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# A Modality Called ‘Negation’

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I propose a comprehensive account of negation as a modal operator, vindicating a moderate logical pluralism. Negation is taken as a quantifier on worlds, restricted by an accessibility relation encoding the basic concept of compatibility. This latter captures the core meaning of the operator. While some candidate negations are then ruled out as violating plausible constraints on compatibility, different specifications of the notion of world support different logical conducts for (the admissible) negations. The approach unifies in a philosophically motivated picture the following results: nothing can be called a negation properly if it does not satisfy (Minimal) Contraposition and Double Negation Introduction; the pair consisting of two split or Galois negations encodes a distinction without a difference; some paraconsistent negations also fail to count as real negations, but others may; intuitionistic negation qualifies as real negation, and classical Boolean negation does as well, to the extent that constructivist and paraconsistent doubts on it do not turn on the basic concept of compatibility but rather on the interpretation of worlds.

When it is asserted that a negative signifies a contrary, we shall not agree, but admit no more than this: that the prefix ‘not’ indicates something different from the words that follow, or rather from the things designated by the words pronounced after the negative.

*Plato, Sophist 257b–c*

## 1. Introduction

Possible worlds semantics has taught analytic philosophers and logicians how to give trustworthy truth conditions for the modal operators: we take them as quantifiers on worlds, restricted by accessibility relations from the standpoint of a given world. By imposing simple algebraic conditions on the relevant accessibility, we validate various modal inferences characteristic of different formal systems. This result, due to the work of Kanger, Kripke, Hintikka, and others, is possibly the most celebrated of twentieth century philosophical logic. It has

helped to dispel traditional Quinean worries about intensional notions; it has provided a natural meaning to a plethora of modal systems which were previously presented merely syntactically, or were endowed with less-than-enlightening algebraic semantics; and it has helped to translate modal questions into terms many find intuitively more manageable (pending an account of the metaphysical nature of worlds, as [David Lewis \(1986\)](#) stressed).<sup>1</sup>

This story is so well known that it hardly needs rehearsing. Indeed, the rehearsal just provided has the purpose of stressing a parallel, whose other side is less known among philosophers: as with the meaning of a modal operator like the box of necessity, the meaning of a negation operator consists in its being a universal quantifier on worlds, restricted by an accessibility relation with a clear-cut intuitive meaning. And, just as for the box, the further logical features of negation can be phrased as features of its accessibility relation.

This is likely to come as news to philosophers exposed only or mostly to classical logic. Here negation is an extensional connective *par excellence*, captured by the familiar truth table representing Boolean complementation. The evaluation of a negated formula (at a world) is not supposed to take into account what happens at other worlds. In Bob Stalnaker's words:

My assumption about the meaning of ' $\sim$ ' is this:  $\sim P$  is true if and only if  $P$  is false. Or in other words, the set of worlds in which  $\sim P$  is true is the complement of the set of worlds in which  $P$  is true. I learned this rule in my first logic class years ago. I suppose that one might use the symbol differently, but it is hard to see how any metaphysical question could turn on whether we stick with the traditional truth-table account of the negation symbol. ([Stalnaker 1996](#), pp. 196–7)

At the other end of the spectrum, those who work on non-classical logics know that virtually every inferential (proof-theoretic or semantic) feature of negation has been disputed by different logico-philosophical parties: (Minimal) Contraposition, Excluded Middle, the Law of Non-Contradiction, various De Morgan laws, Double Negation Elimination and Introduction, not to speak of the very

<sup>1</sup> Let us start by just taking worlds as points at which sentences or formulas can be true. As we will see, that such a minimal characterization can be further specified in different ways is the key insight for the logical pluralism about negation to be presented.

notion that the operator is truth-functional at all, have all been questioned by quantum logicians, paraconsistentists, or intuitionists.<sup>2</sup>

Such disputes are notoriously difficult to tackle. When philosophers disagree on basic logical or metaphysical concepts like *identity*, *existence*, *necessity*, etc., or on the validity of such inferences as Minimal Contraposition, *reductio ad absurdum*, or Disjunctive Syllogism, discussions often face methodological impasses, or turn into hard clashes of intuitions. We cannot inspect such notions as *predication*, *negation*, etc., without resorting to them. It is therefore hard to decide when some party is starting to beg the question, or who carries the burden of proof. It is not easy to tell whether a non-standard explanation of a basic notion involves a real disagreement with a classical account of that notion, or rather whether its principles simply characterize something else under the same symbol. Do supervaluationism and non-adjunctive approaches, like non-truth-functional accounts of conjunction and disjunction, actually describe *conjunction* and *disjunction*?<sup>3</sup> Some consider these puzzles inscrutable.<sup>4</sup>

When Quine made his famous 'change-of-subject' point in *Philosophical Logic*, his target was precisely negation (and, despite not mentioning them, he had paraconsistent logics in his sights):

To turn to a popular extravaganza, what if someone were to reject the law of non-contradiction and so accept an occasional sentence and its negation as both true? An answer one hears is that this would vitiate science. Any conjunction of the form 'p . ~p' logically implies every sentence whatever;

<sup>2</sup> Even the syntactic type(s) to which negation belongs are controversial, witness the challenges of neo-Aristotelian logicians to viewing negation as a sentential functor (see Englebretsen 1981 and, for a rich overview of the history of predicative negation, Horn 1989). An enlightening review of the issue is Wansing 2001. Wansing claims that the neo-Aristotelians' arguments, rather than showing that sentential (as opposed to predicative) negation cannot account for some phenomena concerning natural language, call for a distinction between different sentential negations. I will not discuss the issue in this paper, which is explicitly focused only on sentential negation. Another issue not to be investigated here is the connection between negation and the pragmatic notion of rejection — on which Smiley 1996 is a classic reference. I have worked on this topic in Berto 2008.

<sup>3</sup> Tappenden (1993) and Varzi (2004) talk of the widespread use of the Argument from Italics: 'You claim that "Either A or B" holds, so *either* A *or* B (stamp the foot, bang the table!) must hold!'

<sup>4</sup> 'I take a dim view of the idea that revising our logic entails using so-called logical words with new meanings. Suppose that until now my mathematical proofs used non-constructive principles, but now I announce that I will restrict myself to constructively acceptable proofs. Have I revised my logic, while continuing to mean the same by "not" and "or" or have I decided to use those words with a different meaning? I don't perceive a fact of the matter here' (Resnik 2004, p. 180).

therefore acceptance of one sentence and its negation as true would commit us to accepting every sentence as true, and thus as forfeiting all distinction between true and false. ... My view of the dialogue is that neither party knows what he is talking about. They think that they are talking about negation, ' $\sim$ ', 'not'; but surely the notion ceased to be recognisable as negation when they took to regarding some conjunctions of the form ' $p \cdot \sim p$ ' as true, and stopped regarding such sentences as implying all others. Here, evidently, is the deviant logician's predicament: when he tries to deny the doctrine he only changes the subject. (Quine 1970, p. 81)

When someone says, 'For some A, both A and not-A can be true', many after Quine wonder what is meant by 'not':

The fact that a logical system tolerates  $A$  and  $\sim A$  is only significant if there is reason to think that the tilde means 'not'. Don't we say 'In Australia, the winter is in the summer', 'In Australia, people who stand upright have their heads pointing downwards', 'In Australia, mammals lay eggs', 'In Australia, swans are black'? If 'In Australia' can thus behave like 'not' ... , perhaps the tilde means 'In Australia'? (Smiley 1993, p. 17)<sup>5</sup>

We thus have a curiously split situation: because of their exposure to classical logic, analytic philosophers generally have a univocal, Stalnakerian view of negation as purely extensional Boolean complementation. On the other hand, specialists in non-classical logics, with the accompanying philosophical motivations, may end up either with a plethora of operators de facto called 'negation', tied together at most by family resemblances,<sup>6</sup> or with endless disputes over which among them do or do not deserve to be properly so called.

This work aims to fix this unfortunate situation. None of the technical results presented below is new: I draw on facts established by Kosta Došen (1986, 1999), Michael Dunn (1993, 1996, 1999), Ed Mares (1995), Greg Restall (1999), and Dimitri Vakarelov (1977, 1989). What is new, and thus comprises the main thesis of this paper, is the philosophical claim that a unified approach to negation as a modal operator vindicates a moderate logical pluralism: a view which develops a recent but already well-established pluralist account of logical consequence. This position steers a middle path between Stalnakerian univocism on negation and the fragmented picture emerging from the aforementioned disputes on non-classical logics. In short: not

<sup>5</sup> Peter van Inwagen once told me that the in-Australia-operator joke is to be credited to R. L. Sturch.

<sup>6</sup> For a defence of the view that there is no 'unique, real' negation, motivated by an exploration of the history of logic, see Dutilh Novaes 2007.

everything called negation in the logical literature deserves that name, but more than one item does.

Such a view is pursued by grounding the meaning of negation in a single (albeit twofold) core notion: the concept of *compatibility*, together with its polar opposite, *incompatibility* (I will henceforth use '(in)compatibility' as a shorthand for the twin notion). The features of (in)compatibility set precise constraints on what counts as a negation. The residual differences between the operators that pass the threshold depend on concepts different from the core notion. Specifically: if the meaning of negation consists in its being a *restricted* quantifier on *worlds*, then some candidate negations can be ruled out as violating plausible constraints on the relevant *restriction*, that is, on the relevant accessibility relation which, as we will see, encodes the core notion of (in)compatibility. However, different admissible precisifications of the notion of *world* support different logical conceptions of negation. To the extent that this is the case, pluralism is enforced.

Among the substantive views on negation vindicated by this uniform approach are: that something cannot be called a negation properly if it does not satisfy (Minimal) Contraposition and Double Negation Introduction; that the pair consisting of two so-called split or Galois negations encodes a distinction without a difference; that some paraconsistent negations also fail to qualify as real negations, but others may; that intuitionistic negation counts as a real negation, the *pièce de résistance* in the disagreement between classical logicians and intuitionists (Double Negation Elimination, Excluded Middle) depending on a different concept from the one encoded by the relevant worldly accessibility relation for negation; and that Boolean negation succeeds as well for the same reason, to the extent that constructivist and recent paraconsistent qualms about its acceptability do not turn on the basic idea of (in)compatibility.

## 2. A simple model

We start by introducing some basic formal machinery. We use a standard sentential language,  $L$ , having sentential letters:  $p, q, r, (p_1, p_2, \dots, p_n)$ ; the binary connectives  $\wedge$  and  $\vee$ , the unary connectives  $\Box$  and  $\neg$ , the 0-ary connectives  $\perp$  and  $\top$ ; and round brackets as auxiliary symbols. The sentential letters,  $\perp$ , and  $\top$  are atomic formulas.  $A, B, C, (A_1, \dots, A_n)$  are metavariables for formulas. If  $A$  is a formula, so

are,  $(A \wedge B)$ ,  $(A \vee B)$ ,  $\neg A$ ,  $\Box A$ ; outermost brackets are omitted in formulas.

A *frame* for  $L$  is a quadruple  $\mathfrak{F} = \langle W, R_P, R_N, \sqsubseteq \rangle$ , where  $W$  is a non-empty set;  $R_P, R_N \subseteq W \times W$ , and  $\sqsubseteq$  is a partial ordering on  $W$ . We use  $a, b, c, x, y, z, (x_1, \dots, x_n)$  in the metalanguage as variables ranging on items in  $W$ , as well as the set-theoretic notation and the symbols  $\forall, \exists, \Rightarrow, \Leftrightarrow, \&$ , or, with the usual reading. Intuitively,  $W$  is a set of worlds (I will not speak of *possible* worlds, for reasons that shall soon be clear);  $R_P$  and  $R_N$  are two accessibility relations on worlds: when  $\langle x, y \rangle \in R_P$  we write this as  $xR_P y$  and claim that world  $x$  *positively* accesses world  $y$ ; when  $\langle x, y \rangle \in R_N$  we write this as  $xR_N y$  and claim that world  $x$  *negatively* accesses world  $y$ . We will soon find another denomination for ' $xR_N y$ '; before this, we need to speak of  $\sqsubseteq$ .

This is to be thought of as an information ordering, similar in some respects to the one of the [Kripke 1965](#) semantics for intuitionistic logic, where worlds are interpreted as evidential-epistemic states of the idealized mathematician (Brouwer's 'creating subject'). More generally, ' $x \sqsubseteq y$ ' means here that world  $y$  retains at least all the information in world  $x$ .<sup>7</sup> That some ordinary possible world of standard modal semantics can properly include the information of another possible world does not make much sense, for such worlds are taken as maximal as far as information goes: the only way for  $y$  to retain at least all the information in  $x$  is for  $y$  to *be*  $x$ . However, [Barwise and Perry's \(1983\)](#) situation semantics has already taught us that talk of situations as partial states of reality makes sense and can bring many theoretical benefits. Now a non-trivial information ordering patently makes sense for such partial items: the situation consisting of my kitchen in Amsterdam does not carry information on the current weather in Rotterdam, whereas the current situation of the Netherlands as a whole does, and the latter properly includes the former as far as information goes.<sup>8</sup>

The logical pluralism proposed below is based on the view that classical possible worlds, intuitionistic-friendly epistemic constructions, and situations or information states, are different ways of providing an intuitive understanding of worlds as points in frames at

<sup>7</sup> The Kripke semantics makes of its  $\sqsubseteq$  the accessibility used in the semantics for intuitionistic negation. As we will see later, this can be recaptured in our comprehensive framework as a special case.

<sup>8</sup> Whether information-inclusion is a parthood relation, whether it is so not only for concrete situations as in our example, but also for abstract ones, and which mereological axioms may hold for it, are interesting questions that will not be tackled here.

which formulas can be true, and over which we quantify in our semantics. Such different ways of making the notion of world precise vindicate different formal features for the information ordering  $\sqsubseteq$  between worlds, which in turn allow for the different inferential behaviour of (the admitted) negations.

In possible worlds semantics it is customary to talk of the proposition expressed by a formula  $A$  as  $[A] \in \mathcal{P}(W)$ , the set of worlds at which the formula is true. When worlds can stand in non-trivial information-inclusion relations one had better take as the set of propositions in a frame  $\mathfrak{F}$  a particular set of sets of worlds,  $Prop(\mathfrak{F}) \subseteq \mathcal{P}(W)$ , namely those sets that are closed upwards with respect to  $\sqsubseteq$ :  $X \in Prop(\mathfrak{F})$  just in case  $x \in X$  &  $x \sqsubseteq y \Rightarrow y \in X$  (see Restall 2000, pp. 239–40).

A frame becomes a *model*  $\mathfrak{M} = \langle W, R_P, R_N, \sqsubseteq, \Vdash \rangle$  when it is endowed with an interpretation  $\Vdash$ , a relation between worlds and formulas: we write ‘ $x \Vdash A$ ’ to mean that  $A$  holds at  $x$  ( $x$  forces  $A$ , etc.), and ‘ $x \not\Vdash A$ ’ to mean that  $A$  fails to hold at  $x$ . We will only consider models including *admissible* interpretations: an interpretation is admissible if for each sentential letter  $p$  of  $L$ ,  $[p] = \{x \in W : x \Vdash p\} \in Prop(\mathfrak{F})$  and it satisfies what is often called the *Heredity Constraint* for atomic formulas (see Dunn 1993 and 1996; Priest 2001, p. 105). For all  $x, y \in W$ :

$$(HC) \quad x \Vdash p \text{ \& } x \sqsubseteq y \Rightarrow y \Vdash p$$

The HC makes a lot of sense given the reading of  $\sqsubseteq$  as information-inclusion. If  $y$  retains all the information in  $x$ , then everything holding at  $x$  should be preserved as holding at  $y$ . In order for the HC to be extended to all formulas, we begin by stating the semantic clauses for our connectives. For all  $x \in W$ :

$$(S\wedge) \quad x \Vdash A \wedge B \Leftrightarrow x \Vdash A \text{ \& } x \Vdash B$$

$$(S\vee) \quad x \Vdash A \vee B \Leftrightarrow x \Vdash A \text{ or } x \Vdash B$$

$$(ST) \quad x \Vdash T$$

$$(S\perp) \quad x \not\Vdash \perp$$

$$(S\Box) \quad x \Vdash \Box A \Leftrightarrow \forall y(xR_P y \Rightarrow y \Vdash A)$$

$$(S\neg) \quad x \Vdash \neg A \Leftrightarrow \forall y(xR_N y \Rightarrow y \not\Vdash A)$$

The box will not do much work in the following; the main role of  $(S\Box)$  (as in Došen 1999) is to highlight the intuitive connections

between  $R_N$  and the familiar accessibility of positive modal operators  $R_P$ . In order for  $\sqsubseteq$  to interact properly with the two accessibilities we need the following two conditions, whose intuitive meaning will also be clarified soon. For all  $x, y, x_1, y_1 \in W$ :

$$\text{(Forwards)} \quad xR_P y \ \& \ x_1 \sqsubseteq x \ \& \ y \sqsubseteq y_1 \Rightarrow x_1 R_P y_1$$

$$\text{(Backwards)} \quad xR_N y \ \& \ x_1 \sqsubseteq x \ \& \ y_1 \sqsubseteq y \Rightarrow x_1 R_N y_1$$

Given Forwards and Backwards,<sup>9</sup> an easy induction on the construction of formulas shows that the Heredity Constraint generalizes: for each formula  $A$  of  $L$  and all  $x, y \in W$ ,  $x \Vdash A \ \& \ x \sqsubseteq y \Rightarrow y \Vdash A$ . Also, for each formula  $A$  of  $L$ ,  $[A] = \{x \in W : x \Vdash A\} \in \text{Prop}(\mathfrak{F})$ . Finally, we define logical consequence in a frame  $\mathfrak{F}$  as truth preservation at all worlds  $x$  in  $\mathfrak{F}$  in all admissible interpretations, that is, in all the relevant models based on the frame. Given a set of formulas  $\Sigma$ :

$$\Sigma \models B \Leftrightarrow \text{For all models } \mathfrak{M} \text{ on } \mathfrak{F} (x \Vdash A \text{ for all } A \in \Sigma \Rightarrow x \Vdash B)$$

For single-premiss entailments, we write  $A \models B$  for  $\{A\} \models B$ .

Now that we have the simple machinery in place, we can move on to its philosophical import: (in)compatibility, being the ground of negation, provides the relevant accessibility  $R_N$  with intuitive meaning. The proposal does not come out of the blue, both on the philosophical and on the formal side. Let us start with the former.

### 3. (In)compatibility and pluralism

On the philosophical side, explaining negation in terms of compatibility and incompatibility is promising. If a good explication of a notion consists in grounding it on some other more fundamental notion(s), it is easily seen why such basic concepts as negation are difficult to deal with: How can we dig deeper than that? On the other hand, following, among others, Dunn (1993, 1996), I take (in)compatibility as the *primitive* twofold notion grounding the origins of our concept of negation and of our usage of the natural language expression ‘not’. Explanations stop when we reach concepts that cannot be defined in terms of other concepts, but only illustrated by way of example. A good choice of primitives resorts to notions we have a

<sup>9</sup> Forwards is just a familiar condition on positive modalities in modal logics, as we will see. One can find Backwards in various papers on negation as a modal operator, such as Dunn 1993, 1996, and Dunn and Zhou 2005.

good intuitive grip of—and this is the case, I submit, with (in)compatibility.

It is difficult to think of a more pervasive and basic feature of experience, than that some things in the world *rule out* some other things; or that the obtaining of this *precludes* the obtaining of that; or that something's being such-and-such *excludes* its being so-and-so. Not only rational epistemic agents and speakers of natural languages, but also animals, or sentient creatures generally, are acquainted with (in)compatibility. On the cognitive side, incompatibility shows itself in the most basic ability a new-born can acquire: that of distinguishing objects, recognizing a difference between something and something else. On the practical side, we face choices between doing this and that, and to face a choice is to experience an incompatibility: we cannot have it both ways, and this holds as well for the Stoics' dog who has to choose between going down one path or the other in following a prey. If the awareness of incompatibility is more primitive than the use of any negation, the primary purpose of uttering a 'not' in the history of the world must have been that of recording some perceived incompatibility and of manifesting it to others, from the most primitive animal verse signalling 'no predators there' onwards.

That the core role of negation in the vernacular is to signal incompatibility is hinted at in Plato's passage from the *Sophist*, constituting the epigraph for this work. Plato appears to claim that to say of something that it is not such-and-such is to assert that it is different from being such-and-such, 'difference' meaning here some incompatibility: the being so-and-so of the thing rules out its being such-and-such, which is different from and incompatible with being so-and-so.<sup>10</sup>

In his paper 'Why "Not"?', Huw Price (1990) also grounded the origins of negation in its social and psychological function as an exclusion-expressing device. Here is his hypothetical conversation between you and me in a language that lacks such a device. You are trying to rule out the possibility of Fred's simultaneously being in the kitchen and in the garden:

Me: 'Fred is in the kitchen.' (Sets off for kitchen.)

You: 'Wait! Fred is in the garden.'

<sup>10</sup> Russell also deals with the connection between negation and incompatibility—in the third Lecture of *The Philosophy of Logical Atomism*, in his critical discussion of views on negation advanced by Demos. Thanks to the editor of this journal for the reference.

Me: 'I see. But he is in the kitchen, so I'll go there.' (Sets off.)

You: 'You lack understanding. The kitchen is Fred-free'.

Me: 'Is it really? But Fred's in it, and that's the important thing.'  
(Leaves for kitchen). (Price 1990, p. 224)

What you would need to say is that Fred is somewhere else—in the garden—and his being there is incompatible with his being in the kitchen—that is, Fred is *not* in the kitchen.

Price also claims—and I agree—that, in foundational debates on the meaning of negation, considerations of (in)compatibility ought to be privileged over considerations concerning the behaviour of negation with respect to truth values:<sup>11</sup> for the latter may prejudice too many issues on the basis of antecedent views on truth. It is a common though not uncontroversial thought, for instance, that anything worth calling a negation ought to flip-flop values between truth and falsity.<sup>12</sup> As this would rule out intuitionistic negation, some insist on negation being a contrary-forming operator. However, as Price highlights, we should refrain from expressing incompatibility via the traditional concept of *contrariness*, as this notion is itself usually phrased by relying on controversial insights on truth and falsity. Defining A and B as contraries if and only if  $A \wedge B$  is logically false 'clearly depends on our knowing that truth and falsity are incompatible' (something a strong paraconsistentist may want to deny, for instance, in the light of the Liar). And 'if we do not have a sense of *that*, the truth tables for negation give us no sense of the connection between negation and incompatibility' (Price 1990, p. 226). Similarly, according to Mark Sainsbury, grasp of incompatibility is more vital than grasp of truth-value machinations for our mastery of negation:

Understanding negation involves a sensitivity to incompatibility, but this notion does not have to be specified [by direct reference to truth and falsity]. For instance, one might suggest that the basic notion of incompatibility in directly semantic terms consists in the fact that incompatible sentences must have opposite truth values, which makes true contradictions conjunctions of incompatibles. However, one might prefer to avoid an account of understanding which involved attributing such semantic notions to speakers, for example on the grounds that the

<sup>11</sup> Thanks to an anonymous referee for pressing me on this point.

<sup>12</sup> This thought is not uncontroversial even among logicians belonging to the same broad family: paraconsistent relevant logicians make a lot of it, but other paraconsistentists in the Brazilian tradition disagree (see Carnielli and Coniglio 2013).

account would not be neutral with respect to realist and intuitionist preconceptions. (Sainsbury 1997, p. 224)

What kinds of things can be compatible or incompatible, that is, can stand in (in)compatibility relations? Different items plausibly qualify: concepts, properties, states of affairs, events, propositions. This is good. It makes (in)compatibility a stable notion across different ontological categories, providing evidence for its being one that carves nature at its joints. Incompatibility can be taken as holding between a pair of properties  $P_1$  and  $P_2$  such that whatever has  $P_1$  dismisses any chance of simultaneously having  $P_2$ , for instance. Or one may take it as relating two states of affairs  $s_1$  and  $s_2$ , just in case the occurring of  $s_1$  (in world  $x$ , at time  $t$ , etc.) precludes the possibility that  $s_2$  also occurs (in world  $x$ , at time  $t$ ).<sup>13</sup> We can go from garden incompatibilities, such as my car's being red all over and its being black all over, to refined scientific exclusions, such as incompatible spins or colour charges for quarks, to mathematical ones, such as an algorithm having polynomial vs. exponential complexity. I have elsewhere (Berto 2008, 2014) dubbed such a general metaphysical conception of (in)compatibility as *material*, to highlight that a characterization of negation as based on it is not merely formally, in the sense of logically (truth-conditionally or inferentially) characterized. (In)compatibility is based on the material content of the relevant concepts, properties, etc. And the inferential-logical properties of negation are to *flow* naturally from the intuitive features of the metaphysical (in)compatibility relation grounding it.

Such an approach, as we will see, places constraints on what counts as an admissible negation. But it leaves room for more than one logical characterization of the operator, thus for pluralism. The view is a development of the logical pluralism defended by Jc Beall and Greg Restall in some well-known works (Beall and Restall 2000, 2001, 2006).

<sup>13</sup> Given the semantics above, it is natural to phrase (in)compatibility as a relation between worlds. How to connect this to incompatibilities concerning other metaphysical categories depends on your favourite account of worlds, which, as far as the formalism goes, are just taken as unstructured points in frames at which formulas are evaluated. If (in)compatibility holds first of all between propositions or states of affairs, and your favourite view of worlds takes them as sets of propositions or as inclusive states of affairs of a certain kind, then a certain (in)compatibility between propositions or states induces a corresponding (in)compatibility between worlds including or encompassing them. But other settings are possible, and which one we choose is immaterial for the following.

The Beall-Restall pluralism on logical consequence consists in the following claims:<sup>14</sup>

- (1) The settled core of logical consequence is given by the ‘Generalized Tarski Thesis’ (GTT): *An argument is valid iff, in every world in which the premises are true, so is the conclusion.*
- (2) But GTT is schematic, for ‘world’ is ambiguous there: an instance of GTT comes from disambiguation.
- (3) *Admissible* specifications of GTT satisfy the settled aspects of the notion of consequence (Beall and Restall have: necessity, normativity, and formality).
- (4) A theory of consequence is given by an admissible instance of GTT.
- (5) There is more than one admissible instance of GTT.

Our pluralism on negation consists in the following claims:

- (1) The settled core of negation is given by the clause  $(S\neg)$ :  $\neg A$  is true in a world  $w$  iff, in all worlds compatible with  $w$ ,  $A$  fails to be true.
- (2) But  $(S\neg)$  is schematic, for ‘world’ is ambiguous there: an instance of  $(S\neg)$  comes from disambiguation.
- (3) *Admissible* specifications of  $(S\neg)$  satisfy the settled aspects of the notion of negation: the constraints on (in)compatibility (to be explored below).
- (4) A theory of negation is given by an admissible instance of  $(S\neg)$ .
- (5) There is more than one admissible instance of  $(S\neg)$ .

The Beall-Restall pluralism is based on the view that we can understand worlds (at least) in a classical sense (say, as maximal ways things may be), or in an intuitionistic-friendly sense (say, as constructions), or in a paraconsistent-friendly sense (say, as situations or information states). We can then have, correspondingly, different precisifications

<sup>14</sup> Compare Beall and Restall 2006, p. 35. I have rephrased their formulation for the sake of consistency with my terminology. They adopt the aseptic term ‘case’ whereas I stick to the more traditional ‘world’. But we both mean (at least) *thing at which claims can be true* by those terms.

of the notion of logical consequence coming from the schematic GTT. The negation pluralism to be developed in the following adds that those same specifications of the notion of world (as maximal ways, constructions, information states) mandate different constraints on the relations between worlds, the compatibility relation  $R_N$ , and the information-inclusion relation  $\sqsubseteq$ . And the variation in the properties of these accounts for the different logical behaviour of various negations. The Beall-Restall pluralism on logical consequence and the negation pluralism proposed here, then, are naturally connected, as flowing from a plurality of admissible ways of understanding the notion of *world*.<sup>15</sup>

Before we move on to a detailed presentation, one final, general remark is worth making on the kind of pluralism at issue. The semantics proposed—and, in particular,  $(S\neg)$ —is not intended as schematic also because of different possible ways of making precise the notion *being true at*,  $\Vdash$ . Or, more tersely: the pluralism at issue here is not truth pluralism. For our purposes, what changes depending on how *world* is specified is the way the negation operator behaves, not what it means for a formula or sentence (which may or may not involve negation) to be true at a world. The unsettledness in the picture ends with the notion of world.

#### 4. Entailment U-turn

How does this view of (in)compatibility connect to the semantics proposed above, and specifically to  $(S\neg)$ ? The formal account of negation-as-(in)compatibility closest to ours is one initially introduced in quantum logic: the Birkoff-von Neumann-Goldblatt notion of *ortho negation*. Goldblatt’s semantics for quantum logic was also based on frames constituted by indices and relations on them, but the indices were narrowly interpreted as outcomes of possible experimental measurements, of the kind performed by quantum physicists. One of these relations was incompatibility (often called in the context ‘perp’, or ‘orthogonality’) between indices, capturing the idea of two outcomes precluding one another (see Birkoff and von Neumann 1936, Goldblatt 1974).

Now, Michael Dunn (1996) has claimed that ‘one can define negation in terms of one primitive relation of incompatibility... in a metaphysical framework’ (p. 9) by resorting to ortho-negation-inspired frames. The clause for negation exploited by Dunn, rephrased

<sup>15</sup> Thanks to an anonymous referee for pressing me to make such a connection explicit.

in terms of our semantics above, goes thus (where ‘ $\bullet$ ’ stands for the perp relation):

$$(S_{\text{Perp}}) \quad x \Vdash \neg A \Leftrightarrow \forall y (y \Vdash A \Rightarrow x \bullet y)$$

$\neg A$  holds at a world just in case any world at which  $A$  holds is incompatible with it. Our clause  $(S_{\neg})$  above is obtained by contraposing the right-hand side of  $(S_{\text{Perp}})$  — looking at it again:

$$(S_{\neg}) \quad x \Vdash \neg A \Leftrightarrow \forall y (x R_N y \Rightarrow y \not\vdash A)$$

Our  $R_N$  restricting the quantification on worlds expressed by negation is thus naturally read precisely as compatibility: world  $x$  sees world  $y$  via  $R_N$  just in case  $x$  is compatible with  $y$ ; nothing obtaining at  $x$  precludes anything obtaining at  $y$ ; or, no information ruled out by  $x$  is supported by  $y$ ; or, no proposition in  $x$  is incompatible with any proposition in  $y$ ; etc.

The Forwards and Backwards clauses above make a lot of sense in this context. If  $x R_P y$ , that is, a world  $x$  (positively) modally accesses a world  $y$ , then everything necessary at  $x$  must be true at  $y$  (this is just  $(S_{\Box})$ , the old clause for necessity). But then if  $x_1 \sqsubseteq x$ , everything necessary at  $x$  must already be such at  $x_1$ , for the former preserves any information holding at the latter. And if  $y \sqsubseteq y_1$ , then everything true at  $y$  must be true at  $y_1$  for the same reason. Then everything necessary at  $x_1$  must be true at  $y_1$ , thus  $x_1 R_P y_1$ .

Reciprocally, if  $x R_N y$ , that is, world  $x$  is compatible with world  $y$ , then as mandated by  $(S_{\neg})$  nothing ruled out by  $x$  holds at  $y$ . But then if  $x_1 \sqsubseteq x$ , anything ruled out by  $x$  must already have been ruled out by  $x_1$ , for the former preserves any information holding at the latter. And if  $y_1 \sqsubseteq y$ , then anything ruled out by  $y$  must already have been ruled out by  $y_1$  for the same reason. Then  $x_1$  must be compatible with  $y_1$ ,  $x_1 R_N y_1$ : sub-worlds of compatible worlds must themselves be compatible.<sup>16</sup>

Before we start to single out the logical properties of negation on the basis of the features associated with  $R_N$  interpreted as compatibility, I should mention an approach to negation similar to the one pursued here, and due to David Ripley (MS).<sup>17</sup> Ripley also proposes a general framework to capture the features of negation as a modal operator. But instead of using a binary accessibility relation

<sup>16</sup> See Restall 2000, pp. 240–1, for an outline of this reasoning in the general setting of the frame semantics for substructural logics.

<sup>17</sup> I am grateful to an anonymous referee for pointing me to this work.

representing (in)compatibility, he resorts to neighbourhood semantics:<sup>18</sup> Ripley's frames embed a function,  $N$ , mapping each world  $w$  to a set  $N(w)$  of subsets of  $W$ , which are its neighbourhoods. A modalized formula, say  $\$A$ , is true at world  $w$  iff  $[A] \in N(w)$ .

Our accessibility frames correspond to a special class of neighbourhood frames. I claim that the approach using accessibility may have a philosophical advantage for our purposes: it makes more straightforwardly evident how the features of (in)compatibility make for the core of negation. In Ripley's models, to understand the neighbourhood function  $N$  as having to do with negation one must gloss ' $[A] \in N(w)$ ' as meaning that the proposition expressed by  $A$  is incompatible with  $w$ . Or, one can turn tables around and claim that  $\$A$ , with  $\$$  understood as a negative modal, holds at  $w$  just in case  $[A] \notin N(w)$ , and then gloss  $N(w)$  as picking out the set of compatible worlds. On the other hand, in the binary accessibility framework the compatibility relation between worldly items is represented in the semantics directly by  $R_N$ , which makes it more straightforward to reason, as we are about to do, about the features of the relation itself (to wonder, for instance, whether (in)compatibility is symmetric, etc.).

The analogy between  $R_N$  and the familiar accessibility of positive modals,  $R_p$ , helps a lot. Here is an example. The minimal essential feature of a normal positive modal operator  $m$  consists in its preserving entailment forward (as highlighted again in Restall 2000, Ch. 3):

$$(EF) \quad A \vDash B \Rightarrow mA \vDash mB$$

*Proof for  $m = \Box$ :* Assume  $A \vDash B$ , that is, all the  $A$ -worlds are  $B$ -worlds, and that  $x \Vdash \Box A$ . Then by (S  $\Box$ ), for all  $y$  such that  $xR_p y$ ,  $y \Vdash A$ . But then  $y$  must be a  $B$ -world,  $y \Vdash B$ ; thus by (S  $\Box$ ) again,  $x \Vdash \Box B$ . EF is of course variously recorded in normal modal logics, for instance, as the basic K axiom, under the view that necessary truths have necessary consequences.

But then, reciprocally, the minimal essential feature of a negative modal operator  $n$  must consist in its U-turning entailment backwards. To be a negative modality means to satisfy the Entailment U-Turn principle:

$$(EU) \quad A \vDash B \Rightarrow nB \vDash nA$$

<sup>18</sup> For a classic presentation of neighbourhood semantics, see the minimal models of Chellas 1980, Ch. 7.

*Proof for  $n = \neg$ :* assume  $A \models B$ , that is, all the  $A$ -worlds are  $B$ -worlds, and that  $x \Vdash \neg B$ . Then by  $(S\neg)$ , for all  $y$  such that  $xR_N y$ ,  $y \not\models B$ . But then  $y$  cannot be an  $A$ -world,  $y \not\models A$ ; thus by  $(S\neg)$  again,  $x \Vdash \neg A$ .

Then nothing can be a negation unless it satisfies Minimal Contraposition: if  $A$  entails  $B$ , then  $\neg B$  has to entail  $\neg A$ .<sup>19</sup> In a paper discussing the dispensable and indispensable inferential features of negation, Wolfgang Lenzen (1996) has listed Minimal Contraposition as the first indispensable. And I agree, for such a claim follows straightforwardly from our natural understanding of (in)compatibility.<sup>20</sup> The few objections to it in the literature tend to fall apart when thought through. Da Costa and Wolf (1980) have rejected Contraposition on the ground that, if accepted, it would lead to a collapse of the paraconsistent logical systems of the so-called Brazilian approach, such as the various Logics of Formal Inconsistency (see Carnielli and Marcos 2002), and specifically of the da Costa hierarchies (see da Costa 1974, 1977). Paraconsistent logics reject *ex contradictione quodlibet*, (ECQ), the view that  $\{A, \neg A\}$  entails an arbitrary  $B$ . But the Brazilian logics have the Weakening axiom  $A \rightarrow (B \rightarrow A)$ , which, coupled with a Contraposition axiom  $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ , would deliver  $A \rightarrow (\neg A \rightarrow \neg B)$ , which is close enough to ECQ.<sup>21</sup>

The situation as such may rather invite us to reject Weakening. And this happens in the most developed paraconsistent logics, namely relevant logics. The rejection is independently motivated, for Weakening

<sup>19</sup> For comparison, in Ripley's (MS) neighbourhood approach Contraposition follows from the assumption that the neighbourhood set  $N(w)$  be closed under subsets for each  $w$ :  $Z \in N(w)$  and  $K \subseteq Z$  imply  $K \in N(w)$ .

<sup>20</sup> EU is confirmed also when negation is characterized as 'arrow falsum'—as the entailment of falsity,  $\neg A =_{df} A \rightarrow f$ , as often happens in subclassical logics. From  $A \models B$  and the triviality  $B \rightarrow f \models B \rightarrow f$ , it follows by modus ponens that  $\{B \rightarrow f, A\} \models f$ . Then by Conditional Proof,  $B \rightarrow f \models A \rightarrow f$ . Incidentally: I claim neutrality on the issue whether  $f = \perp$ , that is, whether a *falsum* constant should automatically be identified with what is true at no world whatsoever. Such an equation is not taken for granted in substructural and relevant logic contexts, where  $\perp$  is often called *trivial falsity*. In such contexts, frames typically include so-called non-normal or impossible worlds, taken as worlds where logical truths may fail. Let  $t$  stand for the conjunction of all logical truths; then  $t$  should be distinct from (trivial) truth,  $\top$ , for the latter holds anywhere whereas the former does not. Dually, if  $f$  is the disjunction of all logical falsities,  $\perp$  may be distinct from it.

<sup>21</sup> This is why minimal logic, despite being technically paraconsistent in that it lacks ECQ (as an axiom:  $A \rightarrow (\neg A \rightarrow B)$ ), is not considered *interestingly* paraconsistent because of its including both Weakening and Minimal Contraposition. That from a contradiction follows an entire class of merely syntactically individuated formulas, namely all negations, already goes against the spirit of paraconsistency.

is a paradigm case of irrelevant inference or *non sequitur*: Why would A's holding imply that A is entailed by any unrelated B? One had better claim that the da Costa paraconsistent negation is not a real negation on the ground of its failing to satisfy Contraposition. Later on, we will see further reasons for doubting that the da Costa negation is a real negation.

Another negation in the neighbourhood of paraconsistency that fails to satisfy Contraposition is the one of Nelson's system N4. Typical semantics for Nelson's constructible negation also have worlds taken as information states, partially ordered by a non-trivial information-inclusion relation, and with a form of twofold Heredity Constraint involving both the preservation of truth and that of falsity, as the two are treated even-handedly (see [Wansing 2001](#), pp. 421–7). This separate treatment of truth and falsity explains how Contraposition can fail in the setting:<sup>22</sup> if for every Nelson model and every world  $w$  in it,  $w$  supports the truth of B if it support the truth of A, then there seems to be no reason to assume that for every model and world  $w$  it holds that  $w$  supports the falsity of A if it supports that of B.

However, firstly, there are reasons for privileging an approach to negation in terms of (in)compatibility over one motivated by considerations concerning truth and falsity (in particular considerations to the effect that truth and falsity get *separate* treatments) in foundational debates on the meaning of negation, as we have seen. Secondly, not even all the versions of Nelson negation are ruled out by our approach: [Nelson \(1959\)](#) develops a variant of his system N3 with a strong constructible negation, which does satisfy Contraposition as a rule, and thus complies with the constraint proposed here. This strong N3 negation is non-paraconsistent (it satisfies ECQ), and entails intuitionistic negation.

A different argument against Minimal Contraposition may come from the consideration that it fails for *ceteris paribus* conditionals. Consider [Lewis's \(1973\)](#) famous example:

If Boris had gone to the party, Olga would still have gone  
If Olga had not gone, Boris would still not have gone

This fails in a situation where Olga, who loves Boris without being loved by him in return, would have attended the party even more

<sup>22</sup> As an anonymous referee appropriately pointed out.

gladly had Boris been around; but Boris, despite wanting to go to the party, in the end renounced it just to avoid her.

I reply that one ought to put the blame for such failures entirely on the specific conditional, not on negation. Notoriously, a counterfactual  $\mapsto$  can provide counterexamples to Transitivity (from  $A \mapsto B$  and  $B \mapsto C$  to  $A \mapsto C$ ) and Antecedent Strengthening (from  $A \mapsto B$  to  $A \& C \mapsto B$ ), but one would hardly think that these failures carry over to standard (monotonic) entailment. These failures do not involve negation, which therefore should not take responsibility for the nonmonotonic features of some forms of conditionality.

## 5. Symmetry

The next issue on the agenda: Is (in)compatibility symmetric? If not, we can conjecture the following semantic clause to define a negative modalizer different from  $\neg$ :

$$(S\sim) \quad x \Vdash \sim A \Leftrightarrow \forall y (y R_N x \Rightarrow y \nVdash A)$$

$\sim A$  holds at a world just in case  $A$  fails to hold at all worlds compatible with *it*. This may not be the same as failing to hold at all worlds with which *it* is compatible. The analogy with the positive modal accessibility  $R_p$  prima facie leaves room for such a distinction: for various conceptions of positive modality, my seeing a world does not perforce entail the world's seeing me, and when this happens the Brouwerische axiom ( $A \rightarrow \Box \Diamond A$ ) fails in normal modal logic. This second negation satisfies EU, thus it passes the first test for real negationship: if all the  $A$ -worlds are  $B$ -worlds, and  $x \Vdash \sim B$ , by  $(S\sim)$  for all  $y$  such that  $y R_N x$ ,  $y \nVdash B$ ; then  $y$  cannot be an  $A$ -world, thus by  $(S\sim)$  again,  $x \Vdash \sim A$ . The pair consisting of the two negations  $\langle \neg, \sim \rangle$  has been named *split negation* by [Chrysafis Hartonas \(1997\)](#) and *Galois negation* by [Michael Dunn \(1996\)](#), on the ground that it mirrors a couple of Galois-connected functions when the two operators are linked by the following two-way Galois Connection:

$$(GC) \quad A \models \neg B \Leftrightarrow B \models \sim A$$

Given the trivialities  $\neg A \models \neg A$  and  $\sim A \models \sim A$ , GC gives two Double Negation Introduction (DNI)-like principles:

$$(DNI1) A \Vdash \neg \sim A$$

$$(DNI2) A \Vdash \sim \neg A$$

Galois-split negations are an interesting subject of formal investigation. But one may wonder whether they encode a distinction without a substantive difference. If (in)compatibility is symmetric, they collapse into one. *Proof:* suppose Symmetry, that is,  $xR_Ny \Rightarrow yR_Nx$ , and that  $x \Vdash \neg A$ . Then by (S $\neg$ ), for any  $y$  such that  $xR_Ny$ ,  $y \not\Vdash A$ ; then for any  $y$  such that  $yR_Nx$ ,  $y \not\Vdash A$ , thus by (S $\sim$ )  $x \Vdash \sim A$  as well. So  $x \Vdash \neg A \Rightarrow x \Vdash \sim A$ . Swap ' $\neg$ ' with ' $\sim$ ' and you get the converse. Since  $x$  was arbitrary,  $\sim A$  is equivalent to  $\neg A$ .

If  $\neg = \sim$  (so we can go back to our initial ' $\neg$ ' for both), (DNI1) and (DNI2) also collapse into the familiar Double Negation Introduction Rule of minimal logic:

$$(DNI) A \Vdash \neg\neg A$$

We actually have a correspondence result due, as far as I know, to Restall (2000, pp. 263–4): DNI holds just in case compatibility is symmetric. For assume it is not. Then there is some frame with two worlds,  $a$  and  $b$ , such that  $aR_Nb$ , but it is not the case that  $bR_Na$ . Take an interpretation  $\Vdash$  such that  $x \Vdash p$  just in case  $a \sqsubseteq x$ , that is,  $p$  is only true at anything including the information in  $a$ . Now take a  $y$  such that  $bR_Ny$ ; we cannot have that  $y \Vdash p$ , for then we would have  $a \sqsubseteq y$ , thus  $bR_Na$ , against the hypothesis. Since  $y$  was arbitrary, for all  $y$  such that  $bR_Ny$ ,  $y \not\Vdash p$ , thus by (S $\neg$ )  $b \Vdash \neg p$ . Since  $aR_Nb$ ,  $a \not\Vdash \neg\neg p$ , so  $p \not\equiv \neg\neg p$ .

Now (in)compatibility must be symmetric: whatever ontological kinds  $a$  and  $b$  belong to, it appears that if  $a$  rules out  $b$ , then  $b$  has to rule out  $a$ ; that if  $a$ 's obtaining is incompatible with  $b$ 's obtaining, then  $b$ 's obtaining must also be incompatible with  $a$ 's obtaining; etc. Alleged counterexamples, it seems to me, rely on equivocation, usually importing some asymmetry from causal relations or the temporal ordering of actions and processes. An example by Hartonas and Dunn (see Dunn 1999): the situation consisting of my son playing his saxophone prevents my reading a technical paper; but my reading a technical paper does not prevent my son's playing his saxophone.

I reply that the two situations still count as symmetrically incompatible with each other: for if my son's playing the sax entails that I get too distracted to read the paper, then by the uncontroversial Minimal Contraposition my not being too distracted to read the paper entails that my son cannot be playing the sax. The seeming counterexample builds upon two factors, namely the intuitive asymmetry of causal relations and there being two different processes at issue: one is my little

sax player's noise actively making it impossible for me to focus, the other is my being focused entailing the absence of distracting factors. Only the former is properly characterized via the expression 'preventing'. This does not change the fact that the obtaining of one situation is incompatible with the obtaining of the other and vice versa. Considerations involving asymmetrical causal relations should not sneak into the purity of our intuitions on the symmetry of (in)compatibility.

Our guiding principle that the features of (in)compatibility, that is, the accessibility relation that captures the core meaning of negation, should dictate the inferential behaviour of the connective, therefore, leads us to take DNI as another indispensable principle. This gives further evidence for the claim that the da Costa negation, which fails DNI, ought to be ruled out as not being a real negation; and the same holds for other paraconsistent negations, such as the one encoded by the Belgian adaptive logic CLuN, which basically follows the da Costa approach.<sup>23</sup>

The inferential features established so far for  $\neg$ , together with the standardly behaving conjunction and disjunction of L, give us the three intuitionistically acceptable De Morgan entailments (I will not go through the proofs here):

$$\begin{aligned}\neg A \wedge \neg B &\vDash \neg(A \vee B) \\ \neg(A \vee B) &\vDash \neg A \wedge \neg B \\ \neg A \vee \neg B &\vDash \neg(A \wedge B)\end{aligned}$$

## 6. Reflexivity

Is compatibility reflexive? Can there be self-incompatible worlds? By accepting that all  $x$  see themselves via  $R_N$  we get the following form of *ex contradictione quodlibet*:

$$(ECQ) \quad A \wedge \neg A \vDash \perp$$

<sup>23</sup> See Batens 1980, 1989, Batens and De Clercq 2004. There are further reasons for dissatisfaction with Brazilian-like negations, which I relegate to this footnote because they are not immediately connected to the features of (in)compatibility. Such negations (say, ' $\neg$ ') have a non-truth-functional semantics: when  $A$  is false,  $\neg A$  is true (which validates Excluded Middle), but when  $\neg A$  is true,  $A$  may be true as well as false. One cannot build a Lindenbaum algebra of the da Costa systems, for they fail replacement of equivalent formulas. The other semantic clause for such negations has it that when  $\neg A$  is true,  $A$  also is, which validates Double Negation Elimination. As we shall see, though, DNE is not enforced by the symmetry of (in)compatibility.

*Proof:* Assume that for any  $x$ ,  $xR_Nx$ . Now if  $x \Vdash A$  then  $x \not\Vdash \neg A$ , thus for no  $x$  can we have  $x \Vdash A \wedge \neg A$ . By (S $\perp$ ) we know that (the trivial) *falsum* holds nowhere, so we have  $\perp \Vdash B$  for any  $B$ , hence we get the other version of ECQ by the transitivity of entailment:  $A \wedge \neg A \Vdash B$ .<sup>24</sup> We in fact have another stronger correspondence result, due to [Dunn and Zhou \(2005\)](#): ECQ holds just in case compatibility is reflexive.<sup>25</sup>

The view that worlds must be self-compatible is sufficient to give full-fledged intuitionistic negation. And this seems another strong intuition: What can it mean for a world, or for whatever other kind of entity that can stand in (in)compatibility relations, to be self-incompatible, if not that it just undermines itself, that is, it rules itself out from possible existence? A way a self-undermining world is, is a way the actual world cannot be. Such a view of self-compatibility as a most general, and, in this sense, metaphysical feature of the world, is in sight in the traditional Aristotelian formulations of the Law of Non-Contradiction (LNC) in Book Gamma of the *Metaphysics*:

For the same thing to hold good and not hold good simultaneously of the same thing and in the same respect is impossible. (Arst. *Met.* 1005b 18–21)

' $P$  does *not* hold good of thing  $t$ ' means: 'For some  $Q$ ,  $t$  is  $Q$ , which is a state incompatible with  $t$ 's being  $P$ '. In Greek, 'the impossible' (*adynamaton*) is that which has no *dynamis*, that is: no chance, no power to be. The ultimate ground of the LNC according to Aristotle is metaphysical, not logical: it is based on material incompatibility between things in the world. Exegetically, this would explain why, whereas we find the LNC formulated also in Aristotle's *Organon*, that is, in his works on logic, only in the *Metaphysics* does he provide a justification for the Law. Aristotle also claims such a justification to pertain only to the science of 'being *qua* being'. So while Dummett called for a logical basis for metaphysics, Aristotle seems to have followed the converse path.

But this strong intuition has been strongly countered by paraconsistent logicians. The literature on this issue is burgeoning (see the

<sup>24</sup> Correspondingly, in a Ripley-style neighbourhood approach with the neighbourhood function  $N$  glossed as compatibility one gets ECQ by stipulating that for every set of worlds  $Z$ , if  $w \in Z$ , then  $Z \in N(w)$ .

<sup>25</sup> The Dunn-Zhou results are inspired by [Shramko 2005](#). These works explore a recent development of the theory of negation as a modal operator, namely the dual of Dunn's approach to negation as compatibility referred to above, where negation is taken as 'unnecessity'.

essays gathered in Beall et al. 2004), and I will limit myself to a few remarks. First, strong paraconsistentists like Graham Priest, also called *dialetheists*, believe that the *actual* world is self-incompatible: self-incompatibility does not preclude actual, therefore possible, existence. According to Priest, some contradictions are true *simpliciter*, paradigmatic cases being provided by the various Liar sentences, but also by other phenomena such as the metaphysics of change and becoming and the paradoxes of set theory with an unrestricted Comprehension Principle (see Priest 1987).

However, one does not need to believe that self-incompatible circumstances *can* obtain to be a paraconsistentist. The intuition regarding the self-compatibility of all worlds turns on the interpretation of the worlds themselves. And here the logical pluralism introduced in section 3 comes into play. The key insight in the above outline was that, just as the GTT at the core of the Beall-Restall logical pluralism underdetermines logical consequence, so the semantic clause for negation,  $(S\neg)$ , underdetermines the behaviour of negation. In particular, the worlds the clause refers to (via quantification), minimally characterized as points at which formulas can be true, can be further specified in different ways. Beall and Restall vindicate a pluralism of logical consequence relations by fine-tuning the notion of world (or, as they call it, *case*) in different ways. But the same holds for our negation pluralism. Specifically: if the worlds of our frames are taken as *information states* (which goes hand in hand with their standing in non-trivial information-inclusion relations), they may very well be self-undermining. Inconsistent theories, or sets of belief, or data bases, are ways actuality cannot be, thus complying with the Aristotelian characterization of the *adynaton*. However, we can reason non-trivially about what does and does not follow from the data encoded in such bodies of information. Indeed, this is one main motivation for the paraconsistent rejection of ECQ. The entailment has been contested within such paraconsistent logics as relevant logics as a plain *non sequitur*, on a par with the Weakening principle discussed two sections ago: ECQ fails minimal tests for relevance, for what has A to do with  $\perp$  or B?

If these considerations are taken seriously, then whereas some paraconsistent negations like the da Costa and Belgian ones are to be ruled out as not being real negations, others may pass the test based on the notion of (in)compatibility. Before we get to give a closer look at one such negation, however, we have to take into account the tortuous

route from constructively acceptable negation to classical Boolean negation.

## 7. Maximality, Seriality, and Convergence

Even if we accept ECQ, three main inferential principles are still lacking before we reach Boolean negation: Excluded Middle, Double Negation Elimination, and the final De Morgan entailment. Their failures are notoriously interrelated in intuitionistic logic. Examining the three entailments one by one in the context of our (in)compatibility semantics, though, reveals that the core of their rejection in a constructivist environment does *not* rely on the concept of (in)compatibility as such. Rather, it relies on a different understanding of what *worlds* can be and, consequently, of which features the information-inclusion relation between them is to have.

We validate the Law of Excluded Middle in the following version:

$$(LEM) \quad \top \vDash A \vee \neg A$$

by stipulating that all worlds be maximal as far as compatible information goes — that is: for any  $x$  and  $y$ ,  $xR_N y \Rightarrow y \sqsubseteq x$ . Any world only sees worlds whose truths are already *its* truths or, equivalently, every world is only compatible with worlds already informationally included in it. *Proof:* suppose  $x \not\vDash A$  and that  $xR_N y$ ; given Maximality,  $y \sqsubseteq x$ , therefore  $y \not\vDash A$ ; hence,  $x \vDash \neg A$ ; thus  $x \vDash A \vee \neg A$ ; since  $x$  was arbitrary, we get LEM.

For Double Negation Elimination:

$$(DNE) \quad \neg\neg A \vDash A$$

we need first of all compatibility to be a serial relation, that is, each world  $x$  must be compatible with some world  $y$ : everybody loves somebody. This involves only  $R_N$ , and enforces the entailment  $\neg\top \vDash \perp$ , as is easily seen. But secondly, we need that, for all worlds  $z$  that  $y$  is compatible with, the information in  $z$  be preserved in  $x$  (see [Mares 2004](#), p. 78). Overall, the clause is:  $\forall x \exists y (xR_N y \ \& \ \forall z (yR_N z \Rightarrow z \sqsubseteq x))$ . This is hardly intuitive, but it works. *Proof:* assume the condition, and let  $x \vDash \neg\neg A$ . Seriality delivers a compatible  $y$  which, by (S $\neg$ ), is such that  $y \not\vDash \neg A$ . By Seriality again there is a  $z$  compatible with  $y$  and, by (S $\neg$ ) again,  $z \vDash A$ . Since  $yR_N z \Rightarrow z \sqsubseteq x$ ,  $A$  must already hold at  $x$ :  $x \vDash A$ ; and since  $x$  was arbitrary, we have DNE.

The final De Morgan entailment, that is:

$$\neg(A \wedge B) \models \neg A \vee \neg B$$

follows from a cumbersome condition, namely:  $xR_N y_1 \ \& \ xR_N y_2 \Rightarrow \exists z(y_1 \sqsubseteq z \ \& \ y_2 \sqsubseteq z \ \& \ xR_N z)$  (for the proof that this works, see Restall 2000, p. 261). This condition has, nevertheless, some intuitiveness to recommend it: for any two worlds that  $x$  is compatible with, there must be another compatible world wrapping up whatever holds at those two. The partial ordering of worlds by information-inclusion is *convergent* (see Restall 1999, pp. 62–3).

What matters for our purposes is that all the clauses needed to validate the nonconstructive inferences are not *only about* compatibility: they crucially involve the information ordering  $\sqsubseteq$ . And which properties this is to obey depend on how we specify the notion of world as a thing at which formulas or sentences can be true. This goes some way towards explaining why (in)compatibility may lead us to take as idle the long-standing dispute on which out of classical and constructive negation is the real one.<sup>26</sup> I have claimed that (in)compatibility is the core primitive concept grounding the meaning of negation. Robust insights into the features of (in)compatibility, taken as worldly accessibility, rule out some wannabe-negations that fail to satisfy Minimal Contraposition, or Double Negation Introduction, as candidates for being the real thing, and also rule out split negations as encoding an empty distinction. The same may hold for a negation that does not comply with ECQ—therefore, for any paraconsistent negation—if the intuition of universal self-compatibility cannot be negotiated.

Even when we agree on all these features of  $R_N$ , though, there still are open issues regarding the idea of a non-trivial information-inclusion relation between worlds, and the properties it has to satisfy. We have seen that a plausible principle governs the interaction between compatibility and information-inclusion, namely the Backwards principle: sub-worlds of compatible worlds must be compatible. But this says nothing about the interpretation of the notion *being a(n informationally proper) sub-world* of a given world; for Backwards is

<sup>26</sup> For an approach to the opposition between intuitionistic and classical negation, different from the one adopted here and giving a different verdict, one can read Avron 2002. Avron distinguishes a syntactical viewpoint on negation, whereby the connective is characterized by its inferential behavior, and a semantic one, where truth conditions aim at hooking up to intended or intuitive meanings. According to Avron, classical negation ends up complying with both the syntactical and semantical constraints, while intuitionistic negation does not.

trivially satisfied when the notion boils down to identity:  $xR_Ny$  entails  $x_1R_{N_1}y_1$  when  $x_1$  just is  $x$  and  $y_1$  just is  $y$ .

Thus, I agree with Crispin Wright’s claim (in response to Peacocke 1987) that it is a mistake to suppose that ‘our most basic understanding of negation, as incorporated in [a characterization in terms of (in)compatibility], provides any push in the direction of a distinctively classical conception of that connective’ (Wright 1993, p. 130). The remark is right, in so far as it addresses the opposition between classicists and constructivists on negation. Such an opposition turns on further concepts besides the core notion of (in)compatibility. If satisfying all the principles that are mandated by that notion *alone* is enough to be counted as a negation, then both a constructivist and a classical negation pass the threshold and, to this extent, logical pluralism on negation is enforced: indeed, ‘there is no distinctively classical conception of negation and no distinctively classical conception of incompatibility either’ (Wright 1993, p. 130). The negation pluralism suggested by Wright has been made precise here, along the lines of our development of the Beall-Restall logical pluralism: in the Beall-Restall approach, the GTT schematic clause can deliver classical or intuitionistic consequence depending on our understanding worlds as classical, possible-maximal ways things could be, or as constructions. In the same way, in our approach negation can fail to be distinctively classical, rather than intuitionistic, if both classical ways and intuitionistic constructions make for admissible precisifications of the notion of *world* in the semantic clause for negation ( $S\neg$ ): then different assumptions on the information ordering  $\sqsubseteq$  will be triggered.<sup>27</sup>

<sup>27</sup> I will round up this section with a remark on the connection between our compatibility clause for negation ( $S\neg$ ) and the standard clause for intuitionistic negation (say, ‘ $\neg$ ’) in the Kripke semantics for intuitionistic logic. Here worlds are partially ordered according to what is intuitively taken as a temporal sequence of possible developments in the subject’s cognitive activity (see van Dalen 1986). The information ordering in this context is also the relevant quantification-restricting accessibility:

$$(S\neg) \quad x \Vdash \neg A \Leftrightarrow \forall y(x \sqsubseteq y \Rightarrow y \not\Vdash A)$$

$\neg A$  is true at  $x$  just in case  $A$  is not true at all  $y \sqsupseteq x$ . The form is the same as that of our compatibility clause, and there is a one-way entailment from information-inclusion to compatibility, mandated by universal self-compatibility (which intuitionists accept):  $x \sqsubseteq y \Rightarrow xC_Ny$ . But the converse entailment  $xC_Ny \Rightarrow x \sqsubseteq y$  is turned by Symmetry (also accepted intuitionistically) into  $yC_Nx \Rightarrow x \sqsubseteq y$ . This is simply our Maximality clause, that is, precisely what would be rejected by a constructivist. Even though  $xC_Ny$  is thus not equivalent to  $x \sqsubseteq y$  in this context, as Dunn (1993) has shown, it is equivalent to a clause involving information-inclusion, which is again a form of Convergence:  $xC_Ny \Leftrightarrow \exists z(x \sqsubseteq z \ \& \ y \sqsubseteq z)$ , that is,  $x$  and  $y$  are compatible just in case there is some common (proper) extension for  $x$  and  $y$ .

## 8. Maximal compatibility

The idea of Convergence variously explored above becomes simpler if we take Seriality on board again, that is  $\forall x \exists y (xR_N y)$ , and add that  $\exists y (xR_N y \ \& \ \forall z (xR_N z \Rightarrow z \sqsubseteq y))$ . Mares (2004) calls this last condition the *Star Postulate* for the following reason. The condition tells us that there is always a maximal world compatible with  $x$ . Call it  $x^*$ . Symmetry gives us  $x \sqsubseteq x^*$ ; if we impose the converse condition  $x^{**} \sqsubseteq x$ , we validate DNE; and given the antisymmetry of  $\sqsubseteq$ ,  $x^{**}$  just is  $x$ . Under these conditions, as highlighted by Greg Restall (1999), our compatibility clause for negation simplifies to:

$$(S^*) \quad x \Vdash \neg A \Leftrightarrow x^* \nVdash A$$

For  $x^* \nVdash A$  just in case  $y \nVdash A$  for each  $y$  compatible with  $x$ , because  $xR_N y$  just in case  $y \sqsubseteq x^*$ : indeed, ' $x^*$  is a "cover all" for each state  $y$  compatible with  $x$ ' (Restall 1999, p. 63). The negation so defined satisfies DNI, DNE, and all the De Morgan entailments; it is thus usually called De Morgan negation. But it does not satisfy ECQ. The star is the period-two operation which takes a world to its maximally compatible peer, introduced by Routley and Routley (1972) and developed by Routley and Meyer (1973) in their seminal works on relevant logics.

In the early days of the worlds semantics for relevant logics, the Routley star came under attack: van Benthem (1979), Copeland (1979, 1986), and others complained that there was no intuitive meaning behind  $(S^*)$ , and that the semantics for relevant logics, to the extent that it relied on the star clause, was pure and uninterpreted, not applied semantics.<sup>28</sup> According to Smiley (1993, pp. 17–18), 'By itself this "star rule" is merely a device for preserving a recursive treatment of the connectives ... and it does nothing to explain their tilde until supplemented by an explanation of  $[x^*]$ '.

However, Restall (1999) has objected that the star rule makes perfect intuitive meaning, precisely in so far as it is grounded on our notion of (in)compatibility. As we have made abundantly clear, this is intuitively warranted if any foundation for negation is. The star rule adds to the basic understanding of negation encoded in  $(S\rightarrow)$  the further

<sup>28</sup> The terminology has become mainstream after its adoption in classic textbooks like Haack 1978. Dummett (1978) already talked of a 'merely algebraic notion of logical consequence' as opposed to a 'semantic notion of logical consequence properly so called', and wanted the latter to be 'framed in terms of concepts which are taken to have a direct relation to the use which is made of the sentences in a language' (p. 204).

assumptions of Symmetry, Seriality, and Convergence (in the star postulate version), which allow for the aforementioned simplification.

Two comments are in order on this issue. First, the nonconstructive inferences satisfied by De Morgan negation are intuitionistically controversial for the usual reason: they crucially involve assumptions on  $\sqsubseteq$ . The star clause presupposes again some kind of maximality or maximal extendibility for the information inclusion: that for any world  $x$  there exists a situation  $x^*$  which is maximally compatible with it is a distinctively nonconstructive assumption. I have argued above that different assumptions on the information-inclusion ordering between worlds depend on different ways of specifying the notion of world, and leave largely unsettled the question of which negations count as real, for this is to be addressed by focusing on the primitive idea of (in)compatibility.

But secondly, what Crispin Wright has called ‘our most basic understanding of negation’ incorporated in the idea of (in)compatibility is *not* neutral with respect to the rivalry between De Morgan and classical-Boolean negation, just as it *is* neutral with respect to the rivalry between the latter and intuitionistic negation. For the viability of a paraconsistent negation like  $(S^*)$ , which invalidates ECQ, depends again on our understanding of  $R_N$ , and specifically on our intuitions about Reflexivity, or the universality of self-compatibility. The vindication of De Morgan negation as endowed with intuitive meaning provided by Restall makes sense, but at least one of the relevant intuitions involving (in)compatibility, namely that concerning the admissibility of self-undermining worlds, may be controversial, as we have seen.

## 9. Extensionality regained?

If we now add ECQ again on top of De Morgan negation by forcing the idea of universal self-compatibility, we have that for each world  $x$ ,  $x = x^*$ . And we are back to the familiar ‘Stalnakerian’ territory where the set of worlds where  $\neg A$  is true is the Boolean complement of the set of worlds where  $A$  is true: all worlds are both maximal and consistent. Under these conditions, that is,  $(S^*)$  collapses into Boolean negation:

$$(BN) \quad x \Vdash \neg A \Leftrightarrow x \not\Vdash A$$

And this is demodalized: the truth value of  $\neg A$  at a world just depends on what goes on there—or rather, on what fails to go on there

(of course,  $A$  may include modalizers, hence its truth value at  $x$  may depend on what goes on elsewhere). I have argued that, in so far as we focus on the basic idea of (in)compatibility which makes for the core of negation, there is little to choose between BN and intuitionistic negation.<sup>29</sup> However, a clever argument proposed by Restall (1999, 2000), and Priest (1990, 2006), casts some doubts on the legitimacy of BN as such! I will close this long paper by reporting the argument, but will take no stance on it because it is, in a sense to be explained, beyond the scope of this work.

According to Restall and Priest, once worlds can stand in non-trivial information-inclusion relations, classical negation is not automatically warranted anymore. For in so far as it makes sense to read ' $x \Vdash A$ ' as the claim that world  $x$  *includes*, or *conveys*, or *supports* the information that  $A$ , then it may well not be the case that  $x$  supports the information that  $\neg A$  just because it fails to support the information that  $A$ , as mandated by BN. If we have proper, non-trivial information-inclusion relations such that  $x \sqsubseteq y$  but  $x \neq y$ , it may happen that  $y \Vdash A$ , but  $x \not\Vdash A$ . Then the Boolean negation of  $A$  holds at  $x$ , but it does not hold at  $y$ . This violates the generalized Heredity Constraint of our semantics above, namely that for each formula  $A$  of  $L$  and all  $x, y \in W$ ,  $x \Vdash A$  &  $x \sqsubseteq y \Rightarrow y \Vdash A$ . In general, if  $\neg$  plays by the BN clause, it may not be the case that  $[\neg A] = \{x \in W: x \Vdash A\} \in Prop(\mathfrak{F})$ . The Boolean negation of a proposition may fail to be a proposition, that is, it may not be a hereditary piece of information: there may be no proposition such that *it* is supported by a world just because some proposition is not supported there.<sup>30</sup>

As powerful as this argument may be against Boolean negation, it turns on the idea of non-trivial information inclusions and, therefore, does not stem from the pure features of (in)compatibility. Despite being advanced *de facto* by paraconsistent logicians willing to support

<sup>29</sup> I have not taken into account the perp- or ortho-negation of quantum logic, which was mentioned at the outset as the source of the characterization of negation in terms of (in)compatibility in Dunn's works. Ortho-negation is quite close to Boolean negation, satisfying both DNE and ECQ, but is implemented in the framework of quantum logic, where the Distributivity entailment  $A \wedge (B \vee C) \vDash (A \wedge B) \vee (A \wedge C)$  notoriously fails. The failure of distributivity brings in interesting features for negation in an algebraic approach, for instance the non-uniqueness of complementation in non-distributive lattices. The model-theoretic counterparts of these phenomena have nevertheless been left aside in this paper.

<sup>30</sup> This is mirrored by the behaviour of negation-as-failure in PROLOG and logic programming. Given the query ' $A ?$ ' in a PROLOG data base, if  $A$  is not derivable, then  $\neg A$  is outputted. Then ' $\neg A$ ' intuitively expresses the absence of a piece of information, rather than the presence of a piece of negative information.

De Morgan negation by discrediting its classical rival, it may as well be used by an intuitionist accepting ECQ. No Quinean alleged 'change of subject' is involved here: it is not claimed that classical negation just is a different notion from whatever ought to be properly called negation. What is questioned, rather, is that the Boolean negation of a proposition automatically makes for a proposition in its turn. In this sense, the Priest-Restall argument is beyond the scope of the issues addressed in the current paper: as far as the features of (in)compatibility go, Boolean negation passes the threshold for qualifying as real negation.<sup>31</sup>

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