F. Bou
U. Riviecco

Bilattices with Implications

Abstract. In a previous work we studied, from the perspective of Abstract Algebraic Logic, the implicationless fragment of a logic introduced by O. Arieli and A. Avron using a class of bilattice-based logical matrices called *logical bilattices*. Here we complete this study by considering the Arieli-Avron logic in the full language, obtained by adding two implication connectives to the standard bilattice language. We prove that this logic is algebraizable and investigate its algebraic models, which turn out to be distributive bilattices with additional implication operations. We axiomatize and state several results on these new classes of algebras, in particular representation theorems analogue to the well-known one for interlaced bilattices.

Keywords: Bilattice, Representation of bilattices, Brouwerian lattice, Disjunctive lattice, Generalized Boolean algebra, Category of bilattices, Algebraic logic.

Introduction

Bilattices are algebraic structures introduced by M. Ginsberg [22] as a uniform framework for inference in Artificial Intelligence. Since then they have found a number of applications, sometimes in quite different areas from the original one. The interest in bilattices has thus different sources: among others, computer science and AI (see especially the works of Ginsberg, O. Arieli and A. Avron), logic programming (M. Fitting), lattice theory and algebra [32, 23, 24] and, more recently, algebraic logic [10, 9]. An up-to-date review of the applications of this formalism and also of the motivation behind its study can be found in the PhD dissertation [30].

In the nineties bilattices have been investigated in depth by Arieli and Avron, both from an algebraic [4, 5] and a logical point of view [1, 3]. In [2] they introduced the first bilattices-based logical systems in the standard sense, with the main aim to deal with paraconsistency and non-monotonic reasoning in AI. The simplest of these logics (that we call $\mathcal{LB}$) is defined semantically from a class of matrices called *logical bilattices* and is a conservative expansion of the Belnap-Dunn four-valued logic [17, 6, 7] to the standard language of bilattices. In [2] a Gentzen-style calculus is presented as a syntactic counterpart of $\mathcal{LB}$, and completeness and cut elimination are

Special Issue: Abstract Algebraic Logic
Edited by **Josep Maria Font and Ramon Jansana**

Studia Logica (2013) 101: 651–675
DOI: 10.1007/s11225-013-9494-3

proved. In the same work the authors consider also an expansion of $\mathcal{LB}$, obtained by adding two (interdefinable) implication connectives (we denote the logic in this expanded language by $\mathcal{LB}_{\supset}$). In [2] both a Gentzen- and a Hilbert-style presentation of $\mathcal{LB}_{\supset}$ are given, and completeness and cut elimination for the Gentzen calculus are proved.

The aim of the present work is to complete the algebraic analysis of $\mathcal{LB}$ begun in [10]. While there we focused on the implicationless fragment of the Arieli-Avron logic, here we are going to deal with the full language. Most of the results that we are going to present here were originally obtained, sometimes in a more limited setting, in the dissertation [30], to which we refer for further details.

We proved in [10] that $\mathcal{LB}$ is non-protoalgebraic and non-selfextensional; however, we managed to characterize its reduced models and reduced generalized models (see [8, 19, 15, 21] for all unexplained Abstract Algebraic Logic terminology). In [30] it is proved that the full logic $\mathcal{LB}_{\supset}$ is algebraizable and its equivalent algebraic semantics is determined, axiomatized and investigated. In the present paper we will take a slightly more general approach, introducing a weaker logic than $\mathcal{LB}_{\supset}$ (that we will call $\mathcal{B}$) and studying its algebraic models, so that the already known results on $\mathcal{LB}_{\supset}$ and its associated class of algebras will be obtained as a particular case.

The paper is organized as follows. Section 1 introduces $\mathcal{B}$ by means of a Hilbert-style calculus and proves that it is algebraizable.

In Section 2 we characterize and study the algebraic counterpart of $\mathcal{B}$. Section 2.1 contains some fundamental definitions and results on bilattices, in particular a theorem stating that all bilattices satisfying certain additional conditions can be represented as a special kind of power of a lattice. Section 2.2 introduces Brouwerian bilattices, that are obtained by adding two new operations (implications) to the standard bilattice language; we state some arithmetical properties of this class and a representation theorem analogous to the one for bilattices. Section 2.3 contains the algebraizability results for $\mathcal{LB}_{\supset}$ and $\mathcal{B}$.

1. The logic $\mathcal{B}$

In this section we will consider propositional calculi defined in the usual way over the language $\langle \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$, with the addition of the defined connectives $\equiv, \rightarrow, \leftrightarrow$. That is, the standard language of bilattice logics $\langle \wedge, \vee, \otimes, \oplus, \neg \rangle$, where $\wedge$ and $\otimes$ are both interpreted as conjunctions, $\vee$ and $\oplus$ as disjunctions and $\neg$ as negation, expanded with a new connective $\supset$ interpreted as implication. As usual, the algebra **Fm** of formulas is the free algebra generated by a countable set $\mathcal{V}ar = \{p, q, r, \ldots\}$ of variables using the above-mentioned algebraic language. Lowercase greek letters φ, ψ, ϑ etc. denote formulas, while uppercase Γ, Δ, etc. denote sets of formulas.

DEFINITION 1.1 (Cf. [2]). Let $\mathcal{B} = \langle \mathbf{Fm}, \vdash_{\mathcal{B}} \rangle$ be the sentential logic defined by the Hilbert-style calculus with axioms:

$$
\begin{array}{lll}
(\supset 1) & p \supset (q \supset p) & \\
(\supset 2) & (p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)) & \\
(\wedge \supset) & (p \wedge q) \supset p & (p \wedge q) \supset q \\
(\supset \wedge) & p \supset (q \supset (p \wedge q)) & \\
(\otimes \supset) & (p \otimes q) \supset p & (p \otimes q) \supset q \\
(\supset \otimes) & p \supset (q \supset (p \otimes q)) & \\
(\supset \vee) & p \supset (p \vee q) & q \supset (p \vee q) \\
(\vee \supset) & (p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r)) & \\
(\supset \oplus) & p \supset (p \oplus q) & q \supset (p \oplus q) \\
(\oplus \supset) & (p \supset r) \supset ((q \supset r) \supset ((p \oplus q) \supset r)) & \\
(\neg \wedge) & \neg(p \wedge q) \equiv (\neg p \vee \neg q) & \\
(\neg \vee) & \neg(p \vee q) \equiv (\neg p \wedge \neg q) & \\
(\neg \otimes) & \neg(p \otimes q) \equiv (\neg p \otimes \neg q) & \\
(\neg \oplus) & \neg(p \oplus q) \equiv (\neg p \oplus \neg q) & \\
(\neg \supset) & \neg(p \supset q) \equiv (p \wedge \neg q) & \\
(\neg\neg) & p \equiv \neg\neg p &
\end{array}
$$

where $\varphi \equiv \psi$ abbreviates $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$, and with modus ponens (MP) as the only inference rule:

$$\frac{p \quad p \supset q}{q}$$

We are going to use the following abbreviations:

$$
\begin{aligned}
\varphi \rightarrow \psi &:= (\varphi \supset \psi) \wedge (\neg\psi \supset \neg\varphi) \\
\varphi \leftrightarrow \psi &:= (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi).
\end{aligned}
$$

If we add to $\mathcal{B}$ the following axiom (Peirce's law):

$$(\supset 3) \qquad ((p \supset q) \supset p) \supset p$$

we obtain the calculus corresponding to the Arieli-Avron [2] logic of logical bilattices with implications. As mentioned above, the dissertation [30] contains a study of this extension, from now on called $\mathcal{LB}_{\supset}$, from an algebraic logic point of view. Many of the results that we are going to state here on the logic $\mathcal{B}$ and its algebraic semantics were originally obtained in [30] for $\mathcal{LB}_{\supset}$ and its associated class of algebras.

Remark 1.2. Let $\Gamma \cup \{\varphi\} \subseteq Fm$ be formulas in the language $\{\wedge, \vee, \supset\}$. Denote by $\vdash_{IPC}$ the derivability relation of the corresponding fragment of intuitionistic propositional logic, where the connectives $\{\wedge, \vee, \supset\}$ are interpreted, respectively, as intuitionistic conjunction, disjunction and implication. Then $\Gamma \vdash_{IPC} \varphi$ implies $\Gamma \vdash_{\mathcal{B}} \varphi$. This follows from the fact that the axioms and rules of $\vdash_{\mathcal{B}}$ involving only $\{\wedge, \vee, \supset\}$ constitute an axiomatization of the $\{\wedge, \vee, \supset\}$-fragment of intuitionistic logic (see for instance the one given in [29, p. 236]). The same holds for formulas in the language $\{\otimes, \oplus, \supset\}$ when we interpret these connectives as, respectively, intuitionistic conjunction, disjunction and implication. It is not difficult to prove that the converse also holds, i.e., that $\Gamma \vdash_{\mathcal{B}} \varphi$ implies $\Gamma \vdash_{IPC} \varphi$, i.e. the two fragments coincide. If we add ($\supset$ 3), then we have an axiomatization of the $\{\wedge, \vee, \supset\}$-fragment of classical logic (see for instance [14, p. 182]) and in this case also it can be proven that the two fragments coincide.

Another remarkable feature that $\mathcal{B}$ shares with intuitionistic logic is the classical Deduction-Detachment Theorem (DDT). Indeed, this is known to hold [33, Theorem 2.4.2] for any calculus that has axioms ($\supset$ 1) and ($\supset$ 2) and MP as the only rule.

Theorem 1.3 (DDT). *Let* $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$. *Then*

$$\Gamma, \varphi \vdash_{\mathcal{B}} \psi \quad \textit{iff} \quad \Gamma \vdash_{\mathcal{B}} \varphi \supset \psi.$$

Thus, if we add ($\supset$ 3) to $\mathcal{B}$, we still have the classical DDT. Moreover, *this logic has no consistent extensions*, i.e., the only stronger logic in the same language is the trivial one in which every formula is derivable [30, Proposition 4.3.14]. This fact can easily be proved algebraically, for, as we will see, the equivalent algebraic semantic of this logic is a variety that has no proper non-trivial subquasivarieties.

Let us list some properties of our calculus [30, Proposition 4.2.2], which will be used in our algebraizability proof.

Proposition 1.4. *For all formulas* $\varphi, \psi, \vartheta, \varphi_1, \varphi_2, \psi_1, \psi_2 \in Fm$,

(i) $\varphi \vdash_{\mathcal{B}} \psi \wedge \vartheta \quad \textit{iff} \quad \varphi \vdash_{\mathcal{B}} \psi \ \textit{and} \ \varphi \vdash_{\mathcal{B}} \vartheta \quad \textit{iff} \quad \varphi \vdash_{\mathcal{B}} \psi \otimes \vartheta$

(ii) $\varphi \supset \psi, \psi \supset \vartheta \vdash_{\mathcal{B}} \varphi \supset \vartheta$

(iii) $\vdash_{\mathcal{B}} \varphi \supset \varphi$

(iv) $\vdash_{\mathcal{B}} \neg(\varphi \supset \varphi) \supset \neg\varphi$

(v) $\varphi \vdash_{\mathcal{B}} \varphi \leftrightarrow (\varphi \supset \varphi)$

(vi) $\varphi \leftrightarrow (\varphi \supset \varphi) \vdash_{\mathcal{B}} \varphi$

(vii) $\vdash_{\mathcal{B}} \varphi \leftrightarrow \varphi$

(viii) $\varphi \leftrightarrow \psi \vdash_{\mathcal{B}} \psi \leftrightarrow \varphi$

(ix) $\varphi \leftrightarrow \psi, \psi \leftrightarrow \vartheta \vdash_{\mathcal{B}} \varphi \leftrightarrow \vartheta$

(x) $\varphi \leftrightarrow \psi \vdash_{\mathcal{B}} \neg\varphi \leftrightarrow \neg\psi$

(xi) $\varphi_1 \to \psi_1, \varphi_2 \to \psi_2 \vdash_{\mathcal{B}} (\varphi_1 \wedge \varphi_2) \to (\psi_1 \wedge \psi_2)$

(xii) $\varphi_1 \to \psi_1, \varphi_2 \to \psi_2 \vdash_{\mathcal{B}} (\varphi_1 \vee \varphi_2) \to (\psi_1 \vee \psi_2)$

(xiii) $\varphi_1 \to \psi_1, \varphi_2 \to \psi_2 \vdash_{\mathcal{B}} (\varphi_1 \otimes \varphi_2) \to (\psi_1 \otimes \psi_2)$

(xiv) $\varphi_1 \to \psi_1, \varphi_2 \to \psi_2 \vdash_{\mathcal{B}} (\varphi_1 \oplus \varphi_2) \to (\psi_1 \oplus \psi_2)$

(xv) $\psi_1 \to \varphi_1, \varphi_2 \to \psi_2 \vdash_{\mathcal{B}} (\varphi_1 \supset \varphi_2) \to (\psi_1 \supset \psi_2)$

(xvi) $\varphi, \psi \dashv\vdash_{\mathcal{B}} \varphi \wedge \psi$ *and* $\varphi \wedge \psi \dashv\vdash_{\mathcal{B}} \varphi \otimes \psi$

(xvii) *if* $\varphi \dashv\vdash_{\mathcal{B}} \psi$*, then* $\vdash_{\mathcal{B}} (\varphi \supset \chi) \leftrightarrow (\psi \supset \chi)$ *for all* $\chi \in Fm$

(xviii) *if* $\varphi_1 \vdash_{\mathcal{B}} \psi_1$ *and* $\varphi_2 \vdash_{\mathcal{B}} \psi_2$*, then* $\varphi_1 \vee \varphi_2 \vdash_{\mathcal{B}} \psi_1 \vee \psi_2$

(xix) *if* $\varphi_1 \vdash_{\mathcal{B}} \psi_1$ *and* $\varphi_2 \vdash_{\mathcal{B}} \psi_2$*, then* $\varphi_1 \oplus \varphi_2 \vdash_{\mathcal{B}} \psi_1 \oplus \psi_2$

(xx) $\vdash_{\mathcal{B}} \varphi \to \psi$ *if and only if* $\vdash_{\mathcal{B}} (\varphi \wedge \psi) \leftrightarrow \psi$.

Proof. (i). The rightward implication is easily proved using $(\wedge \supset)$. As to the leftward one, note that by $(\supset \wedge)$ we have $\varphi \vdash_{\mathcal{B}} \psi \supset (\vartheta \supset (\psi \wedge \vartheta))$, so applying MP twice we obtain $\varphi \vdash_{\mathcal{B}} \psi \wedge \vartheta$. The proof for the case of $\otimes$ is similar, we just need to use $(\otimes \supset)$ and $(\supset \otimes)$ instead of $(\wedge \supset)$ and $(\supset \wedge)$.
(ii). By $(\supset 1)$ and MP we have $\psi \supset \vartheta \vdash_{\mathcal{B}} \varphi \supset (\psi \supset \vartheta)$ and by $(\supset 2)$ we have $\vdash_{\mathcal{B}} (\varphi \supset (\psi \supset \vartheta)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \vartheta))$. So, applying MP, we have $\psi \supset \vartheta \vdash_{\mathcal{B}} (\varphi \supset \psi) \supset (\varphi \supset \vartheta)$. Hence, by MP, $\psi \supset \vartheta, \varphi \supset \psi \vdash_{\mathcal{B}} (\varphi \supset \vartheta)$.
(iii). $(\varphi \supset ((\psi \supset \varphi) \supset \varphi)) \supset ((\varphi \supset (\psi \supset \varphi)) \supset (\varphi \supset \varphi))$ is an instance of $(\supset 2)$ and $(\varphi \supset ((\psi \supset \varphi) \supset \varphi))$ and $(\varphi \supset (\psi \supset \varphi))$ are instances of $(\supset 1)$. So applying MP twice we obtain $\vdash_{\mathcal{B}} \varphi \supset \varphi$.
(iv). $\neg(\varphi \supset \varphi) \supset (\varphi \wedge \neg\varphi)$ is an instance of $(\neg \supset)$ and $(\varphi \wedge \neg\varphi) \supset \neg\varphi$ is an instance of $(\wedge \supset)$. So, by (ii), we obtain $\vdash_{\mathcal{B}} \neg(\varphi \supset \varphi) \supset \neg\varphi$.
(v). Taking into account (i), it is sufficient to prove the following: $\varphi \vdash_{\mathcal{B}} \varphi \supset (\varphi \supset \varphi)$, $\varphi \vdash_{\mathcal{B}} \neg(\varphi \supset \varphi) \supset \neg\varphi$, $\varphi \vdash_{\mathcal{B}} (\varphi \supset \varphi) \supset \varphi$ and $\varphi \vdash_{\mathcal{B}} \neg\varphi \supset \neg(\varphi \supset \varphi)$. The first follows immediately from $(\supset 1)$, while the second follows from (iv). The third amounts to $\varphi, \varphi \supset \varphi \vdash_{\mathcal{B}} \varphi$, which is obvious, and the fourth to $\varphi, \neg\varphi \vdash_{\mathcal{B}} \neg(\varphi \supset \varphi)$, which easily follows from $(\neg \supset)$.

(vi). It is sufficient to prove that $(\varphi \supset \varphi) \supset \varphi \vdash_{\mathcal{B}} \varphi$, and this follows from (iii) by MP.
(vii). Follows immediately from (iii).
(viii). Immediate.
(ix). Follows easily, using (i) and (ii).
(x). It is sufficient to prove that $\varphi \supset \psi \vdash_{\mathcal{B}} \neg\neg\varphi \supset \neg\neg\psi$ and $\psi \supset \varphi \vdash_{\mathcal{B}} \neg\neg\psi \supset \neg\neg\varphi$, and this follows easily using $(\neg\neg)$ and the transitivity of $\supset$.
(xi). We will prove that $\varphi_1 \supset \psi_1, \varphi_2 \supset \psi_2 \vdash_{\mathcal{B}} (\varphi_1 \wedge \varphi_2) \supset (\psi_1 \wedge \psi_2)$ and $\neg\psi_1 \supset \neg\varphi_1, \neg\psi_2 \supset \neg\varphi_2 \vdash_{\mathcal{B}} \neg(\psi_1 \wedge \psi_2) \supset \neg(\varphi_1 \wedge \varphi_2)$. The former is equivalent to $\varphi_1 \supset \psi_1, \varphi_2 \supset \psi_2, \varphi_1 \wedge \varphi_2 \vdash_{\mathcal{B}} (\psi_1 \wedge \psi_2)$, which is easily proved. As to the latter, by $(\supset \vee)$ and the transitivity of $\vdash_{\mathcal{B}}$, we have $\neg\psi_1 \supset \neg\varphi_1, \neg\psi_2 \supset \neg\varphi_2 \vdash_{\mathcal{B}} \neg\psi_1 \supset (\neg\varphi_1 \vee \neg\varphi_2)$ and $\neg\psi_1 \supset \neg\varphi_1, \neg\psi_2 \supset \neg\varphi_2 \vdash_{\mathcal{B}} \neg\psi_2 \supset (\neg\varphi_1 \vee \neg\varphi_2)$. Then, using $(\vee \supset)$, we obtain $\neg\psi_1 \supset \neg\varphi_1, \neg\psi_2 \supset \neg\varphi_2 \vdash_{\mathcal{B}} (\neg\psi_1 \vee \neg\psi_2) \supset (\neg\varphi_1 \vee \neg\varphi_2)$. By $(\neg\wedge)$ we have $\vdash_{\mathcal{B}} \neg(\psi_1 \wedge \psi_2) \supset (\neg\psi_1 \vee \neg\psi_2)$ and $\vdash_{\mathcal{B}} (\neg\varphi_1 \vee \neg\varphi_2) \supset \neg(\varphi_1 \wedge \varphi_2)$. So, applying (ii), we obtain the result.
(xii). We will prove that $\varphi_1 \supset \psi_1, \varphi_2 \supset \psi_2 \vdash_{\mathcal{B}} (\varphi_1 \vee \varphi_2) \supset (\psi_1 \vee \psi_2)$ and $\neg\psi_1 \supset \neg\varphi_1, \neg\psi_2 \supset \neg\varphi_2 \vdash_{\mathcal{B}} \neg(\psi_1 \vee \psi_2) \supset \neg(\varphi_1 \vee \varphi_2)$. As to the first, we have that $\varphi_1 \supset \psi_1 \vdash_{\mathcal{B}} \varphi_1 \supset (\psi_1 \vee \psi_2)$ and $\varphi_2 \supset \psi_2 \vdash_{\mathcal{B}} \varphi_2 \supset (\psi_1 \vee \psi_2)$. Now, using $(\vee \supset)$ we obtain $\varphi_1 \supset \psi_1, \varphi_2 \supset \psi_2 \vdash_{\mathcal{B}} (\varphi_1 \vee \varphi_2) \supset (\psi_1 \vee \psi_2)$. As to the second, using (xi) we have that $\neg\psi_1 \supset \neg\varphi_1, \neg\psi_2 \supset \neg\varphi_2 \vdash_{\mathcal{B}} (\neg\psi_1 \wedge \neg\psi_2) \supset (\neg\varphi_1 \wedge \neg\varphi_2)$. Now using $(\neg\vee)$ and transitivity we obtain the result.
(xiii). We will prove that $\varphi_1 \supset \psi_1, \varphi_2 \supset \psi_2 \vdash_{\mathcal{B}} (\varphi_1 \otimes \varphi_2) \supset (\psi_1 \otimes \psi_2)$ and $\neg\psi_1 \supset \neg\varphi_1, \neg\psi_2 \supset \neg\varphi_2 \vdash_{\mathcal{B}} \neg(\psi_1 \otimes \psi_2) \supset \neg(\varphi_1 \otimes \varphi_2)$. A proof of the first one can be obtained from that of (xi), just replacing any occurence of $\wedge$ with $\otimes$ and using the corresponding axioms for $\otimes$. As to the second, it is easy to prove $\neg\psi_1 \supset \neg\varphi_1, \neg\psi_2 \supset \neg\varphi_2, \neg\psi_1 \otimes \neg\psi_2 \vdash_{\mathcal{B}} \neg\varphi_1 \otimes \neg\varphi_2$ and from this, using $(\neg\otimes)$, we obtain the result.
(xiv). We will prove that $\varphi_1 \supset \psi_1, \varphi_2 \supset \psi_2 \vdash_{\mathcal{B}} (\varphi_1 \oplus \varphi_2) \supset (\psi_1 \oplus \psi_2)$ and $\neg\psi_1 \supset \neg\varphi_1, \neg\psi_2 \supset \neg\varphi_2 \vdash_{\mathcal{B}} \neg(\psi_1 \oplus \psi_2) \supset \neg(\varphi_1 \oplus \varphi_2)$. A proof of the first one can be obtained from that of (xii), just replacing any occurence of $\vee$ with $\oplus$ and using the corresponding axioms for $\oplus$. As to the second, it is easy to prove that $\neg\psi_1 \supset \neg\varphi_1, \neg\psi_2 \supset \neg\varphi_2 \vdash_{\mathcal{B}} (\neg\psi_1 \oplus \neg\psi_2) \supset (\neg\varphi_1 \oplus \neg\varphi_2)$ and from this, using $(\neg\oplus)$ and transitivity, we obtain the result.
(xv). We will prove that $\psi_1 \supset \varphi_1, \varphi_2 \supset \psi_2 \vdash_{\mathcal{B}} (\varphi_1 \supset \varphi_2) \supset (\psi_1 \supset \psi_2)$ and $\psi_1 \supset \varphi_1, \neg\psi_2 \supset \neg\varphi_2 \vdash_{\mathcal{B}} \neg(\psi_1 \supset \psi_2) \supset \neg(\varphi_1 \supset \varphi_2)$. The former is equivalent to $\psi_1 \supset \varphi_1, \varphi_2 \supset \psi_2, \varphi_1 \supset \varphi_2, \psi_1 \vdash_{\mathcal{B}} \psi_2$, which is easily proved by transitivity. In order to prove the latter, note that $\psi_1 \supset \varphi_1, \neg\psi_2 \supset \neg\varphi_2, \psi_1, \neg\psi_2 \vdash_{\mathcal{B}} \varphi_1 \wedge \neg\varphi_2$, so $\psi_1 \supset \varphi_1, \neg\psi_2 \supset \neg\varphi_2 \vdash_{\mathcal{B}} (\psi_1 \wedge \neg\psi_2) \supset (\varphi_1 \wedge \neg\varphi_2)$. Now, by $(\neg \supset)$ we have $\vdash_{\mathcal{B}} \neg(\psi_1 \supset \psi_2) \supset (\psi_1 \wedge \neg\psi_2)$ and $\vdash_{\mathcal{B}} (\varphi_1 \wedge \neg\varphi_2) \supset \neg(\varphi_1 \supset \varphi_2)$. So by transitivity we obtain the result.

(xvi). Easy, using using $(\wedge \supset)$, $(\supset \wedge)$, $(\otimes \supset)$ and $(\supset \otimes)$. In the following proofs we will sometimes make use of this property without notice.
(xvii). Assume $\varphi \dashv\vdash_{\mathcal{B}} \psi$. Using (i), we will prove that for all $\chi \in Fm$:

$$\begin{aligned}
&\vdash_{\mathcal{B}} (\varphi \supset \chi) \supset (\psi \supset \chi)\\
&\vdash_{\mathcal{B}} (\psi \supset \chi) \supset (\varphi \supset \chi)\\
&\vdash_{\mathcal{B}} \neg(\varphi \supset \chi) \supset \neg(\psi \supset \chi)\\
&\vdash_{\mathcal{B}} \neg(\psi \supset \chi) \supset \neg(\varphi \supset \chi).
\end{aligned}$$

The first two are equivalent to $\varphi \supset \chi, \psi \vdash_{\mathcal{B}} \chi$ and $\psi \supset \chi, \varphi \vdash_{\mathcal{B}} \chi$, so they are easily proved. As to the second two, they amount to showing that $\neg(\varphi \supset \chi) \dashv\vdash_{\mathcal{B}} \neg(\psi \supset \chi)$. Now note that by $(\neg \supset)$ we have $\neg(\varphi \supset \chi) \dashv\vdash_{\mathcal{B}} \varphi \wedge \neg\chi$ and $\neg(\psi \supset \chi) \dashv\vdash_{\mathcal{B}} \psi \wedge \neg\chi$. So, it is sufficient to prove that $\varphi \wedge \neg\chi \dashv\vdash_{\mathcal{B}} \psi \wedge \neg\chi$ and, using (i) and (xvi), this is easy.
(xviii). Assume $\varphi_1 \vdash_{\mathcal{B}} \psi_1$ and $\varphi_2 \vdash_{\mathcal{B}} \psi_2$. Note that

$$(\varphi_1 \supset (\psi_1 \vee \psi_2)) \supset ((\varphi_2 \supset (\psi_1 \vee \psi_2)) \supset ((\varphi_1 \vee \varphi_2) \supset (\psi_1 \vee \psi_2)))$$

is an instance of $(\vee \supset)$, and that using $(\supset \vee)$ one can easily derive $\vdash_{\mathcal{B}} \varphi_1 \supset (\psi_1 \vee \psi_2)$ from the first assumption and $\vdash_{\mathcal{B}} \varphi_2 \supset (\psi_1 \vee \psi_2)$ from the second. Hence, by MP, we have $\vdash_{\mathcal{B}} (\varphi_1 \vee \varphi_2) \supset (\psi_1 \vee \psi_2)$, which implies $\varphi_1 \vee \varphi_2 \vdash_{\mathcal{B}} \psi_1 \vee \psi_2$.
(xix). The proof can be easily obtained from that of (xviii), just using the corresponding axioms for $\oplus$ instead of those for $\vee$.
(xx). Note that $\vdash_{\mathcal{B}} (\varphi \wedge \psi) \to \varphi$, because $\vdash_{\mathcal{B}} (\varphi \wedge \psi) \supset \varphi$ is an instance of $(\wedge \supset)$ and $\vdash_{\mathcal{B}} \neg\varphi \supset \neg(\varphi \wedge \psi)$ follows from the fact that, by $(\neg\wedge)$, we have $\neg\varphi \vee \neg\psi \vdash_{\mathcal{B}} \neg(\varphi \wedge \psi)$ and, by $(\supset \vee)$, we have $\neg\varphi \vdash_{\mathcal{B}} \neg\varphi \vee \neg\psi$.
So, assuming $\vdash_{\mathcal{B}} \varphi \to \psi$, we only need to prove that $\vdash_{\mathcal{B}} \varphi \to (\varphi \wedge \psi)$. It is not difficult to prove that $\vdash_{\mathcal{B}} \varphi \leftrightarrow (\varphi \wedge \varphi)$, and by (vi) we also have $\vdash_{\mathcal{B}} \varphi \to \varphi$. Now, using (xi) and the assumption, we obtain $\vdash_{\mathcal{B}} (\varphi \wedge \varphi) \to (\varphi \wedge \psi)$. Finally, using (ix), we obtain $\vdash_{\mathcal{B}} \varphi \to (\varphi \wedge \psi)$.
Conversely, assume $\vdash_{\mathcal{B}} \varphi \to (\varphi \wedge \psi)$. Clearly, the same proof of $(\varphi \wedge \psi) \to \varphi$ shows that $\vdash_{\mathcal{B}} (\varphi \wedge \psi) \to \psi$, so we can apply (i) and the transitivity of $\supset$ and the result follows easily. ∎

Proposition 1.5. *For all formulas $\varphi, \psi, \vartheta \in Fm$,*

(i) $\vdash_{\mathcal{B}} \varphi \leftrightarrow (\varphi \wedge \varphi)$

(ii) $\vdash_{\mathcal{B}} (\varphi \wedge \psi) \leftrightarrow (\psi \wedge \varphi)$

(iii) $\vdash_{\mathcal{B}} (\varphi \wedge (\psi \wedge \vartheta)) \leftrightarrow ((\varphi \wedge \psi) \wedge \vartheta)$

(iv) $\vdash_{\mathcal{B}} \varphi \leftrightarrow (\varphi \vee \varphi)$

(v) $\vdash_{\mathcal{B}} (\varphi \vee \psi) \leftrightarrow (\psi \vee \varphi)$

(vi) $\vdash_{\mathcal{B}} (\varphi \vee (\psi \vee \vartheta)) \leftrightarrow ((\varphi \vee \psi) \vee \vartheta)$

(vii) $\vdash_{\mathcal{B}} (\varphi \wedge (\varphi \vee \psi)) \leftrightarrow \varphi$

(viii) $\vdash_{\mathcal{B}} (\varphi \vee (\varphi \wedge \psi)) \leftrightarrow \varphi$

(ix) $\vdash_{\mathcal{B}} \varphi \leftrightarrow (\varphi \otimes \varphi)$

(x) $\vdash_{\mathcal{B}} (\varphi \otimes \psi) \leftrightarrow (\psi \otimes \varphi)$

(xi) $\vdash_{\mathcal{B}} (\varphi \otimes (\psi \otimes \vartheta)) \leftrightarrow ((\varphi \otimes \psi) \otimes \vartheta)$

(xii) $\vdash_{\mathcal{B}} \varphi \leftrightarrow (\varphi \oplus \varphi)$

(xiii) $\vdash_{\mathcal{B}} (\varphi \oplus \psi) \leftrightarrow (\psi \oplus \varphi)$

(xiv) $\vdash_{\mathcal{B}} (\varphi \oplus (\psi \oplus \vartheta)) \leftrightarrow ((\varphi \oplus \psi) \oplus \vartheta)$

(xv) $\vdash_{\mathcal{B}} (\varphi \otimes (\varphi \oplus \psi)) \leftrightarrow \varphi$

(xvi) $\vdash_{\mathcal{B}} (\varphi \oplus (\varphi \otimes \psi)) \leftrightarrow \varphi$

(xvii) $\vdash_{\mathcal{B}} \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$

(xviii) $\vdash_{\mathcal{B}} \neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$

(xix) $\vdash_{\mathcal{B}} \neg(\varphi \otimes \psi) \leftrightarrow (\neg\varphi \otimes \neg\psi)$

(xx) $\vdash_{\mathcal{B}} \neg(\varphi \oplus \psi) \leftrightarrow (\neg\varphi \oplus \neg\psi)$

(xxi) $\vdash_{\mathcal{B}} \varphi \leftrightarrow \neg\neg\varphi$

(xxii) $\vdash_{\mathcal{B}} ((\varphi \supset \varphi) \supset \psi) \leftrightarrow \psi$

(xxiii) $\vdash_{\mathcal{B}} ((\varphi \wedge \psi) \supset \vartheta) \leftrightarrow (\varphi \supset (\psi \supset \vartheta))$

(xxiv) $\vdash_{\mathcal{B}} ((\varphi \wedge \psi) \supset \vartheta) \leftrightarrow ((\varphi \otimes \psi) \supset \vartheta)$

(xxv) $\vdash_{\mathcal{B}} ((\varphi \vee \psi) \supset \vartheta) \leftrightarrow ((\varphi \oplus \psi) \supset \vartheta)$

(xxvi) $\vdash_{\mathcal{B}} \varphi \rightarrow ((\varphi \supset \psi) \supset (\varphi \otimes \psi))$

(xxvii) $\vdash_{\mathcal{B}} (\neg(\varphi \supset \psi) \supset \vartheta) \leftrightarrow ((\varphi \wedge \neg\psi) \supset \vartheta)$

(xxviii) $\vdash_{\mathcal{B}} ((\varphi \vee \psi) \supset \vartheta) \leftrightarrow ((\varphi \supset \vartheta) \wedge (\psi \supset \vartheta))$.

Proof. (i). We have to prove that $\vdash_{\mathcal{B}} \neg(\varphi \wedge \varphi) \supset \neg\varphi$ and $\vdash_{\mathcal{B}} \neg\varphi \supset \neg(\varphi \wedge \varphi)$. As to the former, note that by $(\neg\wedge)$ we have $\neg(\varphi \wedge \varphi) \vdash_{\mathcal{B}} \neg\varphi \vee \neg\varphi$, so it will be enough to prove $\neg\varphi \vee \neg\varphi \vdash_{\mathcal{B}} \neg\varphi$, and this is easily done using $(\vee \supset)$. As to the latter, by $(\supset \vee)$ we have $\vdash_{\mathcal{B}} \neg\varphi \supset \neg\varphi \vee \neg\varphi$ and from this, using $(\neg\wedge)$, we easily obtain the result.

(ii). To prove that $\vdash_{\mathcal{B}} \neg(\varphi \wedge \psi) \equiv \neg(\psi \wedge \varphi)$, using $(\neg\wedge)$, it suffices to show that $\vdash_{\mathcal{B}} (\varphi \vee \psi) \equiv (\psi \vee \varphi)$, and this follows by Remark 1.2.

(iii). To prove that $\vdash_{\mathcal{B}} \neg(\varphi \wedge (\psi \wedge \vartheta)) \equiv \neg((\varphi \wedge \psi) \wedge \vartheta)$, note that by $(\neg\wedge)$ we have, on the one hand, $\vdash_{\mathcal{B}} \neg(\varphi \wedge (\psi \wedge \vartheta)) \equiv \neg\varphi \vee \neg(\psi \wedge \vartheta)$ and, using also Proposition 1.4 (xviii), $\vdash_{\mathcal{B}} \neg\varphi \vee \neg(\psi \wedge \vartheta) \equiv \neg\varphi \vee (\neg\psi \vee \neg\vartheta)$. On the other hand we have $\vdash_{\mathcal{B}} \neg((\varphi \wedge \psi) \wedge \vartheta) \equiv \neg(\varphi \wedge \psi) \vee \neg\vartheta$ and $\vdash_{\mathcal{B}} \neg(\varphi \wedge \psi) \vee \neg\vartheta \equiv (\neg\varphi \vee \neg\psi) \vee \neg\vartheta$. Hence, it suffices to prove that $\neg\varphi \vee (\neg\psi \vee \neg\vartheta) \equiv (\neg\varphi \vee \neg\psi) \vee \neg\vartheta$, and this follows from Remark 1.2.
(iv). To prove that $\vdash_{\mathcal{B}} \neg\varphi \equiv \neg(\varphi \vee \varphi)$, we only need to use $(\neg\vee)$ and (i).
(v). To prove that $\vdash_{\mathcal{B}} \neg(\varphi \vee \psi) \equiv \neg(\psi \vee \varphi)$, we only need to use $(\neg\vee)$ and (ii).
(vi). To prove that $\vdash_{\mathcal{B}} \neg(\varphi \vee (\psi \vee \vartheta)) \equiv \neg((\varphi \vee \psi) \vee \vartheta)$ we can use $(\neg\vee)$ and Proposition 1.4 (xviii).
(vii). To prove that $\vdash_{\mathcal{B}} \neg(\varphi \wedge (\varphi \vee \psi)) \equiv \neg\varphi$, note that by $(\neg\wedge)$ we have $\vdash_{\mathcal{B}} \neg(\varphi \wedge (\varphi \vee \psi)) \equiv \neg\varphi \vee \neg(\varphi \vee \psi))$ and, using $(\neg\vee)$ and Proposition 1.4 (xviii), we have $\vdash_{\mathcal{B}} \neg\varphi \vee \neg(\varphi \vee \psi)) \equiv \neg\varphi \vee (\neg\varphi \wedge \neg\psi)$. Now we may use Remark 1.2 to obtain the result.
(viii). To prove that $\vdash_{\mathcal{B}} \neg(\varphi \vee (\varphi \wedge \psi)) \equiv \neg\varphi$ we may reason as in (vii), just using $(\neg\wedge)$ instead of $(\neg\vee)$.
(ix). To prove that $\vdash_{\mathcal{B}} \neg\varphi \equiv \neg(\varphi \otimes \varphi)$, it is sufficient to observe that by $(\neg\otimes)$ we have $\vdash_{\mathcal{B}} \neg(\varphi \otimes \varphi) \equiv \neg\varphi \otimes \neg\varphi$, so we may use again Remark 1.2.
(x). To prove that $\vdash_{\mathcal{B}} \neg(\varphi \otimes \psi) \equiv \neg(\psi \otimes \varphi)$, it is sufficient to observe that by $(\neg\otimes)$ we have $\vdash_{\mathcal{B}} \neg(\varphi \otimes \psi) \equiv \neg\varphi \otimes \neg\psi$, so we may use again Remark 1.2.
(xi). To see that $\vdash_{\mathcal{B}} \neg(\varphi \otimes (\psi \otimes \vartheta)) \equiv \neg((\varphi \otimes \psi) \otimes \vartheta)$, as in the previous cases, we may use $(\neg\otimes)$ and Remark 1.2.
(xii). We may proceed as in (ix), just using $(\neg\oplus)$ instead of $(\neg\otimes)$.
(xiii). We may proceed as in (x), just using $(\neg\oplus)$ instead of $(\neg\otimes)$.
(xiv). We may proceed as in (xi), using $(\neg\oplus)$ instead of $(\neg\otimes)$, together with Proposition 1.4 (xix).
(xv). To prove $\vdash_{\mathcal{B}} \neg(\varphi \otimes (\varphi \oplus \psi)) \equiv \neg\varphi$, we use $(\neg\otimes)$ to obtain $\vdash_{\mathcal{B}} \neg(\varphi \otimes (\varphi \oplus \psi)) \equiv \neg\varphi \otimes \neg(\varphi \oplus \psi)$. Using $(\neg\oplus)$, it is easy to obtain $\vdash_{\mathcal{B}} \neg\varphi \otimes \neg(\varphi \oplus \psi) \equiv \neg\varphi \otimes (\neg\varphi \oplus \neg\psi)$. Now we may apply Remark 1.2 again to obtain the result.
(xvi). We may proceed as in (xv), using the property stated in Proposition 1.4 (xix).
(xvii). By $(\neg\wedge)$ we have $\vdash_{\mathcal{B}} \neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$. To prove $\vdash_{\mathcal{B}} \neg\neg(\varphi \wedge \psi) \equiv \neg(\neg\varphi \vee \neg\psi)$, observe that, by $(\neg\neg)$, we have $\vdash_{\mathcal{B}} \neg\neg(\varphi \wedge \psi) \equiv (\varphi \wedge \psi)$ and, moreover, $\varphi, \psi \dashv\vdash_{\mathcal{B}} \neg\neg\varphi, \neg\neg\psi$. By Proposition 1.4 (xvi), this means that $\vdash_{\mathcal{B}} (\varphi \wedge \psi) \equiv (\neg\neg\varphi \wedge \neg\neg\psi)$. By $(\neg\vee)$, we have $\vdash_{\mathcal{B}} (\neg\neg\varphi \wedge \neg\neg\psi) \equiv \neg(\neg\varphi \vee \neg\psi)$. Now, using Proposition 1.4 (ii), we obtain the result.
(xviii). We may proceed as in (xvii), just using $(\neg\vee)$ instead of $(\neg\wedge)$ and viceversa.

(xix). By $(\neg\otimes)$ we have $\vdash_{\mathcal{B}} \neg(\varphi \otimes \psi) \equiv (\neg\varphi \otimes \neg\psi)$. To prove that $\vdash_{\mathcal{B}} \neg\neg(\varphi \otimes \psi) \equiv \neg(\neg\varphi \otimes \neg\psi)$, note that by $(\neg\neg)$ we have $\vdash_{\mathcal{B}} \neg\neg(\varphi \otimes \psi) \equiv (\varphi \otimes \psi)$ and by $(\neg\otimes)$ we have $\vdash_{\mathcal{B}} \neg(\neg\varphi \otimes \neg\psi) \equiv (\neg\neg\varphi \otimes \neg\neg\psi)$. Now we apply $(\neg\neg)$ to obtain the result.
(xx). By $(\neg\oplus)$ we have $\vdash_{\mathcal{B}} \neg(\varphi \oplus \psi) \equiv (\neg\varphi \oplus \neg\psi)$. To prove $\vdash_{\mathcal{B}} \neg\neg(\varphi \oplus \psi) \equiv \neg(\neg\varphi \oplus \neg\psi)$ we may proceed as in (xix), using Proposition 1.4 (xix).
(xxi). By $(\neg\neg)$ we have $\vdash_{\mathcal{B}} \varphi \equiv \neg\neg\varphi$, and $\vdash_{\mathcal{B}} \neg\varphi \equiv \neg\neg\neg\varphi$ is also an instance of $(\neg\neg)$.
(xxii). In Proposition 1.4 (iii) we proved that $\vdash_{\mathcal{B}} \varphi \supset \varphi$. From this it follows immediately that

$$(\varphi \supset \varphi) \supset \psi \vdash_{\mathcal{B}} \psi$$

therefore

$$\vdash_{\mathcal{B}} ((\varphi \supset \varphi) \supset \psi) \supset \psi.$$

That $\vdash_{\mathcal{B}} \psi \supset ((\varphi \supset \varphi) \supset \psi)$ holds is also clear, since it is an instance of $(\supset 1)$. To prove that $\vdash_{\mathcal{B}} \neg((\varphi \supset \varphi) \supset \psi) \supset \neg\psi$, it is enough to note that, by $(\neg \supset)$, we have

$$\neg((\varphi \supset \varphi) \supset \psi) \vdash_{\mathcal{B}} (\varphi \supset \varphi) \wedge \neg\psi.$$

So, by transitivity of $\vdash_{\mathcal{B}}$, the result follows. Finally, to prove that $\vdash_{\mathcal{B}} \neg\psi \supset \neg((\varphi \supset \varphi) \supset \psi)$, note that by $(\neg \supset)$ we have

$$(\varphi \supset \varphi) \wedge \neg\psi \vdash_{\mathcal{B}} \neg((\varphi \supset \varphi) \supset \psi),$$

so the result again follows easily.
(xxiii). Using Proposition 1.4 (i), we will prove that

$$\begin{aligned}
&\vdash_{\mathcal{B}} ((\varphi \wedge \psi) \supset \vartheta) \supset (\varphi \supset (\psi \supset \vartheta))\\
&\vdash_{\mathcal{B}} (\varphi \supset (\psi \supset \vartheta)) \supset ((\varphi \wedge \psi) \supset \vartheta)\\
&\vdash_{\mathcal{B}} \neg((\varphi \wedge \psi) \supset \vartheta) \supset \neg(\varphi \supset (\psi \supset \vartheta))\\
&\vdash_{\mathcal{B}} \neg(\varphi \supset (\psi \supset \vartheta)) \supset \neg((\varphi \wedge \psi) \supset \vartheta).
\end{aligned}$$

The first two are easily proved, for they amount to $(\varphi \wedge \psi) \supset \vartheta, \varphi, \psi \vdash_{\mathcal{B}} \vartheta$ and $\varphi \supset (\psi \supset \vartheta), \varphi \wedge \psi \vdash_{\mathcal{B}} \vartheta$. As to the second two, using $(\neg \supset)$, we will prove that $(\varphi \wedge \psi) \wedge \neg\vartheta \dashv\vdash_{\mathcal{B}} \varphi \wedge \neg(\psi \supset \vartheta)$. Applying $(\neg \supset)$ again, it is easy to see that this follows from the fact that $(\varphi \wedge \psi) \wedge \neg\vartheta \dashv\vdash_{\mathcal{B}} \varphi \wedge (\psi \wedge \neg\vartheta)$.
(xxiv). Follows immediately from Proposition 1.4 (xvi) and (xvii).
(xxv). Using Proposition 1.4 (xvii), it will be enough to prove that $\varphi \vee \psi \dashv\vdash_{\mathcal{B}} \varphi \oplus \psi$ for all $\varphi, \psi \in Fm$. We have that

$$(\varphi \supset (\varphi \oplus \psi)) \supset ((\psi \supset (\varphi \oplus \psi)) \supset ((\varphi \vee \psi) \supset (\varphi \oplus \psi)))$$

is an instance of ($\vee \supset$). Now, since $\varphi \supset (\varphi \oplus \psi)$ and $\psi \supset (\varphi \oplus \psi)$ are instances of ($\supset \oplus$), we may apply MP two times to obtain $\vdash_{\mathcal{B}} (\varphi \vee \psi) \supset (\varphi \oplus \psi)$, hence $\varphi \vee \psi \vdash_{\mathcal{B}} \varphi \oplus \psi$. The same reasoning, using ($\oplus \supset$) and ($\supset \vee$) instead of ($\vee \supset$) and ($\supset \oplus$), allows us to conclude that $\varphi \oplus \psi \vdash_{\mathcal{B}} \varphi \vee \psi$.

(xxvi). Clearly $\varphi, \varphi \supset \psi \vdash_{\mathcal{B}} \varphi$ and by MP we have $\varphi, \varphi \supset \psi \vdash_{\mathcal{B}} \psi$. Now, using (i), we obtain $\varphi, \varphi \supset \psi \vdash_{\mathcal{B}} \varphi \otimes \psi$, i.e., $\vdash_{\mathcal{B}} \varphi \supset ((\varphi \supset \psi) \supset (\varphi \otimes \psi))$. To prove that $\vdash_{\mathcal{B}} \neg((\varphi \supset \psi) \supset (\varphi \otimes \psi)) \supset \neg\varphi$, note that by ($\neg \supset$) we have $\neg((\varphi \supset \psi) \supset (\varphi \otimes \psi)) \vdash_{\mathcal{B}} (\varphi \supset \psi) \wedge \neg(\varphi \otimes \psi)$. By ($\wedge \supset$) we have $(\varphi \supset \psi) \wedge \neg(\varphi \otimes \psi) \vdash_{\mathcal{B}} \neg(\varphi \otimes \psi)$ and by ($\neg\otimes$) we obtain $\neg(\varphi \otimes \psi) \vdash_{\mathcal{B}} \neg\varphi \otimes \neg\psi$. Now, since $\neg\varphi \otimes \neg\psi \vdash_{\mathcal{B}} \neg\varphi$ by ($\otimes \supset$), using the transitivity of $\vdash_{\mathcal{B}}$ we obtain $\neg((\varphi \supset \psi) \supset (\varphi \otimes \psi)) \vdash_{\mathcal{B}} \neg\varphi$.

(xxvii). Follows immediately from ($\neg \supset$) and Proposition 1.4 (xvii).

(xxviii). To see that $\vdash_{\mathcal{B}} ((\varphi \vee \psi) \supset \vartheta) \supset ((\varphi \supset \vartheta) \wedge (\psi \supset \vartheta))$, just note that $\varphi \supset (\varphi \vee \psi)$ is an instance of ($\supset \vee$), so by the transitivity of $\supset$ we have $(\varphi \vee \psi) \supset \vartheta, \varphi \vdash_{\mathcal{B}} \vartheta$ and similarly $(\varphi \vee \psi) \supset \vartheta, \psi \vdash_{\mathcal{B}} \vartheta$. Hence, using Proposition 1.4 (i), we obtain the result.

To prove that $\vdash_{\mathcal{B}} \neg((\varphi \supset \vartheta) \wedge (\psi \supset \vartheta)) \supset \neg((\varphi \vee \psi) \supset \vartheta)$, note that $((\varphi \wedge \neg\vartheta) \supset ((\varphi \vee \psi) \wedge \neg\vartheta)) \supset (((\psi \wedge \neg\vartheta) \supset ((\varphi \vee \psi)) \wedge \neg\vartheta) \supset ((\varphi \wedge \neg\vartheta) \vee (\psi \wedge \neg\vartheta)) \supset ((\varphi \vee \psi) \wedge \neg\vartheta))$ is an instance of ($\vee \supset$). It is not difficult to prove that $\vdash_{\mathcal{B}} (\varphi \wedge \neg\vartheta) \supset ((\varphi \vee \psi) \wedge \neg\vartheta)$ and $\vdash_{\mathcal{B}} (\psi \wedge \neg\vartheta) \supset ((\varphi \vee \psi) \wedge \neg\vartheta)$, so applying MP we obtain

$$(\varphi \wedge \neg\vartheta) \vee (\psi \wedge \neg\vartheta)) \vdash_{\mathcal{B}} (\varphi \vee \psi) \wedge \neg\vartheta.$$

Now observe that by ($\neg\wedge$) we have

$$\neg((\varphi \supset \vartheta) \wedge (\psi \supset \vartheta)) \vdash_{\mathcal{B}} \neg(\varphi \supset \vartheta) \vee \neg(\psi \supset \vartheta)$$

and, applying ($\neg \supset$) and (xviii), we obtain

$$\neg(\varphi \supset \vartheta) \vee \neg(\psi \supset \vartheta)) \vdash_{\mathcal{B}} (\varphi \wedge \neg\vartheta) \vee (\psi \wedge \neg\vartheta).$$

Hence, by the transitivity of $\vdash_{\mathcal{B}}$, we have

$$\neg((\varphi \supset \vartheta) \wedge (\psi \supset \vartheta)) \vdash_{\mathcal{B}} (\varphi \vee \psi) \wedge \neg\vartheta.$$

Now, by ($\neg \supset$) we have $(\varphi \vee \psi) \wedge \neg\vartheta \vdash_{\mathcal{B}} \neg((\varphi \vee \psi) \supset \vartheta)$, so the result immediately follows.

To see that $\vdash_{\mathcal{B}} ((\varphi \supset \vartheta) \wedge (\psi \supset \vartheta)) \supset ((\varphi \vee \psi) \supset \vartheta)$, note that using ($\vee \supset$) we obtain $\varphi \supset \vartheta, \psi \supset \vartheta \vdash_{\mathcal{B}} (\varphi \vee \psi) \supset \vartheta$, Hence, by Proposition 1.4 (xvi), the result easily follows.

It remains to prove that

$$\vdash_{\mathcal{B}} \neg((\varphi \vee \psi) \supset \vartheta) \supset \neg((\varphi \supset \vartheta) \wedge (\psi \supset \vartheta)).$$

By $(\neg \supset)$ we have

$$\neg((\varphi \vee \psi) \supset \vartheta) \vdash_{\mathcal{B}} (\varphi \vee \psi) \wedge \neg\vartheta.$$

Using again $(\neg \supset)$ and Proposition 1.4 (xviii), we have

$$(\varphi \wedge \neg\vartheta) \vee (\psi \wedge \neg\vartheta) \vdash_{\mathcal{B}} \neg(\varphi \supset \vartheta) \vee \neg(\psi \supset \vartheta).$$

By $(\neg\wedge)$ we have

$$\neg(\varphi \supset \vartheta) \vee \neg(\psi \supset \vartheta) \vdash_{\mathcal{B}} \neg((\varphi \supset \vartheta) \wedge (\psi \supset \vartheta)).$$

Hence, by the transitivity of $\vdash_{\mathcal{B}}$, it will be enough to prove that

$$(\varphi \vee \psi) \wedge \neg\vartheta \vdash_{\mathcal{B}} (\varphi \wedge \neg\vartheta) \vee (\psi \wedge \neg\vartheta).$$

To see this, note that by $(\supset \vee)$ it is easy to show that $\varphi, \neg\vartheta \vdash_{\mathcal{B}} (\varphi \wedge \neg\vartheta) \vee (\psi \wedge \neg\vartheta)$ and $\psi, \neg\vartheta \vdash_{\mathcal{B}} (\varphi \wedge \neg\vartheta) \vee (\psi \wedge \neg\vartheta)$. Hence we have

$$\varphi \vdash_{\mathcal{B}} \neg\vartheta \supset ((\varphi \wedge \neg\vartheta) \vee (\psi \wedge \neg\vartheta))$$

and

$$\psi \vdash_{\mathcal{B}} \neg\vartheta \supset ((\varphi \wedge \neg\vartheta) \vee (\psi \wedge \neg\vartheta)).$$

Now, using $(\vee \supset)$, we obtain

$$\varphi \vee \psi \vdash_{\mathcal{B}} \neg\vartheta \supset ((\varphi \wedge \neg\vartheta) \vee (\psi \wedge \neg\vartheta)),$$

hence

$$\varphi \vee \psi, \neg\vartheta \vdash_{\mathcal{B}} (\varphi \wedge \neg\vartheta) \vee (\psi \wedge \neg\vartheta).$$

Now from this the result easily follows. ■

At this point we may observe that, while the "weak" implication $\supset$ shares perhaps more properties with the intuitionistic one, it is the "strong" implication $\rightarrow$ that is needed in order to define equivalence of formulas within the logic. The presence in $\mathcal{B}$ of two implications may recall Nelson's logic [25]: the two logics are in fact closely related, as we are going to see below (Section 2.2).

The properties stated in Proposition 1.4 (v) to (xv) are sufficient to establish that the calculus introduced above is algebraizable (Cf. [30, Theorem 4.2.4]):

THEOREM 1.6. *The logic $\mathcal{B}$ is algebraizable with equivalence formula $\varphi \leftrightarrow \psi$ and defining equation $\varphi \approx \varphi \supset \varphi$.*

PROOF. Using the intrinsic characterization given by Blok and Pigozzi [8, Theorem 4.7], it is sufficient to check that the following conditions hold: for all formulas $\varphi, \psi, \vartheta \in Fm$,

(i) $\varphi \dashv\vdash_{\mathcal{B}} \varphi \leftrightarrow (\varphi \supset \varphi)$

(ii) $\vdash_{\mathcal{B}} \varphi \leftrightarrow \varphi$

(iii) $\varphi \leftrightarrow \psi \vdash_{\mathcal{B}} \psi \leftrightarrow \varphi$

(iv) $\varphi \leftrightarrow \psi, \psi \leftrightarrow \vartheta \vdash_{\mathcal{B}} \varphi \leftrightarrow \vartheta$

(v) $\varphi \leftrightarrow \psi \vdash_{\mathcal{B}} \neg\varphi \leftrightarrow \neg\psi$

(vi) $\varphi_1 \leftrightarrow \psi_1, \varphi_2 \leftrightarrow \psi_2 \vdash_{\mathcal{B}} (\varphi_1 \bullet \varphi_2) \leftrightarrow (\psi_1 \bullet \psi_2)$ for all formulas $\varphi_1, \varphi_2, \psi_1, \psi_2$ and any connective $\bullet \in \{\wedge, \vee, \otimes, \oplus, \supset\}$.

And this follows directly from Proposition 1.4 (v) to (xv). ■

This obviously implies that any axiomatic extension of $\mathcal{B}$, for instance $\mathcal{LB}_{\supset}$, is also algebraizable with the same translations (see for instance [8, 21]). Notice that instead of the defining equation $\varphi \approx \varphi \supset \varphi$ we could use $\varphi \approx \varphi \to \varphi$, and instead of the single equivalence formula $\varphi \leftrightarrow \psi$ we could use either of the following sets: $\{\varphi \supset \psi, \psi \supset \varphi, \neg\varphi \supset \neg\psi, \neg\psi \supset \neg\varphi\}$ or $\{\varphi \to \psi, \psi \to \varphi\}$. This indicates that both the $\{\to\}$-fragment and the $\{\supset, \neg\}$-fragment of $\mathcal{B}$ are algebraizable; the equivalent algebraic semantics of the $\{\supset, \neg\}$-fragment of $\mathcal{LB}_{\supset}$ has been characterized and investigated in [30, Section 5.5.] and [31]. The problem of axiomatizing the $\{\to\}$-fragment seems to be particularly challenging (as far as the authors are aware of, this fragment does not coincide with any known logic).

2. Brouwerian bilattices

In this section we are going to introduce a class of algebras having a bilattice reduct which will be proved to be the equivalent algebraic semantics of the logic $\mathcal{B}$. We start by recalling some preliminary definitions and results on bilattices.

2.1. Bilattices

DEFINITION 2.1 ([22]). A *bilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ such that the reducts $\langle B, \wedge, \vee \rangle$ and $\langle B, \otimes, \oplus \rangle$ are both lattices and the *negation* $\neg$ is a unary operation satisfying, for every $a, b \in B$,

(neg 1) if $a \leq_t b$, then $\neg b \leq_t \neg a$

(neg 2) if $a \leq_k b$, then $\neg a \leq_k \neg b$

(neg 3) $a = \neg\neg a$.

The order corresponding to the lattice $\langle B, \wedge, \vee \rangle$, sometimes called the *truth lattice* or *t-lattice*, is denoted by $\leq_t$ and is called the *truth order*, while the order $\leq_k$ associated with $\langle B, \otimes, \oplus \rangle$, sometimes called the *knowledge lattice* or *k-lattice*, is the *knowledge order*.

Usually in the literature it is required that the two lattices be complete or at least bounded, but here none of these assumptions is made. The minimum and maximum element of the lattice $\langle B, \wedge, \vee \rangle$, in case they exist, will be denoted, respectively, by f and t. Similarly, $\bot$ and $\top$ will refer to the minimum and maximum of $\langle B, \otimes, \oplus \rangle$ when they exist.

Perhaps the most interesting subclasses of bilattices are the interlaced and the distributive ones. A bilattice is *interlaced* [18] whenever each one of the four operations $\{\wedge, \vee, \otimes, \oplus\}$ is monotonic with respect to both orders $\leq_t$ and $\leq_k$. That is, when the following quasi-equations hold:

$$x \leq_t y \;\Rightarrow\; x \otimes z \leq_t y \otimes z \qquad\qquad x \leq_t y \;\Rightarrow\; x \oplus z \leq_t y \oplus z$$
$$x \leq_k y \;\Rightarrow\; x \wedge z \leq_k y \wedge z \qquad\qquad x \leq_k y \;\Rightarrow\; x \vee z \leq_k y \vee z.$$

A bilattice is *distributive* [22] when all possible distributive laws concerning the four lattice operations, i.e., all identities of the following form, hold:

$$x \circ (y \bullet z) \approx (x \circ y) \bullet (x \circ z) \qquad \text{for every } \circ, \bullet \in \{\wedge, \vee, \otimes, \oplus\}.$$

All the above-mentioned classes are varieties. Moreover, distributive bilattices form a proper subvariety of the interlaced ones, which form a proper subvariety of the class of all bilattices.

Figure 1 shows the double Hasse diagrams of some of the best-known bilattices (the names adopted here have by now become more or less standard in the literature). They should be read as follows: $a \leq_t b$ means that there is a path from a to b which goes uniformly from left to right, while $a \leq_k b$ means that there is a path from a to b which goes uniformly from the bottom to the top. The four lattice operations are thus uniquely determined by the diagram, while negation corresponds to reflection along the vertical axis connecting $\bot$ and $\top$.

The smallest non-trivial bilattice, $\mathcal{FOUR}$, has a fundamental role among bilattices, both from an algebraic and a logical point of view. $\mathcal{FOUR}$ is distributive and is a simple algebra. It is in fact, up to isomorphism, the only

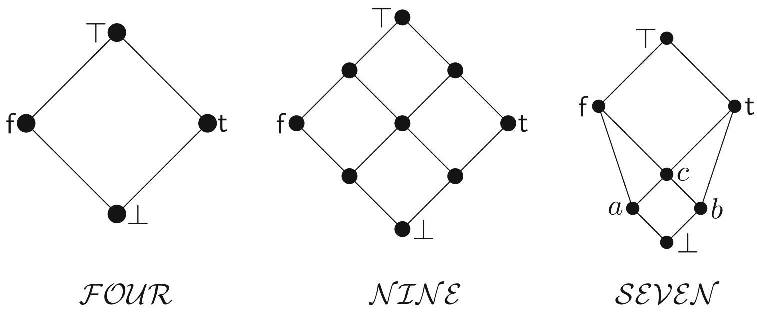

Figure 1. Some examples of bilattices.

subdirectly irreducible distributive bilattice (this was proved for the bounded case in [23], then generalized in [10] to the unbounded).

An important result about interlaced bilattices, proved in full generality in [24, 10], is that they can be represented by means of a special kind of product of two copies of one lattice. Let us introduce the construction (due to Fitting [18]) involved in this result.

Let $\mathbf{L} = \langle L, \sqcap, \sqcup \rangle$ be a lattice with associated order $\sqsubseteq$. The *product bilattice* $\mathbf{L} \odot \mathbf{L} = \langle L \times L, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is defined as follows. For all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L \times L$,

$$\begin{aligned}
\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle &:= \langle a_1 \sqcap b_1,\ a_2 \sqcup b_2 \rangle \\
\langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle &:= \langle a_1 \sqcup b_1,\ a_2 \sqcap b_2 \rangle \\
\langle a_1, a_2 \rangle \otimes \langle b_1, b_2 \rangle &:= \langle a_1 \sqcap b_1,\ a_2 \sqcap b_2 \rangle \\
\langle a_1, a_2 \rangle \oplus \langle b_1, b_2 \rangle &:= \langle a_1 \sqcup b_1,\ a_2 \sqcup b_2 \rangle \\
\neg \langle a_1, a_2 \rangle &:= \langle a_2, a_1 \rangle.
\end{aligned}$$

$\mathbf{L} \odot \mathbf{L}$ is always an interlaced bilattice, and it is distributive if and only if $\mathbf{L}$ is a distributive lattice. From the definition it follows immediately that

$$\begin{array}{lcccc}
\langle a_1, a_2 \rangle \leq_k \langle b_1, b_2 \rangle & \text{iff} & a_1 \sqsubseteq b_1 & \text{and} & a_2 \sqsubseteq b_2 \\
\langle a_1, a_2 \rangle \leq_t \langle b_1, b_2 \rangle & \text{iff} & a_1 \sqsubseteq b_1 & \text{and} & b_2 \sqsubseteq a_2.
\end{array}$$

The above-mentioned representation theorem states that any interlaced bilattice can be represented as a product of this form [10, Proposition 3.13]:

THEOREM 2.2. *Let* $\mathbf{B}$ *be an interlaced bilattice. Then:*

(i) *there is a lattice* $\mathbf{L}$ *such that* $\mathbf{B} \cong \mathbf{L} \odot \mathbf{L}$

(ii) **B** *is distributive if and only if* **L** *is a distributive lattice*

(iii) *the lattice of congruences of* **B** *is isomorphic to the lattice of congruences of* **L**.

Notice that item (iii) of the above theorem has a special interest from a universal algebraic point of view, as it implies that questions on congruence lattices of interlaced bilattices (congruence-permutability, subdirect irreducibility etc.) can be reduced to questions concerning lattices. An application of this is the following result [10, Theorem 3.19]: *the variety of distributive bilattices is generated by its four-element member* $\mathcal{FOUR}$.

The lattice-theoretic notion of filter also has a bilattice counterpart that plays an important role on a logical level.

Given a bilattice **B**, we call a subset $F \subseteq B$ a *bifilter* if F is a filter of both lattice orders of **B**. It is easy to see that bifilters form a closure system, so it is possible to define an operator of bifilter generation just as one does for lattice filters. [10] contains several results on bifilters in interlaced bilattices, among which we just cite the following [10, Proposition 3.18]:

Proposition 2.3. *A subset* $F \subseteq L \times L$ *of an interlaced bilattice* $\mathbf{L} \odot \mathbf{L}$ *is a bifilter if and only if there is a lattice filter* $F' \subseteq L$ *of* **L** *such that* $F = F' \times L$.

2.2. Brouwerian bilattices

In this section we are going to expand the bilattice language considered above with a new binary operation that on a logical level will play the role of an implication. The class of algebras thus obtained, that we define through an equational presentation, will be proved to be the equivalent algebraic semantics of the logic $\mathcal{B}$.

Definition 2.4. An algebra $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$ is a *Brouwerian bilattice* if the reduct $\langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is a bilattice and the following equations are satisfied:

(B1) $(x \supset x) \supset y \approx y$

(B2) $x \supset (y \supset z) \approx (x \wedge y) \supset z \approx (x \otimes y) \supset z$

(B3) $(x \vee y) \supset z \approx (x \supset z) \wedge (y \supset z) \approx (x \oplus y) \supset z$

(B4) $x \wedge ((x \supset y) \supset (x \otimes y)) \approx x$

(B5) $\neg(x \supset y) \supset z \approx (x \wedge \neg y) \supset z$.

As we have done before for propositional formulas, we shall write $x \to y$ as an abbreviation for $(x \supset y) \wedge (\neg y \supset \neg x)$. Brouwerian bilattices obviously form a variety. An interesting subclass of Brouwerian bilattices is the variety of *classical implicative bilattices*[1], defined as the Brouwerian bilattices that additionally satisfy the following (Peirce's) equation:

$$((x \supset y) \supset x) \supset x \approx x \supset x.$$

As we shall explain below, the relation between the above classes is analogous to the relation between (generalized) Heyting algebras and (generalized) Boolean algebras. Let us note that within the context of classical implicative bilattices the two implication operations are interdefinable, for the following equation is valid in this variety [2, Proposition 3.31]:

$$x \supset y \approx y \vee (x \to (x \to y)).$$

We list below some interesting arithmetical properties of Brouwerian bilattices. In order to simplify the notation, we use the following abbreviation: for any element a of a Brouwerian bilattice, we write $E(a)$ to mean that $a = a \supset a$ holds.

The proof of the following proposition is quite long and uses several lemmas. In order to save space, we omit it and refer the interested reader to [30, Propositions 4.3.2, 4.3.3] (some items are also proved in [9, Proposition 4.8]).

PROPOSITION 2.5. *Let* $\mathbf{B}$ *be an Brouwerian bilattice. Then, for all* $a, b, c \in B$*:*

(i) $a \supset (b \supset c) = (a \supset b) \supset (a \supset c)$

(ii) $a \supset (b \wedge c) = (a \supset b) \wedge (a \supset c)$

(iii) $a \supset (b \otimes c) = (a \supset b) \otimes (a \supset c)$

(iv) $a \to b = (a \supset b) \otimes (\neg b \supset \neg a)$

(v) $a \to (b \to c) = b \to (a \to c)$

(vi) $(a \supset a) \oplus \neg(a \supset a) = (b \supset b) \oplus \neg(b \supset b) = \top$

(vii) $E(a)$ *if and only if* $\top \leq_t a$

(viii) *if* $\top \leq_t a$ *and* $\top \leq_t a \supset b$*, then* $\top \leq_t b$

---

[1]These algebras were introduced through a different equational presentation and studied in [28] under the name of "implicative distributive bilattices with regular negation". In [30] they are called just "implicative bilattices", but we consider the terminology adopted in the present paper more adequate, as it suggests the relation between these algebras and the classical implicative lattices of [14] established by our representation result (see Theorem 2.6 below).

(ix) *$a \leq_t a \otimes b$ if and only if $\top \leq_t a \supset b$*

(x) *$a \leq_t b$ if and only if $E(a \supset b)$ and $E(\neg b \supset \neg a)$*

(xi) *$a \leq_k b$ if and only if $E(a \supset b)$ and $E(\neg a \supset \neg b)$.*

Item (vi) of the previous proposition implies that in any Brouwerian bilattice **B** there is an algebraic constant which is moreover the top element of the k-lattice reduct of **B**. It also follows that any Brouwerian bilattice has a minimal non-empty bifilter $F(\top)$ which is generated by $\top$, given by

$$F(\top) = \{a \in B : \top \leq_t a\} = \{a \in B : E(a)\}.$$

The elements of $F(\top)$ play, from a logical point of view, the role of designated elements of our logic: this can be easily guessed if we read (viii) as an algebraic counterpart of modus ponens. Finally, note that (x) and (xi) indicate that both lattice orders are explicitly definable using just $\{\supset, \neg\}$.

In [9] it is proved that the bilattice reduct of any Brouwerian bilattice is interlaced (in fact, it is distributive). This result suggests that it may be possible to represent Brouwerian bilattices by means of a product construction similar to the one introduced above for product bilattices. This is indeed the case and the required construction is just a slight modification of the previous one.

Let $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, 1\rangle$ be a Brouwerian lattice[2], i.e., an algebra such that $\langle L, \sqcap, \sqcup, 1\rangle$ is a lattice with maximum element 1 and the following residuation condition is satisfied: for all $a, b, c \in L$,

$$a \sqcap b \leq c \text{ if and only if } b \leq a\backslash c.$$

We denote by $\mathbf{L}\odot\mathbf{L}$ the algebra $\langle L\times L, \wedge, \vee, \otimes, \oplus, \supset, \neg\rangle$ whose bilattice reduct is the usual product bilattice $\langle L, \sqcap, \sqcup\rangle \odot \langle L, \sqcap, \sqcup\rangle$ and where the operation $\supset$ is defined, for all $a_1, a_2, b_1, b_2 \in L$, as

$$\langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle = \langle a_1\backslash b_1, a_1 \sqcap b_2\rangle.$$

Then $\mathbf{L}\odot\mathbf{L}$ is a Brouwerian bilattice[3]. Conversely, we have the following [9]:

[2] Also called *generalized Heyting algebras* [12], *Brouwerian algebras* [16], implicative lattices [26] or *relatively pseudo-complemented lattices* [29]. Note also that some authors call "Brouwerian lattices" structures that are dual to ours.

[3] The product Brouwerian bilattice construction is very similar to the "twist-structure" used in [26, 27] to represent the algebras there called *N4-lattices*, that constitute an algebraic semantics for the paraconsistent version of Nelson's logic, as special products of two copies of a Brouwerian lattice. In fact, it can be proved that N4-lattices coincide with the $\{\wedge, \vee, \supset, \neg\}$-subreducts of Brouwerian bilattices.

Theorem 2.6. *Let* $\mathbf{B}$ *be a Brouwerian bilattice. Then:*

(i) *there is a Brouwerian lattice* $\mathbf{L}$ *such that* $\mathbf{B} \cong \mathbf{L} \odot \mathbf{L}$

(ii) *the lattice of congruences of* $\mathbf{B}$ *is isomorphic to the lattice of congruences of* $\mathbf{L}$*.*

The required Brouwerian lattice $\mathbf{L}$ can be constructed in several ways. Here we describe one that will be especially useful in the following sections.

Given a Brouwerian bilattice $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$, we define the relation $\sim \subseteq B \times B$ as follows:

$$a \sim b \quad \text{iff} \quad E(a \supset b) \text{ and } E(b \supset a). \tag{1}$$

This relation is compatible with all the operations of the Brouwerian bilattice except the negation [30, Corollary 4.3.6], i.e., it is a congruence of the reduct $\langle B, \wedge, \vee, \otimes, \oplus, \supset \rangle$. We may therefore consider the quotient algebra

$$\langle B/\sim, \wedge, \vee, \otimes, \oplus, \supset \rangle ,$$

in which $\wedge = \otimes$ and $\vee = \oplus$, that turns out to be the Brouwerian lattice we were looking for. Notice that the relation defined above can also be characterized as follows: $a \sim b$ if and only if a and b generate the same bifilter (cf. [10, Section 3.3]).

In the particular case in which $\mathbf{B}$ is a classical implicative bilattice, i.e., a Brouwerian bilattice satisfying Peirce's equation, then the lattice $\mathbf{L}$ such that $\mathbf{B} \cong \mathbf{L} \odot \mathbf{L}$ is indeed a *classical implicative lattice*, i.e., $\mathbf{L}$ additionally satisfies the following equation: $(x \backslash y) \backslash x \approx x$.

The above representation results should clarify why we chose to call the above classes of bilattices with implication "Brouwerian" and "classical implicative". Notice also that classical implicative lattices coincide, up to algebraic signature, with bottom-less subreducts of Boolean algebras (sometimes called *generalized Boolean algebras*) or, equivalently, relatively complemented lattices with top element; similarly, Brouwerian lattices correspond to bottom-less subreducts of Heyting algebras.

In [9] it is proved that Theorem 2.6 may be used to establish a one-to-one correspondence between subquasivarieties of Brouwerian lattices and subquasivarieties of Brouwerian bilattices (the minimal non-trivial ones being classical implicative lattices and classical implicative bilattices), which moreover extends to a categorical equivalence among the corresponding categories.

An easy consequence of Theorem 2.6 is that, since the lattice reduct of any Brouwerian lattice is distributive, we may invoke Theorem 2.2 (ii) to conclude that the bilattice reduct of any Brouwerian bilattice is distributive as well. Notice also that, as happened with interlaced bilattices, the algebraic properties that depend on the lattice structure of congruences are exactly the same in Brouwerian lattices and Brouwerian bilattices. Thus, for instance, the variety of Brouwerian bilattices is the variety of arithmetical, as is Brouwerian lattices. Also, a Brouwerian bilattice is subdirectly irreducible if and only if the corresponding Brouwerian lattice is, and so on.

2.3. Algebraizability of $\mathcal{B}$

In this section we state the announced algebraizability result and characterize the reduced models of our logic. We shall need the following:

LEMMA 2.7. *For every axiom φ of $\mathcal{B}$, the equation $\varphi \wedge \top \approx \top$ (sometimes abbreviated $\top \leq_t \varphi$) is valid in the variety of Brouwerian bilattices.*

PROOF. Let $\mathbf{B}$ be a Brouwerian bilattice. By Theorem 2.6, we can assume without loss of generality that $\mathbf{B} = \mathbf{L} \odot \mathbf{L}$, where $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, 1 \rangle$ is a Brouwerian lattice. In this case $\top = \langle 1, 1 \rangle$ and, for any $\langle a_1, a_2 \rangle \in B$, the condition that $\langle a_1, a_2 \rangle \wedge \top = \top$ is equivalent to $a_1 = 1$. So, for instance, in order to prove the case of ($\supset$ 1), we need to show that, for any $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in B$,

$$\langle 1, 1 \rangle \leq_t \langle a_1, a_2 \rangle \supset (\langle b_1, b_2 \rangle \supset \langle a_1, a_2 \rangle) = \langle a_1 \backslash (b_1 \backslash a_1), a_1 \sqcap b_1 \sqcap a_2 \rangle$$

i.e., that $a_1 \backslash (b_1 \backslash a_1) = 1$, which is true in any Brouwerian lattice (see [29, p. 53]). Similarly, the case of ($\supset$ 2) amounts to proving that, for all $a, b, c \in L$,

$$(a \backslash (b \backslash c)) \backslash ((a \backslash b) \backslash (a \backslash c)) = 1$$

which is also true in Brouwerian lattices, as indeed $a \backslash (b \backslash c) = (a \backslash b) \backslash (a \backslash c)$. All the other cases follow easily from well-known properties of Brouwerian lattices, so we omit the proof. ■

We now have the following:

THEOREM 2.8. *The logic $\mathcal{B}$ is algebraizable with respect to the variety of Brouwerian bilattices, with equivalence formula $\varphi \leftrightarrow \psi$ and defining equation $\varphi \approx \varphi \supset \varphi$.*

PROOF. We will prove that $\mathbf{Alg}^*\mathcal{B}$ is precisely the class of Brouwerian bilattices. By [8, Theorem 2.17], we know that the class $\mathbf{Alg}^*\mathcal{B}$ is axiomatized by the following equations and quasiequations:

(a) $\varphi \approx \varphi \supset \varphi$ for any axiom φ of $\mathcal{B}$

(b) $x \approx x \supset x \;\;\&\;\; x \supset y \approx (x \supset y) \supset (x \supset y) \quad \Rightarrow \quad y \approx y \supset y$

(c) $x \leftrightarrow y \approx (x \leftrightarrow y) \supset (x \leftrightarrow y) \quad \Rightarrow \quad x \approx y.$

In order to prove that the variety of Brouwerian bilattices is contained in $\mathbf{Alg}^*\mathcal{B}$, it is then sufficient to prove that any Brouwerian bilattice $\mathbf{B}$ satisfies (a) to (c). Note that by Proposition 2.5 (vii) we have that, in any Brouwerian bilattice, $x \approx x \supset x$ is equivalent to $\top \leq_t x$. Hence we see that (a) has been proven in Lemma 2.7. As to (b), it follows from Proposition 2.5 (viii), while (c) follows from Proposition 2.5 (x).
In order to prove that $\mathbf{Alg}^*\mathcal{B}$ is a subclass of Brouwerian bilattices, we have to show that any $\mathbf{A} \in \mathbf{Alg}^*\mathcal{B}$ satisfies all equations defining the variety of Brouwerian bilattices, i.e. all equations defining the variety of bilattices plus (B1)-(B5). To see this, using (a) and (c), it is enough to prove that, for any equation $\varphi \approx \psi$ axiomatizing the variety of Brouwerian bilattices, it holds that $\vdash_{\mathcal{B}} \varphi \leftrightarrow \psi$. And this was shown in Proposition 1.5. ■

As a consequence of the previous theorem, we obtain the following [30, Theorem 4.3.10]:

Corollary 2.9. *The logic $\mathcal{LB}_{\supset}$ is algebraizable with respect to the variety of classical implicative bilattices, with equivalence formula $\varphi \leftrightarrow \psi$ and defining equation $\varphi \approx \varphi \supset \varphi$.*

A key notion in algebraic logic is that of *logical filter* of a logic on a given algebra $\mathbf{A}$. It is defined as a subset of A that, for all homomorphisms from the formula algebra to $\mathbf{A}$, contains all the interpretations of the axioms and is closed under the rules of the logic. In the case of our logic $\mathcal{B}$, this amounts to the following: $D \subseteq A$ is a logical filter if, for all homomorphisms $v : Fm \to A$,

(i) $v(\varphi) \in D$ for any axiom φ of $\mathcal{B}$

(ii) $v(\psi) \in D$ whenever $v(\varphi), v(\varphi \supset \psi) \in D$.

This notion is usually compared with a purely algebraic one, that in our case we define as follows:

Definition 2.10. Given a Brouwerian bilattice $\mathbf{B}$, a subset $F \subseteq B$ is a *deductive filter* if and only if $\{a \in B : a \geq_t \top\} \subseteq F$ and, for all $a, b \in B$, if $a \in F$ and $a \supset b \in F$, then $b \in F$.

The relationships among these notions are synthesized in the following [30, Proposition 4.3.9]:

Proposition 2.11. *Let* $\mathbf{B}$ *be a Brouwerian (or a classical implicative) bilattice and* $F \subseteq B$. *The following conditions are equivalent:*

(i) F *is a non–empty bifilter.*

(ii) F *is a deductive filter.*

(iii) F *is a logical filter of* $\mathcal{B}$ *(or of* $\mathcal{B}$ *plus Peirce's law).*

(iv) F *is a lattice filter of the t-order and* $\top \in F$.

(v) F *is a lattice filter of the k-order and* $\{a \in B : a \geq_t \top\} \subseteq F$.

Proof. As before, we assume that $\mathbf{B} = \mathbf{L} \odot \mathbf{L}$ for some Brouwerian lattice $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, 1\rangle$. Then, we know from Proposition 2.3 that $F \subseteq B$ is a bifilter if and only if $F = F' \times L$ where F' is a lattice filter of $\mathbf{L}$.
(i) $\Rightarrow$ (ii). Using the above characterization of bifilters, we have that, for any element $\langle a_1, a_2\rangle \in B$, it holds that $\langle a_1, a_2\rangle \geq_t \top = \langle 1, 1\rangle$ if and only if $a_1 = 1$. Since $1 \in F'$ because F' is a lattice filter, we have that $\langle a_1, a_2\rangle \in F$. Now suppose that $\langle a_1, a_2\rangle, \langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle \in F$. Applying the definitions, the assumptions mean that $a_1, a_1 \backslash b_1 \in F'$. So $a_1 \sqcap (a_1 \backslash b_1) = a_1 \sqcap b_1 \in F'$. This implies that $b_1 \in F'$ and so $\langle b_1, b_2\rangle \in F$. Hence, F is a deductive filter.
(ii) $\Rightarrow$ (iii). Let F be a deductive filter and $v \colon Fm \to B$ a homomorphism. By Lemma 2.7, for any axiom φ of $\mathcal{B}$, it holds that $v(\varphi) \geq_t \top$. So, by the definition of deductive filter, we have that $v(\varphi) \in F$. Similarly, if $v(\varphi), v(\varphi \supset \psi) \in F$, then since $v(\varphi \supset \psi) = v(\varphi) \supset v(\psi)$, we easily conclude that $v(\psi) \in F$ as well. Hence, F is a logical filter of $\mathcal{B}$.
(iii) $\Rightarrow$ (iv). Assume F is a logical filter. It follows from the definition that, for any theorem φ of $\mathcal{B}$ and any homomorphism $v \colon Fm \to B$, we have that $v(\varphi) \in F$. By Proposition 1.4 (iii), we have that $\varphi \supset \varphi$ is a theorem of $\mathcal{B}$. From this and ($\supset \oplus$) we conclude, by MP, that $(\varphi \supset \varphi) \oplus \neg(\varphi \supset \varphi)$ is also a theorem. Then $v((\varphi \supset \varphi) \oplus \neg(\varphi \supset \varphi)) \in F$ as well, and by Proposition 2.5 (vi) we conclude that $v((\varphi \supset \varphi) \oplus \neg(\varphi \supset \varphi)) = \top$. So $\top \in F$. To prove that F is a t-filter, assume $a, b \in F$. Since any interpretation of the axiom ($\supset \wedge$) belongs to F, we have $a \supset (b \supset (a \wedge b)) \in F$, so by MP we obtain $a \wedge b \in F$. Similarly, if $a \in F$ and $a \leq_t b$, then $a \vee b = b$, so we can apply MP to ($\supset \vee$) to conclude that $b \in F$.
(iv) $\Rightarrow$ (v). Let F be a t-filter such that $\top \in F$. It is then immediate to conclude that $\{a \in B : a \geq_t \top\} \subseteq F$. Now assume $a, b \in F$. Then $a \wedge b \in F$ because F is a t-filter. By the interlacing conditions we have that $a \wedge b \leq_t a \otimes b$. Since F is an up-set w.r.t. $\leq_t$, we conclude that $a \otimes b \in F$. If $a \in F$ and $a \leq_k b$, then $a \oplus b = b$. The interlacing conditions imply [5, Theorem 5.3] that $a \oplus b = (a \wedge \top) \vee (b \wedge \top) \vee (a \wedge b)$. Since $a \wedge \top \in F$, we conclude that

$a \oplus b = b \in F$ as well.
(v) $\Rightarrow$ (i). Clearly we only have to prove that the assumptions imply that F is a t-filter. As before, we have that if $a, b \in F$, then $a \otimes b \in F$. By the interlacing conditions, $a \otimes b \leq_k a \wedge b$, so we conclude that $a \wedge b \in F$. Now let $a \in F$ and $a \leq_t b$. By Proposition 2.5 (x), we have that $a \leq_t b$ implies $a \supset b \geq_t \top$. So, from the assumptions it follows that $a \supset b \in F$ and $a \otimes (a \supset b) \in F$. The desired result will then follow from the fact that $a \otimes (a \supset b) \leq_k b$. To see this, let $a = \langle a_1, a_2 \rangle$ and $b = \langle b_1, b_2 \rangle$. Then we have that

$$\begin{aligned}
\langle a_1, a_2 \rangle \otimes (\langle a_1, a_2 \rangle \supset \langle b_1, b_2 \rangle) &= \langle a_1, a_2 \rangle \otimes \langle a_1 \backslash b_1, a_1 \sqcap b_2 \rangle \\
&= \langle a_1 \sqcap (a_1 \backslash b_1), a_2 \sqcap a_1 \sqcap b_2 \rangle \\
&= \langle a_1 \sqcap b_1, a_2 \sqcap a_1 \sqcap b_2 \rangle.
\end{aligned}$$

So the result immediately follows from the definition of $\leq_k$. ■

This result allows to characterize the reduced models of $\mathcal{B}$ as follows:

THEOREM 2.12. *A matrix $\langle \mathbf{A}, D \rangle$ is a reduced model of $\mathcal{B}$ (or of $\mathcal{B}$ plus Peirce's law) if and only if $\mathbf{A}$ is a Brouwerian (or a classical implicative) bilattice and $D = \{a \in A : E(a)\} = \{a \in A : \top \leq_t a\}$.*

Notice that the set D coincides with the minimal non-empty bifilter $F(\top)$ considered in the previous section. In the light of the representation theorem and of Proposition 2.3, we may conclude that, for any Brouwerian bilattice $\mathbf{B} = \mathbf{L} \odot \mathbf{L}$, where $\mathbf{L} = \langle L, \sqcap, \sqcup, \backslash, 1 \rangle$, it holds that $D = \{1\} \times L$.

Acknowledgements. The research of the first author was partially supported by grant 2009SGR-1433 of the AGAUR of the Generalitat de Catalunya and by grant TIN2010-20967-C04-01 of the Spanish Ministerio de Ciencia e Innovación. The research of the second author has received funding from the *People Programme* (Marie Curie Actions) of the European Unions *Seventh Framework Programme* (FP7/2007-2013) under REA grant agreement PIEF-GA-2010-272737.

References

[1] ARIELI, O., and A. AVRON, Logical bilattices and inconsistent data, in *LICS, Proceedings, Ninth Annual IEEE Symposium on Logic in Computer Science, 4-7 July 1994, Paris, France*, 1994, pp. 468–476.

[2] ARIELI, O., and A. AVRON, Reasoning with logical bilattices, *Journal of Logic, Language and Information* 5 (1):25–63, 1996.

[3] Arieli, O., and A. Avron, The value of the four values, *Artificial Intelligence* 102 (1):97–141, 1998.

[4] Avron, A., A note on the structure of bilattices, *Mathematical Structures in Computer Science* 5 (3):431–438, 1995.

[5] Avron, A., The structure of interlaced bilattices, *Mathematical Structures in Computer Science* 6 (3):287–299, 1996.

[6] Belnap, N. D., How a computer should think, in G. Ryle, (ed.), *Contemporary Aspects of Philosophy*, Oriel Press, Boston, 1976, pp. 30–56.

[7] Belnap, Jr., N. D., A useful four-valued logic, in J. M. Dunn, and G. Epstein, (eds.), *Modern uses of multiple-valued logic (Fifth Internat. Sympos., Indiana Univ., Bloomington, Ind., 1975)*, Reidel, Dordrecht, 1977, pp. 5–37. Episteme, Vol. 2.

[8] Blok, W. J., and D. Pigozzi, *Algebraizable logics*, vol. 396 of *Mem. Amer. Math. Soc.*, A.M.S., Providence, 1989.

[9] Bou, F., R. Jansana and U. Rivieccio, Varieties of interlaced bilattices, *Algebra Universalis* 66 (1):115–141, 2011.

[10] Bou, F., and U. Rivieccio, The logic of distributive bilattices, *Logic Journal of the I.G.P.L.* 19 (1):183–216, 2011.

[11] Burris, S., and H. P. Sankappanavar, *A course in Universal Algebra*, the millennium edn., 2000.

[12] Cignoli, R., and A. Torrens, Free Algebras in Varieties of Glivenko MTL-algebras Satisfying the Equation $2(x^2) = (2x)^2$, *Studia Logica* 83 (1–3):157–181, 2006.

[13] Conrad, P. F., and M. R. Darnel, Generalized Boolean algebras in lattice-ordered groups, *Order* 14 (4):295–319, 1997/98.

[14] Curry, H. B., *Foundations of mathematical logic*, Dover Publications Inc., New York, 1977. Corrected reprinting.

[15] Czelakowski, J., *Protoalgebraic logics*, vol. 10 of *Trends in Logic—Studia Logica Library*, Kluwer Academic Publishers, Dordrecht, 2001.

[16] Davey, B. A., Dualities for equational classes of Brouwerian algebras and Heyting algebras, *Transactions of the Americal Mathematical Society* 221 (1):119–146, 1976.

[17] Dunn, J. M., *The algebra of intensional logics*, Ph. D. Thesis, University of Pittsburgh, 1966.

[18] Fitting, M., Bilattices in logic programming, in *Proceedings of the 20th International Symposium on Multiple-Valued Logic*, The IEEE Computer Society Press, Charlotte, 1990, pp. 238–246.

[19] Font, J. M., Belnap's four-valued logic and De Morgan lattices, *Logic Journal of the I.G.P.L.* 5 (3):413–440, 1997.

[20] Font, J. M., F. Guzmán, and V. Verdú, Characterization of the reduced matrices for the $\{\wedge, \vee\}$-fragment of classical logic, *Bulletin of the Section of Logic* 20:124–128, 1991.

[21] Font, J. M., and R. Jansana, *A general algebraic semantics for sentential logics*, vol. 7 of *Lecture Notes in Logic*, second edn., Springer-Verlag, 2009.

[22] Ginsberg, M. L., Multivalued logics: A uniform approach to inference in artificial intelligence, *Computational Intelligence* 4:265–316, 1988.

[23] Mobasher, B., D. Pigozzi, G. Slutzki, and G. Voutsadakis, A duality theory for bilattices, *Algebra Universalis* 43 (2-3):109–125, 2000.

[24] Movsisyan, Y. M., A. B. Romanowska, and J. D. H. Smith, Superproducts, hyperidentities, and algebraic structures of logic programming, *J. Combin. Math. Combin. Comput.* 58:101–111, 2006.

[25] Nelson, D., Constructible falsity, *The Journal of Symbolic Logic* 14:16–26, 1949.

[26] Odintsov, S. P., Algebraic semantics for paraconsistent Nelson's logic, *Journal of Logic and Computation* 13 (4):453–468, 2003.

[27] Odintsov, S. P., On the representation of N4-lattices, *Studia Logica* 76 (3):385–405, 2004.

[28] Pynko, A. P., Functional completeness and axiomatizability within Belnap's four-valued logic and its expansions, *Journal of Applied Non-Classical Logics* 9 (1):61–105, 1999. Multi-valued logics.

[29] Rasiowa, H., *An algebraic approach to non-classical logics*, vol. 78 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1974.

[30] Rivieccio, U., *An Algebraic Study of Bilattice-based Logics*, Ph. D. Dissertation, University of Barcelona, 2010.

[31] Rivieccio, U., Implicative twist-structures, *Algebra Universalis*, To appear.

[32] Romanowska, A., and A. Trakul, On the structure of some bilattices, in *Universal and applied algebra (Turawa, 1988)*, World Sci. Publ., Teaneck, NJ, 1989, pp. 235–253.

[33] Wójcicki, R., *Theory of logical calculi. Basic theory of consequence operations*, vol. 199 of *Synthese Library*, Reidel, Dordrecht, 1988.

Félix Bou
Department of Probability, Logic, and Statistics
Faculty of Mathematics
University of Barcelona
Gran Via de les Corts Catalanes 585
Barcelona E-08071, Spain
bou@ub.edu

Umberto Rivieccio
School of Computer Science
University of Birmingham
Edgbaston B152TT
Birmingham, United Kingdom
u.rivieccio@cs.bham.ac.uk