Modeling future indeterminacy in possibility semantics

Fabrizio Cariani

University of Maryland, College Park

draft of March 21, 2024 comments welcome

Abstract

Possibility semantics offers an elegant framework for a semantic analysis of modal logic that does not recruit fully determinate entities such as possible worlds. The present papers considers the application of possibility semantics to the modeling of the indeterminacy of the future. Interesting theoretical problems arise in connection to the addition of object-language determinacy operator. We argue that adding a two-dimensional layer to possibility semantics can help solve these problems. The resulting system assigns to the two-dimensional determinacy operator a well-known logic (coinciding with the logic of universal modalities under global consequence). The paper concludes with some preliminary inroads into the question of how to distinguish two-dimensional possibility semantics from the more established branching framework.

1 Introduction

Possibility semantics offers an elegant framework for a semantic analysis of modal logic that does not recruit fully determinate entities such as possible worlds.[1] This paper

[0]Thanks to David Boylan, Rohan French, Jeff Horty, Arc Kocurek, Stephen Kuhn, John MacFarlane, Matt Mandelkern, Eric Pacuit, Masayuki Tashiro, and Alessandro Torza, as well as two anonymous referees for conversations, exchanges on early drafts of this paper. Also thanks to the audience at the Maryland Work in Progress Workshop, the Logic, Language, and Cognition (LLC) group at the University of Turin, Virlawp working group, and especially to Ginger Schultheis and Malte Willer whose remarks prompted large revisions. Special thanks to Ilaria Canavotto and Lloyd Humberstone for detailed feedback at different stages of the composition of the paper; to Paolo Santorio for many years of discussions and exchanges of ideas on this topic; and finally to Wes Holliday who came through with multiple suggestions that unlocked significant improvements in the paper.

[1]The phrase "possibility semantics" was coined by Humberstone (1981). The tools undergirding the framework have longer histories, including (Fine, 1975, especially §2), Humberstone (1979), as well as deep

develops the application of possibility semantics to the modeling of the indeterminacy, or openness, of the future, and some other related forms of metaphysical indeterminacy. Possibility semantics is plausibly viewed as an alternative to more established branching-time models (Thomason 1970, 1984, 2007, Belnap *et al.* 2001, MacFarlane 2003, 2014) in which indeterminacy is grounded in the overlap of complete possibilities—sometimes referred to as "histories". The key finding is that interesting technical and conceptual problems arise in connection to the explicit modeling of indeterminacy within the object language.

As understood here, the open-future hypothesis is the claim that some future events and states are objectively, and not merely epistemically, unsettled.[2] It is not assumed here that the unsettledness of the future and quantum indeterminacy are one and the same. The recurring illustrative example will be the proposition that some specific random coin will land heads on its next toss, under the stipulation that the outcome of the coin's toss is not settled by the facts about the past and the present of the tossing apparatus. If in actuality there are no such setups, the case may be entertained as a thought-experiment.

The indeterminacy associated with the future seems unlike other kinds of indeterminacy that have attracted the attention of philosophers. For example, it seems unlike the indeterminacy that some theories associate with vagueness. For one thing, it does not appear to give rise to higher-order indeterminacy. It is generally agreed by those who think that vagueness is grounded in some kind of indeterminacy that it may itself be indeterminate whether Joe is borderline tall. By contrast, it is common to assume that, as far as the unsettledness of the future is concerned, there are no states or events whose determinacy status is itself indeterminate. It might be unsettled whether there will be a sea battle tomorrow, but it cannot be unsettled whether it's unsettled. A second marker of the indeterminacy of the future is that it is not plausibly associated with unusual effects on credence. Many different philosophers have been attracted to the view that there is something non-classical about credence in the contents of vague statements. One

roots in the algebraic logic tradition. For a contemporary and comprehensive introduction, see Holliday (2022). Possibility semantics is one of a variety of styles of theories that do not rely on worlds, but on coarser objects. In addition to possibility semantics, the general family of "pointless" theories includes various kinds of states-based semantic analyses (Aloni 2018, Willer 2018), truthmaker semantics (Fine, 2017b), as well as several varieties of situation semantics (Barwise and Perry, 1981; Kratzer, 2021). It would be desirable to have a comparative study of these frameworks highlighting the commonalities, as well as the differences, between them.

[2]There is much literature on what constitutes the (alleged) openness of the future. The present discussion leans in various ways on Thomason (1970); Belnap and Green (1994); Belnap *et al.* (2001); MacFarlane (2003, 2014); Barnes and Cameron (2009, 2011); Torre (2011); Cariani and Santorio (2018); Cariani (2021b); Todd (2022).

form of this is Field's (2000) claim that vague contents require low credence in certain instances of the law of excluded middle; another is Williams's claim that vague contents seem to require imprecise probability (2014).[3] By contrast, statements about the future appear to be paradigmatic examples for the application of theories of classical credence. In prototypical cases, it seems perfectly warranted to have a sharp credence that the coin will land heads. The fact that the indeterminacy of the future has these characteristics licenses us to theorize about this specific type of indeterminacy on its own (cf. §2.3 of Torza, forthcoming, on pluralism about indeterminacy).[4]

As a last disclaimer, exploring the indeterminacy hypothesis involves no commitment to the claim that the future is open. What we are in fact committed to is the weaker claim that the hypothesis is worth taking seriously. As Stalnaker (2019, p.197) puts it, "You don't have to sign on to this metaphysical theory (as I do not) in order to find it intelligible (as I do) and to use it as a kind of precedent for a case where the thesis of metaphysical indeterminacy may be less controversial."

We lead with a general introduction to possibility semantics for a sentential modal language (§2). The next section focuses on the representation of indeterminacy in possibility semantics (§3). The framework itself already incorporates a representation of indeterminacy in the model theory. However, contrary to the inclination of Humberstone (1981), it seems important to have ways of capturing the notion of indeterminacy in the object language. Unfortunately, it is not possible to add a determinacy operator with the right profile to the system—not at least without other interventions. The main contribution of §3 is an impossibility result to this effect. After considering some theoretical options that would repair the inconsistency by means of local interventions (§4), we consider an attractive solution to the problem, which lies in the integration of possibility semantics with a two-dimensional framework (§5). The last two sections respectively highlight some logical properties of the resulting system (§6) and explore an extension of the approach that incorporates temporal operators (§7).

The insight behind the approach proposed in §5 is owed to remarks in Fine (1975). The Cliffs notes on Fine's paper focus on the fact that it is the first application of super-

[3]The matter is highly complicated, in ways that go beyond the relatively simple demarcation point that is made in this paragraph. For a sophisticated discussion, see Bacon (2018).

[4]It is worth highlighting that that some of the formal discussion to follow is not be restricted in scope to the alleged indeterminacy of the future. It will pertain to any application of possibility semantics to concepts of indeterminacy that do not give rise to higher-order indeterminacy and are not associated with funky effects on credences. As an example, Stalnaker (1984) famously suggests that counterfactual selection results in a kind of indeterminacy, and has more recently suggested that this kind of indeterminacy might be viewed as a 'milder' version of the indeterminacy that is associated with the future (Stalnaker, 2019, p.197-ff).

valuationist techniques to vague language. However, it is also a central juncture for the logical development of semantics based on partial objects, since Fine builds up to the supervaluationist machinery by first analyzing a system in which precisifications of a vague language are viewed as partial. (NB: this account is only considered in passing in Fine 1975, and moreover Fine's theory of vagueness has significantly changed, e.g. in Fine 2017a.) The present ambition is to recast some of those insights about determinacy operators in a different theoretical context, allowing some distinct issues and theoretical choice points to come to light.[5]

2 Background on possibility semantics

The basic ideological tenet of possibility semantics is that formulas are not evaluated against worlds, but against "coarser" objects called *possibilities*. This ideology marks a deviation from the standard account of the indeterminacy of the future—which is broadly within the framework of branching time (Thomason, 1970, 1984, 2007; Belnap *et al.*, 2001). According to the branching time picture, indeterminacy is adequately captured by the overlap of multiple complete possibilities with equal claim to fit the settled facts. The exact details of the analysis here depend on deeper metaphysical commitments. For instance, someone with broadly ersatzist leanings might say that the indeterminate reality is represented by multiple, incompatible perfectly determinate representations (Barnes and Cameron, 2009, 2011; Barnes and Williams, 2011).

Possibility semantics proceeds in a different way. Instead of taking a maximally precise representation as its basic modeling object, it deploys primitive objects that are themselves incomplete. That incompleteness is naturally associated with a concept of indeterminacy: possibilities settle the truth values of some sentences of a language, while leaving others unsettled.

The present formulation of possibility semantics originates from Humberstone (1981). The language is a sentential modal language, whose signature features a non-empty countable set of modal operators. (In later sections, we will add a *determinacy* operator D, in addition to these.) Models for this language are quadruples of the form, $\langle P, \gg, \mathbf{R}, V\rangle$. Here P represents a non-empty set of possibilities; $\gg$ is a refinement relation over the possibilities. Structurally, $\gg$ is a *weak partial order* (thus, it is reflexive, transitive and

[5]The idea of using possibility semantics to model the unsettledness of the future is also explored in a preliminary way in Boylan (forthcoming). However, because Boylan is focused on a different set of problems, he ends up in a theoretical space that is not compatible with the present outlook, especially with regards to the analysis of negation.

antisymmetric). From an intuitive standpoint, $Y \gg X$ holds when everything that is settled as either true or false by X is settled in the same way by Y. In short, Y agrees on all the determinate facts that X settles. (Explicit structural assumptions are needed in order to guarantee that models satisfy this intuition, and they will be provided in short order.) **R** is a non-empty set of accessibility relations, and finally V is a partial valuation function: in this setting a valuation function inputs an atomic formula and a possibility, and, if defined, outputs either 0 or 1. When $V(\mathsf{A}, X)$ is undefined, we write $V(\mathsf{A}, X)\uparrow$, otherwise $V(\mathsf{A}, X)\downarrow$. Occasionally, when it is important to disambiguate, and a model $\mathcal{M}$ is salient in context, a subscripted "$\mathcal{M}$" will be used to indicate its coordinates. For example, "$P_{\mathcal{M}}$" refers to the set of possibilities in $\mathcal{M}$.

Models for this language are ordinarily assumed to satisfy two constraints.

Refinability. For every atomic formula A and possibility X, if $V(\mathsf{A}, X)\uparrow$, then there are Y, Z such that $Y \gg X$ and $Z \gg X$, s.t. $V(\mathsf{A}, Y) = 1$ and $V(\mathsf{A}, Z) = 0$.

Persistence. For all atomic A, if $V(A, X)\downarrow$ then for every $Y \gg X$, $V(\mathsf{A}, X) = V(\mathsf{A}, Y)$.

Persistence says that whenever atomic A is settled at X, it stays settled in the same way through X's refinements. Refinability says that whenever an atomic formula A is unsettled at a possibility X, there are Y and Z—both refinements of X—that settle A as true and false respectively. Refinability is related to, but logically distinct from, the assumption that any partial possibility might be refined all the way to a complete one (which Fine 1975 calls "Completability"). In a language with infinitely many atomic sentences, refinability might be satisfied, without completability being satisfied.

Persistence is required to give formal representation to the intuitive conception of refinement. Indeed, under persistence, it is tempting to think of refinement structures as mirroring the structure of the branching models for future contingency,[6] as illustrated by the diagram in Figure 1.

[6]See Thomason (1970); Belnap *et al.* (2001); MacFarlane (2014), Cariani (2021b, ch.2) for discussion of branching models.

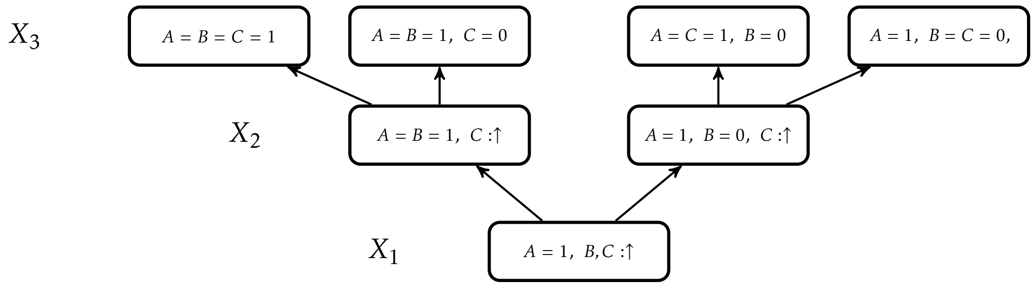

Figure 1: The branching structure of refinements ($X_3 \gg X_2 \gg X_1$)

However, an important lingering difference — which the formal theory ought to help disentangle — is that standard branching models are built on the idea of maximal *histories*, which at any moment assign a definite truth-value to all the formulas of the language. Indeed, the linear paths through the tree can naturally be viewed as temporally structured possible worlds. No such assumption of completeness is imposed on possibility models.

Another important observation is that our assumptions on possibility models do not, by themselves, rule out backwards branching. For example, the following model satisfies our stipulations.

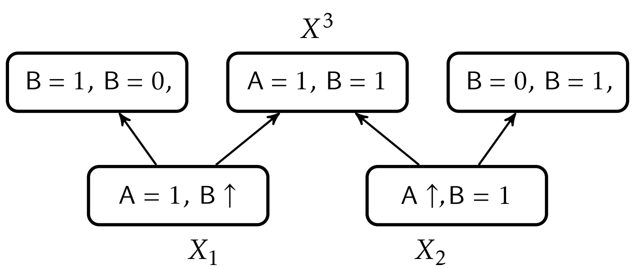

Figure 2: Backwards branching possibility model

The model in Figure 2 shows that Refinability and Persistence are not sufficient to rule out backwards branching. In this model, Refinability and Persistence are both satisfied, and yet, X_3 has two arrows going into it. If we wanted to rule out this possibility, we would need to impose a "no backwards branching" condition, similar to those that are used in defining branching models. Specifically, we would have to stipulate that whenever $X \gg Y$ and $X \gg Z$, then either $Y \gg Z$ or $Z \gg Y$. We will implicitly be restricting our attention to models that satisfy this condition, but none of our results will require imposing it.

Humberstone's semantic entries rely on the idea of using a valuation function defined on the atomic formulas of $\mathcal{L}$ to ground a notion of support between possibilities and formulas of the whole language. They are as follows:

- $\mathcal{M}, X \Vdash p$ iff $V_{\mathcal{M}}(p, X) = 1$
- $\mathcal{M}, X \Vdash \mathsf{A} \wedge \mathsf{B}$ iff $\mathcal{M}, X \Vdash \mathsf{A}$ and $\mathcal{M}, X \Vdash \mathsf{B}$
- $\mathcal{M}, X \Vdash \neg\mathsf{A}$ iff for all $Y \gg X$, $\mathcal{M}, Y \nVdash \mathsf{A}$
- $\mathcal{M}, X \Vdash \Box_i\mathsf{A}$ iff for all $Y \in P$, s.t. $R_i XY$, $\mathcal{M}, Y \Vdash \mathsf{A}$

As for other operators, such as $\vee$, $\rightarrow$, $\Diamond$, a common approach recovers entries by fixing some standard equivalences. In the case of disjunction, one might assume it characterized by conjunction, negation and DeMorgan's laws.[7] This results in the following entry:

- $\mathcal{M}, X \Vdash \mathsf{A} \vee \mathsf{B}$ iff for all $Y \gg X$, there is $Z \gg Y$, s.t. $\mathcal{M}, Z \Vdash \mathsf{A}$ or $\mathcal{M}, Z \Vdash \mathsf{B}$

Another route to the same goal would be to stipulate some general principles about what it takes for various kinds of possibilities to settle a disjunction as true/false (Holliday, 2022).

- A possibility X settles a disjunction $\mathsf{A} \vee \mathsf{B}$ as false iff it settles A as false and settles B as false.

Assume that a possibility settles A as false iff it settles $\neg\mathsf{A}$ as true. Next, note that the entries for negation and conjunction tell us that:

- A possibility X settles a conjunction $\mathsf{A} \wedge \mathsf{B}$ as true iff it settles both A and B as true.
- A possibility X settles A as false iff every refinement of X fails to settle A as true.

These assumptions are sufficient to pin down the same entry for disjunction as above. A similar analysis could be carried out for the other operators.[8]

Lastly we follow Humberstone in defining consequence as preservation of support.

[7] As an alternative, disjunction could be defined instead by the condition $\mathcal{M}, X \Vdash \mathsf{A} \vee \mathsf{B}$ iff $\mathcal{M}, X \Vdash \mathsf{A}$ or $\mathcal{M}, X \Vdash \mathsf{B}$. This would have the effect of making the logic of the sentential sub-language non-classical.

[8] While the analysis of necessity simply lifts Kripke semantics to the level of possibilities, an account of modality also involves the specification of interplay conditions connecting accessibility and refinement. Humberstone proposed:

(uR) for all X, Y, Z, if $Z \gg X$ and RZY, then RXY

(Rd) for all X, Y, Z, if $Z \gg Y$ and RXY, then RXZ

(R) for all X, Y, if RXY then $\exists X' \gg X$, for all $X'' \gg X'$, $RX''Y$

Holliday (2014, forthcoming) noted that condition (R) is overly strong. One suitable weakening is a condition that Holliday calls R-refinability (see Lemma 5.3.7 of Holliday (2022)).

(RR) for all X, Y, if RXY, then $\exists X' \gg X, \forall X'' \gg X', \exists Y' \gg Y, RX''Y'$

In addition to 'RR', the names given here to these conditions are abbreviations of Holliday's names: '(uR)' is for Holliday's 'up-R' for (uR) and '(Rd) is for Holliday's 'R-down'. These conditions will be of relevance in §7.

Definition 1 *$A_1, ..., A_n \Vdash B$ iff for all models $\mathcal{M}$ and any X in $P_{\mathcal{M}}$, if for all i, $\mathcal{M}, X \Vdash A_i$, then $\mathcal{M}, X \Vdash B$.*

It is a well established fact about this formalism that the logic of the sentential sub-language is classical, both in the sense that the set of logical truths coincides with the set of classical tautologies, and in the sense that the class of valid arguments in this sub-language coincides with the class of tautologically valid arguments (Humberstone, 1981, pp.320-321).

3 Adding object language determinacy operators

It is reasonable to claim that possibility semantics incorporates a model of indeterminacy: an atomic formula A is indeterminate at a possibility X when X leaves A undefined. Imagine a possibility X and an atomic formula, *heads*, which we may take as symbolizing the English sentence *The coin will land heads (on a specific toss that will take place tomorrow at noon)*. In a clear sense, the metatheoretic fact that $V_{\mathcal{M}}(heads, X)\uparrow$ represents the relevant indeterminacy from the perspective of the model theory. This warrants the view that indeterminacy is captured in standard possibility semantics at the metatheoretic level.

However, as the system is set up, there is no object language device to express the concept of indeterminacy. We do not have, or have not identified, an operator that can properly express things like *it is determinate that the coin landed heads on today's toss, but it is not determinate that it will land heads tomorrow*. This is unfortunate because, for various modeling purposes, it's important to have determinacy operators in the object language. For example, determinacy operators may help formulate constraints that involve the interaction of indeterminacy with other concepts. To take just one example drawn from the recent literature, Cariani (2021a) explores interactions between (in)determinacy operators and epistemic operators. In this kind of discussion, certain principles become important that can only be formulated with determinacy operators. An example is: $\neg D\mathsf{A} \rightarrow \neg K\mathsf{A}$—the principle that if A is not determinately true, then it is not known. Such principles, and the constraints they impose on models, are best analyzed from the perspective of a formalized language.[9]

Let us then introduce a determinacy operator D to the formal language—with the interpretation that its argument is determinately *true*. Thus $\neg D\mathsf{A}$ is interpreted as claiming

[9] For some additional considerations in favor of introducing object language determinacy operators, see also Barnes and Williams (2011, §5)

that the proposition expressed by A is not determinately true, while leaving it open that it might be determinately false. To express the claim that A is indeterminate, we add an indeterminacy operator I governed by the condition in Definition 2, which is standardly taken to be definitional of indeterminacy (e.g. in Fine, 1975):

Definition 2 $IA =_{df} \neg DA \wedge \neg D \neg A$

In many respects that are going to be relevant, non-determinacy (which is expressed by '$\neg D$') behaves similarly to indeterminacy. It is important however to keep in mind that in the present terminology 'indeterminacy' denotes a two-sided status, in the sense that it requires that both A and its negation fail to be determinate. By contrast, non-determinacy (the obtaining of $\neg D$) is a one-sided status: a proposition may fail to be determinate, while its negation is determinate.

The addition of determinacy operators to the language of possibility semantics should be guided by some key constraints. To start, object language indeterminacy should, in a precise sense, align with metatheoretic indeterminacy. The simplest statement of this constraint is at the level of atomic formulas:

Constraint 1 (Alignment) *For atomic* A, $\mathcal{M}, X \Vdash IA$ *iff* $V_{\mathcal{M}}(A, X) \uparrow$.

Alignment entails a second constraint: formulas expressing non-determinacy (and indeterminacy) claims must violate (a generalization of) persistence. As initially formulated, persistence applies to the atomic formulas of the language, but there is an entirely natural generalization of it involving the concept of support. A possibility X might support that it's indeterminate whether the coin will land heads, while at the same time it could be refinable into a possibility Y that settles that the coin will land heads. The exact principle that follows from this is:

Constraint 2 (Non-persistence of non-determinacy) *There is a formula* A, *and model* $\mathcal{M}$ *with possibilities* $X, Y \in P_{\mathcal{M}}$ *and* $Y \gg X$ *such that* $\mathcal{M}, X \Vdash \neg DA$ *but* $\mathcal{M}, Y \nVdash \neg DA$

Constraint 3 (Non-persistence of indeterminacy) *There is a formula* A, *and model* $\mathcal{M}$ *with possibilities* $X, Y \in P_{\mathcal{M}}$ *and* $Y \gg X$ *such that* $\mathcal{M}, X \Vdash IA$ *but* $\mathcal{M}, Y \nVdash IA$

With enough of the possibility framework on board, the route from the alignment constraint to non-persistence is relatively straightforward.

Fact 1 *Given Definition 2 and Refinability, Alignment entails (i) Non-persistence of indeterminacy and (ii) of non-determinacy.*

> *Proof.* Consider a model $\mathcal{M}$ with two possibilities X and Y drawn from its possibility set, such that $Y \gg X$. Suppose in particular that Y settles some atomic formula A that X leaves unsettled. The existence of such a Y is guaranteed by Refinability. Then $V_{\mathcal{M}}(\mathsf{A}, X) \uparrow$ but $V_{\mathcal{M}}(\mathsf{A}, Y) = 1$ or $V_{\mathcal{M}}(\mathsf{A}, Y) = 0$ and so $\mathcal{M}, X \Vdash I\mathsf{A}$ but $\mathcal{M}, Y \nVdash I\mathsf{A}$. For (ii), exploit Refinability to suppose that Y refines X so that $V_{\mathcal{M}}(\mathsf{A}, Y) = 1$. Definition 2 yields $\mathcal{M}, X \Vdash \neg D\mathsf{A}$, but from the way Y refines X it follows that $\mathcal{M}, Y \Vdash D\mathsf{A}$. □

Alignment provides powerful motivation for Non-persistence. It is nonetheless valuable to keep the claims separate, because Non-persistence is weaker and might be motivated in other ways. Another reason to keep these separate is that there are versions of possibility semantics that drop Refinability (see e.g. the development in Holliday and Mandelkern ms.).

While these constraints seem plausible, important difficulties are lurking under the surface. The just-added ingredients are inconsistent with the framework. In particular, there is tension between the analysis of indeterminacy in Definition 2, the Non-persistence of indeterminacy and the analysis of negation.

Fact 2 *The following are inconsistent (given the framework):*

IN. $IA \equiv_{df} \neg DA \wedge \neg D\neg A$.

NP. *There are* $\mathcal{M}$, A, X, $Y \gg X$ *with* $\mathcal{M}, X \Vdash IA$ *but* $\mathcal{M}, Y \nVdash IA$.

NE. $\mathcal{M}, X \Vdash \neg A$ *iff for all* Y *such that* $Y \gg X$, $\mathcal{M}, Y \nVdash A$

> *Proof.* Consider witnesses, $\mathcal{M}, X, Y, \mathsf{A}$ for NP. By IN, $\mathcal{M}, X \Vdash \neg D\mathsf{A} \wedge \neg D\neg\mathsf{A}$. By the clause for conjunction, $\mathcal{M}, X \Vdash \neg D\mathsf{A}$ and $\mathcal{M}, X \Vdash \neg D\neg\mathsf{A}$. By the clause for negation (NE), $D\mathsf{A}$ and $D\neg\mathsf{A}$ cannot be supported throughout any refinements of X. That is, for all $Z \gg X$, $\mathcal{M}, Z \nVdash D\mathsf{A}$ and $\mathcal{M}, Z \nVdash D\neg\mathsf{A}$. However, since any refinement of Y is a refinement of X, we must also have $\mathcal{M}, Y \Vdash \neg D\mathsf{A}$ and $\mathcal{M}, Y \Vdash \neg D\neg\mathsf{A}$, and hence, by IN, $\mathcal{M}, Y \Vdash I\mathsf{A}$. This contradicts the fact that $\mathcal{M}, \mathsf{A}, X, Y$ were chosen as witnesses for the existential in NP. □

A plausible initial diagnosis is that the problem arises because the negation operator forces persistence. That is to say, the system guarantees that $\neg \mathsf{A}$ must always be persistent, whether A is persistent or not. A consequence of this fact is that the indeterminacy operator I cannot be both defined in terms of negation and also such that formulas like $I\mathsf{A}$ are non-persistent.

This inconsistency is related to a less specific unease with object language indeterminacy operators that is already expressed by Humberstone (1981). Humberstone claims that an indeterminacy operator like the one just introduced would go "against the spirit of the present enterprise, since it would give rise to formulas which were not persistent into refinement [...], and thus undermines the idea of refinements as mere resolvers of indeterminacy". Humberstone's exact concern is hard to pin down, and certainly broader than the inconsistency articulated in Fact 2. (He uses this kind of argument to press against other non-persistent operators, including ones that do not give rise to inconsistencies like the one just identified.) But whatever we may think of the broad concern, the inconsistency does show that adding (in)determinacy operators is not entirely innocent.

4 A preliminary journey around the options

Is there a path for integrating possibility semantics with object language determinacy operators? Evidently, any such path requires giving up one of IN, NP, or NE. In other words, it requires either altering the definition of indeterminacy, or giving up non-persistence or modifying the analysis of negation. The option of giving up IN is a non-starter and may be set aside immediately. The problem is not merely that the definition of indeterminacy captured by Definition 2 is relatively well entrenched, which it is. The real issue is that a version of the inconsistency in Fact 2 arises directly for $\neg D\mathsf{A}$, independently of how $I\mathsf{A}$ is defined.

By contrast, it seems more promising to pursue some version of the second option, and so to deny the non-persistence constraint. One might support a plea for persistence by thinking in terms of temporally indexed indeterminacy operators. [10] To illustrate the essence of the approach, start by noting that, in the relevant applications, there is a connection between refinement and temporality. Specifically, advancing through time along a history should result in encountering more and more refined possibilities. Under this temporal interpretation, it might seem attractive to entertain determinacy operators

[10] I owe this suggestion to Masayuki Tashiro.

that are relativized to a specific point in time. Under this approach, the object language would feature a collection of operators $\{D_t \mid t \in T\}$, where T is a designated set of times in the model.[11] Simplifying somewhat, imagine that the set of times that are distinguished in a given possibility model is finite. Then consider operators $D_0, D_1, D_2, ..., D_n$, each marking what is determinate at a certain time in the development of history, with each D_i anchored to some specific time t_i. To complete the proposal say that the language does not contain any unrelativized determinacy operators, and thus that all determinacy discourse is captured by means of relativized ones.

This model's way out of the inconsistency is to undermine some of the motivation for non-persistence. Suppose again that X_{Mon} represents Monday's possibility, in which the coin has not yet landed heads, and X_{Wed} represents the state of affairs on Wednesday, after the coin has been tossed and has landed heads. In the original approach, with unrelativized determinacy operators, one should approach this by saying that $\neg D(heads)$ is supported at X_{Mon} but unsupported at X_{Wed}. By contrast, the relativized framework opens up a different option: X_{Mon} supports $\neg D_{Mon}(heads)$, while X_{Wed} supports $D_{Wed}(heads)$. Crucially, the formulas $\neg D_{Mon}$ and $\neg D_{Wed}$ can be assumed to be persistent (even when the operator is embedded under negation). The intuitive meaning of $D_{Mon}\mathsf{A}$ would be something like "A is/was settled true on Monday". From Wednesday's point of view—i.e., as far as X_{Wed} is concerned—$\neg D_{Mon}\mathsf{A}$ remains supported. Relatedly, the claim $\neg D_{Monday}(heads) \wedge D_{Wednesday}(heads)$ is perfectly consistent (from any point in time).[12]

This approach is valuable, and the solution offered in this paper incorporates some of the insight that motivates it. However, it also seems unsatisfactory in some respects: it is not especially controversial to claim that people possess an unrelativized concept of indeterminacy — plausibly one that can be captured at the level of the theory by an operator that satisfies the alignment constraint. There is no special reason to think that there are barriers to expressing *that* concept in the object language. It is at the very least worth asking whether such a concept is definable.

Before moving to the positive proposal, let us entertain one last option. The initial hunch concerning the incompatibility in Fact 2 was that it is due to the persistence-forcing

[11] It requires a bit of manipulation to endow standard model of branching time with times. In particular, what is required is a simultaneity relation that connects points on different branches. See chapter 2 of Cariani (2021b) for discussion.

[12] A notational variant of this approach maintains that we can have a single concept of indeterminacy that is *relational*, so that the canonical logical form for determinacy claims is $D(Monday, \mathsf{A})$. From our perspective, this approach is not substantially different from the indexed operator approach, and the critique to be made below applies to both.

effect of negation. The obvious alternative would be to introduce a type of negation that does not force persistence.[13] To this end, introduce '$\sim$' as the connective characterized by the clause: $\mathcal{M}, X \Vdash {\sim}\mathsf{A}$ iff $\mathcal{M}, X \nVdash \mathsf{A}$. This alternate negation operator does not have the effect of transforming a non-persistent claim into a persistent one. Indeed, it would make correct predictions for non-determinacy claims in the proof of Fact 2.

An evident problem with this approach is that '$\sim$' cannot be the correct negation operator for the entire language. Outside of determinacy claims, '$\sim$' conflates non-support with rejection. It is undesirable for *it's not the case that the coin will land heads* to be supported by a possibility that merely fails to settle heads. More generally, '$\sim$' is not the correct negation operator for the sentential sub-language of the language. In response, one might consider a language in which the two negation operators, '$\neg$' and '$\sim$', coexist. Footnote 15 of Humberstone (1981) identifies a minor expressive advantage to having both operators (though Humberstone does not endorse the suggestion currently under consideration): their combination, '$\neg \sim$', is a plausible candidate for a determinacy operator, as it expresses universal quantification over all refinements. (So $\mathcal{M}, X \Vdash \neg \sim \mathsf{A}$ iff all refinements of X support A). However, for the present application, having both operators around is not well-motivated. There is no principled reason for why one negation operator ($\neg$) should apply in the D-free sub-language, while the other operator should apply to formulas involving D. Additionally, any attempt at formulating a generalization concerning which operator is appropriate for a given formula would have to deal with the thorny problem of choosing the correct negation for mixed formulas (like the negation of $\mathsf{A} \wedge D\mathsf{A}$), Ultimately, it is unprincipled to have two negation operators floating around without a systematic account of their distinct roles.

5 Introducing two-dimensional possibility semantics.

This section presents a two-dimensional version of possibility semantics that is capable of addressing the inconsistency.[14] Before presenting it, it will be valuable to collect the desiderata we identified along the way. What is needed is a version of possibility semantics

[13] Humberstone (1979) considers this alternative negation for a similar application. This is also the negation that Boylan (forthcoming) uses in his application of possibility semantics to the future.

[14] For some general surveys on canonical applications of two-dimensional semantics see Humberstone (2004); Kuhn (2013); Schroeter (2021). The suggestion of a two-dimensional treatment of the determinacy operator is first explored in Fine (1975). Fine rightfully questions the ability of such an operator to handle higher-order indeterminacy, but of course this concern is not salient in the present application. The present claim is not that a two-dimensional semantics is anything new, but that it provides an elegant solution to an otherwise extremely thorny puzzle. A slight variation of a two-dimensional determinacy operator is also introduced in Burgess and Humberstone (1987, §6.2).

that incorporates a non-persistent, non-relativized determinacy operator that is "aligned" with the metatheoretic concept of indeterminacy that is ordinarily built into the possibility semantics framework. The logic is to be classical within the sentential sub-language, and the D operator must not trivialize. As a specific litmus test, $\mathsf{A} \vee \neg\mathsf{A}$ is to be valid (because the logic is classical) while $D\mathsf{A} \vee D\neg\mathsf{A}$ is not. Finally, the system must avoid conflating failure to support with rejection.

The opening move in crafting such a framework is to distinguish two dimensions of evaluation. In addition to evaluating at a pair consisting of a model and a possibility, consider evaluating at a triple $\mathcal{M}, X, Y$ consisting of the model and *two* possibilities. Doubling the evaluation possibility allows it to play two separate roles: one coordinate of evaluation is operated on by connectives (call this the 'primary possibility'), while the other is read by the determinacy operator D and left untouched by the connectives (call this the 'secondary possibility'). On the basis of the two-dimensional semantics, we can produce a unidimensional entry according to a standard diagonal principle:

Diagonal principle: $\mathcal{M}, X \Vdash \mathsf{A}$ iff $\mathcal{M}, X, X \Vdash \mathsf{A}$

The conceptual motivation for continuing to value unidimensional evaluation is that we continue to focus on a concept of truth, or support, at a possibility as the ultimate target of the theory. Moreover, thanks to the diagonal principle, the two-dimensional system can inherit the definition of consequence as preservation of support at a model.

Recursive clauses for the connectives and for the determinacy operator are specified at the level of two-dimensional evaluation. Note that the new secondary possibility is largely idle, except for contributing to the interpretation of the determinacy operator.

(i) $\mathcal{M}, X, Z \Vdash p$ iff $V_{\mathcal{M}}(p, X) = 1$

(ii) $\mathcal{M}, X, Z \Vdash \mathsf{A} \wedge \mathsf{B}$ iff $\mathcal{M}, X, Z \Vdash \mathsf{A}$ and $\mathcal{M}, X, Z \Vdash \mathsf{B}$

(iii) $\mathcal{M}, X, Z \Vdash \neg\mathsf{A}$ iff for all $Y \gg X$, $\mathcal{M}, Y, Z \nVdash \mathsf{A}$

(iv) $\mathcal{M}, X, Z \Vdash \Box_i\mathsf{A}$ iff for all $Y \in P$, s.t. $R_i XY$, $\mathcal{M}, Y, Z \Vdash \mathsf{A}$

(v) for $\vee$, $\rightarrow$, $\Diamond$, use standard equivalences to infer clauses.

(vi) $\mathcal{M}, X, Z \Vdash D\mathsf{A}$ iff $\mathcal{M}, Z, Z \Vdash \mathsf{A}$

It is notable that, under this analysis, the determinacy operator resembles an actuality operator in more standard applications of two-dimensional semantics. It evaluates the argument of DA after setting the primary evaluation possibility so as to match the secondary one.

Logical consequence remains defined as preservation of unidimensional support, as per Definition 1. Furthermore, Refinability and Persistence, understood as constraints on atomic formulas, continue to be in place. While they have generalizations for the full language, the status of those generalizations is not settled by the status of their atomic variants. Thus, saying that the complex formula IA is non-Persistent is fully compatible with saying that atomic formulas persist through refinements.

6 Victory lap

This section has two objectives: the broad objective is to illustrate that the system fulfills the main desiderata for adding an object language determinacy operator. More narrowly, once those general desiderata are established, it aims to illustrate that the system incorporates a way out of the central incompatibility identified in Fact 2. A key intermediate step in establishing these objective is the characterization of the logic of the system, identified below as Theorem 1.

As noted, the persistence constraint has a natural generalization concerning arbitrary formulas and involving the notion of support.

Definition 3 *(i) An arbitrary formula A is g-persistent in $\mathcal{M}$ iff for all $X, Y \in P_{\mathcal{M}}$ with $Y \gg X$, $\mathcal{M}, X \Vdash A$ implies $\mathcal{M}, Y \Vdash A$; (ii) A is g-persistent iff for all $\mathcal{M}$, A is g-persistent in $\mathcal{M}$.*

Our previous Constraints 2 and 3 become the claim that $\neg D$A and IA are not persistent in this generalized sense.

Fact 3 (Non-persistence of non-determinacy and indeterminacy) *Let A be an atomic formula. Then $\neg DA$ and IA are not g-persistent.*

> *Proof.* Let A be an atomic formula. We want to identify a model $\mathcal{M}$ in which $\neg D$A and IA are not g-persistent. Consider a "minimal fork" model with three possibilities X_1, X_2 and X_3 with $X_2, X_3 \gg X_1$ and such that $V(\mathsf{A}, X_1)\uparrow$, $V(\mathsf{A}, X_2) = 1$, $V(\mathsf{A}, X_3) = 0$. (See Figure 3.) In the model, $\mathcal{M}, X_1 \Vdash \neg D\mathsf{A}$, but $\mathcal{M}, X_2 \nVdash \neg D\mathsf{A}$ (since $\mathcal{M}, X_2 \Vdash D\mathsf{A}$).[15]

[15] The support conditions for $\neg D$A, when A is atomic are as follows.

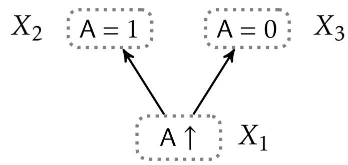

Figure 3: The Minimal Fork Model

The same model also illustrates the non-persistence of IA. □

Violations of g-persistence are limited to the fragment of the language that includes determinacy operators. It is easy to establish by induction that formulas in the D-free fragment are g-persistent.

We can also make quick work of establishing that the present system satisfies the alignment constraint (i.e., Constraint 1).

Fact 4 (Alignment) *$\mathcal{M}, X \Vdash IA$ iff $\mathcal{M}, X \nVdash A$ and $\mathcal{M}, X \nVdash \neg A$*

- $\mathcal{M}, X \Vdash I\mathsf{A} \Leftrightarrow \mathcal{M}, X \Vdash \neg D\mathsf{A} \wedge \neg D\neg\mathsf{A} \Leftrightarrow \mathcal{M}, X, X \Vdash \neg D\mathsf{A} \wedge \neg D\neg\mathsf{A} \Leftrightarrow$
- $\mathcal{M}, X, X \Vdash \neg D\mathsf{A}$ and $\mathcal{M}, X, X \Vdash \neg D\neg\mathsf{A} \Leftrightarrow$
- $\mathcal{M}, X, X \nVdash \mathsf{A}$ and $\mathcal{M}, X, X \nVdash \neg\mathsf{A} \Leftrightarrow$
- $\mathcal{M}, X \nVdash \mathsf{A}$ and $\mathcal{M}, X \nVdash \neg\mathsf{A}$ □

There is no way of setting the accessibility relation R to define a modal operator on the primary evaluation coordinate that is equivalent to the determinacy operator. To see this, note that any modal that operates on the primary evaluation coordinate would collapse the two-dimensional framework into the one-dimensional one. We know from Fact 3 that the two systems do not collapse.

There is, however, an important relationship between the two-dimensional determinacy operator in a language with no other modals, and certain ordinary modals as evaluated in some designated submodels. Given a model $\mathcal{M}$ and possibility X, let $\mathcal{M}_X$ be the submodel of $\mathcal{M}$ that is generated by X. This is the model $\langle P_X, \gg', \mathbf{R}', V' \rangle$ where P_X is the closure of $\{X\}$ under $\gg$ and any accessibility relation in $\mathcal{M}$, and all the other elements of the model are restrictions of the remaining relations and functions in $\mathcal{M}$ to this set. Let

---

- $\mathcal{M}, X \Vdash \neg D\mathsf{A} \Leftrightarrow \mathcal{M}, X, X \Vdash \neg D\mathsf{A} \Leftrightarrow \forall W \gg X : M, W, X \nVdash D\mathsf{A} \Leftrightarrow \forall W \gg X : \mathcal{M}, X, X \nVdash \mathsf{A} \Leftrightarrow \mathcal{M}, X, X \nVdash \mathsf{A} \Leftrightarrow V(\mathsf{A}, X) \neq 1$.

It is easy to check that $\mathcal{M}, X_1 \neg D\mathsf{A}$ holds but $\mathcal{M}, X_2 \Vdash \neg D\mathsf{A}$ does not.

$\Vdash_U$ be the support relation generated by interpreting formulas of our formal language according to the unidimensional rules, while interpreting D as the universal modality in $\mathcal{M}_X$ (i.e. by assuming YRZ for any Y and Z in $P_{\mathcal{M}_X}$.).

Fact 5 *For any refinement Y of X, $\mathcal{M}, Y, X \Vdash \mathsf{B}$ iff $\mathcal{M}_X, Y \Vdash_U \mathsf{B}$.*

> *Proof:* Reason by induction on the complexity of B. If B is atomic, the claim holds because the models have agreeing valuation functions. If B is a negated formula $\neg\mathsf{A}$, and Y is an arbitrary element of P_X, for all $Z \geq Y$, $\mathcal{M}, Z, X \nVdash \mathsf{A}$ iff $\mathcal{M}_X, Z \nVdash_U \mathsf{A}$, but since Z is a refinement of Y, it is also a refinement of X, so this follows from the induction hypothesis. If B is a modal formula $\Box_i\mathsf{A}$, and Y is a refinement of X, $\mathcal{M}, Y, X \Vdash \Box_i\mathsf{A}$ iff for all Y' with YR_iY', $\mathcal{M}, Y', X \Vdash \mathsf{A}$ iff (by induction hypothesis and definition of $\mathcal{M}_X$) for all Y' with $YR_i'Y'$, $\mathcal{M}_X, Y' \Vdash_U \mathsf{A}$ iff $\mathcal{M}_X, Y \Vdash_U \Box_i\mathsf{A}$. Setting aside the trivial case in which B is a conjunction, the remaining case of interest is where $\mathsf{B} = D\mathsf{A}$ for A satisfying the induction hypothesis. Consider Y refining X: then $\mathcal{M}, Y, X \Vdash D\mathsf{A}$ iff $\mathcal{M}, X, X \Vdash \mathsf{A}$ iff $\mathcal{M}_X, X \Vdash_U \mathsf{A}$ iff $\mathcal{M}_Y, X \Vdash_U D\mathsf{A}$.

Fact 5 is the key to characterizing the logic of our determinacy operator, at least in the special case in which the language does not contain other modal operators. Let $\models_{S5}$ denote the S5 consequence relation, and $\models^g_{S5}$ the global consequence relation as characterized on Kripke models (Blackburn *et al.*, 2001, §1.3). The consequence relation on two-dimensional possibility models coincides with the global consequence relation on universal Kripke models.[16]

Theorem 1 *If the original language does not contain modals other than D, $\mathsf{A}_1, \ldots, \mathsf{A}_n \Vdash \mathsf{C}$ iff $D\mathsf{A}_1, \ldots, D\mathsf{A}_n \models_{S5} D\mathsf{C}$ iff $\mathsf{A}_1, \ldots, \mathsf{A}_n \models^g_{S5} \mathsf{C}$*

> *Proof.* We exploit the fact that universal modalities in Kripke frameworks have the same logic (i.e. S5) as universal modalities in the possibility framework. So, let $\Vdash_U$ be the logic of a possibility framework for a language with a single modal D with R_D as the universal relation. So:
>
> $$D\mathsf{A}_1, \ldots, D\mathsf{A}_n \Vdash_U D\mathsf{C} \text{ iff } D\mathsf{A}_1, \ldots, D\mathsf{A}_n \models_{S5} D\mathsf{C}$$

[16] See the appendix of Schulz (2010) for a similar result involving the logic of Yalcin's (2007) semantics for epistemic necessity—albeit one that is presented wholly at the level of worlds-based semantics.

What is left to prove is:

$$\mathsf{A}_1, ..., \mathsf{A}_n \Vdash \mathsf{C} \text{ iff } D\mathsf{A}_1, ..., D\mathsf{A}_n \Vdash_U D\mathsf{C}$$

This is proven by identifying a chain of equivalences between the claim that an arbirary argument has a countermodel in the two-dimensional framework, and the claim that it has a countermodel in the unidimensional framework with D as universal modality. Let $\Gamma = \{\mathsf{A}_1, ..., \mathsf{A}_n\}$, and $D\Gamma = \{D\mathsf{A}_1, ..., D\mathsf{A}_n\}$.

(i) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}}$, s.t. $\mathcal{M}, X \Vdash \Gamma$, but $\mathcal{M}, X \nVdash \mathsf{C}$.

(ii) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}}$, s.t. $\mathcal{M}, X, X \Vdash \Gamma$, but $\mathcal{M}, X, X \nVdash \mathsf{C}$.

(iii) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}}$, s.t. $\forall Z \gg X$, $\mathcal{M}, Z, X \Vdash \Gamma$, but $\mathcal{M}, X, X \nVdash \mathsf{C}$.

(iv) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}_X}$, s.t. $\forall Z \gg X$, $\mathcal{M}_X, Z \Vdash_U \Gamma$, but $\mathcal{M}_X, X \nVdash_U \mathsf{C}$.

(v) $\exists \mathcal{M}$ and $X \in P_{\mathcal{M}_X}$, s.t. $\mathcal{M}_X, X \Vdash_U D\Gamma$, but $\mathcal{M}_Y, X \nVdash_U D\mathsf{C}$

The equivalence between (ii) and (iii) is due to the fact that evaluation along the primary coordinate is persistent even in the two-dimensional system. The equivalence between (iii) and (iv) relies on Fact 5 and persistence in the unidimensional framework. For the equivalence between (iv) and (v), note that if $D\mathsf{C}$ fails at X, C must fail at some possibility in the model—i.e. at some refinement of X. But if so, C must also fail at X (or else persistence would force it to hold throughout the entire model). □

Theorem 1 enables us to ascertain the satisfaction of many of our design principles. The remaining facts in this section are all presented without explicit proof, on the understanding that they are elementary corollaries of Theorem 1. First, notice that it entails that the logic in the sentential sub-language remains classical. It is also a simple corollary that the theorems of the two-dimensional theory with D as the sole modal operators are exactly the theorems of S5.

Fact 6 $\Vdash C$ *iff* $\models_{\mathrm{S5}} DC$ *iff* $\models_{\mathrm{S5}} C$

Next, we notice that there are consistent statements of indeterminacy, i.e.:

Fact 7 (Non-triviality of indeterminacy)

(i) IA is consistent (i.e. $\nVdash DA \vee D\neg A$).

(ii) $D(A \vee B) \not\Vdash DA \vee DB$

The failure of the entailment in part (ii) of Fact 7 is relevant for comparison with an alternate system involving determinacy operators and two-dimensional semantics (i.e., the one on pp. 220-221 of Burgess and Humberstone, 1987).

Per our design specifications, there is no higher-order indeterminacy in this system.

Fact 8 (No higher-order indeterminacy)

(i) $A \Vdash DA$, *and in particular,* $DA \Vdash DDA$ *and* $\neg DA \Vdash D\neg DA$.

(ii) $\not\Vdash A \equiv DA$

(iii) $\Vdash DA \equiv DDA$

(iv) $\Vdash \neg DA \equiv D\neg DA$

Note that establishing the entailments in part (i) is not the same as claiming that truth and settled truth coincide, as the observation in part (ii) highlights. And indeed there is an important difference between higher-order and first-order determinacy claims when it comes to object language collapse facts observed in parts (iii) and (iv)—as contrasted with the non-collapse in part (ii).

At the same time, Theorem 1 illustrates that the system has a familiar non-classical profile when it comes to its meta-rules. Though the extension of the consequence relation in the D-free fragment matches that of classical sentential logic, adding expressive capacity to the language in the form of the D-operator results in some non-classical behavior. One example of this behavior is that the consequence relation does not contrapose over the full language: $A \Vdash DA$ holds, as we noted in Fact 8, but $\neg DA \Vdash \neg A$ does not. This phenomenon mirrors the standard behavior of similar systems based on S5 global consequence relations. For example, it is observed in supervaluationist analyses based on the idea of "global" validity (Fine, 1975; Williamson, 1994; Varzi, 2007; Asher *et al.*, 2009; Bacon, 2018) and also in informational analyses of consequence for languages with epistemic modals, as in Yalcin (2007); Bledin (2014); Schulz (2010); Incurvati and Schlöder (2022). More specifically, Theorem 1 entails the following failures:

Fact 9

- *No Conditional proof :* $A \Vdash DA$, *but* $\not\Vdash A \to DA$

- *No Reductio:* $A \wedge \neg DA \Vdash DA$ *and* $A \wedge \neg DA \Vdash \neg DA$ *but* $\nVdash \neg(A \wedge \neg DA)$

- *No Contraposition:* $A \Vdash DA$ *but* $\neg DA \nVdash \neg A$

Disjunctive syllogism may fail too, depending on its exact characterization.[17]

To conclude this section, it is valuable to reflect on exactly how the system manages to avoid the inconsistency in Fact 2. Recall, that the inconsistency pits the definition of indeterminacy (IN), the claim that indeterminacy is non-persistent (NP) and the analysis of negation (NE) against each other. The technical fact of the matter is that the two-dimensional system avoids the inconsistency by rejecting the negation condition, NE. In particular, in the two-dimensional system, there is no guarantee that if X supports $\neg$A, then X's refinements will fail to support A. This is because unidimensional evaluation is governed by the diagonal principle, and so what's supported at X depends on evaluation triples of the form $\langle M, X, X\rangle$, whereas what's supported at Y depends on evaluation triples of the form $\langle \mathcal{M}, Y, Y\rangle$. These may come apart in ways that undermine the negation clause. Of course, the *effect* of the negation operator is preserved because there is an analogous operator at the level of two-dimensional evaluation. However, that operator only quantifies over refinements along the primary dimension.

This technical gloss is important but it does not illuminate the central mechanics behind the two-dimensional proposal. Instead, the two-dimensional system is better thought of as a more flexible generalization of the idea of indexing determinacy operators. The job of the secondary coordinate of evaluation is to anchor the facts that ground determinacy claims, shielding them from the shifting effects of other operators. The failure of the unidimensional negation clause is a downstream consequence of this intervention.

7 Remarks on adding tense operators.

The framework within which we developed the model of indeterminacy is founded on a distinctive structural assumption concerning the relationship between refinement and the open future. This is the idea that there is a correspondence between the refinement relation and the temporal precedence ordering. More specifically, we have supposed that

[17] We know immediately that the following form of disjunctive syllogism must fail:

If A $\Vdash$ C, B $\Vdash$ C, then A $\vee$ B $\Vdash$ C

If it didn't, then we would have A $\vee \neg$A $\Vdash$ DA $\vee$ $D\neg$A (contradicting Fact 7). Alternatively, it is possible to formulate disjunctive syllogism as follows:

If $\Vdash$ A $\rightarrow$ C, $\Vdash$ B $\rightarrow$ C, then $\Vdash$ (A $\vee$ B) $\rightarrow$ C

It's another consequence of Theorem 1 that this reformulated schema is correct.

a possibility in which the coin toss is indeterminate splits, after a step of refinement, into two possibilities that *immediately* follow the initial one. This assumption has both simplifying and heuristic value, but it may, with reason, be viewed with some suspicion. For example, it obfuscates how one might sensibly add temporal operators to the language. Relatedly, the assumption seems to force an eternalist understanding of propositions—the idea that propositions do not vary in their truth-value from time to time. Eternalism is not obviously mistaken, but it also is not obviously mandated by any arguments established until now. This concluding section explores the prospects for lifting this assumption, and advances formal observations about the shape of a theory of tenses that integrates with two-dimensional possibility semantics. This will serve as proof of concept that the integration is possible but we will stop short of developing the theory in full.

To start, let us follow (Holliday, 2022, §5.3, and specifically example 5.3.9) in severing the connection between refinement and temporality. We are going to add temporal operators $\langle F \rangle$, for *sometime in the future*, and $\langle P \rangle$ for *sometime in the past*, respectively governed by accessibility relations R^F and R^P. (So, for example $R^F XY$ means that Y is in the future of X, and $R^P XY$ means that Y is in the past of X) We assume R^F and R^P to be at least irreflexive, transitive, and asymmetric. For reasons that will become clear momentarily, we do not make the standard assumption that R^P is always the converse of R^F. Suppose 'p' is an atomic formula of our object language, to be interpreted as meaning that it's raining. Then the model in Figure 4 diagrams a situation in which X_1 precedes

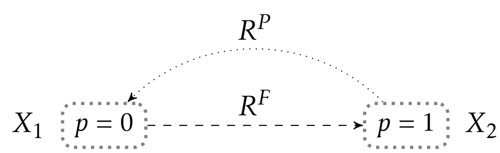

Figure 4: Simple temporal model

X_2, and such that from X_2's perspective it will be raining in the future and from X_2's perspective it was not raining in the past.

A bit of notational convention will help improve readability: introduce the notation $R^F(X)$ to denote $\{Y \mid R^F XY\}$ — the set of possibilities in the future of X (with regards to an accessibility relation, and within a given model). This will give a more natural informal gloss to some formal statements, as we can read $Y \in R^F(X)$ as "Y is in the future of X". It is worth keeping in mind that in certain open future contexts multiple possibilities might be in the future of a given base possibility: see Figure 5 below.

A simple off-the shelf idea is to apply the standard analysis of modal operators to the semantics of $\langle F\rangle$ and $\langle P\rangle$. We begin by adopting the three interplay conditions entertained in §5.3 of Holliday (2022) for each of R^F and R^P. Letting R be some arbitrary accessibility relation, these are:

(uR) for all X, Y, Z, if $Y \gg X$ and $Z \in R(Y)$, then $Z \in R(X)$

(Rd) for all X, Y, Z, if $Y \gg X$ and $X \in R(Z)$, then $Y \in R(Z)$

(RR) for all X, Y, if $Y \in R(X)$, then $\exists X' \gg X, \forall X'' \gg X', \exists Y' \gg Y, Y' \in R(X'')$

Applied to R^F, (uR) says that any possibility Z that is in the future of a refinement of X is also in the future of X; (Rd) says that any possibility Y that refines a possibility that is in the future of Z is also in the future of Z; (RR) states that for any Y in the future of X, there is a refinement X' of X every refinement of which has some refinement of Y in its future.

We treat $\langle F\rangle$ and $\langle P\rangle$ as duals of modals $[F]$ and $[P]$ with the standard semantics from §5. Thus adapting clauses (iv) and (v), we get

- $\mathcal{M}, X, Z \Vdash [F]\mathsf{A}$ iff for all $Y \in R^F(X)$, $\mathcal{M}, Y, Z \Vdash \mathsf{A}$
- $\mathcal{M}, X, Z \Vdash [P]\mathsf{A}$ iff for all $Y \in R^P(X)$, $\mathcal{M}, Y, Z \Vdash \mathsf{A}$
- $\langle F\rangle\mathsf{A} =_{def} \neg[F]\neg\mathsf{A}$
- $\langle P\rangle\mathsf{A} =_{def} \neg[P]\neg\mathsf{A}$

Under condition (Rd), the induced support conditions for $\langle F\rangle$, and $\langle P\rangle$ simplify to (F1) and (P1) below.[18]

(F1) $\mathcal{M}, X, Z \Vdash \langle F\rangle\mathsf{A}$ iff for all $Y \gg X$, $\exists K \in R^F(Y)$, $\mathcal{M}, K, Z \Vdash \mathsf{A}$

(P1) $\mathcal{M}, X, Z \Vdash \langle P\rangle\mathsf{A}$ iff for all $Y \gg X$, $\exists K \in R^P(Y)$, $\mathcal{M}, K, Z \Vdash \mathsf{A}$

Let's call (F1) and (P1) the "unidimensional tenses". A notable — and ultimately, as we will see, problematic — feature of the unidimensional tenses is that, although we did represent their secondary evaluation coordinate, their support conditions are insensitive to it.

[18] See Lemma 5.3.8 in Holliday 2022. In the current setup: $\mathcal{M}, X, Z \Vdash \neg\Box_i\neg\mathsf{A}$ iff $\forall Y \gg X$, $\mathcal{M}, Y, Z \nVdash \Box_i\neg\mathsf{A}$ iff $\forall Y \gg X, \exists K \in R_i(Y), \mathcal{M}, K, Z \nVdash \neg\mathsf{A}$ iff $\forall Y \gg X, \exists K \in R_i(Y), \exists K' \gg K, \mathcal{M}, K', Z \Vdash \mathsf{A}$. Now, fix $Y \gg X$ and $K \in R_i(Y)$. Suppose $K' \gg K$. Then, by (Rd), $K' \in R_i(Y)$. Thus K' would be available as witness to the existential in (F1) and (P1).

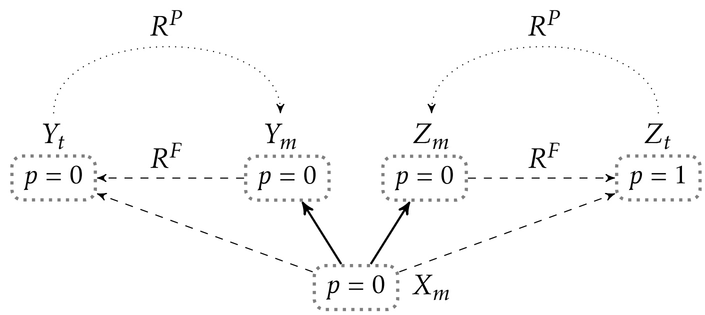

Figure 5: Holliday's model of the Sea Battle puzzle

With these tools in hand, temporality and refinement can be cleanly separated. Refinement relations track the resolution of metaphysical indeterminacy, without shifting the locus of temporal evaluation. This idea may be best illustrated with Holliday's own example of the sea battle puzzle. Suppose that there are two salient times, Monday and Tuesday and that p represents the proposition that there is a sea battle on Tuesday. Let us define a model $\mathcal{SB}$ (for "sea battle") as follows, and as diagrammed in Figure 5. Let X_m be Monday's state of affairs both with regards to categorical facts and also with regards to which facts are determinate. Now, X_m can be refined into two possibilities Y_m, and Z_m without changing the temporal perspective from which we evaluate: both Y_m and Z_m represent the world as it is on Monday. Where they differ is that Z_m has a future in which the sea battle occurs (Z_t), while Y_m has a future (Y_t) in which it does not. In light of condition (uR), both Y_t and Z_t must also be futures of X_m: after all, Z_m (viz. Y_m) refines X_m and Z_t (viz. Y_t) are in the future of Z_m (viz. Z_m).

It is a remarkable fact about this model that R^P is not simply the converse of R^F. Plausibly, Holliday's intuition is that, "looking backwards" from Z_t's perspective, the past has a new veneer of determinacy. Here is Holliday's gloss on this model, with adaptations to our notation in square bracket:

> Thus, the future is presently [i.e. at X_m] open. Yet if there is a sea battle, so [Z_t] is realized, then the past will turn out to be [Z_m], in which there would be a future sea battle, whereas if there is no sea battle, so [Y_t] is realized, then the past will turn out to be [Y_m], in which there would be no future sea battle. Come tomorrow, we might say, "the past is not what it used to be." (Holliday, 2022, §5.3)

This is an interesting assumption, which we will carry along here for the sake of exposition.

It also warrants attention in future work, so as to determine what kinds of theoretical tradeoffs it involves.

Holliday's notes that the model verifies $\mathcal{SB}, X_m \Vdash \langle F\rangle p \vee \neg\langle F\rangle p$ while $\mathcal{SB}, X_m \nVdash \langle F\rangle p$ and $\mathcal{SB}, X_m \nVdash \neg\langle F\rangle p$. Once we add our two dimensional determinacy operator, we can further verify $\mathcal{SB}, X_m \Vdash \neg D\langle F\rangle p$ and $\mathcal{SB}, X_m \Vdash \neg D\neg\langle F\rangle p$.[19]

This is indeed a minimal standard for capturing a concept of openness of the future. However, once we combine the two-dimensional picture of determinacy operators with our temporalist-friendly ideas we run into some initial difficulties. Here is one: the combined theory makes the counterintuitive prediction that determinacy statuses are permanent. Intuitively, if it's determinate today that there is a sea battle it should not follow that in the past it was determinate that there was a sea battle then.

The two-dimensional system augmented with unidimensional tenses incorrectly predicts that $D\mathsf{A}$ entails $\langle P\rangle D\mathsf{A}$ — in fact, it predicts that $D\mathsf{A}$ entails $[P]D\mathsf{A}$! This is because the secondary coordinate of evaluation is not affected in the evaluation of $\langle F\rangle$ and $\langle P\rangle$(or $[F]/[P]$). More generally:

Fact 10 (Determinacy is forever) *For any model $\mathcal{M}$, possibility $X, Z \in P_{\mathcal{M}}$ and formula A,*

(i) $\mathcal{M}, X, Z \Vdash D\mathsf{A}$ *iff* $\mathcal{M}, X, Z \Vdash [P]D\mathsf{A}$

(ii) $\mathcal{M}, X, Z \Vdash D\mathsf{A}$ *iff* $\mathcal{M}, X, Z \Vdash [F]D\mathsf{A}$

Proof: For (i), the left side reduces immediately to $M, Z, Z \Vdash \mathsf{A}$ in light of the support conditions for D. Unpacking on the right side of the biconditional: $\mathcal{M}, X, Z \Vdash [P]D\mathsf{A}$ iff for all $Y \in R^P(Y)$, $\mathcal{M}, Y, Z \Vdash D\mathsf{A}$, which also reduces to $M, Z, Z \Vdash \mathsf{A}$. For (ii), replace '[P]' with '[F]' in this argument.

Informally, because the D operator overwrites the primary coordinate of evaluation with the value of the secondary coordinate, the effect of a temporal operator with higher scope is rendered irrelevant.

There is an obvious idea to work around this problem: modify the semantic entries for the tenses by treating them as genuinely two-dimensional operators.

[19] While Holliday (2022) discusses this model within a unidimensional possibility semantics (with operators $\langle F\rangle$ and $\langle P\rangle$ also being given standard unidimensional entries), his discussion can easily be exported to the two-dimensional setting with no essential alteration. The observation concerning the determinacy operators is not in Holliday (2022), but it is also noted in slides for Holliday's NASSLLI course on possibility semantics. To verify the first of these claims: $\mathcal{SB}, X_m, X_m \Vdash \neg D\langle F\rangle p$ iff $\forall X' \gg X_m$, $\mathcal{SB}, X', X_m \nVdash D\langle F\rangle p$ iff $\mathcal{SB}, X_m, X_m \nVdash \langle F\rangle p$, which is indeed the case since it's not the case that every refinement of X_m has a future in which p holds (e.g. Y_m does not).

(F2) $\mathcal{M}, X, Z \Vdash \langle F\rangle \mathsf{A}$ iff for all $Y \gg X$, $\exists K \in R^F(Y)$, $\mathcal{M}, K, K \Vdash \mathsf{A}$

(P2) $\mathcal{M}, X, Z \Vdash \langle P\rangle \mathsf{A}$ iff for all $Y \gg X$, $\exists K \in R^P(Y)$, $\mathcal{M}, K, K \Vdash \mathsf{A}$

Effectively, the analysis makes $\langle F\rangle$ and $\langle P\rangle$ into complex determinacy operators in their own right. We should expect some new instances of the metarule failures we noted in Fact 9. But, in exchange, we should get a much better behaved picture of the interaction between determinacy and tense operators. Crucially for our purposes, the two-dimensional analysis of tense operators fixes the bug that plagues the unidimensional analysis: the analogue of Fact 10 fails. The model in Figure 5 already illustrates this. Specifically: $\mathcal{SB}, Z_t, Z_t \Vdash Dp$, but $\mathcal{SB}, Z_m, Z_m \Vdash \neg Dp$, and yet $\{Z_m\} = R^P(Z_t)$, and so $\mathcal{SB}, Z_t, Z_t, \Vdash \langle P\rangle \neg Dp$. Happily, determinacy is not forever.

Note that after this modification, the semantics can still meet the minimal benchmarks for the analysis of Holliday's model of the Sea Battle puzzle. In particular, in the Sea Battle model of Figure 5, we continue to have $\mathcal{SB}, X_m \Vdash \langle F\rangle p \vee \langle F\rangle \neg p$, $\mathcal{SB}, X_m \nVdash \langle F\rangle p$, $\mathcal{SB}, X_m \nVdash \langle F\rangle \neg p$. Furthermore, by the same reasoning as in Footnote 19, the two-dimensional tense operators also deliver $\mathcal{SB}, X_m \Vdash \neg D\langle F\rangle p$ and $\mathcal{SB}, X_m \Vdash \neg D\langle F\rangle \neg p$.

The goal of the present section was to highlight a path for relaxing the collapse of refinement and temporal precedence. We have made progress in this direction, but whether the integration pursued here is successful requires substantial additional work. This work should concern how much of open future lore can be retrieved within the theory under development. For example, an open future theorist might want to validate $D\langle P\rangle \mathsf{A} \equiv \langle P\rangle D\mathsf{A}$ without validating $D\langle F\rangle \mathsf{A} \equiv \langle F\rangle D\mathsf{A}$. The rationale might be as follows. For the first equivalence, because the past is settled, some might intuit an equivalence between it being determinately the case that a proposition was settled in the past, and it being the case in the past that a proposition was determinately settled. Open future theorists, however, maintain that the future behaves differently. Indeed, the following line of thought seems consistent:

> Given how things are right now, on Monday, it is objectively possible that, in the future, it will be determinate that there will be a sea battle but also that it's not determinate that in the future there will be a sea battle.

Suppose that the relevant sense of objective possibility is modeled by an operator $\Diamond$ that is analyzed as the dual of D ($\Diamond \mathsf{A} = \neg D \neg \mathsf{A}$). The theoretical end-point of this line of thought is that, in one sense of 'consistent', the formula $\Diamond(\langle F\rangle Dp \wedge \neg D\langle F\rangle p)$ should be consistent. Indeed, it is consistent in branching time semantics.

The question I would like to end on is whether the formula is consistent in our setting. As is typical of two-dimensional systems, we can track two senses of consistency (illustrated here for the specific case of a single formula):

> **Diagonal consistency.** The formula A is diagonally consistent iff there is a model $\mathcal{M}$ and a possibility X within it such that $\mathcal{M}, X \Vdash \mathsf{A}$ (i.e. $\mathcal{M}, X, X \Vdash \mathsf{A}$).
>
> **Weak consistency.** The formula A is weakly consistent iff there is a model $\mathcal{M}$ and a pair of possibilities X and Y within it such that $\mathcal{M}, X, Y \Vdash \mathsf{A}$.

Any diagonally consistent formula is evidently weakly consistent, but the converse is not true (a counterexample is discussed in Footnote 20).

When it comes to the consistency of $\Diamond(\langle F\rangle Dp \wedge \neg D\langle F\rangle p)$ one initial worry is that unlike in branching-time the dual of D is not a diamond-like operator (because D is not a box-like operator). Nonetheless, as it happens, if $\Diamond$ is the dual of the two-dimensional D, the desired consistency claim goes through. Indeed, the formula holds at X_m in the Sea Battle model of Figure 5:[20]

$$\mathcal{SB}, X_m, X_m \Vdash \Diamond[\langle F\rangle Dp \wedge \neg D\langle F\rangle p]$$

To verify this claim, we need to acknowledge a quick fact about the support conditions of $\neg D \neg \mathsf{A}$.

Fact 11 *$\mathcal{M}, X, Z \Vdash \neg D\neg A$ iff $\exists K \gg Z, \mathcal{M}, K, Z \Vdash A$.*

> *Proof* $\mathcal{M}, X, Z \Vdash \neg D\neg \mathsf{A}$ iff $\forall Y \gg X, \mathcal{M}, Y, Z \nVdash D\neg \mathsf{A}$ iff $\forall Y \gg X, \mathcal{M}, Z, Z \nVdash \neg \mathsf{A}$ iff $\mathcal{M}, Z, Z \nVdash \neg \mathsf{A}$ iff $\exists K \gg Z, \mathcal{M}, K, Z \Vdash \mathsf{A}$. The first and fourth biconditionals are justified by the support conditions for negation; the second by the support conditions for D, the third by vacuous quantification.

[20] Incidentally, we can also observe that $\langle F\rangle Dp \wedge \neg D\langle F\rangle p$ is weakly consistent. In particular:

- $\mathcal{SB}, Z_m, X_m \Vdash \langle F\rangle Dp$ iff for all $Y \gg Z_m$, $\exists K \in R^F(Y)$, $\mathcal{SB}, K, K \Vdash Dp$

Since Z_m is its only refinement, the right side is made true by $\mathcal{SB}, Z_t, Z_t \Vdash Dp$. By contrast,

- $\mathcal{SB}, Z_m, X_m \Vdash D\langle F\rangle p$ iff $\mathcal{SB}, X_m, X_m \Vdash \langle F\rangle p$

But we already know this is not the case, since X_m has a refinement, namely Y_m that does not have p supported in any of its futures.

Intuitively, $\Diamond$A is the claim that the secondary possibility of evaluation does not settle $\neg$A as true.

Let us go back to check $\mathcal{SB}, X_m \Vdash \neg D \neg(\langle F\rangle Dp \wedge \neg D\langle F\rangle p)$. In light of Fact 11,

(1) $\quad \mathcal{SB}, X_m, X_m \Vdash \neg D \neg(\langle F\rangle Dp \wedge \neg D\langle F\rangle p)$ iff $\exists Z \gg X_m, \mathcal{SB}, Z, X_m \Vdash \langle F\rangle Dp \wedge \neg D\langle F\rangle p$

To establish the left side of (1), note that Z_m works as a witness for Z. That is:

(2) $\quad \mathcal{SB}, Z_m, X_m \Vdash \langle F\rangle Dp$ and $\mathcal{SB}, Z_m, X_m \Vdash \neg D\langle F\rangle p$

Focus first on the first conjunct. By (F2),

(3) $\quad \mathcal{SB}, Z_m, X_m \Vdash \langle F\rangle Dp$ iff $\forall Y \gg Z_m\ \exists K \in R^F(Y), \mathcal{SB}, K, K \Vdash Dp$

Since Z_m is its only refinement, the question is whether there is a K in the future of Z_m that supports Dp. Indeed, such a K exists — namely Z_t — since $Z_t \in R^F(Z_m)$ and $\mathcal{SB}, Z_t, Z_t \Vdash Dp$.

Let us move to the second conjunct of (2): $\mathcal{SB}, Z_m, X_m \Vdash \neg D\langle F\rangle p$. This is established by reasoning along the same lines as in Footnote 19: $\mathcal{SB}, Z_m, X_m \Vdash \neg D\langle F\rangle p$ iff for all $K \gg Z_m$, $\mathcal{SB}, K, X_m \nVdash D\langle F\rangle p$ iff $\mathcal{SB}, X_m, X_m \nVdash \langle F\rangle p$ iff $\exists Y \gg X_m, \forall K \in R^F(Y), \mathcal{SB}, K, K \nVdash p$. The last claim in the chain clearly holds by selecting Y_m as witness for the existential. Y_m refines X_m, but every possibility in its future fails to support p.

Let us take stock of the dialectical situation. We have proven that the full two-dimensional system — the system that includes two-dimensional tense operators — is better behaved than its variant without the two-dimensional tenses. Moreover, it can yield some core principles of open future lore in that it can distinguish between $\langle F\rangle D$A and $d\langle F\rangle$A. Much more would need to be said to provide a full vindication of the two-dimensional possibility semantics as a model of the open future. For example, we have not commented on the equivalence $D\langle P\rangle$A $\equiv \langle P\rangle D$A. In my view, supporting this equivalence requires important conceptual decisions, and it is not just a matter of nailing down a convenient technical fact. In particular, we can easily compute:

- $\mathcal{M}, X, Y \Vdash D\langle P\rangle$A iff for all $Z \gg Y$, $\exists K \in R^P(Z), \mathcal{M}, K, K \Vdash$ A.
- $\mathcal{M}, X, Y \Vdash \langle P\rangle D$A iff for all $Z \gg X$, $\exists K \in R^P(Z), \mathcal{M}, K, K \Vdash$ A

However, capturing the right sort of link between these claims requires having the correct model of the fixity of the past. This might include exploring generalizations of Holliday's idea concerning the past accessibility relation R^P. A fuller exploration of the prospects for this temporalist-friendly variant of two-dimensional possibility semantics lies ahead for future work.

8 Conclusion

The main conclusions are as follows: there is a clear path for the application of possibility semantics to the metaphysical hypothesis of the open future. That path must include the characterization of object language determinacy operators. Introducing such operators under something like the alignment constraint requires, on pain of inconsistency, some modifications to the original framework. A two-dimensional variant of possibility semantics is one path to relieve this theoretical pressure. In its natural interpretation, the logic of determinacy under the two-dimensional analysis is the global version of S5.

The most immediate development of this idea is feasible under a broadly eternalist conception of propositions, and under the hypothesis that the refinement relation and the (reflexive closure of the) temporal precedence relation collapse. It appears important to explore the prospects for the two-dimensional analysis in a context that does not involve these structural assumptions. Holliday (2022) has already provided key insights for how to think about facets of the open future without collapsing refinement and temporal precedence. We added to this insight that integrating these insights within the two-dimensional framework requires also thinking of tenses as diagonal operators. We also noted the dual of the two-dimensional determinacy operator manages to express an interesting concept of possibility within a possibility model.

Bibliography

Aloni, Maria (2018), 'Fc disjunction in states-based semantics', University of Amsterdam, currently available at `https://www.marialoni.org/resources/Aloni2018.pdf`.

Asher, Nicholas, Dever, Josh and Pappas, Chris (2009), 'Supervaluations debugged', *Mind*, 118(472), 901–933.

Bacon, Andrew (2018), *Vagueness and Thought*, Oxford University Press.

Barnes, Elizabeth and Cameron, Ross (2009), 'The open future: bivalence, determinism and ontology', *Philosophical Studies*, 146, 291–309.

Barnes, Elizabeth and Cameron, Ross (2011), 'Back to the open future', *Philosophical Perspectives*, 25(1), 1–26.

Barnes, Elizabeth and Williams, J.R.G. (2011), 'A theory of metaphysical indeterminacy', *Oxford studies in metaphysics*, 6, 103–148.

Barwise, Jon and Perry, John (1981), 'Situations and attitudes', *The Journal of Philosophy*, 78(11), 668–691.

Belnap, Nuel and Green, Mitchell (1994), 'Indeterminism and the thin red line', *Philosophical perspectives*, 3, 365–388.

Belnap, Nuel, Perloff, Michael and Xu, Ming (2001), *Facing the Future*, Oxford University Press.

Blackburn, Patrick, De Rijke, Maarten and Venema, Yde (2001), *Modal logic*, Cambridge University Press.

Bledin, Justin (2014), 'Logic informed', *Mind*, 123(490), 277–316.

Boylan, David (forthcoming), 'Does success entail ability?', *Noûs*.

Burgess, J. A. and Humberstone, I. L. (1987), 'Natural deduction rules for a logic of vagueness', *Erkenntnis*, 27(2), 197–229.

Cariani, Fabrizio (2021a), 'Human foreknowledge', *Philosophical Perspectives*.

Cariani, Fabrizio (2021b), *The Modal Future*, Cambridge University Press.

Cariani, Fabrizio and Santorio, Paolo (2018), '*Will* done better: Selection semantics, future credence, and indeterminacy', *Mind*, 127(505), 129–165.

Field, Hartry (2000), 'Indeterminacy, degree of belief, and excluded middle', *Noûs*, 34(1), 1–30.

Fine, Kit (1975), 'Vagueness, truth, and logic', *Synthese*, 30, 265–300.

Fine, Kit (2017a), 'The possibility of vagueness', *Synthese*, (10), 3699–3725.

Fine, Kit (2017b), 'Truthmaker semantics', *A Companion to the Philosophy of Language*, 2, 556–577.

Holliday, Wesley H. (2014), 'Partiality and adjointness in modal logic', *Advances in Modal Logic*, 10.

Holliday, Wesley H. (2022), 'Possibility semantics', in Melvin Fitting (ed.), 'Selected Topics from Contemporary Logic (Landscapes in Logic Vol. 2)', 363–476.

Holliday, Wesley H. (forthcoming), 'Possibility frames and forcing for modal logic', *Australasian Journal of Logic*.

Holliday, Wesley H. and Mandelkern, Matthew (ms.), 'The orthologic of epistemic modals', University of California, Berkeley and NYU.

Humberstone, Lloyd (1979), 'Interval semantics for tense logic: Some remarks', *Journal of philosophical logic*, 8(1), 171–196.

Humberstone, Lloyd (1981), 'From worlds to possibilities', *Journal of Philosophical Logic*, 10(3), 313–339.

Humberstone, Lloyd (2004), 'Two-dimensional adventures', *Philosophical Studies*, 118(1), 17–65.

Incurvati, Luca and Schlöder, Julian J. (2022), 'Meta-inferences and supervaluationism', *Journal of Philosophical Logic*, 1549–1582.

Kratzer, Angelika (2021), 'Situations in Natural Language Semantics', in Edward N. Zalta (ed.), 'The Stanford Encyclopedia of Philosophy', Metaphysics Research Lab, Stanford University, Winter 2021 edition.

Kuhn, Steven T (2013), ‘Two-dimensional logic and Two-dimensionalism in philosophy’, in Delia Graff Fara and Gillian Russell (eds.), ‘Routledge Companion to Philosophy of Language’, 640–651, Routledge.

MacFarlane, John (2003), ‘Future contingents and relative truth’, *Philosophical Quarterly*, 53, 321–336.

MacFarlane, John (2014), *Assessment Sensitivity*, Oxford University Press.

Schroeter, Laura (2021), ‘Two-Dimensional Semantics’, in Edward N. Zalta (ed.), ‘The Stanford Encyclopedia of Philosophy’, Metaphysics Research Lab, Stanford University, Fall 2021 edition.

Schulz, Moritz (2010), ‘Epistemic modals and informational consequence’, *Synthese*, 174, 385–395.

Stalnaker, Robert (1984), *Inquiry*, MIT Press (Bradford Books).

Stalnaker, Robert C. (2019), *Knowledge and conditionals: essays on the structure of inquiry*, Oxford University Press.

Thomason, Richmond H. (1970), ‘Indeterminist time and truth-value gaps’, *Theoria*, 18(3), 264–281.

Thomason, Richmond H. (1984), ‘Combinations of tense and modality’, in D. Gabbay and F. Guenthner (eds.), ‘Handbook of philosophical logic (Volume II: Extensions of Classical Logic)’, 135–165, Springer.

Thomason, Richmond H. (2007), ‘Truth after the fact in indeterminist tense logic’, URL `https://web.eecs.umich.edu/~rthomaso/documents/indeterminism/indettns.pdf`, unpublished Manuscript: University of Michigan.

Todd, Patrick (2022), *The Open Future*, Oxford University Press.

Torre, Stephan (2011), ‘The open future’, *Philosophy Compass*, 6(5), 360–373.

Torza, Alessandro (forthcoming), *Indeterminacy in the world*, Cambridge University Press.

Varzi, Achille C (2007), ‘Supervaluationism and its logics’, *Mind*, 116(463), 633–676.

Willer, Malte (2018), ‘Simplifying with free choice’, *Topoi*, 37(3), 379–392.

Williams, J Robert G (2014), 'Decision-making under indeterminacy', *Philosophers' Imprint*, 14, 1–34.

Williamson, Timothy (1994), *Vagueness*, Routledge.

Yalcin, Seth (2007), 'Epistemic modals', *Mind*, 116(4), 983–1027.