

Modal Pluralism and Higher-Order Logic¹

In this article, we discuss a simple argument that modal metaphysics is misconceived, and responses to it. Unlike Quine's, this argument begins with the banal observation that there are different candidate interpretations of the predicate 'could have been the case'. This is analogous to the observation that there are different candidate interpretations of the predicate 'is a member of'. The argument then infers that the search for metaphysical necessities is misguided in much the way the 'set-theoretic pluralist' (Hamkins and Clarke-Doane [2017]) claims that the search for the true axioms of set theory is. We show that the obvious responses to this argument fail. However, a new response has emerged that purports to prove, from higher order logical principles, that metaphysical possibility is the broadest kind of possibility applying to propositions, and is to that extent special. We distill two lines of reasoning from the literature, and argue that their import depends on premises that a 'modal pluralist' should deny. Both presuppose that there is a unique typed hierarchy, which is what the modal pluralist, in the context of higher-order logic, should disavow. In other words, both presupposes that there is a unique candidate for what higher-order claims could mean.² We consider the worry that, in a higher-order setting, modal pluralism faces an insuperable problem of articulation, collapses into modal monism, is vulnerable to the Russell-Myhill paradox, or even contravenes the truism that there is a unique actual world, and argue that these worries are misplaced. We also sketch the

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² We will speak freely of 'typed hierarchies' as if they were things of type e , just as the set-theoretic pluralist speaks freely of 'universes', as if those were sets in V . But it is the multiplicity of candidate interpretations that matters.

bearing of the resulting ‘Higher Order Pluralism’ on the theory of content. One upshot is that, if Higher Order Pluralism is true, then there is no fixed metatheory from which to characterize higher order reality.

I. The Argument from Modal Pluralism

Modal metaphysics is the theory of *how the world could have been*. Could God have failed to exist, if God exists, in fact? Could the mind have been distinct from the body? Could you have had different parents? Quine dismissed modal metaphysics as misconceived [1952]. His criticisms are widely agreed to have turned on confusions -- between necessity and analyticity, and names and definite descriptions, for example. Once these confusions were resolved, there was little to which to object. So say the orthodox (Soames [2005, Part. VII], Williamson [2016]).

However, the orthodox have proceeded under an assumption of their own -- namely, that if modal metaphysics is misconceived, then this is because its questions are unintelligible.³ In recent years, this assumption has come under scrutiny (Cameron [2009], Sider [2011]). The problem that we wish to press (Clarke-Doane [2021]) is that there are a *plurality of candidate interpretations* of the expression ‘could have been’ giving intuitively opposite verdicts on paradigmatic questions of modal metaphysics (‘intuitively’ because of course the questions mean subtly different things under the different interpretations).⁴ All of these interpretations are *counterfactual* -- concerning how the world could have been, as opposed to epistemic (concerning how it might be, for all we know or believe), or deontic (concerning how it is

³ There is another assumption that the orthodox have made, namely, that modal claims purport to state mind-independent facts, or at least facts. See Sidelle [1989], and Blackburn [1986] and Thomasson [2020], respectively, for arguments against this.

⁴ Cameron’s argument requires that modal operators are quantifiers over Lewis’s ‘ersatz worlds’, while Sider’s argument turns on an alleged reduction of modality, and on the availability of a non-modal analysis of logical consequence. The problem that we will consider does not make any of these assumptions.

permissible for it to be, according to some set of norms).⁵ For instance, we can ask how, as a matter of physical possibility, the world could have been, or, apparently, how, as a matter of logical possibility (fixing on a logic), it could have been. The worry is: even if we can *also* ask how the world could have been in a distinctively *metaphysical* sense of ‘could have been’, what is there to learn from this exercise except how select academics use the phrase ‘could have been’?

Let us illustrate. A paradigmatic metaphysical necessity -- i.e., a necessity under the interpretation of ‘necessary’ on which modal metaphysics has focused -- is that you could not have had different parents (Kripke [1980, 113]). Let us grant, then, that the world could not, as a matter of metaphysical possibility, have been such that you had different parents. The problem is that you *could* have had different parents in a more inclusive, logical, sense of ‘could have’. The sentence ‘X’s parents are Donald and Melania Trump’ (where ‘X’ is your name) has a first-order model, after all (more on this below).⁶ So, you could not have had different parents in one sense of ‘could have’ and you could have in another. The question of whether you could have had different parents *period* looks worryingly like that of whether two lines making less than a 180° with a third must intersect, period -- i.e., that of whether the Parallel Postulate is true, understood as a question of pure mathematics.⁷ Yes, they must, as a matter of Euclidean geometry. But, no, they need not, as a matter of, say, hyperbolic geometry. *This* question is certainly misconceived!

⁵ However, there is no *single* sense of how the world could have been, if the pluralist view described below is true.

⁶ As will become clear, we do not have in mind a metalinguistic idea by ‘sense’, notwithstanding the reference to models.

⁷ Compare van Inwagen, who writes that if P is metaphysically possible, then it is possible “tout court. Possible simpliciter. Possible period...possib[le] without qualification [1997, 72].”

Indeed, there is a deep analogy between worries about modal metaphysics and worries about the foundations of mathematics (Clarke-Doane [2019], [2020, C.2], [In Progress]). The latter concerns itself with finding the true axioms of set theory (or a surrogate theory, like Category Theory or Type Theory). Is the Axiom of Choice true? Is the Continuum Hypothesis? What about ‘large’ large cardinal axioms like that of a Measurable Cardinal and beyond? The traditional critique of such questions is that, taken at face-value, they are simply unintelligible. ‘Infinitary’ mathematics makes no sense (Hilbert [1983/1926]). The best that we can do is to ask what follows from different sets of axioms that purport to speak of such objects as all uncountable subsets of the continuum. While this position does require that questions of arithmetic make sense (since questions of what does *not* follow from what are generally Π_1 undecidables, by Gödel’s Second Incompleteness Theorem), it is sometimes argued that the natural number structure is transparent in a way that uncountable structures are not (Feferman [1979, 70]).

However, as in the modal case, there is another critique of the foundations of mathematics that does not turn on considerations of intelligibility, an argument from pluralism. This is that there are a rich plurality of candidate interpretations of the predicate ‘is a member of’ giving intuitively different answers to such questions as ‘does every set have a Choice function?’, ‘is there a bijection between every uncountable subset of the real numbers and all of them?’, and so forth. While the answer might be (determinately) ‘yes’ under the interpretation that the set-theoretic community happens to have adopted, myriad other interpretations are available. According to many of these, the answer will be ‘no’.⁸ Even if such questions *make sense*, they

⁸ We do *not* just mean that one can find a (set) model of the negations of the original sentences, if one can find such models of the sentences! That follows from the Completeness Theorem. We mean that one can find an *intended* (class) model of them. For example, Quine’s *NF* is not just supposed by a pluralist to have an obviously unintended

may be misconceived in much the way that the Parallel Postulate question would be if we were so misguided as to ask it. All we would learn by resolving such questions is something about ourselves, rather than learning what set-theoretic reality contains (Field [1998], Clarke-Doane [2020, Ch. 6]. This is a kind of deflation by inflation. It is the *richness* of reality that undercuts the significance of the universe that happens to dominate the interests of set theorists today.⁹

It is important to emphasize that the pluralist critique does *not* stem from worries about the determinacy of reference.¹⁰ The worry is not that there is nothing in our practice or in the world that could make it the case that we mean *is a member of* by ‘is a member of’. (This was Putnam’s worry in his influential [1980].) That problem can be addressed by a theory of reference, as in Lewis [1983]. On the contrary, even if we all determinately mean sets, by ‘sets’, it would be enough to undercut the search for the true axioms of set theory that *there are* sets, just like sets, but satisfying different sentences -- whether or not anyone happens to refer to them. (This is why arguments that there is a serious question whether the Continuum Hypothesis is true, like Woodin [2010] or Koellner [2014], make no appeal to metasemantics.) The pluralist critique is instead like Einstein’s ‘critique’ of simultaneity. Even if we all refer to the same property with ‘simultaneous’—simultaneous-relative-to-reference-frame-R, say—this does nothing to vindicate the search for what is *really* simultaneous with what. This is so even if there is a mind-and-language independent answer to the question out of our mouths. The problem is

model in the *ZFC* sets (if *ZFC* $\dashv\vdash$ *Con(NF)*), but an intended model, of the sort that Gödel takes *ZFC* itself to have. More on this in Section V.

⁹ Of course, we recognize a use/mention distinction. The question “if you pass two straight lines through another, and on one side of the original line the sum of the angles that the two lines make with the original is less than 180° , must the two lines intersect on the side which sums to less than 180° ?” is not *about* ‘lines’. But, understood as pure mathematics, an answer to it would only inform us about ‘lines’. We already know what geometric spaces there are (among the candidates). We just learn which (classes) of them we are talking about.

¹⁰ We will make claims about the metasemantics of higher order quantifiers. But the argument from pluralism does not depend on them.

metaphysical. There are too many simultaneity-like relations in the neighborhood, giving intuitively opposite verdicts to our question (as is the case with the Parallel Postulate, understood as pure mathematics). Modal pluralism says something similar about ‘could have been the case’.¹¹

II. Responses

How do the orthodox respond to this simple ‘argument from modal pluralism’ (as we will call it)?¹² To the extent that they respond at all, they respond with the following. If a sense of ‘could have been’ is less inclusive than metaphysical possibility, then it is really just another way of talking about what could, as a matter of metaphysical possibility, have been the case had certain conditions obtained. For instance, physical possibility is just metaphysical possibility, given the laws of physics (Williamson [2016, 462]).¹³ The analogous response in the set-theoretic case is to say that talk of less inclusive notions of set is really just talk of the canonical notion of set, restricted somehow. The constructible sets (in the sense of Gödel [1940], for instance, are just (all of) the sets that are also definable in the language of ZF with parameters of a special kind.

What, though, if a sense of ‘could have been’ is *more* inclusive than metaphysical possibility? In that case, the orthodox tell us that it is not *alethic* [Hale, 2013], *real* [Rosen, 2002, 16], *ontic* [Kment, 2014, 31], or *objective* [Williamson, 2016, 459]. A sense of logical possibility according to which Hesperus could have failed to be identical to Phosphorus, for instance, is in

¹¹ We do not offer a full account of when one property is ‘in the neighborhood of’ another here. What will matter is that the candidate interpretations of higher order vocabulary are more like candidate interpretations of set theoretic vocabulary than they are like an interpretation according to which, say, ‘cat’ means dog.

¹² For a fuller treatment of the responses considered in this section, other than the one from higher order logic on which we will focus, see Clarke-Doane [2021].

¹³ That is, for a proposition p to be physically possible is for the proposition $T \& p$ to be metaphysically possible, where T is the conjunction of the laws of nature. There are objections even to this claim (see Fine [2002], but we will not pursue them here.

an important sense a merely *epistemic* or *verbal* kind of possibility. What do the italicized terms mean? Unfortunately, responders rarely tell us, and, when they claim to, their definitions do not rule out broader interpretations of ‘could have been the case’ than metaphysical possibility. For example, logical necessity is certainly alethic in that it may satisfy the axiom (T) $\Box P \rightarrow P$.¹⁴

Strohminger and Yli Vakkuri say that “the most straightforward way to characterize objective modality is negatively: it is what the modal words express when they are *not* used in any epistemic or deontic sense...” [2017, 825, emphasis in original]. But, again, according to this criterion, logical possibility is an objective modality. Williamson adds that objective modalities are also “not sensitive to the guises under which the objects, properties, relations and states of affairs at issue are presented” [2016, 454] and concludes that “identity [and distinctness are] simply objectively necessary...” [2016, 454].¹⁵ However, not even Williamson’s additions give the desired verdicts. Just consider a reading of ‘could have been’ that validates the Necessity of Identity and Distinctness, but according to which there could have been no mathematical entities, you could have had different parents, and so forth (see, e.g., Priest [2008, 16.2 & 16.3]).¹⁶

Indeed, the reader might wonder what is accomplished by such exercises in conceptual explication. Suppose we manage to define ‘objective’ (‘real’, ‘alethic’, etc.) so that it is true that

¹⁴ There is also the problem of what to say about senses of ‘could have been’ that are neither less inclusive nor more inclusive than metaphysical possibility. We will return to problems of incomparability below.

¹⁵ For a sketch of kinds of logical possibility according to which identities can fail see Priest [2008, Ch. 17], and the Appendix. (Did not Kripke prove that identities are necessary, appealing only to the idea of rigid designation [Kripke 1971, 181]? If saying that names are rigid designators is to say that, e.g., ‘Hesperus’ and ‘Phosphorus’ refer to what they actually refer to in every world, then showing that ‘Hesperus’ and ‘Phosphorus’ are rigid designators does not show that the terms *co-refer* at every world. It shows that ‘Hesperus’ refers to Hesperus, and that ‘Phosphorus’ refers to Phosphorus, in every world. If it means that ‘Hesperus’ and ‘Phosphorus’ co-refer in every world, if they do in the actual world, then Kripke assumes what he seeks to prove. See Cameron [2006].)

¹⁶ Williamson is explicit that the mathematical truths are necessary (as is Kripke (1980, 37). For instance, he says that “the structure of the hierarchy of pure sets...seems to be a metaphysically noncontingent matter” (2017, 199). See also his [2016, 454]. (Again, we do not have in mind a metalinguistic idea by ‘reading’. We clarify our meaning below.)

metaphysical possibility is the most inclusive objective interpretation of ‘possible’. The original problem would get transposed. What is special about that interpretation of ‘objective’? So it is not ‘objectively’ possible that there could have failed to be any numbers. Who cares, given that the world *could have been* that way (in some sense)? If we are interested in how the world could have been -- what we are calling *counterfactual possibility* -- then appeal to terms like ‘objective’ only seems to help if we can establish that the objective possibilities exhaust the counterfactual possibilities. (We will examine the prospects for establishing exactly this when we discuss purported proofs of the existence of a broadest kind of objective possibility below.)

A slightly better response appeals to Lewis’s idea of a natural kind -- a property that ‘carves nature at the joints’. As Sider [2011] points out, there is nothing to bar us from applying this idea to quantifiers, connectives, and operators. Perhaps, then, the orthodox could claim that modal metaphysics is not misconceived because it discovers how the world could have been under the most *natural* interpretation of ‘could have been’ [Nolan 2011, 322].¹⁷ But it is hard to think of a reason to regard metaphysical possibility as more natural than, say, logical possibility. Metaphysical possibility looks notoriously gerrymandered by comparison [Sider 2011, Ch. 12]!

Some try to argue that what ties the apparently disparate metaphysical necessities -- including that there are prime numbers, that you could not have had different parents, and that nothing can be two places at once -- together is that they are all ‘grounded in the nature of things’ (Hale [2013], Lowe [2012], Fine [1994], Kment [2014], where grounding and nature are hyperintensional phenomena on which we are supposed to have an independent grip. But, unless, *a la* Anselm, things can exist *of their very natures*, it could at most be grounded in the

¹⁷ Nolan himself seems only to think that the relevant interpretation may be one of many natural interpretations.

nature of things that *if* there are numbers, *then*, e.g., there are infinitely-many prime ones (Kment [2006, 267]). It could not be grounded in the nature of numbers *that* there are infinitely-many prime numbers. Moreover, someone worried that there is nothing special about our interpretation of ‘could have been different’ should just be worried that there is nothing special about our interpretation of ‘nature’ (Clarke-Doane, [2019, Sec. 6]). Along with nature, let us introduce shnature. Even if it is, say, part of your nature that you have the parents that you have, it is no part of your shnature. Our question rearises. Why are not disagreements about *nature* like disagreements about the Parallel Postulate (understood as a pure mathematical conjecture)?¹⁸

The most promising response to the argument from modal pluralism of which we are aware is that any interpretation of ‘could have been’ that is more inclusive than metaphysical possibility must apply to *sentences*, not propositions. (This is one way of giving content to the response that more inclusive interpretations are not real, ontic, objective, etc.) Did we not betray as much when we said that ‘X’s parents are Donald and Melania Trump’ has a model? How the world could have been, understood as the question of which sentences have models (or which sentences have true substitution instances relative to some class of substitutions) is different than how the

¹⁸ It is tempting to claim that the nature of things is special as compared to their shnature because we care about *them!* The problem is that we can all agree on what worlds there are (for the purposes of the critique). We just disagree about what to count as ‘the same thing’ across them -- e.g., about whether to call a world with a lectern just like this one but made of ice *this lectern*. It is hard to see what factual (rather than practical) question could be at stake, if not just one of natural language semantics. See Unger [2014, Ch. 4] for a worry along these lines. Also recall Quine’s complaint: “When modal logic has been paraphrased in terms of such notions as possible world or rigid designator, where the displaced fog settles is on the question when to identify objects between worlds, or when to treat a designator as rigid, or where to attribute meta-physical necessity [Quine 1972, 492-493].” (Note that the ‘shnature’ difficulty is *not* generic, applicable to disagreements in physics, for example. Bracketing grue-like predicates, being a shgraviton is not instanced. By contrast, shnatures are instanced if natures are, where nature and shnature may be much alike, and comparably natural, except that, for example, it is no part of your shnature that you have the parents that you have. One might respond that, even if the shnature difficulty does not arise in physics, it arises in high-level sciences. Indeed, we believe that this is roughly correct. The question of how to carve things up in biology or psychiatry, for example, is practical, depending on our purposes, in a way that it does not in physics.)

world could have been, interpreted at face-value. It is like the difference between the hypothesis that the Parallel Postulate is true in a structure, and the claim that it is true of physical spacetime.

This rejoinder does assume that there is a determinate question as to what propositions, or states of affairs, there are ‘up there’, as it were, and which ones are identical with which. Although one could certainly have doubts about this, we would like to examine what happens if one grants the assumption.¹⁹ Then, if one could argue, from premises that even a ‘modal pluralist’ should accept, that metaphysical possibility is the most inclusive kind of counterfactual possibility applying to propositions, the pluralist critique would arguably be answered. The answer would be much the same as the answer the Gödelian gives to the objection: why search for the size of the continuum when $ZF(C) + V = L$ already proves that the Continuum Hypothesis is true? The answer is that, assuming a Measurable Cardinal, $V = L$ is a *proper* inner model of the universe of all sets.

The caveat ‘arguably’ is needed because even if metaphysical possibility is the broadest kind of counterfactual possibility, it could be *indefinitely extensible* -- just as the cumulative hierarchical conception of set might be the broadest conception while being indefinitely extensible [Shapiro and Wright 2006].²⁰ In that case, there would not be a definite collection of all propositions over which metaphysical possibility operates [Rayo Manuscript], and a version of the pluralist critique would survive. There would be no unique *stage*, as it were, at which metaphysical

¹⁹ At least provisionally. It will emerge that part and parcel to ‘Higher Order Pluralism’ is a higher-order quantifier variantism. (Again, this is *not* to grant that there is a *unique* such question, just as granting that there is a determinate question of what is simultaneous with what is not to grant that there is a unique such question.)

²⁰ Although Shapiro and Wright focus on extensions of the ‘height’ of the cumulative hierarchy, it can also be argued that it is indefinitely extensible by ‘width’ using forcing. In the simplest case, a generic extension, $M[G]$, of a model, M , may add κ -many new subsets of ω to M , without adding (or subtracting) any ordinals (or collapsing cardinals), resulting in a ‘wider’ model of the same height.

possibility has been extended a determinate amount, and is broadest as compared to it at any other stage. This might vindicate the import of questions that are ‘absolute’ as the hierarchy expands. But it would undercut hope of pinning down that hierarchy. We discuss indeterminacy about the class of propositions, and a more radical kind of pluralism about that class, below.

III. Higher-Order Monism: Two Variations

Various authors have argued, or contributed to arguments, that metaphysical possibility is the most inclusive kind of counterfactual possibility applying to propositions (we will not consistently add the qualification ‘counterfactual’ in what follows).²¹ First, various authors have argued that *there is* a broadest, or most inclusive, kind of possibility that operates on propositions.²² Second, some have gone on to identify the broadest kind of possibility with metaphysical possibility.²³ We focus on the first thesis since the second is often (though by no means always) assumed as a matter of definition (Kripke [1980, 99], Stalnaker [2003, 203], Lewis [1986], Williamson [2016, 459]).²⁴ It is worth noting, however, that if ‘metaphysical possibility’ is defined as the broadest kind of possibility, then it could turn out that traditional modal questions are misconceived in another way. It could be that virtually nothing of metaphysical interest is metaphysically necessary (Clarke-Doane [2019], Mortensen [1989], Nolan [2011]).

²¹ Indeed, some of the relevant authors do not seem to recognize a difference between counterfactual kinds of possibility and, say, deontic kinds. This is another obscurity afflicting the arguments that we will not pursue.

²² See Bacon [2018], Bacon and Zeng [Forthcoming], Williamson [2013], Dorr [2016], and Yli-Vakkuri and Goodsell [Manuscript].

²³ See Williamson [2013], Dorr, Hawthorne and Yli-Vakkuri [Forthcoming], and Yli-vakkuri and Goodsell [Manuscript].

²⁴ This is another juncture at which verbal disagreement threatens. Bacon [2020], Nolan [2011], and Mallozzi [2019] are adamant that metaphysical possibility is *not* the broadest kind of possibility! (Another reason we focus on the first claim is that its failure would seem to represent a more serious threat to metaphysics as practiced than the mere failure of the second.)

The arguments that there is a broadest kind of possibility in question takes place in the framework of higher order logic. Higher order languages offer a natural tool for investigating counterfactual possibility (for those -- non-Quineans! -- who have no qualms about higher-order entities). In modal logic, a kind of necessity (or possibility) is typically regimented with a sentential operator representing a phrase like *it is necessary that* (or *it is possible that*). A language with a particular sentential operator is suitable for articulating the theory of a particular kind of necessity. But in order to reason about kinds of necessity, it is useful to be able to quantify into the position of the operators, and to employ expressions with more complicated types, such as expressions which combine with operators to form sentences. These include operator identity statements, and statements in which operators are predicated with features like *being a necessity*.

The new arguments infer from allegedly natural logical or modal axioms that there is a broadest kind of possibility. Let us sketch two variations. The first begins with the assumption that there is one true higher order logic, and it is (an extension of) H_C , the logic Bacon and Dorr call *Classicism*.²⁵ This extends the commonly accepted classical core logic H_0 ²⁶ with, what Bacon

²⁵ The same kind of reasoning applies if the one true higher order logic is assumed to be merely (an extension of) *Booleanism*, which is a light weakening of *Classicism*.

²⁶ H_0 has the following axioms and rules:

PC: All theorems of propositional calculus

UI: $\forall \sigma F \rightarrow F a$

βE : $(\lambda x_1 \dots x_n. A) N_1 \dots N_n \leftrightarrow A[N_1/x_1, \dots, N_n/x_n]$

MP: If $\vdash A \rightarrow B$ and $\vdash A$, then $\vdash B$

Gen: If $\vdash A \rightarrow F x$, then $\vdash A \rightarrow \forall \sigma F$, provided x is not free in A

Ref: $M =_{\sigma} M$

LL: $M =_{\sigma} N \rightarrow A[M/x] \rightarrow A[N/x]$.

and Dorr dubb the Boolean and Adjunctive identities.^{27 28} Higher order theories like Classicism are theories of the so-called ‘fineness of grain’ of higher order domains, including the domain of propositions, and of propositional operators. According to Classicism, the domain of propositions forms a Boolean Algebra. So, propositions p and q are identical in this framework when the proposition that $p \leftrightarrow q$ is identical to the top proposition. Also, the grain and algebraic structure of propositions determine that of other higher order domains.²⁹ Consequently, these domains form Boolean Algebras as well. If one is a ‘monist’ about higher-order entities (more on this below), then one can hope that Classicism gives the correct theory of how fine-grained those entities are.

Suppose now that the Classicist accepts the following definitions. An operator X is a weak necessity just in case (i) X applies to tautologies -- XT , where T is the top proposition; and (ii) X is closed under modus ponens -- $(\forall p)(\forall q)(Xp \rightarrow X(p \rightarrow q) \rightarrow Xq)$. An operator X is a kind of *necessity* just in case for every weak necessity Y , it is Y -necessary that X is a weak necessity. A necessity X is *at least as strict as* another necessity O just in case $Y(Xp \rightarrow Op)$, whenever Y is a kind of necessity. X is the strictest kind of necessity just in case it is at least as strict as every other kind, and no other kind is at least as strict as it. An operator is a kind of *possibility* just in case it is the dual of a kind of necessity. Finally, the *broadest* possibility is a kind of possibility Y such that (i) every proposition that has any kind of possibility has Y , and (ii) it is necessary,

²⁷ H_C is the logic which results from replacing βE in H_0 with $\beta\eta$: $A \leftrightarrow B$ whenever A and B are $\beta\eta$ -equivalent; and by closing under the rules E : If $\ulcorner A \leftrightarrow B$, then $\ulcorner A =_t B$, and ζ : If $\ulcorner Mx =_\tau Nx$, then $\ulcorner M =_\sigma \rightarrow_\tau N$.

²⁸ One gets *Booleanism* by just adding the Boolean, and not the Adjunctive, identities.

²⁹ Provided one only admits relation types into one’s hierarchy.

with the force of every necessity, that (i) holds.³⁰ Given these definitions, the Classicist can prove that there is broadest kind of possibility (formally, $\lambda p(p \neq F)$, where F is the dual of T).³¹

What of the second argument? Recall that in propositional and first-order settings one can extend extensional logic to modal logic by adding a modal sentential operator. One can do something analogous in a higher order setting. One can extend the language of simple-type theory, for instance, by adding a non-logical constant of type $t \rightarrow t$, which is interpreted as a modal operator. Then one can theorize about a specific kind of modality. But one can also extend the language by adding a predicate of operators Nec , of type $(t \rightarrow t) \rightarrow t$, which is interpreted as *being a necessity operator*. This lets one theorize about the kinds of necessity.

We will call such theories about kinds of necessity *modality theories*. An illustrative example is TN , due to [Bacon and Zeng Forthcoming]. This is a theory in the language of simply-typed functional type theory, augmented with a predicate of operators Nec . The theory results from adding two axioms to H_0 , and closing the theory under a rule of necessitation. The first axiom is the axiom of *Necessity*. The idea is that an operator X is a necessity operator when X is *Closed* -- meaning that it satisfies the K axiom of modal logic with the force of every kind of necessity, and when X is *Logical* -- meaning that X obeys a principle analogous to the rule of necessitation

³⁰ These definitions were suggested by Andrew Bacon in personal correspondence.

³¹ Here is the idea. It is part of the definition of a kind of possibility that F does not have any kinds of possibility. So, consider the operator *being distinct from F*. This only applies to every proposition other than F . If one could prove in Classicism that it is a kind of possibility, and that it is necessarily a possibility, for every kind of necessity, then one could prove that being distinct from F is at least as broad as any other broadest kind of possibility. Moreover, given that Classicism proves that kinds of possibility that are exactly as broad as each other are identical, one could prove that it is the broadest kind of possibility. (Bacon establishes this in [2018].) So, relative to these definitions, one could prove in higher-order logic that *being distinct from F* is the broadest kind of possibility. (Bacon actually uses a slight strengthening of H_C , what he calls *HFE*. But he notes that one could carry the proof out in H_C .)

of modal logic. The second is the axiom of *L-Necessity*. This says that $L := \lambda p. \forall X(\text{Nec } X \rightarrow Xp)$ -- the operator of being necessary in every sense of necessary -- is itself a kind of necessity.³²

These axioms are said to express features of the class of all necessity operators. They give a theory of the class of modal operators, as the *ZFC* axioms give a theory of the cumulative hierarchy of sets. We can now argue that, according to this theory, there is a broadest kind of possibility.³³ According to the theory, given any collection of possibility operators, there is a

³² Here are the details:

(i) $K := \lambda X. \forall pq(X(p \rightarrow q) \rightarrow Xp \rightarrow Xq)$ -- An operator X has K just in case it satisfies the K axiom of modal logic, which is just to say that it is closed under Modus Ponens

(ii) $L := \lambda p. \forall X(\text{Nec } X \rightarrow Xp)$ -- A proposition has L just in case it has every kind of necessity; if it is necessary in every sense of necessity.

(iii) *Closed* $:= \lambda X.(KX \wedge LKX)$ -- An operator is Closed just in case it is closed under modus ponens, and that it is closed under modus ponens is necessarily the case for every kind of necessity.

(iv) $N := \lambda X. \forall p(Lp \rightarrow LXp)$ -- An operator X has N just in case if some proposition p is necessary in every sense, then Xp is also necessary in every sense.

(v) *Logical* $:= \lambda X.(NX \wedge LNX)$ -- An operator X is logical just in case it has N , and that it has N is necessary for every kind of necessity.

(vi) $L_W := \lambda p. \forall X(W X \rightarrow Xp)$ -- One can think of L_W as the the collection of all propositions which have all of the operators which have W . So a proposition has L_W just in case it has all of the operators which have W . One can think of L_W as the operator of possessing all of the W -operators.

(1) Necessity: $\text{Nec } X \leftrightarrow \text{Logical } X \wedge \text{Closed } X$ -- The necessity operators are exactly those operators which are (i) closed under modus ponens with the force of every necessity, and (ii) are closed under necessitation for every kind of necessity with the force of every necessity.

(2) L-Necessity: $\text{Nec } L$ -- Being necessary in every sense of necessity is itself a kind of necessity.

(3) Necessitation: If TN proves p , then TN proves $\text{Nec } X \rightarrow Xp$ -- This is just a natural generalization of the familiar rule of necessitation from modal logic. TN proves that L is a kind of necessity. Indeed it proves that L is as broad as every kind of necessity. And so if TN proves p , then it proves Lp .

(4) TN is $H_0 \oplus \text{Necessity} \oplus \text{L-Necessity} \oplus \text{Necessitation}$.

³³ More precisely, according to the theory there are modal operators that are as broad as every kind of necessity, with the force of every necessity, with the force of every necessity, See Bacon and Zeng [Forthcoming] for details.

possibility operator which is the disjunction of these possibility operators, and so, necessarily, in every sense of *necessary*, applies to a proposition just in case some operator in the collection does. This ensures that there is a kind of possibility which is as *broad* as every actual kind of possibility. All that's left are merely possible kinds of possibility. As to these, the theory proves that it is impossible, in every sense of *possible*, that there could be a kind of possibility that is broader than every actual kind of possibility. This ensures that no merely possible kind of possibility is broader than any actual kind of possibility. The upshot is that there is a kind of possibility which is as broad as every kind of possibility, and every merely possible kind of possibility, and every merely possible merely possible kind of possibility, and so on. This argument does not build in assumptions about the grain of propositions. The conclusion that there is a broadest possibility follows just from the axioms about the class of modal operators.

If either of these arguments (or some variation on one) succeeds, then the argument from modal pluralism would seem to fail. There would be a broadest kind of propositional possibility. As the broadest, it would be special in the same way that a broadest kind of set (Gödel [1947]) would be. If this is metaphysical possibility, as it is commonly argued to be,³⁴ then modal metaphysics would be in good order -- at least as far the simple argument from modal pluralism is concerned.

IV. Assessment of the Arguments

Do these new arguments respond to the argument from modal pluralism? *Prima facie*, they do. There are two components to the first. First, there is the thesis that Classicism is the correct higher order logic. Although there are objections to Classicism (Goodman [2019], Dorr [2016]), it turns out that, for a range of higher-order logics, one can prove that there is a broadest kind of

³⁴ Ironically, Bacon and Zeng are not among those who accept this argument!

possibility.³⁵ So, not much seems to turn on this component of the first argument. The second component concerns the definitions of possibility and broadest possibility. Although one can quibble about these, many metaphysicians suppose that there are features that all kinds of possibility share, and that the theorems of the background logic are necessary. So, they grant that there are necessary conditions for being a kind of possibility. One could still doubt that the class of kinds of possibility can be pinned down. Maybe the specifiable classes also contain propositional operators that are not kinds of possibility. However, not much seems to turn on this either. It might still be provable that the broadest element of the class is a kind of possibility.

The second argument, from modality theory, also appears responsive. There are two features of that theory which undergird the broadest possibility. The first is that, given some kinds of possibility, there is a kind of possibility which is *having one of those kinds of possibility*. Kinds of possibility would have to be sparse for this to fail. The second feature is that it is impossible, in every sense of *possible*, for there to be a kind of possibility that is broader than every actual kind of possibility. Although this might seem doubtful, one can make a surprisingly robust case for it. The gist of the case is that one can collect all of the kinds of possibility, the merely possible kinds of possibility, the merely possible merely possible kinds of possibility, and so on

³⁵ In many (and perhaps even all) of the currently popular higher-order logics (setting aside H_0) one can prove this. Many of the currently popular logics are extensions of Classicism, and so prove everything that Classicism proves. And many of the other logics that have been considered, which are weaker than Classicism, such as Goodman's [2019] 'Agglomerative' logic, and the extended version of Dorr's [2016] 'Only Logical Circles' logic, also prove that there is a broadest kind of possibility. This suggests that a logic would need to be quite weak in order *not* to prove that there is a broadest kind of possibility. So, it may be difficult to avoid a logic in which one can prove that there is a broadest kind of possibility when trying to systematically theorize in a higher order framework (Dorr, Hawthorne and Yli-Vakkuri [Forthcoming]. (That said, one *can* construct a weak extension of H_0 in which one can capture some distinctive features of modal pluralism as described below, such as the idea that the notion of absolute possibility is indefinitely extensible.)

for a countable number of stages, and define a broadest kind of possibility in terms of all of these kinds.³⁶

On closer inspection, however, both arguments ring hollow. Their force turns on a kind of ‘higher order monism’ which a modal pluralist may well deny. At first pass: it turns on the assumption that there is a unique hierarchy of typed functions based on a unique collection of individuals and collection of propositions.³⁷ Every propositional operator is supposed to lie in this hierarchy, as are all properties of propositional operators, properties of those, and so on. In particular, all *modal operators* are supposed to lie there. (This is a ‘first pass’ because it abuses quantifiers, speaking as though hierarchies were things of type *e*. We will continue to speak in this way throughout, just as set-theoretic pluralists speak of universes of sets as though they were objects living inside of *V*. But, again, we do not really believe that a hierarchy is an entity of type *e*, any more than the set theoretic pluralist really believes that an intended (class) model of a given set theory is a set that satisfies some such theory³⁸.) So, dropping talk of hierarchies, the monist assumption is that *there is a unique candidate for what higher-order claims could mean*.³⁹

³⁶ Let us say that *Extensibilism* is the view that even though there is a kind of possibility that is as broad as every actual kind of possibility, it is possible in some sense of *possible*, that there could be a kind of possibility that is broader than every actual kind. (Roughly, there is a merely possible kind of possibility that is broader than any actual kind.) The problem with *Extensibilism* is that one can define an operator, in the language of *TN*, that applies to a proposition just in case any of the kinds of possibility do, or any of the merely possible kinds of possibility do, or any of the merely possible merely possible kinds of possibility do, and so on. This operator is at least as broad as any kind of possibility that appears anywhere in the *Extensibilist* modal hierarchy. And there would seem to be a case for its being a kind of possibility. If it were, then this operator would be the broadest kind of possibility. (See Bacon and Zeng [Manuscript] for more on this argument. They call the operator *L**.) However, one could respond to this argument that *Extensibilism* does not go far enough to capture the view that kinds of possibility are indefinitely extensible. To really capture the view one would need to extend the iteration process *indefinitely*, not just for a countable number of stages. If one were to do this, however, then one could no longer define an operator in *TN* in terms of all of the ‘merely possible kinds of possibility’.

³⁷ By ‘functions’ we do not mean first-order functions (which are commonly understood to be sets). Nor do we assume that the higher order entities in all higher order hierarchies are functional, in the sense of being identical just in case they are coextensive. We use the term ‘function’ in a thin sense, to mean something which takes something to something else. We build no assumptions about grain or type into this notion of a function.

³⁸ Bracketing the case of set theories with a universal set!

³⁹ By ‘higher order claims’ we will principally, but not exclusively, have in mind those that only use the standard logical constants. (A different way to put the presupposition might be that there is ‘one true higher-order logic’.

Let us call this assumption *Higher Order Monism*, since there is an obvious analogy between Higher Order Monism and monism about set theory, alluded to above [Gödel 1947].⁴⁰ Whereas set-theoretic monism is the view that there is one hierarchy of sets, Higher Order Monism is the view that there is a unique typed hierarchy of entities based on one collection of individuals and one collection of propositions. Monism is seemingly ubiquitous in the work of higher-order metaphysicians. However, if we are right, Higher Order Monism is the key point of contention.

Let us see how the assumption of Higher Order Monism works in the two arguments. Higher Order Monism says that there is a unique hierarchy of higher order entities which is correctly described by some particular higher order logic. The first argument shows that if Classicism (or some other sufficiently strong logic) correctly describes this hierarchy, then there is a broadest kind of possibility. But what if there are (so to speak) many hierarchies, each containing entities of different grain, which are correctly described by different higher order logics? In other words, what if there are many different candidate interpretations of higher order claims, just like the set-theoretic pluralist says that there are many different candidate interpretations of set-theoretic claims? If such a *Higher Order Pluralism* is true, then there are many hierarchies with various kinds of proposition, propositional operator, properties of propositional operators based on these, and so on, each *correctly* described by a *different* higher order theory -- just like the set-theory

This last statement is natural insofar as a system of higher type theory is a *logic*, unlike a system of set theory, and monists about first-order logic routinely state their view by saying that there is one true logic. But we find this language obscure [Clarke-Doane, In Progress, 4.5]. It is unnatural insofar as higher type theory lacks typical features of logic, such as completeness (with respect to the full semantics, if the proof system is sound), so is, in this way, more like set theory. The fact that it seems no easier to explain our reliability with respect to higher-order truths than with respect to set-theoretic ones is one reason to doubt the import of whether higher type theory counts as logic. See Clarke-Doane [2020, Ch. 5].)

⁴⁰ This is especially clear given that type theory is itself occasionally advocated as a foundation for mathematics, like set theory. See Field [1998], Koellner [2014], Martin [1998] & [2001] and Woodin [2010] for contemporary discussion of monism about set theory.

case. In this scenario, *the Classicist argument at most establishes that, within a fixed class of typed hierarchies, there is a broadest kind of possibility*. It does not establish that in *all* typed hierarchies there is. The argument shows nothing about hierarchies that are not Classicist. The hypothesis that there is a broadest necessity could be true in some hierarchies, but false in others.

The problem with the second argument is similar. It seeks to establish that *TN* is true in *the* typed hierarchy (or under the unique candidate interpretation of the higher-order language). Higher Order Pluralism says that there *is* no unique hierarchy, just as set-theoretic pluralism says that there is no unique universe of sets. So, *TN* may be true in some of these hierarchies, but false in others. Since it does not make assumptions about the grain of propositions, it may be true in a larger portion of the ‘pluriverse’ of hierarchies than Classicism. But there may be hierarchies in which *TN* is false, either because the underlying logic H_o is, or because some of the modal axioms of *TN* are. For instance, it is false in a hierarchy in which the axiom *Necessity* fails because there are operators that have *Nec*, but do not satisfy the *K* axiom of modal logic, and in hierarchies in which the principle of *Extensibility*, considered by Roberts, holds (Roberts [Manuscript]).⁴¹ In the latter kind, there is not a broadest kind of possibility. So, like the first argument, the second at most shows that *in TN hierarchies* there is a broadest kind of possibility.

The import of the two arguments for the existence of a broadest kind of possibility is thus bound up with a monistic conception of the ‘type hierarchy’ (or, more carefully, the interpretation of the higher-order language). Why would this be objectionable to someone moved by the simple argument from modal pluralism with which we began? Because modal pluralism was specifically motivated by analogy to set-theoretic pluralism. It was motivated by analogy to a

⁴¹ Roberts himself does not ultimately endorse this principle.

view about set theory that is the opposite of Higher Order Monism. There are a rich plurality of candidate interpretations of the predicate ‘is a member of’, reflecting different universes of set. Why privilege some one interpretation of ‘is a member of’, but no one interpretation of, say, geometrical terms like ‘point’ and ‘line’? Similar reasoning applies in the modal case. There are a rich plurality of candidate interpretations of the predicate ‘is possible’ reflecting, *in this higher order setting*, different typed hierarchies (or different candidate interpretations of the higher order language). Why privilege one interpretation of this language over others?⁴²

To repeat: we are not saying that the arguments are bound up with the vapid thesis that there are alternative interpretations of higher order language (any more than the set-theoretic pluralist is saying that there are alternative interpretations of ‘is a member of’). The view is not that if a sentence and its negation both have models, then a debate over what it expresses is misconceived! (That would make debate over dark matter misconceived.) Again, the problem in the set-theoretic case is that, if pluralism is true, and CH is true in ‘the’ universe of sets, V , then there is a (class) model ‘just like’ V except $\sim CH$ is true there -- as in the geometric case, with Euclidean and hyperbolic space, say. Likewise, even if we all refer to simultaneity-relative-to-reference-frame- R with ‘simultaneous’, the *metaphysical* fact of Special Relativity that there are myriad simultaneity-like relations (corresponding to all spacelike

⁴² To be clear: none of this shows that the arguments in question are *invalid*. The set-theoretic pluralist critique does not *invalidate* arguments for a broadest concept of set (to be discussed) either. It invites a reconception of their significance. Martin emphasizes that his influential [2001] argument does not have the significance that one might hope that it did, if pluralism about the concept of well-ordering is true. Likewise, we are not saying that modal pluralism is *incompatible* with Higher Order Monism. There is a ‘boring’ version of the view, on analogy with a boring version of set-theoretic pluralism (both discussed in the next section), that a Higher Order Monist could adopt. However, as we will see, Higher Order Pluralism is the natural complement to a more authentic kind of set-theoretic pluralism. Indeed, monism about second order logic already engenders monism about the Continuum Hypothesis! (There is a second-order sentence with a model just in case CH holds, and a sentence with no model just in case it does not.) This brings to mind Woodin’s quip: “[One could always rejoin that] instead of sets we should be studying widgets...[T]he axioms for widgets are obvious and, more-over...resolve the Continuum Hypothesis [2001, 690].”

hypersurfaces) that we could have picked out instead with ‘simultaneous’ is enough to show that questions of what is *really* simultaneous with what is misconceived. (Of course, there may be situations where it matters what is simultaneous-relative-to-reference-frame-R, just as there are situations where it matters what is true in a particular model of set theory. The point is that those are not in general the situations where it was thought to matter what is simultaneous, or what is true in ‘the’ universe of sets, V , period.) The pluralist critique of modal metaphysics is like this.

It is also important to emphasize that a pluralist need not deny that we take *interest* in some interpretations of the higher order language over others. We also take interest in some interpretations of ‘point’ and ‘line’ over others (e.g., those that are useful for modeling physical spacetime). But no mathematical realist would suggest that some interpretations of ‘point’ and ‘line’, construed as pure mathematical entities, are thereby more real than others. There exist hyperbolic points and lines if there exist Euclidean ones. And so on for arbitrarily curved Riemannian (and pseudo-Riemannian) varieties. The set theoretic pluralist says the same thing about sets. And the higher-order modal pluralist says the same thing about higher-order entities.

We have been illustrating how analogies between a monistic conception of the type hierarchy and a monistic conception of sets tend to undercut the import of arguments for a broadest kind of possibility. But Higher Order Monism is in a way even *worse off* than monism about set theory. There is a still more intimate analogy between modal pluralism and a mundane kind of *logical* pluralism. According to this, there is a relation of, say, intuitionistic validity in whatever sense there is a relation of classical validity. This is independent of whether, as a matter of natural language semantics, intuitionistic consequence satisfies the criteria we associate with the word

‘consequence’ (so is independent of whether pluralism in the sense of Beal and Restall [2005] is true). It is also independent of whether we ought to reason using intuitionistic logic. It is a claim of pure metaphysics whose negation is hard to even understand. But logical notions are naturally understood modally, indeed counterfactually. To say that it is logically possible that P is just to say that “things might (logically)...have been” such that P [Rumfitt [2010, fn. 21].⁴³ Moreover, the predicate ‘might logically have been’ admits of myriad interpretations, just like the predicate ‘is a member of’! So, we really have a schema, giving different kinds of logical possibility, depending on the logic. For appropriate substitutions (and refinements),⁴⁴ we get classical possibility, intuitionistic possibility, *LP* possibility, quantum possibility, *FDE* possibility, and so on. Where does it end? There appears to be no principled “place to *stop* the process of generalisation and broadening” the generic notion of logical possibility -- at least among kinds that we can actually *use* in a sustained way [Beall and Restall, 2006, 92, italics in original]. If so, and if logical notions are counterfactual modal notions, then, trivially, there is no broadest kind of possibility. However, even if logical notions are not themselves modal, the intimate relation between logical and modal notions should give a Higher Order Monist pause (and the fact that higher type theory, unlike set theory, is *itself* a kind of logic should give her even longer pause). Why would there be a broadest kind of possibility, but not a most inclusive kind of consistency?

The take-home message is that the two arguments for the existence of a broadest kind of possibility have bite only if one assumes Higher Order Monism. But this is exactly what, in this higher-order setting, a modal pluralist should deny. The arguments at most establish conclusions about *particular kinds* of higher-order entity. They are not even set up to make claims about a

⁴³ Validity is often taken to be a property of sentences. But this is by no means universal.

⁴⁴ We need to decide what counts as a designated value when propositions are allowed to be other than true or false, for instance.

‘pluriverse’ of them, if there is one. So, contrary to our initial assessment, the new response from higher-order logic to the argument from modal pluralism actually appears toothless.

V. What is Higher Order Pluralism?

We have argued that the new responses to the argument from modal pluralism from higher order logic do not stand up to scrutiny. Their significance depends on a kind of Higher Order Monism, which a modal pluralist, in the context of higher order type theory, will reject. What, though, is the alternative? What *is* Higher Order Pluralism? What could it mean to say that ‘there are many typed hierarchies’ -- or, more carefully, that ‘there are many candidate interpretations of higher order claims’? In what language, and against what background theory, do we say this?⁴⁵

It is tempting to respond by simply constructing a higher order theory. We want to formulate pluralism about propositions, propositional operators, properties of propositional operators, and so on. So, we might formulate pluralism about propositions by saying that there are many different (unrestricted) proposition quantifiers, pluralism about about propositional operators by saying that there are many different propositional operator quantifiers, and, in general, pluralism about entities of type σ by saying that there are many different (unrestricted) quantifiers of type $(\sigma \rightarrow t) \rightarrow t$. For example, we might formulate the existence of Fregean propositions by writing: $\forall_t^1 p, q (p \leftrightarrow q) \rightarrow p =_1 q$, and we might formulate the existence of some non-Fregean kind of proposition by writing: $\neg(\forall_t^2 p, q (p \leftrightarrow q) \rightarrow p =_2 q)$. We might then assert our general propositional pluralism by introducing the predicate *is a propositional quantifier*, and maintaining that there are different entities that satisfy this predicate. We could then hope to prove that the conjunction of these claims in $H_1 + H_2 + \dots + H_n$ (the axioms and rules of H stated

⁴⁵ Thanks especially to Andrew Bacon and Alex Roberts for pressing us on this, and for the criticisms to follow.

for all of the sets of quantifiers) is consistent with whatever non-modal truths that we wish to be monists about (surely there are some!). Such a proof would proceed as in the set-theoretic case.

An initial worry with this is that the procedure would need to be generalized in a way that it need not be in the set-theoretic case. One can express set theoretic pluralism ‘all at once’ by quantifying into binary predicate-of-individuals position. Nothing similar is possible in the higher order case -- no quantifier could quantify over all types (although a quantifier might range over everything of finite order). Actually, we see no bar to introducing a cumulative quantifier, or a countable infinity of finite quantifiers at once. But the deeper point is that the disanalogy is premised on monism about predicate-of-individual quantifiers. This is something that no set theoretic pluralist should accept. Monism about second-order quantification is tantamount to monism about ‘all subsets’! So, not even a set-theoretic pluralist can adequately express their view in the envisioned way. A pluralist, whether higher order or set-theoretic, can only ever write down a provisional theory of their pluriverse. As we will see, this is integral to pluralism.⁴⁶

The real problem with the envisioned formulation of pluralism, whether higher order or set-theoretic, is that the pluralist needs distinct unrestricted quantifiers. Otherwise, they are vulnerable to a higher order variant of the ‘collapse argument’.⁴⁷ The Higher Order Pluralist

⁴⁶ Much less radical kinds of pluralism about propositions do not encounter the same problem of articulation. Fritz [Forthcoming], for instance, admits two kinds of proposition. But the resulting view is monist about higher order logic (with a fixed logic of propositions). It simply admits of fine grained proxies for propositions over which fine grained proxies for kinds of possibility could operate. (This is related to a version of set-theoretic pluralism according to which there is a background concept of set with respect to which all others are restrictions. See below.)

⁴⁷ Thanks to Alex Roberts for formulating the following simple pluralist theory, as well as the collapse argument in personal communication. Consider a very weak pluralist theory, according to which, putting it informally, there are just two type hierarchies: one in which entities of all types are individuated by co-extensiveness and one in which entities of all types are individuated by the principles of Classicism (where the type hierarchies are ‘pure’ in the sense that the only basic type is t , the type of formulas). The signature for the theory contains the material conditional $\rightarrow: t \rightarrow (t \rightarrow t)$, negation $\sim: t \rightarrow t$ and two kinds of universal quantifier for each type σ , $\forall(\sigma)_1: (\sigma \rightarrow t) \rightarrow t$ and $\forall(\sigma)_2: (\sigma \rightarrow t) \rightarrow t$. There is an infinite set of variables V which is disjoint from the set of constants and a type assignment function on V which guarantees that there are countably many variables of each type and that no

needs a multi-sorted language in which constants, variables, and perhaps even the truth-functional constants, are *indexed to the quantifiers*. (So, for instance, there will be distinct formulas $(\lambda p^1. p^1 \rightarrow p^1)$ and $(\lambda p^2. p^2 \rightarrow p^2)$ where $p^1, p^2 : t$, and $\forall \sigma^1 (\lambda p^1. p^1 \rightarrow p^1)$ and $\forall \sigma^2 (\lambda p^2. p^2 \rightarrow p^2)$ are well-formed, but $\forall \sigma^2 (\lambda p^1. p^1 \rightarrow p^1)$ and $\forall \sigma^1 (\lambda p^2. p^2 \rightarrow p^2)$ are not.) A pluralist can assert that there are Fregean propositions, and a countable infinity of types of Fregean entities based on these. They can also assert that there are non-Fregean, Classicist propositions, and a countable infinity of types of Classicist entities based on these.⁴⁸ But they cannot ‘put these facts together’. They can say that *there-is¹* the Fregean interpretation, and *there-is²* the Classicist interpretation. But they cannot say ‘there are two candidate interpretations of ‘could

variable of one type is a variable of another type. Given the standard definition of type σ identity in terms of Leibniz equivalence, we have two different relations of identity in this setting:

$$x =^{\sigma^1} y := \forall^1 X, Y (Xx \leftrightarrow Yy), \text{ where } x, y : \sigma \text{ and } X, Y : \sigma \rightarrow t$$

$$x =^{\sigma^2} y := \forall^2 X, Y (Xx \leftrightarrow Yy), \text{ where } x, y : \sigma \text{ and } X, Y : \sigma \rightarrow t$$

The theory then looks like this:

PL: All instances of propositional tautologies.

MP: From A and $A \rightarrow B$, infer B

Gen1: From $A \rightarrow Fx$, infer $A \rightarrow \forall \sigma^1 F$ when x does not occur free in A

Gen2: From $A \rightarrow Fx$, infer $A \rightarrow \forall \sigma^2 F$ when x does not occur free in A

UI1: $\forall \sigma^1 F \rightarrow F a$

UI2: $\forall \sigma^2 F \rightarrow F a$

Extensional β : $(\lambda v_1 \dots v_n. \varphi) a_1 \dots a_n \leftrightarrow \varphi[a_i/v_i]$, where v_1, \dots, v_n are any distinct variables, a_1, \dots, a_n are any terms of types such that $v_i, a_i : \tau_i$ for all $1 < i < n$, and each v_i is substitutable for a_i in φ

Ind1: $\forall^1 X \forall^1 Y \forall^1 x_1 \dots \forall^1 x_n ((Xx_1 \dots x_n \leftrightarrow Y x_1 \dots x_n) \rightarrow X =^1 Y)$

Ind2a: If $A \leftrightarrow B$ is a theorem not derived using Ind1, then so is $A =_i^2 B$

Ind2b: If $Mx =^2 \tau Nx$ (with x not free in M or N) is a theorem not derived using Ind1, then so is $M =^2 \sigma \rightarrow \tau N$

The problem with this theory is that there will be a type of collapse argument which leads to the conclusion that \forall^1 and \forall^2 are identical¹ and identical² (in what follows let x be a variable which does not occur free in F):

- | | |
|--|---|
| (1) $\forall \sigma^1 F \rightarrow F x$ | UI1 |
| (2) $\forall \sigma^1 F \rightarrow \forall \sigma^2 F$ | 1, Gen2 |
| (3) $\forall \sigma^2 F \rightarrow \forall \sigma^1 F$ | UI2, Gen1 |
| (4) $\forall \sigma^1 F =^2 \forall \sigma^2 F$ | 2, 3, Ind2a |
| (5) $\forall \sigma^1 =^2 \forall \sigma^2$ | 4, Ind2b |
| (6) $\forall^1 (\lambda X. (\forall \sigma^2 X \leftrightarrow \forall \sigma^1 X))$ | 2, 3, Extensional β, Gen1 |
| (7) $\forall \sigma^1 =^1 \forall \sigma^2$ | 6, Ind1 |

We expect that this type of problem will afflict any statement of higher order pluralism in a single-sorted language.

⁴⁸ We add the qualification ‘non-Fregean’ because Fregean propositions are, strictly speaking, Classicist propositions, in the sense that Fregeanism is a strengthening of Classicism.

have been”. In order to say this, they would need to mix indices, as in: $\sim \forall^1 \sim ((t \rightarrow t) \rightarrow t) X^1 \sim \forall^2 \sim ((t \rightarrow t) \rightarrow t) X^2 \sim (X^1 =^1 X^2)$. They cannot even say that ‘metaphysical necessity is not the most inclusive necessity’, *if this is taken to quantify over necessities of different kinds*. In a multi-sorted system, one cannot make such comparisons. In the same way, the set-theoretic pluralist can say that *there-is^{NF}* a non-well-founded set and *there-is^{ZF}* not. But they cannot ‘put these claims together’ without doing violence to their meaning. For instance, if *ZFC* is their metatheory, then any ‘non-well-founded’ set witnessing the first claim will, by their own lights, be well-founded.

The underlying difficulty in both cases is that one cannot state pluralism about a kind of potential metatheory, like set theory or type theory, using a theory of that kind. Any metatheory will take itself to be broadest. What makes pluralism about foundational theories interesting is exactly that it makes *metatheoretic* concepts framework-relative. Nobody denies that geometric concepts are framework-relative. This is because they can all be realized in a single metatheory, like set theory. *Set-theoretic* pluralism, by contrast, precludes any such stable background arena. From the standpoint of any given set theory, other ‘universes’ are mere (set) models inside its own *V*.

The upshot is that Higher Order Pluralism, like set-theoretic pluralism, involves a kind of quantifier variantism.⁴⁹ The different quantifiers are *not* restricted. So, there is no formal theory in which the Higher Order Pluralist can assert strictness across hierarchies, just as the set-theoretic pluralist cannot compare broadness of sets across the pluriverse (more on this

⁴⁹ Although quantifier variantism in the first-order setting has been discussed at length, quantifiers variantism about higher-order quantifiers seems not to have received the attention that it deserves.

shortly). The authentic pluralist claims that *whether a quantifier is unrestricted is perspective (metatheory) dependent*. From the perspective of hierarchy H , H 's quantifiers are unrestricted. From the perspective of H^* , H^* 's quantifiers are. It is a practical question whether to reason in one hierarchy or the other. One can also discuss both hierarchies, using a multi-sorted theory comprising the theories of H and H^* . But this cannot be from the *perspective* of H or H^* .⁵⁰

Here is a picturesque way to characterize the pluralist's position (again, on analogy with set-theoretic pluralism). Consider a higher order theory, H . We adopt the language of H . We reason in it. We assert that it is true (in an expanded theory sufficient to define truth in H). We get it right. Then we consider a different theory H^* . We adopt *its* language. We reason in it. We assert that it is true (in an expanded theory). And we get it right again! The idea is that for a vast swath of such theories -- very roughly, theories that are consistent with the class of non-modal claims about which we are monist -- we may do this, and be right each time. So, one can construe our slogan that there are many candidate interpretations of higher order claims as shorthand for the claim that one can jump into different theories and assert that they are true of their intended subject matter, and be right, in this way.⁵¹ Of course, this is nothing like a *formal*

⁵⁰ This kind of meta-theoretic perspectivism is resonant with the ethical relativism of Rovane [2013], and the fragmentalism about time developed in Fine [2006].

⁵¹ Like Balaguer's and Hamkins's formulations of set-theoretic pluralism, our formulation of Higher Order Pluralism leaves open exactly how generous the pluriverse is, or even whether it has an exact extent (setting aside some constraints yet to be introduced) (Balaguer claims that any mathematical structure that could exist does exist, but says incongruent things about what makes for a possible mathematical structure. See Clarke-Doane [2020, n. 4]. Hamkins relies on examples of universes to give a feel for the scope of his pluriverse.) Where to draw the line requires principles. The natural option in both cases is classical consistency -- or, rather, classical consistency with select truths about the actual world. But even here one might wonder why we should preclude non-classically consistent (or non-trivial) theories (Priest [2012]). (Note that, paraconsistent logics of the sort advocated by Priest generate peculiar problems. It is paraconsistent with claims about the actual world with respect to which we want to be monists, like that Biden is president, that Biden is not president. This means, for instance, that if one accepts paraconsistent logics as legitimate, then the 'kernel' of invariant truths must also rule out odd iterations of negations of invariant truths. More on the kernel idea below.) We do not discuss such principles in this article. However, we argue for some elsewhere. See Clarke-Doane [2019 Sec. 8], [2020, 3.5 & 6.2], and [In Progress, 4.4] and McCarthy [In Progress].)

theory. As Balaguer [1998, 6] and Hamkins [2012] emphasize in the set-theoretic case, *part and parcel to pluralism is that it cannot be formalized*. Any formalization would violate the intended meaning, either because it would privilege a universe or because it would invite pluralism about pluralism. (In this respect, pluralism resembles ultrafinitism (Gaifman [2012, Sec. 2.1]. But whereas the pluralist takes formal resources to be *insufficiently* powerful, ultrafinitists take them to be too powerful to be meaningful.) Such a position will be unsatisfying to those with monist sensibilities. But we are not trying to convince modal monists to be pluralists. We are arguing that modal pluralism is ‘a *consistent* position’ [Koellner Manuscript, 22, italics in original].

In sum, we may distinguish four positions:

(I) Higher Order Monism plus modal monism (the claim that there is a broadest kind of necessity).

(II) Higher Order Monism, plus modal pluralism (the claim that there is not a broadest kind of necessity)

(III) Higher Order Pluralism, plus modal monism.

(IV) Higher Order Pluralism, plus 'there is a broadest necessity' does not hold in every hierarchy.

We are defending (IV), and arguing against (I). We claim that (II) does not do justice to modal pluralism (just as set-theoretic pluralism against the background of a weak, even indefinitely extensible, concept of set does not do justice to set-theoretic pluralism). And we claim that (III), while apparently coherent (it could be that 'there is a broadest necessity' holds in every hierarchy), is simply false. It fails, for example, in hierarchies in which the aforementioned Extensibility principle holds.

VI. Objections and Replies

Having elaborated on the content of Higher Order Pluralism, let us consider objections to the view. The obvious objection is that there may be a *broadest hierarchy* in the pluriverse of hierarchies (assuming that these can be ordered by broadness⁵²). In other words, there may be a broadest interpretation of the higher-order vocabulary. This would seem to serve the purpose of the Higher Order Monist's unique interpretation. Indeed, an analog of this worry is familiar from the set-theoretic case. As Martin puts it, "[t]he models postulated by [pluralist] determine a canonical maximal set-theoretic structure, the amalgamation. If one takes those models seriously, then one should regard this canonical structure as the true universe of sets" [2001, 14]. The problem with this argument is that it assumes that we can meaningfully compare competing concepts of set. The difficulty, alluded to above, is that, in order to make the comparison, we must appeal to such a concept [Clarke-Doane In Progress, 4.4]! So, the real conclusion is only that "*within any fixed set-theoretic background concept* [Hamkins, 427, italics in original]" there

⁵² To give a feeling for what this might mean consider the following simple case. Suppose that a pluralist held that there was a definite domain of individuals, and a definite domain of propositions which were shared by all of the typed hierarchies they accepted. Then their pluralism would consist in what higher order entities - typed functions - they took there to be based on these two domains. They would take there to be no 'objective' answer to this (in the sense that there is no objective answer to the Parallel Postulate question). This is analogous to a pluralism about *impure* set theory. In this scenario we might say that a hierarchy is broader than another just in case each of its higher order domains contain the corresponding domains of the other. (We do *not* endorse this kind of pluralism!)

is a broadest concept of set. It is not that there is a broadest concept *independent* of choice of set-theoretic background. The higher order situation is the same. It may fail to be true that there is a broadest interpretation of the higher order language (in the sense of IV above), even if *relative to some chosen one*, there is. Any argument that there is will take place against a background interpretation.^{53 54}

Of course, even if one could meaningfully compare typed hierarchies, it would not follow that there was a broadest such hierarchy for the reason that we pointed out at the end of Section II. Consider the case of logical pluralism. Again, the monist might argue that the union of all logical possibilities, for any kind of logical possibility accepted by the pluralist, gives a most inclusive kind of logical consistency. Even bracketing the problem that one must assume a logic to make the comparison (which is like the one above), the argument assumes that the plurality of accepted kinds of logical possibility is fixed. As the quotation from Beall and Restall illustrates, a pluralist may hold that, for any alleged most inclusive kind of logical consistency, there is a more inclusive one. In that case, while it might be ‘objectively’ true that a given kind of logical consistency is broader than another, there would be no final court of appeals for logical questions. The situation would be like the one in which there is a unique, but indefinitely extensible, hierarchy of sets. A limited version of the pluralist critique would survive.

⁵³ The difficulty is glaring if logical notions are themselves modal. What follows from what even *relative to a logic is itself* relative to a logic (Shapiro [2014, ch. 7]. So, if commonplace logical pluralism is true, and logical notions are modal (as discussed in Section IV), then there is certainly no stable fact as to the breadth of a kind of possibility.

⁵⁴ Again, the objection that Higher Order Pluralism requires a metatheory, and any such theory will itself engender a maximal higher order hierarchy, is question-begging in a similar way. As in the set-theoretic case, any metatheory takes itself to be maximal. Every set theory amounts to a metatheory, a model theory in which to discuss theories, with its own interpretation of, e.g., higher-order quantification. But not even the staunchest critics of set-theoretic pluralism take *that* to show that set-theoretic pluralism is false. As Koellner puts it, “the...argument is circular....[It] just presupposes in the meta-language what one set out to establish” [Manuscript, 11]. Indeed, Koellner has in mind here the most radical form of the view that he considers, according to which, roughly, *every* first-order consistent set-theory is true of its intended subject. This even includes set theories that ‘disagree’ with us about finiteness -- and, hence, consistency, theories themselves, well-formed formulas, and syntax more generally.

Perhaps there is a variation on the above criticism, however.⁵⁵ It might be thought that propositions would have to be incredibly fine grained in order to support all of the kinds of possibility that we believe in. The background logic would be absurdly weak, too weak to prove the claims that we make in this paper. This is a higher order variant of what Koellner [Manuscript] calls the ‘Problem of Articulation’. However, this criticism misunderstands Higher Order Pluralism. It is like interpreting set theoretic pluralism as the view that there is a single background kind of set, and all of the kinds of set that the pluralist talks about are restrictions of it (e.g., Gödel’s constructible sets).⁵⁶ From a Higher Order Monist perspective, adopting higher order logic requires thinking that it governs everything. From a Higher Order Pluralist perspective, adopting such a logic is to ‘jump’ into a hierarchy and reason inside it -- as the set-theoretic pluralist holds that adopting a set theory is to jump into a set theory and reason inside it.⁵⁷ One does not thereby *restrict* the one true hierarchy. There is none! A pluralist ‘living’ in the hierarchy agrees with the monist that its logic governs everything. But there is a difference. A monist can mimic this phenomenon when jumping into hierarchies that are proper parts of theirs, governed by stronger logics. But they cannot adopt *weaker* logics, and jump into ‘outer hierarchies’ of theirs. This is like the fact that the set-theoretic monist can jump into Gödel’s constructible universe, but cannot jump into a forcing extension of Gödel’s ‘one true V’.

From what meta-theoretic perspective, though, do we make the claims we make in this paper?

Well, as we have emphasized, the whole point of pluralism, in both the set-theoretic and

⁵⁵ Thanks to Juhani Yli-Vakkuri for pressing us on this.

⁵⁶ This is, in fact, how Field appears to advocate understanding the view in his [1998]. Nolan [2011] could be interpreted as advocating a similar picture in the modal case.

⁵⁷ Hamkins uses the ‘jump’ language in the set-theoretic case. See, e.g., [2012, 417].

higher-order contexts, is that there is no formalizable theory which gives the one true story of the subject. There is no privileged perspective on the sets or higher-order entities, respectively. Nevertheless, *ZFC* set theory (with classical logic) suffices to make most claims that a Higher-Order Pluralist might want to make (as it does in the set-theoretic case). *ZFC* is an expressively powerful and versatile theory which allows us to talk about set-theoretic representations of a wide swath of higher order typed hierarchies. For the pluralist, discovering that *ZFC* proves that different higher order theories have models tells us more than just that they are consistent (insofar as they think that *ZFC* is). The Higher Order Pluralist, like the set-theoretic pluralist, takes this to show (with the caveats below) that the theories have intended models -- i.e., that they are true of their subject. This is like the conclusion that geometers drew from the realization that hyperbolic geometry could be interpreted in Euclidean geometry. They gave up on the idea that Euclidean space was the ‘one true geometry’, with hyperbolic space just a simulation within it (Hamkins [2012, 425-6]). Both geometries stand on their own, true of their subjects. Geometers now ascribe the same status to geometries generally.

Of course, we do not maintain that the higher order entities in typed hierarchies *are ZFC* sets, any more than the set-theoretic pluralist claims that, say, Quine’s non-well-founded sets are *ZFC* sets. We claim that they have models in *ZFC*, and, therefore, intended ones -- i.e., that they are true of their intended subjects. To be sure, this last claim cannot be made in *ZFC*.⁵⁸ One cannot define truth for a theory in a language other than that of set theory in a set theory. Again, the set-theoretic case is analogous. When the set theoretic pluralist adopts *ZFC* and proves that *NF* has a model (if she can!⁵⁹), concluding that *NF* has an intended model -- that *NF* is true of its

⁵⁸ Thanks to Juhani Yli-Vakkuri for pushing us on this point.

⁵⁹ The relative consistency of *NF* is still officially an open problem. But experts seem to be converging on a view. See: <https://mathoverflow.net/questions/132103/the-status-of-the-consistency-of-nf-relative-to-zf>. (Coincidentally,

subject -- this claim cannot be made in the language of *ZFC*. One cannot define truth for *NF* in *ZFC*, if one thinks that *NF* and *ZFC* are in different languages, as the pluralist does. What the set-theoretic pluralist *can* do is shift their metatheory. They can observe in *ZFC* that $Con(NF)$. They can then jump to *NF+*, where *NF+* is *NF* plus resources sufficient to give a theory of truth for *NF*. In *NF+* they can claim that *NF* is true. (In switching metatheories, they of course use 'true' ambiguously. The truth predicate is different in the different metatheories.) As indicated in Section V, we are doing something analogous. We conclude in *ZFC* that a higher order theory *H* has a model (and satisfies some further constraints to be mentioned). We then jump into *H+*, where *H+* is a theory sufficient to define truth in *H*, and assert in *H+* that *H* is true. This obviously assumes that there are different higher order languages -- different higher order quantifiers and so on. But, again, that is what the Higher-Order Pluralist maintained all along.

A different kind of worry is that there *must* be a broadest hierarchy, on pain of the Russell-Myhill paradox (Russell [1902/1996, Appendix B], Myhill [1958]). This is usually taken to establish that there is an upper bound on the grain of propositions. The thrust of the argument is that the structured view of propositions is the broadest conceivable picture, and that it is classically inconsistent.⁶⁰ ⁶¹ Hence, propositions cannot be structured. In the present context, one might take

consistency-in-a-logic claims are among those abouts which we are monists, bracketing the fact that one must assume a logic to check those claims. So, we are monists about the natural number structure, but logical pluralists. See Clarke-Doane [2020, Sec. 1.6, 3.5, & 6.2] for the rationale.)

⁶⁰ The structured view of propositions says that for all *XY*, for all *xy* ($Xx=Yy \rightarrow X=Y \ \& \ x=y$).

⁶¹ Here is a loose version of the argument, which Dorr presents in [2016]. Choose some arbitrary proposition *p*, say that snow is white. Let a heteropredicative proposition be one that predicates of *p* some property that it itself lacks. Now consider the proposition that *p* is heteropredicative, call it *q*. Is *q* heteropredicative? If not, then *q* must have every property that it predicates of *p*, and in particular the property of being heteropredicative; contradiction. So *q* is heteropredicative: it predicates of *p* some property *f* that it, *q*, lacks. This *f* cannot be the property of being heteropredicative, which, as we have just seen, *q* does not lack. So, there must be two distinct—and indeed non-coextensive—properties which this single proposition *q* predicates of *p*. Dorr offers the following spin on this. “The argument is essentially Cantorian: one can think of the conclusion as saying that the domain of properties of propositions is larger than the domain of propositions, so that there can be no one-one correspondence between the two domains, and in particular the relation of being a property *f* and a proposition *q* such that *q* is *f*(*p*) cannot be one-one as required by Propositional Structure [2016] 28-30.”

this to undergird an argument that there is a broadest hierarchy, if one thinks that the grain of propositions in a hierarchy amounts to its broadness. However, this is not a tenable view. There are many axes other than the grain of proposition along which hierarchies can differ, and these differences would also seem to bear on the relative broadness question. One of these axes is the domain of individuals (things of type e) -- different hierarchies can have distinct domains of individuals. In particular, different hierarchies can differ on what *merely possible* individuals there are. Suppose that H is a predicative hierarchy in which propositions are structured.⁶² Now consider a proper inner hierarchy H^* of H , in which the propositions are also structured, and whose domain of individuals is a proper subset of the domain of individuals of H . It would be very odd to take H^* to be the broadest hierarchy, simply in virtue of its having structured propositions, given that it is a proper inner hierarchy of H ! The pluralist does not believe that there is a unique collection of all possible individuals⁶³ -- there simply is no hierarchy which contains all of the individuals in every other hierarchy. So, there will be another hierarchy $H\#$ which stands to H , as H stands to H^* .⁶⁴ And so on forever. Or suppose that we amend the above example so that H is a full impredicative hierarchy in which propositions are close to being structured. Suppose that H^* is a predicative inner hierarchy of H , in which all of the entities which ‘made’ the propositions in H not fully structured are removed. It would, again, be very strange to conclude that H^* is broader than H just in virtue of its having structured propositions!

To compound matters, note that hierarchies in which propositions are structured may differ on the grain of other higher order entities as well. For instance, consider two predicative hierarchies

⁶² Structured propositions are consistent in a predicative hierarchy. See [Walsh 2016].

⁶³ See Rayo [manuscript] for details on this point.

⁶⁴ We would not even accept that all structured hierarchies are broader than non-structured hierarchies. A hierarchy H which does not have fully structured propositions may have a predicative inner hierarchy H^* in which propositions are fully structured.

H and H^* , in which the propositions are fully structured. In H^* all of the higher order domains are individuated extensionally. In H they are individuated by metaphysical equivalence. Again, it would be bizarre to claim that H^* is broader than H just by virtue of having structured propositions. However, if pluralism is true, then there will be a hierarchy $H\#$ in which higher order entities are individuated by a finer kind of necessary equivalence than metaphysical equivalence, which stands to H , as H stands to H^* . So, structured propositions are not the end all of broadness. *Even if we could meaningfully compare hierarchies* -- something that, to repeat, we cannot see how to usefully do -- having structured propositions would not seem to be definitive.

Perhaps the impulse behind some of the objections above is that Higher Order Pluralism leads to *reality* pluralism. In particular, propositional pluralism seems to engender pluralism about facts. And is that not tantamount to pluralism about whether it is the case that P , for arbitrary P ? It is not. Yes, the Higher Order pluralist accepts a plurality of different interpretations of the higher order language, in which there are a diverse array of modal facts, facts about the grain of propositions and higher order entities, and even some facts about the cardinality of domains. But this does not mean that there are no facts that hold under every interpretation. The pluralist may hold that all of the true higher order theories agree on a kernel of non-modal facts. Those facts may be invariant. What is true is that the non-modal kernel must correspond to propositions of different grains across the pluriverse. For instance, it may be true in every hierarchy that Biden is president and that $2+2=4$. But this will correspond to one proposition in an Extensional hierarchy, and a different proposition in a more fine grained Boolean hierarchy. (This is, again, like the set-theoretic case, where different universes at most agree on *sentences*.) What matters is that there is non-modal agreement among the interpretations. To repeat, such facts cannot be

identified with propositions because these are not shared between interpretations. Indeed, we deny that such facts -- e.g., that Biden is president -- are entities of some type over and above the different kinds of proposition. There is the Fregean proposition that Biden is president, the metaphysical proposition that Biden is president, and so on, not some further entity *that Biden is president*, to which they all bear a relation. Rather, given a non-modal language, like the language of fundamental physics, each of the interpretations satisfies the same sentences. The resulting theory is analogous to that of set-theoretic pluralism in the context of *impure* set theory.⁶⁵

Are there any other objections to modal pluralism, the context of higher order logic? There are certainly surprising *implications* of the view (the view would be quite boring if this were not the case!). Higher Order Pluralism implies a kind of pluralism about content. There are various contents of assertion, belief, doubt, and so forth corresponding to the various typed hierarchies.⁶⁶ We could make this explicit by indexing propositional attitude predicates to different hierarchies. The idea can already be illustrated in the first-order context. Suppose that we think of the content of a sentence as the set of worlds in which it is true. Then, of course, we get different notions of content depending on whether we consider, say, the (first-order) logically possible as opposed to metaphysically possible worlds (holding fixed the modal logic). Orthodoxy has it that the *real* content is the second set, not the first (Stalnaker [1984]). But what turns on this

⁶⁵ According to the development in McCarthy [In Progress], while there is just one Biden, there are many properties of being president. For instance, there is the Fregean property of being president, and the metaphysical (identified up to metaphysical equivalence) property of being president. According to the Fregean picture, being (the US) president (at this moment), and being Joe Biden are the same property. According to the metaphysical picture, they are distinct. However, even if one accepted monism about properties of individuals, one could still be a pluralist about propositions.

⁶⁶ Does this not imply that $\sim(\text{Biden is president} = \text{Biden is president})$? (Thanks to Jin Zeng for the question.) Again, if one wants to talk about different hierarchies, then one had better use a multi-sorted higher order language! Then, on any of the available disambiguations, $\sim(\text{Biden is president} = \text{Biden is president})$ will be false. Of course, for ordinary purposes, one hierarchy suffices, just as one universe of sets suffices for ordinary mathematical purposes.

except how select academics use ‘content’? If there are sets of metaphysically possible worlds (understood as abstract objects), then there are sets of logically possible worlds, and there is nothing to preclude us from introducing terms to single the latter out. We can say $\exists_{\text{Metaphysical}}$ the proposition that snow is white, and \exists_{Logical} the proposition that snow is white. Each of these may be expressed by the sentence ‘snow is white’. In general, for every sentence, we bear relations to countless propositions of as many kinds. This too is part and parcel to Higher Order Pluralism.⁶⁷

VII. Conclusion

We have defended a simple argument that modal metaphysics is misconceived. Unlike Quine’s, this does not require that modal questions are unintelligible. It requires that there are different candidate interpretations of the predicate ‘could have been’ giving intuitively opposite verdicts on modal questions, none of which is broadest, i.e., most inclusive. Modal pluralism is the analog to an increasingly prevalent view about set theory, according to which there are different candidate interpretations of ‘is a member of’, giving intuitively different answers to set-theoretic questions, none of which is broadest. We showed that the obvious responses to the argument from modal pluralism fail. It is no use arguing that metaphysical possibility is the broadest ‘real’ or ‘objective’ kind of possibility, or that it is the most natural kind of possibility, for instance. However, a new response has emerged that purports to prove, in a higher-order logic, that metaphysical possibility is the broadest kind of possibility applying to propositions. We distilled

⁶⁷ Other surprising implications of modal pluralism, whether married to higher type theory or not, include that there are no objective facts as to what supervenes on what, what counts as an intrinsic property, what the state space of a physical system is, and, arguably, whether one event caused another (or whether an arbitrary counterfactual is true). See Clarke-Doane [2019, Sec, 7], and [In Progress, 4.5]. (Note that there are independent reasons to accept many of these conclusions. For instance, a causal model under an interpretation only represents a situation relative to a space of possibilities. So, if we take this appearance at face-value, actual causation must be relative to a background space of possibilities, even given modal monism. (See McDonald [2021].) Of course, the view that objective causal relations are not needed in basic physics is familiar.)

two lines of reasoning from the literature, and argued that their import depends on an assumption that pluralists should deny. It depends on the assumption that there is a unique typed hierarchy (or, more carefully, a unique candidate for what higher-order claims could mean) which a modal pluralist, in the context of higher-order logic, may well reject. It might be worried that modal pluralism, so conceived, faces an insuperable problem of articulation, is vulnerable to the Russell-Myhill paradox, or even contravenes the truism that there is a unique actual world. However, we argued that these worries are misplaced. Most of them are simple applications of arguments familiar from the set-theoretic literature -- arguments that are widely agreed to fail.

One may certainly have doubts about modal pluralism, whether of the higher order variety discussed here or not. Modal pluralism is a radical departure from the established views of metaphysicians. It has ramifications beyond modal metaphysics, e.g., for the theory of content. And, in a higher order context, modal pluralism, like set-theoretic pluralism, introduces a kind of perspectivalism about metatheoretic concepts that have long been assumed to be objective (in the sense that, say, geometric concepts are not). Our purpose has not been to argue *for* modal pluralism. We do that elsewhere (Clarke-Doane [2019], [2020] and [In Progress], McCarthy [In Progress]). Our purpose has been to show that what might have appeared, in a higher-order setting, to be proof that modal pluralism is false is ineffective on inspection, like obvious objections to the view.

Appendix: Logical Possibility

In this Appendix, we sketch two theories of propositional logical possibility, both of which have the consequence that it is logically possible that identities fail.

Our aim with the first theory is to construct a theory of propositions and a kind of logical necessity which stands to those propositions, as validity in propositional $S4$ stands to the sentences which can express them.⁶⁸ The signature of propositional $S4$ contains the sentential operators \neg , \wedge and \Box , as well as a countable infinity of propositional letters $p_1, p_2, \dots, p_n, \dots$.

The propositional modal logic $S4$ is the result of taking as axioms:

- All of the propositional tautologies
- K: $\Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$
- T: $\Box A \rightarrow A$
- 4: $\Box A \rightarrow \Box \Box A$;

And closing these under the rules of modus ponens and necessitation. An algebraic frame is a non-empty set of propositions, together with some operations on those propositions. A Boolean algebraic frame is an algebraic frame, in which there is an element T , and there are two operations, \sim and \wedge , on the propositions such that they satisfy the following equations:

- Commutativity: $p \wedge q = q \wedge p$
- Distribution: $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
- Identity: $p \wedge T = p$
- Contradiction: $p \wedge \sim T = p \wedge \sim p$
- Equivalence: If $p \equiv_L q$, then $p = q$, where \vee and \equiv_L are defined as follows:
- $p \vee q := \sim(\sim p \wedge \sim q)$

⁶⁸ We clarify what we mean by ‘express’ below.

- $p \equiv_L q := p \wedge q = p \vee q$ ⁶⁹

An $S4$ frame is a Boolean algebraic frame in which there is a further operation N on the propositions such that they satisfy the following equations:

- NT
- $N\sim(p \wedge \sim q) < \sim(Np \wedge \sim Nq)$
- $Np < p$, (iv) $Np < NNp$.⁷⁰

An $S4$ algebraic model then is a triple, which contains an $S4$ frame, a function V which assigns truth values to the propositions, and a function A which assigns sentences of the logic to propositions in the algebra. V must meet the following conditions.

- V takes the elements of S , the set of simple propositions, to $[1,0]$.
- $V(\sim B) = 1$ iff $V(B) = 0$. (iii) $V(B \wedge C) = \min(V(B), V(C))$.

We now fix how V behaves for propositions in the range of N . The domain of N contains every proposition in the algebra. For any proposition A , in the algebra, if $V(A) = 0$, then $V(N(A)) = 0$. Then for propositions A such that $V(A) = 1$, V can freely assign true values to $\Box A$, so long as that does not contravene any of the other conditions on V . The function A must meet the following conditions:

- A takes sentence letters to propositions in the algebra

⁶⁹ This axiomatization is from Goodman [2019].

⁷⁰ Where $p < q := p \wedge q = p$.

- $A(\neg B) = \neg A(B)$, (ii) $A(B \wedge C) = A(B) \wedge A(C)$
- $A[\Box B] = N(A(B))$.

Let us say that an assignment function A is *faithful* just in case it respects the equivalence of sentences, the non-equivalence of sentences, and the form of the propositions. More formally, an assignment function A , from the propositional modal logic $S4$ to an algebra of propositions is faithful if only if

- If $S4$ proves $p \leftrightarrow q$, then $A(p) = A(q)$
- If it's not the case that $S4$ proves $p \leftrightarrow q$, then $A(p) \neq A(q)$
- It respects the form of the propositions relative to the logic.

We now make the following assumptions about the actual propositions, so that there can be a faithful assignment from the language of $S4$ to them. First, we assume that the propositions form an $S4$ algebra with respect to negation, conjunction, and necessitation. Second, we assume that propositions have a form that mirrors that of sentences relative to propositional $S4$. Some propositions are simple, or atomic. These simple propositions are algebraically independent of one another. (These are propositions which are atomic ‘from the perspective of propositional $S4$ ’.) So, for example, the propositions that John is English, that $2+2=4$, that everything is self-identical, that Hesperus is identical to Phosphorus, are all atomic. And then there are also negated propositions, conjunctive propositions, and necessitated propositions.

A faithful assignment function A sends the sentence letters to the simple propositions. Also, A must be injective. Given conditions outlined, we obtain that a faithful assignment function sends sentences of the form $\neg p$ to negated propositions, sentences of the form $p \wedge q$ to conjunctive propositions, and sentences of the form $\Box p$ to necessitated propositions. We may then say that a proposition is *expressible* by a sentence in the logic if and only if there is a faithful assignment function which sends the sentence to that proposition. So, the simple propositions are expressible by the sentence letters, conjunctive propositions are expressible by sentences involving \wedge , necessitated propositions are expressible by sentences involving \Box , and so on.⁷¹

We want N to stand to the propositions as validity in $S4$ stands to the sentences which can express them. We, thus, impose the following three conditions on the truth value assignment function, V :

- $V(N(p))$ is true just in case p is expressible by theorems of the logic
- $V(\sim N\sim(p))$ is true just in case p is expressible by consistent sentences of the logic
- $V(\sim N\sim(p))$ is false just in case p is expressible by contradictions of the logic.

This has the consequences that the only logically necessary proposition is the top proposition in the algebra; the only logically impossible proposition is the bottom proposition; and every other proposition in the algebra is possible. This also ensures that p and q are identical just in case it is

⁷¹ There is a complication that we are ignoring, as it does not matter for our purposes here. One might want to say that simple propositions are expressible only by sentence letters, that a proposition of the form $\sim p$ is expressible only by a negated sentence letter, and so on. One might want the form of propositions to mirror the forms of the sentences which can express them. But one cannot quite have this. The complication is that logically equivalent sentences express the same proposition, relative to any faithful assignment function. And so a simple proposition p will be expressible by p , $\sim\sim p$, $p \wedge p$, and so on. The precise account would deal with equivalence classes of sentences relative to the $S4$. We could say that simple propositions are expressible only by sentences in the equivalence classes of sentences based on sentence letters, and so on.

logically necessary that p if and only if q . Then, for a sentence of natural language, s , we see that ‘it is logically necessary that s ’ is true just in case s has the form of a theorem, relative to propositional $S4$; ‘it is possible that s ’ is true just in case s is a consistent sentence, relative to $S4$; and ‘it is impossible that s ’ is true just in case s has the form of an $S4$ contradiction.⁷² The intended model of propositions is an ordered triple containing an algebraic frame of the actual propositions, as well as the actual propositional operators of negation, conjunction and logical necessity, a truth value assignment which sends propositions in the algebra which obtain to true, and ones which do not obtain to false, and a faithful assignment function from the sentences of the logic to the propositions of the algebra.

Now consider the proposition that Hesperus is identical to Phosphorus. This is a simple proposition. It is expressible by sentence letters of propositional $S4$. So, the negation of that

⁷² Although this is a consistent setup when the modal logic in question is $S4$, it is not a consistent one if the logic is $S5$. Consider propositional modal logic $S5$. Any algebraic frame of $S5$ is a Boolean algebra of propositions together with the function N , which sends the top proposition to itself, and every other proposition to the bottom proposition. So, any intended model of $S5$ will be an ordered triple $M = \langle S, V, A \rangle$, where S must contain at least a countable infinity of simple propositions. Fix on the frame S which has exactly a countable infinity of simple propositions. V is a truth value assignment. A is a faithful assignment function. Let M be the dual of N . Now consider some simple proposition p . Say that $A(p) = p$, where $[p]$ is an atomic sentence letter. $[p]$ is consistent. So $V(M(p)) = 1$. Since this is an $S5$ algebra we get that if $V(M(p)) = 1$, then $V(N(M(p))) = 1$. But this can only happen if $M(p)$ is the top proposition. This means that whatever sentence expresses $M(p)$ has to be a theorem of $S5$. But $M(p)$ is expressed by $[\langle p \rangle]$, which is not a theorem of $S5$. (Another way of putting this point is that $M(p) = N(M(p))$. So, it is necessary that it is possible that p . But this means that $M(p)$ is the top proposition, so p is the top proposition. But, by assumption, p is neither the top nor the bottom proposition. This is the contradiction.)

The reason for this discrepancy is that the setup only works for ‘coherent’ modal logics in the sense of Meyer [1971]. Bacon and Fine [2021, 33] write: “Let Δ be a normal modal logic. A function $v : L(\neg \wedge \hat{}) \rightarrow \{0, 1\}$ is a Δ -valuation if and only if (i) $v(A \wedge B) = \min(v(A), v(B))$, (ii) $v(\neg A) = 1 - v(A)$, and (iii) $v(\hat{A}) = 1$ iff $A \in \Delta$. Any modal logic may, of course, be plugged into this definition. But if Δ is supposed to represent the logical truths and v a possible interpretation of $\hat{}$ as logical truth, then we would like the truths under the interpretation to include the logical truths: $v(A) = 1$ whenever $A \in \Delta$. Following Meyer (1971) we say: A modal logic Δ is coherent iff for every $A \in \Delta$ and Δ -valuation v , $v(A) = 1$. Intuitively, a modal logic is coherent when it accommodates an interpretation of $\hat{}$ with its own logic. Not every modal logic is coherent.” However, $S4$ is a coherent modal logic. The assumption that propositions are as fine grained a logically necessary equivalence, where that kind of logical necessity stands to propositions as validity in the propositional modal logic of the sentences which can express those propositions, is consistent for coherent modal logics. Since $S4$ is such a logic, one can consistently make this claim for $S4$.

proposition, the proposition that it is not the case that Hesperus is identical to Phosphorus, is expressible by sentences of the form $\neg p$, where p is a sentence letter. According to our setup, this latter proposition is possible just in case $\neg p$ is a consistent sentence in $S4$. And this is the case. So, it is logically possible, in this framework, that Hesperus is not identical to Phosphorus.

Let us now sketch a second kind of logical possibility according to which identities can fail. This is similar to Carnap's account of propositional logical possibility.⁷³ Much of the setup is the same as the $S4$ case, but there are differences, as $S5$ is not a coherent modal logic (in the sense of Meyer [1971]). The idea is to construct a theory of propositions and a kind of logical necessity which stands to the propositions just as pre-validity stands to sentences of $S5$ which can express them.⁷⁴

An $S5$ frame is a boolean algebra of propositions which one obtains from an $S4$ algebra by adding the following constraint to the operation N :

- $N(T) = T$, and $N(p) = F$, for every other proposition p in the algebra.

Again, one cannot assume that the propositions are identical just in case they are equivalent in $S5$, that a proposition is necessary just in case it is expressible by theorems of $S5$, that a proposition is possible just in case it is expressible by consistent sentences of $S5$, or that a proposition is impossible just in case it is expressible by contradictions of $S5$, as outlined in footnote 71. However, one can make this assumption if one replaces $S5$ validity with $S5$

⁷³ See Cresswell [2013] for a full treatment of Carnap's account.

⁷⁴ We use the locution 'pre-valid' following Bacon and Fine [2021] who introduce and discuss the notion of pre-validity relative to a class of substitutions.

‘pre-validity’ (to be described). So, one may assume that propositions are identical just in case they are pre-equivalent in *S5*, that a proposition is necessary just in case it is expressible by the pre-validities of *S5*, that a proposition is possible just in case it is expressible by pre-consistent sentences of *S5*, and that a proposition is impossible just in case it is expressible by pre-contradictions of *S5*.

What are these pre-validities? Consider the following class of valuation functions *C*. Let a valuation *V* be in *C* just in case it is a function which takes the sentence letters to $[1,0]$, and which behaves in the normal way with respect to the truth functional connectives, and such that:

$$[] \quad V([]A) = 1 \quad \text{if and only if} \quad W(A) = 1 \quad \text{for all valuations } W \text{ in } C$$

$$(\diamond) \quad V(\diamond A) = 1 \quad \text{if and only if} \quad W(A) = 1 \quad \text{for some valuation } W \text{ in } C$$

The pre-validities of *S5* are the sentences which are true according to every valuation *V* in *C*; the pre-consistent sentences of *S5* are the sentences which are true according to some valuation *V* in *C*; the pre-contradictions are the sentences which are false according to every valuation *V* in *C*; and some sentences *p* and *q* are pre-equivalent just in case $p \leftrightarrow q$ is a pre-validity. The distinctive feature of the pre-validities is that they are not closed under substitution.^{75 76}

We may now suppose that the propositions mirror the logic of this class of valuations in form. So, the propositions form an *S5* algebra, and propositions *p* and *q* are identical just in case, for

⁷⁵ This way of presenting the pre-validities of *S5* is based on Burgess’s account of logical possibility in Burgess [2003].

⁷⁶ It is straightforward to see that validity and pre-validity come apart. Just consider $\diamond p$. This is not a theorem of *S5*. However it is a pre-validity. Take an arbitrary valuation *V*. $V(\diamond p) = 1$ if and only if $W(p) = 1$ for some valuation *W*. Since *p* is a sentence letter, there is such a valuation *W*. Hence, $V(\diamond p) = 1$ for every *V*.

some faithful assignment function which sends p to p , and q to q , $p \leftrightarrow q$ is a pre-validity of $S5$.

We then say that

- (i) $V(N(p))$ is true just in case p is expressible by pre-validities of $S5$
- (ii) $V(\sim N\sim(p))$ is true just in case p is expressible by pre-consistent sentences of $S5$
- (iii) $V(\sim N\sim(p))$ is false just in case p is expressible by pre-contradictions of $S5$.

Again, this has the consequences that the only logically necessary proposition is the top proposition in the algebra; the only logically impossible proposition is the bottom proposition; and every other proposition in the algebra is possible. This also ensures that p and q are identical just in case it is logically necessary that $p \leftrightarrow q$.

Now, just as in the case of $S4$ necessity, a sentence of natural language, s , is logically necessary in the present sense when s has the form of a pre-validity of $S5$. Similarly, for possibility and impossibility. So, evidently, in this context too it is logically possible that it is not the case that Hesperus is identical to Phosphorus, as this sentence is expressible by pre-consistent sentences of $S5$.

Note that both of the kinds of logical possibility that we have discussed require that propositions meet very specific conditions -- indeed, conditions which cannot not be jointly met! However, this is no problem from the point of view of a pluralist (rather than monist) about propositions.

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