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Losing Sight of the Forest for the Ψ: Beyond the Wavefunction Hegemony

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Abstract:
Traditionally Ψ is used to stand in for both the mathematical wavefunction (the representation) and the quantum state (the thing in the world). This elision has been elevated to a metaphysical thesis by advocates of the view known as wavefunction realism. My aim in this paper is to challenge the hegemony of the wavefunction by calling attention to a little-known formulation of quantum theory that does not make use of the wavefunction in representing the quantum state. This approach, called Lagrangian quantum hydrodynamics (LQH), is not an approximation scheme, but rather a full alternative formulation of quantum theory. I argue that a careful consideration of alternative formalisms is an essential part of any realist project that attempts to read the ontology of a theory off of the mathematical formalism. In particular, I show that LQH undercuts the central presumption of wavefunction realism and falsifies the claim that one must represent the many-body quantum state as living in a 3*n*-dimensional configuration space. I conclude by briefly sketching three different realist approaches one could take toward LQH, and argue that both models of the quantum state should be admitted. When exploring quantum realism, regaining sight of the proverbial forest of quantum representations beyond the Ψ is just the first step.

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I. Introduction

Second only to experiment, the formalism of a theory is a central avenue through which to explore the ontological implications of a scientific theory. When engaging in such a realist project, it is essential to keep in mind the distinction between the representational vehicle (the mathematics) and the thing in the world it represents. In the context of quantum mechanics however, this obvious distinction has often been ignored. In particular, there has been an elision between the wavefunction (Ψ) and the quantum state (properties of the system in the world) it represents. This practice of using Ψ to stand in for both the mathematical representational vehicle and the thing in the world is problematic for a number of reasons: first, it encourages a reifying of the mathematics; and, second, there may be properties of the representational vehicle that are not properties of the thing in the world. We see these issues particularly vividly in the case of wavefunction realism, which takes the wavefunction to be a concrete physical object, and takes the $3n$-dimensional configuration space on which the wavefunction is defined, to be the space that we (despite appearances to the contrary) actually live in.

A useful antidote to this habit of eliding the mathematics with the thing in the world is a consideration of alternative mathematical formalisms. A careful consideration of alternative formalisms is particularly important in the context of realist projects that try to read the ontology of a theory off of the mathematics. My aim in this paper is to highlight a little-known alternative formalism for quantum theory that does not make use of the wavefunction, Ψ, in representing the quantum state. Instead, it represents the quantum state by means of a displacement function of a continuum of interacting 'particles' following spacetime trajectories, $q_i(a, t)$. The Schrödinger equation is recast as a second-order Newtonian law for this congruence of spacetime trajectories. The congruence of trajectories can be computed independently of the wavefunction, thus providing a "quantum mechanics without wavefunctions." As will become clear, this is a full formulation or representation of quantum mechanics—not an approximation scheme—and also not merely an "interpretation" of quantum mechanics. It is a full alternative formalism for representing quantum theory that is equivalent to the standard formalism. I will refer to this alternative formalism for quantum theory, with its "displacement" or "trajectory" model of the quantum state, as Lagrangian quantum hydrodynamics (LQH). It should be emphasized, however, that 'hydrodynamics' here simply refers to the analogy used in the development of the mathematical formalism, and in no way requires a commitment to anything like a quantum fluid.

LQH is a representation—not interpretation—of quantum mechanics, and all questions related to the measurement problem or interpretations of quantum mechanics will be bracketed for the purposes of this paper. It is not my aim here to offer a realist interpretation of quantum mechanics, that is, to answer the question of what quantum mechanics (on any formalism) tells us is physically real. Rather my aim is to argue that a more careful consideration of alternative possible formalisms is a critical prerequisite to any such realist project that attempts to read the ontological implications of a theory off the formalism.

Exploring alternative formalisms for a theory, such as LQH, is a fruitful endeavor for at least four reasons. First, as has already been emphasized, it undercuts a facile identification of the formalism with the world, bringing to the fore the substantive questions about what sorts of ontological inferences can legitimately be made. Second, it may be that some phenomena or questions are more readily treated in one formalism rather than another (indeed this is typically why such alternative formalisms are developed and widely taught). Third, some formalisms may be more fertile in suggesting future lines of theory development (e.g., think of how the Hamilton-Jacobi formulation of classical mechanics was seminal in the development of quantum mechanics[1]). And, fourth, alternative formalisms can challenge common presuppositions about what features are supposedly demanded by the theory (e.g., the presumption that the quantum state must live in a 3*n*-dimensional configuration space, as will be discussed later on).

The structure of this paper will be as follows: I will begin in Section II by reviewing the claims of wavefunction realism, and then turn to some cautionary tales that are worth recalling when undertaking such a realist project. In order to motivate Lagrangian quantum hydrodynamics and introduce the central conceptual elements needed to understand this formulation, I will review in Section III *classical* hydrodynamics and the fertility of continuum hydrodynamic representations at the classical scale. This review of classical hydrodynamics is helpful both because quantum hydrodynamics is built in close formal analogy with it, and because it provides a broader context in which to think about the ontological implications of LQH. Section IV will then turn to hydrodynamic representations at the quantum scale, beginning with the early "Eulerian" hydrodynamic representation of Erwin Madelung (1927), and then turning to the more recent "Lagrangian" quantum hydrodynamics (LQH) developed by Peter Holland (2005).

Section V will explore some of the broader questions about LQH. In particular, this formulation of quantum mechanics will be clearly distinguished from the de Broglie-Bohm interpretation of quantum mechanics, which also emphasizes trajectories. As we will see, LQH exhibits a novel quantum symmetry, which is obscured on the standard formulation, and this symmetry (by Noether's theorem) corresponds to a set of conservation laws. This section will also explore whether or not, in analogy with arguments that have been made in the classical case,

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[1] See, e.g., Butterfield 2005.

there is some sense in which the Lagrangian formulation of quantum mechanics is more fundamental.

In the final section, VI, I draw out some of the implications of LQH for realist projects in quantum theory. In connection with wavefunction realism, the existence of LQH—a formulation of quantum mechanics without wavefunctions—challenges the hegemony of the wavefunction in representing the quantum state. Moreover, it undermines the supposed necessity of identifying the $3n$-dimensional configuration space of Ψ as the space we live in. I conclude by briefly identifying three interpretive attitudes one can take towards LQH, situating my preferred approach within an alternative tradition in scientific realism that understands realism in terms of the development of fruitful metaphors, rather than literal construals.

II. Quantum Physics Realism Debate

a. Ψ Realism

Traditionally the realist project in quantum theory has been understood as one of trying to read the mathematical formalism of the theory as a *literal depiction* of the world. Interestingly, Bas van Fraassen takes this literalist commitment to be required not just for the realist, but also for his own (antirealist) constructive empiricism. He writes, "The agreement between scientific realism and constructive empiricism is considerable and includes the literal interpretation of the language of science" (1991, p. 4).[2] A prominent thread in this quantum realism project, which has recently experienced a resurgence of interest, is known as "wavefunction realism."

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[2] As will be discussed in Section VI, it is precisely this commitment to a literal construal that I urge the realist should abandon.

Alyssa Ney concisely defines this view as follows: "the view that the wave function is a fundamental object and a real, physical field on configuration space is today referred to as 'wave function realism' " (2013, p. 37). Wavefunction realism has been defended by David Albert since the mid-1990s, and he claims it is a necessary part of the realist project in quantum theory on any interpretation.[3] He writes,

> [I]t has been essential (that is) to the project of quantum-mechanical *realism* (in *whatever* particular form it takes—Bohm's theory, or modal theories, or Everettish theories, or theories of spontaneous localization), to learn to think of wave functions as physical objects *in and of themselves*. (Albert 1996, p. 277; emphasis original)

According to Albert, this reifying of the wavefunction (which we should keep in mind, that as a *function*, is strictly speaking a mathematical object) is both a necessary and obvious consequence of taking quantum mechanics seriously under any interpretation. Of course, what he has in mind here is not a mere Platonism about mathematical objects, but rather a realism about a physical field, which the wavefunction is supposed to represent (e.g., Albert 2013, p. 53).

Furthermore, since the wavefunction (the mathematical object) is defined on a 3*n*-dimensional configuration space (where *n* is the number of particles in the universe) Albert concludes that this astronomically large configuration space must also be interpreted literally as our actual physical space. He writes,

> And of course the space those sorts of objects live in, and (therefore) the space we live in, the space in which any realistic understanding of quantum mechanics is necessarily going to depict the history of the world as playing itself out. . . is configuration space. And whatever impression we have to the contrary (whatever impression we have, say, of living in a three-dimensional space, or in a four-dimensional space-time) is somehow flatly illusory." (Albert 1996, p. 277)

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[3] J.S. Bell defended a precursor of this view in connection with Bohm's theory when he writes, "No one can understand this theory until he is willing to think of ψ as a real objective field rather than just a 'probability amplitude'. Even though it propagates not in 3-space but in 3*n*-space." (1987, p. 128).

Here again, we see the realist project in quantum theory being understood as taking certain features of the mathematical formalism of the theory at face value as a *literal depiction* of what our world is really like.

Despite the enormous amount of ink and intellectual resources that have been poured into wavefunction realism and the interpretation of quantum mechanics, neither has managed to produce a broad consensus, despite the former project going on for over two decades and the latter project nearing a century. When a field has remained in such a holding pattern for an extended period of time, it is reasonable to ask whether the time has come to reframe the realist project in quantum physics and perhaps start asking a different set of questions. Before engaging in such a reframing, however, it is important to recall a few cautionary tales.

b. A Few Cautionary Tales

The first cautionary tale recalls that, despite its fundamental role in quantum mechanics, the wavefunction is both enigmatic and unobservable. As a letter to Nature recently lamented,

> [The wavefunction] is typically introduced as an abstract element of the theory with no explicit definition. Rather physicists come to a working understanding of the wavefunction through its use to calculate measurement outcome probabilities by way of the Born rule. At present, the wavefunction is determined through tomographic methods, which estimate the wavefunction most consistent with a diverse collection of measurements. The indirectness of these methods compounds the problem of defining the wavefunction. (Lundeen et al 2011, p. 188)[4]

In practice, the wavefunction must be estimated and inferred from a collection of various measurements (of quantities that are observable) made on an ensemble of identically prepared

[4] There have been various attempts to measure the wavefunction of a single system, using weak or protective measurements, including the paper cited here. For a recent review see Gao (2017), and for criticisms see, for example, Combes et al. (2017). An alternative way to understand such measurements will be briefly discussed in Section VI.

systems. Even someone like John S. Bell, who interprets the wavefunction in accordance with Bohmian mechanics as a real, physical field, notes its inaccessibility:

> Although Ψ is a real field it does not show up immediately in the results of a single 'measurement,' but only in the statistics of many such results. It is the de Broglie-Bohm variable X that shows up immediately each time. That X rather than Ψ is historically called a 'hidden' variables is a piece of historical silliness. (Bell 1987, pp. 162-163)

The point here, of course, is not that one should never be a realist about unobservables. Rather the point is to recognize that there is an additional level of difficulty involved in such cases, so the evidential bar must be set higher. Extra caution should be exercised when trying to read ontological implications off a formalism when the entity or state in question is unobservable and only indirectly accessible.

The second cautionary tale is that quantum mechanics is not the final, most fundamental, theory of everything, but rather is only an effective theory. The nonfundamental status of quantum mechanics has been emphasized in connection with wavefunction realism by Wayne Myrvold who argues that we should

> bear in mind that quantum mechanics—that is, the nonrelativistic quantum theory of systems of a fixed, finite number of degrees of freedom—is not a fundamental theory, but arises, in a certain approximation, valid in a limited regime, from a relativistic quantum field theory. (2015, p. 3247)

He goes on to note that configuration spaces are not fundamental and that wavefunctions are not really like classical fields.

An effective theory can be understood as a framework for capturing the essential physics within some circumscribed domain, without claiming to describe the one true fundamental ontology. In his book on effective theories, James Wells, describes this recent shift in the physics community away from thinking about the "Theory of Everything", toward thinking of theories in physics as effective. As Wells emphasizes, one of the important lessons to take away from this shift is the recognition of "the power that explicitly agreeing to the Effective Theory mindset can have in developing richer theories of nature and achieving a deeper understanding" (2012, p. *v*). Although the implications of this shift for the philosophy of physics have yet to be fully explored, the suggestion here, which I think is worth exploring, is the idea that a collection of effective theories provides us with a deeper understanding of nature than a single theory of everything does. The cautionary lesson is that it is important to recall the effective nature of quantum mechanics and view the theory within the broader context of other physical theories, rather than trying to draw ontological conclusions from it as if it were the final true theory of everything.

The third cautionary tale, noted at the outset, is that one should always clearly distinguish the mathematical representation from the thing being represented. Buried in a footnote to his discussion of the reality of the wavefunction, Peter Lewis notes,

> strictly speaking the wave function itself is a mathematical representation of an (unnamed) physical state of affairs. . . . However, . . . it is traditional to use the term 'wave function' for both the mathematical representation and the physical object represented. (2016, p. 192)

This habit of the field is problematic for several related reasons: First, failing to distinguish the mathematical representation from the target can lead one to unreflectively reify the mathematics. Second, it can lead to confusion in that there may be properties of the representational vehicle that are not properties of the thing in the world. Third, and perhaps most importantly, there are ways to represent the state of a quantum system other than by means of the wavefunction, as we will discuss in detail. By clearly distinguishing the representation from the thing represented, the conceptual space is opened up to explore alternative representations, which may provide further insights.

The fourth cautionary tale is that mathematical representations do not wear their ontological interpretations on their sleeves. Many equations in physics today are so familiar to us that their physical meaning appears indistinguishable from their formal expression. That this identification is nontrivial, however, is evident to anyone who has studied the history of physics.[5] Moreover, there is often more than one formalism to choose from in representing a particular physical system. As Tim Maudlin rightly emphasizes,

> If one intends to try to read the physical ontology of a theory off of the mathematical structure used to present the theory, then one should give a great deal of consideration to alternative mathematical structures and the reasons for choosing one or another. (2013, p. 136)

It is precisely such a case, where we have more than one formalism for representing a given physical system and a literal reading-off of the ontology is ill advised, that I want to examine here.

As will be discussed in detail in the coming sections, there is a hydrodynamic formulation of quantum mechanics that does not make use of the Ψ in representing the quantum state. This alternative quantum formalism undercuts a facile identification of the Ψ with the quantum state, and highlights the importance of these four cautionary lessons when trying to read realist implications off the formalism of quantum theory. It is my contention that debates about realism in quantum physics have lost sight of the big-picture forest through an excessive focus on the Ψ. My aim is not to replace the Ψ conception of state with the hydrodynamic one, but rather to underscore the importance of recognizing the legitimacy of both. These two conceptions or models of the quantum state, though empirically equivalent, paint very different pictures of the unobservable world. However, neither one should be read as a literal depiction.

[5] See Bokulich (2015) for a discussion of this point, using the example of how Helmholtz and Maxwell, though agreeing on the same formal equation, disagreed about the right way to hook up the physical quantities with the elements of the equation.

In order to properly understand the implications of the hydrodynamic formulation of quantum mechanics, it is helpful to contextualize it within the class of hydrodynamic representations in physics more broadly. Hence, the next step on our path is to regain sight of the proverbial forest is to take a scenic tour of hydrodynamic representations through a variety of length scales.

III. Hydrodynamic Representations at the Classical Scale

Classical hydrodynamics is the study of fluid flow, where the fluid is modeled as a continuum (that is, as a continuous distribution of mass) rather than as being composed of molecules or atoms. On this approach, the macroscopic properties of the fluid, such as density and pressure, are taken to be well defined down to infinitesimal volume elements. Although strictly false as a depiction of real fluids, for many domains the continuum assumption is an example of what I have elsewhere called a "credentialed fiction" (e.g., Bokulich 2016). More generally, hydrodynamics can be understood as an "effective theory," which captures the essential physics at certain length scales (viz., greater than inter-atomic), and it is the appropriate way to represent fluids in fields such as oceanography and meteorology (or what is more generally referred to as geophysical fluid dynamics).[6] The continuum model of hydrodynamics allows one to take advantage of the powerful mathematical framework of continuum mechanics, which provides insights that would be difficult (if not impossible) to achieve on an atomistic approach.

[6] There are, of course, certain situations where one must take care in using continuum representations, such as when it comes to shocks and certain boundary conditions.

There are two different pictures or formulations of (continuum) hydrodynamics: the Eulerian formulation and the Lagrangian formulation.[7] On the Eulerian formulation, one records the evolution of the fluid at each point in space, **x**, and time, *t*, in a fixed inertial coordinate system. It is a field description, where the fluid properties such as density, velocity, and pressure are thought of as fields defined at a particular point as follows:

$$\rho_p(\mathbf{x},t), \quad \mathbf{v}_p(\mathbf{x},t), \quad p_p(\mathbf{x},t).$$

Perhaps counterintuitively, the properties are ascribed to the points in space, and not to some substance moving through that space. An example of an Eulerian measuring device is a probe fixed in space. On the Eulerian picture, to say that a fluid flows, is to say that properties defined at various spatial locations are changing over time.

On the Lagrangian picture, by contrast, one considers the fluid as a dense set of fluid particles or parcels[8], each of which carries its own properties and maintains its identity as it follows a classical trajectory. Each fluid parcel has its own unique particle label, **a** = (*a,b,c*), which can be taken to be the position of the parcel at some initial time, and the trajectories do not cross. The trajectories are expressed by the function **x**(**a**, *t*), which follows in time the position of the fluid parcel initially at **a**. Conceptually, there are two ways to think about the fluid motion on this picture:

> We can think of a label space with coordinates (*a*,*b*,*c*) and a location space with coordinates (*x*,*y*,*z*). Then the fluid motion . . . is a time-dependent mapping between these two spaces. Alternatively, we can think of the label variables (*a*,*b*,*c*) as curvilinear coordinates in location space. Then the fluid motion drags these curvilinear coordinates through location space. (Salmon 2014, p. 5)

[7] Despite its name, the Lagrangian picture was first introduced by Euler in the context of acoustics (see, e.g., Darrigol 2005, p. 29 and references therein)
[8] These parcels or 'particles' are understood as pieces of the continuum.

The flow, described by $\mathbf{x}(\mathbf{a}, t)$, can be considered as a continuous differentiable mapping of the three-dimensional Euclidean space onto itself $\mathbf{x}(\mathbf{a}, t) = \Phi_t(\mathbf{a})$, which allows one to construct the Jacobian matrix of the mapping:

$$\mathbb{J}_{ik}(t) \equiv \left(\frac{\partial x_i}{\partial a_k}\right)(t) = \propto.$$

The familiar continuity equation, which in the Eulerian picture is expressed as

$$\left.\frac{d\rho}{dt}\right|_x + \nabla \cdot (\rho \mathbf{v}) = 0,$$

can be understood as arising from the Lagrangian-picture requirement that fixed volumes in particle-label space always contain the same mass (Roulstone 2015, p. 369).

One can write down the Lagrangian for a perfect fluid as

$$\mathcal{L} = \int d^3\boldsymbol{a}\left[\frac{1}{2}\dot{x}^2 - E(\propto, S(\boldsymbol{a})) - \boldsymbol{\Phi}(\mathbf{x})\right],$$

where $E(\propto, S(\boldsymbol{a}))$ is the internal energy, which is a function of the volume, $\propto$, and entropy, $S(\boldsymbol{a})$, and $\boldsymbol{\Phi}(\mathbf{x})$ is the potential for external forces, with the integration being a measure over particle label space.

Hamilton's principle, which is a variational principle equivalent to Newton's second law, states that the action is stationary for arbitrary variations in $\delta\mathbf{x}(\mathbf{a}, \mathrm{t})$:

$$\delta \int \mathcal{L} dt = 0.$$

Because the particle labels $\boldsymbol{a}(\mathbf{x}, t)$ enter the Lagrangian only through the density $\partial(\mathbf{a})/\,\partial(\mathbf{x})$ and entropy $S(\boldsymbol{a})$, the potential energy terms in the Lagrangian are unaffected by particle-label variations $\delta\mathbf{a}(\mathbf{x}, t)$ that leave the density and entropy unchanged. The physical oceanographer Rick Salmon was the first to recognize in 1982 that this relabeling symmetry corresponds by Noether's theorem to a conservation law, namely, the most general statement of vorticity conservation (Salmon 1988, p. 238):

$$\partial/\partial t(\nabla_a \times \mathbf{A}) = 0,$$

where ∇_a is the gradient operator in particle label space and

$$\mathbf{A} = u\nabla_a x + v\nabla_a y + w\nabla_a z$$

is the projection of the velocity components on the basis of the gradient operator in particle-label space.

From this general statement of vorticity conservation, one can derive the well-known Ertel's theorem of potential vorticity conservation,

$$\frac{\partial}{\partial t}[(\nabla_a \times \mathbf{A}) \cdot \nabla_a \theta] = 0$$

(where θ is any conserved quantity on the flow), Helmholtz's various theorems, and Kelvin's circulation theorem

$$\frac{d}{dt}\oint \mathbf{v} \cdot d\mathbf{x} = 0.$$

This vorticity conservation law, which results from the particle relabeling symmetry of the Lagrangian picture, is scientifically one of the most important results in geophysical fluid dynamics. The meteorologist Peter Névir, for example, writes that the "conservation of potential vorticity is a cornerstone in dynamic meteorology" (2004, p. 486). Similarly, the oceanographer Peter Müller writes "most aspects of large-scale oceanography can be understood in terms of potential vorticity and its evolution, as is stressed in textbooks" (1995, p. 68). Intuitively potential vorticity conservation is the idea that a rotating column of fluid, despite changing shape, will conserve volume and angular momentum.

Although these conservation of vorticity and circulation theorems were known long before the particle-relabeling symmetry was recognized, their derivation from the symmetry property provides a unification and simple explanation that is missing on the standard approaches. Moreover, as Salmon points out, "the symmetry approach shows that vorticity conservation is a consequence of the continuum approximation. It has no analogue in particle mechanics, where the particle labels cannot be varied continuously" (Salmon 1988, p. 241).

The rich representational machinery of classical hydrodynamics is not just useful for understanding ordinary fluids, but can also be used to describe granular materials, which are conglomerations of discrete macroscopic particles, such as piles of sand or grain in a silo. Although granular materials sometimes behave as a solid, they can also behave as a fluid and flow (even though no water is involved) as is familiar in an hourglass. Granular hydrodynamics is the use of classical hydrodynamics to model the flow of granular materials.[9] Granular hydrodynamics is particularly useful for helping geoscientists understand earthquake-induced landslides, for example.

Similarly, classical hydrodynamics representations are extremely fruitful at the cosmological scale and are regularly employed in cosmology and astrophysics. The hydrodynamic equations—in both the Eulerian and Lagrangian formulations—are productively

[9] For a classic introduction see Jaeger, Nagel, and Behringer (1996), and for a more recent review see Trujillos, Sigalotti, and Klapp (2013).

used to understand the formations of stars, galaxies, and large-scale structures in the universe.[10] As with ordinary hydrodynamics, each of these pictures is useful for bringing out certain sorts of insights. As Shy Genel and colleagues write,

> The equations of hydrodynamics are usually solved in astrophysical applications using either . . . particle-based Lagrangian-like schemes such as smoothed particle hydrodynamics (SPH) or mesh-based Eulerian-like schemes such as adaptive mesh refinement (AMR). There are various advantages and shortcoming of each approach. (Genel et al. 2013, p. 1426)

As an example, Genel and colleagues note that one of the most debated questions in galaxy formation is how galaxies get their gas, for which the ability to track the mass flow in a Lagrangian manner is crucial. They further note, however, that using the Lagrangian formulation comes at a price, and hence they explore how the strengths of each representation can be exploited, collectively giving deeper insights into galaxy formation than would be obtained by using one representation alone.

In all of these applications of classical hydrodynamics, from ordinary geophysical fluids in oceanography and meteorology to hydrodynamic representations in astrophysics and cosmology, the interesting question is not whether the continuum representation is a literally true depiction of the world. Rather, the relevant question is what true physical insights and correct inferences does this representation allow us to draw about the world. I urge that we should keep this lesson in mind as we turn to hydrodynamic representations at the quantum scale.

[10] For textbook reviews of hydrodynamic representations at the cosmological scale see, for example, Murkhanov's (2005) *Physical Foundations of Cosmology* or Regev et al.'s (2016) *Modern Fluid Dynamics for Physics and Astrophysics*.

IV. Hydrodynamic Representations at the Quantum Scale

a. Eulerian Quantum Hydrodynamics

Hydrodynamic representations of quantum mechanics are almost as old as quantum mechanics itself (though as we will see, a fuller elaboration of the formal analogy was only carried out quite recently). Within months of Erwin Schrödinger's seminal papers introducing the wave equation, Erwin Madelung published a paper titled "Quantum Theory in Hydrodynamical Form" (*Quantentheorie in hydrodynamischer Form*). Madelung describes as the purpose of this paper to "show that far-reaching analogies with hydrodynamics exist" (Madelung 1927, p. 322). He begins by writing the wavefunction in polar form[11]

$$\psi = \mathrm{R}e^{iS/\hbar}\,, \tag{1}$$

which when substituted into the single-particle Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{2}$$

and separated into real and imaginary parts yields the following two real, coupled partial differential equations:[12]

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + V = 0 \tag{3}$$

and

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[11] I'll be using a slightly different notation here from Madelung's original in order to be consistent with the notation used later in connection with Holland's extensions of the hydrodynamic analogy.

[12] For further details on the derivation in this same notation, see for example Cushing (1994, Chapter 8, Appendix 1).

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left(\frac{R^2 \nabla S}{m}\right) = 0 \quad (4)$$

The heart of this hydrodynamic analogy is to identify $\rho = R^2$ as the fluid density and $v = \nabla S/m$ as the velocity field of this fluid; then the above equation becomes

$$\nabla \cdot (\rho v) + \frac{\partial \rho}{\partial t} = 0\,, \quad (5)$$

which is a continuity equation expressing the conservation of mass of the fluid. The other part (Equation 4), after applying the operator ∇ and using $v = \nabla S/m$ becomes[13]

$$\partial \mathbf{v}/\partial t + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -(1/m)\nabla(V + Q), \quad (6)$$

where

$$Q \equiv -\left(\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\right). \quad (7)$$

Equation (6) is analogous in form to the classical Euler equation:

$$\partial \mathbf{v}/\partial t + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -(1/\rho)\nabla p, \quad (8)$$

where p is the pressure of the fluid, and ρ, recall, is the density.

After deriving the two key equations, Madelung concludes (with our notation in brackets),

> [This equation] has the character of a hydrodynamical equation of continuity when one regards α^2 [i.e., ρ] as a density and φ [i.e., S/m] as the velocity potential of a flow $u = \text{grad}\varphi$ [i.e., **v**]. . . . [The other equation] also corresponds precisely to a hydrodynamical one, namely that of an irrotational flow moving under the action of conservative forces. . . . We therefore see that [the equation from Schrödinger] is

[13] Again, for the explicit intervening steps see Cushing (1994, Chapter 8, Appendix 1) or Holland (1993, Chapter 3).

> completely explainable in terms of hydrodynamics, and that a peculiarity appears only in one term, which represents the internal mechanism of the continuum. (Madelung 1927, p. 323).

The ‘peculiarity’ Madelung notes in one term is what would later, in the context of David Bohm's (1952) interpretation of quantum mechanics, be called the “quantum potential."

Madelung's hydrodynamical model of quantum theory was further developed by Takehiko Takabayasi (1952), who elaborated this 'peculiar' term as an expression of the internal stress in the fluid, which he represents as a stress tensor, σ_{ij}. If we rewrite the classical Euler equation as[14]

$$\partial v_i/\partial t + (\mathbf{v}\cdot\mathbf{\nabla})v_i = -(1/m)\partial_i V - (1/m\rho)\partial_j(p\delta_{ij}) \quad (9)$$

we can see the formal analogy with the quantum version of the Euler equation, which (from Equation 6) can be written

$$\partial v_i/\partial t + (\mathbf{v}\cdot\mathbf{\nabla})v_i = -(1/m)\partial_i V - (1/m\rho)\partial_j(p\sigma_{ij})\ , \quad (10)$$

where the stress tensor is

$$\sigma_{ij} = -(\hbar^2\rho/4m)\partial_{ij}\log\rho. \quad (11)$$

Takabayasi concludes, "This showes [sic] that the motion can just be pictured as that of [a] fluid with (symmetric) stress tensor σ_{ik}[i.e, our σ_{ij}] and that the influence of quantum mechanics is regarded as introducing this 'quantum-theoretical' stress" (Takabayasi 1952, p. 180). As can be seen above (in Equations 9 & 10), the classical Euler and "quantum Euler" equations are

[14] Here I am following the notation of Holland (1993), p. 121 for consistency.

formally identical, apart from the classical pressure tensor, $p\delta_{ij}$, being replaced with the quantum-mechanical stress tensor (Equation 11).[15]

To complete the hydrodynamic analogy, the appropriate boundary and subsidiary conditions need to be imposed on the density and velocity fields ρ and **v** (Holland 1993, p. 121). In ordinary quantum mechanics one imposes the condition that the wavefunction is single-valued, which means that at every instant, each point of space can be assigned a unique value of the function ψ. In the context of the hydrodynamic representation, the velocity field is irrotational (the fluid parcels are not rotating), except at the nodes where it is undefined, and the single-valuedness condition means

$$\oint \mathbf{v} \cdot d\mathbf{x} = nh/m, \quad (12)$$

which is interpreted as the fluid possessing 'quantized vortices.'[16]

Takabayasi takes pains to distinguish this hydrodynamical model of quantum mechanics from David Bohm's (1952) interpretation. While on the hydrodynamic analogy the quantum potential term is an internal stress in the fluid, on Bohm's interpretation it is understood as an external force, $-\nabla Q$. Further differentiating his approach, Takabayasi writes, "Instead of doing this Bohm has reintroduced the quantity ψ not as a mere mathematical tool but as an objectively real field" (Takabayasi 1952, p. 155). Takabayasi further emphasizes the distinction between a *formulation* and an *interpretation.* He writes,

> The hydrodynamic analogy . . . though sometimes useful for the analysis of the Schrödinger equation and rather appropriate to make one visualize the presence of

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[15] As Bohm and Vigier (1954) point out, the quantum stress is unlike the classical stress in that it depends on derivatives of the fluid density (p. 209).

[16] This is reminiscent of the quantum condition of the old quantum theory, though of course is given a different physical interpretation in that context (Holland 1993, p. 72).

> internal force, does not necessarily prove the *reality* of the hydrodynamic picture. (Takabayasi 1952, p. 150)

While Madelung seems somewhat agnostic about how literally it should be interpreted, describing the use of hydrodynamics as an 'analogy', Takabayasi is quite explicit in describing the use of hydrodynamics as just a 'model' or a 'formulation' of quantum mechanics, analogous to Feynman's path integral formulation. By contrast, Bohm and Jean-Paul Vigier (1954), for example, go further in describing it as a theory of a real fluid. They write, "Since the Madelung fluid is being assumed to be some kind of physically real fluid, it is therefore quite natural to suppose that it too undergoes more or less random fluctuations in its motion" (Bohm and Vigier 1954, p. 209). As will be discussed later, these same questions about whether to understand the hydrodynamic analogy in quantum mechanics as an interpretation or a formulation persist to this day.

The hydrodynamic analogy in quantum theory, as developed by Madelung, Takabayasi, and others in the twentieth century, has been exclusively in the Eulerian picture. As discussed in the previous section, hydrodynamics (like quantum mechanics) admits of two different formulations: the Eulerian picture and the Lagrangian picture. Although Takabayasi, in a footnote to his 1953 paper, briefly alludes to the possibility of a self-contained Lagrangian formulation, surprisingly, it would be another fifty years before such a full Lagrangian formulation of quantum hydrodynamics would be developed.

b. Lagrangian Quantum Hydrodynamics

The systematic development of a Lagrangian quantum hydrodynamics, as a full alternative picture or formulation of quantum mechanics, was only carried out recently in a series of papers by Peter Holland beginning in 2005.[17] Recall that the Lagrangian picture treats the fluid as a continuum of fluid particles (or parcels), each of which maintains its identity throughout the evolution of the fluid. Unlike Madelung's Eulerian formulation of quantum hydrodynamics, the Lagrangian formulation introduces new variables (the particle labels a) into the quantum formalism. One begins by introducing a vector (e.g., in three-dimensional Euclidean space) label a_i for each particle, which can be taken to be the position of that fluid particle at $t = 0$. As Holland notes,

> This step is not merely of mathematical significance for the labeling allows us to conceive of fluid functions such as density and pressure in terms of notions not available in the Eulerian picture, namely, interparticle interactions described by the deformation matrix $\partial q_i/\partial a_l$. (Holland 2017, p. 340)

The full congruence of all the particle trajectories is described by the displacement function $q_i(a, t)$. This displacement function provides a new, alternative conception of the quantum state, in place of the wavefunction Ψ. In order to have a complete Lagrangian hydrodynamic representation of QM, one needs an independent way of calculating the congruence of trajectories. Following Holland (2017), if one substitutes $x_i = q_i(a, t)$ into the quantum analog of Euler's force law

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = \frac{1}{m}\frac{\partial}{\partial x_i}(V + V_Q), \quad (13)$$

then one obtains

$$m\frac{\partial^2 q_i(a)}{\partial t^2} = -\frac{\partial}{\partial q_i}(V(x) + V_Q(x))\Big|_{x=q(a,t)} \quad (14)$$

[17] There were also efforts toward a Lagrangian quantum hydrodynamics developed as an *approximation scheme* (not full formulation) in the quantum chemistry community, most notably by Robert Wyatt (2005).

where the derivatives with respect to q_i are defined as

$$\frac{\partial}{\partial q_i} = J^{-1} J_{ij} \frac{\partial}{\partial a_j}$$

and J_{ij} is the adjoint of the deformation matrix $\partial q_i / \partial a_l$. Holland concludes that Equation (14) should be understood as Schrödinger's equation in the form of Newton's second law. To have a flow representative of Schrödinger evolution, one must restrict the admissible solutions to what Holland calls a "quasi-potential" flow.[18] He concludes,

> The quantum state is now represented by the 'displacement amplitude' $q_i(a, t)$ encoding the history of an infinite ensemble of particles whose interaction is described by the derivatives of q_i with respect to a_i. . . . With the appropriate initial conditions the vector $q_i(a)$ determines the motion completely, without reference to ψ(x). (Holland 2017, p. 342)

The Lagrangian model of quantum hydrodynamics provides a complete formulation of quantum mechanics and an alternative representation of the quantum state. It is thus a formulation of *quantum mechanics without wavefunctions*.[19] Furthermore, beginning with the congruence of trajectories it is possible to deduce the time-dependence of the wavefunction.[20] In other words,

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[18] The restriction to quasi-potential flow means the initial velocity field is of the form $\frac{\partial q(a)}{\partial t} = \frac{1}{m}\frac{\partial S(a)}{\partial a}$, but the flow is not irrotational everywhere since the quantum phase obeys the quantization condition. Vortices occur only in the nodal regions, where the density vanishes.

[19] Indeed this is the title of the paper Schiff and Poirier (2012) where they further emphasize that what we are here calling Lagrangian quantum hydrodynamics is a "standalone reformulation of quantum mechanics, that neither relies on the TDSE [time-dependent Schrödinger equation] nor makes any mention of any external constructs such as Ψ" (p. 031102-1).

[20] Although the Eulerian variables ρ, ν are not canonical, Holland also shows how one can set up a canonical formulation of Eulerian QH by introducing potentials for the velocity, which result in a Clebsch-like representation. One can define new dependent variables Q, P_i which are canonically conjugate (defining 3 pairs position & momentum variables for each space point, whose temporal development is governed by Hamilton's equations) which become the new descriptors of the state. Hamilton's equations plus initial conditions imply the Schrödinger

the congruence of trajectories can be understood as the fundamental quantum state entity through which to understand the time evolution of the system.

Although the hydrodynamic formulation of quantum mechanics was initially developed for a single-body, spin-0 system (Holland 2005a, b), Holland shows how it can be readily extended to an n-body system by allowing the indices *i, j, …* to range over $3n$ values and the congruence of trajectories in configuration space can be mapped into ensembles of interlacing trajectories in 3-space (2017, p. 343). Holland (2006) further extends the Lagrangian hydrodynamic model to spin-1/2 systems by allowing the fluid particles to have an internal rotational freedom. The standard angular momentum approach, where the rotational degrees of freedom appear as discrete indices in the wavefunction, does not provide sufficient information (since the fluid quantities are "averaged" over these indices), so Holland instead uses the angular *coordinate* representation, where the spin degrees of freedom are represented by the continuous parameter Euler angles, α. To construct a fully general Lagrangian hydrodynamic model, Holland generalizes to an arbitrary-dimensional Riemannian manifold, M, with a static metric $g_{\mu\upsilon}(x), \mu, \nu, \ldots = 1, \ldots, N$, where the history of the fluid is encoded in the positions $\xi(\xi_0, t)$ of the distinct fluid elements at time *t*. The result is a simpler and physically clearer hydrodynamic model embracing both fermions and bosons, that is a straight-forward generalization of the spin-0 theory (Holland 2006, p. 371).[21]

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equation via the QH equations (continuity and Euler-type force law). Thus, the trajectory formulation of QM can be obtained by a canonical transformation of wave mechanics when the latter is formulated in terms of the hydrodynamic phase space variables or gauge potentials $Q_i(x), P_i(x)$. See Holland 2017, Section 4 for details.

[21] Holland also notes that this provides an alternate method of quantization: from the single particle case one can pass to a continuum of particles, introduce the interparticle interaction, generalize to Riemann space with external scalar and vector potentials, and finally pass to the Eulerian description (Holland 2011, p. 85).

Holland similarly uses this method to construct a full (Eulerian and Lagrangian) hydrodynamic "fluid" model of the electromagnetic field (a feat which famously eluded the 19th century ether theorists, though of course here without the literal physical interpretation). More specifically he shows how "the relativistic spin-1 field obeying the source-free Maxwell equations can be computed from the Lagrangian trajectories" (Holland 2005b, p. 3660). As in the spin-1/2 case, one endows the fluid particles with a rotational degree of freedom and makes use of the generalized Riemannian manifold construction, from which electromagnetic theory can be extracted as a special case.[22] Thus he provides a

> continuum mechanics model from which we may deduce Maxwell's equations as the Eulerian counterpart to the equation of motion of the Lagrangian trajectories, and in particular with an algorithm to compute the electromagnetic field from the latter. (Holland 2005b, p. 3671)

He further argues that this method of construction can be extended to produce a Lagrangian trajectory formulation of any generic field theory (Holland 2012, 2013). This work thus provides researchers with new conceptual and computational resources for exploring field theories.

V. Exploring Lagrangian Quantum Hydrodynamics

With the details of this hydrodynamic formulation of quantum mechanics in hand, let us now turn to exploring some of the broader questions about this formalism. In particular we, first, distinguish the LQH formulation from the de Broglie-Bohm interpretation, which similarly uses trajectories. Second, we examine the novel particle-relabeling symmetry and conservation laws that arise in the LQH formulation. And third, we explore whether there is any sense in which the

[22] This method makes use of the fact that Maxwell's 'curl' equations are a form of Schrödinger's equation (with a constraint from the divergence equations) and that the corresponding wave equation has a continuous representation in the Euler angles (Holland personal communication).

LQH formulation of quantum mechanics should be thought of as more fundamental than the Ψ representation.

a. LQH is not the de Broglie-Bohm Interpretation

It is important to clearly distinguish between Lagrangian quantum hydrodynamics (LQH) and the de Broglie-Bohm (dBB) interpretation of quantum mechanics. The dBB interpretation differs from LQH in five respects. First, dBB attributes ontological status to the wavefunction, while LQH is a formulation of quantum mechanics without wavefunctions. Second, dBB depends on the wavefunction to obtain the trajectories; that is, one first solves the time-dependent Schrödinger equation to obtain the wavefunction, and then uses this to derive the quantum trajectories. On LQH, by contrast, one obtains the trajectories directly by solving the second-order Newtonian-type equation (Equation 14). Third, unlike LQH, dBB postulates the existence of a corpuscle of mass *m* following one of the trajectories. As one researcher notes, Bohm was interested in "corpuscle propagation using information gleaned from precomputed wave functions" (Wyatt 2005, p. 2), while LQH dispenses with both the corpuscle and the wavefunction, and instead determines the fluid trajectories directly.

The similarity to note between LQH and deBB is that one of the paths of the fluid parcels coincides with that of the dBB corpuscle. However, as Holland notes, “The corpuscle is . . . to be distinguished from a fluid particle both in its mass and in its dynamical behavior” (Holland 2005a, p. 525). This points to a fourth difference between LQH and dBB: On dBB, the motion of the corpuscle is determined by the external quantum potential, while on LQH approach the trajectories are determined by the interactions between the fluid parcels themselves. This arises because the LQH introduces new variables (the particle labels) into the quantum formalism, while the dBB interpretation does not. This means that the LQH fluid parcels do not suffer from the dBB problem of no back-reaction (i.e., the asymmetry of the wavefunction, via the quantum potential, acting on the particle, but the particle not back-reacting on the wavefunction).[23]

Finally, the fifth and perhaps most salient difference between dBB and LQH is that the former is an *interpretation* of quantum mechanics, and the latter is best understood as a *formulation* of quantum mechanics. While the dBB interpretation seeks to solve the measurement problem by underpinning the indeterministic and statistical story told by standard quantum mechanics with a completely deterministic, causal story of individual quantum systems and processes, the LQH formulation, on its own, has no such interpretive ambitions.[24]

b. Symmetries in Quantum Hydrodynamics

The Lagrangian formulation of quantum hydrodynamics provides new insights into symmetries and conservation laws in quantum mechanics. Most strikingly, it allows for the discovery of a novel quantum symmetry that is obscured on the standard formulation. Recall that the Lagrangian formulation of quantum hydrodynamics introduces new variables into the quantum formalism, namely the particle labels, a_i. Moreover, no particular choice of particle label is preferred. This freedom corresponds to a new quantum symmetry or gauge freedom, namely the infinite-parameter particle relabeling group, with respect to which the Eulerian variables of position, density, and velocity, are invariant. As Holland explains,

> The origin of the relabelling symmetry is that the deformation coefficients (derivatives of the current position with respect to the label) appear in the field equations only through

[23] See, for example, Holland 1993, Section 3.3.2 for a discussion of this feature of de Broglie-Bohm.

[24] Not surprisingly, some have tried to turn the LQH approach into an interpretation of quantum mechanics, such as in what is known as the "many interacting worlds" (MIW) interpretation (Hall et al. 2014). I will return to briefly discuss this interpretation in the final section.

> the Jacobian. . . . [This is] a characteristic feature of fluid mechanics that is not displayed in other continuum theories. (Holland 2013, p. 57)

By Emmy Noether's (first) theorem, we know that this relabeling symmetry implies a conservation law. Indeed this symmetry is the quantum analog of the classical symmetry in ordinary hydrodynamics, examined earlier, which is responsible for the vorticity and circulation conservation theorems of Helmholtz, Kelvin, and Ertel. In the Lagrangian picture one can, for example, derive the *quantum* version of Kelvin's conservation of circulation theorem (Holland 2017, p. 342):

$$\frac{\partial}{\partial t}\oint \dot{q}_i \, dq_i = 0 \ , \tag{15}$$

which states that the closed loop of particles remains closed during the flow due to the continuity of the function $q_i(a)$. There is also a conserved density and current associated with the relabeling symmetry (Holland 2013, p. 71):

$$P(a,t) = -m\rho_0 \frac{\partial q_i}{\partial t}\frac{\partial q_i}{\partial a_j}\xi_i, \qquad J_i(a,t) = \rho_0 \xi_i \left(\frac{1}{2}m\frac{\partial q_i}{\partial t}\frac{\partial q_i}{\partial t} - V(q(a,t)) - V_Q\right) ,$$

which also have analogs in classical hydrodynamics. Although the importance of these conservation theorems is indisputable in the classical context, their significance in non-relativistic quantum mechanics is less clear. Moving to the relativistic context, however, Poirier (2017) has argued that the conservation law associated with the particle relabeling symmetry is what allows one to define global simultaneity manifolds, restoring absolute simultaneity, and allowing a foliation into a 3+1 space and time.

The Lagrangian hydrodynamic formulation of electromagnetism, mentioned earlier, also provides an alternative perspective on the Lorentz covariance of the theory. The fluid paths in the Lagrangian hydrodynamic formulation do not form a Lorentz covariant structure. In that respect they are similar to the electric and magnetic field lines, or the energy flow lines derived from Poynting's vector, which, though not a covariant structure themselves, are derived from the Lorentz covariant electromagnetic theory. Holland argues that the fact that the

> non-covariant hydrodynamic ones [trajectories] may be employed as a basis from which to derive the relativistic theory. . . . suggests that [the Eulerian field variables] $\rho(x)$and $S(x)$ (and hence $E(x)$ and $B(x)$) may be regarded as 'collective coordinates'—functions that describe the bulk properties of the system without depending on the complex details of the particulate substructure. Features peculiar to the Eulerian picture, such as Lorentz covariance, may therefore be viewed as collective rather than fundamental properties. (Holland 2005b, p. 3678)

This question of whether the Lagrangian formulation should be considered as, in some sense, more fundamental than the Eulerian formulation is an interesting issue that comes up in the context of classical hydrodynamics as well, and will be briefly examined next.

c. Is the Lagrangian Picture More Fundamental?

The Lagrangian and Eulerian formulations of hydrodynamics are taken to be—in some not yet philosophically precise sense—equivalent.[25] It is important to note, however, that they are not merely coordinate transformations of each other, but rather are two different ways of specifying the fluid flow (i.e., one can use the Eulerian and Lagrangian formulations in any frame of reference and using any coordinate system). A recent textbook describes the relation between the Eulerian and Lagrangian formulations as follows:

> [The] two approaches ultimately describe physical objects that are equivalent to one another, as well-defined mathematical manipulations transform one perspective into the other. In certain practical situations, however, one approach may be superior to the other . . . both in terms of the mathematical formulation of a given problem and its interpretation. (Regev et al. 2016, p. 2)

[25] There is actually a substantive philosophical debate about what it means for two theories, models, or formalisms to be "theoretically equivalent" (see, for example, Nguyen 2017 and references therein).

In other words, although they are two ways of describing the same physical system, one formulation may be superior to another in a given situation in terms of the calculational tractability or the physical insight it provides.

Despite the empirical equivalence and inter-transformability of the Lagrangian and Eulerian descriptions, there are two (related) arguments one finds in the classical fluid dynamics literature for why the Lagrangian formulation should be thought of as the more fundamental description.[26] The first argument comes from the perspective of geometric mechanics and reduction theory. Reduction is a powerful tool in the study of mechanical systems, whereby one can exploit conserved quantities and symmetry groups to reduce the dimensions of a phase space.[27] We can think of a Hamiltonian system as consisting of a phase space and two geometrical objects: the Poisson bracket $\{,\}$ and the Hamiltonian, *H*. When the Poisson bracket is *singular* there exist a set of what are called Casimir functions $\{C\}$ for which $\{C, A\} = 0$ for every function *A*, including, H, meaning *C* is conserved by the dynamics. As Salmon points out,

> Singular Poisson brackets typically arise from a transformation from canonical coordinates (in which the bracket is nonsingular) to a reduced set of (fewer) coordinates in which the dynamics comprises a fewer number of equations but is nevertheless closed. Then . . . the Casimirs are the conserved quantities corresponding to the symmetries that permit the reduction. (Salmon 2014, p. 339)

In fluid mechanics, the reduction is precisely the transformation from the (canonical) Lagrangian variables to the non-canonical Eulerian variables. There is thus a sense in which the Lagrangian formulation is a more complete description of the fluid. As Salmon notes,

> If we know [the Eulerian variables of velocity, density, and entropy at fixed locations], then we know everything we need to compute the Hamiltonian . . . but our knowledge of

[26] As we will see, the appropriate notion of "fundamentality" still needs to be elaborated.
[27] The inverse of reduction is known as reconstruction, whereby one can get back the dynamics of the full Hamiltonian system from the reduced system; holonomies, related to phenomena such as Berry's phase and A-B effect, can be viewed as an instance of reconstruction (Marsden and Ratiu 1999, p. 256).

> the fluid motion is incomplete; we cannot say which fluid particle went where. (Salmon 2014, p. 337)

The symmetry that permits the reduction from Lagrangian variables (the velocities and locations of marked fluid particles) to the Eulerian variables—and thus what is responsible for there being a closed Eulerian formulation of fluid mechanics—is the relabeling symmetry.

This last point brings us to the second, related, argument for why the Lagrangian formulation can be thought of as more fundamental: The general law of vorticity conservation is a consequence of the relabeling symmetry, which can only be articulated on the Lagrangian formulation. As one researcher remarks, despite the central importance of the Eulerian description,

> the general vorticity law cannot be formulated without referring to the positions of marked fluid particles; an example proving that the continuum model of the fluid particles and the Lagrangian description are in a sense more fundamental than the Eulerian description. (Sieniutycz 1994, p. 55)

The physical oceanographer Peter Müller likewise notes,

> The [usual analysis of vorticity conservation] is unsatisfactory since it neither reveals the underlying cause for the material conservation of potential vorticity nor offers any explicit expressions for homentropic and homogeneous fluids. These issues become resolved in a Lagrangian description of the fluid motion. (Müller 1995, p. 72)

And Salmon similarly writes,

> [T]he general vorticity law cannot be stated without referring to the locations of marked fluid particles. This is but one of several important examples in which the greatest simplicity and generality are achieved only by considering the complete set of Lagrangian fluid variables. These examples suggest that the primitive picture of a fluid as a continuous distribution of massive particles is in some sense the more fundamental, and that the simplicity of the conventional Eulerian description has been purchased at a definite price. (Salmon 1988, p. 226)

Interestingly the relevant notion of 'fundamentality' being used here is not that of being less idealized or closer to the truth. Both the Eulerian and Lagrangian formulations are continuum approximations for fluids that are ultimately understood to be composed of discrete molecules.

Yet, as Salmon notes, "the particle-relabeling symmetry property is unique to fluid mechanics. It has no analogue in discrete-particle mechanics, where the particle labels cannot be varied continuously" (Salmon 2014, p. 304).

Returning to the quantum context, one might similarly argue that the Lagrangian formulation of quantum hydrodynamics is more fundamental, where again 'fundamental' need not be construed as 'closer to the truth'. Recall that on the Lagrangian quantum hydrodynamics approach, the quantum state is represented by the full congruence of fluid particle (parcel) trajectories described by the displacement function $q_i(a, t)$. As discussed earlier, these trajectories can be calculated directly (by means of Equation 14) and provide a self-contained formulation of quantum mechanics, without reference to the wavefunction. Moreover, as Holland shows, if one wishes, one can then derive the time-dependent wavefunction from these independently calculated trajectories. Hence, "one may make the displacement function of the collective the basis of the quantum description with the wavefunction being regarded as a derived quantity" (Holland 2017, p. 334). In other words, rather than viewing the wavefunction as the fundamental entity and the trajectories as a derived or interpretive overlay, on the Lagrangian approach it is natural to view the trajectories as the more fundamental description. In analogy with the classical hydrodynamics case, on the Eulerian formulation of quantum hydrodynamics (e.g., of Madelung), one may view the Eulerian functions $\rho(x)$ and $S(x)$ as the "'collective coordinates'—functions that describe the bulk properties of the system without depending on the complex details of the particulate substructure" (Holland 2006, p. 384).

Given the equally idealized and effective nature of both the Eulerian and Lagrangian formulations (of either classical hydrodynamics or quantum mechanics), their physical equivalence, and their intertransformability, I am not sure these arguments about fundamentality are likely to convince anyone not already sympathetic to the view. However, what I think these arguments do succeed in doing is shifting the burden of proof: there are no longer grounds for automatically assuming that the Eulerian field picture and the wavefunction representation of the quantum state (or relatedly the $\rho(x)$ and $S(x)$ representation of the quantum state) are privileged.[28]

VI. Beyond the Wavefunction Hegemony

With this alternative formulation of quantum mechanics in hand, let us return to drawing out some of the preliminary implications of LQH for various realist projects in quantum theory. While a full exploration of these implications is not possible here, some promising avenues for future work are indicated. Most immediately, the LQH formulation, with its displacement representation of the quantum state, $q_i(a, t)$, challenges the hegemony of the wavefunction. Unlike the de Broglie-Bohm interpretation of quantum mechanics, which has a dual ontology of trajectories and wavefunctions (or pilot waves), the LQH dispenses with the wavefunction entirely; the quantum evolution is borne by the congruence of trajectories alone. Hence it shows how it is possible to do "quantum mechanics without wavefunctions." The two pictures (Ψ and q_i) are, of course, inter-transformable, and share the common mathematical data of the initial conditions. That is, the initial density and velocity of the congruence is equal to the initial squared magnitude and phase gradient of Ψ_0. Not assuming one picture to be more fundamental than the other, one could say that certain mathematical functions (initial data) may be ascribed

[28] That, of course, doesn't mean that one formulation might not be pragmatically preferred in a given situation, just like one may find Feynman's path integral formulation of quantum mechanics more useful in some situations.

different interpretations.[29] Although the trajectory formulation does not disprove wavefunction realism, it does shift the burden of proof, raising substantive questions about what ontological conclusions can legitimately be read from the formalism, given the non-necessity of the wavefunction concept of state.

Not only does LQH show that one can formulate quantum mechanics without wavefunctions, but it also provides a different perspective on experiments that claim to measure the wavefunction of a single system as an extended object. In a well-known paper, Yakir Aharonov and colleagues argue that so called "protective measurements", which use a suitable adiabatic interaction to measure the expectation values of operators without appreciably disturbing a quantum state, provides evidence that the wavefunction is ontologically real (Aharonov et al. 1993). Holland (2017) shows, however, what is directly measured in such cases are the hydrodynamic variables (ρ and v), and not the wavefunction itself. Since these (Eulerian) hydrodynamic variables can be used to construct either the $\psi\,(x,t)$ or $q_i\,(a,t)$, the protective measurement scheme cannot be used to privilege the wavefunction conception of the quantum state as being more ontologically real than the displacement conception.

As noted in the introduction, a consideration of alternative formalisms is not only helpful in avoiding an elision between the mathematical formalism and what it represents, but can also challenge the presumption that some feature of the mathematical representation is also necessarily a feature of the world. A striking example of this is the claim by wavefunction realists that because the wavefunction Ψ lives in a $3n$-dimensional configuration space (where n is the number of particles in the universe), we too must live in this high-dimensional abstract space, and the three-dimensional physical space of our experience is in fact an illusion (e.g.,

[29] I owe this way of expressing it to Holland (personal communication).

Albert 1996, p. 277). This view, sometimes referred to as "configuration-space realism," arises from a literalist approach to scientific realism, as others have noted (e.g., Dorato and Laudisa 2015, p. 120).

Against configuration-space realism, the LQH formalism shows that it is possible to represent the quantum state of a many-body system as a set of states in ordinary, three-dimensional physical space. As is well known, the many body wavefunction for a system of n particles with masses m_r with $r = 1, ..., n$ is $\psi(x_1, ..., x_n)$, which is defined in a $3n$-dimensional configuration space. In the equivalent LQH representation, such an n-body quantum state is represented as a single-valued congruence of curves q_{ri} $(a_1, ..., a_n)$ in the $3n$-dimensional configuration space, where the indices r,i collectively range over the $3n$ values. The $a_1, ..., a_n$, recall, uniquely label the initial positions $q_{r0i} = a_{ri}$. As Holland emphasizes in a recent paper,

> From the grouping of the indices, we see immediately that in this picture each configuration space trajectory is composed of n trajectories in three-dimensional physical space, the r^{th} trajectory being given by the position vector q_{ri}. The whole nondenumerable configuration space congruence is therefore composed of n families of trajectories in 3-space. *The n-body quantum state may be represented as a collection of n states in 3-space*. (Holland 2018, p. 269-3; emphasis original)

In other words, on the LQH formulation, the many-body quantum state can be represented in ordinary physical 3-space, consistent with our experience—no grand illusions required.

In the generalization of the previously-discussed one-body case to a many-body system, the Schrödinger equation can be cast as a set of n Newtonian-type equations describing the coupled evolution of the set of n displacement 3-vectors:

$$m_r \frac{\partial^2 q_{ri}(a_i, \cdots, a_n)}{\partial t^2} = -\frac{\partial}{\partial q_{ri}} \left[V(x_i, \cdots, x_n) + V_Q(x_i, \cdots, x_n) \right] \bigg|_{x_r = q_r(a_1, \cdots, a_n, t)}$$

As Holland explains, from this equation we see that the trajectory q_{ri} is generally coupled with all the other current locations, such that if the r^{th} family of trajectories is acted upon by an external force, the whole congruence will respond. In this way, nonlocality is still captured in this trajectory formulation, as one would expect. In sum, the LQH formulation undermines configuration-space realism by showing how one can represent the full many-body quantum state as living in ordinary 3-space, while still capturing the essential feature of nonlocality.

As emphasized at the outset, LQH is a full mathematical representation or formulation of quantum mechanics—not an interpretation. Nonetheless in the context of realist explorations of quantum theory, one can ask what the further project of interpreting this LQH formalism might look like. In other words, what hints might this formalism give us about the way the world really is? So far, three different philosophical approaches towards LQH have emerged, which I will refer to as the interpretations approach, the duality approach, and the inferential realist approach.

The first option is to turn LQH into an interpretation of quantum mechanics. Examples of this approach include the "Many Interacting Worlds (MIW)" interpretation of Bill Poirier (2010) and Michael Hall and colleagues (Hall et al. 2014) or the "Newtonian QM" of Chip Sebens (2015). In contrast with Bohmian mechanics, which takes only one trajectory to be real, Poirier notes that on this alternative interpretation,

> one might prefer to regard *all* trajectories in the quantum ensemble as equally valid and real. It is hard to imagine how this could be achieved without positing that each trajectory inhabits a separate world. . . . this version of the many worlds interpretation would be *very* different from the standard form. (Poirier 2010, p. 14)

Similarly Sebens describes this interpretation as a novel no-collapse interpretation that

> combines elements of Bohmian mechanics and the many-worlds interpretation to form a theory in which there is no wave function. . . . Unlike the many worlds of the many-worlds interpretation, these worlds are fundamental, not emergent; they are interacting, not causally isolated; and they never branch. (Sebens 2015, p. 267)

To make this interpretation fly, however, one must assume a finite number of worlds. As Sebens notes,

> The meaning of ρ becomes unclear if we move to a continuous infinity of worlds since we can no longer understand ρ as yielding the proportion of all worlds in a given volume of configuration space upon integration over that volume. (Sebens 2015, p. 283)

Michael Hall and colleagues similarly try to avoid the "ontological difficulty" of a continuum of worlds by "replacing the continuum of fluid elements in the Holland-Poirier approach by a huge but finite number of interacting 'worlds'" (Hall et al. 2014, p. 1).[30] By moving to a finite number of trajectories or worlds, this is strictly speaking a break from the full LQH formulation of standard QM. Recall that LQH is based on a continuum mechanics approach which, like classical hydrodynamics, rests on an infinite continuum of "fluid" parcel trajectories. So the MIW interpretation is only equivalent to standard quantum dynamics in the limit where the number of worlds becomes uncountably infinite.

Instead of turning this formulation of quantum mechanics into an interpretation, the second approach takes a step back and asks what realist implications might follow from the fact that quantum mechanics admits of both a wavefunction and a trajectory concept of state. Holland himself sees this as indicative of a new kind of wave-particle duality. He writes,

> The full hydrodynamic model of quantum mechanics therefore provides an interpretation of two pictures—the wave-mechanical (Eulerian) and the particle (Lagrangian), and the latter is just as valid a representation of quantum processes as the former. . . . This mapping therefore gives a new and mathematically precise meaning to the notion of "wave-particle duality." (Holland 2005, p. 508)

These two pictures each emphasize and bring out different aspects of the quantum state. For example, the second order Newtonian equation of the trajectory picture, discussed above, highlights the force behind quantum propagation. Holland moreover argues that this wave-particle duality is not unique to quantum theory, but rather is a generic feature of field theories which can admit of Lagrangian trajectory formulations, such as in the case for electromagnetism

[30] Hall et al. note their approach is "broadly similar" to that of Sebens (2015).

(Holland 2005b) and relativity (Holland 2012). Whether this duality is to be understood simply as a feature of our mathematical representations, or whether it is to be read as revealing an ontological duality inherent in the world remains unclear.

A third approach might be along the lines of what I call inferential realism. Similar to the duality approach, this approach emphasizes the importance of a plurality of representations, though it does not read these two pictures of the quantum state as a joint depiction of some more fundamental ontological duality. Inferential realism shifts the focus of the realism question from 'what there is,' to 'what true things can we learn.'[31] Inferential realism rejects the literalist approach to scientific realism characteristic of van Fraassen, and instead traces its roots back to Ernan McMullin's (1984) understanding of realism as the development of scientifically fruitful metaphors.

In the heyday of the realism-antirealism debate in the 1980s there were two diametrically opposed conceptions of scientific realism. One conception, articulated by the antirealist van Fraassen, construed scientific realism as follows:

> *Science aims to give us, in its theories, a literally true story of what the world is like, and acceptance of a scientific theory involves the belief that it is true.* This is the correct statement of scientific realism. (van Fraassen 1980, p. 8; emphasis original)

Although van Fraassen claimed that this is a minimal construal that any realist would assent to, McMullin, one of the chief defenders of scientific realism at the time, flatly rejected this way of conceiving the realist project. In his paper "A Case for Scientific Realism", McMullin outlines a very different conception of realism:

> Science aims at fruitful metaphor and at ever more detailed structure. . . . The realist would not use the term 'true' to describe a good theory. He [or she] would suppose that

[31] Although these questions are often linked, they can also come apart in significant ways, as discussed in Bokulich (2016).

> the structures of the theory give some insight into the structures of the world. (McMullin 1984, p. 35)

Here we see McMullin rejecting the view that realism is committed to a literal construal of scientific theories, and instead conceiving of the realist project as one of developing fruitful metaphors, analogies and models. On this alternative view, the realism comes in the new discoveries, insights, and deeper understanding that these metaphors, analogies and models enable, rather than in their interpretation as a literal depiction of world.[32]

In this tradition, inferential realism is not about finding the one true depiction of the world, but rather about developing a plurality of fertile representations. Representations, like depictions, are aimed at and purport to tell us about the world, but their connection is often looser involving pragmatic elements; hence, a greater caution is required in drawing ontological conclusions. Moreover, different representations can be more or less useful in different contexts and domains. This view is nonetheless committed to realism because it holds that these representations yield genuine knowledge and advance our understanding of the actual world.

When trying to decide what philosophical attitude to take towards LQH, it is helpful to contextualize it within the broader class of Lagrangian hydrodynamic approaches in the physical sciences (e.g., in geophysical fluid dynamics and astrophysics), as I have done here. Within this broader context, it would be odd to say that the Eulerian formulation, for example, is the only legitimate mathematical formulation of classical hydrodynamics (as one might with the standard Ψ formulation of QM); or at the other extreme, it would be odd to allow the Lagrangian formulation only if it is taken as a literal depiction of real fluids (as for example in the MIW interpretation). In the classical case, we clearly see the enormous fertility and explanatory power

[32] For a discussion of how metaphors, analogies and models, despite being not literally true, can nonetheless give true insights into the physical systems they represent, again see "Fiction as a Vehicle for Truth" (Bokulich 2016).

of the hydrodynamic analogy (and formalism of continuum mechanics more broadly) as a non-literal representation of our world. The credentials of classical hydrodynamics come not from its status as a literal depiction of the world, but rather from the many correct inferences it licenses.

In the context of quantum theory, the hydrodynamic analogy has given rise to the discovery of a new way of representing and time-evolving the quantum state. Although it is an empirically equivalent formulation of quantum mechanics, it nonetheless has profound implications for the quantum realism debate, as we have seen. In particular, the LQH formulation, with its displacement or trajectory representation of the quantum state, challenges the hegemony of the wavefunction. By showing that the wavefunction is neither a necessary—nor even the most fundamental—representation of quantum systems, LQH undercuts the central argument for wavefunction realism. Just as strikingly, it falsifies the claim that configuration space realism is a necessary consequence of any realist understanding of quantum mechanics, by showing how one can represent an *n*-body quantum system as a set of *n* states in ordinary three-dimensional space. To reiterate, it has not been my contention that we should replace the Ψ conception of state with the trajectory one—we should admit both. Regardless of what philosophical attitude one takes towards the LQH formulation, regaining sight of the proverbial forest of quantum representations beyond the Ψ is an essential step in exploring the realist implications of quantum theory.

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