Countabilism and Maximality Principles

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Abstract

It is standard in set theory to assume that Cantor's Theorem establishes that the continuum is an uncountable set. A challenge for this position comes from the observation that through forcing one can collapse any cardinal to the countable and that the continuum can be made arbitrarily large. In this paper, we present a different take on the relationship between Cantor's Theorem and extensions of universes, arguing that they can be seen as showing that every set is countable and that the continuum is a proper class. We examine several principles based on maximality considerations in this framework, and show how some (namely *Ordinal Inner Model Hypotheses*) enable us to incorporate standard set theories (including **ZFC** with large cardinals added). We conclude that the systems considered raise questions concerning the foundational purposes of set theory.

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Introduction

The notion of maximality has been mobilised in favour of several axiom systems *extending* **ZFC**. However it has been little considered whether the notion of maximality might *contradict* **ZFC**.[1]

In this paper we provide exposition of some systems of set theory based on an interpretation of maximality on which every set is countable and the continuum is a proper class. We identify some pleasing features of the view and argue that it does not prevent set theory from fulfilling its foundational roles. This presents a challenge for those who think that **ZFC**-based set theory is true and that maximality considerations should figure into the justification of set-theoretic axioms, since there appear to be legitimate perspectives on maximality in set theory which violate the **ZFC** axioms.

Here's the plan: After these introductory comments, in §1 we provide some philosophical and historical remarks outlining why it might not be such a heresy to hold the countabilist position that every set is countable, and describe some challenges for the view concerning how set theory is meant to provide a foundation. In §2 we set up some preliminaries regarding removing the Powerset Axiom, before (§3) examining the possibility of simply brute-forcing large cardinal strength. §4–§7 examine more natural maximality principles, culminating in a theory with substantial large cardinal strength up to the level of $0^\sharp$. §4 considers a formal system *Forcing Saturated Set Theory* (or **FSST**) which uses inspiration from maximality in the context of forcing axioms to develop an axiom which implies that every set is countable and the continuum is a proper class. We then show that the axiom is comparatively weak—it is consistent relative to **ZFC** - Powerset and consistent with $V = L$. Next (§5) we consider a principle of *absoluteness* we call the *Axiom of Set-Generic Absoluteness*. This axiom implies that $V \neq L$ but nonetheless is still consistent relative to **ZFC** - Powerset. §6 builds on this idea by developing extreme versions of the Inner Model Hypothesis—however we show that they go too far and conflict with reasonably weak theories (Theorem 25). In §7 we consider weakenings of these assumptions—*Ordinal Inner Model Hypotheses*—and show that one such (i) implies that every set is countable and the continuum is a

[1]One exception is [Holmes, 2017] who considers a set theory on which every set is countable that he calls *Pocket Set Theory*:

> We do remark that it is not necessarily the case that the hypothetical advocate of pocket set theory thinks that the universe is small; he or she might instead think that the continuum is very large... [Holmes, 2017]

proper class, (ii) is consistent relative to **ZFC** + PD (Theorem 28), and (iii) implies that "$0^\sharp$ exists" (Theorem 31) and hence implies the existence of inner models satisfying **ZFC** with large cardinals. From this we argue for our main claim:

Main Claim. There are well-motivated perspectives on maximality in set theory on which every set is countable and set theory is able to do its usual foundational duties.

This presents a challenge for those who wish to say that **ZFC** is true—there seem to be perfectly good interpretations of set-theoretic discourse on which the Powerset Axiom is false. Whilst we do not repudiate **ZFC**-based set theory, the current paper presents a challenge as to what we require from a satisfactory set-theoretic axiomatisation. Should it merely be a theory for interpreting mathematics? Or is it necessary for it to provide a hierarchy of actual uncountable transfinite infinities?

1 Desiderata on a set-theoretic foundation

In this article, we will consider set theories that imply the following position:

> **Countabilism.** Every set is countable.

We'll refer to the position that there are uncountable sets as uncountabilism, and the proponents of the two positions as the countabilist and uncountabilist respectively. Given the contemporary perspective, one might regard countabilism as anathema to the practice of set theory. After all, isn't a lot of set-theoretic discourse directed at the study of uncountable sets? It is one objective of this article to argue that this might not be the case. To begin, we provide some philosophical motivation to convince the reader that countabilism is a perspective worth studying, and that's what we'll do in this section.

We start with a puzzle that we'll call the *Cohen-Scott Paradox*. It depends on three observations:

Observation A. (Cantor's Theorem) The Powerset Axiom implies that there are uncountable sets.[2]

[2]Obviously one needs to work over some suitably strong base theory to prove this result. In this article, we won't work with anything weaker than the family of theories obtained from removing Powerset from **ZFC** (see §2 for some discussion of these theories).

Observation B. Given any model of set theory M, and any M-cardinal κ, there is forcing partial order $Col(\omega, \kappa)$ which forces κ to be countable in the extension.

Observation C. Given any model of set theory M and any M-cardinal κ, there is a forcing $Add(\mathcal{P}^M(\omega, \kappa^+))$ that pushes the value of the continuum above κ in the extension.[3]

If we think of forcing as a way of adding subsets, and we think that the universe contains all possible subsets, we can generate a paradox as follows:

The Cohen-Scott Paradox. We think that there are uncountable sets (in particular the set of all real numbers) by Cantor's Theorem. But by Observation B, we (in some sense) 'could' collapse any set x to the countable by adding a surjection $f : \omega \to x$, and in particular (by Observation C) we 'could' make the reals bigger than x. On the assumption that the universe should contain all possible sets, we have a puzzle. On the one hand we think that the V should contain uncountable sets, but on the other hand if the universe does contain uncountable sets it appears to be missing all sorts of collapsing generics for partial orders, and in particular the reals seem smaller than they might have been.

Of course the standard response to the Cohen-Scott Paradox is that it shows that various kinds of models are *unintended* in some sense—

[3]This is what perhaps influenced Cohen, in the Conclusion to his seminal [Cohen, 1966] (in which he presents forcing), to write:

> "A point of view which the author feels may eventually come to be accepted is that CH [the continuum hypothesis] is obviously false. The main reason one accepts the Axiom of Infinity is probably that we feel it absurd to think that the process of adding only one set at a time can exhaust the entire universe. Similarly with the higher axioms of infinity. Now $\aleph_1$ is the set of countable ordinals and this is merely a special and the simplest way of generating a higher cardinal. The set C [the continuum] is, in contrast, generated by a totally new and more powerful principle, namely the Power Set Axiom. It is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach C. Thus C is greater than $\aleph_n$, $\aleph_\omega$, $\aleph_\alpha$ where $\alpha = \aleph_\omega$ etc. This point of view regards C as an incredibly rich set given to us by one bold new axiom, which can never be approached by any piecemeal process of construction. Perhaps later generations will see the problem more clearly and express themselves more eloquently." ([Cohen, 1966], p. 151, underline original)

for example they may be Boolean-valued or countable—and the sense in which the reals 'might' have been larger than a particular cardinal is only with respect to some unintended interpretation. In many cases the kinds of models considered in forcing are somehow different from the uncountabilist's universe, e.g. the usual Boolean-valued models are not even two-valued and countable models miss out a whole bunch of sets.[4] The uncountabilist thus rejects our interpretation of Observations B and C as showing that there 'could' have been more sets than there actually are if **ZFC** is true—this modal claim only makes sense if our possible set-theoretic worlds include unintended models.

Note, however, that this is just *one* of the available responses. A different approach would be to say that it is the interpretation of Cantor's Theorem that is at fault. An (unattractive) possibility is to say that Cantor's Theorem is *false*. This looks problematic; Cantor's argument for the uncountability of the reals depends only upon being able to talk about enumerations of real numbers and then given any one such enumeration, use the standard diagonal reasoning to generate a real not on the list. Insofar as it shows that there are no bijections between the natural numbers and the reals, Cantor's reasoning seems impeccable.

A different and more plausible option is to deny the Powerset Axiom and hold that every set is countable and the continuum is a proper class. Whilst the position is controversial, it is not without precedent. Similar ideas have been considered by [Hallett, 1984][5] [Holmes et al., 2012], [Meadows, 2015], [Friedman, 2016], [Pruss, 2019], and [Scambler, 2021].[6] However, Scott (in a forward to Bell's textbook on Boolean-valued models[7]) presents the earliest consideration of this suggestion that we are aware of:

> I see that there are any number of contradictory set theories, all extending the Zermelo-Fraenkel axioms: but the models are all just models of the first-order axioms, and first-order logic is weak. I still feel that it ought to be possible

[4]See [Koellner, 2013] and [Barton, 2020] for some discussion of this issue.

[5]In particular, he remarks the following after appreciatively quoting Cohen and Scott:

> Thus, the continuum evades all our attempts to characterize it by size (Cohen), so maybe we should start with this transcendence as a datum (Scott). ([Hallett, 1984], p. 208)

[6][Friedman, 2016] (pp. 529–530) and [Scambler, 2021] (throughout) in particular, strongly emphasise the modal point concerning forcing.

[7]See [Bell, 2011] for the third edition.

> to have strong axioms, which would generate these types of models as submodels of the universe, but where the universe can be thought of as something absolute. Perhaps we would be pushed in the end to say that all sets are countable (and that the continuum is not even a set) when at last all cardinals are absolutely destroyed. But really pleasant axioms have not been produced by me or anyone else, and the suggestion remains speculation. A new idea (or point of view) is needed, and in the meantime all we can do is to study the great variety of models. ([Scott, 1977], p. xv)

Scott's request for 'pleasant axioms' on the countabilist perspective is pertinent. Many countabilist perspectives offered thus far are essentially multiversist or potentialist in spirit, obtaining a modal form of countabilism in which any set (at some world) *could* be countable in a larger world. For example, [Scambler, 2021] shows how a modalised version of set theory interprets **ZFC**$-$Powerset $+$"Every set is countable" under a modal translation, and [Meadows, 2015] directly imports some of his framework from the multiversist [Steel, 2014]. In this paper we want to consider what the universe might look like *non*-modally and in which these potentialist and multiversist accounts appear as substructures within the universe.

Whilst it is most likely a matter of taste what counts as 'pleasant' (as Scott requests) there are several foundational jobs that set theory has been seen to do, as recently made precise by Penelope Maddy (in [Maddy, 2017] and [Maddy, 2019]). One important one is:

Generous Arena. Find *representatives* for our usual mathematical structures (e.g. $\mathbb{N}$, $\mathbb{R}$) in our theory of sets.

This is often what is meant when it is said that set theory is 'foundational'—all mathematical objects can be regarded as encoded within set theory if one wishes. Closely linked is the idea of:

Shared Standard. Provide a standard of correctness for proof in mathematics.

The idea being that if we can code all mathematical objects as sets (as in **Generous Arena**) then (if needed) we could view all proofs in mathematical theories as proofs about the sets. Of course, this practice would be anathema to the working mathematician, who should feel free to work with the more fluid language of the relevant field.[8] However in the case of disagreement, mathematicians could in principle reduce everything to a proof in set theory. Indeed, we can study (within set theory) proofs *themselves*, in particular proving relative consistency via the study of *models* of set theory, providing us with:

Metamathematical Corral. Provide a theory in which metamathematical investigations of relative provability and consistency strengths can be conducted.

Of course we do not *need* the sledgehammer of set theory to do this work, usually some (weak) theory of arithmetic will do. However, the *natural* theory in which this is study conducted is set theory. Here various different models can be easily studied and compared, as the enormous literature of independence results using set-theoretic resources testifies. This ability to study the consistency strengths of various theories and the fact that they are embedded within a common framework that is well-understood plausibly provides:

Risk Assessment. Provide a degree of confidence in theories commensurate with their large cardinal strength.

Risk Assessment leads naturally to the idea that we should want to consider theories that maximise consistency strength and (if possible) do so in a *well-motivated* way. It is one thing to show that some theory or other can be calibrated to have a certain large cardinal strength, and another to increase our confidence in the theory by motivating the idea that the relevant large cardinal axiom is consistent. For this reason we add the following[9]:

Motivational Challenge. Motivate a theory with a substantial degree of large cardinal strength on the basis of an account of the global nature of the universe.

Before we continue we should make a remark about just how weak this challenge is. We are not asking for full *justification* of some axiomatic system or other, but rather merely that it responds to some natural intuitive ideas concerning the nature of the universe. We are responding to Scott's request for 'pleasant' axioms, rather than engaging

[8]Indeed, this is how *set theorists* operate—no-one is churning out derivations in first-order **ZFC**.

[9]It should be noted that [Maddy, 2019] also addresses this point, referring to the rough intuitive picture behind the iterative conception. For us it will be important, and so we separate it out as a separate challenge.

in the Gödelian search for well-justified axioms. Nonetheless, whilst it is weak, it is not merely a challenge that is satisfied by any kind of axiom whatsoever with a merely ad hoc or post hoc explanation. We want some *global* description of what the universe of sets is *like*. To see this a little more clearly, it is useful to consider the **ZFC**-context. Many motivations there relate to the following idea:

Maximality. The universe of sets should be as *large* as possible.

Of course, the notion of **Maximality** is *very* vague. A recent survey of just how many sharpenings there are of the notion is available in [Incurvati, 2017]. For our purposes it is enough to note that one popular motivation for the *uncountabilist* has been the use of *Reflection Principles*, and these shall form our primary point of comparison in the present paper. In fact, part of our conclusion will be that the axioms we provide put the countabilist in a similar position to the uncountabilist when the latter is viewed as motivating large cardinal axioms on the basis of reflection principles. These formalise in various ways the idea that the universe contains so many sets that there are initial segments that resemble the universe, and have been used to motivate the existence of large cardinals directly. Depending on how the principles are calibrated, this can be up to the level of large cardinals consistent with $V = L$ (see here [Koellner, 2009] for a thorough examination), or possibly even many Woodin or extendible cardinals (see here [Welch and Horsten, 2016] and [Roberts, 2017]). They are global principles about the nature of the set-theoretic universe.[10] Contrast these then with the 'axioms' "the value of the continuum is exactly $\aleph_{9001}$" or "there is an inaccessible cardinal". These axioms do not clearly capture what the universe of sets is like as a whole (even if the latter is implied by many reflection principles).

Given this response to the **Motivational Challenge**, let us pause just for a minute to reflect on how superbly **ZFC**-based set theories perform with respect to Maddy's criteria with the **Motivational Challenge** added:

Regarding **Generous Arena**, it is of course slightly unclear what is meant by 'usual' mathematical structures. However it is one of the

[10] Other candidates for principles about the nature of the set-theoretic universe that might be taken to answer the **Motivational Challenge** include various inner model axioms (including the idea of Ultimate-L, see [Woodin, 2017]) and the study of maximality principles in the context of the Hyperuniverse Programme. Indeed, this latter approach is *designed* to examine such competing 'pictures' of the way the universe might be. A review of some of these options is available in [Friedman, 2016] (p. 519) which contends that no first-order axiom will ever receive consensus support—rather we need to move to a context with *higher-order* resources.

salient features of **ZFC**-based set theory that the flexibility afforded by the Powerset Axiom yields natural representatives for both the natural numbers and the continuum, and in fact almost all mathematics could be conducted in the first few levels above V_ω if desired. This then easily gives us **Shared Standard**—a proof is correct just in case there is a proof about the relevant entities from the axioms of **ZFC** (or maybe some extension thereof). **Metamathematical Corral** barely needs mentioning—**ZFC** *just is* the standard theory we usually use for studying independence. Regarding **Risk Assessment**; whilst **ZFC** is quite weak, it can be naturally extended with large cardinal axioms, allowing us to pinpoint reasonably accurately the consistency strength of new pieces of mathematics. (e.g. Whilst it has been subsequently weakened[11] the original proof of Fermat's Last Theorem required the addition of inaccessible cardinals.) As noted above, one can provide answers to the **Motivational Challenge** in underwriting these large cardinal axioms (and in particular we have focussed on the specific example of reflection principles as a point of comparison).

The challenge facing the countabilist is thus great if we also want to have 'pleasant axioms'. Whilst **Metamathematical Corral** seems unproblematic—we can always conduct this study by looking at countable models of various theories—the other foundational roles are not clearly satisfied. **Generous Arena** seems particularly problematic since the countabilist does not have the Powerset Axiom to generate their representatives. This in turn calls into question **Shared Standard** since without the representatives it is unclear when we should regard a proof about those representatives as 'correct'. In turn **Risk Assessment** is difficult to ascertain—without an answer to the **Motivational Challenge** it is unclear why we should have (relative) confidence in strong theories for the countabilist. In the remainder of this paper we show one way for the countabilist to respond to these issues. Our core strategy is to consider principles of 'width absoluteness'—axioms that imply that every set is countable via a strong saturation of V under sets that 'could' exist. We will see (§§4–5) that although some of these options are quite weak, and (§6) it is possible to go too far, there is nonetheless (§7) a relatively strong axiom that is consistent relative to large cardinals and can be motivated along these lines. Our contention is that this puts the countabilist in a somewhat similar position to the uncountabilist with respect to foundational roles.

[11]See here [McLarty, 2010].

2 Removing Powerset

One issue that needs to be dealt with before we get into the main part of our proposal is: What do we take the countabilist's base theory to be?

A natural answer: It is the theory **ZFC** with the Powerset Axiom removed (possibly with "Every set is countable" added, but the axioms we will consider all imply this). However, as is now well known (especially since [Zarach, 1996] and [Gitman et al., 2011]) various equivalences one normally has in the presence of the Powerset Axiom disappear once it is removed. In particular, simply deleting the Powerset Axiom and keeping Replacement does not preserve Collection, and various versions of the Axiom of Choice become non-equivalent.[12] We therefore need to set up what we mean by various theories lacking the Powerset Axiom:

Definition 1. We distinguish between the following theories:

(1.) $\mathbf{ZFC}-$ is $\mathbf{ZFC}$ with the Powerset Axiom Removed and AC formulated as the claim that every set can be well-ordered.

(2.) $\mathbf{ZFC}^-$ is $\mathbf{ZFC}-$ with the Collection and Separation Schema substituted for the Replacement Scheme.

(3.) $\mathbf{ZFC}^-_{Ref}$ is $\mathbf{ZFC}^-$ with the following schematic reflection principle added (for any ϕ in the language of set theory):

$$\forall x \exists A(x \in A \wedge \text{"}A \text{ is transitive"} \wedge \phi \leftrightarrow \phi^A)$$

i.e. for any set x there is a transitive set A such that $x \in A$ and ϕ is absolute between A and the universe. We will refer to this principle as the *First-Order Reflection Principle* (or just '*Reflection*').

(4.) By $\mathbf{NBG}-$, $\mathbf{NBG}^-$, and $\mathbf{NBG}^-_{Ref}$ we mean the corresponding versions of $\mathbf{NBG}$, with two sorts of variables and any corresponding schema replaced by single second-order (predicative) axioms.

It is known that the three theories $\mathbf{ZFC}-$, $\mathbf{ZFC}^-$, and $\mathbf{ZFC}^-_{Ref}$ are distinct in the sense that the classes of their models are pairwise non-identical, and hence that the obvious inclusions are proper (easy arguments show that there are inclusions, see [Gitman et al., 2011], §1). The

[12]For instance, the existence of a Choice function picking a member from every element of a choice set does not imply that every set can be well-ordered. See here [Zarach, 1982] who also shows several other results. The inequivalence of versions of *global* choice also appear in the literature on second-order logic and choice principles in that context, see here [Shapiro, 1991], §5.1.3.

result that there are models of $\mathbf{ZFC}-$ that are not models of $\mathbf{ZFC}^{-}$ was shown by [Zarach, 1996] and further explored in [Gitman et al., 2011]. Recently [Friedman et al., F] showed that there are models of $\mathbf{ZFC}^{-}$ in which $\mathbf{ZFC}^{-}_{Ref}$ fails (and indeed in which all sets are countable). We introduce the class theories $\mathbf{NBG}-$, $\mathbf{NBG}^{-}$, and $\mathbf{NBG}^{-}_{Ref}$ because we will want to use proper classes in talking about the reals (which may form a proper class). The observation that for any $(M, \in) \models \mathbf{ZFC}-$ (respectively $\mathbf{ZFC}^{-}$, $\mathbf{ZFC}^{-}_{Ref}$) we have $(M, \in, Def(M)) \models \mathbf{NBG}-$ (respectively $\mathbf{NBG}^{-}$, $\mathbf{NBG}^{-}_{Ref}$) shows that the obvious inclusions between the classes of models of $\mathbf{NBG}-$, $\mathbf{NBG}^{-}$, and $\mathbf{NBG}^{-}_{Ref}$ are also proper. In light of these results, one might think of the study of these Powerset-free theories as concerned with trying to include as many natural principles as possible (without adding strength) in order to facilitate mathematical reasoning in the absence of the Powerset Axiom. This naturally motivates $\mathbf{ZFC}^{-}_{Ref}$ and $\mathbf{NBG}^{-}_{Ref}$ as the natural choices of base theory for the countabilist, particularly one who has mathematical applications in mind. Consider, for example, Gitman, Hamkins and Johnstone who write (concerning the comparison of $\mathbf{ZFC}-$ and $\mathbf{ZFC}^{-}$):

> The main point of this paper, therefore, is to reveal what can go wrong when one naively uses $\mathbf{ZFC}-$ in a set-theoretic argument for which one should really be using $\mathbf{ZFC}^{-}$, and to point out that if one indeed would use $\mathbf{ZFC}^{-}$, then all standard arguments carry through as expected. In other words, our point is that $\mathbf{ZFC}-$ is the wrong theory, and in almost all applications, set theorists should be using $\mathbf{ZFC}^{-}$ instead. ([Gitman et al., 2011], p. 3)

As long as there are interesting applications using Reflection, this point naturally extends to a motivation for $\mathbf{ZFC}^{-}_{Ref}$ (and not just $\mathbf{ZFC}^{-}$) too. In the uncountabilist context, first-order reflection is implied by the usual Lévy-Montague reflection principle (since, in particular, any V_α to which we reflect is transitive). However in the $\mathbf{ZFC}^{-}$ context, First-Order Reflection is equivalent to the following choice-like principle (see [Friedman et al., F] for the result):

Definition 2. ($\mathbf{ZFC}^{-}$) The *Dependent Choice Scheme* (we will also use the terms 'DC-Scheme' and 'DCS') is the scheme of assertions claiming that for each formula $\phi(x, y, z)$ and parameter a, if for every x there is a y such that $\phi(x, y, a)$ holds, then there is an ω-sequence $\langle x_n | n \in \omega \rangle$ such that for all n, $\phi(x_n, x_{n+1}, a)$ holds. (i.e. If a definable relation has no terminal nodes, we can make ω-many dependent choices on its basis.)

Thus $\mathbf{ZFC}^-_{Ref}$ is equivalent (modulo $\mathbf{ZFC}^-$) to $\mathbf{ZFC}^-$ with the DC-scheme added (mutatis mutandis for $\mathbf{NBG}^-_{Ref}$ and $\mathbf{NBG}^-$ with a single axiom in place of the DC-Scheme). This relationship between DCS and Reflection further supports choosing $\mathbf{ZFC}^-_{Ref}/\mathbf{NBG}^-_{Ref}$ as our base theory (if one is motivated by mathematical expedience). This is especially so given that the current context that can be thought of through the lens of second-order arithmetic. A folklore result[13] shows that second-order arithmetic and $\mathbf{ZFC}^-$ + "Every set is countable" are bi-interpretable and so one can consider the present study as investigating models of second-order arithmetic as well as the countabilist theories we will propose, and certainly DCS is a useful principle there. With this in hand, let us consider some ways we might begin to bolster these base theories for the countabilist.

3 The Brute Force Strategy

For the sake of brevity we will often abbreviate the axiom "Every set is countable" by Count. The first point to be made is that there *are* statements of set theory concerning second-order arithmetic (and hence $\mathbf{ZFC}^-$ + Count) that yield substantial large cardinal strength, and indeed large cardinals in inner models of models satisfying $\mathbf{ZFC}^-$ + Count.[14]

The core observation is that much recent work in set theory has involved building inner models for large cardinals from principles of second-order arithmetic. This is done via an ultrapower construction (often using so-called mice) and since $\mathbf{ZFC}^-$ and $\mathbf{ZFC}^-_{Ref}$ provide us with the resources to construct ultrapowers, we are often able to build the relevant models.[15] Some care is required here, however. For example, in $\mathbf{ZFC}$ "$0^\sharp$ exists" has many equivalent formulations, which don't all work in $\mathbf{ZFC}^-$. To see this, note that one formulation is that the uncountable cardinals are indiscernible in L, but in $\mathbf{ZFC}^-$ we have no guarantee that there are any uncountable cardinals (in fact we have the negation given Count). We therefore take the following formulation:

Definition 3. ($\mathbf{ZFC}^-/\mathbf{NBG}^-$) "$0^\sharp$ exists" will be taken to mean that there is a definable club of L-indiscernibles.

[13]See §5.1 of Regula Krapf's PhD thesis [Krapf, 2017].

[14]We are very grateful to Sandra Müller and Chris Scambler for some discussion of ideas in this section.

[15]See [Schimmerling, 2001] for a pleasant explanation of mice, and [Schindler, 2014], §10.2 (esp. Def. 10.37) how $0^\sharp$ can be used in this context.

A tempting complaint concerning this definition is to point to use of the term "definable" (a notion which is not first-order). There are two ways this might play out depending on whether we are in the $\mathbf{NBG}^-$ or $\mathbf{ZFC}^-$ context. In the former case one can just omit the word "definable" and talk about the kinds of closed unbounded classes that exist. Alternatively (in both the $\mathbf{ZFC}^-$ and $\mathbf{NBG}^-$ context) one can appeal to the work of Silver who shows that if there is a definable club of L-indiscernibles then there is a unique such club (the club of Silver indiscernibles) which generates L (i.e. its Skolem hull in L is all of L), and this club is Δ_2-definable. Therefore we can replace the word "definable" by "Δ_2-definable" in Definition 3 and obtain a first-order definition. Moreover Silver's result (see [Jech, 2002], Ch. 18) works in the absence of the Powerset Axiom. By shifting Silver indiscernibles we can easily obtain many elementary embeddings of L to L. We can then find various strong theories under the countabilist perspective. For example:

Fact 4. $\mathbf{ZFC}^-$+"$0^\sharp$ *exists" implies that there is a definable inner model (i.e. a transitive model containing all the ordinals) for* $\mathbf{ZFC}$+*"There exists a proper class of inaccessible cardinals".*

Proof. Every Silver indiscernible is L-inaccessible, and so the existence of $0^\sharp$ implies the existence of a proper class of inaccessible cardinals in L. Moreover each Silver indiscernible α is such that L_α is elementary in L, and so $L \models \mathbf{ZFC}$. □

A theme that will repeatedly emerge is: One can always build inner models (e.g. L) within models of theories extending $\mathbf{ZFC}^-$, and in the presence of suitable principles we obtain $\mathbf{ZFC}$ with large cardinals there (even though V thinks that every set is countable). Moreover, one can easily go beyond $0^\sharp$. From $\undertilde{\Pi}^1_1$-Determinacy and the $\undertilde{\Pi}^1_2$-Perfect Set Property one can obtain inner models of $\mathbf{ZFC}$+"Every set of ordinals has a sharp" (and indeed that this implication can be reversed from a model of $\mathbf{ZFC}$+"Every set of ordinals has a sharp").[16] With Projective Determinacy one gets inner models with Woodin cardinals (in particular n-many for every $n \in \mathbb{N}$).[17] In this context, whilst some inner models have $\mathbf{ZFC}$ (and indeed much more) they are impoverished with respect to the functions they can see (in particular they are blind to all sorts of collapsing functions).

Thus we can see how the countabilist perspective is compatible with principles asserting a significant degree of large cardinal strength,

[16] Regula Krapf's dissertation [Krapf, 2017], Ch. 5 shows this equivalence.

[17] See [Koellner and Woodin, 2010] for a description of how to get models of large cardinal axioms from determinacy hypotheses.

responding to the challenge of providing a strong theory suggested by the desideratum of **Risk Assessment**. However, given the **Motivational Challenge**, we might worry about whether baldly asserting the existence of various non-trivial regularity properties in second-order arithmetic is a satisfactory response given that they do not easily conform to any intuition about the nature of the set-theoretic universe or relate to considerations of **Maximality**.[18] In this respect, the countabilist is not necessarily in a worse position than her uncountabilist counterpart since the motivational story for him is also unclear beyond $0^\sharp$. This is especially so given that some authors (especially [Koellner, 2009] and [Friedman and Honzik, 2016]) hold that no reflection principle is likely to deliver this level of large cardinal strength. There are some counterarguments here (notably [Welch and Horsten, 2016], [Welch, 2017], and [Roberts, 2017]) which attempt to provide reflection-style arguments for large cardinals up to (and beyond) the level of many Woodin cardinals, however their status as reflection principles is somewhat open. It is thus fair to say that the situation is also murky for the uncountabilist with respect to principles past $0^\sharp$.[19] In the rest of the paper, we will show how to motivate the existence of $0^\sharp$ using **Maximality** on the countabilist perspective.

4 Forcing Saturated Set Theory

Our suggestion is to view the maximality of the universe via different kinds of *saturation* under possible sets. This is a somewhat vague motivation but admits of various formalisations, similar to how the uncountabilist wishes to postulate the existence of 'large' ordinals and can formalise this idea via large cardinals/reflection principles. We'll approach this gradually; the current section (§4) will provide a principle consistent with $V = L$, §5 will go beyond $V = L$ but not provide any large cardinal strength, §6 will consider a principle that is *too* strong, before we isolate a principle in §7 that is consistent relative to standard large cardinals but also implies the existence of $0^\sharp$.

The first principle we shall examine stems from consideration of *forcing axioms*. A forcing axiom expresses the idea that the universe has already been saturated under forcing for certain partial orders and

[18]For example, Martin (concerning Projective Determinacy) writes: "Is PD true? It is certainly not self-evident." ([Martin, 1977], p. 813). Examples of this kind can be multiplied, for example [Martin, 1976], p. 90 [Moschovakis, 1980], p. 610. [Maddy, 1988] provides a good survey of the terrain here.

[19]Often these principles are justified by their 'fruitfulness' or via 'extrinsic' considerations (e.g. [Maddy, 2011]). We return to this issue in §8.

families of dense sets. For example we have the following axiom:

Definition 5. Let κ be an infinite cardinal. $\mathsf{MA}(\kappa)$ is the statement that for any forcing poset $\mathbb{P}$ in which all antichains are countable (i.e. $\mathbb{P}$ has the countable chain condition), and any family of dense sets $\mathcal{D}$ such that $|\mathcal{D}| \leqslant \kappa$, there is a filter G on $\mathbb{P}$ such that if $D \in \mathcal{D}$ is a dense subset of $\mathbb{P}$, then $G \cap D \neq \emptyset$.

Definition 6. *Martin's Axiom* (or just MA) is the statement that for every κ smaller than the cardinality of the continuum, $\mathsf{MA}(\kappa)$ holds.

In this way MA asserts that the universe has been saturated under forcing for certain partial orders and families of dense sets. Stronger forcing axioms such as the Proper Forcing Axiom (PFA) and Martin's Maximum (MM) have been isolated, and some set theorists have linked these kinds of axiom to maximality ideas. Magidor, for example, writes:

> "Forcing axioms like Martin's Axiom (MA), the Proper Forcing Axiom (PFA), Martin's Maximum (MM) and other variations were very successful in settling many independent problems. The intuitive motivation for all of them is that the universe of sets is as rich as possible, or at the slogan level: A set [whose] existence is possible and there is no clear obstruction to its existence [exists]...
>
> ...What do we mean by "possible"? I think that a good approximation is "can be forced to [exist]"...
>
> I consider forcing axioms as an attempt to try and get a consistent approximation to the above intuitive principle by restricting the properties we talk about and the the forcing extensions we use. ([Magidor, 2012], pp. 15–16)

Forcing axioms can thus be seen as an attempt to get a grip on the notion of maximising the subsets available. However, as will be well known to specialists, there are usually some limitations as to how far one can go. For instance, consider the following facts:

Fact 7. *(**ZFC**) Letting $\mathfrak{c}$ denote the cardinality of the continuum,* $\mathsf{MA}(\mathfrak{c})$ *is inconsistent with **ZFC**.*[20]

Fact 8. *(**ZFC**) In **ZFC** there is a non-countable-chain-condition $\mathbb{P}$ such that for a $(\leqslant \aleph_1)$-sized family of dense subsets $\mathcal{D}$ of $\mathbb{P}$, there is no filter G on $\mathbb{P}$ intersecting every member of $\mathcal{D}$ (i.e. $\mathsf{MA}_{\mathbb{P}}(\aleph_1)$ is false).*[21]

[20] See [Kunen, 2013], p. 175, Lemma III.3.13.
[21] See [Kunen, 2013], pp. 175–176, Lemma III.3.15.

It seems then that there are some limitations on what generics one can have. Given **ZFC**, we cannot just assert the existence of generic sets in a careless manner. However, if we accept the possibility that every set might be countable and drop the Powerset Axiom, more options are open to us. We can then consider a forcing axiom that allows us to have a generic for *any* set-sized family of dense sets:

Definition 9. ($\mathbf{ZFC}^-$) *The Forcing Saturation Axiom* (or FSA). If $\mathbb{P}$ is a forcing poset, and $\mathcal{D}$ is a set-sized family of dense sets, then there is a filter $G \subseteq \mathbb{P}$ intersecting every member of $\mathcal{D}$. The theory of *Forcing Saturated Set Theory* or **FSST** comprises $\mathbf{ZFC}^- +$ FSA.

FSST implies that every set is countable, as we show below. However, it is also weak, as is shown by the following fact:

Fact 10. *($\mathbf{ZFC}^-$)* **FSST** *is equivalent to the theory* $\mathbf{ZFC}^-$*+"Every set is countable".*

Proof. (1.) **FSST** $\Rightarrow$ $\mathbf{ZFC}^-$+"Every set is countable".

To see that **FSST** implies that every set is countable, let α be the order-type of a well-ordering of an arbitrary set x (α is our putative 'uncountable' cardinal). Then, the poset $Col(\omega, \alpha)$ is obtainable by taking definable powersets. (Note that in $\mathbf{ZFC}^-$ the *definable* powerset of any set still exists.) We can now define a family of dense sets in order to get generic encoding a surjection from ω to α. At each $\beta < \alpha$ we can define the following set:

$$D_\beta = \{p \in Col(\omega, \alpha) | \beta \in dom(p)\}$$

Clearly each D_β is dense since we can always extend above any condition q to obtain an r with $\beta \in dom(r)$. Collection and Separation then yields the family $\mathcal{D} = \langle D_\beta | \beta < \alpha \rangle$. Using the Forcing Saturation Axiom, there is a generic G for this family coding a surjection from ω to α.

(2.) $\mathbf{ZFC}^-$+"Every set is countable" $\Rightarrow$ **FSST**.

To obtain the Forcing Saturation Axiom from the axiom that every set is countable, let $\mathbb{P}$ be a forcing poset and $\mathcal{D}$ be a family of dense subsets of $\mathbb{P}$. Since every set is countable, we can enumerate $\mathcal{D}$ in order-type ω. So, without loss of generality, $\mathcal{D} = \langle D_n | n \in \omega \rangle$. Since every set is countable, $\mathbb{P}$ can also be enumerated in order-type ω, let 'f' denote the relevant enumerating function. We can then define via recursion (and using the parameter f) the following function π from $\mathcal{D}$ to $\mathbb{P}$:

$$\pi(D_0) = \text{“The } f\text{-least } p \in D_0\text{”}$$

$$\pi(D_{n+1}) = \text{“The } f\text{-least } p \in D_{n+1} \text{ such that } p \leqslant_{\mathbb{P}} \pi(D_n)\text{”}$$

Effectively π successively picks elements of each member of $\mathcal{D}$, ensuring that we always go below our previous pick in the forcing order (this is allowed because each $D \in \mathcal{D}$ is dense in $\mathbb{P}$, and so such a p always exists). By Replacement $ran(\pi)$ exists, and the object obtained generates a generic for $\mathcal{D}$ (namely the set of q such that p is below q for some $p \in ran(\pi)$), and so the Forcing Saturation Axiom holds. □

By Fact 10 we have the immediate:

Corollary 11. **FSST** *is consistent relative to the theory* $\mathbf{ZFC}^-$*.*

Proof. Take any model M of $\mathbf{ZFC}^-$. $H(\omega_1)$ of the model satisfies $\mathbf{ZFC}^- + \mathsf{Count}$ and hence **FSST** (and in the case where $\omega_1 = Ord$ we can simply say that M itself satisfies **FSST**). □

This shows that **FSST** is not just weak in consistency strength, but also has minimal consequences. It is reasonable to expect that maximality principles should destroy $V = L$. Here we have the immediate easy corollary:

Corollary 12. **FSST** *is consistent with* $V = L$*.*

Proof. In any given model of $\mathbf{ZFC}^-$, $(H(\omega_1))^L \models V = L + \mathbf{FSST}$. □

Thus whilst **FSST** does imply countabilism through some sort of saturation idea, it fails to break $V = L$. We should not necessarily view this fact as a deathblow to **FSST**, however. We might rather view **FSST** as an initial stepping stone to stronger theories, much like how **ZFC** is consistent with $V = L$ but can be strengthened using large cardinals to theories that break $V = L$. As we'll see in the following sections, there are strengthenings of **FSST** that do just this.

5 The Axiom of Set-Generic Absoluteness

In this section, we provide an exposition of the *Axiom of Set-Generic Absoluteness*, an axiom that implies that every set is countable and that $V \neq L$. Whilst it is still consistent relative to $\mathbf{ZFC}^-$, its consideration will be useful for setting up the general ideas behind stronger axioms later.

The way that the FSA postulated the saturation of the universe under forcing was somewhat brutal; we simply asserted the existence of the relevant generics for partial orders and families of dense sets. Somewhat similar considerations apply here as in the brute assertion of determinacy in generating large cardinal strength—it is unclear why the assertion of combinatorial statements about the existence of generics relates to more intuitive ideas concerning the nature of the universe. Whilst the FSA does perhaps mesh better with **Maximality** than determinacy hypotheses, it would nonetheless be preferable if the countabilist could assert something more natural about the universe of sets which implies that every set is countable (and, for that matter, breaks $V = L$).

We can come to slightly more elegant axioms via the use of *absoluteness principles*. An absoluteness principle asserts that if some formula is satisfied in an extension of the universe, then it is satisfied within the universe (in some appropriate context). Absoluteness principles have already been found for characterising some standard forcing axioms. For example:

Definition 13. [Bagaria, 1997] *Absolute*-MA. We say that $V \models \mathbf{ZFC}$ satisfies *Absolute*-MA iff whenever $V[G]$ is a generic extension of V by a partial order $\mathbb{P}$ with the countable chain condition in V, and $\phi(x)$ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula (i.e. a Σ_1-formula containing only parameters from $\mathcal{P}(\omega_1)$), if $V[G] \models \exists x \phi(x)$ then there is a y in V such that $\phi(y)$.

Further, we can characterise the Bounded Proper Forcing Axiom (BPFA) as follows:

Definition 14. [Bagaria, 2000] *Absolute*-BPFA. We say that $V \models \mathbf{ZFC}$ satisfies *Absolute*-BPFA iff whenever ϕ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula, if ϕ holds a forcing extension $V[G]$ obtained by proper forcing, then ϕ holds in V.

These principles turn out to be equivalent to MA and BFPA respectively. Bagaria also sees motivation for them as springing from the idea that the universe has been saturated under the existence of possible kinds of sets:

> In the case of MA and some weaker forms of PFA and MM, some justification for their being taken as true axioms is based on the fact that they are equivalent to principles of generic absoluteness. That is, they assert, under certain restrictions that are necessary to avoid inconsistency, that everything that might exist, does exist. More precisely, if

some set having certain properties could be forced to exist over V, then a set having the same properties already exists (in V). ([Bagaria, 2008], pp. 319–320)

Bagaria's point concerning 'certain restrictions' is pertinent: In the **ZFC**-context, a careful calibration is required between the complexity of sentences figuring into the principle and the kinds of parameters allowed when formulating absoluteness principles. For example, if we allow ω_1 as a parameter and all forcings that collapse cardinals, we would obtain a contradiction in **ZFC**, since ω_1 would then be countable in V. However, since we are in the $\mathbf{ZFC}^-$ context, we have no such obstacles:

Definition 15. ($\mathbf{ZFC}^-$) We say that V, a model of $\mathbf{ZFC}^-$, satisfies the *Weak Axiom of Set-Generic Absoluteness* (WASGA) iff whenever $\phi(\vec{a})$ is a Σ_1-formula in the language of set theory in the parameters $\vec{a} \in V$, if $\mathbb{P} \in V$ is a forcing partial order, G is V-generic in the sense that it intersects **every** dense set in V, and $\phi(\vec{a})$ holds in $V[G] \models \mathbf{ZFC}^-$, then $\phi(\vec{a})$ holds in V.

This axiom generalises the absoluteness of Σ_1-formulas to include unrestricted parameters and and arbitrary forcings. Unfortunately this buys us no additional strength beyond **FSST**:

Fact 16. *($\mathbf{ZFC}^-$) The* WASGA, FSA, *and* Count *are equivalent (modulo $\mathbf{ZFC}^-$).*

Proof. WASGA $\Rightarrow$ Count is obvious, since for any particular set x "x is countable" is a Σ_1-formula in the parameter x. Hence x is countable in an outer model (since any set can be collapsed in a forcing extension), thus x is countable in an inner model of V, and (by the upwards-absoluteness of countability) is countable in V. Count $\Rightarrow$ WASGA follows from Lévy Absoluteness which tells us that if a Σ_1-formula with real parameters holds in an outer model of $\mathbf{ZFC}^-$ then it holds in V. So if V satisfies $\mathbf{ZFC}^-$ + Count then WASGA will hold for Σ_1-formulas (since under Count *every* set is coded by a real). □

The WASGA thus just gives us an equivalent formulation (over $\mathbf{ZFC}^-$) of **FSST**. In this way, whilst it might be somewhat more natural, it does nothing to assuage worries concerning the lack of consistency strength or other consequences (e.g. it fails to break $V = L$).

In order to obtain a principle with more consequences we will need to go further. A natural target is the complexity of the formulas allowed in the absoluteness claims. To this end, we can fomulate the following axiom:

Definition 17. ($\mathbf{ZFC}^-$) We say that V, a model of $\mathbf{ZFC}^-$, satisfies the *Axiom of Set-Generic Absoluteness* (or ASGA) iff whenever $\phi(\vec{a})$ is a sentence in the language of set theory *in the parameters* $\vec{a} \in V$, if $\mathbb{P} \in V$ is a forcing partial order, G is V-generic in the sense that it intersects *every* dense set in V, and $\phi(\vec{a})$ holds in $V[G] \models \mathbf{ZFC}^-$, then $\phi(\vec{a})$ holds in V.

The ASGA is an absoluteness principle stating that first-order sentences of arbitrary complexity with arbitrary parameters holding in set forcing extensions are true in V. We can very quickly show that:

Fact 18. *Over the theory* $\mathbf{ZFC}^-$ *the* ASGA *implies the* FSA*.*

Proof. The cardinality of any set can be collapsed to ω in some extension $V[G]$, and hence by the ASGA every set is countable in V, which in turn is equivalent to the FSA. □

However the ASGA also goes substantially further than the FSA, as shown by the following:

Fact 19. $\mathbf{ZFC}^- +$ ASGA *implies that* $V \neq L$*.*

Proof. Since we can force the existence of a non-constructible real in some $V[G]$, by the ASGA a non-constructible real exists in V, and hence $V \neq L$. □

Thus the ASGA goes substantially beyond the FSA in terms of consequences. At this point, we might worry about its consistency. Normally generic absoluteness says that if there is a set in a forcing extension that satisfies an absolute property then there is such a set in the ground model. Typically the underlying absolute property is Δ_0, hence many generic absoluteness axioms (e.g. Absolute-MA, Absolute-BPFA) postulate Σ_1-absoluteness. If one then wants to postulate Σ_2-absoluteness, then one might think that we should ensure that we have a situation in which we already have Σ_1-absoluteness (else it is not even clear that if a Σ_2-property holds in the ground model then it would continue to hold in the forcing extension). Typically one doesn't expect Σ_2-absoluteness to be consistent between models that violate Σ_1-absoluteness for this reason. Indeed (in the $\mathbf{ZFC}$-context) both Absolute-MA and Absolute-BPFA become inconsistent if Σ_2-formulas are allowed instead of only Σ_1-formulas (since, for instance, both CH and $\neg$CH can be given a Σ_2 formulation). The following fact shows (somewhat surprisingly) that the ASGA is actually very *weak* in terms of consistency strength:

Fact 20. $\mathbf{ZFC}^-$ + ASGA *is consistent relative to* $\mathbf{ZFC}^-$*.*

Proof. We begin with a model M of $\mathbf{ZFC}$, and explain how the strength can be reduced later. Begin by forcing using an $\aleph_1$-product of Cohen forcings with finite support (call this forcing $\mathbb{P}$), to form an extension $M[G]$.

We claim that $H(\omega_1)^{M[G]}$ satisfies $\mathbf{ZFC}^-$+ASGA. The fact that $\mathbf{ZFC}^-$ holds is immediate, since the $H(\omega_1)$ of any model of $\mathbf{ZFC}$ satisfies $\mathbf{FSST}$. It just remains to argue that $H(\omega_1)^{M[G]}$ satisfies the ASGA. To see this, we begin by noting that any finite sequence of parameters $\vec{a}$ from $H(\omega_1)^{M[G]}$ appears at some stage of the iteration. In other words, if we let G_α be the first α-many Cohen reals added by G, then $\vec{a}$ appears in $V[G_\alpha]$.

Since $\vec{a}$ is hereditarily countable, it can be coded by some real r. Moreover, r must belong to $V[G_\alpha]$ for some countable α. This is because $\mathbb{P}$ has the countable chain condition, which in turn implies that any real added by G has a countable $\mathbb{P}$-name, and hence, letting $\mathbb{P}_\alpha$ be the finite support α-length product of Cohen forcing, r has a $\mathbb{P}_\alpha$-name. In other words, any real r added by G is already added for some G_α, for countable α. Letting $G_{\alpha\rightsquigarrow}$ be the Cohen reals added after G_α by $\mathbb{P}$, we can then view $H(\omega_1)^{M[G]}$ as $H(\omega_1)^M[G_\alpha][G_{\alpha\rightsquigarrow}]$, where $G_{\alpha\rightsquigarrow}$ is $H(\omega_1)^M[G_\alpha]$-generic for the ω_1-many Cohen forcings after the α^{th} stage of the iteration given by $\mathbb{P}$.

Now suppose that there is a countable forcing $\mathbb{Q}$ in $H(\omega_1)^{M[G]} = H(\omega_1)^M[G_\alpha][G_{\alpha\rightsquigarrow}]$, and generic $G_\mathbb{Q}$ such that $H(\omega_1)^{M[G]}[G_\mathbb{Q}] \models \phi(\vec{a})$ where $\vec{a} \in H(\omega_1)^{M[G]}$. To show that the ASGA is satisfied by $H(\omega_1)^{M[G]}$, we just have to show that $H(\omega_1)^{M[G]} \models \phi(\vec{a})$. Since $G_\mathbb{Q}$ is generic over $H(\omega_1)^{M[G]}$ for a countable forcing (i.e. $\mathbb{Q}$), we can assume without loss of generality that $G_\mathbb{Q}$ is generic for Cohen forcing, since Cohen forcing is the only countable forcing up to forcing-equivalence. Thus, since $H(\omega_1)^{M[G]} = H(\omega_1)^M[G_\alpha][G_{\alpha\rightsquigarrow}]$, we know that $H(\omega_1)^{M[G]}[G_\mathbb{Q}] = H(\omega_1)^M[G_\alpha][G_{\alpha\rightsquigarrow}][G_\mathbb{Q}]$, and hence that $\phi(\vec{a})$ becomes true after forcing with the finite support product over $H(\omega_1)^{M[G_\alpha]} = H(\omega_1)^M[G_\alpha]$, adding $G_{\alpha\rightsquigarrow}$ and $G_\mathbb{Q}$, i.e. adding $(\omega_1 + 1)$-many Cohen reals (which is just ω_1-many Cohen reals). It follows (using the homogeneity of Cohen forcing) that $H(\omega_1)^M[G_\alpha][G_{\alpha\rightsquigarrow}] = H(\omega_1)^M[G] \models \phi(\vec{a})$, as required.

To reduce the strength of our initial assumption to $\mathbf{ZFC}^-$, we cannot simply use an $\aleph_1$-product of Cohen forcings with finite support, since we have no guarantee that $\aleph_1$-exists. Supposing that it does not, we can force with the finite support product of Ord-many Cohen forcings (i.e. Ord now plays the role of ω_1). This is a class forcing, but it is $\mathbf{ZFC}^-$ preserving and we can run the same argument as above. □

Thus we see (surprisingly given the strength of the generic absoluteness postulated) that the ASGA is consistent relative to $\mathbf{ZFC}^-$. The reason for consistency (as shown by the previous proof) is extremely special to the countabilist context: If every set is countable then all set-forcings are equivalent to Cohen forcing.

Whilst the ASGA goes some way to providing a perspective that seems to capture the idea of maximality under countabilism, it is still weaker than we would like in terms of consistency strength. Without further argumentation, we have not yet adequately answered the **Motivational Challenge**. In the next two sections (§6, §7) we consider some possibilities for strengthening our theory.

6 The Extreme Inner Model Hypothesis

We would now like to try developing absoluteness principles that imply that every set is countable, but have greater strength than either the FSA or ASGA. Fact 20 was revealing in that it showed us how absoluteness for formulas of arbitrary complexity could be combined with the countabilist perspective consistently. We can now note that the ASGA only appealed to set forcing, and set forcing is only one model building construction among many. We also have class forcing, ultrapowers, and so on. Perhaps then we should insist that a higher degree of absoluteness be present on the countabilist picture, not just with respect to set forcings, but beyond.

A principle that does so in the Powerset-based context is the Inner Model Hypothesis, proposed in [Friedman, 2006]. The original Inner Model Hypothesis is stated as follows:

Definition 21. [Friedman, 2006] The *Inner Model Hypothesis* (or IMH) states that if a parameter-free first-order sentence ϕ is true in an inner model of an outer model of V, then ϕ is already true in an inner model of V.

The idea is that any sentence which can be 'dreamed' to be true (consistent with V's initial starting structure) is true in some inner model context. The IMH is inherently higher-order in character, and depends upon access to a suitable coding of the outer models of V. This can be done either by formulating the IMH as about countable transitive models or by using a class theory in which satisfaction in outer models of V can be coded.[22] (Analogously, the uses of set forcing in the absoluteness characterisations of forcing axioms can be coded

[22][Antos et al., 2021] shows that such a coding is possible within a variant of Morse-Kelley class theory.

away using the relevant forcing relation, keeping in mind that the analogy is not perfect—set-forcing can be coded within $\mathbf{ZFC}$ and the forcing language, the situation with arbitrary extensions is more subtle.) Moreover, the IMH has substantial large cardinal strength; it implies that there are arbitrarily large measurable cardinals in inner models, and its consistency is provable from the existence of a Woodin cardinal with an inaccessible above. It also has significant anti-large cardinal features—it implies that there are no inaccessible cardinals in V.

A large part of developing inner model hypotheses has been the introduction of parameters.[23] In the $\mathbf{ZFC}$-context, this is tricky since a naive introduction of parameters without care allows ω_1 to be collapsed in an inner model—an impossibility given the upwards-absoluteness of countability. If we accept the perspective offered by countabilism though, we are not bound by any such restrictions. We can therefore propose:

Definition 22. *Extreme Inner Model Hypotheses.* The *Extreme Inner Model Hypothesis for* $\mathbf{T}$ or $\mathsf{EIMH}^{\mathbf{T}}$ states that if a first-order sentence $\phi(\vec{a})$ in the parameters $\vec{a}$ in V is true in a definable inner model $I^* \models \mathbf{T}$ of an outer model $V^* \models \mathbf{T}$ of V obtained by a definable pretame class forcing, then $\phi(\vec{a})$ is already true in a definable inner model $I \models \mathbf{T}$ of V. We shall use EIMH^- and EIMH^-_{Ref} to denote the EIMH for $\mathbf{ZFC}^-$ and $\mathbf{ZFC}^-_{Ref}$ respectively.

Several remarks concerning these principles are in order before we proceed. First, there are several places where we might make different choices. We might, for instance, move to a case where the ground model and the relevant outer models should satisfy one of $\mathbf{NBG}^-$ or $\mathbf{NBG}^-_{Ref}$ (we still let the inner models satisfy only $\mathbf{ZFC}^-$ or $\mathbf{ZFC}^-_{Ref}$). Here 'definable inner model' can be replaced by 'inner model', but this expressive ease is paid for by the fact that the ground model and outer models are of the form (M, C) where C is an interpretation of the second-order variables satisfying $\mathbf{T}$. This adds additional complications to our proofs (we will indicate where and how our results extend in due course). Moreover, we might consider arbitrary extensions of V rather than those obtainable by definable pretame class forcings. Again, this level of open-endedness would result in more complicated proofs and so we simply restrict to pretame class forcings and $\mathbf{ZFC}^-$ / $\mathbf{ZFC}^-_{Ref}$ satisfaction for the time being.

Less trivially, there is a substantial question of how to formulate the principles EIMH^- and EIMH^-_{Ref}. We wish to emphasise: **These princi-**

[23]See, for example, the remarks on introducing parameters into the Strong Inner Model Hypothesis in [Friedman et al., 2008].

ples are essentially higher-order in character. We want to say that if $\phi(\vec{a})$ holds in a definable inner model of a pretame definable forcing extension then it holds in a definable inner model of V. The hypothesis thus essentially involves an infinite disjunction of statements of the following form (where $\mathbb{P}_\psi$ is a pretame class forcing defined by ψ and M_χ is the inner model of the $\mathbb{P}_\psi$-generic extension defined by χ):

"$\mathbb{P}_\psi$ is pretame and forces that M_χ satisfies $\phi(\vec{a})$"

Our first 'problem' is that we want to require M_χ to satisfy $\mathbf{ZFC}^-$ or $\mathbf{ZFC}^-_{Ref}$, which cannot be expressed with a single sentence. Our second 'problem' is that for the conclusion of the relevant $\mathsf{EIMH}^{\mathbf{T}}$ we want to say (where N_ξ is the inner model (of V) defined by the formula ξ):

"For some ξ, N_ξ satisfies $\phi(\vec{a})$"

Again we can't express that N_ξ is a model of $\mathbf{ZFC}^-$ or $\mathbf{ZFC}^-_{Ref}$ with one sentence. And we are now quantifying over ξ existentially, so the conclusion is an infinite disjunction. The natural formulation of the EIMH^- and EIMH^-_{Ref} is thus not first-order and is not even given by a first-order scheme (i.e. infinite conjunction of first-order sentences). Instead it is an infinitary Boolean combination of first-order sentences of low infinitary rank.

The choice of using NBG^- or NBG^-_{Ref} appears to look better (ignoring the earlier mentioned proof-based complications) as one can replace "definable inner model" by "inner model". But still one is stuck with formulating "M is an inner model of $\mathbf{ZFC}^-$ / $\mathbf{ZFC}^-_{Ref}$". In the presence of the Powerset Axiom, i.e. in the context of $\mathbf{ZFC}$, to say that a definable transitive proper class is a model of $\mathbf{ZFC}$ is easy because it's enough for it to be a model of $\mathbf{ZFC}_n$ (with Replacement restricted to Σ_n-formulas for large enough n), since using the von Neumann hierarchy Full Collection in V implies Full Collection in transitive inner models of $\mathbf{ZFC}_n$. But in our Powerset-free theories we don't have a substitute for the von Neumann hierarchy in general. It is thus unclear how to formulate these principles, even when employing $\mathbf{NBG}^-$ / $\mathbf{NBG}^-_{Ref}$. Only with $\mathbf{MK}^-$-based theories (i.e. where the relevant predicative second-order axioms are replaced by impredicative ones) are we in good shape because we have truth predicates available and can use them to easily express "M is an inner rmodel of $\mathbf{ZFC}^-$ / $\mathbf{ZFC}^-_{Ref}$". Note, however, the uncountabilist supporter of reflection principles for large cardinals in the $\mathbf{ZFC}$-context is in a similar position. An easy observation (often credited to Bernays) shows that over $\mathbf{NBG}$ the second-order reflection principle immediately implies full Morse-Kelley class theory (since any $(V_\alpha, \mathcal{P}(V_\alpha))$ satisfies impredicative comprehension).

Thus, whilst the higher-order character of these principles might be viewed as a cost, in the present context it is not so important that they be first-order (schematically) expressible. They are expressible by infinitary sentences of low infinitary rank, and we will show relative consistency and inconsistency results by implicitly using this formulation and interpreting the principles over countable transitive models (any results with no formal theory specified should be understood this way, over a 'strong enough' base theory).

Let's start with a couple of easy observations. We immediately have the result that the EIMH^- is a natural continuation of the Forcing Saturation Axiom:

Fact 23. $\mathbf{ZFC}^- + \mathsf{EIMH}^-$ *implies* Count *(equivalently the* FSA *and the* WASGA*).*

Proof. Immediate since the EIMH since any set can be collapsed in an extension and thus is countable in V. □

However, we also get some further consequences out of the EIMH:

Fact 24. $\mathbf{ZFC}^- + \mathsf{EIMH}^-$ *implies that* $V \neq L$.

Proof. As before, we can force the existence of a non-constructible real in an outer model, and so we have non-constructible reals in V. □

Moving on to less trivial matters: One very salient question is whether or not we can prove that the various hypotheses are consistent relative to the existence of large cardinals. As it turns out, at least the EIMH^-_{Ref} is inconsistent:

Theorem 25. $\mathbf{ZFC}^-_{Ref}$ + EIMH^-_{Ref} *is inconsistent.*

Proof. We will show that there is no transitive model of $\mathbf{ZFC}^-_{Ref}$ + EIMH^-_{Ref}. The proof will in fact show that $\mathbf{ZFC}^-_{Ref}$ + EIMH^-_{Ref} proves there is no transitive model of a particular finite subtheory $\mathbf{T}$ of $\mathbf{ZFC}^-_{Ref}$ + EIMH^-_{Ref}—as $\mathbf{ZFC}^-_{Ref}$ + EIMH^-_{Ref} proves that $\mathbf{T}$ has a transitive model (by Reflection/DCS) we infer the inconsistency of $\mathbf{ZFC}^-_{Ref}$ + EIMH^-_{Ref}.

Suppose that V is a transitive model of $\mathbf{ZFC}^-_{Ref}$ + EIMH^-_{Ref} of ordinal height α, we may assume that V is countable. Note that V satisfies Count because any set x in V can be made countable in a set-forcing extension of V, where $\mathbf{ZFC}^-_{Ref}$ must still hold, and therefore by the

EIMH^-_{Ref}, x is countable in an inner model of V and therefore in V. Now as in the proof of Theorem 3.8 of [Antos and Friedman, 2017], we can produce an outer model of V satisfying $\mathbf{ZFC}^-_{Ref}$ which is of the form $L_\alpha[r_0]$ for some real r_0. And as in the proof of Theorem 4.1 of [Antos and Friedman, 2017], we can enlarge further to a model of $\mathbf{ZFC}^-_{Ref}$ of the form $L_\alpha[r]$ for some real r such that for every ordinal $\beta < \alpha$, $L_\beta[r]$ fails to satisfy Collection. Applying the EIMH^-_{Ref}, there is such a real, which we denote by r', in the original model V.

For each $\beta < \alpha$ let $f(\beta)$ be the least n so that Σ_n-Collection fails in $L_\beta[r']$. Now as in Proposition 3.5 of [Friedman, 2000], for each n we can force over V to add a club C_n consisting of ordinals $\beta < \alpha$ such that $f(\beta)$ is at least n. And again as in the proof of Theorem 3.8 of [Antos and Friedman, 2017], we can with further forcing add a real s_n which codes C_n. By the EIMH^-_{Ref} in V, there are such reals s'_n in V, coding corresponding clubs C'_n. But taking some β belonging to the intersection of all the various C'_n, we have that $f(\beta)$ is at least n for each n, a contradiction. □

Remark 26. The above proof can be modified to a version of the EIMH formulated as concerned with $\mathbf{NBG}^-_{Ref}$ models (i.e. where the outer model $(V[G], C[G])$ of (V, C) also has to satisfy $\mathbf{NBG}^-_{Ref}$, but the inner models (which need not be definable) satisfy $\mathbf{ZFC}^-_{Ref}$). For the inconsistency of $\mathbf{NBG}^-_{Ref}$ with this version of the EIMH we need one more fact. We start with a model (V, C) of this theory and first enlarge it to a model (V^*, C^*) where $V^* = L[A]$ for some single class A and C^* consists only of the (V^*, A)-definable classes. It is a result due independently to Friedman and Kossak-Schmerl that this can be done (see here Theorem 15 of [Hamkins et al., 2013]). We can then apply Jensen coding to force further to get A to be coded by a real x. Now one can complete the proof as above for the $\mathbf{ZFC}^-_{Ref}$ context, by ensuring that for some real r, Collection fails in $L_\beta[r]$ for each $\beta < \alpha = Ord(V)$.

The fact that an EIMH-style principle (namely EIMH^-_{Ref}) is incompatible with even very mild forms of Dependent Choice (equivalently Reflection) in the class theory is troubling.

Firstly, where possible, we would like to be as open-minded as possible about the class theory to be adopted, and EIMH-principles put severe constraints on the theories we can have. Secondly, we might think that these conflicts with First-Order Reflection put us in direct conflict with **Maximality**. Whilst we know that the level of width absoluteness we are asking for is incompatible with the existence of uncountable sets (and thus with all standard reflection principles), we might want to incorporate as much 'height' absoluteness as possible given the current picture. Thirdly, the fact that the level of Choice/Reflection required is so minimal raises doubts as to whether the EIMH^- (or the $\mathsf{EIMH}-$ for that matter) is consistent at all, and certainly there is no clear route to a relative consistency proof.

One might try to make a kind of 'bad company' objection to the maximality as (width) absoluteness idea, by arguing that it extends naturally to inconsistency. We think this argument is not especially convincing when used by the uncountabilist against the countabilist, since almost *all* motivations for set-theoretic axioms extend to inconsistency when taken far enough. The same is true (within $\mathbf{ZFC}$) for forcing axioms (e.g. $\mathsf{MA}(\mathfrak{c})$), and reflection principles (which are inconsistent at the level of third-order), or the direct postulation of large cardinals (as indicated by the Kunen Inconsistency). The natural route to take instead is to consider slight weakenings of EIMH-principles. As we'll see shortly we can obtain a principle that is consistent relative to large cardinals but which can also be used to generate substantial strength.

7 Ordinal Inner Model Hypotheses

We are now in a position where we would like to weaken the EIMH but still go beyond the ASGA. A natural choice here is to restrict the parameters allowed:

Definition 27. *Ordinal Inner Model Hypotheses.* The *Ordinal Inner Model Hypothesis for* $\mathbf{T}$ or $\mathsf{OIMH}^{\mathbf{T}}$ states that if a first-order sentence $\phi(\vec{a})$ with **ordinal** parameters $\vec{a}$ in V is true in a definable inner model $I^* \models \mathbf{T}$ of an outer model $V^* \models \mathbf{T}$ of V obtained by a definable pretame class forcing, then $\phi(\vec{a})$ is already true in a definable inner model $I \models \mathbf{T}$ of V. We shall use OIMH^- and OIMH^-_{Ref} to denote the OIMH for $\mathbf{ZFC}^-$ and $\mathbf{ZFC}^-_{Ref}$ respectively.

We should note that exactly the same remarks concerning formalisations of versions of the EIMH apply to versions of the OIMH. Namely: (1.) One can also formulate versions of the OIMH where we insist that both the ground model and outer models satisfy $\mathbf{NBG}^-$ or $\mathbf{NBG}^-_{Ref}$, and (2.) The OIMH^- and OIMH^-_{Ref} are *not* first-order expressible.

Ordinal Inner Model Hypotheses clearly imply that every set is countable in the presence of the Axiom of Choice (since the cardinality of any set can always be collapsed in an extension). However, they do not clearly contradict the Dependent Choice Scheme since in Theorem 25 we depended on the use of unrestricted *real* parameters. In fact we can prove:

Theorem 28. $\mathbf{ZFC}^-_{Ref}$ + OIMH^-_{Ref} *is consistent relative to the theory* **ZFC** + PD.

Proof. The strategy of the proof is to work in a model of **ZFC** + PD and use the structure of Turing degrees given by PD to ensure that we can find models with the right behaviour.

For any set x of ordinals let $M(x)$ denote the least transitive model of $\mathbf{ZFC}^-$ containing x as an element (such a model is of the form $L_\beta[x]$ for some β and satisfies the DC-scheme (and hence $\mathbf{ZFC}^-_{Ref}$) in virtue of $L_\beta[x]$'s definable global well-order).

We now define a function that will be useful in finding models with the same theory (for extracting the inner models required for the OIMH^-_{Ref} later). For each countable ordinal α let $f(\alpha)$ be a real r_α such that α is countable in $M(r_\alpha)$ and for all y in which r_α is recursive[24] we have that $M(y)$ has the same theory with parameter α as $M(r_\alpha)$.

We use PD to check that f is well-defined: First note that PD implies (by Martin's Cone Lemma in [Martin, 1968]) that any projective set of reals closed under Turing equivalence either contains or is disjoint from a Turing cone. Also (in **ZFC** alone) the intersection of countably many Turing cones contains a Turing cone. Now for each sentence ϕ in the language of set theory with parameter α, let X_ϕ be the set of reals x such that α is countable in $M(x)$ and $M(x)$ satisfies ϕ. The X_ϕ are closed under Turing-equivalence since if x_0 and x_1 are Turing equivalent then $M(x_0) = M(x_1)$ (just by unfolding computations in the relevant minimal model). Moreover each X_ϕ is projective (indeed Δ^1_2).[25]

Next, for each ϕ choose a Turing cone inside either X_ϕ or $X_{\neg\phi}$ and let y be in the intersection of these Turing cones. Note that α is countable in $M(y)$ as one of these Turing cones only has reals with α countable. Furthermore, if y is recursive in z it follows that $M(y)$ and $M(z)$ have the same theory with parameter α (again just by unfolding the relevant computations). So we obtain f by picking a unique such r_α for each countable ordinal α (using AC, or if one wishes to do so definably, Projective Uniformisation).

Let N^* be a countable elementary submodel of some large $H(\theta)$ with θ regular containing f as an element, and let N be $N^* \cap H(\omega_1)$ (the sets in N^* which are hereditarily countable in N^*). Equivalently, N is the $H(\omega_1)$ of the transitive collapse of N^*. As $H(\omega_1)$ satisfies the

[24] i.e. r_α is recursive when using y as an oracle.

[25] Each X_ϕ is Δ^1_2 because x belongs to X_ϕ iff $\exists T(T = M(x) \wedge$ "α is countable in T" $\wedge$ "T satisfies ϕ"). This statement is Σ_1 in a real coding α, so in terms of the projective hierarchy it is Σ^1_2 and its complement is also Σ^1_2 and thus X_ϕ is Δ^1_2.

DCS (given that we are now in **ZFC**), so does its image under transitive collapse, which is N.

We now use N to find our model of the OIMH^-_{Ref}. Let β denote the ordinal height of N. Similarly to Theorem 25, we use Theorem 3.8 of [Antos and Friedman, 2017] and Theorem 4.1 of [Antos and Friedman, 2017] to force to add a real y so that $N[y] = L_\beta[y]$ is the least model of $\mathbf{ZFC}^-$ (and indeed $\mathbf{ZFC}^-_{Ref}$) containing y, i.e. $N[y] = M(y)$.

We claim that $M(y)$ satisfies the OIMH^-_{Ref}. Suppose that ϕ with parameter α (for $\alpha < \beta$) is satisfied in a inner model M_0 of an outer model M of $M(y)$. We will find a definable (with parameters) inner model of $M(y)$ satisfying ϕ. We first enlarge M (again using the methods of Theorem 25) to a model of the form $M(z)$ (for z a real) in which M is a definable inner model. Since M_0 is definable in M and M is definable in $M(z)$, we know that M_0 is a definable inner model of $M(z)$ (by the transitivity of definable inner models).

Choose n such that M_0 is a Σ_n-definable inner model of $M(z)$, and let ψ be the sentence: "There is a Σ_n-definable inner model satisfying ϕ". ψ is a sentence with parameter α true in $M(z)$, i.e. ψ belongs to the theory of $M(z)$ with parameter α.

We now pick a z^* in $M(z)$ that is Turing-above both z and $f(\alpha)$. (For concreteness, we could just let z^* be the join of z and $f(\alpha)$.) Now, we know that z^* belongs to $M(z)$ (by assumption) and that $z \in M(z^*)$ (since z is Turing-below z^*). We then have that $M(z) = M(z^*)$ since in general $x_0 \in M(x_1)$ implies that $M(x_0) \subseteq M(x_1)$.

We know that ψ holds in $M(z^*)$ simply because $M(z^*) = M(z)$ and ψ holds in $M(z)$. Recalling the definition of $f(\alpha)$, we note that $f(\alpha)$ was chosen specifically so that for any x, $M(f(\alpha))$ and $M(x)$ have the same theory with parameter α for any x that are Turing-above $f(\alpha)$. Since z^* is Turing-above $f(\alpha)$, we know that ψ holds in $M(f(\alpha))$. We also know that $f(\alpha)$ belongs to $M(y)$ (since $f(\alpha)$ belongs to N), and so we can choose a real y^* in $M(y)$ that is Turing-above both y and $f(\alpha)$. Then, as before, $M(y^*) = M(y)$ (since $y^* \in M(y)$ and $y \in M(y^*)$). But now, since y^* is Turing-above $f(\alpha)$, $M(y^*)$ has the same theory with parameter α as $M(f(\alpha))$, and so ψ holds in $M(y^*) = M(y)$. But ψ exactly says that ϕ holds in a Σ_n-definable inner model, and so ϕ holds in a definable inner model of $M(y)$ as desired. □

Remark 29. The above proof can be modified to fit the version of the OIMH^-_{Ref} that has ground model and outer models satisfying $\mathbf{NBG}^-_{Ref}$. In that context, the model we should take is $(M(y), C)$ where C consists of the $M(y)$-definable classes, and the outer model M should be a model of $\mathbf{NBG}^-_{Ref}$ (and hence of the form (M, C)). By assuming that ϕ holds in an inner model M_0 of (M, C), we can assume that in fact

C consists only of the classes definable over (M, A) for a single class A (so that M_0 is definable over (M, A)). Then we enlarge (M, A) to a model of the form $(M(z), C(z))$, z a real, where $C(z)$ consists only of the $M(z)$-definable classes (via Jensen coding (M, A)) and over which (M, A) is definable.

Theorem 28 represents a substantial improvement over the EIMH, showing that the OIMH^-_{Ref} is consistent relative to large cardinals, and moreover with the DC-Scheme.

However, to answer the Motivational Challenge, we would like to derive the consistency of large cardinals from the OIMH^-_{Ref}. As it turns out, $\mathbf{ZFC}^-_{Ref}$ with the OIMH^-_{Ref} added proves the existence of $0^\sharp$. Before we prove this we need a short lemma concerning the formulation of $0^\sharp$ we will use in this context. In particular we will need to ensure that we can treat Σ_1-L-indiscernibles as full L-indiscernibles:

Lemma 30. *($\mathbf{ZFC}^-_{Ref}$) Suppose that C is a club of Σ_1-indiscernibles for L (i.e. for a Σ_1-formula ϕ, $\phi(\vec{x})^L$ iff $\phi(\vec{y})^L$ for increasing tuples $\vec{x}$, $\vec{y}$ from C of the same length). Then C consists of Σ_ω- indiscernibles for L, i.e. for any ϕ, $\phi(\vec{x})^L$ iff $\phi(\vec{y})^L$ for increasing tuples $\vec{x}$,$\vec{y}$ from C of the same length.*

Proof. First we show that if α belongs to C then L_α is Σ_n-elementary in L for each n. Because C is a club and the class of α such that L_α is Σ_n-elementary in L is also a (definable) club, there are unboundedly many α in C such that L_α is Σ_n-elementary in L. In particular there are $\alpha < \beta$ in C such that L_α is Σ_n-elementary in L_β. But then by Σ_1-indiscernibility, L_α is Σ_n-elementary in L_β for all $\alpha < \beta$ in C, since "L_α is Σ_n-elementary in L_β" is a Σ^L_1-statement about the pair $\langle\alpha, \beta\rangle$. It follows that for each α in C, L_α is Σ_n-elementary in L because L is the limit of the Σ_n-elementary chain of L_α for α in C.

Now suppose that ϕ is arbitrary and $\vec{x}$, $\vec{y}$ are tuples in C of the same length. Choose α in C greater than $\vec{x}$, $\vec{y}$. Now $\phi(\vec{x})^L$ is equivalent to $\phi(\vec{x})^{L_\alpha}$ because L_α is Σ_n-elementary in L. Moreover $\phi(\vec{x})^{L_\alpha}$ is equivalent to $\phi(\vec{y})^{L_\alpha}$ because $\langle\vec{x}, \alpha\rangle$ and $\langle\vec{y}, \alpha\rangle$ are increasing tuples from C of the same length and "$\phi(\vec{x})^{L_\alpha}$" is a Σ^L_1-statement about $\langle\vec{x}, \alpha\rangle$, and the same goes for "$\phi(\vec{y})^{L_\alpha}$". Finally, $\phi(\vec{y})^{L_\alpha}$ is equivalent to $\phi(\vec{y})^L$ because L_α is Σ_n-elementary in L. In conclusion, $\phi(\vec{x})^L$ iff $\phi(\vec{y})^L$, showing that C consists of Σ_ω-indiscernibles for L. □

We can now prove:

Theorem 31. *Suppose that V satisfies $\mathbf{ZFC}^-_{Ref} + \mathsf{OIMH}^-_{Ref}$. Then V satisfies "$0^\sharp$ exists".*

Proof. Suppose that V satisfies $\mathbf{ZFC}^{-}_{Ref}$ + OIMH^{-}_{Ref}. By preparatory forcing (exactly as in Theorems 25 and 28) we can choose an outer model of V satisfying $\mathbf{ZFC}^{-}_{Ref}$ of the form $L[x]$ for a real x, in which every set is countable and the OIMH^{-}_{Ref} holds. (Note that if V is a *definable* inner model of W and V satisfies the OIMH^{-}_{Ref} then so does W, because definable inner models of V are also definable inner models of W. Fortunately, when we force to turn V into $L[x]$ for a real x, V will be a definable inner model of $L[x]$, and so the OIMH^{-}_{Ref} indeed holds there.) We'll show that in $L[x]$ there is a real y coding a Δ_1-definable club of Σ_1-indiscernibles for L (when we say that a real y codes a Δ_1-definable club C we mean that C is Δ_1-definable with parameter y over $L[y]$). Then it follows from the OIMH^{-}_{Ref} over V that there is such a real in V, completing the proof.

We begin our journey in $L[x]$, with the following:

Lemma 32. *Work in $L[x]$. Suppose that ϕ is a parameter-free formula with one free variable. Then for some Δ_1-definable (with real parameter) club C, either $\phi(\alpha)$ holds in L for all α in C or $\phi(\alpha)$ fails in L for all α in C.*

Proof of Lemma 32. Without loss of generality suppose that the class X of α such that $\phi(\alpha)$ holds in L is definably-stationary in $L[x]$ (i.e.X hits every $L[x]$-definable club). (Note that either X or its complement must be definably-stationary in $L[x]$, as otherwise we would obtain a contradiction from the existence of two disjoint clubs definable in $L[x]$.)

Then over $L[x]$ we can force a club C through X such that $(L[x], C)$ satisfies $\mathbf{ZFC}^{-}_{Ref}$: Conditions in $\mathbb{P}$ are closed subsets of X, ordered by end-extension. The forcing is ω-distributive, i.e. if $\langle D_i | i < \omega \rangle$ is a definable sequence of open dense classes, any condition p can be extended to a condition q belonging to each D_i. This is because by Reflection in $L[x]$ there is a definable club of ordinals C' such that for every $\alpha \in C'$, $\langle D_i \cap L_\alpha[x] | i \in \omega \rangle$ is dense in $\mathbb{P} \cap L_\alpha[x]$. By the definable stationarity of X we can choose such an α in X; then extend p ω-many times to conditions in $L_\alpha[x]$, hitting the various D_i. The union p_ω of these conditions together with α on top is a condition since α belongs to X and because taking the union of end-extending closed sets yields a closed set provided you add the relevant supremum (namely the supremum of the union).

But as every set is countable (by the OIMH^{-}_{Ref} in $L[x]$) this shows that $\mathbb{P}$ is $(< Ord)$-distributive. This distributivity yields pretameness and therefore $\mathbf{ZFC}^{-}_{Ref}$ preservation (and indeed $\mathbf{ZFC}^{-}_{Ref}$ preservation relative to the generic club added).

We can now further force over $(L[x], C)$ to add a real y so that C is Δ_1-definable over $L[y]$ with parameter y. This can be done with almost disjoint coding. To do the coding we need a definable class X such that each ordinal α is not only countable, but countable in $L[X \cap \alpha]$. But in the present setting, this is trivial as we can take the class X to simply be the real x. Moreover, in a general setting, to code a class X by a real with almost disjoint coding (when every set is countable) we need a sequence of distinct reals $\langle r_\alpha | \alpha \in Ord\rangle$ where each r_α can be defined just from the data given by $X \cap \alpha$. So if we have "decoded" $X \cap \alpha$ we can find r_α and then "decode" $X \cap (\alpha + 1)$, and then one can inductively "decode" all of X. In the present setting we can assume that C consists only of infinite ordinals and take X to be $x \cup C$ and take r_α to be the α-th real in the canonical well-order of $L[x]$. Then for all (infinite) α, $X \cap \alpha$ gives us x and therefore r_α. To code X by a generic real y, we replace each r_α by the set of codes for its finite initial segments (so that the various r_α are pairwise almost disjoint) and force the existence of a y with the property that α belongs to X iff y is almost disjoint from r_α. We now have an extension $L[x, y]$ in which C is Δ_1-definable from x and y. If desired, x and y can be combined into a single real z, with C Δ_1-definable in the parameter z over $L[z]$. All that needs to be checked (before we can pull back the inner model from $L[z]$ to $L[x]$ using the OIMH^-_{Ref} in $L[x]$) is that the forcing to add y over $L[x]$ preserves $\mathbf{ZFC}^-_{Ref}$. But this follows from the fact that the almost disjoint coding has the Ord-chain condition, proving Lemma 32. □

We can now use Lemma 32 to show the existence of the indiscernibles required for Theorem 31. Using Lemma 32, for each Σ_1-formula ϕ with one free variable choose a Δ_1-definable (in some real parameter) club $C(\phi)$ so that either $\phi(\alpha)$ holds in L for all α in $C(\phi)$ or $\phi(\alpha)$ fails in L for all α in $C(\phi)$. Note that these choices can be made definably so the intersection C_1 of the various $C(\phi)$ is a definable club of ordinals with $\phi^L(\alpha)$ iff $\phi^L(\beta)$ for all $\alpha, \beta \in C_1$ and all Σ_1-formulas ϕ with one free variable. To describe these classes of indiscernibles the following definition will be useful:

Definition 33. A class X of ordinals is Σ_m-n-indiscernible for L if for any two increasing n-tuples $\vec{\alpha}$, $\vec{\beta}$ from X and any Σ_m-formula ϕ with n-many free variables:

$$\phi^L(\vec{\alpha}) \Leftrightarrow \phi^L(\vec{\beta})$$

Using this terminology, we can describe C_1 as a club of Σ_1-1-indiscernibles for L.

Again, using the methods of Theorem 25 and 28, we can force to make C_1 Δ_1-definable in a real and by the OIMH^-_{Ref} we have a Δ_1-definable club of Σ_1-1-indiscernibles for L in $L[x]$ (we'll also denote this club by 'C_1' for the sake of convenience).

Now we want to go to more free variables before we intersect the clubs together to get the full L-indiscernibles required for $0^\sharp$. For each Σ_1-formula ϕ with parameter α in *two* free variables use Lemma 32 choose a Δ_1-definable (in a real parameter) club $C_1(\alpha, \phi)$ so that either $\phi(\alpha, \beta)$ holds in L for all β in $C_1(\alpha, \phi)$ or $\phi(\alpha, \beta)$ fails in L for all β in $C_1(\alpha, \phi)$. In the former case we say that α is ϕ-positive and in the latter case ϕ-negative. Either the first case holds for stationary-many γ or the second case holds for stationary-many γ (or both). By shooting a club we can ensure that either the first case holds for a club or the second case holds for a club (in either case, let the relevant club be C_2'). We thin this club C_2' further by intersecting with the diagonal intersection of the various $C_1(\alpha, \phi)$ i.e. we take all β in C_2' which belong to $C_1(\alpha, \phi)$ for all $\alpha < \beta$ and all Σ_1-formulas ϕ with parameter α. Call this club C_2. Now if $\alpha < \beta$ and $\alpha^* < \beta^*$ are in C_2 and ϕ is a Σ_1-formula with two free variables we have:

$$\phi^L(\alpha, \beta) \Leftrightarrow \phi^L(\alpha, \beta^*)$$

This holds because both β and β^* belong to $C_1(\alpha, \phi)$ iff $\phi^L(\alpha^*, \beta^*)$ (which in turn holds because either both α and α^* are ϕ-positive or both α and α^* are ϕ-negative). Thus, C_2 is a class of Σ_1-2-indiscernibles. Again applying the OIMH^-_{Ref} we can assume that C_2 is Δ_1-definable (in a real) in $L[x]$.

We then repeat this to get Δ_1-definable clubs of Σ_1-3-indiscernibles by choosing $C_2(\alpha, \phi)$ to be a Δ_1-definable club such that either $\phi^L(\alpha, \beta, \gamma)$ holds for all $\beta < \gamma$ in $C_2(\alpha, \phi)$ or $\phi^L(\alpha, \beta, \gamma)$ fails for all $\beta < \gamma \in C_2(\alpha, \phi)$ and proceed as in the previous step to get a club C_3 which is Δ_1-definable in a real, consisting of Σ_1-3-indiscernibles for L. By repeating this procedure we get Δ_1-definable clubs of Σ_1-4-indiscernibles, Σ_1-5-indiscernibles and so on in $L[x]$. We continue this for ω-many steps and produce a definable sequence $\langle C_n | n < \omega\rangle$ of clubs which are Δ_1-definable in a real so that C_n consists of Σ_1-n-indiscernibles for L. Then the intersection $\bigcap_{n \in \omega} C_n$ is a definable club of Σ_1-indiscernibles for L, and by Lemma 30, this club is also fully Σ_ω-indiscernible for L, and thus we have $0^\sharp$ in $L[x]$. Using the OIMH^-_{Ref} over V we pull $0^\sharp$ back into V, completing the proof. □

To sum up, we are now in a position where:

(1.) $\mathbf{ZFC}^-_{Ref}$ + OIMH^-_{Ref} is provably consistent from **ZFC** + PD (Theorem 28), and

(2.) $\mathbf{ZFC}^-_{Ref}$ + OIMH^-_{Ref} proves "$0^\sharp$ exists" (Theorem 31) and thus that **ZFC** with many large cardinal axioms added holds in L.

The obvious question now becomes: How does this affect the prospects for countabilist foundations regarding the roles identified in §1? We will now turn to this issue.

8 Set theory as a foundation under countabilism

Before we dive right in to the relevant constraints, let's review how the uncountabilist and countabilist view each other's perspective. One naive kind of criticism of the countabilist would be to say that they have an 'impoverished' perspective because they do not consider uncountable sets. But this criticism fails to take the countabilist perspective seriously. *Both* the uncountabilist and the countabilist look at one another and think that the other misses out sets. The uncountabilist thinks that the countabilist stops at $H(\omega_1)$, or lives in some countable transitive model of $\mathbf{ZFC}^-$ + Count. The countabilist on the other hand looks at the uncountabilist and thinks that they fail to consider all the available collapses that should exist, and lives in some impoverished inner model satisfying **ZFC** (possibly with large cardinals added).[26] The idea that countabilism is in some sense 'impoverished' is thus fundamentally question-begging.

For this reason, the foundational desiderata from §1 are especially salient in comparing the two perspectives. As noted in §1, **ZFC**-based set theory is able to afford a flexible way of understanding mathematics that fulfils the roles of **Generous Arena**, **Shared Standard**, **Metamathematical Corral**, and **Risk Assessment**, and do so whilst providing responses to the **Motivational Challenge** on the basis of **Maximality**. How does the countabilist fare? In this section, we argue that the countabilist responds reasonably well to these constraints, though

[26] This observation can, in fact, be mobilised in favour of a Maddy-style (based on the analysis in [Maddy, 1998]) argument that it is the *un*countabilist who has the restrictive position—note that the countabilist has **ZFC** plus large cardinals in *inner* models, whereas the uncountabilist only has models of countabilist theories in *transitive* models (with height at most ω_1). There are, in fact, a whole gamut of different restrictiveness conditions we might consider. We defer examination of this philosophically complex issue to future work.

in a manner somewhat different from the uncountabilist. However, we'll argue that **Risk Assessment** remains somewhat contentious, and could form the basis of future research into countabilist accounts of set theory.

Let us start with **Generous Arena** since it shows perhaps the starkest distinction between uncountabilist and countabilist perspectives. It first bears mentioning that some work has already been done here. Holmes, when considering a system he calls Pocket Set Theory (that includes an axiom that every set is countable) writes:

> It is well-known that coding tricks allow one to do classical mathematics without ever going above cardinality $\mathfrak{c}$: for example, the class of all functions from the reals to the reals, is too large to be even a proper class here, but the class of continuous functions is of cardinality $\mathfrak{c}$. An individual continuous function f might seem to be a proper class, but it can be coded as a hereditarily countable set by (for example) letting the countable set of pairs of rationals (p, q) such that $p < f(q)$ code the function f. In fact, it is claimed that most of classical mathematics can be carried out using just natural numbers and sets of natural numbers (second-order arithmetic) or in even weaker systems, so pocket set theory (having the strength of third order arithmetic) can be thought to be rather generous. ([Holmes, 2017], §9.1)[27]

So we can code a lot of mathematics relatively easily on the countabilist perspective. The reals are a proper class and talk of many uncountable entities (like the continuum) should be understood as concerning the classes (probably in an appropriate extension of $\mathbf{NBG}^-_{Ref}$). There is a sense though on which the perspective considered is somewhat revisionary, entities that standardly have more objects than there are real numbers (e.g. the function space on the reals) do not exist. In a

[27]Similar remarks are available in [Holmes et al., 2012]:

> The collection of all functions from the reals to the reals is too large, but notice that the collection of continuous functions from the reals to the reals is of size $\mathfrak{c}$ and can be represented in fairly natural ways, and in general the constructions actually needed in mathematical physics (or any mathematics short of set theory and shorn of excessive levels of abstraction) do not transcend the cardinality $\mathfrak{c}$. Points of Hilbert space are countable sequences of real numbers (thus sets) and continuous functions on Hilbert space are representable just as continuous functions on the reals are representable, and so forth. ([Holmes et al., 2012], pp. 581–582)

similar way, the uncountabilist must draw a line—for her the function space on the classes of ordinals does not exist. The countabilist simply thinks that those two problems come down to the same thing. Two points are in order:

First, for certain uncountable structures we can use elementary equivalence to export results from small structures to large ones. For example, when proving facts about the reals (conceived of as a field), we can use the fact that the algebraic reals (of which there are only countably many) form an elementary substructure of the class of all reals.[28] But this is a point about the *theory* of $(\mathbb{R}, +, \times, 0, 1, <)$ and does not depend on uncountable sets in any way. Thus, if we want to know what holds in $\mathbb{R}$ we can also examine the smaller object, and subsequently export results back via the elementary equivalence. Of course this does not deal with *every* situation in which we talk about the reals (for example often we want to talk about the transcendental numbers) but for a wide class of applications it does suffice.

The second point is simpler: Since we have provided a perspective on which we have inner models of **ZFC** (and much more) we can provide *some* interpretation of all the objects that normally exist in the universe under **ZFC** simply by interpreting them as concerned with objects in some (impoverished) inner models of **ZFC**. Moreover, these models provide a very natural place for the uncountabilist to work: The structures look *very much* like the uncountabilist's world in that they contain all ordinals, are transitive, and we have **ZFC** there. They are only deficient in failing to take into account every collapsing function that exists in the universe. This situation chimes well with Scott's remarks concerning the Cohen-Scott Paradox; we have "pleasant axioms" (namely the OIMH^{-}_{Ref}) that generates many of the usual models as "submodels of the universe" with the continuum "not even a set" and where all (set) cardinals are "absolutely destroyed".

Thus **Generous Arena** is only a problem if we *assume* that our talk about uncountable structures with cardinality above the reals is transparent (in that these objects *really* do have the properties they have under **ZFC**) thereby begging the question against the countabilist. The countabilist can perfectly well interpret much classical mathematics in her class theory, and what goes beyond $\mathbf{NBG}^{-}_{Ref}$ + Count can be interpreted in inner models satisfying **ZFC**. This then closely links in with **Shared Standard**—we can have a shared standard for interpreting proofs in set theory, it is just that the role of **ZFC**-based proofs is somewhat different and is to be interpreted as proving theorems about sets in some impoverished universe. **Metamathematical Corral** is also

[28] We thank Rodrigo Freiere for pointing this out to us.

unaffected—we can study models of set theory (including **ZFC** models) exactly as before, either in the wider world of $\mathbf{NBG}^-_{Ref}$ or within some other ambient model of **ZFC**.

Risk Assessment is somewhat more complicated. Whilst we have answered the **Motivational Challenge** via the idea of the universe being saturated under 'possible' sets (in line with **Maximality**) it should be noted that **ZFC**-based set theory has something more—a clear intuitive description of an underlying structure given by the iterative conception. This is partly (along with the long and unsuccessful attempt to find a contradiction) what convinces many that **ZFC** and its extensions is consistent.

There are a few points to note here, and things are subtle. The first is that, strictly speaking, the countabilist can simply piggy-back off the **Risk Assessment** provided by the iterative conception. This conception, she can contend, should indeed convince us that **ZFC** embodies a consistent conception of set. Unfortunately, is just that **Maximality** (for her) tells against the *truth* of **ZFC**. So, for the countabilist, the coherent picture provided by the iterative conception convinces her that **ZFC** is consistent. Indeed, part of the strategy outlined earlier is to use **ZFC** plus large cardinals to provide a *consistency proof* for her favoured theory of sets (namely $\mathbf{ZFC}^-_{Ref}$ + OIMH^-_{Ref}, possibly with a class theory added on top). This is not incoherent behaviour, but rather simply using a theory that is believed to be false but consistent to prove that a theory that is believed to be true is consistent.

This response, whilst coherent, is somewhat unsatisfying.[29] One might rightly complain that **ZFC**-based set theory still has a clear underlying conception where the countabilist perspective does not. This, one might think, speaks (*ceteris paribus*) in favour of **ZFC**. If one is moved by this criticism, there are two main responses one could muster here:

(1.) Come up with some *non-iterative* conception to underwrite countabilism and (extensions of) $\mathbf{ZFC}^-$ / $\mathbf{ZFC}^-_{Ref}$.

(2.) Modify or reinterpret the *iterative* conception to make it work on a countabilist perspective.

(1.) we leave as an open question in the final section, we do not see an easy strategy here. However (2.) is more promising, though several details remain open.

One way of addressing (2.) is *modally*. The iterative conception has been given a 'height potentialist' formulation in [Linnebo, 2013].

[29]We thank Leon Horsten for some helpful discussions here.

Even if one thinks that there is one maximal universe of sets, one can view [Linnebo, 2013] as giving a modal description of a set construction process, increasing our confidence in the claim that whatever is described is consistent. The fact that Linnebo's modalised theory interprets **ZFC** under a modal translation of **ZFC**'s quantifiers (replacing $\forall$ and $\exists$ with $\Box\forall$ and $\Diamond\exists$) increases our confidence in the consistency of **ZFC** by giving us something of an underlying conception—we know that there is a modal picture of how the universe is obtained. Recently Chris Scambler (in [Scambler, 2021]) has shown that there is a kind of modal theory that generalises this 'upward' modal picture of the iterative conception by incorporating vertical modality (adding ranks) *and* horizontal modality (adding forcing generics). He shows that the resulting theory interprets $\mathbf{ZFC}^-$ + Count under the same modal translation (and, in turn the theory is consistent relative to **ZFC** with the existence of a Mahlo cardinal).[30] This gives us confidence that there is a coherent underlying conception to be had for $\mathbf{ZFC}^-$ + Count—we can describe a modal picture of the universe that interprets the right theory under the natural modal translation.

A second alternative is to reinterpret the notion of what it means to say *all possible sets* in articulating the iterative conception. Normally what is required for the iterative conception is that when we grab "all possible subsets" at a successor stage, "all possible" coincides with "absolutely all". Instead, the countabilist wants a picture on which we grab at most countably-many at successor stages (assuming we want to keep the stages set-sized). In order for all sets to be included in this iterative process, we need to have the universe well-ordered in order type On. In that case we could define (letting R denote this well-order) the countabilist hierarchy V^C as follows:

Definition 34. The *Countabilist Iterative Hierarchy* is defined as follows:

$V^C_0 = \emptyset$

$V^C_{\alpha+1} = Def(T_\alpha) \cup \{x\}$, where x is the R-least set not in V^C_α.

$V^C_\lambda = \bigcup_{\beta<\lambda} V^C_\beta$

$V^C = \bigcup_{\alpha\in On} V^C_\alpha$

This would provide us with a hierarchy that stratifies the countabilist's universe in much the same way as the V_α hierarchy does for

[30] [Scambler, 2021] actually shows this for $\mathbf{ZFC}-$, but an easy modification to his proof (namely by changing his formulation of **ZFC** to include Collection and Separation rather than Replacement) yields $\mathbf{ZFC}^-$. We thank Dr. Scambler for some discussion concerning these details.

the uncountabilist. Unfortunately, this hierarchy is somewhat unsatisfying as is. First, we have no idea what R is like and indeed it is an additional commitment (one that seems to come from nowhere) beyond the normal prerequisites for an iterative hierarchy. Whilst the **ZFC** theorist needs the ordinals to generate the relevant V_α, this presentation of a countabilist hierarchy needs the ordinals *and* R. Second, the existence of such an R in the countabilist context is equivalent to CH. This, one might think, is an excessively specific assumption that needs to be used to generate a reasonable hierarchy.[31]

More promising might be the following idea: We iteratively add in collapses of the various sets to ω. The main problem here is to select the right collapse, but if one can stomach a Boolean universe one could build up a canonical universe $V^{\mathbb{B}}$ consisting of Boolean-valued names so that for every name for a set there is also a name for a collapse of that set to ω. This leads to the idea that the universe looks like $V^{Col(\omega,<Ord)}$ where V is a inner model of **ZFC** (without names) and then the hierarchy is $V^{Col(\omega,<Ord)} = \bigcup_{\alpha\in Ord} V_\alpha^{Col(\omega,<\alpha)}$.[32]

One might object that this only produces models of $\mathbf{ZFC}^-$ + Count of a special form—those obtained from a model of **ZFC** by forcing with $Col(\omega, < Ord)$—but it does at least produce the germ of an idea for developing non-arbitrary iterative hierarchies for the countabilist. Nonetheless, it is unclear whether the countabilist's appeal to a different interpretation of what collecting together 'all possible' sets at successor stages entails (and whether the possible responses suggested here have substantial mileage) is satisfactory. Exactly how to resolve these questions we leave open, but it is at least clear that there are *options* for modifying the iterative conception to suit the countabilist, and it seems too quick to dismiss their position on these grounds.

9 Conclusions and Open Questions

We have argued that there are perspectives on **Maximality** in set theory on which every set is countable and on which set theory can perform many, if not all, of its usual foundational roles. Moreover, this view motivates theories with a non-trivial degree of consistency

[31]There is a kind of perverse argument one might give here to the effect that we have solved the continuum problem: The countabilist might contend that since $\mathbf{NBG}^-_{Ref}$ is *true* and since the price of stratification is the existence of such an R, then we should accept CH on extrinsic grounds. Of course, this argument is unlikely to convince anyone of the truth of CH.

[32]This idea is somewhat reminiscent of [Steel, 2014]'s consistency proof of his multiverse theory. See [Maddy and Meadows, 2020] for a recent presentation.

strength. In this way, it represents a completely different perspective compared to standard **ZFC**-based set theory, but one that is nonetheless able to do many of the same jobs. There are, however, several open questions for the proposal. An important observation is that a key difference between the countabilist and uncountabilist—one the uncountabilist might use to apply philosophical pressure—is that the underlying conception for the countabilist is unclear. However, more work needs to be done on behalf of the countabilist before this conclusion can carry the required philosophical weight—after all we have spent a *long* time studying the iterative picture intertwined with **ZFC**-based uncountabilist set theories in comparison to countabilist perspectives. We therefore ask:

Question. Are there *non-iterative* underlying conceptions that validate the countabilist perspective?

Further, if an iterative strategy is in fact desired:

Question. What are the available options for stratifying the hereditarily countable sets in order to provide an iterative picture for the countabilist, and how might they be philosophically motivated?

The next question concerns the mathematics of how the principles we have examined might be developed:

Question. Is it possible to find natural principles (e.g. by modifying the relevant absoluteness principle in question) that increase the large cardinal strength further (other than the brute force strategy)? Can this be done so as to yield more inner models with stronger large cardinals?

We want to close on one final important (but more nebulous) challenge. We hope to have shown that there are perspectives on maximality on which every set is countable. Moreover, this view performs well with respect to the usual jobs we expect from set theory and in particular can be used to motivate a theory with reasonably high interpretative strength. This raises deep questions for the claim that **ZFC** and its extensions are our best-justified theories of sets, given that the countabilist can provide a well-motivated foundational theory. This raises the more general question:

Question. What do we really want set theory for? Is it merely a tool designed to fulfil certain foundational goals? Or is it part and parcel of our conception of set that it provide a study of *many* uncountably infinite cardinals?[33]

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[33] [Friedman, 2016], for example, argues that there are three main roles for set theory: (1.) it is a branch of mathematics, (2.) it is a foundation for mathematics, and (3.) it provides a study of the set-concept. If we accept that each provides evidence for set-theoretic truth, progress could be made on the dialectic between the countabilist and uncountabilist (e.g. the conflict might be grounded in one of the two parties prioritising a different kind of evidence). We leave examination of this question to future work.

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