Mathematical Explanation: A Pythagorean Proposal

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Mathematics appears to play an explanatory role in science. This, in turn, is thought to pave a way toward mathematical Platonism. A central challenge for mathematical Platonists, however, is to provide an account of how mathematical explanations work. I propose a property-based account: physical systems possess mathematical properties, which either guarantee the presence of other mathematical properties and, by extension, the physical states that possess them; or rule out other mathematical properties, and their associated physical states. I explain why Platonists should accept that physical systems have mathematical properties, and why a property based account is better than existing accounts of mathematical explanation. I close by considering whether nominalists can accept the view I propose here. I argue that they cannot.

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1. Introduction

A mathematical explanation in science (MES) is a scientific explanation in which mathematics is playing a substantive explanatory role.[1] Debate continues over whether there are any genuine MESs. The case for the affirmative is based largely on the presentation of putative examples.[2] Against such cases, a number of philosophers have argued that mathematics is playing a merely representational role in science, and is not doing any explanatory work.[3] The presence of genuine MESs paves a road to mathematical Platonism via the enhanced indispensability argument (Baker

[1]See (Andersen [2016]; Baker [2005]; Baker [2009]; Baron *et al.* [2017]; Colyvan [2001]; Jansson and Saatsi [2019]; Knowles and Saatsi [2019]; Lange [2016]; Leng [2010]; Lyon [2012]; Pincock [2012]; Povich [2021]; Saatsi [2011]).

[2]See (Baker [2005]; Batterman [2006]; Colyvan [2001]; Lange [2016]; Lyon [2012]; Pincock [2012]) for cases.

[3]See (Bueno [2012]; Leng [2010]; Melia [2000]; Saatsi [2011]; Yablo [2012]).

[2009]). In this paper, I will assume that there are some genuine MESs. My goal is to present an account of how MESs work, given Platonism. The account I offer is property-based: physical systems share intrinsic properties with mathematical objects, and these shared properties either guarantee the presence of other mathematical properties and, by extension, the physical states that possess them; or rule out other mathematical properties, and any associated physical states. The central 'mechanism' of an MES is a relation between mathematical properties that are physically instantiated.

The assumption that physical objects possess mathematical properties involves commitment to a partial form of Pythagoreanism: the belief that the universe is 'made' of mathematics. I thus begin, in Section Two, by explaining why Platonists should accept Partial Pythagoreanism. In Section Three I briefly motivate my account of MESs by showing why Platonists should prefer it to existing accounts. This sets the stage for Section Four, in which I outline my account of mathematical explanation. I will close, in Section Five, by responding to an objection: namely, that my account cannot support the indispensability argument for Platonism.

2. Partial Pythagoreanism

In this paper, I will talk of physical and mathematical properties. I won't provide an account of what I mean by 'physical' here, deferring instead to whatever the best account of this notion happens to be. A physical property is any intrinsic property that satisfies the best account of what it is to be physical. A mathematical property, by contrast, is any intrinsic property possessed by a mathematical object, or any property possessed by a mathematical object that supervenes on intrinsic properties. A mathematical object is an abstract, non-causal, non-spatiotemporal entity.

There is a stronger notion of a mathematical property that I will mention only to set it aside: a strongly mathematical property is an intrinsic property that only mathematical objects possess. Some properties of mathematical objects are like this. For instance, the property of being prime is a property that only a number can have. Strongly mathematical properties are not the focus of this paper.

Platonists should accept that physical and mathematical objects share intrinsic properties. I call this view 'Partial Pythagoreanism' on the grounds that physical and mathematical objects are not entirely distinct. They overlap in terms of their qualitative aspects. In this, fairly weak sense, the physical world is partly mathematical in nature. The intrinsic properties that are shared between physical and mathematical objects include properties of having a certain structure, such as having a particular geometry or being ordered in a certain way (as in a linear or partial ordering). Mathematical and physical objects can also share other properties in virtue of having a shared structure. Such properties can include cardinality properties like being finite or infinite, or, as we shall see, certain pathing properties.

Partial Pythagoreanism is recommended by our best account of applied mathematics. Platonists are united in the belief that mathematical objects play a modelling role in science. A common analogy used for the modelling role of mathematics in science is a map (Bueno and Colyvan [2011], p. 346; Saatsi [2011], p. 146). A map is an object that depicts the world in a specific way, for a specific purpose. When the map is accurate, we learn a great deal about the world just by reading the map. A mathematical model, like a map, is an object that can provide us with information about the world. Model and map thus play what Saatsi ([2012], p. 579) calls a 'knowledge-conferring' role.

A long-standing view is that the relation between a mathematical model M and its target T is a mapping relation, in the sense of a morphism.[4] The strongest mapping relation is an isomorphism. When M and T are isomorphic, they share the same structure. From this it follows that M and T share a property in common. At the very least, they share the property of having a specific structure. Call this property P.

P is an intrinsic property. According to Lewis ([1983], p. 197), intrinsic properties are those preserved under duplication. So, for instance, take a particular graph. Suppose the graph has a specific structure consisting of adjacency relations between vertexes. If we duplicate the graph then we must duplicate the structure as well. We cannot copy the graph without giving it the same structure of adjacency relations. By the same token, a physical system, like the famous Bridges of Königsberg has a specific structure, involving the adjacency relations between the various bridges. Just like the graph, we cannot duplicate the entire bridge system without duplicating the structure.

Of course, some mathematical models have more structure than the physical systems they represent; others have less. Accordingly, partial isomorphisms, homorphisms, epimorphisms and monomorphisms are also used (Bueno and French [2011]; Bueno and Colyvan [2011]). In each case, the mapping relation is structure-preserving. Given the presence of the mapping there is some aspect of structure shared between the model and its target. The fact that physical entities share some structure with mathematical objects is enough to establish shared properties. Physical and mathematical entities share structural properties in common, even if the structure specified by the property does not exhaust the structure of the entity in question (mathematical or physical).

3. Motivation

In the next section I will use the shared properties of physical and mathematical objects to give an account of MESs. First, however, some motivation. Various accounts of MESs are already available, why should Platonists prefer mine? The answer is that existing accounts are controversial, either on their own or when coupled with Platonism.

For instance, Baron *et al.*'s ([2017]) account requires that some counterfactuals with impossible antecedents are non-trivially true. However, the status of such counterfactuals is hotly debated (see the exchange between (Berto *et al.* [2018] and Williamson [2018]). Baron *et al.*'s account also likely requires a non-classical logic to support the relevant counterfactuals, namely a logic that can tolerate contradictions. Platonists, however, may reasonably wish to remain classical.[5] Two

[4] See (Krantz *et al.* [1989], Pincock [2004] and Bueno and Colyvan [2011]).

[5] Accounts given in (Knowles and Saatsi [2019]; Jansson and Saatsi [2019]) avoid these commitments, but are nominalistic.

other views incur similar commitments. Povich's ([2021]) ontic counterfactual account uses similar counterfactuals to Baron *et al.*'s account, and so is controversial in a similar way. Baron's ([2020]) hybrid counterfactual-unificationist account is in the same boat, adding that unification is necessary for explanation, which is a contested assumption in its own right (Gijsbers [2007]).

Counterfactual views are also controversial in another respect. Counterfactual dependence generally holds in virtue of some underlying connection between whatever enters into the relevant counterfactual relationship at issue. For instance, if it is true that had Suzy not thrown the rock, the window would not have smashed, this is generally because of some underlying causal or nomic connection between the throwing and the smashing. For a Platonist, then, there needs to be a connection between mathematical and physical objects in virtue of which one counterfactually depends on the other. It remains unclear, however, what the relevant connection might be. The connection is certainly not the causal connection implicated in interventionist theories of explanation (Woodward [2003]).

Povich ([2021]) offers two accounts of the connection. First, he proposes grounding. Grounding, however, is a metaphysical dependence relation of relative fundamentality such that x grounds y only if x is more fundamental than y (Schaffer [2009]). However, the idea that the physical is less metaphysically fundamental than the mathematical is not obviously something that Platonists should accept. Second, Povich considers an instantiation relation between physical objects and mathematical properties. As we shall see, however, MESs can be accounted for using relations between the very same physically instantiated mathematical properties. A property-based account is therefore simpler as it does not require an added layer of counterfactual structure (Baron *et al.* appeal to morphisms to underwrite counterfactuals, but the moral is the same).

Baron's ([2019]) DN account avoids controversial counterfactuals, but still employs a non-classical logic, namely relevance logic (though he does suggest a way to make the view classical). Baron's picture is also open to symmetry problems due to its reliance on the standard machinery of DN accounts (Povich [2021]). More importantly, on Baron's view mathematical facts manage to carry information about physical systems. As he notes, there is a question concerning how this happens, and there is a worry that, for a Platonist, some connection between physical and mathematical objects will be needed to explain the informational relationship at issue. The answer that Baron gives appeals to mapping relations and shared structure: mathematical facts carry information about physical systems because there are morphisms between mathematical and physical objects, and physical systems sometimes possess mathematical properties. Again, as we shall see, these shared properties are enough for MESs, and so there is no need for the DN framework.

Lange's ([2013; 2016]) constraint-based approach does better, requiring no controversial counterfactuals or non-classical logic. For Lange, mathematical facts play an explanatory role by constraining the world in much the way laws do, except that the constraints exerted by mathematics are stronger than any physical law. Now, Lange never frames his account in terms of mathematical objects. Indeed, he deems the question of whether mathematical objects exist "irrelevant" (Lange [2013], p. 492) to his account. What happens, then, when we try to combine Lange's account with a Platonistic picture of MESs in which mathematical objects play a role? For a Platonist, it seems, mathematical objects must be implicated in the constraints that form the core of Lange's account.

A Platonist who endorses Lange's view thus seems to require that mathematical objects somehow constrain the physical world. This is similar to the need for a connection between mathematical and physical objects that underwrites counterfactual dependence, and is controversial in the same way: how, exactly, do mathematical objects exert this kind of influence?

This brief survey is by no means exhaustive. But it is enough to show that recent accounts of MESs carry some surprising commitments for Platonists. Specifically, one needs either (i) non-trivially true counterfactuals with impossible antecedents; (ii) non-classical logic; (iii) a connection between physical and mathematical objects that underwrites counterfactual dependence (or information sharing); or (iv) a way for mathematical objects to constrain the physical world. Commitments (i) and (ii) are independent of Platonism, whereas commitments (iii) and (iv) arise when the Platonist adopts certain existing views. My account of MESs, by contrast, avoids all of these commitments, and uses only what Platonists already accept given the best account of applied mathematics. In this way, the view boasts a *prima facie* advantage for Platonists over current offerings.

4. Mathematical Explanation

My view is similar to Lyon's ([2012]) programming account. Before outlining my account, then, it is useful to briefly examine Lyon's account and show why there is a need to go beyond it.

4.1. Program and Process

Lyon's programming account rests on a distinction between program and process explanations. A process explanation is an explanation in terms of the low-level causal facts that bring about a particular event. A program explanation, by contrast, is an explanation in terms of high-level facts that ensure a certain causal fact must obtain. Lyon's view is that mathematical facts sometimes play a programming role by ensuring the presence of certain causal facts that, in turn, bring about particular explananda. One way of understanding this idea appeals to properties: the high-level mathematical properties of a physical system program for low-level causal properties.

Lyon's account over-emphasises causation. According to Lyon, MESs that fit the programming account:

> ...cite properties and/ or entities which are not causally efficacious but nevertheless program the instantiation of causally efficacious properties and/or entities that causally produce the explanandum. (Lyon [2012], p. 567)

On this picture, a given explanandum in an MES is always causally produced, and it is the cause of the explanandum that is programmed by high-level mathematical properties. Mathematical explanation, on the programming account, is causally mediated.

To see why the emphasis on causation is a problem, consider the well-known Bridges of Königsberg case. It is impossible to walk across the seven bridges of Königbserg crossing each bridge exactly once. Why? The answer lies in graph theory. The seven bridges and the landmasses they connect can be represented by a graph: a mathematical object consisting of vertexes connected by edges. A graph lacks an Eulerian circuit just in case there is no path through the graph that crosses every edge exactly once. The graph corresponding to the seven bridges lacks an Eulerian circuit. Thus, it follows that there is no way to cross the seven bridges exactly once.

The Bridges of Königsberg is one of the core examples that Lyon uses to motivate the programming account. With respect to this case, however, it is important to differentiate between two explananda:

> Why has no one ever continuously walked over Königsberg's seven famous bridges passing over each bridge exactly once?
>
> Why can't anyone walk over Königsberg's seven famous bridges exactly once?

Mathematical facts about Eulerian graphs are relevant to both explananda. The fact that the bridges have the same structure as a non-Eulerian graph explains why, as a matter of fact, no-one has walked over the seven bridges. But these structural facts also explain why no-one can walk over the bridges in the required manner. Now, causal facts are clearly relevant to the first explanandum. The fact that no-one has ever walked over the seven bridges, passing over each bridge exactly once, can be explained in terms of a series of causal histories, each of which starts with a walker and ends in failure. The mathematical facts thus program these causal histories by ensuring that each occurs and, in this way, program the explanandum.

However, the causal facts seem to be irrelevant to the second explanandum. Even if no-one has ever or will ever attempt to cross the seven bridges, still it would be impossible to do so. In so far as the mathematics has an explanatory role to play in such a situation, it cannot be because it ensures the existence of some property or entity that causally produces the explanandum. That's because the second explanandum does not seem to be causally produced in any obvious sense. The explanandum is not an effect, nor even the absence of an effect. It is the impossibility of a physical event of a certain type.

Because of its emphasis on causation, Lyon's account is unfit to handle a range of MESs that Lange ([2013]) calls 'distinctively mathematical explanations'. These are explanations in which a given explanandum is explained entirely in terms of mathematical facts without any reference to causation. Of course, Lyon admits that his account may not be fully general, and so may not extend to every MES. But the lack of generality is a symptom of a deeper concern. In Lyon's account, mathematics is subordinate to causal and physical explanations. This seems to blunt the explanatory potential of mathematics, particularly once it is realised that many of the cases Lyon focuses on feature a modally strengthened explanandum like the one considered above.

A second reason to be unsatisfied with Lyon's account is offered by Saatsi ([2012]). Because of the mediating role that causation plays in Lyon's account, mathematical properties must be shown to ensure or otherwise generate causal properties. Saatsi's worry is that there is no metaphysical, nomic, or logical connection between mathematical and physical properties in virtue of which mathematical properties might generate causal ones. Here's Saatsi:

> What kind of metaphysical connection or law of nature could in this way link a mathematical property to causally efficacious properties? It beats me, and I am not opti-

> mistic about the prospects of making sense of a mathematical property entering into a kind of programming relation. (Saatsi [2012], pp. 582–583)

Saatsi's worry is strongest when we assume that mathematical properties are possessed only by mathematical objects. Then it would seem we need some account of how it is that properties possessed by mathematical objects 'reach into' the physical realm and bring about causal facts. To some extent the worry can be addressed by coupling Lyon's account to Partial Pythagoreanism. For then, at least, the mathematical properties that ensure the various causal properties of physical systems are properties of those systems. There is less of a metaphysical gap to be bridged. Even if we assume Partial Pythagoreanism, however, Saatsi's objection is not completely answered. For it remains unclear exactly how mathematical properties force certain causal properties to obtain. Lyon doesn't tell us how programming works, only that it does. What we need, as Saatsi points out, is an analysis of the programming relation itself.

4.2. Mathematical Explanation

Like Lyon's account, my account focuses on the mathematical properties of physical systems. However, the account is designed to overcome the limitations of Lyon's account. Let us assume, as seems plausible, that a single physical system can enter into multiple distinct physical states. Suppose also that physical systems have global properties—properties that the system has regardless of its state. And also that there are state-specific properties: properties that a system would have, were it to enter a specific physical state. Then:

> An MES is any case in which a system has global mathematical properties that relate to state-specific mathematical properties in such a way that either (i) the system is forced to enter a physical state because that state has certain mathematical properties that are required by the global mathematical properties of the system or (ii) the system is forbidden from entering a certain physical state because the mathematical properties of the state conflict with the global mathematical properties of the system.

In general, MESs are a function of exclusion and requirement relations between global and state-specific mathematical properties. Because the state-specific mathematical properties are always possessed by some physical state or other, certain physical states are forced or forbidden in virtue of the exclusion and requirement relations between their mathematical properties. The explanatory work, however, is always being done by the relationships between mathematical properties in the first instance. Mathematical properties don't ensure physical properties directly; only indirectly in virtue of the enforcement or exclusion of a particular state's properties.

MESs, on my view, are not really that different to a certain class of non-mathematical explanations. Many explanations within science involve showing how a certain physical system cannot enter into a particular state, because the specific physical properties it would gain by entering that state conflict with the global physical properties of the system.

For instance, consider a spatiotemporally located physical system consisting of Sara and Nox. Sara and Nox both hold matches, and are at a spacelike distance from one another. Suppose that only Sara strikes her match and that the system undergoes a state change as a result. It cannot be the case that Nox's match lights as a result of Sara's striking. That state—the state of both matches having lit—is inaccessible from a state in which just Sara strikes her match. It is inaccessible because the system is a spatiotemporal one, and spatiotemporal systems forbid faster than light processing. For Nox's match to light because of Sara's strike some faster than light processing would be needed from Sara's striking to Nox's lighting (assuming that no other causes are present within or impinging on the system). This example is one in which a state is out of bounds for a system. But it is not difficult to imagine a case in which a specific physical state of a system is forced instead, given the global physical properties of the system.

It is tempting to add that the mathematical properties of physical systems always supervene on physical properties. If this is assumed, then an incompatibility between mathematical properties will generally correspond to an incompatibility between subvenient physical properties. My account does not, however, require the supervenience of mathematical properties on physical ones. It could be that the mathematical properties supervene on physical ones, but it could also be that some physical properties supervene on mathematical ones. Or there may be no supervenience relation at all.

The nature of the relationship between physical and mathematical properties is not important. The core of an MES is not a matter of mathematical properties interacting with physical ones. It is a matter of mathematical properties interacting with other mathematical properties. So long as there is some reason to think that physical systems have certain mathematical properties—and I have argued that, for a Platonist, there is—then that is enough to get the account off the ground. Of course, it must also be assumed that physical systems have both global and state-specific mathematical properties. But since it is possible to represent physical systems in full generality using mathematical models, and since it is also possible to represent specific physical states mathematically, we (Platonists) have reason to believe that there are physically instantiated mathematical properties of both kinds. In this way, the requirements of the account are quite minimal. All that the account of explanation needs is whatever we already gain from the modelling role of mathematics: namely, property sharing between physical and mathematical objects.

In a sense, mathematical properties can be said to 'program' physical systems on my account. At least, this is true in so far as 'programming' just means that the mathematical properties of physical systems force them to enter or avoid certain physical states. The picture is not the standard programming account, which brings me back to the difference between my account and Lyon's. The difference is represented in Figure 1. On the left-hand side, we have the standard programming relation. The explanatory power of mathematics is mediated by causal properties. On the right-hand side, by contrast, we have the relationship between mathematical properties that characterises my account.

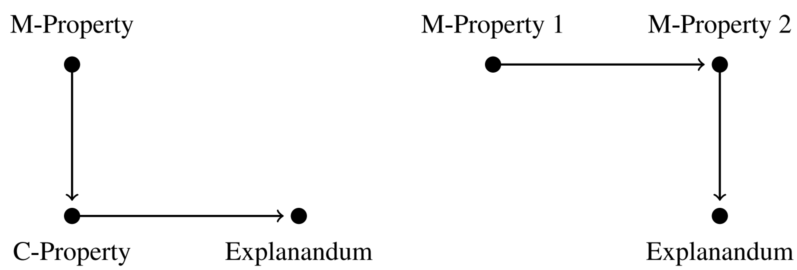


Figure 1: M-properties are mathematical properties, C-properties are causally efficacious properties.

In the picture on the right, the explanatory power of mathematics is not mediated by any causal properties. Rather, mathematics explains why some physical system is in a certain state by directly ruling in or ruling out other mathematical properties possessed by a state. Because the explanatory power of mathematical properties is not mediated by the causal properties of physical systems, there can be a mathematical explanation of some fact even when there is no causal explanation of that fact available (unlike Lyon's account). There is also no need to further explain how mathematical properties bring about or otherwise ensure causally efficacious physical properties (as per Saatsi's objection to Lyon's account).

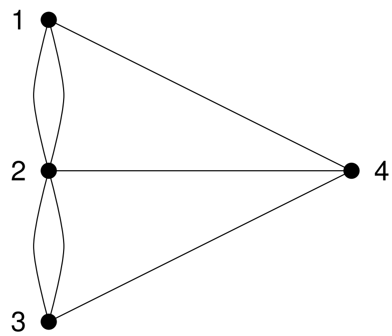
Nonetheless, like Lyon's account, my account provides a way for mathematics to ensure that a certain physical explanandum occurs. But there is no mystery as to how this happens: it happens because physical systems have certain global mathematical properties, and these properties are incompatible with the mathematical properties that the system would gain were it to enter certain physical states. All that the account requires is for physical entities to have these global and state-specific mathematical properties. The exclusion and requirement relations between the mathematical properties then do all of the explanatory work.

4.3. Concrete Examples

To make this all a bit more concrete, let's take a look at some specific examples. I will focus on two: the Bridges of Königsberg case already discussed, and the cicada case outlined by Baker ([2005]). I will assume that these are genuine MESs, and thus that mathematics has an explanatory role to play. Nothing I say here is supposed to constitute an argument for the explanatory power of mathematics in these cases.

A note on methodology: in each case I will start from the assumption that various aspects of physical systems can be modelled mathematically. I will then use the modelling relation as a basis for identifying properties that are shared between physical systems and mathematical objects. These properties will then be used to interpret the cases so that they fit the account of explanation outlined above. The advantage of proceeding in this way is that the properties I appeal to are properties that the Platonist has reason to believe in.

Let us begin with the Bridges of Königsberg. As already discussed, the seven bridges have the same structure as a non-Eulerian graph. It is possible to abstract away from the specific details of bridges or graphs and talk about the structure they share in general terms. From this abstract standpoint, the structural property at issue is a matter of featuring certain adjacency relations between points. Different things can play the role of points or adjacency relations. In the case of a graph, the points are vertexes and the adjacency is mediated by edges. In the case of the bridge system, the points are land-masses and the adjacency is mediated by bridges. In the most general terms, the structural property shared between the seven bridges and certain graphs can be represented by an adjacency matrix: an $n \times n$ two-dimensional matrix with i rows and j columns where the ij-th entry is the number of connections between adjacent points i and j (see Figure 2). In the case of a graph, the entries in the matrix correspond to edge connections between nodes. In the case of a structure of bridges, the entries in the matrix correspond to bridge connections between land-masses. Call the structural property represented by the adjacency matrix in Figure 2, M_1.



$$\begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Figure 2: The matrix represents the structure of the graph and of the bridges.

M_1 is only part of the story. In addition, we must also look at the properties of a path. A path is an abstract way of talking about the way in which the points in a structure represented by an adjacency matrix may be traversed in a specific sequence. By filling in particular details of the kind of object whose structure is being represented, we manage to isolate the kinds of things that paths can be. In the case of a graph, a path is a line that runs across some sequence of edges and vertexes. In the case of a bridge system, a path is a worldline that covers some sequence of bridges and land-masses. Paths through graphs model paths across bridges. In virtue of this modelling relationship, the two types of path share properties with one another, just like the bridges and the graphs themselves.

One such property is the property of being complete. A path through a certain structure (such as any object with the M_1 structural property) is complete when that path crosses every point in the relevant structure of adjacency relations exactly once. Lines through graphs can be complete in this sense. A line through a graph is complete just when it is an Eulerian circuit. Worldlines can also be complete in this sense. A worldline across the seven bridges is complete in so far as it crosses every bridge exactly once. The reason why the completeness property can be had by lines and worldlines alike is just that the structural properties they relate to can be had by mathematical objects and physical objects alike. In both cases, there is good sense to be made of a total tour of the relevant structure.

Call the completeness property of a path M_2. For any object that has the M_1 structural property, no path through that object can have the M_2 completeness property. This is the reason why there are no Eulerian circuits through certain graphs (because these would be paths that are complete, and thus would be ones that possess the M_2 property). The graph-theoretic case is, however, just an instance of the more general conflict between M_1 and M_2, a conflict that can arise for any structure and any path through that structure. This more general conflict is represented by a general result for adjacency matrixes, namely that for any adjacency matrix in which the number of columns or rows that sum to an odd number is greater than two, there will be no complete path through an object that possesses the structure represented by that matrix (regardless of whether the object is a graph or some other object).

It is the general conflict between M_1 and M_2 that explains what is happening in the Bridges of Königsberg case. The seven bridges can be treated as a single physical system. The starting state of the system, T_1, is a state in which no-one has crossed any bridges. Let T_2 be a state in which a walker tours all seven bridges without doubling back over a single bridge and let T_3 be a state in which a walker tours all seven bridges and doubles back at least once. The M_1 property is a global property of the bridges: a property it has regardless of the state it is in. Any worldline across the bridges that traverses every bridge exactly once would possess the M_2 completeness property. M_2 is thus a state-specific property that the system would gain were it to enter T_3. Because no system with an M_1 structure can enter a state that gives it the M_2 property, the bridge system cannot enter the T_3 state. That, in turn, explains why the system never enters the T_3 state, regardless of how many times a walker tries to cross the bridges.

The Bridges of Königsberg case involves a conflict between two very generic properties: the property of having a certain structure, possessed by some physical systems, and the property of being complete, possessed by some worldlines through those systems. The same conflict arises for graphs, because graphs are capable of possessing the same generic properties. The conflict itself, however, has little to do with bridges or graphs *per se*, since it arises at a more general level of adjacency relations, a level at which we can discern a shared structure between bridges and graphs.

4.4. The Cicadas

This brings us to the second example: Baker's cicada case. The magicicada spends most of its life in the ground, in its larval form. After a certain period of time, magicicadas arise as a swarm for around two weeks during which they eat, breed, die and repeat the cycle. The sub-species of magicicada spend thirteen or seventeen years underground respectively. Why these life-cycles? Why not twelve and fourteen? Or sixteen and eighteen?

Here's one answer. Suppose that, in the past, the cicadas were predated by organisms that have two, three, four, five, six, seven, eight and nine-year life-cycles respectively. The optimal way for the cicadas to avoid predation is for them to come out of the ground when their predators are still lying dormant. More generally, if the cicadas can minimise the frequency with which both they and their predators are out of the ground, then they will have the best chance at long-term evolutionary success. Now, suppose that the cicadas are constrained to have life-cycles within the twelve to eighteen year range for ecological reasons (an assumption that Baker maintains is plausible). Within that range, the optimal way to avoid predation is to adopt a thirteen or a seventeen year life cycle.

This explanation relies on two results in number theory concerning the lowest common multiple (LCM) of two numbers. Baker ([2005]) outlines the two results as follows:

> **Lemma 1:** the lowest common multiple of m and n is maximal if and only if m and n are coprime.
>
> **Lemma 2:** a number, m, is coprime with each number $n < 2m, n \neq m$ if and only if m is prime.

Take the numbers two, three, four, five, six, seven, eight, nine and compare them to the numbers twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen. By Lemma 2, only thirteen and seventeen are coprime with each of two to nine. By Lemma 1, the lowest common multiple of each of two to nine is maximal with respect to thirteen and seventeen only. Now, the frequency with which the cicadas intersect with their predators is a function of the lowest common multiple of cicada and predator life-cycle length. The frequency of intersection is minimal when the lowest common multiple of life-cycle lengths is maximal. Assuming that cicadas should evolve toward the optimal strategy for predator avoidance, it follows that they should have prime life-cycles, and thirteen and seventeen are their only options.

In order to reconstruct this explanation using the account offered above, we must identify a feature of a physical system consisting of the cicadas and their environment that can be modelled mathematically. One way forward is to focus on time. The temporal structure of the cicada system is a crucial aspect of the case, and time is something that can be quite easily modelled using mathematics. Cicada life-cycles are expressed in physical quantities of time, in particular: intervals of years. Intervals of time form a cumulative series. First, a year passes. After a year has passed and another year has passed, two years have passed. After another year has passed, three years will have passed and so on. This temporal series of years can be mapped to a number series. Specifically, it can be mapped to $\mathbb{N}_1$: the natural numbers without zero (aka the whole numbers).

A time-series of years, and the series $\mathbb{N}_1$ are both linear orders and both series are discrete. More importantly, for present purposes, both series are also additive. In the numerical case, when we take two numbers and combine them, we get a third number that is greater than the first two. Similarly, when we take two intervals of years (say three years and four years) and combine them, we get a third interval of years that is larger than the first two. The notion of 'combining' in the case of years corresponds to the combination of quantities: when we combine two physical quantities of time we get a third quantity of time. If the notion of 'combining' for years seems too abstract, we can focus, instead, on the passage of time. A two-year passage of time is made up of two one-year passages of time, in the sense note above: if one year passes, and then another year passes, two years will have passed.

Despite the similarities between a time-series of years and $\mathbb{N}_1$, we cannot transport Baker's two lemmas over to the time-series. For it is unclear that multiplication makes sense for quantities of time. What would it mean to multiply one quantity of time by another? To be sure, we can take the numbers that represent the quantities, and multiply those. But this is not the same thing as multiplying the temporal quantities themselves. Without an obvious notion of multiplication it is difficult to make sense of the notions of lowest common multiple, primality and coprimality. These are properties of structures for which multiplication is well-defined.

Given the differences between a time-series of years and $\mathbb{N}_1$, a straightforward interpretation of the cicada case is unavailable. A straightforward interpretation of that case attributes the mathematical properties of primality and coprimality to intervals of years directly, and then seeks to show that a prime life-cycle is required for the cicadas to optimally avoid their predators. It is not at all obvious, however, that we can take intervals of years to have mathematical properties that are reserved for numbers.

There are, however, nearby properties that are held in common between the natural number series $\mathbb{N}_1$ and a time-series of years. Using these properties, it is possible to reformulate the cicada case. First of all, there is a notion of primality, call it primality*. Primality* can be defined as follows:

> **Primality*** For any element x either in $\mathbb{N}_1$ or in a time-series of years τ, x is prime* iff x cannot be produced by combining equal multiples of elements of the series except by combining equal multiples of the first element.

Note that 'multiple' in the above definition does not denote multiplication. A multiple is just a matter of having many elements; an equal multiple is a matter of having many of the same elements. So, for instance, in the case of a time-series, three one-year intervals is an equal multiple of elements of that series. We can produce other members of the time series by combining equal multiples. Indeed, every member of the time-series can be produced by combining equal multiples of the first element of the series—a one-year interval. Which is to say that the passage of any amount of years is equivalent to the passage of some multiple of one-year intervals.

Prime numbers in $\mathbb{N}_1$ are also prime*. 3 is prime. But it is also prime*, since it cannot be produced by combining equal multiples of elements of $\mathbb{N}_1$ except by combining equal multiples of one (that is, one plus one plus one). Similarly, thirteen is a prime number. But it is also prime*. We cannot, for instance, produce thirteen by adding equal multiples of twos, threes, fours, fives or sixes and so on. Elements in the time series can be prime* as well. A thirteen-year interval cannot be produced by combining equal multiples of two-year intervals, three-year intervals or four-year intervals and so on. It can only be produced by combining thirteen one-year intervals. If you want to wait out thirteen years, you can only do so by waiting out one year thirteen times. You can't do it by waiting out two years any number of times, or by waiting out three years any number of times.

In addition to primality*, we can also define a notion that is analogous to LCM. Here's the definition:

> **LCM*** For any elements x and y that appear together either in $\mathbb{N}_1$ or in a time-series of years τ, the LCM* of x and y is the smallest element of the series that can be produced both by combining equal multiples of x and by combining equal multiples of y.

So, for example, consider a ten-year interval and a four-year interval. The LCM* of these two intervals is the smallest interval of years that can be produced both by combining equal multiples of ten-year intervals and by combining equal multiples of four-year intervals. In this case, the LCM* is a twenty-year interval. Similarly, consider the numbers 4 and 10 in $\mathbb{N}_1$. Their LCM* is twenty, since that is the smallest number that can be produced both by adding multiples of four and by adding multiples of ten. The key difference between LCM* and LCM is that LCM* is defined in terms of an addition-like property of combining elements, whereas LCM is generally defined in terms of multiplication. Since multiplication cannot be defined in terms of addition, the two properties are not the same. They do, however, agree for all numbers in $\mathbb{N}_1$. The LCM* of two numbers is always their LCM and vice versa.

Finally, we can define a notion analogous to coprimality, as follows:

> **Coprimality*** For any elements x and y that appear together either in $\mathbb{N}_1$ or in a time-series of years τ, x and y are coprime* iff the only element of the series that can be combined in equal multiples to produce both x and y is the first element.

A two-year interval and a five-year interval are coprime*, since the only element of the time-series that can be combined in equal multiples to produce both intervals, is a one-year interval of time. Simarly, the numbers two and five are coprime* because the only number that can be combined in equal multiples to produce two and five is the number one. For any two numbers in $\mathbb{N}_1$, if those two numbers are coprime then they will be coprime* and vice versa. Coprimality and coprimality* are, however, different notions: elements of a time series can be coprime* with one another, but they can't be coprime (because coprimality, like primality, is standardly defined in terms of numbers and multiplication).

With the notions of primality*, coprimality* and LCM* in hand, we can now state versions of Lemma 1 and Lemma 2 above, that are true for both numbers in $\mathbb{N}_1$ and years. Now, strictly speaking we don't need anything quite as strong as Lemma 1. Rather we can get by with Lemma 1*:

> **Lemma 1*:** For any elements x and y that appear together either in $\mathbb{N}_1$ or in a time-series of years τ such that y is close to x, the average LCM* for x and each element smaller than x in the series is higher than the average LCM* for y and each element smaller than y in the series iff x is coprime* with every element smaller than x and y is not coprime* with every element smaller than y.

Where y is 'close' to x if y is smaller than the element produced by combining x with itself.

Lemma 1* is a bit of a mouthful. What it basically does is capture the idea that the LCM* between an element x and other nearby elements in $\mathbb{N}_1$ or in a time-series of years is generally higher when x is coprime* with all of those elements. So, for instance, consider a thirteen-year interval and compare it to a fourteen-year interval. The LCM* between a thirteen-year interval and each interval of years smaller than it is, on average, higher than the LCM* between a fourteen-year interval and each interval of years smaller than it. This is predicted by Lemma 1*, since a thirteen-year interval is coprime* with each interval smaller than it, and a fourteen-year interval is not.

Note that the average is calculated by taking each LCM* and dividing it by the number of LCM*s. The average LCM* is not, itself, a member of the time-series of years at issue, but is a statistical measure (just as the average person is not part of a population that is measured statistically).

In addition to Lemma 1*, we need Lemma 2*:

> **Lemma 2*:** For any element x either in $\mathbb{N}_1$ or in a time-series of years τ, x is coprime* with every element in the series that is smaller than the element produced by combining x with itself iff x is prime*.

Lemma 2* tells us, basically, that when an element of $\mathbb{N}_1$ or a time-series of years is prime* then that is necessary and sufficient for it to be coprime* with every element smaller than double its size. Lemma 2* thus captures a connection between coprimality* and primality* that is analogous to the connection between coprimality and primality in Lemma 2.

Using Lemma 1* and 2* we can now formulate the cicada case in terms of the shared properties of primality*, coprimality* and LCM*. We start with a physical system composed of the cicadas, their environment and a temporal structure of years. The temporal structure of the system imbues intervals of years with a range of global properties: properties that intervals have regardless of the state the system enters into. In particular, the structure of time dictates that only certain intervals are prime*. Any interval of years that can be represented by a prime number will be a prime* interval. Because cicada life-cycles just are intervals of years, a cicada life-cycle will also be prime* just in case that interval can be represented by a prime number.

Now, the system begins in a state, T_1, where the cicadas have life-cycle lengths of, let us say, twelve years. Predators with life-cycles of two to nine years are then added to the system, which triggers a state change. For simplicity, let us suppose that there are two states for the system to enter into: T_2, in which the cicadas have thirteen or seventeen-year life-cycles, and T_3 in which the cicadas have some other life-cycle in the range of twelve to eighteen years.

Let us assume for a moment that the system is deterministic, in this sense: if predators are added to the environment of the cicadas, then anything other than the optimal strategy for predator avoidance will result in cicada extinction. Then in order for the system to enter T_3—a state in which the cicadas survive and they have life-cycles other than thirteen or seventeen—some other life-cycle in the twelve to eighteen year range would need to be the optimal way to avoid predator avoidance. Call this life-cycle L. L can only be the optimal way to avoid predator avoidance if the frequency of intersection between L and the life-cycles of predators is minimised compared to all other life-cycles in the twelve to eighteen year range.

However, the frequency of intersection between two life-cycles—which, again, are just intervals of years—is a function of their LCM*: the higher the LCM*, the lower the frequency of intersection over time. Thus, in order for L to be optimal, its LCM* with all smaller life-cycles would, on average, need to be higher than the average LCM* for all other life-cycles. By Lemma 1*, this can only be true if L is coprime* with every life-cycle smaller than it; which in turn requires, by Lemma 2*, that L is prime*. It is, however, impossible for L to be prime*, given the structure of the time-series at issue. L's being prime* is ruled out by the mathematical structure of time. It follows, then, that there is a conflict between the global mathematical properties of the system—the mathematical properties of the time series—and the state-specific mathematical properties L would have were the system to enter T_3. It is this conflict that forces the system to enter T_2.

What if the system is not deterministic? Then we can still model the explanation, so long as the cicadas are likely to evolve toward the optimal method for predator avoidance. For, again, L cannot be the optimal method for avoiding predation without being prime*. So if the system is likely to be in an optimal state, then it is likely to be in a state where the cicadas have thirteen and seventeen year life cycles, since that is the only situation mathematically compatible with optimality.

In sum, the mathematical properties of time can be used to explain why it is that cicadas have evolved to avoid predators in a certain way. In particular, there are properties that time shares with the number series $\mathbb{N}_1$ that are sufficient to explain why the cicadas have a certain life-cycle length. These are not properties that we are generally interested in within mathematics. We are usually interested in primality not primality*, and with LCMs not LCM*s. Nonetheless, $\mathbb{N}_1$ has the starred properties, and these are properties that time has as well.

Once we see how the account applies to the cicada case and to the Bridges of Königsberg case, it is not difficult to see how it generalises. Why do hivebees have hexagonal honeycomb? Because hexagons are the most efficient shape for a nectar-storing honeycomb cell (Lyon [2012]). This, in turn, is because honeybee hives share geometric properties with hexagons, and the face of a honeybee hive shares properties in common with the Euclidean plane. Why do sharks in the open sea adopt certain paths to find prey? Because those paths are optimal in a prey-scarce environment (Baron [2014]). This, in turn, is because those paths share properties in common with random walks, a particular kind of mathematical path through a space, and the open sea shares properties in common with a 3-geometry.

4.5. False Positives

So far I have shown that key MESs can be captured within my account. In addition, a viable account of MESs should not classify cases in which mathematics is playing a merely representational role as genuine MESs. So, for instance, consider Baron's train case:

> Suppose we want to explain why it is that a train T arrives at a station, S, at 3:00 pm. The explanation is as follows: T left another station, S*, 10 kilometres away at 2:00 pm and headed towards S at 10 kph. ... Numbers are used to state the distance between stations as well as the speed of the train and a very basic mathematical calculation is deployed, namely: 10/10 = 1. However, the mathematics itself does not do any explanatory work. (Baron [2016], pp. 459–460)

In order to work out whether the train case is classified as an MES by my account, we need to determine the mathematical properties of the train system. For that, we need a model. In order to model the train case we need to represent the spatiotemporal structure of the system. This can be done by using a flat, four-dimensional geometry in which the train's path is represented as a line connecting two points. The final states of the system can be represented as different geometrical configurations involving different lines. A short line represents a quick journey, a long line represents a long journey. For simplicity, let us compare just two states: T_2, in which the train arrives at 3:00pm, and T_3 in which the train arrives at some other time (the starting state of the system is T_1).

By attending to the model, we can see that the train system does indeed possess mathematical properties. It has global mathematical properties, which correspond to the global geometry of the system, and it has state-specific properties, which correspond to the particular geometric configurations of the T_2 and T_3 states. The trouble is that there does not seem to be any conflict between the state-specific mathematical properties of the system and the global geometry. The global geometry permits both of the configurations at issue. There is, in particular, nothing like the mathematical property of completeness for paths (as in the Bridges of Königsberg case) or the primality* of years (as in the cicada case) that would rule out some geometric configurations, thereby ensuring a certain time of arrival for the train. It is only once we specify a range of physical properties for the train, that its arrival time is determined. In particular, it is only when we specify that the train is travelling at a particular speed, in a certain direction that we can then see why the train arrives at 3:00pm. These properties are not properties that any mathematical object can possess and so there is no plausible way to describe the train case as a conflict between mathematical properties of the system.

Now, one might worry that my model does not capture the full structure of the case. In addition to specific geometry, the train case involves a particular relationship between time travelled, velocity and distance covered. The relationship is hydraulic, in this sense: fix any one of these features, and the other features will be such that as one diminishes, the other increases and *vice versa*. A mathematical function with the same hydraulic structure exists (but where it is a function of three variables x, y and z that can be filled in by numbers). One might therefore argue that the train system shares a structural property in common with some mathematical functions. This structural property, one might continue, can be used to generate a conflict with state-specific properties. Holding fixed a particular velocity for the train, and a particular distance to cover, the train cannot enter into T_3 since this would violate the hydraulic relationship at issue.

There are two things to say here. First, the relationship between time travelled, velocity and distance covered is a law of nature. It is, however, the law that shares mathematical structure with some function. While the law might govern a physical system, it is not clear that any of the physical objects that make up such a system thereby possess the mathematical structure of the law as an intrinsic property. On my account, explanations involve the intrinsic structural properties of physical objects. So it is not clear that the law at issue gives the system global properties of the right kind. Second, the state-specific property that generates a conflict with the relevant law doesn't seem to be a mathematical property. The state-specific property that conflicts with the law is just the property of having a certain arrival time, namely one that is not 3:00pm. The conflict arises because the law, the arrival time, the velocity and the distance travelled are mutually inconsistent. But, as before, the relevant state-specific property is not shared by any mathematical objects, since mathematical objects lack temporal properties.

What emerges from the train case is a general picture: mathematics is playing a merely representational role when it is used to model a physical system, but where there is no detectable conflict between global and state-specific intrinsic mathematical properties. A counterexample to my account is a case in which such a conflict exists, but the mathematics is not intuitively explanatory. While I cannot rule such cases out, I find it hard to see how there could be any. Our intuitions about MESs appear to be driven by the sense that mathematics sometimes forces physical phenomena to happen. That's exactly what occurs when the global mathematical properties of a system exclude or require state-specific mathematical properties. My bet is that the kind of conflict underlying my account, once identified for a case, is likely to generate an intuition that mathematics is explanatory in that case. Whether that's right, however, can only really be determined by applying the account to a wider range of examples.

5. Back to Indispensability

My account involves structural properties shared between physical and mathematical objects. But, one might argue, a nominalist could just take these properties to be physical structural properties—properties that don't involve mathematical objects. If nominalists can accept the account as it stands, however, then a case cannot be made in favour of Platonism via MESs given my view. Platonists should thus reject my account because it undermines the case for Platonism.

It is, however, far from obvious that nominalists can endorse the structural properties needed for my account. I take a structural property to be the property of having a particular structure. But I follow Resnik in thinking that:

> Structures are abstract entities; they can exist without being instantiated in space-time and they are independent of our knowledge and beliefs about them. (Resnik [1985], p. 177)

A physical system has a particular structure when the system instantiates it. Structural properties on my account make indispensable reference to abstract objects.

A nominalist hoping to adopt my account has two choices. She can either show that the notion of instantiating a structure is nominalistically acceptable, or she can replace structural properties with properties of her own. The first option does not seem very promising. The main challenge is to say what structures are. Initially, this looks easy: a structure, a nominalist might argue, is just a collection of physical objects that stand in certain relations to one another. But this won't do. Structures are multiply realisable: many different collections of physical objects can share the same structure. A structure therefore cannot be identified with any particular collection of physical objects.

Perhaps, then, structures are collections of collections of physical objects. However, a structure cannot be any collection of collections of actual objects. For there are merely possible instances of any given structure (perhaps no-one has arranged train tunnels in the same manner as the Bridges of Königsberg but one certainly could). It seems, then, that structures have to be collections of collections of actual and possible objects. But this account requires possibilia, which are not nominalistically acceptable either.

Now, a nominalist might resist the need to nominalise structures, by denying that they exist. If structures don't exist, however, then no physical system can literally instantiate one. This is fine, one might think: we can pretend as if a given physical system has a certain structure and adopt the explanations outlined in §3 accordingly. It is not clear, however, that this will work. Suppose we pretend as if the Bridges of Königsberg have a certain structure, and then we use that structure to explain something about the bridges. If the bridges don't in fact have that structure, then it is hard to see how the explanation we've provided is a genuine explanation. It's like explaining why a window smashed by pretending that Suzy threw a rock at it. If Suzy didn't in fact throw the rock, then we've just failed to explain the smashing.

Without a better account of what structures are, a nominalist who accepts my account must replace structural properties with something else. Again, this might seem straightforward. For surely it is nominalistically acceptable to say that physical objects stand in relations. Moreover, we can treat a description of some objects and the relations in which they stand as a property. Call this: a characterising property. Characterising properties, one might argue, can play the same role in my account as structural properties.

However, characterising properties are not enough; structural properties are essential to the account. For suppose we introduce a characterising property, P, which just describes each of the seven Bridges of Königsberg and each relation between them. We can then introduce another nominalistically acceptable property, C, which is a property of wordlines: namely the property of crossing every bridge exactly once, possesed by any worldline that does so. Now, using the framework of my account, the proposed nominalistic explanation would have to go as follows: the P-property is a global property of a physical system involving the bridges, one it has regardless of which state the physical system is in. Because P-properties of bridges exclude C-properties of worldlines across those bridges, the bridge system cannot enter into a state in which a worldline has the C-property. So it is impossible to cross the seven bridges, passing over each bridge exactly once.

This, however, leaves it mysterious as to why the P-properties of bridges exclude C-properties of worldlines. The only way that I can see to explain this further fact involves an appeal to structures. Any physical system with the P-property is an instance of a particular structure S, namely: a structure of adjacency relations described by a certain matrix. It is a general fact about structures of this kind, that they do not allow for paths through them that are complete (in the manner discussed in §3). Because a worldline across all seven bridges would be complete in the relevant sense, there can be no such worldline through any physical system that is an instance of the relevant structure.

Without structural properties we lack an account of why the exclusions core to my account hold. The exclusion relations between properties are explained by general features of instantiated structures, which is key to understanding. Of course, a nominalist could develop a new account of MESs using only characterising properties, one that doesn't involve the particular relationship between global and state-specific properties seen in my account. If a nominalist wants to accept the approach to MESs developed here, however, then she must first nominalise structures.

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