Frege's *Basic Law (V)* and *Cantor's Theorem*

A Case Study in Rejecting some Axiom

The following essay reconsiders the ontological and logical issues around Frege's *Basic Law (V).* If focuses less on Russell's Paradox, as most treatments of Frege's *Grundgesetze der Arithmetik* (GGA)[1] do, but rather on the relation between Frege's *Basic Law (V)* and *Cantor's Theorem* (CT). So for the most part the inconsistency of Naïve Comprehension (in the context of standard Second Order Logic) will not concern us, but rather the ontological issues central to the conflict between (BLV) and (CT). These ontological issues are interesting in their own right. And if and only if in case ontological considerations make a strong case for something like (BLV) we have to trouble us with inconsistency and paraconsistency. These ontological issues also lead to a renewed methodological reflection what to assume or recognize as an axiom.

§1 Value ranges and extensions

Frege's *Basic Law (V)*[2]

(BLV) $(\forall F, G)(\acute{e}F(e) = \acute{e}G(e) \equiv (\forall x)(F(x) \equiv G(x))$

For any two concepts[3] it is true that their respective *value ranges* are identical if and only if their applications to any objects are equivalent. As is well known value ranges are the new ingredient in GGA that supplements the system of the *Begriffsschrift*. The logic of (the book)

1 I will use "GGA" as abbreviation both for the book and the logical system developed in the book and disambiguate only if a context makes this necessary.

2 Standard symbols are used here instead of Frege's *Begriffsschrift* notation. In contrast to Frege identity between objects ("a = b") is distinguished from equivalence between statements ("F(a) ≡ G(a)"). The debate about statements as names for truth values does not affect the issues discussed here.

3 (BLV) holds in GGA also for any *functions* whatsoever. We consider only *concepts* (i.e. functions from objects to truth values). For our purposes here we do not consider relations either. Relations could be reduced to concepts anyway (e.g. simply by working with concepts like "( ) being the father of George" instead of the relation "( ) being the father of ( )*").

Begriffsschrift ist Second Order Logic (SOL) without comprehension principles. With GGA Frege takes 'value range' as a basic concept, which can be illustrated by considering the graph of a function. Value ranges thus resemble sets of ordered pairs, but as a basic concept 'value range' is undefined and could thus itself be used in introducing ordered pairs. Frege also often speaks of the 'extension' of a concept, as if "extension" meant almost the same as "value range". Extensions understood as sets can be gained from value ranges by considering only those objects that are mapped under a concept to the truth value TRUE (i.e. by considering a projection on the corresponding value range)[4]. Thus, although Frege does not use set theory in the standard sense of ZFC and related systems, but refers only to his value range objects, we may talk about extensions as sets and (BLV) as stating an identity condition on extensions (and thus sets).

Frege defines a relation ("$x \cup \acute{e}F(e)$") of an object and a value range that outputs on the input of an object and a value range the value the objects is mapped to under that value range. So if we consider the function 'father-of( )' and George the relation in question outputs, say, Lloyd. If we consider concepts, say 'is-prime( )', and do the same with 'is-prime( )' and the object 2 we get TRUE. When we only consider concepts the relation in question between a value range/an extension of a concept and an object outputs TRUE if and only if the object is mapped by the concept to TRUE, i.e. if and only if the concepts applies to the objects. Thus, concerning concepts and talking of extensions as sets the relation defined by Frege corresponds to set membership (expressed by "$\in$") and given (BLV) the following abstraction principle holds

(NCF1) $(\forall F)(\forall x)(x \in \acute{e}F(e) \equiv F(x))$

This resembles λ-conversion and is similar to the common expression of *Naïve*

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4 "TRUE" is used here as referring to the truth value TRUE, a basic object. Frege assumes truth values as basic objects. For mostly technical reasons he identifies them with some extensions (ultimately their own singletons) in GGA. We follow Frege's commitment to truth values here as nothing in the argument *here* depends on the issue whether statements refer to truth values or just are evaluated with respect to their truth (and refer to something else or do not refer at all). The whole debate could be restated with statements referring to facts or the world. I prefer understanding sentences as referring to facts/situations, cf. "Die Welt ist die Gesamtheit der Tatsachen, nicht der Dinge", *Grazer Philosophische Studien*, 1999, pp. 111-32.

Comprehension, as we develop more closely in §2. Although Frege uses a relation wider in definition than elementhood – as he used with 'value range' a concept wider than 'extension' – what elementhood does is present in GGA, and thus we can freely use "$\in$" and set abstracts (like $\{x \mid F(x)\}$) without doing anything that could not be done in GGA.

§2 Abstraction and Comprehension in *Basic Law (V)*

With (BLV) GGA adds an *abstraction principle* to the system of the book *Begriffsschrift* (SOL). We abstract from the different ways we come to or compute some value range by using some concepts to the value range that these concepts share. Concepts taken thus are concepts in extension (extensional concepts).[5]

From (BLV) we get by substitution the same concept twice

(1) $\acute{e}F(e) = \acute{e}F(e) \equiv (\forall x)(F(x) \equiv F(x))$

as the right hand side is a logical truth we can detach to

(2) $\acute{e}F(e) = \acute{e}F(e)$

and then, as extensions/value ranges are first order objects, existentially generalize to

(3) $(\exists x)(x = \acute{e}F(e))$

and then again generalize on the second order constant to get *Naïve Comprehension* as an existence claim

(NCF2) $(\forall F)(\exists x)(x = \acute{e}F(e))$

In more modern notation and using the presence of elementhood, as expressed in (NCF1), we can write more generally

(NC2) $(\forall F)(\exists x)(\forall y)(y \in x \equiv F(y))$ ["x" not free in "F( )"][6]

the first order, schematic version of which is

(NC1) $(\exists x)(\forall y)(y \in x \equiv \varphi(y))$ ["x" not free in φ]

With the help of set abstracts we can write also

(NCF2') $(\forall F)(\exists x)(x = \{y \mid F(y)\})$

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5 Frege has no use for intensional entities (and thus for concepts in intension) in GGA.

6 Unless mentioned we understand this requirement to be expressed and fulfilled.

We will refer to these comprehension principles, if no distinction between these last versions is relevant, just as "(NC)". As (BLV) leads to (NC) we can also say that GGA extends SOL by (unrestricted) comprehension principles.

(NC) expresses the idea that for every concept there is an extension (to that concept).[7]

§3 Comprehension and *Cantor's Theorem*

(BLV) ensures that two extensionally distinct concepts have distinct extensions. (NC2) then expresses a correspondence between the domain of quantification of the upper case letters and a subdomain (proper or not) of the domain of quantification of the lower case letters; the converse would express that every object is the extension of some concept, a principle which Frege actually supports in GGA, which is concerned only with truth values, which Frege identifies with some extensions (namely their own singletons), and mathematical objects like numbers, which Frege constructs as extensions, but this converse of (NC2), of course, cannot be accepted for a general application of logic.

Even for extensional concepts – intensional concepts could only introduce more distinctions – there have to be as least as many first order objects as there are second order objects. Since any two distinguished extensional concepts are distinguished in their application to some object at least, so that they have different extensions by (BLV).

This stands in direct contradiction to Cantor's Theorem, that there have to be more sets of order n corresponding to the propositional functions (concepts) on objects of order n-1 than objects of order n-1; in its general form

(CT) $(\forall x)(|\{0,1\}^x| > |x|)$

The cardinality of the (set of) functions from some set x to {0,1}, which may be thought of as concepts mapping objects from x to TRUE or FALSE, is strictly larger than the cardinality of

7 As often retold, (NC) leads to *Russells Paradox*, since it assures us that (i) the set $\{x \mid x \notin x\}$ has to exist as the concept '$x \notin x$' exists, and (ii) that this set is a first order object itself, able to fall under that very concept, which leads to the contradiction (of it being a member of itself and not being a member of itself). As also often retold Russell discovered his paradox by working through the proof Cantor gave of (CT); in effect Russell's argument is only a special case of Cantor's more general argument. And Russell told Frege about this in a letter, more of which later [§4].

the set x in question. A more familiar expression talks instead of these functions of the subsets of x they generate, so that we can write – using the concept of 'powerset':

(CT') $(\forall x)(|\wp(x)| > |x|)$

Looking at (CT) and its talk of function we can say: There are *more* propositional functions/concepts applicable to objects than there are objects. A rephrasing directly relevant and in contradiction to (NC2).

We can so immediately recognize that combining (NC) with a system including set theory enough to yield (CT) has to be inconsistent.

§4 Frege and *Cantor's Theorem*

Of course, Frege was surprised by *Russell's Paradox* and he did not see the problems engendered for referentiality that come with (BLV) [see §5]. He explicitly, however, derives his form of (NCF1), and thus would have supported (NC2). One of his tactical manoeuvres consists in exploiting (BLV) and (NC2) to substitute talk about extensions (i.e. first order objects) for talk about concepts. He can avoid higher order functions of degree more than second order in GGA by substituting for a concept as argument of a higher order function the extension of that concept in a closely related first order function (GGA §§35, 37). Thus he doubly requires a correspondence both between concepts and their extensions as well as between second order and first order functions, violating (CT) twice over.

Why did Frege not see the supposedly obvious contradiction between (CT) and (NC2)?

Frege knew Cantor's work in general, but could he have overlooked or not known (CT)?

Cantor develop (CT) stepwise. Around 1873 he discovered the version dealing with the case of real numbers and published it in 1874 and 1878 (the famous "Beitrag zur Mannigfaltigkeitslehre"), the powers of sets are developed in 1883 (in the even more famous *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*), and the diagonalization argument, which later inspired Russell to his paradox, was presented publicly and published in 1891 ("Über eine elementare Frage der Mannigfaltigkeitslehre"), the explicit argument in terms of the powerset he pointed out to Dedekind in a letter only.[8]

Frege mentions the *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* in his own *Grundlagen der Arithmetik* (GLA) with respect to Cantor's theory of the infinite cardinalities, but they then quarrel only in an exchange about Cantor's 1885 review of GLA about the question whether Cantor's cardinalities ('Mächtigkeiten') are the same as Frege's ('Anzahlen'). Cantor in fact warned to take extensions as a basis of a theory of numbers, but the idea that not all collections are sets, but that some of them are too large, came to Cantor only around 1890, from which time on he knew about the paradoxes of the universal set and – supposedly – the *Burali-Forti Paradox*, but he did not publish these insights.[9]

In the second volume of GGA Cantor is targeted for his formal imprecision and supposed confusion between sign and abstract object, and is supported again concerning the existence of the actual infinite. Cantor's major work in set theory Frege does not discuss in any detail, and so one may think that he just missed the discovery of *Cantor's Theorem*, however difficult to believe this sounds. Frege showed no interest in general set theory beyond the use of extensions at the foundation of arithmetic and reserved judgement on the role larger ordinals may play. Frege had to know *Cantor's Theorem* before the publication of the second volume of GGA, if only because of Bertrand Russell.

Russell in his correspondence with Frege explains his discovery of his paradox (in a letter from June 24th 1902) by his study of Cantor's proof that there is no largest cardinal number, which he supposes Frege knew.[10] In another letter to Frege (July 24th 1902) Russell explicitly says that one can easily prove that there is *no correspondence* between all objects and all functions and in a letter from September 29th 1902 even outlines his formal version of (CT) mentioning again Cantor's claim about powersets.[11] Frege first doesn't reply to the allusions

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8 Cf. Dauben, Joseph. *Georg Cantor*. His Mathematics and Philosophy of the Infinite. Princeton, 1979. Zermelo in 1908 called (CT) "Cantor's Theorem".

9 Cf. Hallet, Michael. *Cantorian Set Theory and Limitation of Size*. Oxford, 1984, pp.126-28, 165-75.

10 Given Cantor's diagonal proof that there can exist no correspondence between a set x and its powerset, *Russell's Paradox* just results as the special case of considering the universal set and the identity function as correspondence.

11 Cf. Gabriel, Gottfried/Kambartel, Friedrich/Thiel, Christian (Hg.). *Gottlob Freges Briefwechsel*. Mit D. Hilbert, E. Husserl, B. Russell, sowie ausgewählte Einzelbriefe Freges. Hamburg, 1980.

to Cantor and comes up (in a letter from August 3rd 1902) with the astonishing remark that he considers the proof that there can be no correspondence between all objects and all functions as ‘questionable’ (‘bedenklich’). He argues that the very idea of correspondence and uniqueness presupposes the notion of identity, and identity is a first order concept! (Remember that GGA and SOL use “=” only for objects and concepts are compared only with respect to co-extensionality by means of universal quantified equivalences using “≡”.)

This is astonishing in two respects: Firstly, it seems that Frege flatly denies (CT) or at least Cantor’s proof! Secondly: even if concepts do not enter in identity statements, Frege’s formal system allows for mixed relations (having as one argument an object and as another argument a concept) and Frege uses such mixed relations himself (the most crucial being, of course “´” which relates a concept to its extension). In terms of such mixed relation a correspondence between all objects and all functions can be considered without obviously presupposing identity. The ‘uniqueness’ presupposed in a correspondence (in the sense of equinumerosity) is functionality, which makes use of identity between objects (like in Frege’s own definitions in GGA §§37-40):

(4) $\text{Funk}(F) \stackrel{\text{def}}{=} (\forall x,y,z)(F(x,y) \land F(x,z) \supset y = z)$

Defining a correspondence strictly in that fashion it seems one is forced to apply identity to concepts[12]

(5*) $\text{Funk}(M_{xy}) \stackrel{\text{def}}{=} (\forall F,G,H)(M_{xy}(F,G) \land M_{xy}(F,H) \supset G = H)$

This reasoning seems to lay behind Frege’s remark to Russell. Obviously, however, there is a simple solution to that problem: Let *R* be a mixed relation having objects as first argument and concepts as second argument, we then define

(6) $\text{Correspondence}(R_y) \stackrel{\text{def}}{=}$

(i) $(\forall x,G,H)(R_y(x,G) \land R_y(x,H) \supset (\forall z)(G(z) \equiv H(z))$

12 We use „M_{xy}” to indicate that M is a second order relation, the common practice of using “F” etc. just like ordinary arguments (i.e. not in their Fregean form of “F( )”) hides their possession of argument places, which for Frege constitutes the crucial difference between concepts and objects. Writing just “M(F)” at least depends on a common understanding that we cannot have also “M(a)”.

(ii) $\text{Funk}(R_y^{-1})$

As Frege thinks only of concepts in extension two concepts equivalent in their application to objects have to be identified – or whatever one may call that relation between them.

Starting with (6) we can proceed on the lines of Cantor's proof.[13] Frege's remark thus seems widely off the mark.

Frege having not seen the incompatibility between (NC2) and (CT) should not stop us, however, from further investigating the issues involved in that conflict. Still we put the inconsistency of (NC) itself (in combination with standard SOL or FOL) to the side for the moment and focus on the ontological questions involved.

§5 Determinacy of concepts and reference and (BLV)

As the universe of GGA is flat and not stratified we can neither assume that some objects are being created later than others nor that they are acceptable by being structurally proper placed on a level of stratification which presupposes another level. In the cumulative hierarchy associated with ZFC the sets in higher ranks are not created later than those on lower ranks either, but they are structurally properly placed as all their members are on ranks below them. Thus taking the iterative hierarchy to be in place all at once – not being the result of some mystical temporal process of construction – does not exclude structural dependencies that resemble dependencies in a process of generation. The introduction of value ranges/extensions corresponding to all propositional functions/concepts as expressed in (NC2) thus poses a difficulty given Frege's principle that any concept has to be defined for all objects (as otherwise *tertium non datur* will not hold and his principle excluding stepwise definitions can be violated). For later reference let us call this principle

(DET) $(\forall F)(\forall x)(F(x) \vee \neg F(x))$

As value ranges are introduced as objects concepts have to be applied to them as well, to all of

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13 We can run an argument resembling Cantor's and Russell's argument with respect to Frege's correspondence ´ between concepts and their extensions: We start with $r \overset{\text{def}}{=} \acute{e}((\forall F)(e = \acute{a}F(a) \supset e \notin e)$ and re-run Russell's argument with respect to: $r \in r$.

them. If we now introduce an extension $\acute{e}F(e)$ all concepts have to be applied to it, including F itself. How then can "F( )" have satisfied (DET) before? If again we think of the relation of concepts to their extensions not as a temporal proceeding we can suppose that with a concept we also already have the determination of the application of that concept to its own extension given. But given that we can proceed further and define propositional functions involving parameters for extensions the application of (BLV) as determining the identity of extensions runs into trouble.[14] If we define:

(7) $G(\;) \stackrel{def}{\equiv} (\;) = \acute{e}F(e)$

something being the extension of concept F. G has an extension, and we can ask whether it is identical to the extension of any other concept. If we consider F we have to determine

(8) $\acute{e}G(e) = \acute{e}F(e)$

But when we now look at the following instance of (BLV)

(9) $\acute{e}G(e) = \acute{e}F(e) \equiv (\forall x)(G(x) \equiv F(x))$

we will find ourselves in the situation to determine (as an instance of the right hand side)

(10) $G(\acute{e}G(e)) \equiv F(\acute{e}G(e))$

which requires to evaluate

(11) $G(\acute{e}G(e))$

which brings us by the definition of "G( )" full circle to

(7) $\acute{e}G(e) = \acute{e}F(e)$

Thus we meet an object the identity of which is indeterminable. This in itself is not a contradiction, but it renders, for a start, Frege's attempted proof that every name in GGA has a proper reference useless. If every name (including sentences for Frege) had a proper reference, then GGA would be consistent (arguing on the stepwise increase in the complexity of names). Frege's attempted proof (GGA §31) fails as we cannot determine/proof proper referentiality (even if it obtained).

14 Cf. von Kutschera, Franz. *Gottlob Fege*. Berlin/New York, 1989, pp.127-29. Frege's fixture of (BLV) in the appendix to the second volume of GGA tries to solve this very problem by exempting the extensions themselves from the verification of conceptual equivalence. The way Frege put this implies unfortunately that there are only two objects. It makes extension as well to second-class objects.

Frege's preference for (BLV) over other abstraction principles like (HP)[15] and his argument around the 'Julius Caesar problem' are based on identity determination by (BLV) being generally feasible. As this is now falsified we have another reason (in addition to the inconsistency stemming from (BLV)) to switch to a reformed system like so-called “Frege Arithmetic” (i.e. GGA without (BLV), but with (HP) as additional axiom). Frege Arithmetic (FA) is consistent and can derive Peano Arithmetic.[16]

FA still has a flat universe of objects, but does not entail (NC2) – as otherwise it couldn't be consistent. We can thus no longer be sure that every concept has an extension. More strongly put: We now know, since (NC2) cannot be true, that there are concepts without extensions. Can we accept this? [More on this later, §7]

A flat universe in itself is consistent, but only as long as we avoid new comprehension schemas. Once we introduce even limited comprehension like in the Axiom (Schema) of Separation, as is done in ZFC, we run into further difficulties like now being able to proof the inexistence of the universal set and so the inexistence of absolute complements[17], both of which are crucial ingredients in GGA (and FA).

§6 Where to put the blame?

As the concept of subset is elementarily tied to the concept of set the Powerset Axiom (of ZFC) is beyond reproach. Simple observations within finite set theory corroborate that there are more subsets to a non empty set than one for each element, thus the set of the subsets of a set exceeds that set in cardinality (in finite set theory at least). These observations are not sufficient for infinite sets (as enumerations of ¦ show), and here (CT) comes in.

Proving (CT) works by diagonalization or with an indirect argument. This argument need not be valid in non-standard logics, but is in SOL and in GGA. (NC) might so – presumably – be combinable with a non-standard logic which does not allow deriving (CT).

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15 (HP) states that the cardinalities of two concepts coincide if and only if there is a correspondence between their extensions (if and only if they are equinumerous).

16 Cf. Heck, Richard. *Frege's Theorem*. Oxford, 2011.

17 The proof as outlined in many set theory *text* books being again a *variation* on *Russell's Paradox*.

One idea might be to avoid the additional content of (NC) of a 1st and 2nd- order correspondence (additional to the idea of every concept having an extension) and *only* claim that there is an extension to any concept. These extensions then cannot be 1st-order objects given (CT) and SOL. Where to put them? Full SOL contains a Comprehension Schema[18]

(NC23) $(\exists X)(\forall y)(X(y) \equiv \varphi(y))$ ["X" not free in φ]

This asserts that there exists a second-order entity the application of which corresponds to the application of a propositional function of the full language. "(NC23)" means here a variant of (NC) in which the crucial places of (NC) are second-order and at least second order. The later point distinguishes (NC23) from (NC2) as the bound variable "F" in (NC2) has to be second-order (with an argument place for a first-order objects), although it may impredicatively contain second-order quantification (i.e. *bound* second-order variables). As φ is understood to be a propositional function for first-order arguments, GGA does not contain third-order quantification, and given φ was third-order, we had to discuss just the analogue issue of 2nd-3rd-order correspondence, we revert, for now, to

(NC22) $(\forall F)(\exists X)(\forall y)(X(y) \equiv F(y))$

If we now understand "X(y)" as functional application, then (NC22) becomes almost vacuous: "X( )" is just another – or even the same – first order propositional function/concept as "F( )". If we understand "X(y)" as short for "y∈X" we get

(NC22*) $(\forall F)(\exists X)(\forall y)(y \in X \equiv F(y))$

We can take this as the claim that to every concept (propositional function of the language, with the usual restriction on "X" in "F( )") there exists an extension as a second-order *object*. (NC22) claims a correspondence between a type of function on first-order objects (i.e. concepts) as denizens of the second level and objects on the second level. This does not contradict (CT) making no cross-level correspondence claims. We can therefore combine versions of (NC) with (CT) as long as the versions of (NC) move extensions out of the range of the quantifier over the objects comprehended. Such a version of (NC), like (NC22*), will

18 Cf. for instance Shapiro, Stewart. *Foundations without Foundationalism*. A Case for Second-Order Logic. Oxford, 1991, p.66.

be stratified, but can still be impredicative. Frege, however, could not accept (NC22*): In (NC22*) we have second-order quantification, *and* the second-order quantifiers range *both* over functional entities (namely the concepts) and objects (namely the extensions). (NC22*) so violates one of Frege's most important distinctions: that between function and object. Frege's ontological dualism only knows objects (including truth values) and functions (including concepts). Functions combine with their arguments to yield sentences. Two objects do not combine to a sentence. A sort of quantification covering both functions and objects generates thus syntactic non-well-formed expressions. This is unacceptable. If we want to keep the idea of extensions as higher order entities we have to introduce a second form of second-order quantification, say "⋁" and postulate:

(NC22') $(\forall F)(\bigvee X)(\exists y)(y \in X \equiv F(y))$

We maintain thus the distinction between concepts and objects, but now one wants to know what our ontological model has become. We have a level of first-order entities (ordinary objects), a level of second-order entities taking first-order entities as arguments (i.e. concepts) and a level of second-order entities standing to the first-order entities in the elementhood relationship and to the second-order entities of the first kind in the being-the-extension-of relationship (i.e. extensions). Extension names are like ordinary names (singular terms) saturated (i.e. refer to an object), but refer to an object of another kind. They can be quantified over, but they cannot be comprehended with the objects into a single domain of quantification, on pains of introducing the contradiction (again). As they are objects one may suppose that concepts can apply to them. Frege postulated (DET): Concepts have to be determined everywhere. We can, of course, keep (DET) and all concepts can satisfy (DET) by being applicable to first-order objects. It seems we keep (DET) then only in letter and not in spirit: We have objects (i.e. extensions) now for which concepts are not defined. Syntactically, however, concepts better be defined not for them as otherwise they would have to take object variables of first and second order – syntactic garbage again. There can be mixed concepts, of course, relating first and second-order objects (like the elementhood relation, expressed by “∈”).

(CT) may apply to the domain of first-order variables, the powerset of that domain being beyond the reach of first-order quantifiers. In full SOL the members of this powerset are the values of the second order variables. The powerset itself is not an object of the theory. Given the preceding ontology with a second kind of second-order entities the members of the powerset of the domain of first-order variables are the values of the second-order object expressions. What about concepts then? What values to assign to them? Concepts may be taken not as sets/extensions but as functions, taking functions to be a *basic* sort of entity (i.e. not taking them as sets). One may also assign *both* concept- and extension-expressions members of the powerset of the domain of first-order quantification; they are thought to correspond anyway. The first solution, although not usual today, meets Frege’s insistence on distinguishing objects of any kind from functions. The semantics of the reformed system would contain then two domains of objects and a domain of functions/concepts.

On closer inspection, our problems reoccur. There should be concepts pertaining to extensions (i.e. our second-order objects). What about the extensions of *these* concepts? If they are again second-order objects (i.e. fall within the domain of quantification of extensions) we are back to our old troubles. If they are not second-order objects but objects of a further kind we are forced to introduce a hierarchy of objects (i.e. first a third domain of third-order objects as extensions of concepts of second-order objects, and so forth …).[19] As GGA does not use third-order quantification one may just forsake higher order concepts. This, however, is both unnatural (as there are such higher order concepts even if not treated in one’s system) and misleading anyway, as “∈” is already in use. What is the extension of “a∈( )” for some constant “a”? It has to be a collection of extensions (i.e. a third type of object). Once we have this extension the story can be developed further and further.

Blame, so it seems, thus has to be laid on Frege’s idea of an unstratified universe. Combining SOL with ZFC (i.e. ZFC2) allows distinguishing quantification of objects from quantification

19 This reminds us of the fact that first-order ZFC can be carried out completely on first-order variables. So distinctions of size corresponding to (CT) occur within the domain of first-order variables.

of functions/concepts. ZFC and ZFC2 come with the cumulative hierarchy as model for the domain of first-order quantification. The cumulative hierarchy (the universe V) itself is not an object of ZFC and the powerset of all sets of first order entities is not an entity of ZFC2. So a new ontological conundrum (or some variant of it) raises its head: What is the status of V? Frege's ontology maintained a universal set and avoided *this*, at least.

ZFC also abandons (NC) altogether in favour of the Separation Axiom (schema), and ZCF2 restricts comprehension to first-order objects. There are strong Fregean arguments in favour of (NC), however.

§7 A Fregean transcendental argument for (NC)

Frege committed himself to (NC2) as he is committed to extensions as logical objects. Extensions of uncontroversial basic concepts serve as first encounter with logical objects of the kind numbers turn out to be. Their claim to be 'logical objects' (i.e. being forced upon us by mere logic) rests on their close association with concepts: As we have a logical grasp of concepts we have a grasp of their extensions. From our logical knowledge alone we should expect that there is one extension for every concept – the very claim expressed in (NC2). As logical objects are objects – and numbers are the very paradigm of logical objects – these extension are *objects*, thus within the range of the first-order quantifiers, as (NC2) has it.

One can extract from Frege's philosophy a transcendental argument for (NC2), or at least for some version of (NC).[20] That arguments runs somewhat like this:

1. We distinguish sentences from lists of words.
2. There has to be something giving unity to the sentence.
3. This is the general term (i.e. the word expressing a concept) as it applies to singular terms by having at least one slot for arguments.[21]
4. The work done is not due to the ink marks or sounds but due to the reference of the

20 The argument is 'transcendental' as it ties (NC) to our ability to understand sentences. If (NC) belongs to the preconditions and background of that ability we cannot question it without raising doubts about understanding sentences, which quickly can prove self-destructive.

21 For the moment we only consider non-relational expressions.

general term: the concept.

5. The concept thus applies to the reference of the singular term to yield the reference of the sentence (as preserved under translation).[22]
6. In this the concept maps some objects to TRUE and some to FALSE.
7. In the way of this mapping (extensional) concepts can be distinguished from one another.
8. This mapping is their *value range* (their graph).
9. Thus every concept has a value range (even if several general terms expressing the same concept can share a value range).
10. So, as we have to attest concepts as necessary conditions for sentential content (and structure), we have to assume value ranges.
11. If we define the collection of those objects mapped by a concept to TRUE as the *extension* of the concept, every concept has an extension.

Where should this argument be blocked?

If we do not want to endorse that every concept has an extension we need to claim either

i) Some concepts do their work of mapping without a map.

or

ii) Some sentence like linguistic entities which appear like sentences are not sentences at all, since they lack a concept in their meaning.

Claim (i) is hardly comprehensible. It might be understood as claiming incomplete maps not being maps at all, and thus ultimately may lead to a 3-valued map, which again would modify but not suspend the fundamental picture of concepts coming with a map. Even the concept mapping every object to FALSE is a proper concept.

Claim (ii) poses the difficult problem of distinguishing proper from apparent sentences. And even if this could be done (without ingenuity), we still have the situation that concepts are not

22 We consider in this argument only reference, but a similar principle of compositionality works at the level of meaning (in the narrow sense of intensional meaning). The argument only gains strength in being non-committal about intensional entities, which play no role in GGA.

to blame as they are absent from the meaning of these misbehaving linguistic entities. For concepts (NC2) still holds. Every concept (even 'x ≠ x') has an extension then (even if it is ∅). One may question whether every sentence associates with a corresponding concept, but once a sentence is well-formed it contains a well-formed general term, which should at least correspond to the 'empty' concept (i.e. a map having ∅ as extension).

If we have no better ideas (like accepting (NC2) and moving to a non-standard logic) the only way out is to deny straightforwardly that extensions are objects (first order entities in the basic case). We can then accept the transcendental argument, but deny the crucial step of including extensions into the domain of first-order objects and first-order quantification.

Frege took the idea of logical objects literally and considered them to be not just logical items of thought (after all a concept is something we think about and cannot be an object – the very idea of this bordering on the inexpressible in Frege's ontological framework). Frege moves from 'logical object' to 'object' in the sense of being a member of the first-order domain. There are no other *objects* for Frege – again: since concepts are of a different ontological category. Frege also maintains no distinctions of levels in the realm of individuals. (NC) by itself is not inconsistent (not just with (CT) also by itself) but only in combination with a single sorted universe of objects. As we have seen [in §6], however, introducing a stratified universe means introducing the iterative hierarchy, which at least in its meta-theory contains inaugurating a further ontological categorical distinction: between sets and proper classes (or whatever might be the category V belongs to). Given his ontological categorical duality and the transcendental argument Frege had no option, but to fully endorse (NC2).

§8 The issue of abstract entities

The problems around (BLV) have been diagnosed sometimes as going back to Frege's plan to justify the assumption of abstract entities.[23] Dummett traces (BLV) to Frege's strategy of employing his *Context Principle* to justify reference to extensions (and thus abstract objects): As we understand expressions in the context of complete statements, so we are justified in

23 Cf. for instance Dummett, Michael. *Frege*. Philosophy of Mathematics. London, 1991, pp. 209-40.

assuming those entities (referred to by expressions in these statements) occurring in the truth conditions of these statements. The contradiction thus 'refuted the context principle, as Frege had used it'.[24] The explanatory failure goes back to the circularity explained above [in §5] of spelling out the truth conditions of sentences about extensions with sentences involving universal quantification about first-order entities thus referring us back to the extensions themselves (as supposed first-order entities). Well-founded epistemology has to proceed by first identifying the domain of first-order quantification and continuing with (DET) to extensions of concepts. The circularity prevents us from computing the identity of extensions using (BLV). This computational and epistemological failure *by itself*, however, does neither imply the inconsistency of (BLV) nor that the first-order domain cannot contain abstract entities. The failure shows the failure of Frege's *epistemological* project to explain our reference to extensions as abstract objects. Frege's epistemological project in the philosophy of mathematics rested less in establishing the existence of numbers, then explaining *how* they are given to us. The *Context Principle* (especially in combination with definitions which equate the content of two sentences) linked numbers to more accessible content (like in HP numbers are linked to equinumerosity). The failure of this project due to the circularity involved [as seen in §5] does not show by itself the failure of (BLV) or (NC) as *ontological* structural truths.

Succeeding or failing to introduce numbers as extensions, in any case, does not exhaust the issue of abstract entities. Extensions go back to concepts as the prime logical objects [as we saw in §7]. Concepts are abstract entities and already the system of *Begriffsschrift* quantifies over them. (NC) is impredicative (the concepts comprehended may contain bound second-order variables) and does not develop the domain of second-order quantification piecemeal, but that only concerns constructivists – like Dummett. The system of *Begriffsschrift*, SOL without (NC), is consistent, as Frege himself established there by arguing for its correctness. Thus for a realist – like Frege – SOL and so concepts as abstract entities are justified with the

24 Dummett, p. 225. Dummett endorses the *Context Principle* if epistemologically proper employed (ibid p.235-40).

system of *Begriffsschrift*.

A 'no class'-theory denying the existence of extensions (set/class-like entities) would be overkill on the other hand. The late Frege himself resorted to the idea that numbers are not extensions at all, but that numerals occur only within wider sentential contexts which create the impression that these were referring singular terms.[25] But even if it was feasible to substitute predication for elementhood statements and quantification over concepts for quantification over collections, which may not be easy to achieve, still there is the need for domains of quantification. There simply are collections of objects. If our symbolism cannot deal with them, so the worse for our symbolism. Most fundamentally our universe of discourse is a collection. Quantification essentially considers items collected to state something about *their* entirety or some *out of* them.[26] Once we acknowledge the domain of quantification it is nothing but natural – comprehension again – to acknowledge subcollections. There are kinds of entities. And kinds and collections are not nothings, so we should be able to talk about them, if not about *all of them*, then at least those we have concepts for. We arrive full circle at (NC2). With this we also justify second-order quantification, even if not full Second Order Logic.[27]

§9 Believing (BLV) as axiom?

Frege himself famously reflects on the status of his axioms in the preface of GGA, and admits that the only axiom one may doubt to be a logical truth is (BLV), which he himself, however, believes to be a logical truth. The neo-Fregean debate around FA examined whether we

25 Cf. Parsons, Charles. „Some Remarks on Frege's Conception of Extension", in: Schirn, Matthias (Ed). *Studien zu Frege*. Stuttgart, 1976, Vol. 1, pp.265.77. A recent position resembling this attitude one finds in: Philip Hugly and Charles Sayward. *Arithmetic and Ontology*. A Non-Realist Philosophy of Arithmetic. Amsterdam/New York, 2006.

26 Plural quantification, if feasible, may reduce second-order quantification, but its semantics still presupposes subsets of the domain as collecting the entities of a kind.

27 Using full SOL (in the usual sense of being contrasted to SOL with a Henkin-style semantics, which reduces to many-sorted FOL [cf. Shapiro, pp.70-76, 88-95]) we would employ a formal system which validates inferences (like the ω-rule) which we as finite beings cannot draw. Thus we end up either as essentially logically incomplete (as our derivational system only captures a subsystem of the valid inferences of the logic in questions) or we have to possess a non-computational faculty of ω-rule like reasoning.

should take some version of (HP) as analytic.[28] Is (BLV) analytic? This seems dubious as it engenders the contradiction (given standard logic as background logic). The transcendental argument above [§7] elucidates its intuitive appeal. What can it mean to believe it as an axiom? Frege himself stresses in the outlines of his general approach that all our confidence in our results has to rest in the correctness of the rules applied to logical truths as axioms.

Let us call logical, semantic and mathematical axioms which are not conceptual truths 'structural truths' (for lack of a better unifying term). 'Conceptual truths' are truths going back to the definitions of the concepts involved or going back to the meaning of undefined basic concepts.

Our cognition ('mind/brain') comes equipped with a conceptual framework; for well-known reasons the mind cannot be a blank slate.[29] If within that framework some concepts have definitions we know *a priori* some conceptual truths, and we may re-capture them in our formal systems. There is no problem with being certain about these conceptual truths as they repeat in object language terms some stipulations about the use of a term. But definitions, of the stipulative type (of the type considered appropriate by Frege in GGA) if non-circular depend on undefined basic concepts.[30]

That some concept is undefined means we can give no non-circular definitions of it. It does not mean that we cannot elucidate in some fashion how we employ that concept, and in which sentential contexts it is used. With respect to ourselves we may not need such elucidations at all – we just use the concept. In case of built-in concepts we come equipped with them as working cognitive tools. Only when reflecting on our concepts we might need to elucidate what to do with them. An example are basic propositional connectives. In stating their truth conditions we merely elucidate their meanings, because in stating the truth conditions we

28 Cf. Heck, pp. 156-79.

29 Cf. Chomsky, Noam. *Knowledge of Language*. New York, 1986.

30 Circular definitions may be acceptable in some form of semantic holism, the circularity not being vicious if the circle is large enough and we do not have to follow it every time we use one of the defined concepts (employing some burden of proof routine to stop following definitional links). Given such a form of semantic holism all axioms may turn out to conceptual truths. As semantic holism of this type does not enjoy massive support – to say the least – we stick to Frege's picture of definition, and suppose that there are undefined structural concepts.

already presuppose these meanings. Stating a rule or a truth condition for a conditional, for example, presupposes being able to understand conditional reasoning. Given these basic meanings the axioms of propositional logic are conceptual truths. The same might be said about quantificational rules. So there are conceptual truths at the foundations of logic, but already definite descriptions provide a non clear cut case, as witnessed by quite different approaches to them (in standard or, say, in Free Logics).[31]

Nothing forbids or excludes that our cognitive framework contains principles involving undefined concepts in non-definitional relations (i.e. containing axioms, traditionally labelled 'synthetic *a priori* truths'). If φ is such a statement we will be certain of it, as it is part of our cognitive equipment. Synthetic *a priori* truths are not true by meaning (as not being analytic). Their preferential status in comparison to empirical synthetic truths resides in their foundtional character: They set the frame for empirical theories. In semantic terms we have to say they are true in all models (all possible worlds) relative to our cognitive framework.

When considering a candidate *a priori* statement, how can we distinguish between conceptual truths and synthetic axioms or structural truths then? If we can look-up definitions in a lexicon they can be separated from structural truths. In case of a re-construction of our conceptual framework we may declare something as a definition to capture our mental lexicon. Using definitions we have to ask whether they come with a specific epistemic quality to them. That quality might be a combination of (a) our certainty in them and (b) our knowledge of being able (in principle) to expand our definitional system, set the conceptual truth thus apart from the axioms. Once we know in general that there are definitions we have formed or just possess the concept of our definitional capacities. Conceptual truths therefore carry a mode of certainty that relies on our implicit access to our mental lexicon and the corresponding differentiation between defined and undefined concepts.

As there can be different modes of certainty we may well be able to distinguish structural truths from conceptual truths. Structural truths then are built-in principles we consider to be certain and not to be conceptually true. We are sure of them because they come to us as built

31 Frege's definition of definite descriptions depends on the prior introduction of value ranges.

into our cognitive framework. Once we acknowledge this thesis of built-in principles nothing depends anymore on claiming mathematics to be analytic. The logicist's claims concerning the analytic character of mathematics were important only given the prior rejection of the synthetic *a priori* truths.[32]

So, whether (BLV) and (NC) are analytic or synthetic does not decide about their foundational character. The intuitive appeal that comes with them may point to the fact that they are part of our cognitive framework.

If axioms are synthetic they cannot be re-constructed by conceptual analysis, there has to be more to the methods of philosophy. We may recognize an axiom by coming to see its role as an axiom of an area in question, founding the theorems there (in conjunction with definitions).[33] In that sense (BLV) and (NC) can obviously play a foundational role for set theory and mathematics, if only because their inconsistency allows deriving everything; but – to repeat – that inconsistency also depends on the background logic. (CT) in contrast (i.e. (CT) in the general case beyond finite set theory) *lacks* this intuitive support. Cantor famously remarks on his first proof of the uncountability of the reals that he sees it but cannot believe it. It took sometime for Cantorian set theory to be accepted. (CT) carries *surprising* synthetic content. This might point to its not being *a priori*. (CT) in the powerset version relates to the Powerset Axiom. The Powerset Axiom has immediate appeal, even may be a conceptual truth, and follows from (NC) anyway. The crucial form of (CT), however, requires an additional ingredient: the Axiom of Infinity. The Axiom of Infinity is surely synthetic as it claims the *existence* of at least one infinite set:

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32 That rejection stemmed from a narrow conception of the synthetic *a priori* as involving intuition. Thus the 'semantic traditon' (cf. Coffa, Albert. *The Semantic Tradition from Kant to Carnap*. To the Vienna Station. Cambridge, 1993) went off the wrong track. Frege does not explicitly reject the synthetic *a priori* in GLA, and at the end of his life toyed with the idea of a geometric (intuition involving) foundation of mathematics. One should also not reject something as 'a priori' because one confuses 'a priori' with 'non revisable', since this is misleading as well. Of course we may be forced to revise our theories about what the structural truths are. To classify something as 'a priori' means to consider it to be a framework assumption in our currently best re-construction of our cognitive framework.

33 Which formulas we take as axioms and which as theorems depends – besides the intuitive appeal a formula may possess – on general issues of methodology in developing formal systems, which we will not enter into here.

(INF) $(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \supset y \cup \{y\} \in x))$

This synthetic character already worried Russell as a supposed violation of logicism. The real conflict, therefore, may be that between Naïve Comprehension and the Axiom of Infinity (given standard FOL or SOL). *Finite* set theory supports (CT), and even given standard logic some version of Naïve Comprehension is *consistent* over the finite sets (i.e. if the comprehended sets are finite).[34] But the relevant version of Naïve Comprehension has to be (NC22'). If the domain consists only of the finite sets, then (NC2) is simply wrong, as not all comprehending collections are finite sets. If we modify (NC2) to

(NC22") $(\forall F)(\exists x)(\forall y)(\text{Finite}(y) \supset (y \in x \equiv F(y)))$

we have introduced a distinction between two types of collections within the range of one sort of quantification, thus, in effect, working with a way of comprehension that mirrors the Axiom of Separation (in this case separating subsets within the set of finite sets). (NC22') poses the problem of introducing collections which cannot be collected in the formal system, but as finite sets can be members of finite sets the mere presence of "a∈( ) does not yield the severe problems discussed above [cf. §6] as the argument has to be typed (i.e. *can* be typed to objects of the domain of the first order quantifiers).

Finitism, thus, may resolve the conflict. $ZFC^{-\infty}$ is ZFC with an Axiom of Finitude (just the negation of the Axiom of Infinity):

(FIN) $\neg(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \supset y \cup \{y\} \in x))$

$ZFC^{-\infty}$ can do what Peano Arithmetic does (e.g. proving Gödel's and Tarski's metalogical theorems).[35] Finitism raises a couple of interesting questions, but may contain enough of mathematics for all practical purposes.[36] Given finite set theory we are left still (given

34 *Russell's Paradox* proves then that the *Russell Set* is not a finite set, which is should not given the Axiom of Foundation. Note that the collection of all finite set is infinite, but not an object of the theory. Proving that every number has a successor (i.e. that for every set with some cardinality there is a set with an additional member), as required in Peano Arithmetic, proves that there is an unlimited supply of numbers, it does not prove that this supply is a set (of the theory).

35 On further details cf. Fitting, Melvin. *Incompleteness in the Land of Sets*. London, 2007. On finite set theory in general see: Mahler, Laurence. *Finite Sets*. Theory, Counting, and Applications. Columbus, 1968. Willard van Orman Quine showed in his *Set Theory and Its Logic* that one can do arithmetic with the set of natural numbers being a virtual set only.

36 Cf. Bremer, Manuel. "Varieties of Finitism", *Metaphysica*, 2007.

standard logic) with the question what the status of the domain of sets is (the domain now being Vω, in the light of the original iterative hierarchy), as that domain cannot be an objects of the theory on pains of *Russell's Paradox* and its lot.

All of this, of course, does not decide in itself that (CT) is false and (NC) true. In general evolutionary and pragmatist considerations support the thesis that built in cognitive assumptions are at least partially adaptive or associated (as saltations) with such adaptations and so fit to structures in reality. They are *a priori* only from our ontogenetic subjective perspective, but acquires phylogenetically. Our mind could come with principles which are not strictly true (i.e. true in all cases). Evolutionary and pragmatist considerations also support the idea that our cognitive framework has only to be a good as is crucial in the majority of our (cognitive) endeavours. Seen in this light a rule of thumb version of (NC) could be part of our cognitive framework, as the difficulties appear only in very specific contexts (usually of direct or indirect self-reference). Similar remarks could be made about general principles of a truth predicate and the *Liar Paradoxes*. Therefore the transcendental argument [of §7] may trace and support the embeddedness of (NC) within our cognitive framework, in this case our basic meta-linguistic knowledge of concepts. Even this, however, does not decide the conflict in favour of (BLV) and (NC) in their conflict with (CT).

Conclusion

The conflict between (BLV) and (CT) leads to some major issues in formal ontology. It raises the interesting historical question how much Frege understood about (CT) and how seriously he took Cantor’s diagonal proof. It seems that Frege did not recognize the force of (CT) and the obvious conflict to (BLV). Although the conflict commonly is settled to the disadvantage of (BLV) and in favour of (CT) and ZFC, the arguments in favour of (NC) and thus (BLV) are, nonetheless, strong as ever. One option sees us turning to finitism. (BLV), (NC) and (CT) arre compatible then. This option, however, deviates massively from standard mathematics, and certainly Frege would heap scorn on it.

The ultimate option to resolve the conflict lays in changing the logic underlying set theory or a theory of extensions. One option consists in endorsing (BLV) and (NC2) and changing the underlying logic of GGA to a paraconsistent logic. One then uses (NC2) freely and assumes a flat universe, the universal set being an object of the theory. Seen from the arguments in favour of (NC) and the difficult meta-theoretic questions concerning the set-theoretic universe V this seems very advantageous. One may very well lose (CT) however, which loss – leaving us just with one level of infinity – will be the end of cardinal arithmetic. Paraconsistent set theories and inconsistent mathematics are too recent a development to see clearly where this leads and what part of standard mathematics can be regained.[37]

Preserving both standard logic and set theory (BLV) and (NC) cannot be maintained – this, however, is just tautological. "Revision of logic", on the other hand, sounds anathema given Frege's insistence on logic being the core of reason. But 'revision of logic' misleads anyway. One can give this option a Fregean twist.

When scientists get into trouble with their theories it is theories which are revised, not reality. If your biological account of an organ, say the kidney, provides no coherent explanation of the data you cannot revise the kidney, your account of its structure and function has to adapt. The same applies to the organ brain ('mind/brain' as is sometimes said). Seen from this perspective the very phrase "revision of logic" has a misleading tone to it. Compare the case of languages: You can chose to talk German if you are able to when doing business in Germany; you can chose to speak Esperanto to impress your peers; but you cannot chose to have no natural language at all. Despite differences in approach and detail linguists agree that humans possess a language faculty, which is uniform species wide. The mind is not a blank slate. The language faculty has an initial state containing principles and parameters to be set. From this perspective (you may call it the 'cognitive science perspective' or the 'Chomskyan perspective') the same applies to logic. Humans possess – besides or as a part of – the

37 On the issue of V and paraconsistent set theory see: Bremer, Manuel. *Universality in Set Theories*. A Study in Formal Ontology. Frankfurt a.M., 2010. On inconsistent mathematics see: Mortensen, Chris. *Inconsistent Mathematics*. Dordrecht, 1995, and his more recent work.

language faculty a logic faculty or module that comes with a certain structure of principles. This structure is as it is, there is no room for 'logical pluralism' here. Frege agrees to this: For Frege the law of logic are not some conventions of some formal system, they are stated as basic laws in a (at least partially) formalized canonical representation of our (scientific) knowledge. Theories of logic share the fate of linguistic theories: they have to be revised if incoherent in face of the data. Theories of logic are revised, logic isn't. Evolution might revise logic, logicians revise logical theories.

For Frege a crucial difference rests in logic being normative. Following proper logical rules helps to infer true (or in any other way designated) statements from the other true (or in any other way designated) statements. Some rules may be more appropriate in some contexts than other rules. Thus we come to see some formal system ('a logic') to be used on some occasion and not on an other. Logic thus seems up to choice. Call this the 'logical positivist' or 'Carnapian' perspective on logic. Frege does not agree, and given our theory about our cognitive framework we cannot agree either: Choosing logic cannot be regarded as the whole truth for the simple, but fundamental, reason that in choosing a logic the mind cannot be a blank slate. Some core principles have to be operational in deciding on an applied logic. This core may be the logic faculty. Further on, normativity does not stand in conflict with explanatory theories. Compare linguistics again: Norms do not cease to be norms just because you describe their structure and give a (coherent) account of their function and what following them achieves. Thus there is room for Carnap's 'Principle of Tolerance' in choosing applied/regional logics, but behind and besides this we can study the core logic of the logic faculty. And the conflict we considered here pertains to the core of our logical faculties. Paradoxes can be considered as a *heuristic* to assess the coherence of a theory of logic, respectively its accompanying set of rules/axioms. A paradox/antinomy shows that a set of rules/axioms is not maximally coherent, has limited application. We find us in this situation with respect to (BLV) and (CT) given standard set theoretic assumptions (like infinitude), and standard FOL or SOL. Something has to give. This essay argued that it is not obviously

(BLV) and (NC) which have to be given up.

Manuel Bremer

(draft 15/06/2012)