the open-ended and ever-improving nature of her inquiry. In reply to this worry, we claimed that our procedural framework would be able to supply a concrete guideline on how second-philosophical inquiries can be analyzed, while at the same time being flexible enough not to impede the nature of the Second Philosopher's project. In this section we want to argue that we have succeeded on both accounts.

Changes in the framework can occur on different levels: For one, the content of the criteria can change. For example, foundational goals could change, as indeed happened with Maddy's **Final court of appeal**[17] that later seemed too restrictive

[17]"Provide decisive answers to questions of ontology and proof: if you want to know whether or

and were revised into **Shared Standard** and **Generous Arena**.[18] Similarly, practice could produce a number of new results, showing a method to be productive that beforehand was not.

We could also add or remove criteria or make the proposed criteria more fine-grained. What our procedural framework does not allow for, however, is to get rid of the evaluative apparatus altogether (as it is an integral part of the framework). But this restriction does not present a problem for the Second Philosopher. She is not only interested in describing the practice, but in evaluating it in a certain way, separating relevant from irrelevant parts of the practice; this separation has to follow some criteria, therefore there has to be an evaluative apparatus.

The same holds in an even stronger sense for the core procedure, as it contains the methods used in the Second Philosopher's inquiry. Still, we have the possibility of introducing changes to the core procedure. Following Maddy's interpretation of the Second Philosopher's inquiry, it only contains the method of analysis via means-ends relations that is conducted through historical analysis and consideration of contemporary practice. The way in which we choose the means and ends is directed by the evaluative apparatus; any changes here would reflect back to changes in the evaluative apparatus, which we discussed above. This still leaves us with the possibility of adding new methods. Indeed, in the case of science, the Second Philosopher already has more than one method available to him (observation, hypothesis making and testing etc.), so it is reasonable to assume that she could come up with more than one way of analysing set-theoretic methodology. In the following we will provide an argument outline for why adding a new method (or at least adapting the method of means-ends relations) might be necessary.

We start with the following observation: There is one disadvantage to the dependence on historical analysis, namely that we cannot use it when considering very recent developments in set theory. Looking at the literature, historical analysis of

not a so-and-so exists, see whether one can be found in V; if you want to know whether or not such-and-such is provable, see whether it can be derived from the axioms of set theory" [Maddy, 2017, 296].

[18]For a more detailed discussion, see Maddy [2017].

set theory becomes rapidly scarcer for the time frame after the 1950s and 1960s.[19] This is certainly not a result of historical oversight. Instead, if events are too recent, historical analyses are not forthcoming. It might simply not be methodologically possible at this point, as many historical sources are not yet available to the historian.[20] For the perspective of an overall history of mathematics, this might not be a problem; from the perspective of longue durée, fifty years may not even be a proper historical subject matter. But if we want to consider contemporary practice, fifty years is enough to drastically change a mathematical area both in its methods and its research goals (for example, consider the time between the 1850s and the 1900s in algebra).

As we have seen, the correct evaluation of the means-ends relation relies heavily on historical analysis. But if we cannot address (roughly) the last fifty years, we lose much insight into contemporary practice (especially in a discipline that is only little more than one hundred years young). Can this restriction be overcome in some way? Can recent set theory be made accessible to a second-philosophical analysis? We want to give on outline for an affirmative answer—at least up to a certain degree.

The main role of the historical analysis in the procedural schema is to create a test case for our intended means-ends relations. Here we can use a historical account of the practice to search for suitable methods and goals and test them against the criteria of the evaluative apparatus. We also use contemporary practice as a control case, for example to check whether the criteria still hold, whether the historical practice is still relevant, and so on. But if we don't have historical analyses available, we lose our primary source of test cases. To counteract this, we could try to use contemporary practice as the test case. This is difficult, however, as we will lack historical hindsight. For example, it will be more difficult to judge consensus-based criteria, like naturalness, for very new methods, as naturalness is (at least to a degree) connected to the community getting familiar with a new method. Content-based criteria are also difficult to assess, as a method can show much promise but

[19] For example Ferreirós [2007] ends with a short outlook on the time immediately after World War II; work in the 1960s is already scarcer, exceptions are scholarly work by Moore [1987], Kanamori [2008].

[20] For example access to Nachlässe, correspondences, etc.

de facto has not produced relevant results yet. And subject-based criteria are even more difficult to judge: When mathematicians are in the process of proving and formulating hypotheses, they heavily rely on heuristics and intuitions; some distance to the work is needed to see what was actually mathematically necessary and what was a crutch for thought that can be thrown away afterwards.

Nonetheless, we speculate that there is some work that can be done starting from the vantage point of analysis of contemporary practice, namely by considering future developments of the practice. The most immediate example in which considerations regarding the future come up are mathematical hypotheses. To be sure, they don't have the same status as scientific predictions, as the eventual assessment of their truth or falsity does not confirm or refute mathematics as a theory. But they can help provide a future frame of reference to test means-ends relations.

Let us consider the following example: in Maddy's case study of determinacy, she quotes a speculation of Yiannis Moschovakis [1980] about the future developments of determinacy hypotheses:

> In his 1980 state-of-the-art compendium on the subject, Moschovakis observed that 'no one claims direct intuitions ... either for or against determinacy hypotheses', that 'those who have come to favor these hypotheses as plausible, argue from their consequences' (Moschovakis [1980], p. 610). At that time, he concluded:
>
> > At the present state of knowledge only few set theorists accept [$AD^{L(\mathbf{R})}$] as highly plausible and no one is quite ready to believe it beyond a reasonable doubt; and it is certainly possible that someone will simply refute [it] in ZFC. On the other hand, it is also possible that the web of implications involving determinacy hypotheses and relating them to large cardinals will grow steadily until it presents such a natural and compelling picture that more will succumb. (Moschovakis [1980], pp. 610–611)
>
> Here Moschovakis displays impressive foresight, as more have succumbed in recent decades, on the basis of new discoveries.

We can interpret this as presenting the following means-ends relation: The method is identified with the adoption of (some version of) determinacy as an axiom; the goals are *future* mathematical consequences of this axiom. We will be able to use future content-based criteria ("argue[ing] from the consequences") and future consensus-based criteria ("natural and compelling picture") to judge this means-ends relation.

Given this setup, can we use contemporary practice to build a test case for a means-ends relation and then check it on the basis of a future test case? Unfortunately, this is not so simple, as building future test cases will always commit us to certain *ceteris paribus* assumptions: We have to assume that in the meantime nothing changes the general picture, for example that no different axiom with similar productive power gets accepted. Still, we have hope that such general issues can be overcome in some way. This hope relies on the fact that it is already part of mathematical practice to create such future test cases. One example for this is the recent work by Bagaria, Koellner, and Woodin [2019], in which the authors formulate two possible ways set theory can develop depending on which horn of the HOD dichotomy will hold. Also, in general mathematics it is common practice to explore the consequences of some mathematical hypothesis when a proof is still lacking.

This is certainly only an outline of the idea of using future developments of mathematics in the second-philosophical analysis. In order to develop a reliable method, this outline has to be worked out in much more detail. We will have to leave this as future work to be done.

In this section we described how the formulation of a general framework can be used to analyze the work of the Second Philosopher. We have shown that the framework is flexible enough to not impede her work and indeed provides a platform on which we can clearly formulate possible limitations of her methodology. Finally, we have given an outline of how the limitation of the reliance of historical analysis could be overcome.

Concluding, let us note that we motivated the changes that can be implemented in the procedural framework with the Second Philosopher's motivation to have the possibility to change her methodology. This also provides us with the possibility to analyze a separation of Maddy's interpretation from Second Philosophy itself. We could, for example, not accept the criteria that are associated with the "no heuristics" argument and try to argue that we are still conducting a second-philosophical analysis. Of course, such an argument would have to involve a detailed analysis into the contents of the procedural framework, but the fact remains that the framework provides a platform for this kind of analysis to occur.

5 Applying the general framework

In this last section we want to show how the procedural framework can be applied to new cases of set-theoretic methodology. We will give a short and concise example in which we can watch our procedural framework at work. We will then conclude by pointing out directions of further work that could be done by applying the procedural framework in Second Philosophy and discussing how this can be used to address questions in the philosophy of set theory. In this section, we will therefore address point (C) of our general goals in section 1.

The constructible universe L is a fundamental object in contemporary set theory. Introduced by Gödel to show the consistency of the Axiom of Choice and the Continuum Hypothesis, and further investigated by Jensen in the 1970s, generalizations of it are studied in inner model theory. L is sometimes seen as a very natural model of set theory, due to its canonical structure. Based on its importance for set theorists, the Second Philosopher is interested in L and wants to understand its role in set-theoretic methodology.

The natural historical case study for such an investigation is Gödel's own work at the end of the 1930s, such as Gödel [1939] and Gödel [1940]. Of course, Gödel's work has been studied extensively in the philosophical literature, however here we will mainly follow [Kanamori, 2007] as we want to focus on his mathematical work from a historical perspective.

Having established the setting for the historical case study, we want to figure out which method to choose. When looking at Gödel's works, we have an abundance of possibilities: the method could be foundational, like "adopting $V = L$ as a new axiom" with a foundational or mathematical goal to match; the method could be more local, like "using Gödel's Condensation Lemma" with the goal to prove GCH in L; it could be more broad like "minimality arguments" from the fact that L is the smallest inner model of ZF used in an abundance of mathematical theorems; and so on. Not all of these methods will lead to a satisfactory means-ends relation. As Maddy points out repeatedly, the method to adopt $V = L$ is be excluded from her analysis of set-theoretic methodology, as it contradicts foundational goals that are connected to maximality considerations (in the form of large cardinals) [Cf. Maddy, 1997, III.6]. The historical case study could suggest a method like "work in (some variant of) NBG set theory", as Gödel indeed used NBG in Gödel [1940]. However, when referring to contemporary practice, it turns out that this method should not be considered as it is not methodologically relevant or part of the evidential structure of the subject, therefore not fulfilling crucial content-based criteria.

Following Kanamori, we want to look at the method that is

> arguable the central feature of the construction of L: [...] L is a class definable in set theory via a transfinite recursion that could be based on the formalizability of def(x) [...]. Though understated in Gödel's writing, his great achievement here [...] is the submergence of metamathematical notions into mathematics. [Kanamori, 2007, 161]

Let us abbreviate this method as "the definability construction"; the goal suggested by our case study is a mathematical one, namely "proving the consistency of AC and GCH". First, we have to check if we pass the criteria of the evaluative apparatus.

The historical case study presents us with the curious fact that Gödel did use this method in his [1939], but in the longer exposition [1940] he uses the method of Gödel operations. Without going too much into the interesting relation between these two accounts,[21] we see with respect to contemporary practice that the definability construction remained methodologically relevant, as it gave rise to a great number of similar constructions (relative constructibility, ordinal definability and much more general studies of inner models). It has therefore become an integral part of the

[21] For more on this, see Kanamori [2007].

evidential structure of set theory. Additionally to these content-based criteria, the method and goal conforms to consensus-based criteria (building a model up in the way of the definability construction certainly counts as a natural construction). The subject-based criteria seem to be fulfilled as well, as both method and goal are mathematical.[22]

In comparison with the historical case study and contemporary practice, it turns out that our candidate passes the criteria of the evaluative apparatus. Further, we see that the definability construction is an effective means towards the end of proving the consistency of AC and GCH (as well as other mathematical goals) and therefore should be included in set-theoretic methodology.

Starting from this observation, the Second Philosopher could now turn towards the general question of the role of metamathematics in set-theoretic methodology. A first clue on this was given by Kanamori in the quote above. Baldwin [2018] gives an account of a shift in model theory that is connected to the introduction to metamathematical methods. To see if the same holds for set theory, we could therefore use our previous observation as first evidence for a comparable shift in contemporary set theory. Such an investigation would certainly be of importance to the Second Philosopher when turning to her questions about the nature of set-theoretic activity.

We see a further point where it could be interesting to follow the work on metamathematical methods: Maddy [2011] wants to explain how the Second Philosopher inquires into mathematics [cf. Maddy, 2011, 39]. She focuses on set theory instead of general mathematics because she wants to investigate *pure* mathematics [Maddy, 2011, Section I] and assumes set theory to be a good foundation for it. We think that the viewpoint of metamathematical methodology could be an interesting starting point to revisit the connection between mathematical and set-theoretical methodology, as metamathematical methods seem much more prevalent in the latter than the former. This will also be important if we want to transfer insights from the Second Philosopher's inquiry into the nature of set-theoretic activity to the nature

[22]Though it is interesting to note that this has not always have been perceived in this manner, see for instance Cohen's impressions recounted in Cohen [2002, 1087].

of mathematical activity in general.

Last, in the current debate in philosophy of set theory, there are several examples that can be studied with the above developed procedural schema—the most prominent of them being connected to the universe/multiverse debate. Maddy [2017] defends a universist stance by arguing with the methods of Second Philosophy. Ternullo [2019] argues to the contrary, mainly by addressing Maddy's foundational goals that she develops in [Maddy, 2017]. We think that by using the procedural framework we can find some middle ground in the debate. Let us give a very short outline of how this could be used:[23] One important point in the debate concerns the status of models of set theory. Very roughly: Multiversist positions tend to emphasize their status as new objects in set theory brought about by techniques like forcing [See for example Hamkins, 2012]. Formulating this in the "not philosophically loaded" manner of the Second Philosophy framework, we could start an investigation into the possible adoption of the method "models of set theory are introduced as new entities". An investigation into this method will then occur in the setting of the historical case study of the introduction of forcing by Cohen. We then choose a goal that he pursued there, for example the proof of the independence of Axiom of Choice and the Continuum Hypothesis. The main task will be to show that this formulation of method and goal passes the criteria in the evaluative apparatus and that in the end the method is judged to be effective towards this goal. Then it can be inferred that the method of introducing models of set theory as new entities should be included into set-theoretic methodology. One could subsequently investigate what this new picture of set-theoretic methodology tells us about further philosophical considerations, be it connected to Maddy's positions of Thin Realism and Arealism or her foundational goals. We think that such an investigation has the potential to build a bridge between the principles from which Maddy draws her form of universism and the focus on models of set theory from the multiverse side.

In this section we have shown how our general framework can be applied to

[23] To make the full case a detailed analysis is required that is too extensive for this paper. However, the first author of this paper is currently preparing an article concerned with this issue [see Antos, in preparation].

investigate set-theoretic methodology in a second-philosophical manner. We have also given hints on how such an investigation can be used for the discussion of broader philosophical questions. Combining this with the possibility of changing aspects of the Second Philosopher's methodology itself, as developed in section 4.3, we hope that this article contributes to further applications of Second Philosophy to the study of set theory.

References

Carolin Antos. A second philosphy analysis of models of set theory. in preparation.

Joan Bagaria, Peter Koellner, and W. Hugh Woodin. Large cardinals beyond choice. *Bulletin of Symbolic Logic*, 25(3):283–318, 2019.

John T. Baldwin. *Model Theory and the Philosophy of Mathematical Practice: Formalization without Foundationalism*. Cambridge University Press, 2018.

George Boolos. The iterative conception of set. *Journal of Philosophy*, 68(8):215–231, 1971. doi: 10.2307/2025204.

Cantor. Über die ausdehnung eines satzes aus der theorie der trigonometrischen reihen. *Mathematische Annalen*, 5:123–132, 1872.

Paul Cohen. The discovery of forcing. *Rocky Mountain Journal of Mathematics*, 32 (4):1071–1100, 2002.

José Ferreirós. *Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics*. Springer, Berlin, 2007.

Jose Ferreirós. *Mathematical Knowledge and the Interplay of Practices*. Princeton, USA: Princeton University Press, 2016.

Karen Francois and Jean Paul Van Bendegem. Revolutions in mathematics. more than thirty years after crowe's "ten laws". a new interpretation. *Philosophia Mathematica*, 18(3):107–120, 2010.

Kurt Gödel. Consistency-proof for the generalized continuum-hypothesis. *Proceedings of the National Academy of Sciences*, 25(4):220–224, 1939.

Kurt Gödel. *The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis with the Axioms of Set Theory*. Princeton University Press, 1940.

Joel D. Hamkins. The set-theoretic multiverse. *The Review of Symbolic Logic*, 5: 416–449, 2012.

Reuben Hersh. Mathematics has a front and a back. *Synthese*, 88(2):127–133, 1991.

Akihiro Kanamori. Gödel and set theory. *Bull. Symbolic Logic*, 13(2):153–188, 06 2007.

Akihiro Kanamori. Cohen and set theory. *Bulletin of Symbolic Logic*, pages 351–378, 2008.

Penelope Maddy. Believing the axioms. i. *Journal of Symbolic Logic*, 53(2):481–511, 1988.

Penelope Maddy. *Naturalism in Mathematics*. Oxford University Press, Oxford, 1997.

Penelope Maddy. *Second Philosophy: A naturalistic method*. Oxford University Press, Oxford, 2007.

Penelope Maddy. *Defending the Axioms: On the Philosophical Foundations of Set Theory*. Oxford University Press, Oxford, 2011.

Penelope Maddy. Set-theoretic foundations. In Andrés Eduardo Caicedo, James Cummings, Peter Koellner, and Paul B. Larson, editors, *Foundations of mathematics. Essays in honour of W. Hugh Woodin's 60th birtsday*, pages 289–322. American Mathematical Society, 2017.

Gregory H. Moore. The Origins of Forcing. In F.R. Drake and J.K. Truss, editors, *Logic Colloquium '86 Proceedings of the Colloquium held in Hull*, volume 124 of *Studies in Logic and the Foundations of Mathematics*, pages 143—-173. Elsevier, 1987.

Yiannis N. Moschovakis. *Descriptive set theory.* North-Holland Pub. Co. ; Sole distributors for the U.S.A. and Canada, Elsevier-North Holland Amsterdam ; New York : New York, 1980.

Alison Pease, Andrew Aberdein, and Ursula Martin. Explanation in mathematical conversations: An empirical investigation. *Philosophical Transactions of the Royal Society A*, 377, 2019.

Colin Jakob Rittberg. *Methods, Goals and Metaphysics in Contemporary Set Theory.* dissertation, University of Hertfordshire, 2016. URL `http://hdl.handle.net/2299/17218`.

Claudio Ternullo. *Maddy On The Multiverse*, pages 43–78. Springer International Publishing, Cham, 2019.

Ernst Zermelo. Neuer beweis für die möglichkeit einer wohlordnung. *Mathematische Annalen*, 65:107–128, 1908a.

Ernst Zermelo. Untersuchungen über die mengenlehre. i. *Mathematische Annalen*, 65(2):261 – 281, 1908b.