Theory Dec. (2018) 84:61–82
https://doi.org/10.1007/s11238-017-9636-6

Risk attitudes in axiomatic decision theory: a conceptual perspective

Jean Baccelli[1]

Published online: 24 October 2017

Abstract In this paper, I examine the decision-theoretic status of risk attitudes. I start by providing evidence showing that the risk attitude concepts do not play a major role in the axiomatic analysis of the classic models of decision-making under risk. This can be interpreted as reflecting the neutrality of these models between the possible risk attitudes. My central claim, however, is that such neutrality needs to be qualified and the axiomatic relevance of risk attitudes needs to be re-evaluated accordingly. Specifically, I highlight the importance of the *conditional variation* and the *strengthening* of risk attitudes, and I explain why they establish the axiomatic significance of the risk attitude concepts. I also present several questions for future research regarding the strengthening of risk attitudes.

Keywords Risk aversion · Conditional certainty equivalent · Allais paradox · Non-expected utility · Rank-dependent utility · Cautious expected utility

1 Introduction

This paper focuses on choice under risk. In decision-theoretic terminology, *risk* refers to when the decision-maker faces options that constitute random prospects on a given set of possible results, and the prospects follow a known probability distribution. This is exemplified in games of chance such as dice, cards, or roulette. Playing such games, the decision-maker wins or loses money randomly (unlike in choice under certainty). However (unlike in choice under uncertainty), her odds follow known probability distributions. They are determined by the particular chance mechanism which

✉ Jean Baccelli
jean.baccelli@gmail.com

[1] Munich Center for Mathematical Philosophy, Ludwigstrasse 31, 80539 Munich, Germany

she is confronted with, i.e., the number of faces on the dice, the number and kinds of available cards, or the roulette table layout.

It is well known that the attitude towards risk is one of the topics considered in this branch of decision theory. In decision-theoretic terminology, *risk attitude* refers to technical concepts capturing parts of our intuitive psychology regarding the various temperaments that can be exhibited in situations like the ones above. For instance, some love gambling despite the eventuality of going bankrupt, others prefer to play it safe whenever possible, and still others act as if they were altogether insensitive to any such feature of their choice situation. In the present paper, I assess the significance of the technical concepts echoing those intuitive ideas in decision theory. I offer a new conceptual perspective on pre-existing results, which I select, bring together, and interpret. I also articulate some of the open questions which these results lead to.

Whatever their domain of interest (be it certainty, risk, or uncertainty), decision theorists are primarily concerned with analyzing decision models. A decision model can be thought of as an algorithm for evaluating options. In the case of risk, examples include computing the expected value of some utility function on the set of possible results, calculating this expectation with respect to some transformation of the known probabilities, or proposing some way of combining both the expectation and the variance of the utility values. More often than not, decision theorists introduce or even discover models directly from such a numerical perspective. However, their specific task is to characterize each numerical form of evaluation by a few basic properties, namely, those displayed by the preferences of a decision-maker to whom the examined model would apply. This requires, if possible, proving a representation theorem showing how the numerical evaluation reflects structural aspects of the underlying preferences. To this extent, decision theory is essentially a development of representational measurement theory. More generally, it is an application of the axiomatic method.[1] *Axiomatic analysis*, as I will henceforth summarize the decision theorists' task, enables decision models to be compared with one another. Rigorously speaking, the numerical forms of evaluation which they are associated with are not directly comparable. However, once they are translated into the common language of preference, one can identify the true differences between them. Thus, for decision theorists, the most significant properties of preference are those on the basis of which decision models can be axiomatically distinguished from one another.

In this paper, I apply the axiomatic criterion of significance introduced above to assess the status of the risk attitude concepts in decision theory. I start by recalling their technical definitions in a preliminary section. Then, following a deliberately naïve baseline analysis, I provide evidence showing that these concepts do not play any axiomatic role in the theory of decision-making under risk. At this juncture, I stress that the risk attitude concepts do not seem able to account for the fundamental divide between expected utility and the non-expected utility models. This fact is recognized in the current literature, but it deserves to be better highlighted and detailed. To this end, I provide an illustrative discussion of the Allais paradoxes (Allais 1953), which I enrich in the subsequent sections of my paper. Next, I show that following a second,

[1] For the authoritative exposition of the representational theory of measurement, see the *Foundations of Measurement* trilogy (starting with Krantz et al. 1971).

more thorough analysis, different and less familiar conclusions prevail. Specifically, I show that in at least two respects, which pertain to what I call, respectively, the *conditional variation* and the *strengthening* of risk attitudes, axiomatic analysis can rely on risk attitudes to distinguish decision models from one another.

In providing this more thorough analysis of risk attitudes, my paper relies on two strands of literature. The first strand of literature analyzes decision models by means of so-called conditional certainty equivalents (see especially Machina 1982; Chew and Epstein 1989; Chew et al. 1993). Unlike the previous contributors to this literature, I show its conceptual importance for assessing the status of risk attitudes in axiomatic decision theory. The second strand of literature aims at algebraically characterizing each of the risk attitudes, when the numerical framework of a given decision model is taken for granted (for a review of such results, see, e.g., Chateauneuf et al. 1997). More systematically than the previous contributors to this literature, I show that such algebraic characterizations lead to a general axiomatic typology of the existing decision models. In particular, in this context, I offer a new discussion of the recent cautious expected utility model (Cerreia-Vioglio et al. 2015), which proves instrumental in establishing the generality of the typology in question. This discussion also provides new evidence in support of the exceptional flexibility of the rank-dependent utility model (first introduced in Quiggin 1982). These two strands of literature lead to the most accurate assessment of the decision-theoretic status of the risk attitudes concepts. Admittedly, this final assessment amounts to more of a qualification than a rejection of the naïve baseline assessment which I am going to sketch first. However, as I will show, it opens many new theoretical perspectives that, to the best of my knowledge, are yet to be explored.

2 The risk attitude concepts

2.1 The underlying framework

The mathematical framework underlying the present paper is as follows. Take a real interval C.[2] For convenience, I will assume that it is closed and bounded. Let $\Delta(C)$ be the set of all finite-support probability distributions on C. For any $c \in C$, denote by δ_c the degenerate probability distribution with support $\{c\}$. Call C the set of *results* and $\Delta(C)$, the set of *lotteries*. Notice that, in this framework, any lottery $P \in \Delta(C)$ has a well-defined mathematical expectation, denoted here by $E(P)$, and that under the form of the degenerate lottery $\delta_{E(P)}$, $E(P)$ is itself one of the lotteries. Take $\succcurlyeq$, a binary *preference* relation on $\Delta(C)$. Let $\succ$ and $\sim$ be the strict preference and the indifference subrelations, respectively. Assume that $\succcurlyeq$ is complete, transitive, and continuous in the topology of weak convergence. Assume also that $\succcurlyeq$ is strictly increasing in C, interpreted by default as a set of monetary amounts, and identi-

[2] For simplicity, in this paper, C will always be taken in $\mathbb{R}^+$. This is to set aside a specific set of issues, namely, the possible asymmetries between how decision-makers consider gains, and how they consider losses (making the conventional assumption that the former should be mapped onto $\mathbb{R}^+$, and the latter onto $\mathbb{R}^-$). Such asymmetries have been discussed at length with respect to the risk attitudes (see in particular the so-called fourfold pattern of risk attitudes emphasized in, e.g.,Tversky and Kahneman 1992, p. 306).

fied with the set of degenerate lotteries δ_c, and that $\succcurlyeq$ respects first-order stochastic dominance.[3] For the sake of brevity, I will call any such preference relation *classic*.

There are of course more general conditions than the ones given above. First, the theory of decision-making under risk can be developed with respect to arbitrary sets of results, such as finite sets of non-numerical entities. For instance, the options could be lotteries over heterogeneous discrete consumer goods. This implies that, in general, the decision theorist's lotteries have no well-defined mathematical expectation, and that even when they do, the mathematical expectations are not among the options over which the decision-maker's preferences are defined. Second, and independently of the previous point, although the preferences underlying most models of decision-making under risk are classic ones, such is not the case of all. Setting apart the fact that preferences are monotonically increasing in C, there are major exceptions pertaining to each of the properties introduced above.[4]

However restrictive, these conditions fit the purposes of the discussion of my paper. They imply the following fact, on which I will rely in subsequent analysis: for any lottery P, there is a unique result c such that $P \sim \delta_c$. Call this result c the *certainty equivalent* of P for the decision-maker characterized by the preference relation $\succcurlyeq$, which can be written as $CE(P) = c$. Given a preference relation $\succcurlyeq$, define a *certainty equivalent function* as a function $CE : \Delta(C) \to C$, such that $CE(P) \geq CE(Q) \Leftrightarrow P \succcurlyeq Q$, for all $P, Q \in \Delta(C)$, and $CE(\delta_c) = c$, for any $c \in C$. Any classic preference relation can be associated with exactly one certainty equivalent function.[5]

2.2 Definitions of risk attitudes

Thanks to the framework sketched in the previous section, one can introduce the prevailing concepts of risk attitude.[6] What is under discussion is in fact a family of definitions. However, all the definitions follow the same logical pattern. First, there is the specification of one kind of *risk reduction* that can be offered to the decision-maker. Second, there is a classification of decision-makers according to whether they opt for or against such risk reductions, whenever offered.

[3] For any lottery P, let $F_P : \mathbb{R} \to [0, 1]$ be its cumulative distribution function. I will say that P dominates Q in the sense of first-order stochastic dominance if $F_P(x) \leq F_Q(x)$ for all $x \in \mathbb{R}$, and this inequality is strict for some x. The fact that $\succcurlyeq$ respects first-order stochastic dominance thus defined (in the sense that $P \succ Q$ whenever P dominates Q in the sense of first-order stochastic dominance) entails that $\succcurlyeq$ is strictly increasing in C (i.e., for all $c, c' \in C$, if $c > c'$, then $\delta_c \succ \delta_{c'}$). This is a natural assumption to make if c refers to money or, more generally, any continuous good.

[4] For instance, the preferences underlying the models of choice under risk based on regret are intransitive, and, therefore, non-classic (see, e.g., Fishburn 1982; Loomes and Sugden 1982).

[5] For a standard proof, see, e.g., Cerreia-Vioglio et al. 2015, Appendix B, Step 1. This result is essentially an application of Debreu's theorem (Debreu 1964) regarding the existence of a continuous utility function. As I will highlight later, a certainty equivalent function is nothing but a particular utility function representing $\succcurlyeq$.

[6] A standard exposition of these definitions can be found in Cohen 1995, Sect. 1.

The most intuitive kind of risk reduction is *total* risk reduction. I will say that this kind of risk reduction is available to the decision-maker whenever she is offered to choose between a lottery, P, and its mathematical expectation given as a riskless option, $\delta_{E(P)}$. Accordingly, call a decision-maker *weakly risk averse* if $\delta_{E(P)} \succcurlyeq P$, for any P. In the special cases in which, in addition, one such preference is strict for some lottery P, call the decision-maker weakly *strictly* risk averse.[7] Alternatively, call her *weakly risk seeking* if $\delta_{E(P)} \preccurlyeq P$ (with a transparent strict variant), and *weakly risk neutral* if $\delta_{E(P)} \sim P$, for any P. As this paper focuses on classic preferences, one could also introduce these concepts by comparing mathematical expectations with certainty equivalents. By the definition of the certainty equivalent function, $\delta_{E(P)} \succcurlyeq P$ if and only if $\delta_{E(P)} \geq CE(P)$. Therefore, weak risk aversion obtains if and only if the decision-maker's preferences are such that, for any P, $\delta_{E(P)} - CE(P) \geq 0$. Similar inequalities hold for the other two attitudes defined above.

The risk attitudes defined above are deemed weak, because they pertain to very specific risk reductions, namely, total ones. To introduce stronger risk attitudes, one must first provide a general definition of risk reductions, that covers potentially partial risk reductions as well. I will say that P is related to Q by an *arbitrary* risk reduction (i.e., that Q is a so-called mean-preserving spread of Q), which can be symbolized by writing $P\ RR_A\ Q$, if the following condition obtains.[8] For any enumeration $\{c_1, \ldots, c_n\}$ of the support of P, there exists a corresponding set $\{L_1, \ldots, L_n\}$ of lotteries, such that Q can be identified as follows:

$$Q = \sum_{i=1}^{n} P(c_i) L_i \quad \text{with} \quad E(L_i) = c_i \quad \text{for any} \quad i = 1, \ldots, n. \tag{1}$$

For instance, $P = (\frac{1}{2} : 15, \frac{1}{2} : 5)$ is related to $Q = (\frac{1}{4} : 25, \frac{3}{4} : 5)$ in this way: the lotteries $L_1 = (\frac{1}{2} : 25, \frac{1}{2} : 5)$ and $L_2 = (1 : 5)$ lead to the desired identification.[9] Total risk reduction, henceforth symbolized by RR_T, corresponds to the particular case, where $P = \delta_{E(Q)}$, with $L = Q$ being the only auxiliary lottery, as when Q is compared with $P' = \delta_{E(Q)} = (1 : 10)$. Call a decision-maker *strongly risk averse* if $P \succcurlyeq Q$ whenever $P RR_A Q$. This also covers cases of partial risk reductions, hence the fact that this form of risk aversion is considered to be stronger than weak risk aversion. Alternatively, call the decision-maker *strongly risk seeking* if $P \preccurlyeq Q$, and

---

7 An alternative definition would require that $\delta_{E(P)} \succ P$, not for some P, but for all P.

8 One can also say that P dominates Q in the sense of second-order stochastic dominance, applied to lotteries having the same mathematical expectation. In economics, the concept of mean-preserving spread dates back to Rothschild and Stiglitz 1970. The definition in (1), which corresponds to the simplest case possible, compares most directly to the one presented in Rothschild and Stiglitz 1970, I.1 and III.4. While the two definitions are equivalent, the one given here is simpler, and also less restrictive in that it is compatible with C being taken in $\mathbb{R}^+$, i.e., having no element in $\mathbb{R}^-$ (more on this in footnote 2).

9 The so-called principle of reduction of compound lotteries is assumed in (1) and, more generally, throughout my paper. In particular, I will not investigate whether some risk attitude patterns can be related to patterns of violations of the principle of reduction of compound lotteries. Such violations have been systematically explored in the literature (see, e.g., Segal 1990, and more recently Dillenberger and Segal 2015). However, to my knowledge, they have never been explored with reference to the risk attitude concepts introduced in this section. This would be a particularly interesting (and challenging) research topic.

strongly risk neutral if $P \sim Q$, in each case for all P, Q, such that $P\,RR_A\,Q$. Again, definitions involving certainty equivalents are available. For instance, strong risk aversion obtains if and only if $CE(P) - CE(Q) \geq 0$ whenever $P\,RR_A\,Q$.

Many conceivable risk reductions are more specific than the arbitrary ones, but more general than the total ones. For the purposes of this paper, it suffices to consider only one such intermediate case, which requires that the risk reduction be *monotonic*, symbolized by RR_M. Intuitively, this kind of risk reduction is especially homogeneous. Technically, given two lotteries P and Q such that $P\,RR_A\,Q$, the additional requirement is that each interquantile interval of P be no greater than that of Q. This implies that Q cannot have any hedging properties with respect to P, which can occur when P is a non-monotonic risk reduction of Q.[10] For instance, taking the preceding lotteries P and Q, one can check that P is not monotonically less risky than Q, whereas it would be so compared to $Q' = (\frac{1}{2} : 20, \frac{1}{2} : 0)$. The generic definition in (1) would then apply with the lotteries $L_1 = (\frac{3}{4} : 20, \frac{1}{4} : 0)$ and $L_2 = (\frac{1}{4} : 20, \frac{3}{4} : 0)$, and the monotonicity condition on the interquantile intervals would be satisfied. In this paper, call a decision-maker *moderately risk averse* if $P \succcurlyeq Q$ whenever $P\,RR_M\,Q$. Likewise, refer to the corresponding attitudes as *moderate risk seeking* and *moderate risk neutrality*.[11] Again, alternative definitions involving certainty equivalents are available.

The various kinds of risk attitudes thus introduced form a united family of concepts. The following two remarks highlight this point in different ways. First, these concepts pertain to various kinds of risk reduction that are logically related to one another. Total risk reduction is a refinement of monotonic risk reduction, which is itself a refinement of arbitrary risk reduction. As a result, the various strengths of risk aversion or risk seeking are similarly related from a logical point of view. Any strongly risk averse decision-maker must also be moderately so, and any moderately risk averse decision-maker must also be weakly so. However, as will be emphasized later, the converse implications do *not* hold in general, i.e., they do not hold in all decision models. It is clear from the definitions above that one can consistently accept all total risk reductions and refuse some monotonic ones, or accept all monotonic risk reductions and refuse some arbitrary ones. Thus, the various strengths of risk aversion are logically related exactly as follows:

$$\begin{gathered}\text{STRONG AVERSION} \Rightarrow \text{MODERATE AVERSION} \Rightarrow \text{WEAK AVERSION},\\ \text{STRONG AVERSION} \not\Leftarrow \text{MODERATE AVERSION} \not\Leftarrow \text{WEAK AVERSION}.\end{gathered} \tag{2}$$

The same analysis applies to the various strengths of risk seeking. Finally, all brands of risk neutrality are equivalent provided that indifference is transitive. Classic pref-

[10] The defining condition on interquantile intervals is as follows. For any P, let $F_P^{-1} : (0, 1) \to \mathbb{R}$ be its (generalized) inverse distribution function. Then, $P\,RR_M\,Q$ if $P\,RR_A\,Q$ and, for all $p, q \in (0, 1)$ such that $p < q$, $F_P^{-1}(q) - F_P^{-1}(p) \leq F_Q^{-1}(q) - F_Q^{-1}(p)$. One can also say that Q is more dispersed than P in the sense of Bickel and Lehman, applied to lotteries having the same mathematical expectation. This kind of risk reduction is called "monotonic", because, making explicit an underlying state space, P and Q can be related to "co-monotonic" Savagian acts (for a definition, see Schmeidler 1989, p. 575). For more details on this kind of risk reduction, see, e.g., Chateauneuf et al. 1997, p. 29.

[11] Unlike "monotonic", "moderate" fits the prevailing terminology regarding the "weak" and the "strong" attitudes, and it makes transparent the logical links given in (2) above.

erences have transitive indifference subrelations. Accordingly, from now on, in this paper, I will discuss risk neutrality without further qualification, i.e., without specifying any underlying kind of risk reduction.

Second, the concepts above are traditionally presented as pertaining to *absolute* risk attitude. This is by way of contrast with the more general concepts of *comparative* risk attitude, that are also examined in the literature.[12] The comparative concepts of risk attitude can be used not only to introduce the absolute ones, but also to highlight their systematic unity. In the framework of the present paper, I will say that decision-maker D is *more risk averse* than decision-maker D' if, for any P, $CE_D(P) \leq CE_{D'}(P)$.[13] The intuition is that D is more risk averse than D' if, for any lottery P, D is ready to accept at most as much as D' to avoid the intrinsic riskiness of P by receiving, instead of P, some sure amount $CE(P)$. Whatever the underlying kind of risk reduction, a risk averse (respectively, seeking) decision-maker can be characterized by her being more (respectively, less) risk averse than a risk neutral decision-maker. Denote by D_0 a risk neutral decision-maker. Recall that, by definition, $CE_{D_0}(P) = E(P)$ for any P. Observe that a weak risk averter is a decision-maker D_- such that, for any P, $CE_{D_-}(P) \leq CE_{D_0}(P)$. Likewise, a weak risk seeker is a decision-maker D_+ such that, for any P, $CE_{D_+}(P) \geq CE_{D_0}(P)$. To this extent, in all that precedes, there are in fact only two basic ideas from which all the other ones can be derived, namely, risk neutrality and comparative risk aversion (or seeking). However, as will be emphasized later, a decision-maker D can be more (or less) risk averse than decision-maker D', even though neither D nor D' display *any* of the absolute risk attitudes previously listed. Indeed, it might be that for all P, $CE_D(P) \leq CE_{D'}(P)$, but with $CE_D(Q) \leq CE_{D'}(Q) < E(Q)$ for some Q and $E(R) < CE_D(R) \leq CE_{D'}(R)$ for some R. This would illustrate comparative risk aversion while also excluding any form of risk seeking or risk aversion. This explains why the concepts of comparative risk attitude are taken to be more general and flexible than those of absolute risk attitude.

3 Axiomatic analysis versus risk attitude analysis

On the face of it, the axiomatic analysis of decision-making under risk does not rely on the risk attitude concepts introduced in the previous section. As the present section details, this is what one can conclude from examining the traditional representation theorems of the theory of choice under risk.

Most of these theorems can be understood as proceeding in two steps. The first step is common across models. It consists in building a generic utility representation, like the ones examined in the theory of riskless choice. This step can be taken as

[12] The contrast is also with the more specialized concepts of *relative* risk attitudes (in which case, "relative" means "relative to a given underlying wealth of the decision-maker").

[13] Comparative risk aversion can be given a much more general definition (see Yaari 1969 and, building on this contribution, Bommier et al. 2012, Sect. 3). This more general definition, which is the truly fundamental one, can be introduced even when one cannot introduce certainty equivalent functions or, indeed, the absolute risk attitudes themselves.

given in this paper. For any classic preference relation $\succcurlyeq$, there exists a utility function $v : \Delta(C) \rightarrow \mathbb{R}$, such that the following holds:

$$P \succcurlyeq Q \Leftrightarrow v(P) \geq v(Q), \quad \forall P, Q \in \Delta(C). \tag{3}$$

For instance, the certainty equivalent function associated with $\succcurlyeq$ can serve as one such generic representation.

The second step introduces the analytical form of v specific to each model. The best-known specification is that of the *expected utility* model. Given any pair of lotteries $P, Q \in \Delta(C)$, and any number $\alpha \in [0, 1]$, denote by $\alpha P + (1-\alpha) Q$ the convex combination of P and Q with respective weights α and $(1-\alpha)$. Notice that $\Delta(C)$ is closed under this operation of convex combination. In expected utility, the proposed analytical form is that v in (3) be linear in probabilities, i.e., for all $P, Q \in \Delta(C)$ and any $\alpha \in [0, 1]$, we have $v[\alpha P + (1-\alpha) Q] = \alpha v(P) + (1-\alpha) v(Q)$. For this functional form to hold, it is necessary and sufficient to require that classic preferences respect so-called von Neumann–Morgenstern (VNM) independence, i.e., for all $P, Q, R \in \Delta(C)$ and any $\alpha \in (0, 1]$, $P \succcurlyeq Q$ if and only if $\alpha P + (1-\alpha) R \succcurlyeq \alpha Q + (1-\alpha) R$.[14] The other models of decision-making under risk are the *non-expected utility* models, in which v in (3) is not linear in probabilities. These models predominantly rely on weakened forms of VNM independence (now short for: the respect of VNM independence). For instance, the condition can be imposed only when $R = P$ or $R = Q$, which defines the so-called betweenness property. This weakening underlies, e.g., the much-discussed disappointment aversion model. Alternatively, the condition can be restricted to when the convex combination with R preserves the preferential ranks of the elements in the support of P and Q, which defines the so-called co-monotonic weakening of VNM independence. This weakening is characteristic of the empirically prevailing rank-dependent utility model. Among many conceivable generalizations, these two generalizations of VNM independence have received the most interest in the literature.[15]

From an axiomatic point of view, this second step is crucial. Its function is to shed light on how the various decision models differ from one another. It is apparent from

[14] Notice, however, that together with the other defining properties of classic preferences, the respect of VNM independence entails that of first-order stochastic dominance. For a proof of the von Neumann–Morgenstern theorem, see, e.g., Fishburn 1970, Chapter 8.

[15] Betweenness is satisfied if for all $P, Q \in \Delta(C)$, and any $\alpha \in (0, 1]$, $P \succcurlyeq Q$ if and only if $P \succcurlyeq \alpha P + (1-\alpha) Q \succcurlyeq Q$. On the betweenness branch of non-expected utility theory, see Dekel 1986 and Chew 1989. On the disappointment aversion model in particular, see Gul 1991. Next, some notation is needed to introduce the co-monotonic branch of non-expected utility theory. Given a lottery P, denote by $\{c_1^{\mathrm{P}}, \ldots, c_n^{\mathrm{P}}\}$ its support, ordering it (without loss of generality) so that $\delta_{c_1^{\mathrm{P}}} > \cdots > \delta_{c_n^{\mathrm{P}}}$, and accordingly, let P denote $(c_1^{\mathrm{P}}, p_1; \ldots; c_n^{\mathrm{P}}, p_n)$. Take $Q = (c_1^{\mathrm{Q}}, p_1; \ldots; c_i, p_i; \ldots; c_n^{\mathrm{Q}}, p_n)$, $R = (c_1^{\mathrm{R}}, p_1; \ldots; c_i, p_i; \ldots; c_n^{\mathrm{R}}, p_n)$, $Q' = (c_1^{\mathrm{Q}}, p_1; \ldots; c_i', p_i; \ldots; c_n^{\mathrm{Q}}, p_n)$, $R' = (c_1^{\mathrm{R}}, p_1; \ldots; c_i', p_i; \ldots; c_n^{\mathrm{R}}, p_n)$ (i.e., the common result c_i is replaced by the common result c_i' at the same i-th preferential rank). The co-monotonic weakening of VNM independence requires that, for all such Q, R, Q', R', $Q \succcurlyeq R$ if and only if $Q' \succcurlyeq R'$. On this property and the rank-dependent utility model, see, e.g., Chateauneuf 1999. For a classic decision model contained in neither the betweenness, nor the co-monotonic branch of non-expected utility theory, see, e.g., the recent model in Cerreia-Vioglio et al. 2015 (which I will discuss in Sect. 4.2).

the above that the risk attitude concepts do not appear in this key step—at least as it is construed in the traditional axiomatizations. These concepts also seem unlikely to appear in any alternative axiomatizations that, contrary to the traditional ones, would try to emphasize the role of risk attitudes. Indeed, there seems to be no systematic link between any particular model of decision-making under risk and any particular risk attitude. Yet, such a link seems necessary for any such alternative analysis to be carried out successfully. Admittedly, there is one remarkable implication. Take a decision-maker with classic preferences. If she is risk neutral, then she is an expected utility maximizer; equivalently, if she is a non-expected utility maximizer, then she cannot be risk neutral.[16] However, the converse implication does not hold, as illustrated by the well-known possibility that an expected utility maximizer be strictly risk averse or risk seeking. More generally, it is a fact that no known model of decision-making under risk imposes any of the attitudes previously listed. Even when those attitudes can be introduced, as is the case in the framework of the present paper, all classic models of choice under risk can accommodate *non-classifiable* preferences, i.e., preferences that would display none of the introduced attitudes. This can be established by algebraic examples, by building on the available results regarding the characterization of the various risk attitudes within each decision model. It suffices to pick, for each model, a particular functional form that does not display any of the properties identified in those results.[17]

This apparent mismatch between what axiomatic analysis needs and what the risk attitude concepts have to offer can be illustrated more concretely. Consider the fundamental divide between expected and non-expected utility. It is customary to introduce it through the Allais paradoxes, which are combinations of preferences that seem internally consistent but are inconsistent with VNM independence. Non-expected utility models explore various ways of weakening this property, partly because they want to allow for these combinations of preferences. The *paradoxical preferences*, as I will call them here, illustrate how restrictive expected utility really is, and they inspire the various non-expected utility models. Were the risk attitude concepts axiomatically relevant, they should be able to shed light on the paradoxical preferences, displayed next in Figs. 1 and 2.[18]

[16] Given how risk neutrality, the certainty equivalent function, and mathematical expectations are defined, one can check that, for all $P, Q, R \in \Delta(C)$ and any $\alpha \in (0, 1]$, $P \succcurlyeq Q$ if and only if $E(P) \geq E(Q)$ if and only if $E[\alpha P + (1-\alpha)R] \geq E[\alpha Q + (1-\alpha)R]$ if and only if $\alpha P + (1-\alpha)R \succcurlyeq \alpha Q + (1-\alpha)R$. In the above formulation of the contrapositive form of this implication, one should interpret "non-expected utility" in the exclusive sense.

[17] For a review of such results, see, for instance, Chateauneuf et al. 1997, Sect. 3.2.

[18] See Allais 1953 (and empirical data in, e.g., Camerer 1992, 1995, Section III.D). The results of the lotteries displayed must be interpreted in significant units of money (e.g., thousands of dollars).

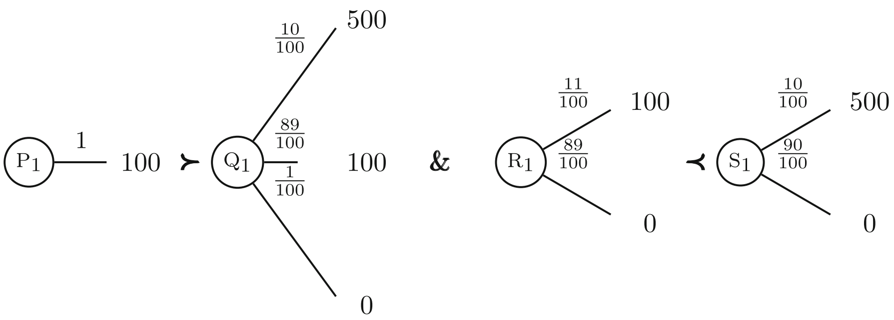

Fig. 1 The first Allais paradox

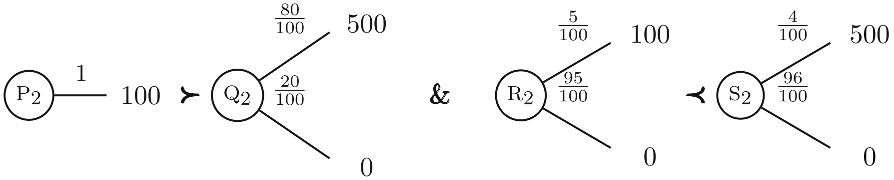

Fig. 2 The second Allais paradox

On the face of it, there does not seem to be any interesting connection between the paradoxical preferences above and the risk attitudes previously defined. Admittedly, if $\succcurlyeq$ is classic, then either $P_1 \succ Q_1$ or $P_2 \succ Q_2$ excludes any form of risk seeking.[19] However, this does not get to the point of the Allais paradoxes, namely, the combinations of preferences $P_1 \succ Q_1$ and $R_1 \prec S_1$, and $P_2 \succ Q_2$ and $R_2 \prec S_2$. These combinations prove compatible with the presence of strong strict risk aversion as well as the absence of weak strict risk aversion. These are the two extremes of what remains possible in the risk attitude spectrum, given the exclusion of risk seeking (and that of risk neutrality, which has already been explained).[20] This is a direct illustration of the fact that there is no compelling link between the paradoxical preferences and any particular risk attitude. As a result, while decision theorists are accustomed to describing the Allais paradoxes in terms of risk attitudes, their descriptions do not seem to

[19] Consider, e.g., the second paradox. Recalling (2), assume by way of contradiction that the decision-maker is weakly risk seeking. Then, by definition of weak risk seeking, $Q_2 \succcurlyeq \delta_{E(Q_2)}$. From this, the fact that $E(Q_2) = 400$, $P_2 \succ Q_2$, and transitivity, it follows that $\delta_{100} \succ \delta_{400}$. This contradicts the fact that $\succcurlyeq$ is strictly increasing in C.

[20] This can be established by algebraic examples. Let D and D' be two decision-makers to whom the rank-dependent utility model applies. Let them be characterized by the same utility function, $u(c) = \sqrt{c}$, together with the probability weighting functions $w_D(p) = 1 - \sqrt{1-p}$ and $w_{D'}(p) = \sqrt{p}/(\sqrt{p} + \sqrt{1-p})^2$, respectively ($w_D$ comes from Segal 1987, p. 149, while $w_{D'}$, with a typical inverse-S shaped graph, comes from Tversky and Kahneman 1992, p. 309). The functional form of rank-dependent utility (which I will recall in footnote 34) implies that both D and D' have the paradoxical preferences. However, it can be proved that D is strongly risk averse (see Chew et al. 1987, Corollary 2), while D' is not even weakly so (see Chateauneuf and Cohen 1994, Corollary 1).

stand up to scrutiny—at least when these descriptions are examined with reference to the prevailing technical definitions.[21] Variations on the Allais paradoxes could permit extending this discussion of the divide between expected and non-expected utility to a similar discussion of the divide between the various non-expected utility models. The latter discussion would likewise lead to apparently negative conclusions on the axiomatic significance of risk attitudes.

The most convincing interpretation of this negative evidence is that the decision models are by and large *neutral* between the various risk attitudes. The history of expected utility theory puts this neutrality in an interesting perspective. First, the expected utility model was introduced, in the wake of the discussions of the St. Petersburg paradox, essentially to allow for not only risk neutrality, but also risk aversion. Risk neutrality is imposed by the model in which decision-makers maximize the expectation of their gains, rather than, more generally, the expectation of the utility of their gains. The less general model was the main source of the St. Petersburg paradox.[22] Second, one important breakthrough made possible by the late axiomatization of the expected utility model was the realization that this model could accommodate not only risk aversion, but also risk seeking. Some scholars, under the confusing influence of ideas of decreasing marginal utilities, had previously claimed the contrary. However, once a representation theorem is available, it becomes clear that only the expectation formula is essential to the model, unlike any particular property of the utility function of which the expectation is taken. In the expected utility model, this is tantamount to showing that risk seeking is possible.[23] Similar remarks could put the history of non-expected utility theory in similar perspective. The neutrality of the decision models between the various risk attitudes is one thread in the history of decision theory at large.

4 Risk attitude analysis in axiomatic analysis

However well founded, the preceding remarks need to be qualified. As I now show, there is more to the risk attitude concepts than these remarks suggest.

4.1 The conditional variation in risk attitude

Consider again the idea that each lottery has a certainty equivalent. As illustrated in Sect. 2.2, this idea can be used to introduce all the risk attitude concepts, starting with

[21] See, e.g., Kahneman and Tversky 1979, p. 267 or Loomes and Sugden 1982, p. 806.

[22] The above is meant not as a precise historical statement, but as a suggestive rational reconstruction. Seidl (Seidl 2013) reviews the history of the St. Petersburg paradox up to the present.

[23] This is due to the characterization of absolute risk attitude in expected utility (which primarily leads back to Rothschild and Stiglitz 1970). Shortly after von Neumann and Morgenstern's groundbreaking axiomatization of expected utility (von Neumann and Morgenstern, 1947, Appendix), Friedman and Savage (Friedman and Savage 1948) were among the first to stress the compatibility between expected utility and risk seeking. This compatibility had previously been denied, most prominently by Marshall (see, e.g., Marshall 1890, Mathematical Appendix, Note IX).

the fundamental concept of comparative risk aversion. Certainty equivalents naturally generalize to *conditional certainty equivalents*.[24] I will denote conditional certainty equivalents by $CCE(P, \cdot)$, and I will define them as follows: $CCE(P, R) = c$ if, for some $\alpha \in [0, 1]$ and some $Q, R \in \Delta(C)$ such that $R = \alpha P + (1 - \alpha)Q$, it is the case that $\alpha P + (1 - \alpha)Q \sim \alpha\delta_c + (1 - \alpha)Q$. In words, instead of identifying a certainty equivalent for P only when P is considered in isolation, do so also when P is one of two lotteries, the convex combination of which forms some lottery R. Any classic preference relation can be associated with exactly one conditional certainty equivalent function. In general, the conditional certainty equivalent assigned by this function to a lottery P can vary with both Q, the lottery with which P is combined, and α, the weight of P in the combination. Clearly, such variations are excluded if VNM independence is respected. In this case, one need not distinguish between conditional certainty equivalents and *unconditional certainty equivalents*, as the traditional certainty equivalents might now be called. The distinction is necessary, however, whenever VNM independence is violated. Those are the cases that will garner the most attention here.

Equipped with the concept of conditional certainty equivalent, one can revisit the Allais paradoxes. They are said to display, respectively, the so-called common consequence and common ratio effects. This is because the lotteries giving rise to the paradoxical preferences can be decomposed as shown in Figs. 3 and 4.

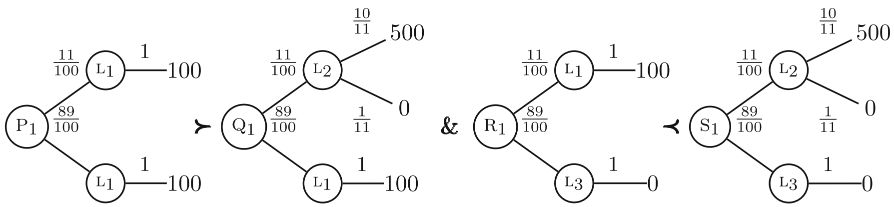

Fig. 3 The first Allais paradox analyzed

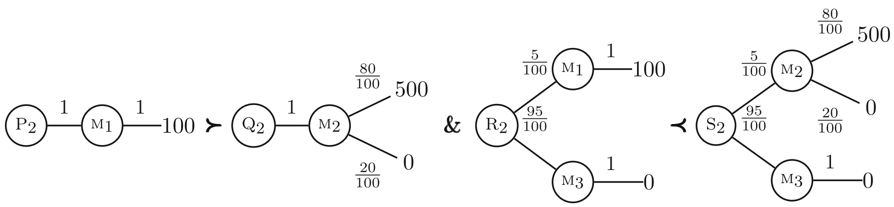

Fig. 4 The second Allais paradox analyzed

24 To my knowledge, Machina was the first to explicitly introduce conditional certainty equivalents (Machina 1982, p. 288).

In the first paradox, L_1 is a "common consequence" of both P_1 and Q_1, and L_3 is a common consequence of both R_1 and S_1. In the second paradox, 1 and .05 constitute a "common ratio" by which both M_1 and M_2 are combined with M_3 to yield, respectively, P_2 and Q_2, and R_2 and S_2. A preference reversal occurs with a change in either the other lottery entering the combination (the common consequence) or the combination weight (the common ratio). Both cases are excluded by the respect of VNM independence.[25]

From the first set of paradoxical preferences (and the fact that classic preferences respect first-order stochastic dominance), one can infer that $CCE(L_2, Q_1) < 100 < CCE(L_2, S_1)$. The conditional certainty equivalent of L_2 varies depending on whether L_2 is combined with L_1, which offers the decision-maker a significant gain (100), or with L_3, which offers her nothing at all (0). This variation occurs, while all the combination weights are the same. The decision-maker has an identical probability of .89 of being offered, respectively, the significant gain or nothing. More specifically, comparing those conditional certainty equivalents indicates that the paradoxical decision-maker is *more risk averse* in the first case than in the second. This follows from the definition of comparative risk aversion, applied not to two decision-makers facing the same choice situation, but to one decision-maker facing two different choice situations (as micro-economists do when they study the effects of income variations on individual risk attitudes).

Similarly, one can deduce from the second set of paradoxical preferences that $CE(M_2) = CCE(M_2, Q_2) < 100 < CCE(M_2, S_2)$. The unconditional certainty equivalent of M_2 differs from the conditional certainty equivalent of M_2 taken as one of the convex components of S_2. Again, the paradoxical decision-makers are more risk averse in the first case than in the second. This second variation occurs, while all the combined lotteries are the same. In the first case, the prevailing combination weights highlight the sure prospect of a significant gain (100), while in the second case, they highlight the quasi-certainty of ending up with nothing (0).

It is noteworthy that, according to a widely held intuition, the decision-makers in the Allais paradoxes act more like gamblers in the second cases than in the first cases, because in the second cases, they have nothing to lose and everything to gain. As illustrated in the two previous paragraphs, the concept of conditional certainty equivalent captures this intuition in terms of the prevailing technical concepts of risk attitude.[26]

Based on the preceding analysis, I propose the following terminology. The Allais paradoxes display a form of *conditionally varying risk attitude*. This idea sheds light not only on the Allais paradoxes and the divide between expected and non-expected utility. This idea illuminates the theory of decision-making under risk as a whole. In the case of classic preferences, it proves equivalent to impose the respect of VNM independence or the constancy of conditional certainty equivalents. The easy direction of this

25 Admittedly, Machina (Machina 1983) and others have a more restrictive definition of the common consequence and the common ratio effects. The simpler definition above suffices for my purposes. Machina (Machina 1982, p. 288) sketches the subsequent analysis of the Allais paradoxes.

26 In particular, the descriptions of the Allais paradoxes referred to in footnote 21 can be made precise using the concept of conditional certainty equivalent.

equivalence (the necessity claim) has already been mentioned, the other direction (the sufficiency claim) requires a proof that can be found in the literature.[27] The authors of this proof show more than this equivalence. They establish that the two main branches of non-expected utility theory, i.e., the previously mentioned betweenness and co-monotonic branches, can also be cast in this axiomatic mold. They show that the various ways in which the models in those branches of non-expected utility theory constrain classic preferences are various ways of restraining the possible variation of conditional certainty equivalents, i.e., the conditional variation in risk attitude. They highlight that their results allow for further unifying the axiomatic theory of choice under risk. I, for my part, stress that such unification is made possible by one risk attitude concept. With the various possible patterns of conditional variation in risk attitude, the risk attitude concepts come into play at the key step of the decision theorists' representation theorems.

However, notice that this step pertains to *comparative* risk attitude only. The underlying absolute risk attitudes are irrelevant. Indeed, what is true of unconditional certainty equivalents is also true of conditional certainty equivalents. A decision-maker might prove to be more risk averse in a first situation than in a second one, more risk seeking in a second situation than in a third one, and so on. Yet, those conditional comparative attitudes need not take the form of any of the canonical absolute risk attitudes. In particular, the underlying preferences might be non-classifiable.

4.2 The strengthening of risk attitudes

The present subsection goes one step further than the previous one by stressing that even the *absolute* risk attitude concepts play a significant axiomatic role. The absolute risk attitudes will even be seen to open more new perspectives than the comparative risk attitudes, discussed in the previous subsection. Recall the various kinds of risk reduction. For the sake of concreteness, consider decision-makers who systematically opt for (rather than against) at least one kind of risk reduction, say, the simplest of all, namely, total risk reduction. Recall that always opting for total risk reductions does not preclude one from consistently opting against some merely partial risk reductions. There is more to this fact, however, than a definitional remark. To see why, consider the variation on the second Allais paradox in Fig. 5. Assume that the paradoxical decision-maker is weakly risk averse.[28]

[27] See Chew and Epstein 1989 (corrected by Chew et al. 1993). Thus, Machina is justified in calling the independence axiom "the requirement of constant conditional certainty equivalents" (Machina 1982, p. 298). Although it is not fully axiomatized (but see the recent results in Cerreia-Vioglio et al. 2016), the decision model proposed by Machina would also support the present analysis. Its key assumption (Machina 1982, Hypothesis II, p. 300) is about comparative risk attitude.

[28] Kahneman and Tversky (Kahneman and Tversky 1979, p. 267) consider a similar variation, yet without making explicit the link with the absolute risk attitude concepts.

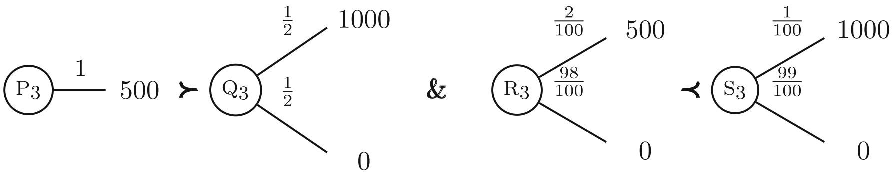

Fig. 5 Variation on the second Allais paradox

One can check that $P_3\ RR_A\ Q_3,\ R_3\ RR_A\ S_3$, and $P_3\ RR_T\ Q_3$, but not $R_3\ RR_T\ S_3$. The decision-maker opts for all total risk reductions (whence her preferring P_3 to Q_3), but nonetheless opts against some merely partial ones (since she prefers S_3 to R_3). I propose to say that a decision-maker displays an *intermediate risk attitude* if, as in the present example, she systematically accepts (respectively, refuses) the risk reductions of a given kind while refusing (respectively, accepting) some risk reductions of a less restrictive kind. In this particular instance, the intermediate risk attitude consists in the fact that the decision-maker is weakly but not strongly risk averse. The point of couching this case as an Allais paradox is to make vivid that this intermediate risk attitude is incompatible with VNM independence, i.e., the constancy of conditional certainty equivalents. Notice, indeed, that for the decision-maker in Fig. 5, $CE(Q_3) < 500 < CCE(Q_3, S_3)$.

Importantly, the particular intermediate risk attitude thus illustrated is not the only one to clash with expected utility. To see this, consider the variation on the first Allais paradox in Fig. 6. Assume that the paradoxical decision-maker is not only weakly, but also moderately risk averse.[29]

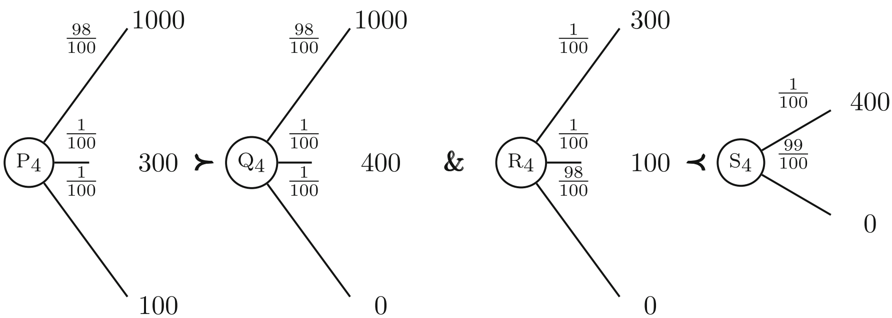

Fig. 6 Variation on the first Allais paradox

29 Notice that unlike any of the Allais cases considered hitherto, this particular variation features no certain option (like $P_1, P_2, or\ P_3$).

One can check that $P_4\ RR_A\ Q_4$, $R_4\ RR_A\ S_4$, and $P_4\ RR_M\ Q_4$, but not $R_4\ RR_M\ S_4$. Even though the decision-maker opts for all monotonic risk reductions (as corroborated by her preferring P_4 to Q_4), she opts against some more arbitrary ones (since she prefers S_4 to R_4). Accordingly, she is moderately, but not strongly risk averse. As this variation illustrates, the above intermediate risk attitude also induces a violation of VNM independence, i.e., it is inconsistent with expected utility maximization.

Admittedly, the various forms of intermediate risk attitude are sufficient but not necessary for VNM independence to be violated.[30] They deserve to be highlighted nonetheless, as they illustrate a topic of general axiomatic interest. I propose to refer to this topic as the *strengthening of risk attitudes*. The expected utility model illustrates the most extreme form of such strengthening. In this model, any weakly risk averse (respectively, seeking) decision-maker must also be moderately so, and any moderately risk averse (respectively, seeking) decision-maker must also be strongly so.[31] Because of the linearity in probabilities imposed by VNM independence, the chain of simple implications in (2) becomes a series of equivalences. To the best of my knowledge, no other classic model of choice under risk displays such rigidity. Accordingly, it might be that in the case of classic preferences, this rigidity characterizes expected utility under a condition that would be interestingly weaker than VNM independence.[32] This raises the following question. Assume that a classic preference relation $\succcurlyeq$ satisfies the following property:

$$\Big(\forall P \in \Delta(C), \delta_{E(P)} \succcurlyeq P\Big) \Rightarrow \Big(\forall P, Q \in \Delta(C), P RR_A Q \Rightarrow P \succcurlyeq Q\Big),$$
$$\Big(\forall P \in \Delta(C), P \succcurlyeq \delta_{E(P)}\Big) \Rightarrow \Big(\forall P, Q \in \Delta(C), P RR_A Q \Rightarrow Q \succcurlyeq P\Big). \quad (4)$$

The above axiom excludes any intermediate risk attitude. It effectively partitions the set of all preference relations into only four categories, corresponding to the following possibilities: non-classifiability, risk neutrality, strong strict risk aversion, or strong strict risk seeking. To the best of my knowledge, the following question has not been studied in the literature. What are the necessary and sufficient conditions ensuring that a classic preference that respects the axiom in (4) also respects VNM independence? Answering this question would better our understanding of the expected utility model and shed light on the divide between expected and non-expected utility.

Expected utility is the only model known to impose the most extreme form of risk attitude strengthening. However, less extreme forms thereof are relevant for

[30] Recall the algebraic examples in footnote 20. D illustrates that one can fail to respect VNM independence while displaying all the degrees of risk aversion distinguished hitherto.

[31] It follows from Jensen's inequality that, in expected utility, a decision-maker is weakly risk averse if and only if the utility function $u : C \to \mathbb{R}$, the expectation of which represents her preferences, is concave. Rothschild and Stiglitz (Rothschild and Stiglitz 1970) show the less immediate result that, in expected utility, a decision-maker is strongly risk averse if and only if $u : C \to \mathbb{R}$ is concave. Then, given (2), all the risk aversion concepts must be equivalent.

[32] Clearly, this rigidity alone cannot characterize expected utility. Consider, e.g., the class of all preference relations respecting the co-monotonic weakening of VNM independence. Building on the results mentioned in footnote 20, one could easily define a subclass of classic preference relations displaying the two following properties. First, all relations respect the axiom in (4). Second, no relation respects VNM independence.

understanding other, non-expected utility models. One model belonging to the comonotonic branch of non-expected utility theory deserves particular attention. It is the so-called dual model of expected utility. The duality in question consists in the following fact. The expected utility model characterizes decision-makers merely by a transformation of the numerical results and leaves the probability values untransformed. The dual model characterizes decision-makers merely by a transformation of the probability values and leaves the numerical results untransformed. In the dual model, any weakly risk averse (respectively, seeking) decision-maker must also be moderately so, but a moderately risk averse (respectively, seeking) decision-maker needs *not* be strongly so.[33] To this extent, the dual model proves partially more flexible than expected utility as regards the intermediate risk attitudes. Incidentally, I am unaware of any decision model that has been proved to impose a form of partial risk attitude strengthening symmetric to the one above, i.e., such that the weak risk attitudes do not necessarily generalize to the moderate ones, but the moderate attitudes generalize to the strong ones.

The partial flexibility of the dual model is informative. It indicates that the rigidity of expected utility with respect to the absolute risk attitudes cannot boil down to the fact that it is a one-parameter model. Such is the case of the dual model (with one function on probabilities, rather than on results), that nonetheless displays greater flexibility, as I have just detailed.

The most significant results available to date on the strengthening of risk attitudes concern the rank-dependent utility model, i.e., the general framework of which both expected utility and its dual model are special cases. This is a two-parameter framework, featuring both a function transforming the numerical results and a function transforming the probability values. It has been shown that in this model, the weak risk attitudes need not generalize to the moderate ones, and the moderate risk attitudes need not generalize to the strong ones.[34] Thus, the chain of simple implications in (2) can be verified as is. In other words, there is no automatic strengthening of the absolute risk attitudes. In particular, unlike any model discussed hitherto, the rank-dependent utility model can accommodate decision-makers that are *exactly* weakly risk averse (respec-

---

[33] The dual model is due to Yaari (Yaari 1987). Its transformation function concerns not the direct probability values, but the decumulative ones. The dual model is a variant of the more general rank-dependent utility model that generalizes expected utility, as described in Sect. 3. On the absolute risk attitudes in the dual model, see, e.g., Chateauneuf et al. 1997, p. 35. Segal and Spivak (Segal and Spivak 1990) shed light on why the dual model is more flexible than expected utility, despite its being, like expected utility, a one-parameter model.

[34] Recall the notation of footnote 15. Denote by G_P the decumulative distribution function of lottery P. The axioms of rank-dependent utility are satisfied if and only if there exist two strictly increasing functions $u : C \to \mathbb{R}$ and $w : [0, 1] \to [0, 1]$, the latter normalized at 0 and at 1, such that one can analyze v in (3) as $v(P) = \sum_{i=1}^{n} \big[w(G_P(c_{i+1})) - w(G_P(c_i))\big]u(c_i)$. Expected utility corresponds to when $w(p) = p$, for all $p \in [0, 1]$. The dual model corresponds to when $u(c) = c$, for all $c \in C$. For a review on the absolute risk attitudes in rank-dependent utility, see Chateauneuf et al. 1997, and see Ryan 2006 for further results. Unlike the strong risk attitudes (Chew et al. 1987) and the moderate risk attitudes (Chateauneuf et al. 2005), the weak risk attitudes have not yet been characterized in the rank-dependent utility model. However, the partial results available (Chateauneuf and Cohen 1994—see also Cohen and Meilijson 2014) suffice to establish that any intermediate risk attitude can be accommodated.

tively, seeking), i.e., those that opt for (respectively, against) any total risk reduction while nonetheless opting against (respectively, for) some merely monotonic ones.

The present state of the literature leaves it unclear, however, whether this remarkable flexibility is due to the specific features of the rank-dependent utility model. To clarify this, one would need to know whether there is any model of choice under risk that (unlike rank-dependent utility) would impose some form of risk attitude strengthening while being (like rank-dependent utility) endowed with more than one parameter. I stress that such clarification is not provided by the advocates of rank-dependent utility who stress the flexibility described above. This is a missing argument in their defense. Indeed, risk attitude strengthening might prove characteristic of one-parameter models, such as expected utility or its dual model, and disappear in any multi-parameter model. In that case, rank-dependent utility should not be credited specifically, among all multi-parameter models, for the flexibility described above. More generally, the topic of risk attitude strengthening would prove of more limited axiomatic significance than my paper suggests.

Nevertheless, as I now show, the missing argument can be provided. Consider the so-called cautious expected utility model, a multi-parameter model which is contained neither in the betweenness, nor in the co-monotonic branch of non-expected utility theory. In this model, decision-makers are characterized not by one utility function on the set of results, like in the expected utility model, but by a set of such functions. Decision-makers are represented as choosing cautiously in the following sense: given any lottery, they compute its expected utility according to each of their possible utility functions, and they assign to the lottery the minimum of these expected utilities. This decision model rests on a surprisingly simple weakening of VNM independence.[35] Although the authors who recently axiomatized this model do not consider this proposition, it can be proved that in this model, any weakly risk seeking decision-maker must also be strongly (and hence moderately) risk seeking, while weakly risk averse decision-makers need not be strongly risk averse (whether or not they need to be moderately risk averse is an open question).[36] To my knowledge, both the fact that one model would strengthen risk aversion and risk seeking asymmetrically, and the fact that one model would strengthen one risk attitude from weak to strong outside the expected utility realm, are exceptional in the literature.[37]

[35] The cautious expected utility model is due to Cerreia-Vioglio and co-authors (Cerreia-Vioglio et al. 2015). Its key axiom is the following weakening of VNM independence: for any $c \in C$, all $P, R \in \Delta(C)$, and any $\alpha \in (0, 1]$, $P \succcurlyeq \delta_c$ if and only if $\alpha P + (1-\alpha)R \succcurlyeq \alpha\delta_c + (1-\alpha)R$.

[36] Notice this implication of the axiom in footnote 35: for all $P, R \in \Delta(C)$, and any $\alpha \in (0, 1]$, $P \succcurlyeq \delta_{E(P)}$ if and only if $\alpha P + (1-\alpha)R \succcurlyeq \alpha\delta_{E(P)} + (1-\alpha)R$. In the case of classic preferences, the preference on the right-hand side proves equivalent to strong risk seeking (see Chew and Mao 1995, p. 413). Thus, in the cautious expected utility model, weak risk seeking and strong risk seeking are equivalent. However, for a decision-maker to be weakly risk averse in this model, it suffices that one of her utility functions is concave, while strong risk aversion requires that all of them are concave (Cerreia-Vioglio et al. 2015, Theorem 3).

[37] Recently, Dean and Ortoleva (Dean and Ortoleva 2017, p. 386–389) have introduced a new model that is to some extent dual to the cautious expected utility model. In this new model, decision-makers are characterized by one utility function and a set of probability weighting functions. They choose cautiously in the sense above, but with reference to rank-dependent utility, instead of expected utility. The properties of this model as regards the absolute risk attitudes are currently unknown. It would be particularly interesting to investigate them.

The case of the cautious expected utility model is thus particularly instructive. First, it highlights one rarely discussed aspect of the rigidity of the expected utility model and its dual model, namely, that these models strengthen risk aversion and risk seeking symmetrically. Second, it establishes that the flexibility of the rank-dependent utility model cannot boil down to the fact that it is a multi-parameter model. Such is the case of the cautious expected utility model (with a set of utility functions, rather than a pair constituted by a utility function and a probability weighting function), that nonetheless displays greater rigidity, as I have just detailed.

This example, however instructive, does not fully clarify the role of the specific features of the rank-dependent utility model in the remarkable flexibility which it displays as regards the absolute risk attitudes. This is an unsettled matter partly because of the lack of characterization results on the moderate risk attitudes in the decision models that do not belong to the co-monotonic branch of non-expected utility theory. In particular, I am unaware of any such result for the multi-parameter models in the betweenness branch of non-expected utility theory.[38] For instance, the influential disappointment aversion model features both a utility function and some proposed coefficient of disappointment aversion. It would be instructive to know how this model accommodates the chain of implications in (2), i.e., whether it imposes any form of risk attitude strengthening.

There seems to be a second way of trying to better our understanding of the moderate risk attitudes, which prove central to the study of risk attitude strengthening. A remarkable result in the literature characterizes the strong risk attitudes for arbitrary classic preferences. This result is exceptional in its applying at the level of generality of (3), i.e., across the variety of decision models compared hitherto.[39] More results of this kind would be invaluable for the discussion of the present paper. Characterizing the weak risk attitudes for arbitrary classic preferences appears to be a particularly difficult task. The more restrictive case of the moderate risk attitudes constitutes, by contrast, a natural next step. New light could be shed on risk attitude strengthening by combining such general results, which pertain to any classic preference, and the various characterizations of the decision models, which isolate particular subclasses of classic preferences.

The discussion above illustrates that new results are called for. For all that, the results currently available are sufficient to illustrate the axiomatic relevance of the absolute risk attitude concepts. These available results—apart from those on the exceptional

[38] Such results would be particularly instructive, because the betweenness and the co-monotonic branches of non-expected utility theory are disjoint, in the sense of having in common only the expected utility model itself (see, e.g., Chew and Epstein 1989, p. 208).

[39] See Chew and Mao 1995 (with slightly different continuity requirements than those assumed here). The key property in this characterization is Schur-concavity (see, e.g., Marshall et al. 2010, Chapter 3), displayed by the function v in (3) when restricted to a distinguished subset of lotteries, namely, the set of all equiprobable lotteries. Apart from the characterization results in Chateauneuf and Lakhnati 2007, I am aware of only one other result established at a similar level of generality, which is to be found in Cerreia-Vioglio et al. 2016, Proposition 2 (the equivalence between (ii) and (iv)). There, it is established that the strong risk aversion of a classic preference relation is equivalent to the weak risk aversion of one of its distinguished subrelations, namely, its largest subrelation respecting VNM independence (together with transitivity and continuity, but not necessarily completeness). The authors of this result do not compare it with that of Chew and Mao.

Table 1 The intermediate risk attitudes in various decision models

Models	Risk attitudes	
	Weak, but not moderate	Moderate, but not strong
Expected utility	✗	✗
Dual expected utility	✗	✓
?	✓	✗
Rank-dependent utility	✓	✓

case of the cautious expected utility model—are summarized in Table 1. To read this table, it is helpful to recall the chain of model-free implications in (2).

The intermediate risk attitudes give rise to several elegant *impossibility results*. These results establish that whenever preferences form such or such simple pattern of risk attitude, there can be no functional form of such or such type representing them. Consequently, as shown in Table 1, a typology of the classic decision models can be proposed, based on their capacity to allow for more or less refined risk attitudes, i.e., on their imposing more or less radical forms of risk attitude strengthening. Thus, the concepts of absolute risk attitude allow for systematic distinctions between decision models, which is the primary purpose of axiomatic analysis.

5 Conclusion

On the face of it, the decision-theoretic concepts of risk attitude seem unable to account for the structural differences between the various models of decision-making under risk. To this extent, these concepts do not seem axiomatically significant. However, taking into account the conditional variation and the strengthening of risk attitudes leads to substantial qualifications of this negative assessment. The decision models are only partially neutral between the various risk attitudes, whence the axiomatic relevance of the concepts introduced to describe those attitudes. Assessing the axiomatic status of the risk attitude concepts also leads to the identification of several interesting open questions, especially in connection with the strengthening of risk attitudes. For a better assessment of the status of risk attitudes in axiomatic decision theory, one would need not only to address these questions, but also to consider other conditions than the ones focused on here. Specifically, it would be necessary to discuss the risk attitude concepts also in the context of non-classic preference relations (i.e., when neither unconditional nor conditional certainty equivalent functions can be defined) and when the lotteries offered to the decision-maker are defined over arbitrary sets of results (in which case, the absolute risk attitudes cannot be defined in the usual way). These are topics for future research.

Acknowledgements For helpful comments or discussions, I am thankful to two reviewers, Mohammed Abdellaoui, Richard Bradley, Michèle Cohen, Mikaël Cozic, Eric Danan, David Dillenberger, Louis Eeckhoudt, Raphaël Giraud, Philippe Mongin, Pietro Ortoleva, and Fanyin Zheng. All errors and omissions are mine. Research for this paper was funded by the Ecole Normale Supérieure-Ulm, the Université Cergy-Pontoise, and the Wagemann Foundation.

References

Allais, M. (1953). Le comportement de l'homme rationnel devant le risque: Critique des postulats et axiomes de l'école américaine. *Econometrica*, *21*(4), 503–546.
Bommier, A., Chassagnon, A., & Le Grand, F. (2012). Comparative risk aversion: A formal approach with applications to saving behavior. *Journal of Economic Theory*, *147*(4), 1614–1641.
Camerer, C. (1992). Recent tests of generalizations of expected utility theory. In W. Edwards (Ed.), *Utility Theories: Measurements and Applications* (pp. 207–251). New York: Springer.
Camerer, C. (1995). Individual decision-making. In J. Kagel, A. Roth, & J. Hey (Eds.), *The handbook of experimental economics* (Vol. II, pp. 587–703). Princeton: Princeton University Press.
Cerreia-Vioglio, S., Dillenberger, D., & Ortoleva, P. (2015). Cautious expected utility and the certainty effect. *Econometrica*, *83*(2), 693–728.
Cerreia-Vioglio, S., Maccheroni, F., & Marinacci, M. (2016). Stochastic dominance analysis without the independence axiom. *Management Science*, *63*(4), 1097–1109.
Chateauneuf, A. (1999). Comonotonicity axioms and rank-dependent expected utility theory for arbitrary consequences. *Journal of Mathematical Economics*, *32*(1), 21–45.
Chateauneuf, A., & Cohen, M. (1994). Risk seeking with diminishing marginal utility in a non-expected utility model. *Journal of Risk and Uncertainty*, *9*(1), 77–91.
Chateauneuf, A., Cohen, M., & Meilijson, I. (1997). New Tools to better model behavior under risk and uncertainty: An overview. *Finance*, *18*(1), 25–46.
Chateauneuf, A., Cohen, M., & Meilijson, I. (2005). More pessimism than greediness: A characterization of monotone risk aversion in the rank-dependent expected utility model. *Economic Theory*, *25*(3), 649–667.
Chateauneuf, A., & Lakhnati, G. (2007). From sure to strong diversification. *Economic Theory*, *32*(3), 511–522.
Chew, S. H. (1989). Axiomatic utility theories with the betweenness property. *Annals of Operations Research*, *19*(1), 273–298.
Chew, S. H., & Epstein, L. (1989). A unifying approach to axiomatic non-expected utility theories. *Journal of Economic Theory*, *49*(2), 207–240.
Chew, S. H., Epstein, L., & Wakker, P. (1993). A unifying approach to axiomatic non-expected utility theories: Correction and comment. *Journal of Economic Theory*, *59*(1), 183–188.
Chew, S. H., Karni, E., & Safra, Z. (1987). Risk aversion in the theory of expected utility with rank dependent probabilities. *Journal of Economic Theory*, *42*(2), 370–381.
Chew, S. H., & Mao, M. H. (1995). A Schur concave characterization of risk aversion for non-expected utility preferences. *Journal of Economic Theory*, *67*(2), 402–435.
Cohen, M. (1995). Risk-aversion concepts in expected- and non-expected-utility models. *The Geneva Papers on Risk and Insurance Theory*, *20*(1), 73–91.
Cohen, M., & Meilijson, I. (2014). Preference for safety under the Choquet model. In search of a characterization. *Economic Theory*, *55*(3), 619–642.
Dean, M., & Ortoleva, P. (2017). Allais, Ellsberg, and preferences for hedging. *Theoretical Economics*, *12*(1), 377–424.
Debreu, G. (1964). Continuity properties of Paretian utility. *International Economic Review*, *5*(3), 285–293.
Dekel, E. (1986). An axiomatic characterization of preferences under uncertainty: Weakening the independence axiom. *Journal of Economic Theory*, *40*(2), 304–318.
Dillenberger, D., & Segal, U. (2015). Recursive ambiguity and Machina's examples. *International Economic Review*, *56*(1), 55–61.
Fishburn, P. (1970). *Utility theory for decision making*. New York: Wiley.
Fishburn, P. (1982). Nontransitive measurable utility. *Journal of Mathematical Psychology*, *26*(1), 31–67.
Friedman, M., & Savage, L. (1948). The utility analysis of choices involving risk. *The Journal of Political Economy*, *56*(4), 279–304.
Gul, F. (1991). A theory of disappointment aversion. *Econometrica*, *59*(3), 667–686.
Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, *47*(2), 263–291.
Krantz, D., Luce, R., Suppes, P., & Tversky, A. (1971). *Foundations of measurement, volume I: Additive and polynomial representations*. New York: Academic Press.
Loomes, G., & Sugden, R. (1982). Regret theory: An alternative theory of rational choice under uncertainty. *The Economic Journal*, *92*(368), 805–824.

Machina, M. (1982). Expected utility analysis without the independence axiom. *Econometrica*, *50*(2), 277–323.

Machina, M. (1983). Generalized expected utility analysis and the nature of observed violations of the independence axiom. In B. Stigum & F. Wenstop (Eds.), *Foundations of utility and risk theory with applications* (pp. 263–293). Dordrecht: D. Reidel.

Marshall, A. (1890). *The principles of economics*. London: Macmillan and Co., Ltd.

Marshall, A., Olkin, I., & Arnold, B. (2010). *Inequalities: Theory of majorization and its applications*. New York: Springer.

Quiggin, J. (1982). A theory of anticipated utility. *Journal of Economic Behavior & Organization*, *3*(4), 323–343.

Rothschild, M., & Stiglitz, J. (1970). Increasing risk: I. A definition. *Journal of Economic Theory*, *2*(3), 225–243.

Ryan, M. (2006). Risk aversion in RDEU. *Journal of Mathematical Economics*, *42*(6), 675–697.

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica*, *57*(3), 571–587.

Segal, U. (1987). Some remarks on Quiggin's anticipated utility. *Journal of Economic Behavior & Organization*, *8*(1), 145–154.

Segal, U. (1990). Two-stage lotteries without the reduction axiom. *Econometrica*, *58*(2), 349–377.

Segal, U., & Spivak, A. (1990). First order versus second order risk aversion. *Journal of Economic Theory*, *51*(1), 111–125.

Seidl, C. (2013). The St. Petersburg Paradox at 300. *Journal of Risk and Uncertainty*, *46*(3), 1–18.

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, *5*(4), 297–323.

von Neumann, J., & Morgenstern, O. (1947). *The theory of games and economic behavior*. Princeton: Princeton University Press.

Yaari, M. (1969). Some remarks on measures of risk aversion and on their uses. *Journal of Economic Theory*, *1*(3), 315–329.

Yaari, M. (1987). The dual theory of choice under risk. *Econometrica*, *55*(1), 95–115.