Paraconsistent Logics for Knowledge Representation and Reasoning: advances and perspectives

Walter Carnielli[1,2,3,4] , **Rafael Testa**[1] ,
[1]Centre for Logic, Epistemology and the History of Science
[2]Institute for Philosophy and Human Sciences
University of Campinas (UNICAMP)
[3]Advanced Institute for Artificial Intelligence (AI2)
[4]Modal Institute
walterac@unicamp.br, rafaeltesta@gmail.com

Abstract

This paper briefly outlines some advancements in paraconsistent logics for modelling knowledge representation and reasoning. Emphasis is given on the so-called Logics of Formal Inconsistency (**LFIs**), a class of paraconsistent logics that formally internalize the very concept(s) of consistency and inconsistency. A couple of specialized systems based on the **LFIs** will be reviewed, including belief revision and probabilistic reasoning. Potential applications of those systems in the AI area of KRR are tackled by illustrating some examples that emphasizes the importance of a fine-tuned treatment of consistency in modelling reputation systems, preferences, argumentation, and evidence.

1 Introduction

Non-classical logics find several applications in artificial intelligence, including multi-agent systems, reasoning with vagueness, uncertainty, and contradictions, among others, mostly akin with the area of knowledge representation and reasoning (Thomason 2020). Regarding this latter, there is a plethora of aims and applications in view when representing a knowledge of an agent, including fields beyond AI like software engineering, databases and robotics. Several logics have been studied for the latter purposes, including non-monotonic, epistemic, temporal, many-valued and fuzzy logics. This paper highlights the use of paraconsistent logics in some inconsistency-tolerant frameworks, introducing the family of Logics of Formal Inconsistency (**LFIs**) (advanced in the literature to be presented) for representing reasoning that makes use of the very notion of consistency and inconsistency, suitably formalized within the systems.

2 Reasoning under contradiction

2.1 The informative power of contradictions

Contradictory information is not only frequent, and more so as systems increase in complexity, but can have a positive role in human thought, in some cases not being totally undesirable. Finding contradictions in juridical testimonies, in statements from suspects of a crime or in suspects of tax fraud, for instance, can be an effective strategy – contradictions can be very informative in those cases (Carnielli and Coniglio 2016).

Indeed, the so called Bar-Hillel-Carnap paradox has already suggested half century ago the collapse between the notions of contradiction and semantic information: the less probable a statement is, the more informative it is, and so contradictions carry the maximum amount of information (Carnap and Bar-Hillel 1952). However, and in the light of standard logic, contradictions are "too informative to be true" as a famous quote by the latter has it.

To face the task of reasoning under contradictions, a field where human agents excel, is a difficult philosophical problem for standard logic, which is forced to equate triviality and contradiction, and to regard all contradictions as equivalent. However, skipping all technicalities in favour of a clear intuition (technical details can be found in (Mendonça 2018)), the Bar-Hillel-Carnap observation is not paradoxical for **LFIs**.

2.2 The beginings of Paraconsistent Logics (modern era)

The idea of a non-Aristotelian logic was advanced in a lecture in 1919 by Nicolai A. Vasiliev, where he proposed a kind of reasoning free from the laws of excluded middle and contradiction – called *Imaginary Logic* as an analogy with Lobachevsky's imaginary geometry. Such a logic would be valid, as the former has it, only for reasoning in "imaginary worlds" (Vasiliev 1912).

A more concrete example of a system for reasoning with contradictions can be found in the *Discussive Logic* (Jaśkowski 1948), advanced as a formal answer to the puzzling situation posed by J. Łukasiewicz: which logic applies in the situation where one has to defend some judgement *A*, also considering *not-A* for the sake of the argument? Jaśkowski's strategy is to avoid the combination of conflicting information by blocking the *rule of adjunction*. The idea is making room for A and $\neg A$ without entailing $A \wedge \neg A$, since the classic *explosion* actually still holds in the form of $A \wedge \neg A \nvdash B$. In terms of reasoning, it has a straightforward meaning: each agent must still be consistent! Jaśkowski's intuitions contributed to the proposal of the *society semantics* and to general case, the *possible-translations semantics*. A discussion on some conceptual points involving society semantics and their role on collective intelligence can be found in (Carnielli and Lima-Marques 2017; Testa 2020).

Another precursor, with a multi-valued approach, is the *Logic of Nonsense* (Halldén 1949) that, despite its name, captured a meaningful form of reasoning – aiming in studying logical paradoxes by means of 3-valued logical matrices (closely related to the *Nonsense Logic* introduced in 1938 by A. Bochvar). An analogous approach is made by F. Asenjo, who introduced a 3-valued logic as a formal framework for studying antinomies by means of 3-valued Kleene's truth-tables for negation and conjunction, where the third truth-value is distinguished (Asenjo 1966). The same logic has been studied by G. Priest, from the perspective of matrix logics, in the form of the so-called *Logic of Paradox* (LP) (Priest 1979).

With respect to a constructive approach to intuitionistic negation, D. Nelson proposed an extension of positive intuitionistic logic with a *strong negation* – a connective designed to capture the notion of "constructible falsity". By eliminating the explosion, Nelson obtained a (first-order) paraconsistent logic (Nelson 1959).

Focusing on the status of contradictions in mathematical reasoning, N. da Costa advanced a hierarchy of paraconsistent systems C_n (for $n \geq 1$) tolerant to contradictions, where the *consistency* of a formula A (in his terminology, the 'well-behavior' of A) is defined in C_1 by the formula $A^\circ = \neg(A \wedge \neg A)$. Let $A^1 =_{def} A^\circ$ and $A^{n+1} =_{def} (A^n)^\circ$. Then, in C^n, the following holds: (i) the well behaviour is denoted by $A^{(n)} =_{def} A^1 \wedge \cdots \wedge A^n$; (ii) $A, \neg A \nvdash B$ in general, but $A^{(n)}, A, \neg A \vdash B$ always holds; and (iii) $A^{(n)}, B^{(n)} \vdash (A \# B)^{(n)}$ and $A^{(n)} \vdash (\neg A)^{(n)}$.

By concentrating on the *non-triviality* of the systems rather than on the absence of contradictions, da Costa defined a logic to be paraconsistent with respect to $\neg$ if it can serve as a basis for $\neg$-contradictory yet non-trivial theories (da Costa 1974):

Definition 1. $\exists \Gamma \exists \alpha \exists \beta (\Gamma \vdash \alpha$ *and* $\Gamma \vdash \neg\alpha$ *and* $\Gamma \nvdash \beta)$

2.3 Motivations: main approaches

Preservationism Similar to the way discussive logic has it, there is a clear distinction between an inconsistent data set, like $\{A, \neg A\}$ (which is considered tractable), with a contradiction in the form $A \wedge \neg A$ (intractable). Thus, given an inconsistent collection of sentences (in an already defined logic **L**, usually classical logic), one should not try to reason about that collection as a whole, but rather focus on internally consistent subsets of premises. (Schotch, Brown, and Jennings 2009).

Relevant Logics Relevant logics are mainly concerned with a meaningful connection between the premises and the conclusion of an argument, thus not accepting for example inferences like $B \vdash A \rightarrow B$. This strategy induces a paraconsistent character in the resulting deductions, since A and $\neg A$, as premisses, do not necessarily have a meaningful connection with an arbitrary conclusion B (Anderson, Belnap, and Dunn 1992).

Adaptive Logics Human reasoning can be better understood as endowed with many dynamic consequence relations. Adaptive reasoning recognizes the so-called *abnormalities* to develop formal strategies to deal with them: for instance, an abnormality might be an inconsistency (inconsistency-adaptive logics), or it might be an inductive inference, and a strategy might be excluding a line of a proof (by marking it), or to change an inference rule. (Batens 2001).

Dialetheism A *dialetheia* is a sentence A, such that both it and its negation $\neg A$ are true. Assuming that falsity is the truth of negation, a dialetheia then is a sentence which is both true and false. Dialetheism, accordingly, is the metaphysical view that there are dialetheia, i.e., that there are true contradictions. As such, dialetheism opposes the Law of Non-Contradiction in the forma of $\neg(A \wedge \neg A)$ (Priest 1987). A system admitting 'both' as a truth-value, for instance, is the aforementioned Logic of Paradox.

Inconsistent (or rather Contradictory) Formal Systems The main idea is that there are situations in which contradictions can, at least temporarily, be admissible if their "behaviour can be somehow controlled", as da Costa has it (op. cit.). Contemporaneously, (Carnielli and Marcos 2002) extended and further generalized such notions, giving rise to the so called Logics of Formal Inconsistency, to be presented in the next section.

3 Logics of Formal Inconsistency- LFIs

3.1 Contradiction, consistency, inconsistency, and triviality

LFIs are a family of paraconsistent logics designed to express the notion(s) of consistency and inconsitency (sometimes defining one another, sometimes taken as primitive, depending on the strength of the axioms) within the object language by employing a connective "$\circ$" (or "$\bullet$"), in which $\circ\alpha$ means that "α is consistent" (and $\bullet\alpha$ means that "α is inconsistent"), further expanding and generalizing da Costa's hierarchy of C systems. Accordingly, the principle of explosion is not valid in general, although this law is not abolished but restricted to the so-called "consistent sentences", a feature captured by the following law, which is referred to as the "principle of Gentle Explosion" (PGE):

$$\alpha, \neg\alpha, \circ\alpha \vdash \beta, \text{ for every } \beta, \text{ but } \alpha, \neg\alpha \nvdash \beta \text{ for some } \beta \quad (1)$$

In formal terms, we have the following (Carnielli and Coniglio 2016):

Definition 2 (A formal definition of LFI). *Let* **L** *be a Tarskian logic with a negation* $\neg$. *The logic* **L** *is a LFI if there is a non-empty set* $\bigcirc(p)$ *of formulas in some language* $\mathbb{L}$ *of* **L** *which depends only on the propositional variable p, satisfying the following:*

a. $\exists\alpha\exists\beta(\neg\alpha, \alpha \nvdash \beta)$
b. $\exists\alpha\exists\beta(\bigcirc(\alpha), \alpha \nvdash \beta)$
c. $\exists\alpha\exists\beta(\bigcirc(\alpha), \neg\alpha \nvdash \beta)$
d. $\forall\alpha\forall\beta(\bigcirc(\alpha), \alpha, \neg\alpha \vdash \beta)$

For any formula α, the set $\bigcirc(\alpha)$ is intended to express, in a specific sense, the consistency of α relative to the logic **L**. *When this set is a singleton, it is denoted by $\circ\alpha$ the sole element of $\bigcirc(\alpha)$, thus defining a consistency operator.*

The connective "$\circ$", as mentioned, is not necessarily a primitive one. Indeed, LFI is an umbrella definition that covers many paraconsistent logics of the literature.

Remark 3 (**Some notable LFIs**)**.** *Following definition 2, it can be easily proved that some well-known logics in the literature are* **LFIs***, including the aforementioned Jaśkowski's Discussive logic, Halldén's nonsense logic and, as expected, da Costa's C-systems (Carnielli and Coniglio 2016; Carnielli, Coniglio, and Marcos 2007; Carnielli and Marcos 2002).*

It is worth observing that each one of the aforementioned logics has their own motivations and particularities - being Remark 3 to be understood as a logic-mathematical reminder that those logics share some common results and properties.

3.2 A family of LFIs

It should be clear that the notions of consistency and non-contradiction are not coincident in the **LFIs**, and that the same holds for the notions of inconsistency and contradiction. There is, however, a fully-fledged hierarchy of **LFIs** where consistency is gradually connected to non-contradiction.

Starting from positive classical logic plus *tertium non datur* ($\alpha \vee \neg\alpha$), **mbC** is one of the basic logics intended to comply with definition 2 in a minimal way: an axiom schema called (bc1) is added solely to capture the aforementioned principle of gentle explosion.

Definition 4 (**mbC**(Carnielli and Marcos 2002))**.** *The logic* **mbC** *is defined over the language $\mathcal{L}$ (generated by the connectives $\wedge, \vee, \rightarrow, \neg, \circ$) by means of a Hilbert system as follows:*

Axioms:

(A1) $\alpha \rightarrow (\beta \rightarrow \alpha)$
(A2) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow (\beta \rightarrow \delta)) \rightarrow (\alpha \rightarrow \delta))$
(A3) $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$
(A4) $(\alpha \wedge \beta) \rightarrow \alpha$
(A5) $(\alpha \wedge \beta) \rightarrow \beta$
(A6) $\alpha \rightarrow (\alpha \vee \beta)$
(A7) $\beta \rightarrow (\alpha \vee \beta)$
(A8) $(\alpha \rightarrow \delta) \rightarrow ((\beta \rightarrow \delta) \rightarrow ((\alpha \vee \beta) \rightarrow \delta))$
(A9) $\alpha \vee (\alpha \rightarrow \beta)$
(A10) $\alpha \vee \neg\alpha$
(bc1) $\circ\alpha \rightarrow (\alpha \rightarrow (\neg\alpha \rightarrow \beta))$

Inference Rule:

(Modus Ponens (MP)) $\alpha, \alpha \rightarrow \beta \vdash \beta$

(A1)-(10) plus (MP) coincides with Baten's paraconsistent logic **CLuN** – it is worth mentioning that a nonmonotonic characterization of the Ci-hierarchy (presented in section 6) can be found in (Batens 2009). Furthermore, (A1)-(A9) plus (MP) defines positive classical propositional logic $\mathbf{CPL}^{+}$.

mbC can be characterized in terms of valuations over $\{0, 1\}$ (also called *bivaluations*), but cannot be semantically characterized by finite matrices (cf. (Carnielli, Coniglio, and Marcos 2007)). Surprisingly, however, **mbC** can be characterized by 5-valued non-deterministic matrices, as shown in (Avron 2005) (details also in Example 6.3.3 of (Carnielli and Coniglio 2016)).

Definition 5 (Valuations for **mbC**)**.** *A function $v : \mathbb{L} \rightarrow \{0, 1\}$ is a* valuation for **mbC** *if it satisfies the following clauses:*

(Biv1) $v(\alpha \wedge \beta) = 1 \iff v(\alpha) = 1$ *and* $v(\beta) = 1$
(Biv2) $v(\alpha \vee \beta) = 1 \iff v(\alpha) = 1$ *or* $v(\beta) = 1$
(Biv3) $v(\alpha \rightarrow \beta) = 1 \iff v(\alpha) = 0$ *or* $v(\beta) = 1$
(Biv4) $v(\neg\alpha) = 0 \implies v(\alpha) = 1$
(Biv5) $v(\circ\alpha) = 1 \implies v(\alpha) = 0$ *or* $v(\neg\alpha) = 0$.

The semantic consequence relation associated to valuations for **mbC** is defined as expected: $X \models_{\mathbf{mbC}} \alpha$ iff, for every **mbC**-valuation v, if $v(\beta) = 1$ for every $\beta \in X$ then $v(\alpha) = 1$.

Definition 6 (Extensions of **mbC** (Carnielli and Marcos 2002; Carnielli, Coniglio, and Marcos 2007; Carnielli and Coniglio 2016))**.** *Consider the following axioms:*

(ciw) $\circ\alpha \vee (\alpha \wedge \neg\alpha)$
(ci) $\neg\circ\alpha \rightarrow (\alpha \wedge \neg\alpha)$
(cl) $\neg(\alpha \wedge \neg\alpha) \rightarrow \circ\alpha$
(cf) $\neg\neg\alpha \rightarrow \alpha$
(ce) $\alpha \rightarrow \neg\neg\alpha$

Some interesting extensions of **mbC** *are the following:*

mbCciw = **mbC**+*(ciw)*
mbCci = **mbC**+*(ci)*
bC = **mbC**+*(cf)*
Ci = **mbC**+*(ci)*+*(cf)* = **mbCci**+*(cf)*
mbCcl = **mbC**+*(cl)*
Cil = **mbC**+*(ci)*+*(cf)*+*(cl)* = **mbCci**+*(cf)*+*(cl)* = **mbCcl**+ *(cf)* + *(ci)* = **Ci**+*(cl)*

The semantic characterization by bivaluations for all these extensions of **mbC** can be easily obtained from the one for **mbC** (see (Carnielli, Coniglio, and Marcos 2007; Carnielli and Coniglio 2016)). For instance, **mbCciw** is characterized by **mbC**-valuations such that $v(\circ\alpha) = 1$ if and only if $v(\alpha) = 0$ or $v(\neg\alpha) = 0$ (if and only if $v(\alpha) \neq v(\neg\alpha)$).

Notation 7 (derived bottom particle and strong negation)**.** $\bot =_{def} \alpha \wedge \neg\alpha \wedge \circ\alpha$ *and* $\sim \alpha =_{def} \alpha \rightarrow \bot$ *(for any α).*

It is then clear that the **LFIs** are at the same time subsystems and extensions of **CPL**. They can be seen as classical logic extended by two connectives: a paraconsistent negation and a consistency connective (or an inconsistency one, dual to it). In formal terms, consider **CPL** defined over the language $\mathcal{L}_0$ generated by the connectives $\wedge, \vee, \rightarrow, \neg$, where $\neg$ represents the classical negation instead of the paraconsistent one. If $Y \subseteq \mathcal{L}_0$ then $\circ(Y) = \{\circ\alpha : \alpha \in Y\}$. Then, the following result can be obtained:

Observation 8 (Derivability Adjustment Theorem (Carnielli and Marcos 2002))**.** *Let $X \cup \{\alpha\}$ be a set of formulas in $\mathcal{L}_0$. Then $X \vdash_{CPL} \alpha$ if and only if $\circ(Y), X \vdash_{mbc} \alpha$ for some $Y \subseteq \mathcal{L}_0$.*

4 Paraconsistent Belief Change

Belief Change in a wide sense has been subject of philosophical reflection since antiquity, including discussions about the mechanisms by which scientific theories develop and proposing rationality criteria for revisions of probability assignments (Fermé and Hansson 2018). Contemporaneously, there is a strong tendency towards confluence of the research traditions on the subject from philosophy and from computer research (Hansson 1999).

The most influential paradigm in this area of study is the AGM model (Alchourrón, Gärdenfors, and Makinson 1985), in which epistemic states are represented as theories – considered simply as sets of sentences closed under logical consequence. Three types of *epistemic changes* (or operations) are considered in this model: *expansion*, the incorporation of a sentence into a given theory; *contraction*, the retraction of a sentence from a given theory; and *revision*, the incorporation of a sentence into a given consistent theory by ensuring the consistency of the resulting one.

Notably, given the possibility of reasoning with contradictions (as paraconsistent logics have it), as well as the aforementioned scrutiny on the very concept of "consistency", the definition of revision can be refined. Indeed, there are some investigations in the literature alongside this direction:

Based on the four-valued relevant logic of first-degree entailment, (Restall and Slaney 1995) defines an AGM-like contraction without satisfying the *recovery* postulate. Revision is obtained from contraction by the Levi identity (to be introduced).

Also based on the first-degree entailment, (Tamminga 2001) advances a system that put forth a distinction between *information* and *belief*. Techniques of expansion, contraction an revisions are applied to information (which can be contradictory), while other kind of operations are advanced for extracting beliefs from those information. The demanding for consistency (i.e. non-contradictoriness) is applied only for those beliefs.

(Mares 2002) proposes a model in which an agent's belief state is represented by a pair of sets – one of these is the belief set, and the other consists of the sentences that the agent rejects. A belief state is coherent if and only if the intersection of these two sets is empty, i.e. if and only if there is no statement that the agent both accepts and rejects. In this model, belief revision preserves coherence but does not necessarily preserve consistency.

Also departing from a distinction between consistency and coherence, (Chopra and Parikh 1999) advances a model based on Belnap and Dunn's logic that preserves an agent's ability to answer contradictory queries in a coherent way, splitting the language to distinguish between implicit and explicit beliefs.

In (Priest 2001) and (Tanaka 2005), it is suggested that revision can be performed by just adding sentences without removing anything, i.e, revision can be defined as a simple expansion. Furthermore, Priest first pointed out that in a paraconsistent framework, revision on belief sets can be performed as external revision, defined with the reversed Levi identity as advanced for belief bases (Hansson 1993) .

The fact is that there are in the literature several systems that could be understood as endowing a certain paraconsistent character, each one based on distinct strategies and motivations (see for instance (Fermé and Wassermann 2017) for an Iterated Belief Change perspective). An approach of Belief Change from the perspective of inconsistent formal systems was conceptually suggested by (da Costa and Bueno 1998). Departing from the technical advances of **mbC** and its extensions, (Testa, Coniglio, and Ribeiro 2017) goes further in this direction, defining *external* and *semi-revisions* for belief sets, as well as consolidation (operations that were originally presented for belief bases (Hansson 1993)(1997). By considering consistency as an epistemic attitude, and allowing temporary contradictions, the informational power of the operations are maximized (as it argued by (Testa 2015)).

It is worth mentioning that, as proposed by Priest and Tanaka (op. cit.), paraconsistent revision could be understood as a plain expansion. As it is explained by (Testa et al. 2018), to equate paraconsistent revision with expansion it is necessary to assume that consistency is necessarily equivalent to non-triviality in a paraconsistent setting and, furthermore, that all paraconsistent logics do not endow a bottom particle (primitive or defined). As this paper intends to highlight, neither assumption is true.

Remark 9. *From now on, let us assume a* **LFI**, *namely* **L**$=\langle\mathcal{L},\vdash_{\mathbf{L}}\rangle$, *such that* **L** *is* **mbC** *or some extension as presented above. Since the context is clear, we will omit the subscript, and simply denote $\vdash_{\mathbf{L}}$ by $\vdash$ and, accordingly, the respective closure by Cn.*

4.1 Revisions in the LFIs

In (Testa, Coniglio, and Ribeiro 2017) the so-called AGMp system is proposed, in which it is shown that a paraconsistent revision of a belief set K by a belief-representing sentence α (the operation $K * \alpha$) can be defined not only by the *Levi identity* as in classical AGM (that is, by a prior contraction by $\neg\alpha$ followed by a expansion by α) but also by reversed Levi identity and other kind of constructions where contradictions are temporarily accepted. Formally, we have the following:

Let $K = Cn(K)$. The expansion of K by α ($K + \alpha$) is given by

Definition 10. $K + \alpha = Cn(K \cup \{\alpha\})$

There are several constructions for defining a contraction operator. The one adopted is the *partial meet contraction*, constructed as follows (Alchourrón, Gärdenfors, and Makinson 1985):

1. Choose some maximal subsets of K (with respect the inclusion) that do not entail α.
2. Take the intersection of such sets.

The *remainder* of K and α is the set of all maximal subsets of K that do not entail α.

Definition 11 (Remainder)**.** *The set of all the maximal subsets of K that do not entail α is called the* remainder set *of K by α and is denoted by $K\perp\alpha$, that is, $K' \in K\perp\alpha$ iff:*

(i) $K' \subseteq K$.

(ii) $\alpha \notin Cn(K')$.

(iii) *If* $K' \subset K'' \subseteq K$ *then* $\alpha \in Cn(K'')$.

Typically $K \perp \alpha$ may contain more than one maximal subset. The main idea constructing a contraction function is to apply a *selection function* γ which intuitively selects the sets in $K \perp \alpha$ containing the beliefs that the agent holds in higher regard (those beliefs that are more entrenched).

Definition 12 (selection function)**.** *A selection function for* K *is a function* γ *such that, for every* α*:*

1. $\gamma(K \perp \alpha) \subseteq K \perp \alpha$ *if* $K \perp \alpha \neq \emptyset$.
2. $\gamma(K \perp \alpha) = \{K\}$ *otherwise.*

The *partial meet contraction* is the intersection of the sets of $K \perp \alpha$ selected by γ.

Definition 13 (partial meet contraction)**.** *Let* K *be a belief set, and* γ *a selection function for* K*. The partial meet contraction on* K *that is generated by* γ *is the operation* $-_\gamma$ *such that for all sentences* α*:*

$$K -_\gamma \alpha = \bigcap \gamma(K \perp \alpha).$$

The distinct revisions are then defined as follows:

Definition 14. Internal revision $(K - \neg\alpha) + \alpha$

External revision $(K + \alpha) - \neg\alpha$

Semi-revision $(K + \alpha)!$

The aforementioned operator "!", originally advanced for belief bases (Hansson 1997), is a particular case of contraction – called consolidation. In Hansson's original presentation, this operator is defined as a contraction by "$\perp$". In the context of **LFIs**, it is defined as the contraction by $\Omega_K = \{\alpha \in K :$ exists $\beta \in \mathbb{L}$ such that $\alpha = \beta \wedge \neg\beta\}$. The technical details of those operations, alongside a presentation through postulates and their respective representation theorems can be found in the references.

4.2 Reasoning with consistency and inconsistency

Each of the **LFIs** in the aforementioned family (recall definition 6) captures distinct properties regarding the notion of formal consistency. For instance, **mbC** separates the notions of consistency from non-contradictoriness ($\circ\alpha \vdash \neg(\neg\alpha \wedge \alpha)$, but the converse does not hold), and also separates the notions of inconsistency from contradictoriness ($\alpha \wedge \neg\alpha \vdash \neg\circ\alpha$, but the converse does not hold). In **Ci** inconsistency and contradictoriness are identified ($\neg\circ\alpha \dashv\vdash \alpha \wedge \neg\alpha$) and, in **Cil** consistency and non-contradictoriness are identified ($\circ\alpha \dashv\vdash \neg(\alpha \wedge \neg\alpha)$).

This cautious way of dealing with the formal concept of consistency allows the modeling of significant forms of reasoning, as it is illustrated by the following example adapted from (Hansson 1999). In Hansson's original presentation, it was intended to show a case of an external partial meet revision that is not also an internal partial meet revision – indeed, neither one can be subsumed under the other. In our analysis, the same conclusion applies: the avoidance of contradictions in every step of the reasoning refrain the revision to adduce the following significant results.

Let $\neg\circ\alpha =_{def} \bullet\alpha$, and let us consider **Ci** as the underlying logic.

Example 1. *A man has died in a remote place in which only two other persons, Adam and Bob, were present. Initially, the public prosecutor believes that neither Adam nor Bob has killed him. Thus her belief state contains* $\neg A$ *(Adam has not killed the deceased) and* $\neg B$ *(Bob has not killed the deceased). For simplicity, we may assume that her belief state is* $K_0 = Cn(\{\neg A, \neg B\})$.
Case 1: The prosecutor receives a police report saying (1) that the deceased has been murdered, and that either Adam or Bob must have done it; and (2) that Adam has previously been convicted of murder several times. After receiving the report, she revises her belief set by $(A \vee B)$ *and by the assumption that Bob's innocence is indeed consistent* $\circ\neg B$*, i.e. she revises her initial belief set by* $(A \vee B) \wedge \circ\neg B$.
Case 2: differs from case 1 only that it is Bob who has previously been convicted of murder. Thus, the new piece of information consists of $(A \vee B) \wedge \circ\neg A$.

Internal Revision approach: If represented as an internal partial meet revision, when the first suboperation is performed (namely, contraction by $\neg((A \vee B) \wedge \circ\neg B)$ and $\neg((A \vee B) \wedge \circ\neg A)$ respectively in case 1 and case 2), we have that

$$K_0 \perp (\neg((A \vee B) \wedge \circ\neg B)) = K_0 \perp (\neg((A \vee B) \wedge \circ\neg A)).$$

The subsequent expansion does not necessarily add nor delete Adam's or Bob's guilty/innocence in both cases, since the previous contraction could indiscriminately delete Adam's or Bob's innocence – not taking profit of the new piece of information as a whole.

External Revision approach: If represented as an external partial meet revision, we have the following.
Case 1: The police report brings about the expansion of K to $K_1 = Cn(K + (A \vee B) \wedge \circ\neg B)$. Notably, $A \in K_1$ (on the grounds that $\circ\neg B, \neg B, A \vee B \vdash \circ\neg B, \neg B, A \vee \neg\neg B \vdash A$). In plain English, Adam is now proven to be guilty. Moreover, $\bullet\neg A \in K_1$ (for $A \wedge \neg A \vdash \neg\neg A \wedge \neg A \vdash \bullet\neg A$) i.e., the initial assumption about Adam's innocence is logically proven to be inconsistent. The subsequent contraction thus has means to delete the initial supposition about Adam's innocence.
Case 2: Mutatus mutandis.

Semi-revision approach: The semi-revision approach is analogous to the external-revision, with the distinction that the second suboperation (namely, contraction) does not necessarily delete Adam's and Bob's innocence (respectively in case 1 and case 2) but, rather, gives the option for deleting the new piece of information given by the police report.

4.3 Formal consistency as an epistemic attitude

An alternative system considered in (Testa, Coniglio, and Ribeiro 2017), called AGM$\circ$, relies heavily on the formal consistency operator. This means that the explicit constructions themselves (and accordingly the postulates) assume that such operator plays a central role. In a static paradigm (i.e., when the focus is the logical consequence relation) this is already the case. Assuming the consistency of the sentence involved in a contradiction entails a trivialization (as elucidated in the gentle explosion principle) – which somehow captures and describes the intuition of the expansion.

The main idea of $AGM\circ$ is to also incorporate the notion of consistency in the contraction. In this case, it is interpreted that a belief being consistent means that it is not liable to be removed from the belief set in question, adducing that the contraction endows the postulate of *failure* (namely, that if $\circ\alpha \in K$ then $K - \alpha = K$).

The strategy is to incorporate the idea of non-revisibility in the selection function – the consistent belief remains in the epistemic state in any situation, unless the agent retract the very fact that such belief is consistent.

Definition 15 (**selection function for AGM∘ contraction**). *A selection function for K is a function γ' such that, for every α:*

1. $\gamma'(K, \alpha) \subseteq K\perp\alpha$ *if* $\alpha \notin Cn(\emptyset)$ *and* $\circ\alpha \notin K$.
2. $\gamma'(K, \alpha) = \{K\}$ *otherwise.*

Contraction, thus, is defined as definition 13.

In short, the seven epistemic attitudes defined in $AGM\circ$ are:

Definition 16 (Possible epistemic attitudes in AGM∘, see figure 1 (Testa, Coniglio, and Ribeiro 2017; Testa 2014))**.** *Let K be a given belief set. Then, a sentence α is said to be:*

Accepted *if* $\alpha \in K$.
Rejected *if* $\neg\alpha \in K$.
Under-determined *if* $\alpha \notin K$ *and* $\neg\alpha \notin K$.
Over-determined *if* $\alpha \in K$ *and* $\neg\alpha \in K$.
Consistent *if* $\circ\alpha \in K$.
Boldly accepted *if* $\circ\alpha \in K$ *and* $\alpha \in K$.
Boldly rejected *if* $\circ\alpha \in K$ *and* $\neg\alpha \in K$ *(i.e.* $\sim G \in K$*).*

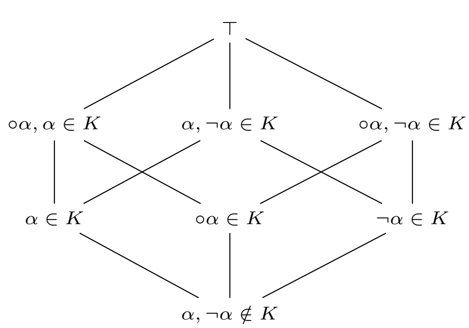

Figure 1: Epistemic attitudes in AGM∘

The following examples illustrate an important feature of human belief that, in classical AGM, has no room in a model solely based on contractions and revisions – the stubbornness of human belief. Instead of introducing the notions of necessity and possibility on the metalanguage, as suggested by (Hansson 1999), it is possible to capture such notions based on the concept of bold-acceptance. Indeed, as interpreted by (Testa 2014), this fact illustrate a well-studied feature regarding the proximity of **LFIs** with modal logics.

Example 2. *Adapted from (Hansson 1999)*

1. *Doris is not religious, but she has religious leanings. She does not believe that God exists ($G \notin K$), but it is possible for her to become a believer ($\sim G \notin K$).*
2. *Ellen, on the other hand, is a believer ($G \in K$). However, it may very well happen that she loses her faith so definitely that she can never become a believer in God again ($\circ\neg G \in K$).*
3. *Florence is an inveterate doubter. Nothing can bring her to a state of firm (irreversible) belief ($\circ G \notin K$) and neither can she be brought to a state of firm disbelief ($\circ\neg G \notin K$)*

Paraconsistent Belief Revision based on the LFIs are an important step for further advancements on systems for detecting and handling with contradictions, mostly if combined with tools for expressing probabilistic reasoning. Some progress in this direction are overviewed in the following sections.

5 Sound probabilistic reasoning under contradiction

This section briefly surveys the research initiative on paraconsistent probability theory based on the **LFIs** and its consequences, which makes it possible to treat realistic probabilistic reasoning under contradiction.

Paraconsistent probabilities can be regarded as degrees of belief that a rational agent attaches to events, even if such degrees of belief might be contradictory. Thus it is not impossible for an agent to believe in the proposition α and $\neg\alpha$ and to be rational, if this belief is justified by evidence, as argued in (Bueno-Soler and Carnielli 2016).

A quite general notion of probability function can be defined, in such a way that different logics can be combined with probabilistic functions, giving rise to new measures that may reflect some subtle aspects of probabilistic reasoning.

Definition 17. *A probability function for a language $\mathcal{L}$ of a logic* **L***, or a* **L***-probability function, is a function $P : \mathcal{L} \mapsto \mathbb{R}$ satisfying the following conditions, where $\vdash_L$ stands for the syntactic derivability relation of* **L***:*

1. *Non-negativity:* $0 \leq P(\varphi) \leq 1$ *for all* $\varphi \in \mathcal{L}$
2. *Tautologicity: If* $\vdash_L \varphi$*, then* $P(\varphi) = 1$
3. *Anti-tautologicity: If* $\varphi \vdash_L$*, then* $P(\varphi) = 0$
4. *Comparison: If* $\psi \vdash_L \varphi$*, then* $P(\psi) \leq P(\varphi)$
5. *Finite additivity:* $P(\varphi \vee \psi) = P(\varphi) + P(\psi) - P(\varphi \wedge \psi)$

This collection of meta-axioms, by assuming appropriate $\vdash_L$ (for instance, by taking the classical, intuitionistic or paraconsistent derivability relation) defines distinct probabilities, each one deserving a full investigation. In particular, for the sake of this project, we have in mind paraconsistent probability theory based on the Logics of Formal Inconsistency, as it has been treated in (Bueno-Soler and Carnielli 2016),(2017).

Several central properties of probability are preserved, as the notions of paraconsistent updating which is materialized through new versions of Bayes' theorem for conditionalization. Other papers already proposed connections between non-classical logics and probabilities and even for the paraconsistent case (references can be found in the aforementioned works), recognizing that some non-classical logics are better suited to support uncertain reasoning in particular domains. The combinations between probabilities and **LFIs** deserves to be emphasized, as they offer a quite natural and intuitive extension of standard probabilities which is useful and philosophically meaningful.

The following example uses the system **Ci**, a member of the **LFI** family with some features that make it reasonably close to classical logic (recall definition 6); it is appropriate, in this way, to define a generalized notion of probability strong enough to enjoy useful properties.

Observation 18 (Paraconsistent Bayes' Conditionalization Rule (PBCR) (Bueno-Soler and Carnielli 2016))**.**
If $P(\alpha \wedge \neg\alpha) \neq 0$, then:

$$P(\alpha/\beta) = \frac{P(\beta/\alpha) \cdot P(\alpha)}{P(\beta/\alpha) \cdot P(\alpha) + P(\beta/\neg\alpha) \cdot P(\neg\alpha) - \delta_\alpha}$$

where $\delta_\alpha = P(\beta/\alpha \wedge \neg\alpha) \cdot P(\alpha \wedge \neg\alpha)$ is the 'contradictory residue' of α.

It is clear that this rule generalizes the classical conditionalization rule, as it reduces to the classical case if $P(\alpha \wedge \neg\alpha) = 0$ or if α is consistent: indeed, in the last case, $P(\beta \wedge \circ\alpha) = P(\beta \wedge \circ\alpha \wedge \alpha) + P(\beta \wedge \circ\alpha \wedge \neg\alpha)$ since $P(\circ\alpha \wedge \alpha \wedge \neg\alpha) = 0$.

We can interpret (PBCR) as Bayes' ruke taking into account the likelihood relative to the contradiction. It is possible, however, to formulate other kinds of conditionalization rules by combining the notions of conditional probability, contradictoriness, consistency and inconsistency.

Example 3. *As an example, suppose that a doping test for an illegal drug is such that it is 98% accurate in the case of a regular user of that drug (i.e., it produces a positive result, showing "doping", with probability 0.98 in the case that the tested individual often uses the drug), and 90% accurate in the case of a non-user of the drug (i.e., it produces a negative result, showing "no doping", with probability 0.9 in the case that the tested individual has never used the drug or does not often use the drug).*

Suppose, additionally, that: (i) it is known that 10% of the entire population of all athletes often uses this drug; (ii) that 95% of the entire population of all athletes does not often use the drug or has never used it; and (iii) that the test produces a positive result, showing "doping", with probability 0.11 for the whole population, independent of the tested individual.

Let the following be some mnemonic abbreviations:

D : *the event that the drug test has declared "doping" (positive) for an individual;*

C : *the event that the drug test has declared "clear" or "no doping" (negative) for an individual;*

A : *the event that the person tested often uses the drug;*

$\neg A$: *the event that the person tested does not often use the drug or has never used it.*

We know that $P(A) = 0.1$ and $P(\neg A) = 0.95$. The situation is clearly contradictory with respect to the events A and $\neg A$, as they are not excludent. Therefore, by finite additivity, $P(A \vee \neg A) = 1 = (P(A) + P(\neg A)) - P(A \wedge \neg A)$, and thus, $P(A \wedge \neg A) = (P(A) + P(\neg A)) - 1 = 0.05$

Furthermore, as given in the problem, $P(D/A) = 0.98$, $P(C/\neg A) = 0.9$ and $P(D) = 0.11$. The results of the test have no paraconsistent character, since the events D ('doping') and C ('no doping') exclude each other. Thus, $P(D/\neg A) = 1 - P(C/\neg A) = 0.1$ and $P(C/A) = 1 - P(D/A) = 0.02$.

Suppose someone has been tested, and the test is positive ("doping"). What is the probability that the tested individual regularly uses this illegal drug, that is what is $P(A/D)$?

By applying the paraconsistent Bayes' rule:

$$P(A/D) = \frac{P(D/A) \cdot P(A)}{P(D/A) \cdot P(A) + P(D/\neg A) \cdot P(\neg A) - \delta_A}$$

where $\delta_A = P(D/A \wedge \neg A) \cdot P(A \wedge \neg A)$ since $P(A \wedge \neg A) \neq 0$.

All of the values are known, with the exception of $P(D/A \wedge \neg A)$. Since:

$$P(D/A \wedge \neg A) = \frac{P(D \wedge A \wedge \neg A)}{P(A \wedge \neg A)}$$

it remains to compute $P(D \wedge A \wedge \neg A)$. It follows directly from some easy properties of probability that $P(D \wedge A \wedge \neg A) = P(D \wedge A) + P(D \wedge \neg A) - P(D) = P(D/A).P(A) + P(D/\neg A).P(\neg A) - P(D) = 0.083$. Therefore, by plugging in all of the values, it follows that $P(A/D) = 51.9\%$[1].

This example suggests, as argued below, that the paraconsistent Bayes' conditionalization rule is more robust than traditional conditionalization, as it can provide useful results even in the case the test could be regarded as ineffective due to contradictions. The following table compares the paraconsistent result with the results obtained by trying to remove the contradiction involving the events A (the event that the person tested often uses the drug) and $\neg A$ (the event that the person tested does not often use the drug or has never used it), that is by trying to make them "classical".

Since A and $\neg A$ overlap by 5%, we might consider reviewing the values, by 'removing the contradiction' according to three hypothetical scenarios: an *alarming scenario*, by lowering the value of $\neg A$ by 5%; a *happy scenario*, by lowering the value of A by 5%; and a *cautious scenario*, by dividing the surplus equally between A and $\neg A$ and computing the probability $P(A/D)$ that the tested individual regularly uses this illegal drug.

Table 1: Removing the contradiction

Alarming Scenario	**Cautious Scenario**	**Happy Scenario**
$P(A) = 10\%$	$P(A) = 7.5\%$	$P(A) = 5\%$
$P(\neg A) = 90\%$	$P(\neg A) = 92.5\%$	$P(\neg A) = 95\%$
$P(D/A) = 98\%$	$P(D/A) = 98\%$	$P(D/A) = 98\%$
$P(D/\neg A) = 10\%$	$P(D/\neg A) = 10\%$	$P(D/\neg A) = 10\%$
Result	**Result**	**Result**
$P(A/D) = 52\%$	$P(A/D) = 44\%$	$P(A/D) = 34\%$

[1]The values correct some miscalculations in (Bueno-Soler and Carnielli 2016).

Using paraconsistent probabilities, one obtains, in the case of this example, a value close (even if a bit inferior) to the "alarming" hypothetical scenario, helping to make a decision even if the contradictory character would make it be seen as ineffective. In other words, the presence of a contradiction does not mean that we need to discard the test, if we have reasoning tools that are sensitive and robust enough.

6 Possibility and necessity measures

Possibility theory is a generalization of (or an alternative to) probability theory devoted to deal with certain types of uncertainty by means of possibility and necessity measures.

As aforementioned, it is well recognized that reasoning with contradictory premises is a critical issue, since large knowledge bases are inexorably prone to incorporate contradictions. Contradictory information comes from the fact that data is provided by different sources, or by a single source that delivers contradictory data as certain.

The connections between the possibilistic and the paraconsistent paradigms are complex and various forms of contradiction can be accommodated into possibilistic logic, defining concepts such as 'paraconsistency degree' and 'paraconsistent completion' (Dubois and Prade 2015). Paraconsistent logics offer simple and effective models for reasoning in the presence of contradictions, as they avoid collapsing into deductive trivialism by a natural logic machinery. Taking into consideration that it is more natural and effective to reason from a contradictory information scenario than trying to remove the contradictions involved, the investigation of credal calculi concerned with necessity and possibility is naturally justified.

On one hand, possibility theory based on classical logic is able to handle contradictions, but at the cost of expensive manoeuvres (Dubois and Prade 2015). On the other hand, paraconsistent logics cannot easily express uncertainty in a gradual way. The blend of both via the **LFIs**, in view of the operators of consistency and inconsistency, offers a simple and natural qualitative and quantitative tool to reason with uncertainty.

The idea of defining possibility and necessity models, dubbed as *credal calculi*, based on the Logics of Formal Inconsistency, takes advantage of the flexibility of the notions of consistency "$\circ$" and inconsistency "$\bullet$". Some basic properties of possibility and necessity functions over the Logics of Formal Inconsistency have been investigated in (Carnielli and Bueno-Soler 2017), making clear that paraconsistent possibility and necessity reasoning can, in general, attain realistic models for artificial judgement.

A generic notion of logic-dependent necessity measures is given by the conditions below.

Definition 19 ((Carnielli and Bueno-Soler 2017)). *A necessity function (or measure) for a language $\mathcal{L}$ in an* **LFI***, called an LFI-necessity function, is a function $N : \mathcal{L} \mapsto \mathbb{R}$ satisfying the following conditions, where $\vdash_L$ stands for the syntactic derivability relation of* ***L****:*

1. *Non-negativity:* $0 \leq N(\varphi) \leq 1$ *for all* $\varphi \in \mathcal{L}$
2. *Tautologicity: If* $\vdash_{\mathbf{L}} \varphi$*, then* $N(\varphi) = 1$
3. *Anti-Tautologicity: If* $\varphi \vdash_{\mathbf{L}}$*, then* $N(\varphi) = 0$
4. *Comparison: If* $\psi \vdash_{\mathbf{L}} \varphi$*, then* $N(\psi) \leq N(\varphi)$
5. *Conjunction:* $N(\varphi \wedge \psi) = min\{N(\varphi), N(\psi)\}$
6. *Metaconsistency:* $N(\bullet\alpha) + N(\circ\alpha) = 1$

A condition $N(\alpha) = \lambda$ can be understood as expressing that 'α is certain to degree λ' (in all normal states of affairs).

Possibilistic measures are also useful when representing preferences expressed as sets of prioritized goals, as e.g. some lattice-valued possibility measures studied in the literature instead of real-valued possibility measures. The parameter **L** in the above definition can be **Cie**, or the three-valued logic **LFI1**(see references for details).

Analogously to the necessity function, a generic notion of logic-dependent possibility measure (dual to a necessity function) is defined as follows:

Definition 20. *A possibility function (or measure) for the language $\mathcal{L}$ of* ***Cie****, or a* ***Cie****- possibility function, is a function $\Pi : \mathcal{L} \mapsto \mathbb{R}$ satisfying the following conditions:*

1. *Non-negativity:* $0 \leq \Pi(\varphi) \leq 1$ *for all* $\varphi \in \mathcal{L}$
2. *Tautologicity: If* $\vdash_{\mathbf{L}} \varphi$*, then* $\Pi(\varphi) = 1$
3. *Anti-Tautologicity: If* $\varphi \vdash_{\mathbf{L}}$*, then* $\Pi(\varphi) = 0$
4. *Comparison: If* $\psi \vdash_{\mathbf{L}} \varphi$*, then* $\Pi(\psi) \leq \Pi(\varphi)$
5. *Disjunction:* $\Pi(\varphi \vee \psi) = max\{\Pi(\varphi), \Pi(\psi)\}$
6. *Metaconsistency:* $\Pi(\bullet\alpha) + \Pi(\circ\alpha) = 1$

Standard necessity and possibility measures do not cope well with contradictions, since they treat contradictions in a global form (even if in a gradual way). This is the main reason to define new forms of necessity and possibility measures based upon paraconsistent logics; although they lack graduality, **LFIs** offer a tool for handling contradictions in knowledge bases in a local form, by locating the contradictions on critical sentences. Yet, the combination of them reaches a good balance: the paraconsistent paradigm by itself does not allow for any fine-grained graduality in the treatment of contradictions, which may lead to some loss of information when contradictions appear in a knowledge base. When enriched with possibility and necessity functions, however, a new reasoning tool emerges.

It is possible to define a natural non-monotonic consequence relation on databases acting under some of the logic **L** as above. Non-monotonic logics are structurally closed to the internal reasoning of belief revision, as argued in (Gärdenfors 1990), where it is shown that the formal structures of the two theories are similar. The resulting logic systems have a great potential to be used in real-life knowledge representation and reasoning systems.

Another important concept that can be advantageously treated by the paraconsistent paradigm is the concept of evidence. The paper (Rodrigues, Bueno-Soler, and Carnielli 2020) introduces the logic of evidence and truth LET_F as an extension of the Belnap-Dunn four-valued logic *FDE*. LET_F is equipped with a classicality operator $\circ$ and its dual to non-classicality operator $\bullet$. It would be interesting to define possibility and necessity measures over LET_F, generalizing the probability measures defined over LET_F and to further investigate the connections between the formal notions of evidence and the graded notions of possibility and necessity.

7 Other applications and further work

Description Logics (DLs) play an important role in the semantic web domain and in connections to computational ontologies, and incorporating uncertainty in DL reasoning has been the topic of lively research. DLs can expanded with paraconsistent, probabilistic and possibilistic tools, or with their combinations (one example toward the relevance of paraconsistent reasoning for the Semantic Web can be found in (Zhang, Lin, and Wang 2010)). Enhancing DLs with **LFI**-probabilities and possibility measures is a research in progress, and will represent a considerable step forward to DLs in regard to the representation of more realistic ontologies.

A second problem concerns clarifying the concept of evidence. As mentioned, (Rodrigues, Bueno-Soler, and Carnielli 2020) introduces the logic of evidence and truth LET_F, a Logic of Formal Inconsistency and Undeterminedness that extends Belnap–Dunn four-valued logic, formalizes a notion of evidence as a concept weaker than truth in the sense that there may be evidence for a proposition α even if α is not true.

The paper proposes a probabilistic semantics for $\mathbf{LET}_F$ taking into account probabilistic and paracomplete scenarios (where, respectively, the sum or probabilities for α and $\neg\alpha$ is $P(\alpha) + P(\neg\alpha)$, is greater or less than 1). Classical reasoning can be recovered when consistency and inconsistency behave within normality, that is, then $P(\circ\alpha) = 1$ or $P(\bullet\alpha) = 0$. In this way it is possible to obtain some new versions of standard results of probability theory. By relating the concepts of evidence and coherence, it may be possible to obtain an enhanced version of the model proposed in (Chopra and Parikh 1999). This may represent an important leap forward into the clarification of the notion of evidence, each time more demanded in AI and KR.

Paraconsistent Bayesian networks is another topic with great interest. Bayesian Networks are indispensable tools for expressing the dependency among events and assigning probabilities to them, thus ascertaining the effect of changes of occurrence in one event given the others.

Bayesian Networks can be (roughly) represented as nodes an annotated acyclic graph (a set of direct edges between variables) that represents a joint (paraconsistent) probability distribution over a finite set of random variables $V = \{V_1 \cdots , V_n\}$. The praxis usually supposes that each variable has only a finite number of possible values (though this is not a mandatory restriction – numeric or continuous variables that take values from a set of continuous numbers can also be used.

For such discrete random variables, conditional probabilities are usually represented by a table containing the probability that a child node takes on each of the values, taking into account the combination of values of its parents, that is, to each variable V_i with parents $\{B_1, \cdots , B_{n_i}\}$ there is attached a conditional probability table relating V_i to its parents (regarded as "causes")

Paraconsistent Bayesian networks, notably when combined with paraconsistent belief revision (including (Testa, Coniglio, and Ribeiro 2017)) and with belief maintenance systems can lead to a new approach to detecting and handling contradictions, and producing explanations for its conclusions. This is naturally relevant, for instance, in medical diagnosis, natural language understanding, forensic sciences and other areas where evidence interpretation is an important issue.

Again, this is work in progress, but it seem clear that paraconsistent Bayesian networks may be useful and stimulating in a series of circumstances where contradictions are around.

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