

# On Magnetic Forces and Work

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(Dated: July 28, 2023)

We address a long-standing debate over whether classical magnetic forces can do work, ultimately answering the question in the affirmative. In detail, we couple a classical particle with intrinsic spin and elementary dipole moments to the electromagnetic field, derive the appropriate generalization of the Lorentz force law, show that the particle's dipole moments must be collinear with its spin axis, and argue that the magnetic field does mechanical work on the particle's elementary magnetic dipole moment. As consistency checks, we calculate the overall system's energy-momentum and angular momentum, and show that their local conservation equations lead to the same force law and therefore the same conclusions about magnetic forces and work. We also compute the system's Belinfante-Rosenfeld energy-momentum tensor.

## I. INTRODUCTION

Textbook treatments and research articles on classical electromagnetism, such as [1, 2], often suggest that magnetic fields cannot do mechanical work. However, everyday examples of bar magnets lifting other bar magnets would seem to suggest otherwise. In this paper, we show that there exists a classical way to understand how magnetic fields can indeed do work.<sup>1</sup>

We start in Section II with a review of the kinematics of classical relativistic point particles with intrinsic spin and permanent, elementary dipole moments, arguing that these dipole moments should be collinear with the particle's spin. In Section III, we couple a particle of this kind to the electromagnetic field and derive its dynamics, showing, in particular, that magnetic forces can classically do work on the particle via its elementary magnetic dipole moment, and verifying the self-consistency of the condition that the particle's dipole moments are collinear with its spin. In Section IV, we derive expressions for the overall system's energy-momentum and angular momentum, and show that their associated conservation laws lead to the same equations of motion as before, thereby providing further confirmation that magnetic fields can do work on a particle with elementary dipole moments. We conclude with one more new result by calculating the system's Belinfante-Rosenfeld energy-momentum tensor.

## II. THE KINEMATICS OF A RELATIVISTIC ELEMENTARY DIPOLE

To start, we will need a relativistic description of the kinematics of a classical point particle with intrinsic spin.

### A. The Phase Space for a Relativistic Massive Particle with Spin

The treatment of such particles has a long history—see, for example, [4–8]. Following [9–11], we model the particle's kinematics using spacetime coordinates  $X^\mu = (cT, \mathbf{X})^\mu$ , relativistic-kinetic energy  $E$ , four-momentum  $p^\mu = (E/c, \mathbf{p})^\mu$ , positive inertial mass  $m > 0$ , and anti-symmetric spin tensor  $S^{\mu\nu}$  by identifying the particle's phase space as a transitive group action (or homogeneous space) of the orthochronous Poincaré group.<sup>2</sup>

In detail, the states in this phase space take the form  $(X, p, S)$  and are each obtained from the unique reference state

$$(0, (mc, \mathbf{0}), S_0) \quad (1)$$

by an appropriate Poincaré transformation  $(a, \Lambda) \in \mathbb{R}^4 \ltimes O(1, 3)$  according to

$$(X, p, S) = (a, \Lambda(mc, \mathbf{0}), \Lambda S_0 \Lambda^T). \quad (2)$$

Here the coordinates  $X^\mu = a^\mu$  and the Lorentz-transformation matrix  $\Lambda^\mu{}_\nu$ , which all vary along the particle's worldline, are treated as the particle's fundamental phase-space variables, with the condition that  $\Lambda^T \eta \Lambda = \eta = \text{diag}(-1, +1, +1, +1)$ . As explained in [9–11], the invariance of the quantities  $p^2 \equiv -m^2 c^2$  and  $s^2 \equiv (1/2)S_{\mu\nu}S^{\mu\nu}$  further requires the auxiliary phase-space condition

$$p_\mu S^{\mu\nu} = 0. \quad (3)$$

In the reference state (1), the particle's four-momentum has the value  $p_0^\mu = (mc, \mathbf{0})^\mu = mc \delta_t^\mu$ , where  $\delta_\nu^\mu$  is the four-dimensional Kronecker delta. In accordance with the formula (2) for all the other states in the

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<sup>1</sup> For a more extensive treatment of the results in this paper, see [3].

<sup>2</sup> This group-theoretic definition of the particle's phase space is the classical counterpart to Wigner's classification [12] of quantum particle-types based on irreducible Hilbert-space representations of the Poincaré group.

particle's phase space, the particle's four-momentum in general states is therefore given in terms of the variable Lorentz-transformation matrix  $\Lambda^\mu{}_\nu$  by  $p^\mu = mc \Lambda^\mu{}_t$ .

Together with the reference value  $p_0^\mu = (mc, \mathbf{0})^\mu$  of the particle's four-momentum, the self-consistency condition (3) tells us that the value  $S_0^{\mu\nu}$  of the particle's spin tensor in the reference state (1) satisfies  $mc S_0^{t\nu} = 0$ . This equation, in turn, implies that the reference value of the spin tensor takes the specific form

$$S_0^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S_{0,z} & -S_{0,y} \\ 0 & -S_{0,z} & 0 & S_{0,x} \\ 0 & S_{0,y} & -S_{0,x} & 0 \end{pmatrix}^{\mu\nu} \quad (4)$$

for a three-dimensional pseudovector

$$\mathbf{S}_0 \equiv (S_{0,x}, S_{0,y}, S_{0,z}) \quad (5)$$

whose direction can be chosen for convenience. As an immediate consequence, we see that the particle's reference state (1) spontaneously breaks the full three-dimensional rotation group down to the symmetry subgroup of rotations around the axis defined by  $\mathbf{S}_0$ .

## B. Charge and Elementary Dipole Moments

We can couple the particle to the electromagnetic field by assigning the particle a purely electric monopole charge  $q$  and an antisymmetric elementary dipole tensor  $m^{\mu\nu}$  encoding both electric and magnetic dipole moments. The particle is then an electrically charged elementary dipole.

Note, in particular, that the elementary magnetic dipole moments considered in this paper are neither of the Ampère model, which would instead consist of loops of moving electric monopoles, nor of the Gilbert model, which would instead consist of pairs of hypothetical magnetic monopoles or dyons. Elementary magnetic dipoles represent a classical extension of Maxwell's original theory of electromagnetism, as Maxwell's theory includes magnetic dipoles only of the Ampère type.<sup>3</sup>

Note also that because elementary magnetic dipoles do not arise from magnetic monopoles or dyons, they will not end up altering the homogeneous Maxwell equations. Indeed, we will see that the elementary dipoles studied in this paper correspond to derivative terms in the charge and current densities that appear in the inhomogeneous Maxwell equations.

We let  $u^\mu \equiv dX^\mu/d\lambda$  denote the particle's four-velocity and  $\gamma \equiv u^0/c$  denote the particle's associated Lorentz factor, where  $u^\mu$  is not generically normalized to  $u^2 = -c^2$  unless the worldline parameter  $\lambda$  is taken to be the particle's proper time  $\tau$ .<sup>4</sup> In terms of  $\gamma$ ,  $c$ , and the particle's three-velocity  $\mathbf{v} \equiv d\mathbf{X}/dt$ , the particle's four-velocity takes the form

$$u^\mu = (\gamma c, \gamma \mathbf{v})^\mu. \quad (6)$$

The particle then has four-dimensional electric-monopole current density

$$j_e^\nu(\mathbf{x}, t) = (\rho_e(\mathbf{x}, t)c, \mathbf{J}_e(\mathbf{x}, t))^\nu = qu^\nu \frac{1}{\gamma} \delta^3(\mathbf{x} - \mathbf{X}) \quad (7)$$

and elementary-dipole density

$$M^{\mu\nu} = m^{\mu\nu} \frac{1}{\gamma} \delta^3(\mathbf{x} - \mathbf{X}), \quad (8)$$

which appears as a derivative contribution to the particle's total current density,

$$j^\nu(\mathbf{x}, t) = j_e^\nu(\mathbf{x}, t) + \partial_\mu M^{\mu\nu}(\mathbf{x}, t), \quad (9)$$

where  $(1/\gamma)\delta^3(\mathbf{x} - \mathbf{X})$  is the Lorentz-invariant form of the three-dimensional Dirac delta function.

It follows immediately from (7) that the particle's electric-monopole density  $\rho_e = j_e^t/c$ , its electric-monopole current density  $\mathbf{J}_e = (j_e^x, j_e^y, j_e^z)$ , and its three-velocity  $\mathbf{v} \equiv d\mathbf{X}/dt$  satisfy the basic relationship

$$\mathbf{J}_e = \rho_e \mathbf{v}. \quad (10)$$

We emphasize that no such relationship holds for the particle's elementary dipole moments, which, again, are not assumed to arise as in the Ampère model from any underlying motion of electric monopoles.

As in [2], by introducing suitable four-vectors  $\pi^\mu$  and  $\mu^\mu$  and antisymmetric tensors

$$\pi^{\mu\nu} \equiv \frac{1}{mc} (p^\mu \pi^\nu - p^\nu \pi^\mu), \quad (11)$$

$$\mu^{\mu\nu} \equiv \frac{1}{mc} \epsilon^{\mu\nu\rho\sigma} p_\rho \mu_\sigma, \quad (12)$$

we can write the particle's elementary dipole tensor in terms of an electric part  $\pi^{\mu\nu}$  and a magnetic part  $\mu^{\mu\nu}$  as

$$m^{\mu\nu} = \pi^{\mu\nu} + \mu^{\mu\nu}. \quad (13)$$

<sup>3</sup> We thank David Griffiths for pointing out that a defining feature of Maxwell's original theory is the inclusion of magnetic dipoles solely of the Ampère type, without magnetic monopoles, dyons, Gilbert dipoles, or elementary magnetic dipoles.

<sup>4</sup> For maximum generality and to avoid introducing any unnecessary constraints into the particle's Lagrangian formulation, it is convenient to wait until after deriving the particle's equations of motion before imposing the simplifying condition that  $\lambda$  is the particle's proper time  $\tau$ .

Here  $\epsilon^{\mu\nu\rho\sigma}$  is the four-dimensional Levi-Civita symbol (with  $\epsilon_{txyz} \equiv +1$ ), and the four-vectors  $\pi^\mu$  and  $\mu^\mu$  are related to their values in the particle's reference state (1) and to the variable Lorentz-transformation matrix  $\Lambda^\mu_\nu$  according to

$$\pi^\mu \equiv \Lambda^\mu_\nu \pi_0^\nu, \quad (14)$$

$$\mu^\mu \equiv \Lambda^\mu_\nu \mu_0^\nu. \quad (15)$$

We can define a three-dimensional electric-dipole vector  $\boldsymbol{\pi} = (\pi_x, \pi_y, \pi_z)$  and a three-dimensional magnetic-dipole pseudovector  $\boldsymbol{\mu} = (\mu_x, \mu_y, \mu_z)$  in terms of components of the elementary dipole tensor (13) according to

$$m^{\mu\nu} \equiv \begin{pmatrix} 0 & c\pi_x & c\pi_y & c\pi_z \\ -c\pi_x & 0 & -\mu_z & \mu_y \\ -c\pi_y & \mu_z & 0 & -\mu_x \\ -c\pi_z & -\mu_y & \mu_x & 0 \end{pmatrix}^{\mu\nu}. \quad (16)$$

Then the values  $\boldsymbol{\pi}_0$  of the electric-dipole vector and  $\boldsymbol{\mu}_0$  of the magnetic-dipole pseudovector in the reference state (1) are related to their four-vector counterparts  $\pi_0^\mu$  and  $\mu_0^\mu$  according to

$$\pi_0^\mu \equiv (0, \boldsymbol{\pi}_0)^\mu, \quad (17)$$

$$\mu_0^\mu \equiv (0, \boldsymbol{\mu}_0)^\mu. \quad (18)$$

Recalling that the particle's reference state (1) is unique and is symmetric under the subgroup of rotations around the axis defined by the particle's spin pseudovector  $\mathbf{S}_0$  from (5), we see that  $\boldsymbol{\pi}_0$  and  $\boldsymbol{\mu}_0$  must be collinear with  $\mathbf{S}_0$  (and therefore must also be collinear with each other). Otherwise rotations around  $\mathbf{S}_0$  would alter either  $\boldsymbol{\pi}_0$  or  $\boldsymbol{\mu}_0$  (or both) and thereby lead to an infinite degeneracy incompatible with the structure of the particle's phase space.<sup>5</sup>

Hence, for two constants of proportionality,  $\Xi$  and  $\Gamma$ , we must have the relations

$$\boldsymbol{\pi}_0 = \frac{1}{c} \Xi \mathbf{S}_0, \quad (19)$$

$$\boldsymbol{\mu}_0 = \Gamma \mathbf{S}_0. \quad (20)$$

Because the electric dipole moment  $\boldsymbol{\pi}_0$  is a proper vector and  $\mathbf{S}_0$  is a pseudovector, the first of these constants,  $\Xi$ , must be a pseudoscalar. The magnetic dipole moment  $\boldsymbol{\mu}_0$ , by contrast, is a pseudovector, so the other constant,  $\Gamma$ , must be a proper scalar. Given its relationship to  $\boldsymbol{\mu}_0$  and  $\mathbf{S}_0$ , we can interpret  $\Gamma$  as the particle's gyromagnetic ratio, whose specific value is not fixed by our group-theoretic arguments here.

### III. THE DYNAMICS OF A RELATIVISTIC ELEMENTARY DIPOLE

Next, we turn to a discussion of the particle's dynamics.

#### A. The Action Functional for a Relativistic Massive Particle with Spin

In the absence of intrinsic spin, [9, 10] show that one can always rewrite the canonical, manifestly covariant action functional for a free relativistic particle,

$$S_{\text{no spin}}[X, \Lambda] = \int d\lambda p_\mu \dot{X}^\mu, \quad (21)$$

in the alternative form

$$S_{\text{no spin}}[X, \Lambda] = \int d\lambda \frac{1}{2} L_{\mu\nu} \dot{\theta}^{\mu\nu}, \quad (22)$$

up to irrelevant boundary terms, where  $\lambda$  is a smooth and monotonic but otherwise arbitrary parameter along the particle's worldline, and where

$$L_{\mu\nu} \equiv X_\mu p_\nu - X_\nu p_\mu \quad (23)$$

is the particle's orbital angular-momentum tensor.

Here  $\theta^{\mu\nu}$  is an antisymmetric tensor of six independent angular and boost variables, with  $\theta^{yz}, \theta^{zx}, \theta^{xy}$  respectively referring to the particle's angular degrees of freedom around the  $x, y, z$  axes, and with  $\theta^{tx}, \theta^{ty}, \theta^{tz}$  respectively referring to rapidities along the  $x, y, z$  axes.<sup>6</sup> These six angular and boost variables are canonically conjugate to the corresponding six independent components of the orbital angular-momentum tensor  $L_{\mu\nu}$ , with  $L_{yz}, L_{zx}, L_{xy}$  respectively describing the  $x, y, z$  components of the particle's three-dimensional angular momentum, and with  $L_{tx}, L_{ty}, L_{tz}$  respectively describing the  $x, y, z$  coordinates of the particle's center of mass.

The inclusion of intrinsic spin entails the replacement

$$L_{\mu\nu} \mapsto J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \quad (24)$$

where  $J_{\mu\nu}$  is the particle's total angular-momentum tensor. Continuing to assume the absence of external interactions, [9–11] show that we can then encode the dynamics of a particle with intrinsic spin in terms of the manifestly covariant action functional

$$S_{\text{particle}}[X, \Lambda] = \int d\lambda \frac{1}{2} J_{\mu\nu} \dot{\theta}^{\mu\nu} = \int d\lambda \left( p_\mu \dot{X}^\mu + \frac{1}{2} \text{Tr}[S \dot{\Lambda} \Lambda^{-1}] \right), \quad (25)$$

where we again ignore irrelevant boundary terms.

<sup>5</sup> The quantum-mechanical analogue of this classical collinearity condition follows from the Wigner-Eckart theorem.

<sup>6</sup> Explicitly, we have  $\dot{\theta}^{\mu\nu} = (i/2) \text{Tr}[\sigma^{\mu\nu} \dot{\Lambda} \Lambda^{-1}]$ , where  $[\sigma^{\mu\nu}]^\alpha_\beta = -i\eta^{\mu\alpha} \delta^\nu_\beta + i\delta^\mu_\beta \eta^{\nu\alpha}$  are the generators of the Lorentz group.

## B. The Particle's Equations of Motion

Our next step will be to couple the particle to the electromagnetic field and obtain the particle's equations of motion, from which we will be able to infer the appropriate generalization of the Lorentz force law.

Given the charge and elementary dipole moments outlined above, the overall action functional for the elementary dipole and the electromagnetic field is given by

$$\begin{aligned}
S[X, \Lambda, A] &\equiv S_{\text{particle}}[X, \Lambda] + S_{\text{field}}[A] + S_{\text{int}}[X, \Lambda, A] \\
&= \int d\lambda \left( p_\mu \dot{X}^\mu + \frac{1}{2} \text{Tr}[S\dot{\Lambda}\Lambda^{-1}] \right) & (S_{\text{particle}}) \\
&\quad + \int dt \int d^3x \left( -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \right) & (S_{\text{field}}) \\
&\quad + \int dt \int d^3x j^\nu A_\nu & (S_{\text{int}}),
\end{aligned} \tag{26}$$

where  $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$  is the standard Faraday tensor and  $j^\nu = j_e^\nu + \partial_\mu M^{\mu\nu}$  is the particle's total current density (9). The interaction term in the final line ensures that extremizing the action functional with respect to the electromagnetic gauge field  $A_\mu$  yields the Maxwell equations in their usual form, with unmodified homogeneous equations  $\nabla \cdot \mathbf{B} = 0$  and  $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ , and with the charge and current densities appearing in the inhomogeneous Maxwell equations determined by (9). The first line in this action functional ( $S_{\text{particle}}$ ) is fixed by group theory, the second line ( $S_{\text{field}}$ ) defines the vacuum in the pure Maxwell theory, and the third line ( $S_{\text{int}}$ ) provides the canonical coupling between the particle and the electromagnetic field in a manner consistent with the Maxwell equations and the particle's features as laid out in the previous section.

After an integration by parts, we can write the interaction term in the final line as

$$S_{\text{int}}[X, \Lambda, A] = \int dt \int d^3x \left( j_e^\nu A_\nu - \frac{1}{2} M^{\mu\nu} F_{\mu\nu} \right). \tag{27}$$

Collecting together all the terms that involve the particle's degrees of freedom, we obtain

$$\begin{aligned}
S_{\text{particle+int}}[X, \Lambda, A] &= \int d\lambda \left( p_\mu \dot{X}^\mu + \frac{1}{2} \text{Tr}[S\dot{\Lambda}\Lambda^{-1}] \right) \\
&\quad + \int dt \int d^3x j_e^\nu A_\nu + \int dt \int d^3x \left( -\frac{1}{2} \right) M^{\mu\nu} F_{\mu\nu},
\end{aligned} \tag{28}$$

which we can further reduce to the form

$$S_{\text{particle+int}}[X, \Lambda, A] = \int d\lambda \mathcal{L}_{\text{particle+int}}, \tag{29}$$

for a manifestly covariant Lagrangian defined by

$$\begin{aligned}
\mathcal{L}_{\text{particle+int}} &\equiv p_\mu \dot{X}^\mu + \frac{1}{2} \text{Tr}[S\dot{\Lambda}\Lambda^{-1}] \\
&\quad + q\dot{X}^\nu A_\nu - \frac{1}{2c} \sqrt{-\dot{X}^2} m^{\mu\nu} F_{\mu\nu}.
\end{aligned} \tag{30}$$

It follows from a straightforward calculation that the particle's equations of motion, expressed in terms of the particle's proper time  $\tau$ , are then

$$\begin{aligned}
\frac{dp^\mu}{d\tau} &= -qu_\nu F^{\nu\mu} - \frac{1}{2} m^{\rho\sigma} \partial^\mu F_{\rho\sigma} - \frac{1}{2c^2} \frac{d}{d\tau} (u^\mu m^{\rho\sigma} F_{\rho\sigma}) \\
&= -qu_\nu F^{\nu\mu} - \frac{1}{2} m^{\rho\sigma} (\eta^{\mu\nu} + \frac{1}{c^2} u^\mu u^\nu) \partial_\nu F_{\rho\sigma} \\
&\quad - \frac{1}{2c^2} \frac{d}{d\tau} (u^\mu m^{\rho\sigma}) F_{\rho\sigma},
\end{aligned} \tag{31}$$

as obtained in [11, 13, 14], and

$$\frac{dS^{\mu\nu}}{d\tau} = -(u^\mu p^\nu - u^\nu p^\mu) - (m^{\mu\rho} F^\nu{}_\rho - m^{\nu\rho} F^\mu{}_\rho), \tag{32}$$

which generalizes the results of [4, 11, 13].

## C. The Non-Relativistic Limit with Time-Independent External Fields

In the non-relativistic limit for time-independent fields, and ignoring self-field effects—so that we can regard the overall electric and magnetic fields as external fields  $\mathbf{E}_{\text{ext}}$  and  $\mathbf{B}_{\text{ext}}$ —the equations of motion (31) and (32) reduce to

$$\frac{d\mathbf{E}}{dt} \approx \frac{d}{dt} (-q\Phi_{\text{ext}} + \boldsymbol{\pi} \cdot \mathbf{E}_{\text{ext}} + \boldsymbol{\mu} \cdot \mathbf{B}_{\text{ext}}), \tag{33}$$

$$\frac{d\mathbf{p}}{dt} \approx q(\mathbf{E}_{\text{ext}} + \mathbf{v} \times \mathbf{B}_{\text{ext}}) + \nabla(\boldsymbol{\pi} \cdot \mathbf{E}_{\text{ext}} + \boldsymbol{\mu} \cdot \mathbf{B}_{\text{ext}}), \tag{34}$$

$$\frac{d\mathbf{J}}{dt} \approx \mathbf{X} \times \frac{d\mathbf{p}}{dt} + \boldsymbol{\pi} \times \mathbf{E}_{\text{ext}} + \boldsymbol{\mu} \times \mathbf{B}_{\text{ext}}. \tag{35}$$

Here the electric field  $\mathbf{E}_{\text{ext}}$  is given in terms of the scalar potential  $\Phi_{\text{ext}}$  according to the usual formula  $\mathbf{E}_{\text{ext}} = -\nabla\Phi_{\text{ext}}$  appropriate to the static case, the particle's four-momentum in this limit is

$$p^\mu = (E/c, \mathbf{p})^\mu \approx (mc + (1/2)m\mathbf{v}^2/c, \mathbf{p})^\mu, \tag{36}$$

and the particle's total angular-momentum pseudovector  $\mathbf{J}$  is made up of orbital and spin contributions according to

$$\mathbf{J} \equiv \mathbf{L} + \mathbf{S} = (L^{yz}, L^{zx}, L^{xy}) + (S^{yz}, S^{zx}, S^{xy}). \tag{37}$$

The dynamical equation (33) for the rate of change of the particle's relativistic-kinetic energy  $E$  describes conservation of the particle's total energy, provided that we identify the combination

$$V = q\Phi_{\text{ext}} - \boldsymbol{\pi} \cdot \mathbf{E}_{\text{ext}} - \boldsymbol{\mu} \cdot \mathbf{B}_{\text{ext}} \tag{38}$$

as the particle's potential energy. Meanwhile, the dynamical equation (34) tells us that the electromagnetic force on the particle is

$$\mathbf{F} = q\mathbf{E}_{\text{ext}} + q\mathbf{v} \times \mathbf{B}_{\text{ext}} + \nabla(\boldsymbol{\pi} \cdot \mathbf{E}_{\text{ext}}) + \nabla(\boldsymbol{\mu} \cdot \mathbf{B}_{\text{ext}}). \quad (39)$$

We observe that the usual Lorentz force law,  $q\mathbf{E}_{\text{ext}} + q\mathbf{v} \times \mathbf{B}_{\text{ext}}$ , is enhanced in the presence of the particle's elementary dipole moments by the appearance of two additional dipole terms  $\nabla(\boldsymbol{\pi} \cdot \mathbf{E}_{\text{ext}})$  and  $\nabla(\boldsymbol{\mu} \cdot \mathbf{B}_{\text{ext}})$ . The magnetic field therefore contributes to the work done by the external electromagnetic field,  $W \equiv \int d\mathbf{X} \cdot \mathbf{F}$ :

$$W = \int_A^B dt (q\mathbf{v} \cdot \mathbf{E}_{\text{ext}}) + \Delta(\boldsymbol{\pi} \cdot \mathbf{E}_{\text{ext}}) + \Delta(\boldsymbol{\mu} \cdot \mathbf{B}_{\text{ext}}). \quad (40)$$

We have reached the key conclusion of this paper—namely, that magnetic forces can do work on classical particles with elementary dipole moments.<sup>7</sup> We next turn to a detailed treatment of self-consistency conditions on the particle's dynamics, as well as obtain the necessary formulas for determining the particle's four-velocity  $u^\mu$  in the presence of a nonzero electromagnetic field. Later on, we will analyze electromagnetic forces and work done on the particle from the standpoint of local conservation laws.

#### D. Implications of Self-Consistency

Taking a derivative of the phase-space condition  $p_\mu S^{\mu\nu}$  from (3) yields the self-consistency requirement

$$\frac{dp_\mu}{d\tau} S^{\mu\nu} + p_\mu \frac{dS^{\mu\nu}}{d\tau} = 0.$$

Together with (32), this self-consistency requirement entails that the particle's four-momentum  $p^\mu$  and its four-velocity  $u^\mu = dX^\mu/d\tau$  (now normalized to  $u^2 = -c^2$ ) are related by

$$p^\mu = m_{\text{eff}} u^\mu + b^\mu. \quad (41)$$

Here  $m_{\text{eff}}$ , which plays the role of an effective mass, is defined by

$$m_{\text{eff}} \equiv -\frac{m^2 c^2}{p \cdot u}, \quad (42)$$

<sup>7</sup> We thank Sebastiano Covone for suggesting the relevance of these results to the Bohr-van Leeuwen theorem [15, 16]. The Bohr-van Leeuwen theorem assumes the original Lorentz force law without contributions from elementary dipole moments, and asserts that a non-rotating system of particles, when treated classically, always has a vanishing average magnetization at thermal equilibrium. The theorem's implication is that phenomena like diamagnetism can only be understood in terms of quantum effects, a view challenged by our results, at least in principle.

and the four-vector  $b^\mu$ , which is orthogonal to the particle's four-momentum,  $b \cdot p = 0$ , is given by

$$b^\mu \equiv \frac{1}{p \cdot u} \left( \frac{dp_\nu}{d\tau} S^{\nu\mu} - p_\nu (m^{\nu\rho} F^\mu{}_\rho - m^{\mu\rho} F^\nu{}_\rho) \right). \quad (43)$$

As in [11], we regard (41) as an implicit formula for determining the behavior of the particle's four-velocity  $u^\mu$  as a function of the proper time. This formula ensures, in particular, that the particle's four-momentum  $p^\mu$  has constant norm-squared  $p^2 = -m^2 c^2$ .

For vanishing field,  $F_{\mu\nu} = 0$ , the relationship (41) reduces to the familiar equation  $p^\mu = mu^\mu$ , as expected. By contrast, when the electromagnetic field is nonzero,  $F_{\mu\nu} \neq 0$ , (41) has the form

$$p^\mu = mu^\mu + (\text{terms of order } 1/c^2). \quad (44)$$

This relation ensures that there is no ambiguity over whether we should identify the particle's relativistic-kinetic energy  $E$  as  $p^t c$  or  $u^t m c^2$  for the purposes of quantifying the work done by the field on the particle in the non-relativistic regime.

Invoking the spin tensor's equation of motion (32), together with the phase-space condition (3),  $p_\mu S^{\mu\nu} = 0$ , and the constancy of the particle's spin-squared scalar  $s^2 \equiv (1/2)S_{\mu\nu}S^{\mu\nu}$ , we find

$$\begin{aligned} \frac{d}{d\tau}(s^2) &= \frac{d}{d\tau} \left( \frac{1}{2} S_{\mu\nu} S^{\mu\nu} \right) \\ &= (S^\rho{}_\mu m^{\mu\sigma} - S^\sigma{}_\mu m^{\mu\rho}) F_{\rho\sigma} = 0, \end{aligned} \quad (45)$$

which yields the condition

$$S^\rho{}_\mu m^{\mu\sigma} = S^\sigma{}_\mu m^{\mu\rho}. \quad (46)$$

In the particle's reference state (1), this equality produces the relations

$$\boldsymbol{\pi}_0 \times \mathbf{S}_0 = 0, \quad (47)$$

$$\boldsymbol{\mu}_0 \times \mathbf{S}_0 = 0, \quad (48)$$

which ensure self-consistency with our requirement that the particle's elementary electric and magnetic dipole moments must be collinear with the particle's spin pseudovector  $\mathbf{S}_0$ . Physically speaking, we can understand the self-consistency conditions (47) as telling us that if the particle's elementary-dipole vectors were not collinear with the particle's spin axis, then torques exerted on the particle by the electromagnetic field would cause the particle's spin to speed up or slow down, in violation of the constancy of  $s^2$ .

#### IV. CONSERVATION LAWS

For completeness, we verify that the equations of motion (31) and (32) also follow from local conservation of

energy-momentum and angular momentum. To begin, we recall the relevant version of Noether's theorem (see, for instance, [3]), which states that if a system's dynamics has a continuous symmetry,

$$\begin{aligned} q_\alpha &\mapsto q'_\alpha = q_\alpha + \delta_\epsilon q_\alpha, \\ \delta_\epsilon q_\alpha &= \sum_b g_{q_\alpha, b} \epsilon_b, \end{aligned} \quad (49)$$

where the quantities  $\epsilon_b$  parameterize the symmetry and the quantities  $g_{q_\alpha, b}$  characterize its precise form, then we have the following conservation law:

$$Q_b \equiv \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} g_{q_\alpha, b} - f_b, \quad \frac{dQ_b}{dt} = 0. \quad (50)$$

Here  $Q_b$  are a set of conserved quantities,  $L$  is the system's Lagrangian,  $q_\alpha$  are its degrees of freedom, and the functions  $f_b$  are related to the change in the Lagrangian according to

$$\begin{aligned} L &\mapsto L + \delta_\epsilon L, \\ \delta_\epsilon L &= \frac{d}{dt} \left( \sum_b f_b \epsilon_b \right) = \sum_b \frac{df_b}{dt} \epsilon_b. \end{aligned} \quad (51)$$

### A. Local Conservation of Energy-Momentum

In order to employ Noether's theorem to obtain the overall system's energy-momentum tensor, we examine the behavior of the system under a translation in space-time by an infinitesimal four-vector  $\epsilon^\mu$ . The particle's phase-space variables transform as

$$\left. \begin{aligned} X^\mu(\lambda) &\mapsto X'^\mu(\lambda) \equiv X^\mu(\lambda) + \epsilon^\mu, \\ \Lambda^\mu{}_\nu(\lambda) &\mapsto \Lambda'^\mu{}_\nu(\lambda) \equiv \Lambda^\mu{}_\nu(\lambda), \end{aligned} \right\} \quad (52)$$

and the electromagnetic gauge potential transforms as

$$\begin{aligned} A_\mu(x) &\mapsto A'_\mu(x) \equiv A_\mu(x - \epsilon) \\ &= A_\mu(x) - \partial_\nu A_\mu(x) \epsilon^\nu. \end{aligned} \quad (53)$$

By an application of Noether's theorem to the particle's manifestly covariant Lagrangian  $\mathcal{L} \equiv \mathcal{L}_{\text{particle+int}}$  defined by (29) and the Lagrangian density  $\mathcal{L}$  for the overall system defined in terms of the action functional  $S[X, \Lambda, A] \equiv \int dt \int d^3x \mathcal{L}$  from (26), one finds that the overall system's conserved four-momentum is expressible as

$$\begin{aligned} P_\nu &= \frac{\partial \mathcal{L}}{\partial \dot{X}^\rho} g_{X^\rho, \nu} + \int d^3x (-n_\mu) \frac{\partial \mathcal{L}}{\partial (c \partial_\mu A_\rho)} g_{A_\rho, \nu} - f_\nu \\ &= p_\nu + q A_\nu + \frac{1}{2c^2} u_\nu m^{\sigma\tau} F_{\sigma\tau} \\ &\quad + \frac{1}{c} \int d^3x (-n_\mu) \left( H^{\mu\rho} \partial_\nu A_\rho - \delta_\nu^\mu \left( \frac{1}{4\mu_0} F^{\rho\sigma} F_{\rho\sigma} \right) \right) \\ &= \frac{1}{c} \int d^3x (-n_\mu) T_{\text{can}, \nu}^\mu, \end{aligned} \quad (54)$$

where  $n_\mu \equiv (-1, \mathbf{0})_\mu$  is a unit timelike four-vector orthogonal to the three-dimensional spatial hypersurface of integration. In this expression, the overall system's canonical energy-momentum tensor is given by

$$T_{\text{can}}^{\mu\nu} = T_{\text{can}, \text{particle}}^{\mu\nu} + T_{\text{can}, \text{field}}^{\mu\nu}, \quad (55)$$

with the contributions from the particle and the field given respectively by<sup>8</sup>

$$T_{\text{can}, \text{particle}}^{\mu\nu} \equiv u^\mu p^\nu \frac{1}{\gamma} \delta^3(\mathbf{x} - \mathbf{X}) \quad (56)$$

and

$$\begin{aligned} T_{\text{can}, \text{field}}^{\mu\nu} &\equiv H^{\mu\rho} F^\nu{}_\rho - \eta^{\mu\nu} \frac{1}{4\mu_0} F^2 \\ &\quad + \frac{1}{2c^2} u^\mu u^\nu m^{\rho\sigma} F_{\rho\sigma} \frac{1}{\gamma} \delta^3(\mathbf{x} - \mathbf{X}) \\ &\quad + \partial_\rho (H^{\mu\rho} A^\nu). \end{aligned} \quad (57)$$

Here  $H^{\mu\nu}$  is the auxiliary Faraday tensor:

$$\begin{aligned} H^{\mu\nu} &\equiv \frac{1}{\mu_0} F^{\mu\nu} + M^{\mu\nu} \\ &= \frac{1}{\mu_0} F^{\mu\nu} + m^{\mu\nu} \frac{1}{\gamma} \delta^3(\mathbf{x} - \mathbf{X}). \end{aligned} \quad (58)$$

The last term in (57) is a total spacetime divergence with vanishing divergence  $\partial_\mu \partial_\rho (H^{\mu\rho} A^\nu) = 0$  on its  $\mu$  index, and its temporal component  $\partial_\rho (H^{t\rho} A^\nu)$  has vanishing integral over three-dimensional space under the assumption that the fields go to zero sufficiently rapidly at spatial infinity. We emphasize that in our approach, all the terms in the overall system's canonical energy-momentum tensor follow from the systematic application of Noether's theorem to the relevant action functionals.

We can integrate the local conservation law  $\partial_\mu T_{\text{can}}^{\mu\nu} = 0$  over three-dimensional space to compute the time derivative of the particle's four-momentum  $p^\nu$ :

$$\begin{aligned} \frac{dp^\nu}{dt} &= \frac{1}{c} \frac{d}{dt} \int d^3x T_{\text{can}, \text{particle}}^{t\nu} \\ &= -\frac{1}{c} \frac{d}{dt} \int d^3x T_{\text{can}, \text{field}}^{t\nu} \\ &= \int d^3x \left( -\partial_\mu \left( H^{\mu\rho} F^\nu{}_\rho - \eta^{\mu\nu} \frac{1}{4\mu_0} F^2 \right) \right) \\ &\quad - \frac{1}{2c^2} \frac{d}{dt} (u^\nu m^{\rho\sigma} F_{\rho\sigma}) \\ &= -q u_\mu F^{\mu\nu} + m_{\rho\mu} \partial^\mu F^{\nu\rho} - \frac{1}{2c^2} \frac{d}{dt} (u^\nu m^{\rho\sigma} F_{\rho\sigma}). \end{aligned}$$

<sup>8</sup> The authors of [2] decompose the overall energy-momentum tensor by including the interaction terms with the energy-momentum tensor for the *particle*, an approach that obscures the work being done by the electromagnetic field on the particle.

After invoking the electromagnetic Bianchi identity  $\partial^\mu F^{\nu\rho} + \partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu} = 0$ , we obtain the equation of motion (31).

Our formulas above for the overall system's canonical energy-momentum tensor are new results. By replicating the particle's equation of motion (31), they provide further support for the key claim of this paper—that magnetic forces can classically do work on particles with elementary dipole moments.

## B. Local Conservation of Angular Momentum

Next, we use Noether's theorem to examine the overall system's angular momentum and its local conservation. Under an infinitesimal Lorentz transformation

$$\Lambda_{\text{inf}} = 1 + \frac{i}{2}\epsilon^{\rho\sigma}\sigma_{\rho\sigma}, \quad (59)$$

the particle's phase-space variables transform as

$$\left. \begin{aligned} X^\mu(\lambda) &\mapsto X'^\mu(\lambda) \equiv (\Lambda_{\text{inf}} X(\lambda))^\mu \\ &= X^\mu(\lambda) + \frac{i}{2}\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\mu{}_\nu X^\nu(\lambda), \\ \Lambda^\mu{}_\nu(\lambda) &\mapsto \Lambda'^\mu{}_\nu(\lambda) \equiv (\Lambda_{\text{inf}} \Lambda(\lambda))^\mu{}_\nu \\ &= \Lambda^\mu{}_\nu(\lambda) + \frac{i}{2}\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\mu{}_\lambda \Lambda^\lambda{}_\nu(\lambda). \end{aligned} \right\} \quad (60)$$

The second of these two transformation laws is equivalent to the following transformation rule for the particle's Lorentz parameters  $\theta^{\mu\nu}(\lambda)$ :

$$\theta^{\mu\nu}(\lambda) \mapsto \theta'^{\mu\nu}(\lambda) \equiv \theta^{\mu\nu}(\lambda) + \epsilon^{\mu\nu}. \quad (61)$$

Meanwhile, the gauge field  $A_\mu(x)$  transforms as

$$\begin{aligned} A_\mu(x) &\mapsto A'_\mu(x) \equiv (A(\Lambda_{\text{inf}}^{-1}x)\Lambda_{\text{inf}}^{-1})_\mu \\ &\equiv A_\lambda((1 - (i/2)\epsilon^{\rho\sigma}\sigma_{\rho\sigma})x)(\delta_\mu^\lambda - (i/2)\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\lambda{}_\mu) \\ &= A_\mu(x) - \partial_\nu A_\mu(x)(i/2)\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\nu{}_\lambda x^\lambda \\ &\quad - A_\lambda(x)(i/2)\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\lambda{}_\mu. \end{aligned} \quad (62)$$

Noether's theorem (50) then yields the system's overall

angular-momentum tensor, up to an overall minus sign:

$$\begin{aligned} -J_{\nu\rho} &= \frac{\partial\mathcal{L}}{\partial\dot{X}^\alpha}g^{X^\alpha,\nu\rho} + \frac{1}{2}\frac{\partial\mathcal{L}}{\partial\dot{\theta}^{\alpha\beta}}g^{\theta^{\alpha\beta},\nu\rho} \\ &\quad + \int d^3x(-n_\mu)\frac{\partial\mathcal{L}}{\partial(c\partial_\mu A_\alpha)}g_{A_\alpha,\nu\rho} - f_{\nu\rho} \\ &= -\left(p_\alpha + qA_\alpha - \frac{1}{2}(-u_\alpha/c^2)m^{\sigma\lambda}F_{\sigma\lambda}\right)(X_\nu\delta_\rho^\alpha - X_\rho\delta_\nu^\alpha) \\ &\quad - S_{\nu\rho} \\ &\quad - \frac{1}{c}\int d^3x(-n_\mu)\left(H^{\mu\alpha} - \delta_\sigma^\mu\left(\frac{1}{4\mu_0}F^2\right)\right) \\ &\quad\quad \times \partial_\sigma A_\alpha(x_\nu\delta_\rho^\sigma - x_\rho\delta_\nu^\sigma) \\ &\quad - \frac{1}{c}\int d^3x(-n_\mu)(H^\mu{}_\nu A_\rho - H^\mu{}_\rho A_\nu) \\ &= -\int d^3x(-n_\mu)\mathcal{J}_{\text{can},\nu\rho}^\mu. \end{aligned} \quad (63)$$

Here we have identified the system's canonical angular-momentum flux tensor as

$$\mathcal{J}_{\text{can}}^{\mu\nu\rho} = \mathcal{L}^{\mu\nu\rho} + \mathcal{S}^{\mu\nu\rho}, \quad (64)$$

with orbital contribution

$$\mathcal{L}^{\mu\nu\rho} \equiv x^\nu\frac{1}{c}T_{\text{can}}^{\mu\rho} - x^\rho\frac{1}{c}T_{\text{can}}^{\mu\nu} \quad (65)$$

and spin contribution

$$\mathcal{S}^{\mu\nu\rho} = \frac{1}{c}u^\mu S^{\nu\rho}\frac{1}{\gamma}\delta^3(\mathbf{x} - \mathbf{X}) + \frac{1}{c}(H^{\mu\nu}A^\rho - H^{\mu\rho}A^\nu). \quad (66)$$

We naturally read off the spin flux tensors for the particle and the field respectively as

$$\mathcal{S}_{\text{particle}}^{\mu\nu\rho} = \frac{1}{c}u^\mu S^{\nu\rho}\frac{1}{\gamma}\delta^3(\mathbf{x} - \mathbf{X}), \quad (67)$$

$$\mathcal{S}_{\text{field}}^{\mu\nu\rho} = \frac{1}{c}(H^{\mu\nu}A^\rho - H^{\mu\rho}A^\nu). \quad (68)$$

Integrating the local conservation law  $\partial_\mu\mathcal{J}_{\text{can}}^{\mu\nu\rho} = 0$  over three-dimensional space and taking advantage of the local conservation  $\partial_\mu T_{\text{can}}^{\mu\rho} = 0$  of the overall canonical energy-momentum tensor  $T_{\text{can}}^{\mu\rho}$ , we can compute the time derivative of the particle's spin tensor as follows:

$$\begin{aligned} \frac{dS^{\nu\rho}}{dt} &= \frac{d}{dt}\int d^3x\mathcal{S}_{\text{particle}}^{\nu\rho} \\ &= -\frac{d}{dt}\int d^3x\frac{1}{c}(x^\nu T_{\text{can}}^{t\rho} - x^\rho T_{\text{can}}^{t\nu} + H^{t\nu}A^\rho - H^{t\rho}A^\nu) \\ &= -\int d^3x\partial_\mu(x^\nu T_{\text{can}}^{\mu\rho} - x^\rho T_{\text{can}}^{\mu\nu} + H^{\mu\nu}A^\rho - H^{\mu\rho}A^\nu) \\ &= -\frac{1}{\gamma}(u^\nu p^\rho - u^\rho p^\nu) - \frac{1}{\gamma}(m^{\nu\sigma}F^\rho{}_\sigma - m^{\rho\sigma}F^\nu{}_\sigma). \end{aligned}$$

We therefore see that local conservation of angular momentum yields the equation of motion (32).

### C. The Belinfante-Rosenfeld Energy-Momentum Tensor

The overall system's canonical energy-momentum tensor (55) is not symmetric on its two indices, a feature that is required of the energy-momentum tensor that locally sources the gravitational field in general relativity. To conclude this paper, we follow the standard Belinfante-Rosenfeld construction<sup>9</sup> to construct a properly symmetric energy-momentum tensor, which will likewise represent a new result.

We start by introducing a new tensor

$$\begin{aligned} \mathcal{B}^{\mu\rho\nu} &\equiv \frac{c}{2}(\mathcal{S}^{\mu\nu\rho} + \mathcal{S}^{\nu\mu\rho} + \mathcal{S}^{\rho\mu\nu}) \\ &= -H^{\mu\rho}A^\nu + \frac{1}{2}(u^\mu S^{\nu\rho} + u^\nu S^{\mu\rho} + u^\rho S^{\mu\nu})\frac{1}{\gamma}\delta^3(\mathbf{x} - \mathbf{X}). \end{aligned} \quad (69)$$

We then obtain a symmetric, locally conserved energy-momentum tensor  $T^{\mu\nu}$  for the overall system from the relation  $T^{\mu\nu} = T_{\text{can}}^{\mu\nu} + \partial_\rho \mathcal{B}^{\mu\rho\nu}$ .<sup>10</sup>

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{2}(u^\mu p^\nu + u^\nu p^\mu)\frac{1}{\gamma}\delta^3(\mathbf{x} - \mathbf{X}) \\ &\quad + \frac{1}{2}H^{\mu\rho}F^\nu{}_\rho + \frac{1}{2}H^{\nu\rho}F^\mu{}_\rho - \eta^{\mu\nu}\frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma} \\ &\quad + \frac{1}{2c^2}u^\mu u^\nu m^{\rho\sigma}F_{\rho\sigma}\frac{1}{\gamma}\delta^3(\mathbf{x} - \mathbf{X}) \\ &\quad + \frac{1}{2}\partial_\rho(\mathcal{S}_{\text{particle}}^{\mu\nu\rho} + \mathcal{S}_{\text{particle}}^{\nu\mu\rho}). \end{aligned} \quad (70)$$

In the free-field limit—meaning in the absence of the particle—this energy-momentum tensor reduces to the standard gauge-invariant Maxwell energy-momentum tensor, as expected:

$$T^{\mu\nu} = \frac{1}{\mu_0}F^{\mu\rho}F^\nu{}_\rho - \eta^{\mu\nu}\frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma}. \quad (71)$$

### ACKNOWLEDGMENTS

J. A. B. has benefited from personal communications with Gary Feldman, Howard Georgi, Andrew Strominger, Bill Phillips, David Griffiths, David Kagan, David Morin, Logan McCarty, Monica Pate, Alex Lupasca, and Sebastiano Covone.

<sup>9</sup> For a review, see [17].

<sup>10</sup> This formula differs from the corresponding result in [13], whose energy-momentum tensor yields the correct equations of motion for the particle only after an unjustified four-dimensional integration by parts.

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