A Clifford Algebraic Analysis and Explanation of Wave Function Reduction in Quantum Mechanics

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Abstract : proof is given of a theorem on Clifford algebra . Its implications are examined in order to give a Clifford algebraic formulation and explanation of the process of wave function reduction in quantum mechanics.

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Let us state a proper definition of Clifford algebra.

The Clifford (geometric) algebra $Cl_{3,0}$ is an associative algebra generated by three vectors $e_1, e_2,$ and e_3 that satisfy the orthonormality relation

$$e_j e_k + e_k e_j = 2\delta_{jk} \qquad (1.1)$$

for $j,k \in [1,2,3]$.

That is,

$e_j^2 = 1$ and $e_j e_k = -e_k e_j$ for $j \neq k$

Let a and b be two vectors spanned by the three unit spatial vectors in $Cl_{3,0}$. By the orthonormality relation the product of these two vectors is given by the well known identity: $ab = a \cdot b + i(a \times b)$ where $i = e_1 e_2 e_3$ is an imaginary number that commutes with vectors.

To give proof, let us follow the approach that, starting with 1981, was developed by Y. Ilamed and N. Salingaros [1].

Let us consider three abstract basic elements, e_i, with $i = 1,2,3$, and let us admit the following two assumptions:

a) it exists the scalar square for each basic element:

$$e_1 e_1 = k_1 \; , \; e_2 e_2 = k_2 \, , \; e_3 e_3 = k_3 \; \text{ with } \; k_i \in \Re \; . \qquad (1.2)$$

In particular we have also that

$e_0 e_0 = 1$.

b) The basic elements e_i are anticommuting elements, that is to say:

$$e_1 e_2 = -e_2 e_1 \; , \; e_2 e_3 = -e_3 e_2 \, , \; e_3 e_1 = -e_1 e_3 \, . \qquad (1.3)$$

In particular it is

$e_i e_0 = e_0 e_i = e_i$.

Consider the general multiplication of the three basic elements $e_1, e_2, e_3,$ using scalar coefficients $\omega_k, \lambda_k, \gamma_k$ pertaining to some field:

$$e_1 e_2 = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \; ; \; e_2 e_3 = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \; ; \; e_3 e_1 = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 \, . \qquad (1.4)$$

Let us introduce left and right alternation:

$$e_1 e_1 e_2 = (e_1 e_1) e_2 \, ; \; e_1 e_2 e_2 = e_1 (e_2 e_2) \, ; \; e_2 e_2 e_3 = (e_2 e_2) e_3 \, ; \; e_2 e_3 e_3 = e_2 (e_3 e_3) \, ; e_3 e_3 e_1 = (e_3 e_3) e_1 \, ;$$
$$e_3 e_1 e_1 = e_3 (e_1 e_1) \, . \qquad (1.5)$$

Using the (1.3) in the (1.5) it is obtained that

$$k_1 e_2 = \omega_1 k_1 + \omega_2 e_1 e_2 + \omega_3 e_1 e_3 \, ; \qquad k_2 e_1 = \omega_1 e_1 e_2 + \omega_2 k_2 + \omega_3 e_3 e_2 \, ;$$
$$k_2 e_3 = \lambda_1 e_2 e_1 + \lambda_2 k_2 + \lambda_3 e_2 e_3 \, ; \qquad k_3 e_2 = \lambda_1 e_1 e_3 + \lambda_2 e_2 e_3 + \lambda_3 k_3 \, ;$$
$$k_3 e_1 = \gamma_1 e_3 e_1 + \gamma_2 e_3 e_2 + \gamma_3 k_3 \, ; \qquad k_1 e_3 = \gamma_1 k_1 + \gamma_2 e_2 e_1 + \gamma_3 e_3 e_1 \; . \qquad (1.6)$$

From the (1.6), using the assumption (b), we obtain that

$$\frac{\omega_1}{k_2}e_1e_2+\omega_2-\frac{\omega_3}{k_2}e_2e_3=\frac{\gamma_1}{k_3}e_3e_1-\frac{\gamma_2}{k_3}e_2e_3+\gamma_3\,;$$

$$\omega_1+\frac{\omega_2}{k_1}e_1e_2-\frac{\omega_3}{k_1}e_3e_1=-\frac{\lambda_1}{k_3}e_3e_1+\frac{\lambda_2}{k_3}e_2e_3+\lambda_3\,;$$

$$\gamma_1-\frac{\gamma_2}{k_1}e_1e_2+\frac{\gamma_3}{k_1}e_3e_1=-\frac{\lambda_1}{k_2}e_1e_2+\lambda_2+\frac{\lambda_3}{k_2}e_2e_3 \qquad (1.7)$$

By the principle of identity , we have that it must be

$$\omega_1=\omega_2=\lambda_2=\lambda_3=\gamma_1=\gamma_3=0 \qquad (1.8)$$

and

$$-\lambda_1k_1+\gamma_2k_2=0 \qquad \gamma_2k_2-\omega_3k_3=0 \qquad \lambda_1k_1-\omega_3k_3=0 \qquad (1.9)$$

The (1.9) is an homogeneous algebraic system admitting non trivial solutions since its determinant $\Lambda=0$, and the following set of solutions is given:

$$k_1=-\gamma_2\omega_3,\; k_2=-\lambda_1\omega_3\;, k_3=-\lambda_1\gamma_2 \qquad (1.10).$$

Admitting $k_1=k_2=k_3=+1$, it is obtained that

$$\omega_3=\lambda_1=\gamma_2=i \qquad (1.11)$$

In this manner, using the (1.2) and the (1.3), as a theorem, the existence of such algebra is proven. The basic features of this algebra are given in the following manner

$$e_1^2=e_2^2=e_3^2=1 \quad ; \quad e_1e_2=-e_2e_1=ie_3 \quad ; \quad e_2e_3=-e_3e_2=ie_1; \quad e_3e_1=-e_1e_3=ie_2 \quad ; i=e_1e_2e_3 \qquad (1.12).$$

The content of this statement is thus established: given three abstract basic elements as defined in (a) and (b), an algebraic structure is established as in (1.12) with four generators (e_0, e_1, e_2, e_3).

The previous Clifford (geometric) algebra $Cl_{3,0}$ admits idempotents. Let us consider two of such idempotents:

$$\psi_1=\frac{1+e_3}{2} \quad \text{and} \quad \psi_2=\frac{1-e_3}{2} \qquad (1.13)$$

It is easy to verify that $\psi_1^2=\psi_1$ and $\psi_2^2=\psi_2$.

Let us examine now the following algebraic relations:

$$e_3\psi_1=\psi_1e_3=\psi_1 \qquad (1.14)$$

$$e_3\psi_2=\psi_2e_3=-\psi_2 \qquad (1.15)$$

Similar relations hold in the case of e_1 or e_2. The given algebraic structure $Cl_{3,0}$, with reference to the idempotent ψ_1, see the (1.14), relates to e_3 the numerical value of $+1$ while the (1.15), with reference to ψ_2, relates to e_3 the numerical value of -1 .

With relation to $e_3\to+1$, from the (1.12) we have that

$$e_1^2=e_2^2=1\,, i^2=-1\,; e_1e_2=i\,, e_2e_1=-i\,, e_2i=-e_1\,, ie_2=e_1\,, e_1i=e_2\,, ie_1=-e_2 \qquad (1.16)$$

with three new basic elements (e_1, e_2, i) instead of (e_1, e_2, e_3) .

In other terms, in the case $e_3\to+1$, a new algebraic structure arises with new generators whose rules are given in (1.16) instead of in (1.11). Therefore, the arising central problem is to proof the real existence of such new algebraic structure. Note that, in the case of the starting algebraic structure ,we showed that it exists in the following manner

$$e_1^2=e_2^2=e_3^2=1\,;$$

$$e_1e_2=-e_2e_1=ie_3\;;\; e_2e_3=-e_3e_2=ie_1;\; e_3e_1=-e_1e_3=ie_2\;; i=e_1e_2e_3 \qquad (1.17).$$

In the present case, ($e_3\to+1$), we have to show that it exists in the following manner

$$e_1^2=e_2^2=1\,;\;\; i^2=-1\,;$$

$e_1e_2 = i$, $e_2e_1 = -i$, $e_2i = -e_1, ie_2 = e_1, e_1i = e_2$, $ie_1 = -e_2$ (1.18)

In this manner we arrive to proof a theorem that, given the algebraic structure A, fixed as in the (1.17), under the condition, $e_3 \to +1$, it exists an algebraic structure B with basic elements (generators) given in (1.18). To proof, rewrite the (1.4) in our case, and performing calculations we arrive to the solutions of the corresponding homogeneous algebraic system that in this new case are given in the following manner:

$k_1 = -\gamma_2\omega_3$; $k_2 = -\lambda_1\omega_3; k_3 = -\lambda_1\gamma_2$ (1.19)

where this time it must be $k_1 = k_2 = +1$ and $k_3 = -1$. It results

$\lambda_1 = -1; \gamma_2 = -1$; $\omega_3 = +1$ (1.20)

and the proof is given.

The theorem also holds in the case in which we relate to e_3 the numerical value of -1. It is

$e_3 \to -1$

and

$e_1^2 = e_2^2 = 1$; $i^2 = -1$;

$e_1e_2 = -i$, $e_2e_1 = i$, $e_2i = e_1, ie_2 = -e_1, e_1i = -e_2$, $ie_1 = e_2$ (1.21)

The solutions of the (1.19) are given in this case by

$\lambda_1 = +1; \gamma_2 = +1$; $\omega_3 = -1$ (1.22).

In a similar way it is obtained the proof when considering the cases of e_1 or of e_2.

Of course, the Clifford algebra given in the (1.18) and in the (1.21) are well known. They are the dihedral Clifford algebra N_i (for details, see ref.2 page 2093 Table II).

The Possible Implications for Quantum Mechanics.

For the problem under consideration, we consider the representation, called the density operator formulation, in which a quantum system is represented by a positive definite Hermitean operator of unit trace known as the density operator.

The density operator ρ of a system described by the state vector $|\psi\rangle$ is simply the projection operator $|\psi\rangle\langle\psi|$. In general, a density operator has the form

$$\rho = \sum_i p_i |i\rangle\langle i| \quad (1.23)$$

in some particular basis $\{ |i\rangle \}$. In this basis, it is immediately evident that the eigenstates of ρ are just the states $|i\rangle$ and the probabilities p_i are the corresponding eigenvalues.

To sketch the problem. Consider the most general form of the state vector of an arbitrary two state quantum system

$$|\psi\rangle = a|+\rangle + be^{i\delta}|-\rangle \quad (1.24)$$

where, without loss of generality, a, b, δ are considered here to be all real. The density operator of this system is

$$\rho = \begin{pmatrix} a^2 & abe^{-i\delta} \\ abe^{i\delta} & b^2 \end{pmatrix} \quad (1.25)$$

There is no nontrivial choice of a, b, and δ that could cause ρ to be diagonal in this basis and still satisfy the requirements of a density operator, and in particular the unit trace requirement. The only way to obtain zeros on the off diagonals is to specify that δ is completely undetermined, and that we must therefore average over all possible values of δ. By this requirement, the averaging turns the complex exponential to zero, giving a diagonal matrix. However, there is no way to

accurately specify δ as a completely undetermined quantity in a manner that allows for rigorous calculations.
Consider a two state quantum system S with connected quantum observable σ_3. We have

$$|\psi\rangle = c_1|\varphi_1\rangle + c_2|\varphi_2\rangle \quad \text{with} \quad \varphi_1 = \begin{pmatrix}1\\0\end{pmatrix} \ , \ \varphi_2 = \begin{pmatrix}0\\1\end{pmatrix} \tag{1.26}$$

and

$$|c_1|^2 + |c_2|^2 = 1 \tag{1.27}$$

Let us represent the state of such system by a density matrix ρ given in the following terms

$$\rho = a + be_1 + ce_2 + de_3 \tag{1.28}$$

with

$$a = \frac{|c_1|^2 + |c_2|^2}{2} \ , \ b = \frac{c_1^* c_2 + c_1 c_2^*}{2} \ , \ c = \frac{i(c_1 c_2^* - c_1^* c_2)}{2}, \ d = \frac{|c_1|^2 - |c_2|^2}{2} \tag{1.29}$$

where in matrix notation, e_1 , e_2, and e_3 are the well known Pauli matrices

$$e_1 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \ , \ e_2 = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \ , \ e_3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \tag{1.30}$$

Of course, the (1.28) is an element of the Clifford algebra as given in the (1.17). As Clifford algebraic element the (1.28) satisfies the requirement to be $\rho^2 = \rho$ and $\mathrm{Tr}(\rho) = 1$ under the conditions $a = 1/2$ and $a^2 - b^2 - c^2 - d^2 = 0$ (null norm of (1.28) algebraic element) as shown in detail in [2]. In the algebraic framework previously outlined, let us admit that we relate $e_3 \to +1$ (that is to say that the quantum observable σ_3 assumes the value +1) or $e_3 \to -1$ (that is to say that the quantum observable σ_3 assumes the value –1). As previously shown, the algebra given in (1.18) and the (1.21)will now hold, respectively. To examine the consequences, starting with the algebraic element (1.28) , write the two equivalent algebraic forms

$$\rho = \frac{1}{2}(|c_1|^2 + |c_2|^2) + \frac{1}{2}(c_1 c_2^*)(e_1 + e_2 i) + \frac{1}{2}(c_1^* c_2)(e_1 - ie_2) + \frac{1}{2}(|c_1|^2 - |c_2|^2)e_3 \tag{1.31}$$

and

$$\rho = \frac{1}{2}(|c_1|^2 + |c_2|^2) + \frac{1}{2}(c_1 c_2^*)(e_1 + ie_2) + \frac{1}{2}(c_1^* c_2)(e_1 - e_2 i) + \frac{1}{2}(|c_1|^2 - |c_2|^2)e_3 \tag{1.32}$$

Let us consider now when we relate $e_3 \to +1$. The (1.18) now hold in the (1.31) that becomes

$$\rho_M = |c_1|^2 \times I$$

Let us consider now when we relate $e_3 \to -1$. The (1.21) now hold in the (1.32) that becomes

$$\rho_M = |c_2|^2 \times I$$

being I the unity matrix.
The quantum interference terms now disappear.

References

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[2] E. Conte, Physics Essays, **6**, 4, (1994)