AGM-Like Paraconsistent Belief Change

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Abstract

Two systems of belief change based on paraconsistent logics are introduced in this paper by means of AGM-like postulates. The first one, AGMp, is defined over any paraconsistent logic that extends classical logic such that the law of excluded middle holds w.r.t. the paraconsistent negation. The second one, AGM∘, is specifically designed for paraconsistent logics known as *Logics of Formal Inconsistency* (**LFI**s), which have a formal *consistency* operator which allows to recover all the classical inferences. Besides the three usual operations over belief sets, namely *expansion*, *contraction* and *revision* (which is obtained from *contraction* by the Levi identity), the underlying paraconsistent logic allows us to define additional operations involving (non-explosive) contradictions. Thus, it is defined *external revision* (which is obtained from *contraction* by the *reverse Levi identity*), *consolidation* and *semi-revision*, all of them over belief sets. It is worth noting that the latter operations, introduced by S. Hansson, involve the temporary acceptance of contradictory beliefs, and so they were originally defined only for belief bases. Unlike to previous proposals in the literature, only defined for specific paraconsistent logics, the present approach can be applied to a general class of paraconsistent logics which are *supraclassical*, thus preserving the spirit of AGM. Moreover, representation theorems w.r.t. constructions based on selection functions are obtained for all the operations.

Keywords: Paraconsistent belief revision, AGM belief revision, paraconsistency, belief change, logics of formal inconsistency, contradiction.

1 Introduction

Belief Revision studies the dynamics of agents' epistemic states. The most influential paradigm in this area of study is the AGM model, introduced by C. Alchourrón, P. Gärdenfors and D. Makinson [1], in which epistemic states are represented as theories – considered simply as sets of sentences closed under logical consequence. Three types of *epistemic changes* (or operations) are considered in this model: *expansion*, the incorporation of a sentence into a given theory; *contraction*, the retraction of a sentence from a given theory; and *revision*, the incorporation of a sentence into a given consistent theory by ensuring the consistency of the resulting one.

From the definition of *revision* the role of consistency in AGM can be perceived. This role is grasped by the so-called *principle of consistency.* The reason for this is that the underlying logic is supraclassical (that is, it is an expansion of propositional classical logic) and therefore satisfies the *explosion principle.* If *explosion* holds, given a negation $\neg$ in a logic $\mathbf{L}$, from a contradiction $\{\beta, \neg\beta\}$ any sentence may be derived in $\mathbf{L}$. Such negation is called an *explosive* one. In this case there is only one inconsistent belief set, namely the whole language, so that given an inconsistent belief set all distinctions are lost.

This means that the belief system must dismiss contradictions in all operations. But this is not how cognitive agents behave in practice. It can be argued that real agents can accept contradictory statements without believing everything and losing all the distinctions (see, for instance, [7], [17] and [18, chapters 7 and 8]). In order to circumvent *explosion*, and define a more realistic model of belief revision, it is possible to weaken the logical closure minimally by assuming a non-explosive negation, i.e. a paraconsistent one. There are some investigations in the literature in this direction, which we briefly describe below (see [29] for a survey on AGM systems of belief change based on non-classical logics, including paraconsistent logics).

G. Restall and J. Slaney [20], based on the four-valued relevant logic of first-degree entailment, define an AGM-like contraction without satisfying the *recovery* postulate. As usual, revision is defined from contraction by the Levi identity. Based on the same logic, A. Tamminga [23] proposes a belief change system which uses finite representations of epistemic states, and which can deal with contradictory beliefs. Additionally, he analyzes the subject of belief change from the epistemological point of view. Also based on a four-valued logic (namely, the related Belnap and Dunn's logic), S. Chopra and R. Parikh [5] propose a model for belief revision that preserves an agent's ability to answer contradictory queries in a coherent way. In a conceptual paper, N. da Costa and O. Bueno [7] suggest a slight modification of the AGM postulates in order to deal with paraconsistent logics, in particular da Costa's C-systems C_n (for $n \geq 1$). E. Mares [16] developed a model in which an agent's belief state is represented by a pair of sets. One of these is the belief set, and the other consists of the sentences that the agent rejects. A belief state is coherent if and only if the intersection of these two sets is empty, i.e. if and only if there is no statement that the agent both accepts and rejects. In this model, belief revision preserves coherence but does not necessarily preserve consistency. G. Priest ([19] and [18, Chapter 8]) and K. Tanaka [24] suggested that, under a paraconsistent logic, revision can be performed just by adding sentences without removing anything. That is, revision could be defined as a simple expansion. Moreover, the question of defining an AGM-like revision operator for paraconsistent logics which differs from expansion is considered as an open problem in [11]. Furthermore, Priest [19] pointed out that in a paraconsistent framework, revision on belief sets can be performed as external revision, defined with the reversed Levi identity as suggested by S. Hansson [13] for bases. However, this idea was not technically developed.

The present paper goes further in this direction, by effectively defining such revisions with full technical details by means of an AGM-like system called AGMp. This belief change system can be defined not only for a specific paraconsistent logic (as in the previous approaches in the literature), but for *any* paraconsistent logic extending classical logic (thus being faithful to the *supraclassicality* desideratum of AGM) in which the paraconsistent negation satisfies the law of excluded middle. A second AGM-like system for belief change, called AGM$\circ$, is also introduced in this paper. It is specifically designed for the class of paraconsistent logics called Logics of Formal Inconsistency (**LFI**s) [4, 3, 2], in which a *consistency* operator $\circ$ allows to recover all the classical inferences (including the *explosion law*) within the logic, in a controlled way. Given that both AGMp and AGM$\circ$ are based on paraconsistent logics, they allow us to give a precise account of *external revision* for belief sets. Recall that the external revision, based on the *reverse Levi identity*, was proposed by Hansson in [13] only for belief bases (that is, sets of formulas which are not closed by logical consequences). Finally, additional change operations (*consolidation* and *semi-revisions*) proposed by Hansson in [14] to deal with belief bases are also extended to belief sets. These constructions can be applied to belief sets over any supraclassical paraconsistent logic (as the ones considered for AGMp) in which at least one contradiction (w.r.t. the paraconsistent negation) is not a theorem, which is quite a reasonable assumption. The richness of the language of the logics considered in this paper (in which there are two negations, a paraconsistent one and a classical one) allows us to consider two Levi identities (one for each negation), enlarging even more the expressive power of the two proposed paradigms. In particular, all the constructions can be applied to the **LFI**s studied in [4, 3, 2], which are decidable. Some of these logics are 3-valued and so it is possible to construct systems of belief change over them with potential concrete applications. It is important to notice that all the belief change operations presented in this paper are given by means of postulates, and fully characterized by concrete constructions through representation theorems.

The structure of the paper is as follows: Section 2 introduces the classical AGM model of belief change. Section 3 briefly discusses the Logics of Formal Inconsistency (**LFI**s), the class of paraconsistent logics adopted in this paper. The system AGMp is introduced in Section 4. The system AGM$\circ$ is presented in Section 5, in which the main features of **LFI**s play an important role. In Section 6 the operations of *consolidation* and *semi-revision*, introduced by S.

Hansson for belief bases, are extended to belief sets. In Section 7, AGMp and AGM$\circ$ are enlarged with a new revision operator defined by means of the Levi indentity w.r.t. the classical negation, and some examples show that revision does not necessarily collapse with expansion in a paraconsistent environment, as suggested by some authors. Finally, some concluding remarks are given in Section 8.

2 On AGM and Levi identities

The AGM model describes an idealized reasoner, called *agent*, with a family of (potentially infinite) sets of beliefs closed under logical consequences, called *epistemic states*. An agent can be a human, a computer program or any system able to subscribe beliefs and whose behaviour can be expected to be rational (see [9]). Those criteria can be summarized as follows (see, for instance, Gärdenfors and Rott [10]):

logical closure Any sentence logically entailed by beliefs in an epistemic state should be included in the epistemic state;

success (i) A sentence to be added is included in the outcome; (ii) A sentence to be contracted is not included in the outcome;

consistency preservation Where possible, epistemic states should remain consistent;

minimal change When changing epistemic states, loss of information should be kept to a minimum.

Those criteria are used to define rationality postulates for each operation. Together with these criteria, additional ones can be required, for instance

entrenchment Beliefs held in higher regard should be retained in favour of those held in lower regard.

Besides presenting postulates, it is possible to define the operations constructively. The result showing that certain explicit construction is fully characterized by a set of postulates is central in AGM theory and is called *representation theorem*. Once a representation theorem is proved one can examine the construction by studying the postulates that characterize it and, in this way, the details of implementation can be abstracted away.

As required in standard AGM, it is assumed that epistemic states are closed under logical consequences. This presupposes the existence of an underlying logic $\mathbf{L}$. The logic $\mathbf{L} = \langle \mathbb{L}, Cn \rangle$, where $\mathbb{L}$ is its language and Cn is its logical closure, is assumed to be *tarskian*, *finitary* and *structural*, that is, it is *standard* (see Appendix). Additionally, it is assumed *supraclassicality*, that is, the logic is closed under all the cassical connectives, being so an expansion of propositional classical logic. Moreover, the (classical) implication connective $\to$ is still *deductive*, i.e., it satisfies the *deduction-detachment* metatheorem:

$$\alpha \to \beta \in Cn(X) \text{ if and only if } \beta \in Cn(X \cup \{\alpha\}).^1$$

As usual, a closed theory of **L**, that is, a set of formulas K such that $K = Cn(K)$ will be called a *belief set*. Following that notation, a sentence α is said to be:

accepted if $\alpha \in K$.

rejected if $\neg\alpha \in K$.

indeterminate if $\alpha \notin K$ and $\neg\alpha \notin K$.

Expansion is certainly the simplest operation and can be easily achieved by the following equation:

$$K + \alpha =_{def} Cn(K \cup \{\alpha\})$$

The next subsections will briefly describe the more interesting operations of classical AGM, namely *contraction* and *revision*.

2.1 AGM contraction

In this section, the standard AGM postulates for contraction and a construction, called *partial meet contraction*, are presented. Both were originally presented in [1].

Contraction consists in the retraction of a belief-representing sentence α from K so as to ensure that: (i) the input in the new epistemic state $K \div \alpha$ should not be accepted, and (ii) the change should ensure *minimal change* – the fourth rationality criterion presented above. In order to achieve *consistency preservation*, *contraction* should retract some other sentences – namely those that entail α. To guarantee *minimal change*, this operation depends on extra-logical compounds for deciding which previous beliefs should be retracted – the *entrenchment* rationality criterion.

The *minimal change* criterion, although it is consensus, depends on the exact interpretation of "minimal". But this is far from a consensus view. There are several heuristics in order to measure the loss of information, as well as different postulates (to be presented below) intending to capture the intuition of information lost in different ways. The main reason for this debate is the fact that the logical form of the operations are not enough to express what must be abandoned in a belief change. Thus, as noted by Gärdenfors [9], extra-logical information is needed.

Let **L** be a logic as mentioned at the beginning of this section. The (basic) postulates for AGM contraction are the following:

[1] Note that, by adding rules to a logic which have a deductive implication $\to$, in the resulting logic the implication $\to$ could be non-deductive. Of course this never happens with axiomatic extensions. This is why we use in the general case the term 'still'.

Definition 2.1 (Postulates for AGM contraction) *An* AGM contraction *over* $\mathbf{L}$ *is a function* $\div : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *satisfying the following postulates:*

(÷closure) $K \div \alpha = Cn(K \div \alpha)$.[2]

(÷success) *If* $\alpha \notin Cn(\emptyset)$ *then* $\alpha \notin K \div \alpha$.

(÷inclusion) $K \div \alpha \subseteq K$.

(÷relevance) *If* $\beta \in K \setminus (K \div \alpha)$ *then there exists* X *such that* $K \div \alpha \subseteq Cn(X) \subseteq K$ *and* $\alpha \notin Cn(X)$*, but* $\alpha \in Cn(X) + \beta$.

(÷extensionality) *If* $(\alpha \leftrightarrow \beta) \in Cn(\emptyset)$ *then* $K \div \alpha = K \div \beta$

As a direct consequence of ÷*relevance* and ÷*inclusion*, it is immediate to see that ÷ also satisfies the following postulate:

(÷vacuity) If $\alpha \notin K$ then $K \div \alpha = K$.

Remark 2.2 *The* ÷relevance *postulate was introduced by Hansson in [12]. In the standard presentation of AGM, the* ÷recovery *postulate*

(÷recovery) $K \subseteq (K \div \alpha) + \alpha$

is placed instead of ÷relevance. *Among the AGM postulates for contraction,* ÷recovery *is the most controversial one. It guarantees that if* α *is contracted from* K *then the new belief set* $K \div \alpha$ *should retain enough information from* K *so that, if it is expanded by* α*, then it recovers every information from* K*. Despite being an important minimality criterion, several authors criticizes* ÷recovery *given its apparently strong assumption. However, as proved in [12],* ÷recovery *is equivalent to* ÷relevance *in the presence of the other postulates, which is arguably a much more intuitive property.*

In [1] the authors present other two postulates, called supplementaries, concerning contraction of sentences of the form $\alpha \wedge \beta$. These will not be discussed in this paper since they are not the focus of this work.

2.1.1 AGM partial meet contraction

The literature on belief revision advances several constructions for contraction that satisfy the AGM postulates. Since we are interested in minimal modifications, as stated by the rationality criteria and captured by the postulates above, the construction should capture that fact.

This can be noted by the explicit construction that defines a contraction. We describe here the one introduced in [1], called *partial meet contraction*, constructed as follows:

[2] Rigorously speaking, this postulate is redundant since by definition the co-domain of the function ÷ is Th($\mathbf{L}$). However, in order to keep closer to the classical AGM presentation, we decide to maintain this postulate in all the operations presented here.

1. Choose some maximal subsets of K (with respect to the inclusion) that do not entail α.

2. Take the intersection of such sets.

The *remainder* of K and α is the set of all maximal subsets of K that do not entail α. Formally the definition is the following:

Definition 2.3 (Remainder [1]) *Let K be a belief set, and let α be a formula. A set $X \subseteq \mathbb{L}$ is a maximal subset of K that does not entail α if and only if:*[3]

(i) *$X \subset K$.*

(ii) *$\alpha \notin Cn(X)$.*

(iii) *If $X \subset X' \subseteq K$ then $\alpha \in Cn(X')$.*

The set of all the maximal subsets of K that do not entail α is called the remainder set of (K, α)*, and is denoted by $K \perp \alpha$.*

Observe that, if $\alpha \notin K$ or $\alpha \in Cn(\emptyset)$ then $K \perp \alpha = \emptyset$ (the converse also holds, see Corollary E in the Appendix). Typically $K \perp \alpha$ may contain more than one maximal subset. The main idea constructing a contraction function is to apply a *selection function* γ which intuitively selects the sets in $K \perp \alpha$ containing the beliefs that the agent holds in higher regard (the beliefs more epistemically entrenched). Such qualitative distinction between beliefs is the extra-logical factor previously mentioned.

Definition 2.4 (AGM selection function) *An* AGM selection function *in* $\mathbf{L}$ *is a function $\gamma : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ such that, for every K, α and β:*

1. *$\gamma(K, \alpha) = \gamma(K, \beta)$ if $(\alpha \leftrightarrow \beta) \in Cn(\emptyset)$;*

2. *$\gamma(K, \alpha) \subseteq K \perp \alpha$ if $K \perp \alpha \neq \emptyset$;*

3. *$\gamma(K, \alpha) = \{K\}$ otherwise.*

Observe that, if $(\alpha \leftrightarrow \beta) \in Cn(\emptyset)$ then $K \perp \alpha = K \perp \beta$, for any belief set K, and so the notion of AGM selection function is well-defined.

The *AGM partial meet contraction* is the intersection of the sets selected by the AGM selection function:

$$K \div_\gamma \alpha =_{def} \bigcap \gamma(K, \alpha).$$

Notice that, if $(\alpha \leftrightarrow \beta) \in Cn(\emptyset)$ then, by definition, $\gamma(K, \alpha) = \gamma(K, \beta)$, for any belief set K. Being so, the $\div$*extensionality* postulate holds for every partial meet contraction operator.

The following classical result (as presented in [1]) holds:

[3] The usual definition requires in item (i) that $X \subseteq K$. We decided to slightly modify item (i) by requiring instead that $X \subset K$.

Theorem 2.5 (Representation for AGM contraction) *An operation* $\div$: $Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *satisfies the postulates of AGM contraction (see Definition 2.1) iff there exists an AGM selection function* γ *in* $\mathbf{L}$ *such that* $K \div \alpha = \bigcap \gamma(K, \alpha)$, *for every* K *and* α.

2.2 AGM Revision

Relative to *revision*, the two major tasks of this operation are (i) adding the new belief α to the belief set K (ii) ensuring that the resulting belief set $K * \alpha$ is consistent (unless α is inconsistent itself).[4] The first task can be accomplished by expanding K by α, that is, $K + \alpha$. The second can be accomplished by prior contracting of K by the (explosive) negation of α, that is, $K \div \sim\alpha$, recalling that the underlying logic $\mathbf{L}$ is supraclassical. The composition of the mentioned sub-operations gives rise to the following definition of *revision* (Levi Identity) [15]:

Definition 2.6 (Internal revision defined by Levi identity) $K * \alpha =_{def} (K \div \sim\alpha) + \alpha$

Alternatively, as suggested by Hansson [13], the two sub-operations may take place in reverse order (reverse Levi identity) but this latter is, classically, only possible if the belief set is not closed under logical consequences (namely, belief base theory).

Definition 2.7 (External revision defined by reverse Levi identity) $K \circledast \alpha =_{def} (K + \alpha) \div \sim\alpha$

External and *internal revision* differ in their logical properties and neither of them can be subsumed under the other. Intuitively, *external revision* by α has an intermediate contradictory state in which both α and $\sim\alpha$ are accepted, whereas *internal revision* has an intermediate non-committed state in which neither α nor $\sim\alpha$ are accepted. Of course *external revision* lies outside the scope of classical AGM for belief sets. In the next sections it will be shown how to overcome this limitation by means of a paraconsistent negation.

Now, let us return to the standard AGM system, in which the (explosive) negation of the underlying supraclassical logic $\mathbf{L}$ is denoted by $\sim$.

Definition 2.8 (AGM (internal) revision) *An* AGM revision *over* $\mathbf{L}$ *is an operation* $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *satisfying the following postulates:*

($*$closure) $K * \alpha = Cn(K * \alpha)$.

($*$success) $\alpha \in K * \alpha$.

[4] Observe that, in the standard AGM model, a consistent belief set is a set K of sentences closed by logical consequences, which is non-trivial – that is, there is some sentence which does not belong to K. Equivalently, a consistent belief set is a closed set of sentences which does not contain any contradiction. Otherwise, it is inconsistent. Of course there is just one inconsistent belief set, namely the set $\mathbb{L}$ of all the sentences of the underlying logic.

($*$inclusion) $K * \alpha \subseteq K + \alpha$.

($*$vacuity) *If* $\sim\alpha \notin K$ *then* $K + \alpha \subseteq K * \alpha$.

($*$non-contradiction) *If* $\sim\alpha \notin Cn(\emptyset)$ *then* $\sim\alpha \notin K * \alpha$.

($*$relevance) *If* $\beta \in K \setminus (K * \alpha)$ *then there exists* X *such that* $K \cap (K * \alpha) \subseteq Cn(X) \subseteq K$, $\sim\alpha \notin Cn(X)$ *and* $\sim\alpha \in Cn(X) + \beta$.

($*$extensionality) *If* $\alpha \leftrightarrow \beta \in Cn(\emptyset)$ *then* $K * \alpha = K * \beta$.

Remark 2.9 *The* $*$non-contradiction *postulate was slightly modified with respect to the usual presentations of AGM, by replacing '*$K * \alpha \neq \mathbb{L}$*' by '*$\sim\alpha \notin K * \alpha$*'. In view of the* $*$success *postulate and the fact that* $\sim$ *is explosive, both conditions are clearly equivalent. This alternative presentation of the postulate will allow the generalization of the notions to paraconsistent logics. As a consequence of this move, two versions of the postulate are admitted, by considering the explosive negation* $\sim$ *and the paraconsistent one, say* $\neg$*, respectively. This possibility will be explored in Subsection 7.1.*

As it might be expected, AGM (internal) revisions defined from partial meet contractions are fully characterized by the postulates of Definition 2.8.

Theorem 2.10 *An operation* $* : Th(\mathbf{L}) \times \mathbb{L} \to Th(\mathbf{L})$ *is an AGM revision over* $\mathbf{L}$ *iff it is an internal partial meet revision operator over* $\mathbf{L}$*, that is: there is an AGM selection function* γ *in* $\mathbf{L}$ *such that* $K * \alpha = \left(\bigcap \gamma(K, \sim\alpha)\right) + \alpha = (K \div_\gamma \sim\alpha) + \alpha$*, for every* K *and* α*.*

3 On Paraconsistency

As observed in [3], contradictoriness (the presence of contradictions in a theory) and triviality (the fact that such a theory entails all possible consequences) are assumed inseparable in classical logic. This is a consequence of a metalogical property known as *explosiveness* (*ex falso quodlibet* or *ex contradictione sequitur quodlibet*). According to this principle, from a contradiction everything is logically derivable. Therefore classical logic (like many other logics with an explosive negation) identify 'consistency' with 'freedom from contradictions'.

Paraconsistency is the study of logic systems having (at least) a negation which is non-explosive. Within these logics, 'consistent theory' is no longer synonymous of 'non-contradictory theory' (at least w.r.t. the paraconsistent negation). Equivalently, in a paraconsistent logic the notions of 'inconsistent theory' and 'contradictory theory' (w.r.t. the paraconsistent negation) do not necessarily coincide. Thus, the pragmatic point of paraconsistency is not whether contradictory theories exist, but how to deal with them. These distinctions will be fundamental in order to consider systems of paraconsistent belief dynamics.

3.1 The Logics of Formal Inconsistency

The Logics of Formal Inconsistency (**LFI**s) [4, 3, 2] constitute a class of paraconsistent logics that can internalize the meta-theoretical notions of consistency and inconsistency by means of formulas of the object language. As a consequence, despite constituting fragments of consistent logics, the **LFI**s can canonically be used to faithfully encode all the consistent inferences.

Roughly speaking, the idea is to express the meta-theoretical notions of consistency and inconsistency at the object language level by adding a new connective $\bullet$ to the language with the intended meaning of "being inconsistent". However, it is the dual $\circ$ expressing "being consistent" that is more frequently used. In this way one can limit the applicability of the explosion principle to the case when α is consistent and so:

(1) $\alpha, \neg\alpha \vdash \beta$ is not the case in general

(2) $\alpha, \neg\alpha, \circ\alpha \vdash \beta$ is always the case.

Condition (2) is usually called the *Gentle Explosion Principle*, in contrast to the usual *Explosion Principle* which states that everything follows from a plain contradiction.

The two systems to be presented here are defined over **LFI**s, but the constructions of the second are specially related to the formal consistency operator.[5] Specifically, we define the constructions over a particular class of **LFI**s, developed by Carnielli, Coniglio and Marcos [3] (see also [2]), in which the formal consistency is taken as a primitive operator. The most basic **LFI** considered there is the propositional logic **mbC**. The language $\mathbb{L}$ of **mbC** is generated by the connectives $\wedge, \vee, \rightarrow, \neg, \circ$.

Definition 3.1 (mbC [3]) *The logic* **mbC** *is defined over the language* $\mathbb{L}$ *by means of a Hilbert system as follows:*

Axioms:

(A1) $\alpha \rightarrow (\beta \rightarrow \alpha)$

(A2) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow (\beta \rightarrow \delta)) \rightarrow (\alpha \rightarrow \delta))$

(A3) $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$

(A4) $(\alpha \wedge \beta) \rightarrow \alpha$

(A5) $(\alpha \wedge \beta) \rightarrow \beta$

(A6) $\alpha \rightarrow (\alpha \vee \beta)$

[5] Notably the terms consistency and inconsistency captures a more sensible definition in the **LFI**s. In order to avoid misunderstanding, in this paper it will be used, for those logics, specifically the terms *formal consistency* and *formal inconsistency*. So the terms *consistency* and *inconsistency* will maintain the usual interpretation, namely *non-triviality* and *triviality*, respectively.

(A7) $\beta \to (\alpha \vee \beta)$

(A8) $(\alpha \to \delta) \to ((\beta \to \delta) \to ((\alpha \vee \beta) \to \delta))$

(A9) $\alpha \vee (\alpha \to \beta)$

(A10) $\alpha \vee \neg\alpha$

(bc1) $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$

Inference Rule:

(Modus Ponens) $\alpha, \alpha \to \beta \vdash \beta$

It is worth noticing that (A1)-(A9) plus *Modus Ponens* constitutes an axiomatization for the classical positive logic $\mathbf{CPL}^+$. The *falsum* (or bottom) is defined in **mbC** by means of the formula $\bot_\beta =_{def} \beta \wedge \neg\beta \wedge \circ\beta$, for any formula β. From this, the classical negation (or strong negation) is defined in **mbC** by $\sim_\beta\alpha =_{def} (\alpha \to \bot_\beta)$. Since $\bot_\beta$ and $\bot_{\beta'}$ are interderivable in **mbC**, for any β and β', then $\sim_\beta\alpha$ and $\sim_{\beta'}\alpha$ are also interderivable. Hence, the strong negation of α will be denoted by $\sim\alpha$, while $\bot$ will denote any formula $\bot_\beta$. As usual, $\alpha \leftrightarrow \beta$ is an abbreviation for $(\alpha \to \beta) \wedge (\beta \to \alpha)$.

Since **mbC** is an axiomatic extension of $\mathbf{CPL}^+$, and a classical negation $\sim$ can be defined in **mbC**, it is clear that **mbC** can be seen as an expansion of the classical propositional logic **CPL** by adding a paraconsistent negation $\neg$ and a consistency operator $\circ$ satisfying certain axioms. Thus, **mbC** is *standard*, *supraclassical* and *deductive* (see Section 2). An interesting way to recover the classical logic inside **mbC** is by assuming the consistency of suitable sets of premises. In formal terms, consider **CPL** defined over the language $\mathbb{L}_0$ generated by the connectives $\wedge, \vee, \to, \neg$ (observe that, now, $\neg$ represents the classical negation instead of the paraconsistent negation of **mbC**). Let us denote by $\vdash_{\mathbf{CPL}}$ the consequence relation in **CPL**, while $\vdash_{\mathbf{mbC}}$ will denote the corresponding one in **mbC**. If $Y \subseteq \mathbb{L}_0$ then $\circ(Y) = \{\circ\alpha \ : \ \alpha \in Y\}$. Then, the following result can be obtained:

Theorem 3.2 (Derivability Adjustment Theorem [3]) *Let $X \cup \{\alpha\}$ be a set of formulas in $\mathbb{L}_0$. Then $X \vdash_{\mathbf{CPL}} \alpha$ if and only if $\circ(Y), X \vdash_{\mathbf{mbC}} \alpha$ for some $Y \subseteq \mathbb{L}_0$.*

For instance $\alpha \to \beta \nvdash_{\mathbf{mbC}} \neg\beta \to \neg\alpha$ despite $\alpha \to \beta \vdash_{\mathbf{CPL}} \neg\beta \to \neg\alpha$. However, $\circ\beta, \alpha \to \beta \vdash_{\mathbf{mbC}} \neg\beta \to \neg\alpha$ is always the case. In particular, **CPL** can be obtained from **mbC** by adding $\circ\alpha$ as an axiom schema (see [3]). The logic **mbC** (as well as every extension of it considered in the literature) is semantically characterized by valuations over $\{0, 1\}$ also called *bivaluations*).

Definition 3.3 (Valuations for mbC) *A function $v : \mathbb{L} \to \{0, 1\}$ is a* valuation for **mbC** *if it satisfies the following clauses:*

(vAnd) $v(\alpha \wedge \beta) = 1 \quad \Longleftrightarrow \quad v(\alpha) = 1$ *and* $v(\beta) = 1$

(vOr) $v(\alpha \vee \beta) = 1 \iff v(\alpha) = 1$ *or* $v(\beta) = 1$

(vImp) $v(\alpha \to \beta) = 1 \iff v(\alpha) = 0$ *or* $v(\beta) = 1$

(vNeg) $v(\neg\alpha) = 0 \implies v(\alpha) = 1$

(vCon) $v(\circ\alpha) = 1 \implies v(\alpha) = 0$ *or* $v(\neg\alpha) = 0$.

The semantical consequence relation associated to valuations for **mbC** is defined as expected: $X \models_{\mathbf{mbC}} \alpha$ iff, for every **mbC**-valuation v, if $v(\beta) = 1$ for every $\beta \in X$ then $v(\alpha) = 1$. The following result is well-known:

Theorem 3.4 (Adequacy of mbC w.r.t. bivaluations [3]) *For every set of formulas $X \cup \{\alpha\}$: $X \vdash_{\mathbf{mbC}} \alpha$ if and only if $X \models_{\mathbf{mbC}} \alpha$.*

Remark 3.5 (Relationship between the two negations in mbC) *As observed above,* **mbC** *have two distinct negations: the paraconsistent (and primitive) $\neg$, and the (derived) classical negation $\sim$. As one could expect, $\sim\alpha \vdash_{\mathbf{mbC}} \neg\alpha$ (and so $\sim\alpha \to \neg\alpha$ is a theorem of* **mbC**, *given that* **mbC** *satisfies the deduction-detachment metatheorem), but the converse does not hold in general.*[6] *This means that, given a belief set K in* **mbC**, *$\sim\alpha \in K$ implies that $\neg\alpha \in K$ (but the converse is not necessarily true). As a consequence of this, if $\neg\alpha \notin K$ then certainly $\sim\alpha \notin K$. This implies that $\sim\alpha \notin K \div \neg\alpha$ (K contracted by $\neg\alpha$), unless $\neg\alpha$ is a theorem, and so $(K \div \neg\alpha) + \alpha$ is not trivial. Since $K * \alpha = (K \div \neg\alpha) + \alpha$ by Levi identity w.r.t. $\neg$ (recall Subsection 2.2), this feature of* **mbC** *is extremely relevant. The Levi identity in paraconsistent belief revision will be additionally analyzed in Subsection 7.1.*

Different **LFI**s generate distinct logical consequences and therefore substantially alter the rationality captured by the *principle of deductive closure.*

Definition 3.6 (Extensions of mbC [2]) *Consider the following axioms:*

(ciw) $\circ\alpha \vee (\alpha \wedge \neg\alpha)$

(ci) $\neg\circ\alpha \to (\alpha \wedge \neg\alpha)$

(cl) $\neg(\alpha \wedge \neg\alpha) \to \circ\alpha$

(cf) $\neg\neg\alpha \to \alpha$

(ce) $\alpha \to \neg\neg\alpha$

(cc) $\circ\circ\alpha$

Some interesting extensions of **mbC** *are the following:*

---

[6]In order to prove this, observe that $\sim\alpha, \alpha \vdash_{\mathbf{mbC}} \neg\alpha$ (since $\sim$ is an explosive negation), while obviously $\sim\alpha, \neg\alpha \vdash_{\mathbf{mbC}} \neg\alpha$. Then $\sim\alpha, \alpha \vee \neg\alpha \vdash_{\mathbf{mbC}} \neg\alpha$ (by the properties of $\vee$) and so $\sim\alpha \vdash_{\mathbf{mbC}} \neg\alpha$, since $\alpha \vee \neg\alpha$ is a theorem of **mbC**. To prove that $\neg\alpha \nvdash_{\mathbf{mbC}} \sim\alpha$ it is enough to consider a valuation v such that $v(\alpha) = v(\neg\alpha) = 1$.

mbCciw = **mbC** + *(ciw)*

Cbr = **mbC** + *(ciw)* + *(ce)* + *(cf)* = = **mbCciw** + *(ce)* + *(cf)*[7]

mbCci = **mbC** + *(ci)* = **mbCciw** + *(cc)* *(see [2, Proposition 3.1.10(3)])*

bC = **mbC** + *(cf)*

Ci = **mbC** + *(ci)* + *(cf)* = **mbCci** + *(cf)*

mbCcl = **mbC** + *(cl)*

Cil = **mbC**+*(ci)*+*(cf)*+*(cl)* = **mbCci**+*(cf)*+*(cl)* = **mbCcl**+*(cf)*+*(ci)* = **Ci**+*(cl)*

The semantical characterization by bivaluations for all these extensions of **mbC** can be easily obtained from the one for **mbC** (see [3, 2]). For instance, **mbCciw** is characterized by **mbC**-valuations such that $v(\circ\alpha) = 1$ if and only if $v(\alpha) = 0$ or $v(\neg\alpha) = 0$ (if and only if $v(\alpha) \neq v(\neg\alpha)$).

The technical details of these logics as well as a taxonomy of **LFI** systems can be found in the references [4, 3, 2].

4 The AGMp system

In this section it will be introduced AGMp, the first of the AGM-like systems of belief dynamics based on paraconsistent logics proposed in this paper.

The main difference of AGMp relative to standard AGM is that epistemic states are represented by logically closed sets of sentences over a given paraconsistent logic expanding classical logic (in order to ensure *supraclassicality*). In particular, it applies to any **LFI** which is an extension of **mbC**. The constructions of AGMp are based on the same postulates of AGM by considering suitable modifications. It is important to notice that the consistency operator $\circ$ will not be explicitly considered in the postulates. A second proposal, AGM$\circ$ (see Section 5), is fully oriented to **LFI**s and so the consistency operator will play a fundamental role in that system.

4.1 Formal Preliminaries

Let us assume a standard propositional logic, namely $\mathbf{L} = \langle \mathbb{L}, Cn \rangle$, which expands classical propositional logic **CPL** with a paraconsistent negation $\neg$ such that $\alpha \vee \neg\alpha \in Cn(\emptyset)$ for every sentence α and where the implication $\rightarrow$ is deductive. For instance, **L** could be an axiomatic extension of **mbC**. In the terminology of AGM, this means that **L** is *supraclassical* (recall Section 2). The deductively closed theories of **L** are called belief sets (or epistemic states) of **L**. The set of belief sets of **L** is denoted by $Th(\mathbf{L})$. Assume that the language $\mathbb{L}$ of **L** contains the connectives $\wedge, \vee, \rightarrow, \neg$. If **L** is an **LFI** as the ones considered in Subsection 3.1, the language also contains a consistency operator $\circ$ such that the strong (explosive) negation is defined by $\sim\alpha =_{def} (\alpha \rightarrow \bot)$. The consequence relation of **L** will be denoted by $\vdash$.

[7]This system was not considered in [4, 3, 2], and it is originally presented in this paper.

Remark 4.1 *As observed in [2, Theorem 3.4.10], any logic* **L** *such as the ones considered above turns out to be an* **LFI** *by defining a consistency operator as* $\circ'\alpha =_{def} {\sim}(\alpha \wedge \neg\alpha)$*, where* $\sim$ *and* $\neg$ *denote the explosive and paraconsistent negation respectively. Moreover, the axiom (ciw) holds for* $\circ'$ *in* **L**. *The only difference between* **L** *and the* **LFI***s extending* **mbC** *is that in the latter the consistency operator* $\circ$ *is primitive. Hence,* **mbC** *and the* **LFI***s extending it have two different consistency operators: a primitive* $\circ$*, which can satisfy or not axiom (ciw), and a derived* $\circ'$ *which always satisfies (ciw).*

4.2 AGM-compliance

An interesting idea proposed by Flouris [8] is to elucidate the applicability of the AGM system in several non-classical logics, a concept called *AGM-compliance*. An AGM-compliant logic is simply one in which it is possible to completely characterize *contraction* via classical postulates.

It is known that standard, supraclassical and deductive logics, like the ones considered in AGMp, are AGM-compliant. Furthermore, in this kind of logic the properties of $\div$*recovery* and $\div$*relevance* are still equivalent. Hence, although this equivalence is not valid in general (see [21, 22]), both postulates can be used indistinguishably for the logics considered here. By coherence with the presentation of the standard AGM system given in Section 2, in what follows the $\div$*relevance* postulate will be considered instead of $\div$*recovery*.

4.3 (Non-extensional) AGMp contraction

As mentioned in the previous section, AGMp *contraction* can be defined as in the classical AGM system one due to the AGM-compliance of the involved logics. Since this operation depends on extra-logical components, it is defined indirectly via the set of rationality postulates from AGM. However, *extensionality* is problematic in this context, and several modifications are required in order to retain it (see Subsection 4.5).

Given the AGM-compliance of the logics considered here, it is possible to define an AGMp contraction by means of AGM selection functions, which guarantee the satisfaction of the $\div$*extensionality* postulate. However, when defining a revision $*$ from a contraction $\div$ by Levi identity (recall Definition 2.6), but now with respect to the paraconsistent negation $\neg$, some problems can arise. Indeed, in the paraconsistent setting, the paraconsistent negation $\neg$ does not preserve, in general, logical equivalences. Thus, assume that α is logically equivalent to β. Since an internal revision $K * \alpha$ pressuposes a previous contraction $K \div \neg\alpha$, and since there is no guarantee of the equivalence between $\neg\alpha$ and $\neg\beta$ from the given hypotesis (and, conversely, the equivalence between $\neg\alpha$ and $\neg\beta$ does not imply the equivalence between α and β), the belief sets $K * \alpha$ and $K * \beta$ should not necessarily coincide. That is, the revision should not satisfy the $*$*extensionality* postulate in general.

As a consequence of this, in the proof of the representation theorem for revision, the selection function defined from a given AGMp revision $*$ could not satisfy the first condition of AGM selection function, and so the induced contraction could not satisfy $\div$*extensionality*. This is why $\div$*extensionality* (and, in consequence, $*$*extensionality*) is dropped in the definition of the basic AGMp operators. In Subsection 4.5 below we will show how to define contractions and revisions in AGMp satisfying a suitable form of extensionality.

Definition 4.2 (Postulates for AGMp contraction) *An* AGMp contraction *over* $\mathbf{L}$ *is a function* $\div : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *satisfying all the postulates of an AGM contraction (see Definition 2.1) with the exception of* $\div$extensionality*.*

As in the case of AGM contraction, an AGMp contraction $\div$ also satisfies $\div$*vacuity*. As a consequence of Definition 4.2, the selection functions of AGM must be weakened in order to generate AGMp contractions.

Definition 4.3 (AGMp selection function) *An* AGMp selection function *in* $\mathbf{L}$ *is a function* $\gamma : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ *such that, for every* K*,* α *and* β*:*

1. $\gamma(K, \alpha) \subseteq K \bot \alpha$ *if* $K \bot \alpha \neq \emptyset$*;*

2. $\gamma(K, \alpha) = \{K\}$ *otherwise.*

The *AGMp partial meet contraction* is the intersection of the sets selected by the AGMp selection function:

$$K \div_\gamma \alpha =_{def} \bigcap \gamma(K, \alpha).$$

By simplifyng the proof for standard AGM, the following result follows easily:

Theorem 4.4 (Representation for AGMp contraction) *An operation* $\div : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *satisfies the postulates of AGMp contraction iff there exists an AGMp selection function* γ *in* $\mathbf{L}$ *such that* $K \div \alpha = \bigcap \gamma(K, \alpha)$*, for every* K *and* α*.*

4.4 (Non-extensional) AGMp revisions

As discussed in Subsection 4.3, an important difference between AGMp and classical AGM is that the $*$*extensionality* postulate will not be valid in general. The corresponding operation satisfying a variant of this postulate will be presented in Subsection 4.5.

Definition 4.5 (AGMp internal revision) *An* AGMp internal revision *over* $\mathbf{L}$ *is an operation* $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *satisfying all the postulates of an AGM revision (see Definition 2.8) with the exception of* $*$extensionality*, and where the explosive negation* $\sim$ *is replaced by the paraconsistent negation* $\neg$*.*

In other words, an AGMp internal revision is an operation satisfying the $*$*closure*, $*$*success* and $*$*inclusion* postulates of Definition 2.8, plus the following:

(**∗vacuity**) If $\neg\alpha \notin K$ then $K + \alpha \subseteq K * \alpha$.

(**∗non-contradiction**) If $\neg\alpha \notin Cn(\emptyset)$ then $\neg\alpha \notin K * \alpha$.

(**∗relevance**) If $\beta \in K \setminus (K * \alpha)$ then there exists X such that $K \cap (K * \alpha) \subseteq Cn(X) \subseteq K$, $\neg\alpha \notin Cn(X)$ and $\neg\alpha \in Cn(X) + \beta$.

As in the case of classical AGM, AGMp internal revisions defined from partial meet contractions are fully characterized by the postulates of Definition 4.5.

Theorem 4.6 (Representation for AGMp internal revision) *An operation* $* : Th(\mathbf{L}) \times \mathbb{L} \to Th(\mathbf{L})$ *is an AGMp internal revision over* $\mathbf{L}$ *iff it is an internal partial meet revision operator over* $\mathbf{L}$*, that is: there is an AGMp selection function* γ *in* $\mathbf{L}$ *such that* $K * \alpha = \left(\bigcap \gamma(K, \neg\alpha)\right) + \alpha = (K \div_\gamma \neg\alpha) + \alpha$*, for every* K *and* α*.*

The most interesting case to be analized in this section is the definition, by means of the reverse Levi identity with respect to the paraconsistent negation, of an *external revision* for theories in $\mathbf{L}$ (recall Definition 2.7), extending the results of Hansson for belief bases (see [13]). Just as with AGMp internal revision, the AGMp external revision will not satisfy the ∗*extensionality* postulate in general. An extensional version of AGMp external revision will be defined in Subsection 4.5.

Definition 4.7 (AGMp external revision) *An* AGMp external revision *over* $\mathbf{L}$ *is an operation* $\circledast : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *satisfying the following postulates:*

(**⊛closure**) $K \circledast \alpha = Cn(K \circledast \alpha)$.

(**⊛success**) $\alpha \in K \circledast \alpha$.

(**⊛inclusion**) $K \circledast \alpha \subseteq K + \alpha$.

(**⊛non-contradiction**) *If* $\neg\alpha \notin Cn(\emptyset)$ *then* $\neg\alpha \notin K \circledast \alpha$.

(**⊛relevance**) *If* $\beta \in K \setminus (K \circledast \alpha)$ *then there exists* X *such that* $K \circledast \alpha \subseteq Cn(X) \subseteq K + \alpha$, $\neg\alpha \notin Cn(X)$ *and* $\neg\alpha \in Cn(X) + \beta$.

(**⊛pre-expansion**) $(K + \alpha) \circledast \alpha = K \circledast \alpha$.

Using the ⊛*pre-expansion*, ⊛*relevance* and ⊛*inclusion* postulates, it is easy to prove that any external revision $\circledast$ also satisfies the following postulate:

(**⊛vacuity**) If $\neg\alpha \notin K$ then $K \circledast \alpha = K + \alpha$.

From this, it can be observed that the differences between an internal and an external revision in AGMp lie in the particular form of the *relevance* postulate, and in the fact that the latter satisfies additionally the ⊛*pre-expansion* postulate.

By reverse Levi identity (see Definition 2.7) we use the AGMp *partial meet contraction* to define a construction for an *external revision* operator defined over belief sets instead of belief bases:

$$K \circledast_\gamma \alpha =_{def} (K + \alpha) \div_\gamma \neg\alpha = \bigcap \gamma(K + \alpha, \neg\alpha).$$

As expected, *external partial meet revision* is fully characterized by the postulates of Definition 4.7.

Theorem 4.8 (Representation for AGMp external revision) *An operation $\circledast : Th(\mathbf{L}) \times \mathbb{L} \to Th(\mathbf{L})$ is an AGMp external revision over $\mathbf{L}$ iff it is an external partial meet revision operator over $\mathbf{L}$, that is: there is an AGMp selection function γ in $\mathbf{L}$ such that $K \circledast \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$, for every K and α.*

Remark 4.9 *The logical possibility of defining external revision challenges the necessity of a prior contraction, as in the internal revision. Thus, it is possible to interpret the contraction underlying an internal revision as an unnecessary retraction and therefore as a violation of the principle of minimality. On the other hand, if we consider the non-contradiction principle as a priority, then internal revision remains the only rational option. This illustrates the clear opposition between the principle of non-contradiction and that of minimality, as explored by R. Testa in [26].*

4.5 AGMp contractions and revisions with extensionality

In order to guarantee a suitable form of extensionality for the contraction and revision operators in AGMp, more general selection functions must be considered. Additionally, it must be required that, in the underlying logic $\mathbf{L}$, α and $\neg\neg\alpha$ be equivalent, for every sentence α.

From now on, given a standard logic $\mathbf{L}$, and two sentences α and β, $\alpha \equiv_{\mathbf{L}} \beta$ means that $\alpha \leftrightarrow \beta \in Cn(\emptyset)$.

The (non-extensional) belief change operations defined in sections 4.3 and 4.4 can thus be modified as follows:

Definition 4.10 (Extensional AGMp contraction) *Let $\mathbf{L}$ be a logic as described in Subsection 4.1 such that, in addition, $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ for every sentence α. An* extensional AGMp contraction *over $\mathbf{L}$ is an AGMp contraction $\div : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ (see Definition 4.2) which additionally satisfies the following postulate:*

($\div$extensionality) *If $\alpha \equiv_{\mathbf{L}} \beta$ and $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$ then $K \div \alpha = K \div \beta$.*

Definition 4.11 (Extensional AGMp internal revision) *Let $\mathbf{L}$ be a logic as described in Subsection 4.1 such that $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ for every sentence α. An* extensional AGMp internal revision *over $\mathbf{L}$ is an AGMp internal revision $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ (see Definition 4.5) which additionally satisfies the following postulate:*

($*$extensionality) *If* $\alpha \equiv_{\mathbf{L}} \beta$ *and* $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$ *then* $K * \alpha = K * \beta$.

Definition 4.12 (Extensional AGMp external revision) *Let* $\mathbf{L}$ *be a logic as described in Subsection 4.1 such that* $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ *for every sentence* α. *An* extensional AGMp external revision *over* $\mathbf{L}$ *is an AGMp external revision* $\circledast : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *(see Definition 4.7) which additionally satisfies the following postulate:*

($\circledast$extensionality) *If* $\alpha \equiv_{\mathbf{L}} \beta$ *and* $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$ *then* $K \circledast \alpha = K \circledast \beta$.

As mentioned above, the construction of extensional AGMp contraction and revisions requires a wider notion of selection function.

Definition 4.13 (General AGMp selection function) *A* general AGMp selection function *in* $\mathbf{L}$ *is a function* $\gamma : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ *such that, for every* K *and* α*:*

1. $\gamma(K, \alpha) = \gamma(K, \beta)$ *if* $\alpha \equiv_{\mathbf{L}} \beta$ *and* $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$*;*
2. $\gamma(K, \alpha) \subseteq K \perp \alpha$ *if* $K \perp \alpha \neq \emptyset$*;*
3. $\gamma(K, \alpha) = \{K\}$ *otherwise.*

As in the case of AGM selection functions, the notion is well-defined.

The extensional AGMp *partial meet contraction* generated by a general AGMp selection function γ is the intersection of the sets selected by γ:

$$K \div_{\gamma} \alpha =_{def} \bigcap \gamma(K, \alpha).$$

Theorem 4.14 (Representation for extensional AGMp contraction) *Let* $\mathbf{L}$ *be a logic as described in Subsection 4.1 such that* $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ *for every sentence* α. *An operation* $\div : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *is an extensional AGMp contraction iff there exists a general AGMp selection function* γ *in* $\mathbf{L}$ *such that* $K \div \alpha = \bigcap \gamma(K, \alpha)$*, for every* K *and* α.

Analogous results can be obtained for the other belief change operators:

Theorem 4.15 (Representation for extensional AGMp internal revision) *Let* $\mathbf{L}$ *be a logic as described in Subsection 4.1 such that* $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ *for every sentence* α. *An operation* $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *is an extensional AGMp internal revision iff there exists a general AGMp selection function* γ *in* $\mathbf{L}$ *such that* $K * \alpha = \big(\bigcap \gamma(K, \neg\alpha)\big) + \alpha$*, for every* K *and* α.

Theorem 4.16 (Representation for extensional AGMp external revision) *Let* $\mathbf{L}$ *be a logic as described in Subsection 4.1 such that* $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ *for every sentence* α. *An operation* $\circledast : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *is an extensional AGMp external revision iff there exists a general AGMp selection function* γ *in* $\mathbf{L}$ *such that, for every* K *and* α, $K \circledast \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$.

5 The AGM∘ system

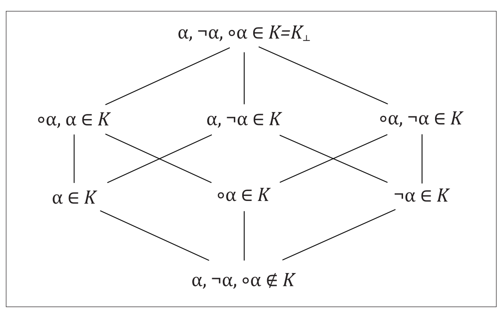
In this section, a second AGM-like system of paraconsistent belief change will be introduced, called AGM∘. The name is motivated by the fact that, different to AGMp, the consistency operator $\circ$ of **LFI**s will play a fundamental role in the postulates and the corresponding constructions.

The main idea of this system is incorporating the notion of formal consistency in the contraction operator, thus being trasferred to revisions by means of the (direct or reverse) Levi identity. The intuition behind AGM∘ is that a belief α being consistent in K (that is, $\circ\alpha \in K$) means its unavailability for removal from the belief set K.

5.1 Epistemic states

From now on, **L** will denote any **LFI** extending **mbC**. In particular, if we allow to consider the consistency operator as a derived (not primitive) one, all the logics in the scope of AGMp lie also in the scope of AGM∘ since they are, as observed in Remark 4.1, extensions of **mbCciw**, where $\circ\alpha =_{def} {\sim}(\alpha \wedge \neg\alpha)$.

When consider an **LFI**, that is, a paraconsistent logic with a consistency operator $\circ$, additional epistemic attitudes can be considered, because of the richness of the language. Figure 1(a) helps to better understand the role of the underlying language of **L** in the definition of the system's epistemic states. It shows how **LFI**s give rise to different scenarios of epistemic attitudes (here, $K_\perp$ denotes the *trivial* belief set $\mathbb{L}$):



(a) Possible scenarios of epistemic attitudes in AGM∘

The link between the scenarios indicates proper inclusion, relative to K, from bottom up.

We can now distinguish three groups of epistemic attitudes:

Propositional: Regarding the acceptance of a belief in the epistemic state.

Quasi-Modal: Regarding the entrenchment of a belief expressed by the consistency operator.

Modal: Regarding the mode one accepts a belief in the epistemic state.

5.1.1 Propositional epistemic attitudes

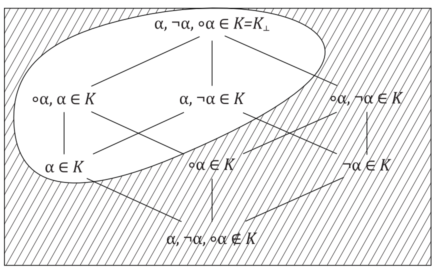
Four epistemic propositional attitudes are considered in relation to a sentence $\alpha \in \mathbb{L}$ (which make sense in any paraconsistent logic, not only **LFI**s). Let K be a given belief set. Then, a sentence α is said to be:

accepted if $\alpha \in K$.

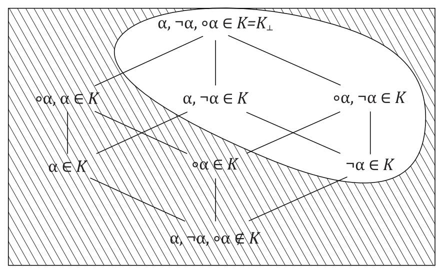
rejected if $\neg\alpha \in K$, i.e., $\neg\alpha$ is accepted in K.

under-determined (or indeterminate) if $\alpha \notin K$ and $\neg\alpha \notin K$, i.e., neither α nor $\neg\alpha$ are accepted in K.

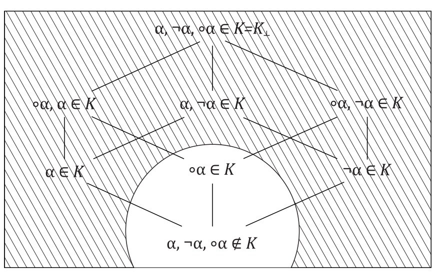
over-determined (or contradictory) if $\alpha \in K$ and $\neg\alpha \in K$, i.e., both α and $\neg\alpha$ are accepted in K.



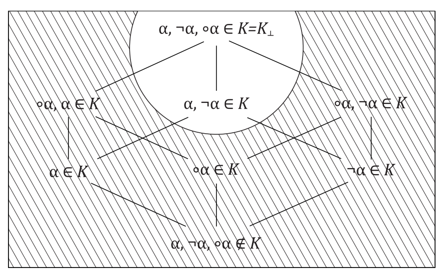
(b) accepted



(c) rejected



(d) underdetermined



(e) overdetermined

In classical AGM over-determination is not considered as an epistemic attitude since it is the trivial case. However, in a paraconsistent paradigm apprehended by AGMp and AGM∘, an agent simultaneously accepting and rejecting a sentence is possible, i.e., it is not incoherent and does not generate trivialization. Consider the following example.

Example 5.1 *I believe in the existence of Poseidon ($p \in K$). I will also accept, because of your argument, your claim that that there is no such thing as Poseidon ($\neg p \in K$) in order to better reflect on the issue.*

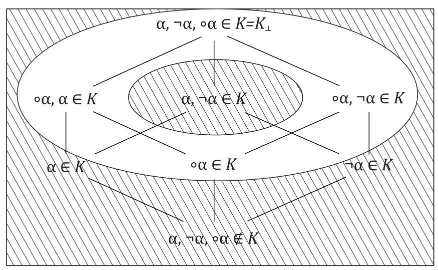
The attitude of not accepting nor rejecting a sentence, despite being already possible in classical AGM, deserves a prominent role in our system because of its duality with over-determination. The agnostic situation of example 5.2 illustrates that fact.

Example 5.2 *I do not accept the existence of Poseidon ($p \notin K$). Nevertheless, I also do not reject it ($\neg p \notin K$).*

5.1.2 Quasi-modal epistemic attitude

Now, let us turn the attention to **LFI**s. Let K be a given belief set in an **LFI** extending **mbC**. Then, a sentence α is said to be:

consistent if $\circ\alpha \in K$, that is, if $\circ\alpha$ is accepted in K (independently of the acceptance or rejection of α).



(f) consistent

The consistency of α in K, illustrated by Figure 1(f), means that any propositional epistemic attitude about it is irrefutable. If the agent accepts or rejects such belief-representing sentence, K will be non-revisable, respectively, by $\neg\alpha$ and α. Furthermore, the sentence will be so entrenched in the epistemic state that to exclude it is not even a possibility (those cases are the modal epistemic attitudes described below).

Such entrenchment may be due to different factors such as, for example, preferences or due to a hierarchy deliberately set by a programmer in a database. Another example is a normative system in which certain norms are not liable to be retracted from the system. Moreover, the consistency may also indicate that the belief in question is not likely to be refuted because the agent believes that there are no arguments against it. Example 5.3 helps us to describe that fact.

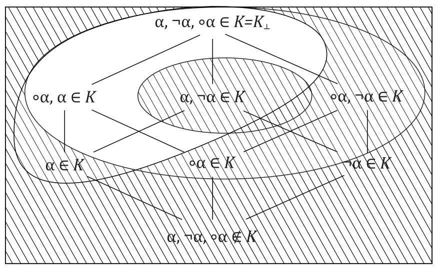
Example 5.3 *I neither accept nor reject the existence of Poseidon. Furthermore, I believe that it is impossible to discuss the existence of Poseidon ($\circ p \in K$) simply because that's a question that transcends any rational argumentation.*

5.1.3 Modal epistemic attitudes

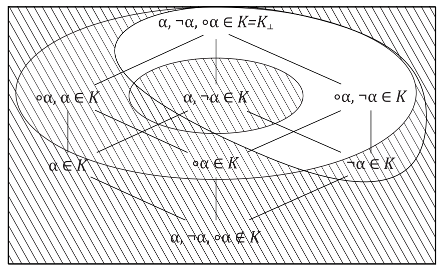
Let K be a given belief set in an **LFI** extending **mbC**. Then, a sentence α is said to be:

boldly accepted if $\alpha \in K$ and $\circ\alpha \in K$, i.e., if α is consistent and it is accepted in K.

boldly rejected if $\neg\alpha \in K$ and $\circ\alpha \in K$, i.e., if α is consistent and it is rejected in K.



(g) boldly accepted



(h) boldly rejected

Remark 5.4 *It is worth noting that in* **mbC** *a sentence α being boldly rejected in K entails that $\sim\alpha \in K$, but the converse is not necessarily true. However, the converse holds in* **mbCciw** *and all of its extensions.*

A sentence α boldly accepted in K means that α is accepted in K, and this set is not liable to be contracted by α. Furthermore K is not revisable by $\neg\alpha$. Consider the following example.

Example 5.5 *I boldly believe in the existence of Poseidon ($\circ p, p \in K$). Therefore for the price of coherence I cannot accept your idea that there is no Poseidon (i.e., $\neg p$), not even for the sake of argument.*

Conversely, a sentence α being boldly rejected means that the belief set is not revisable by α (due the presence of $\sim\alpha$, entailed by the joint presence of $\circ\alpha$ and $\neg\alpha$), as in the following example.

Example 5.6 *I boldly reject the existence of Poseidon ($\circ p, \neg p \in K$). Therefore for the price of coherence I cannot accept your idea that Poseidon exists (i.e., p), not even for the sake of argument.*

It would be possible to define an eighth epistemic attitude, describing the situation in which a sentence is indeterminate and also not consistent – a specific subset of under-determination. But every under-determined sentence is regarded as inconsistent unless its consistency is stated. This is a consequence of an important feature of **LFI**s – there are no theorems of the form $\circ\alpha$ in most of **LFI**s, like **mbC**, **mbCciw** and **mbCcl** (see [3, 2]). This means that the consistency of a sentence is a non-logical (i.e., non-tautological) belief of the agent.

In short, the seven epistemic attitudes defined in $AGM\circ$ are:

Definition 5.7 (Epistemic attitudes of AGM∘) *Let K be a given belief set. Then, a sentence α is said to be:*

Accepted *if $\alpha \in K$.*

Rejected *if $\neg\alpha \in K$.*

Under-determined *(or indeterminate) if $\alpha \notin K$ and $\neg\alpha \notin K$.*

Over-determined *(or contradictory) if $\alpha \in K$ and $\neg\alpha \in K$.*

Consistent *if $\circ\alpha \in K$.*

Boldly accepted *if $\circ\alpha \in K$ and $\alpha \in K$.*

Boldly rejected *if $\circ\alpha \in K$ and $\neg\alpha \in K$.*

Remark 5.8 *Observe that, in* **mbC** *and several of its extensions, $\circ\alpha$ and $\circ\neg\alpha$ are, in general, logically independent. As a consequence of this, the epistemic attitudes of boldly reject α and boldly accept $\neg\alpha$ are, in general, unrelated. These attitudes are equivalent if and only if $(\circ\alpha \leftrightarrow \circ\neg\alpha) \in Cn(\emptyset)$ in the given* **LFI***, for every sentence α. An extension of* **mbC** *in which the validity of the latter condition is guaranteed is* **Cbr***, the system obtained from* **mbCciw** *by adding axioms (ce) and (cf) (that is, the axiom schema $(\alpha \leftrightarrow \neg\neg\alpha)$, see Subsection 3.1). Indeed, if v is a valuation for* **Cbr** *then $v(\alpha) = v(\neg\neg\alpha)$ for every α. Thus : $v(\circ\alpha) = 1$ if and only if $v(\alpha) \neq v(\neg\alpha)$ if and only if $v(\neg\neg\alpha) \neq v(\neg\alpha)$ if and only if $v(\circ\neg\alpha) = 1$.*

5.2 (Non-extensional) AGM∘ contraction

Like in AGM and AGMp, *expansion* is univocally defined in terms of the consequence operator. In order to define *contraction*, a set of postulates will be presented, as usual in AGM like systems of belief dynamics. As asserted before, AGM∘ takes into account the properties of the formal consistency operator in the definitions and, henceforth, the postulates must be redefined accordingly. As in the case of AGMp, the basic operators of contraction (and the induced internal and external revisions) will not satisfy *extensionality.* In Subsection 5.5 it is shown how to obtain *extensionality* by considering additional assumptions, as it was done with AGMp.

5.2.1 Postulates for AGM∘ contraction

As usual, *closure* ensures that the outcome is closed under logical consequences and *success* ensures the sentence's removal. However the removal fails if the sentence is assumed to be consistent. This fact is captured by *failure.* That postulate is the main difference between AGM$\circ$ and AGMp (and also AGM, of course). *Failure* reflects the intuitive notion of formal consistency to be captured. These assumptions, altogether, illustrate the difficulties faced in a *contraction*: we want to remove the sentence from the belief set by preventing the removal of consistent sentences. Furthermore, it is also necessary to respect logical closure. Not to mention that the operation must ensure that the changes are minimal. The latter stricture is exactly what it is required by the other postulates altogether.

The AGM$\circ$ system can be defined over any **LFI**, say **L**, with a (primitive or defined) consistency operator $\circ$, which is *standard* and *supraclassical* (recall Section 2), such that the paraconsistent negation satisfies the law of excluded middle (recall Subsection 4.1 and Remark 4.1). For instance, **L** could be any of the paraconsistent systems C_n of da Costa (see [2, Subsection 3.7]).

Just to fix ideas, assume from now on that **L** is an axiomatic extension of **mbC**. Thus, the language $\mathbb{L}$ of **L** is (at least) generated by the connectives $\wedge, \vee, \to, \neg, \circ$, and the strong (explosive) negation is defined by ${\sim}\alpha =_{def} (\alpha \to \bot)$ (recall Section 3).

The main novelty with respect to AGM and AGMp is that the basic postulates for *contraction* are modified, in order to guarantee that consistent sentences (that is, sentences α such that $\circ\alpha \in K$) cannot be removed, as happens in AGM when $\alpha \in Cn(\emptyset)$.

Definition 5.9 (Postulates for AGM$\circ$ contraction) *An* AGM$\circ$ contraction *over* **L** *is a function* $\div : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *satisfying the following postulates:*

($\div$closure) $K \div \alpha = Cn(K \div \alpha)$.

($\div$success) *If* $\alpha \notin Cn(\emptyset)$ *and* $\circ\alpha \notin K$ *then* $\alpha \notin K \div \alpha$.

($\div$inclusion) $K \div \alpha \subseteq K$.

($\div$failure) *If* $\circ\alpha \in K$ *then* $K \div \alpha = K$.

($\div$relevance) *If* $\beta \in K \setminus (K \div \alpha)$ *then there exists* X *such that* $K \div \alpha \subseteq Cn(X) \subseteq K$ *and* $\alpha \notin Cn(X)$, *but* $\alpha \in Cn(X) + \beta$.

As in the case of AGMp contractions, it is easy to see that any operator $\div$ satisfying the $\div$*relevance* and $\div$*inclusion* postulates of Definition 5.9 also satisfies the following postulate:

($\div$vacuity) If $\alpha \notin K$ then $K \div \alpha = K$.

5.2.2 AGM$\circ$ partial meet contraction function

Regarding the *partial meet* construction, since AGM$\circ$ requires the *failure* postulate, this feature must be taken into account in the *selection function*. This strategy proves to be quite natural when it is considered that, in fact, consistent beliefs are not an option in the retraction – even if they were retracted as the last option such as the more entrenched beliefs. Rather, the consistent belief remains in the epistemic state in any situation, unless the agent retract the own fact that such belief is consistent.[8] Additionally, *extensionality* is not required, analogously with AGMp selection functions.

Definition 5.10 (AGM∘ selection function) *An* AGM∘ selection function *in* **L** *is a function* $\varsigma : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ *such that, for every* K *and* α*:*

1. $\varsigma(K, \alpha) \subseteq K \bot \alpha$ *if* $\circ\alpha \notin K$ *and* $K \bot \alpha \neq \emptyset$.
2. $\varsigma(K, \alpha) = \{K\}$ *otherwise.*

The partial meet contraction is the intersection of the sets selected by the choice function:

$$K \div_{\varsigma} \alpha =_{def} \bigcap \varsigma(K, \alpha).$$

Theorem 5.11 (Representation for AGM∘ contraction) *An operation* $\div : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *is an AGM∘ contraction iff there exists an AGM∘ selection function* ς *in* **L** *such that* $K \div \alpha = \bigcap \varsigma(K, \alpha)$*, for every* K *and* α.

5.3 (Non-extensional) AGM∘ internal Revision

Recall from Definition 2.6 that the *internal revision* is defined in AGM by Levi identity as $K * \alpha = (K \div \neg\alpha) + \alpha$. Although internal revision is defined from contraction in AGM∘ in the same way than in AGM and AGMp, i.e by Levi identity (w.r.t. the paraconsistent negation $\neg$), the new postulates and construction for contraction impose the definition of a new set of postulates and construction for *internal revision*:

Definition 5.12 (Postulates for internal AGM∘ revision) *An* AGM∘ revision *over* **L** *is an operation* $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *satisfying the following:*

(∗closure) $K * \alpha = Cn(K * \alpha)$.

(∗success) $\alpha \in K * \alpha$.

(∗inclusion) $K * \alpha \subseteq K + \alpha$.

(∗vacuity) *If* $\neg\alpha \notin K$ *then* $K + \alpha \subseteq K * \alpha$.

(∗non-contradiction) *If* $\neg\alpha \notin Cn(\emptyset)$ *and* $\circ\neg\alpha \notin K$ *then* $\neg\alpha \notin K * \alpha$.

(∗failure) *If* $\circ\neg\alpha \in K$ *then* $K * \alpha = K + \alpha$

[8] Given the formal consistency propagation (cf. [3]) or a deliberate incorporation of iterated consistency (the consistency of a consistent belief, for instance, by considering axiom (cc), see Definition 3.6), sometimes an agent, in order to retract the fact that a belief is consistent, it is necessary to first retract the consistency of consistency, and so on. But that's not possible in some **LFI**s given the referred consistency propagation. Those **LFI**s capture an interesting reasoning rationality – that all the accepted beliefs are irrefutable, that is, once a belief is accepted, it is taken for granted.

($*$relevance) *If $\beta \in K \setminus (K * \alpha)$ then there exists X such that $K \cap (K * \alpha) \subseteq Cn(X) \subseteq K$ and $\neg\alpha \notin Cn(X)$, but $\neg\alpha \in Cn(X) + \beta$.*

It is worth noting that the $*$*failure* postulate illustrates the case in which the negation of the sentence to be incorporated is consistent in K and thus the prior removal is not possible due to the $\div$*failure* postulate for the contraction operator $\div$ in AGM∘.

By Levi identity, as in the classical model, we use the *partial meet contraction* to define a construction for *AGM∘ internal revision*:

$$K *_{\varsigma} \alpha =_{def} (K \div_{\varsigma} \neg\alpha) + \alpha = \left(\bigcap \varsigma(K, \neg\alpha)\right) + \alpha.$$

Theorem 5.13 (Representation for AGM∘ internal revision) *An operation $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ over $\mathbf{L}$ is an AGM∘ internal revision if and only if there exists an AGM∘ selection function such that $K * \alpha = \left(\bigcap \varsigma(K, \neg\alpha)\right) + \alpha$, for every K and α.*

5.4 (Non-extensional) AGM∘ external Revision

Recall from Definition 2.7 that the *external revision* is defined by reverse Levi identity as $K \circledast \alpha = (K+\alpha) \div \neg\alpha$. Taking this identity into account, the proposed postulates that characterize an external revision in AGM∘ are the following:

Definition 5.14 (Postulates for external AGM∘ revision) *An* AGM∘ external revision *over $\mathbf{L}$ is a function $\circledast : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ satisfying the following postulates:*

($\circledast$closure) *$K \circledast \alpha = Cn(K \circledast \alpha)$.*

($\circledast$success) *$\alpha \in K \circledast \alpha$.*

($\circledast$inclusion) *$K \circledast \alpha \subseteq K + \alpha$.*

($\circledast$non-contradiction) *If $\neg\alpha \notin Cn(\emptyset)$ and $\circ\neg\alpha \notin K + \alpha$ then $\neg\alpha \notin K \circledast \alpha$.*

($\circledast$failure) *If $\circ\neg\alpha \in K + \alpha$ then $K \circledast \alpha = K + \alpha$*

($\circledast$relevance) *if $\beta \in K \setminus (K \circledast \alpha)$ then there exists X such that $K \circledast \alpha \subseteq Cn(X) \subseteq K + \alpha$ and $\neg\alpha \notin Cn(X)$, but $\neg\alpha \in Cn(X) + \beta$.*

($\circledast$pre-expansion) *$(K + \alpha) \circledast \alpha = K \circledast \alpha$.*

By $\circledast$*pre-expansion*, $\circledast$*relevance* and $\circledast$*inclusion*, it is immediate to prove that any AGM∘ external revision $\circledast$ also satisfies the following postulate:

($\circledast$vacuity) If $\neg\alpha \notin K$ then $K \circledast \alpha = K + \alpha$.

Observe that the *relevance* postulate for AGM∘ external revision coincides with the corresponding one for AGMp external revision (recall Definition 4.7). The *non-contradiction*, *failure* and *pre-expansion* postulates highlight the main feature of an external revision in the AGM∘ paradigm.

By Hansson's reverse Levi identity, the *partial meet contraction* of AGM∘ can be used to obtain another construction for an *external revision* operator defined over belief sets, instead of belief bases (as done in [13]):

$$K \circledast_\varsigma \alpha =_{def} (K + \alpha) \div_\varsigma \neg\alpha = \bigcap \varsigma(K + \alpha, \neg\alpha).$$

Theorem 5.15 (Representation for AGM∘ external revision) *An operation* $\circledast : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *over* $\mathbf{L}$ *satisfies the postulates for AGM∘ external revision iff there is an AGM∘ selection function* ς *in* $\mathbf{L}$ *such that, for every* K *and* α, $K \circledast \alpha = \bigcap \varsigma(K + \alpha, \neg\alpha)$.

5.5 AGM∘ contractions and revisions with extensionality

With the aim of satisfy a suitable form of extensionality for the contraction and revision operators in AGM∘, a weaker notion of AGM∘ selection functions must be considered. This strategy is similar to the one adopted for AGMp in Subsection 4.5. Once again, the underlying logic $\mathbf{L}$ must satisfy that $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ for every sentence α.

In order to be extensional, the AGM∘ belief change operations can be modified as follows:

Definition 5.16 (Extensional AGM∘ contraction) *Let* $\mathbf{L}$ *be a logic as described in Subsection 5.1 such that, additionally,* $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ *for every sentence* α. *An* extensional AGM∘ contraction *over* $\mathbf{L}$ *is an AGM∘ contraction* $\div : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *(see Definition 5.9) which additionally satisfies the following postulate:*

(÷extensionality) *If* $\alpha \equiv_{\mathbf{L}} \beta$, $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$ *and* $\circ\alpha \equiv_{\mathbf{L}} \circ\beta$ *then* $K \div \alpha = K \div \beta$.

Definition 5.17 (Extensional AGM∘ internal revision) *Let* $\mathbf{L}$ *be a logic as in Subsection 5.1 such that* $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ *for every* α. *An* extensional AGM∘ revision *over* $\mathbf{L}$ *is an AGM∘ revision* $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *which additionally satisfies the following postulate:*

(∗extensionality) *If* $\alpha \equiv_{\mathbf{L}} \beta$, $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$ *and* $\circ\neg\alpha \equiv_{\mathbf{L}} \circ\neg\beta$ *then* $K * \alpha = K * \beta$.

Definition 5.18 (Extensional AGM∘ external revision) *Let* $\mathbf{L}$ *be a logic as in Subsection 5.1 such that* $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ *for every* α. *An* extensional AGM∘ external revision *over* $\mathbf{L}$ *is an AGM∘ external revision* $\circledast : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *which additionally satisfies the following postulate:*

(⊛extensionality) *If* $\alpha \equiv_{\mathbf{L}} \beta$, $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$ *and* $\circ\neg\alpha \equiv_{\mathbf{L}} \circ\neg\beta$ *then* $K \circledast \alpha = K \circledast \beta$.

The construction of extensional AGM$\circ$ contraction and revisions requires a wider notion of selection function.

Definition 5.19 (General AGM$\circ$ selection function) *A* general AGM$\circ$ selection function *in* $\mathbf{L}$ *is a function* $\varsigma : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ *such that, for every* K *and* α*:*

1. $\varsigma(K, \alpha) = \varsigma(K, \beta)$ *if* $\alpha \equiv_{\mathbf{L}} \beta$*,* $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$ *and* $\circ\alpha \equiv_{\mathbf{L}} \circ\beta$*;*
2. $\varsigma(K, \alpha) \subseteq K \bot \alpha$ *if* $K \bot \alpha \neq \emptyset$*;*
3. $\varsigma(K, \alpha) = \{K\}$ *otherwise.*

As in the case of general AGMp selection functions, the notion above is well-defined.

The extensional AGM$\circ$ *partial meet contraction* generated by a general AGM$\circ$ selection function ς is defined a usual:

$$K \div_{\varsigma} \alpha =_{def} \bigcap \varsigma(K, \alpha).$$

Theorem 5.20 (Representation for extensional AGM$\circ$ contraction) *Let* $\mathbf{L}$ *be a logic as described in Subsection 5.1 such that* $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ *for every sentence* α*. An operation* $\div : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *is an extensional AGM$\circ$ contraction iff there exists a general AGM$\circ$ selection function* ς *in* $\mathbf{L}$ *such that, for every* K *and* α*,* $K \div \alpha = \bigcap \varsigma(K, \alpha)$*.*

Analogous results hold for the other belief change operators:

Theorem 5.21 (Representation for extensional AGM$\circ$ internal revision) *Let* $\mathbf{L}$ *be a logic as described in Subsection 5.1 such that* $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ *for every sentence* α*. An operation* $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *is an extensional AGM$\circ$ internal revision iff there exists a general AGM$\circ$ selection function* ς *in* $\mathbf{L}$ *such that, for every* K *and* α*,* $K * \alpha = \left(\bigcap \varsigma(K, \neg\alpha)\right) + \alpha$*.*

Theorem 5.22 (Representation for extensional AGM$\circ$ external revision) *Let* $\mathbf{L}$ *be a logic as described in Subsection 5.1 such that* $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ *for every sentence* α*. An operation* $\circledast : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ *is an extensional AGM$\circ$ external revision iff there exists a general AGM$\circ$ selection function* ς *in* $\mathbf{L}$ *such that, for every* K *and* α*,* $K \circledast \alpha = \bigcap \varsigma(K + \alpha, \neg\alpha)$*.*

Remark 5.23 *Let* $\mathbf{L}$ *be a logic as the ones considered in this section. Suppose that, additionally, (ciw) holds in* $\mathbf{L}$*. In particular,* $\mathbf{L}$ *could be* **Cbr** *or an axiomatic extension of it (see Definition 3.6). By Remark 5.8,* $\circ\alpha \equiv_{\mathbf{L}} \circ\neg\alpha$ *for every sentence* α*. It is easy to see that, if* $\alpha \equiv_{\mathbf{L}} \beta$ *and* $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$*, then* $\circ\alpha \equiv_{\mathbf{L}} \circ\beta$ *and* $\circ\neg\alpha \equiv_{\mathbf{L}} \circ\neg\beta$*. From this, all the definitions and constructions introduced in this section can be simplified accordingly. For instance,* $\div$extensionality *and* $*$extensionality *coincide with the ones considered in Subsection 4.5.*

An interesting example of a logic with the specifications mentioned above is the well-known da Costa-D'Ottaviano's 3-valued paraconsistent logic **J3***, which* *was independently proposed by several authors under different names in different contexts (see [2, Subsections 4.4.3 and 4.4.7]). The advantages to considering systems of (paraconsistent) belief change based on finitely-valued logics is that they are better suitable to build concrete computational implementations of them (see, for instance, [5]).*

6 Consolidation and Semi-revision in AGMp and AGM∘

In the literature on belief bases, the notion of *consolidation* of a belief base was proposed by Hansson (see [14]). Given a possibly inconsistent (i.e., possibly contradictory w.r.t. the classical negation) belief base, the result of its *consolidation* is a consistent belief base obtained from the previous one by doing minimal changes. Of course this operation is senseless in classical AGM theory since, classically, there is only one inconsistent belief set. In the present framework, however, this operation is meaningful as long as the paraconsistent negation is considered instead of the classical one, and corresponds to the *contraction* of every over-determined sentence. This means that the constructions to be presented in this section can be done both in AGMp and in AGM∘. The only additional requirement on the paraconsistent underlying logic is that no contradiction (w.r.t. the paraconsistent negation) be a theorem. This is usually the case for most paraconsistent logics and, in particular, with the **LFI**s considered here.

Definition 6.1 (Normal paraconsistent logics) *A paraconsistent logic* $\mathbf{L}$ *(w.r.t. a negation* $\neg$*) is* normal *if there is no formula* α *such that* $\alpha \wedge \neg\alpha$ *is a theorem of* $\mathbf{L}$ *(here,* $\wedge$ *denotes a classical conjunction, see Lemma A in the Appendix).*

Given a normal paraconsistent logic $\mathbf{L}$ (w.r.t. a negation $\neg$), a set of formulas $X \subseteq \mathbb{L}$ is *contradictory* if $\alpha \wedge \neg\alpha \in Cn(X)$ for some formula α.

Remark 6.2 *It should be noticed that all the* **LFI***s considered here and, in general, the* **LFI***s studied in [4, 3, 2], are normal paraconsistent logics. Thus, the constructions to be defined below can be applied to them.*

Definition 6.3 (Postulates for AGMp/AGM∘ consolidation) *Let* $\mathbf{L}$ *be a normal paraconsistent logic. An* AGMp/AGM∘ consolidation *over* $\mathbf{L}$ *is an operation* $! : Th(\mathbf{L}) \longrightarrow Th(\mathbf{L})$ *satisfying the following postulates:*

(closure) $K! = Cn(K!)$.

(inclusion) $K! \subseteq K$.

(non-contradiction) *If* $K \neq \mathbb{L}$*, then* $K!$ *is not contradictory (w.r.t. the paraconsistent negation).*

(failure) *If* $K = \mathbb{L}$*, then* $K! = \mathbb{L}$*.*

(relevance) *If* $\beta \in K \setminus K!$ *then there exists* X *such that* $K! \subseteq Cn(X) \subseteq K$ *and* X *is not contradictory, but* $Cn(X) + \beta$ *is contradictory (w.r.t. the paraconsistent negation).*

Note that *consolidation* is a particular case of *contraction*, so it is natural that many of its postulates and the explicit construction follow that operation.

As in the case of *contraction*, a choice function over a remainder set will be used for each *consolidation* operator. The particularity of the definition of remainder sets is that, in the case of *consolidation*, these sets are defined over collections of belief sets.

Definition 6.4 (Remainder for sets [14]) *Let* K *be a belief set in* $\mathbf{L}$ *and* $\emptyset \neq A \subset \mathbb{L}$*. The set* $K \bot_P A \subseteq \wp(\mathbb{L})$ *is such that for every* $X \subseteq \mathbb{L}$*,* $X \in K \bot_P A$ *iff the following is the case:*[9]

(i) $X \subset K$

(ii) $A \cap Cn(X) = \emptyset$

(iii) *If* $X \subset X' \subseteq K$ *then* $A \cap Cn(X') \neq \emptyset$*.*

Clearly the remainder for sets generalize the notion of remainder for formulas (recall Definition 2.3). Indeed, $K \bot \alpha = K \bot_P \{\alpha\}$, for every K and α. Moreover, the elements of $K \bot_P A$ are closed theories (see Lemma F in the Appendix). As in the case of reminder for formulas, it is easy to prove that, if $A \cap K = \emptyset$ or $A \cap Cn(\emptyset) \neq \emptyset$ then $K \bot_P A = \emptyset$. The converse also holds, see Corollary H in the Appendix.

Consolidation considers a specific subset A, that is, the one that represents the totality of contradictory sentences in K, defined as follows:

Definition 6.5 (Set of contradictory sentences) *Let* K *be a belief set in a normal paraconsistent logic* $\mathbf{L}$*. The set* Ω_K *of contradictory sentences of* K *is defined as follows:*

$$\Omega_K = \{\alpha \in K \ : \ \text{there exists } \beta \in \mathbb{L} \text{ such that } \alpha \equiv_{\mathbf{L}} \beta \wedge \neg\beta\}$$

where, for every formulas α *and* β*,* $\alpha \equiv_{\mathbf{L}} \beta$ *means that* $\alpha \leftrightarrow \beta$ *is a theorem of* $\mathbf{L}$*.*

Definition 6.6 (Consolidation function) *Let* $\mathbf{L}$ *be a normal paraconsistent logic. A* consolidation function *in* $\mathbf{L}$ *is a function* $\gamma^f : Th(\mathbf{L}) \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ *such that, for every belief set* K *in* $\mathbf{L}$*:*

1. $\gamma^f(K) \subseteq K \bot_P \Omega_K$ *if* $K \neq \mathbb{L}$ *and* $K \bot_P \Omega_K \neq \emptyset$*.*
2. $\gamma^f(K) = \{K\}$ *otherwise.*

[9] Item (i) was changed as in Definition 2.3.

The consolidation operator defined by a consolidation function γ^f is then defined as follows: for every belief set K in $\mathbf{L}$,

$$K!_{\gamma^f} =_{def} \bigcap \gamma^f(K)$$

Theorem 6.7 (Representation of consolidation) *An operation* $! : Th(\mathbf{L}) \longrightarrow Th(\mathbf{L})$ *over a normal paraconsistent logic* $\mathbf{L}$ *satisfies the postulates of Definition 6.3 iff there exists a consolidation function* γ^f *in* $\mathbf{L}$ *such that* $K! = \bigcap \gamma^f(K)$ *for every belief set* K *in* $\mathbf{L}$.

From the definition of *external revision*, it is possible to define an operation that challenges the *principle of primacy of new information*, tacitly accepted in internal and external *revisions*. In the context of belief bases it is called *semi-revision* by Hansson (see [14]), which is characterized by the *expansion-consolidation* scheme.

In the same way, *semi-revision* for belief sets can be defined as a generalization of *external-revision* in which the choice for the removal is left to the selection function:

$$K?_{\gamma^f}\alpha =_{def} (K + \alpha)!_{\gamma^f}$$

7 Does paraconsistent belief revision coincide with expansion?

In a conceptual essay, G. Priest[17] suggests that the *revision* operation should coincide with *expansions* in an AGM-like system of belief change based on paraconsistent logics. The main reason is that the presence of contradictions do not trivialize, in general, a theory in such logics:

> "If we are allowing for the possibility of inconsistent beliefs, why should revising our beliefs with new information ever cause us to reject anything from our belief-set at all? Why not simply add the belief to our belief-set, and leave it at that?"
> (G. Priest, [17, page 219])

Because of this and other considerations, he proposed an alternative system for belief change by modifying the AGM postulates, in order to deal with paraconsistent logics. Other authors, like P. Girard and K. Tanaka, disagree with Priest's perspective about the identification of revision with expansion in a paraconsistent environment:

> "Hence, even though paraconsistency does not force one to always resolve contradictory beliefs, it is important to distinguish between expanding one's beliefs by accepting contradictory beliefs and revising them by resolving contradictory beliefs. There is thus still a need

for revision in paraconsistent logic." (P. Girard and K. Tanaka, [11, page 2])

However, because of some limiting results obtained in [24], the authors consider that the definition of a paraconsistent belief revision different than expansion constitutes an open problem (see Subsection 7.2).

7.1 Two negations, two Levi identities

Recall from Definition 2.6 that an internal revision $*$ is defined from a contraction $\div$ by means of the Levi identity as follows: $K * \alpha =_{def} (K \div \neg\alpha) + \alpha$, where $\neg$ is some negation defined in the underlying logic **L**. Now, under the hypothesis assumed in Subsection 4.1, there are (at least) two negations in **L**, namely, the paraconsistent $\neg$ and the explosive $\sim$. Being so, two revision operators can be generated from contraction $\div$ by means of Levi identity, one for each negation. Thus, consider, besides Definition 2.6, the following notion:

Definition 7.1 (Classical Internal Revision) $K \bar{*} \alpha =_{def} (K \div {\sim}\alpha) + \alpha$.

The operation $\bar{*}$ is called 'classical' for two reasons: on the one hand, it is defined by means of the 'classical' negation $\sim$. On the other hand, it coincides with 'classical' AGM since $\sim$ should be the main (and only?) negation to be considered in such framework, given the *supraclassicality* assumption. Observe that, as in the case of classical AGM systems, it makes no sense the definition of a classical external revision (that is, an external revision with respect to $\sim$) given that the negation $\sim$ is explosive.

As observed in the beginning of Subsection 4.3, the AGMp contraction $\div$ could be defined in terms of AGM selection functions (because of the AGM-compliance), and so it would satisfy $\div$*extensionality*. In such a case, the classical internal revision $\bar{*}$ would also satisfy $\bar{*}$ *extensionality*.

In terms of the AGM-like systems of paraconsistent belief change proposed here (AGMp and AGM∘), the use of Levi identity with respect to $\neg$ (internal belief revision $*$) or $\sim$ (classical internal belief revision $\bar{*}$) corresponds to quite different approaches to internal belief revision. Indeed, assume in AGMp that $\neg\alpha \notin Cn(\emptyset)$. Hence, ${\sim}\alpha \notin Cn(\emptyset)$ (recall Remark 3.5). In order to revise $K \neq \mathbb{L}$ by α using $*$, firstly the belief set K is contracted by the sentence $\neg\alpha$ and then α is finally added by expansion. Thus, $K * \alpha$ is never contradictory with respect to α. From this perspective, if $\neg\alpha \in K$ and α is received as a new piece of information, then α is never over-determined in $K * \alpha$ (even if this epistemic attitude is possible in a paraconsistent logic). That is, in order to accept α, the (possible) belief in $\neg\alpha$ must be abandoned. On the other hand, if K is revised by α using $\bar{*}$, the belief set K is now contracted by the sentence ${\sim}\alpha$ and then α is added by expansion. But, in this way, it is a valid possibility to keep $\neg\alpha$ in $K \div {\sim}\alpha$ (even if $\neg\alpha \notin Cn(\emptyset)$). In this scenario, $K \bar{*} \alpha$ is contradictory with respect to α. Under this (maybe more conservative) perspective, the contraction with ${\sim}\alpha$ is performed in order to accomodate the new information α avoiding the logical collapse, as required by an AGM-like revision. However, the (possible) belief in $\neg\alpha$ is maintained, for the sake of informational economy. A consequence (desirable or not) of this choice is that the new piece of information α is not merey accepted in $K \bar{*} \alpha$, but it is over-determined.

The same argument as above can be applied to AGM$\circ$, but in this case it must be additionally assumed (besides $\neg\alpha \notin Cn(\emptyset)$) that $\circ{\sim}\alpha \notin K$ and $\circ\neg\alpha \notin K$. Notice that the reminder sets $K \bot {\sim}\alpha$ and $K \bot \neg\alpha$ are incomparable in general, and then so are $K \bar{*} \alpha$ and $K * \alpha$ (in both AGMp and AGM$\circ$).

Example 7.2 *Let* **L** *be* **mbC** *or any extension of it among the ones in Definition 3.6. Let p, q and r be three different propositional variables, and consider $K = Cn({\sim}p, \neg p \to q, q \to r\})$. Then $\neg p \in K$, $q \in K$ and $r \in K$. Let $X_0 = \{\neg p, \neg p \to q, q \to r\} \subseteq K$. Then ${\sim}p \notin Cn(X_0)$, as the following* **L***-valuation v shows: $v(p) = 1$ (hence $v({\sim}p) = 0$), $v(\neg p) = 1$, $v(q) = 1$ (hence $v(\neg p \to q) = 1$) and $v(r) = 1$ (hence $v(q \to r) = 1$). Additionally, $\circ{\sim}p \notin Cn(X_0)$ and $\circ\neg p \notin Cn(X_0)$. In order to see this, consider* **L***-valuations v' and v'' such that $v'(p) = 0$ (hence $v'({\sim}p) = v'(\neg p) = 1$), $v'(q) = 1$ (hence $v'(\neg p \to q) = 1$), $v'(r) = 1$ (hence $v'(q \to r) = 1$) and $v'(\neg{\sim}p) = 1$ (hence $v'(\circ{\sim}p) = 0$); and $v''(p) = v''(\neg p) = v''(\neg\neg p) = v''(\neg\neg\neg p) = v''(q) = v''(r) = 1$ (hence v'' satisfies X_0 but $v''(\circ\neg p) = 0$).*

By Lemma D (see Appendix) there exists $X \in K \bot {\sim}\alpha$ such that $X_0 \subseteq X$ and so the set $R_0 = \{X \in K \bot {\sim}\alpha \ : \ X_0 \subseteq X\}$ is non-empty. Either by definition of AGMp selection function (if we are working in AGMp), or by definition of AGM$\circ$ selection function (if we are working in AGM$\circ$), it can be defined a selection function ϱ such that $\emptyset \neq \varrho(K, {\sim}p) \subseteq R_0$ and so $X_0 \subseteq K \div {\sim}p$. Hence $\{p, \neg p, q, r\} \subseteq K \bar{*} p$, for some classical internal revision $\bar{*}$ in AGMp or AGM$\circ$. Note that $\bar{*}$ can satisfy $\bar{*}$extensionality*, as observed above.*

Now, let $Y_0 = \{q, r\} \subseteq K$. A similar argument can be applied to show that it is possible to have that $Y_0 \subseteq K \div \neg p$ (thus $\{p, q, r\} \subseteq K * p$) for some internal revision $*$ in AGMp or AGM$\circ$ (which can satisfy $*$extensionality*, if desired) . However, $\neg p \notin K * p$ for every internal revision $*$ in AGMp or AGM$\circ$, by Lemma B (see Appendix).*

7.2 Paraconsistent belief revision *does not* coincide with expansion!

In [24], Tanaka proposed Grove's sphere systems in order to deal with an AGM-like paraconsistent belief revision theory for some paraconsistent logics, including da Costa's paraconsistent logics C_n, for $n \geq 1$ (see [6]). For the latter, he shows a kind of trivializing property, which can be reformulated as follows. Observe that each calculus C_n is an special case of **LFI**s in which the consistency operator $\circ_n$ is not primitive, but it is defined by means of a suitable combination of conjunctions and negations (see [2, Subsection 3.7]). For instance, $\circ_1\alpha =_{def} \neg(\alpha \wedge \neg\alpha)$ in C_1. Thus:

Lemma ([24, Lemma 4])
Let K be a belief set in C_n, and let α be a sentence. If $\circ_n\alpha \notin K$ then $K * \alpha =$

$K + \alpha$.

This means that, in the really interesting situations, that is, when the sentence α to be added is not consistent (or it is not 'well-behaved', according to da Costa's terminology) in the given belief set K, then the proposed revision method coincides with expansion. Similar results of collapse of revision with expansion holds for the other paraconsistent logics analyzed in [24]. This is in line with Priest's considerations mentioned at the beginning of this section and motivates [11] to claim that the definition of a paraconsistent belief revision system which does not collapse with expansion constitutes a challenge to be tackled:

> "Defining a distinct [to expansion] revision operation using paraconsistent logic, thus, remains an open question."
> (P. Girard and K. Tanaka, [11, page 3])

The limiting result pointed out by [24] in the Lemma above does not hold, in general, in our setting: let **L** be **mbC** or any extension of it as in Definition 3.6. Let α be a sentence and let K be a belief set in **L** such that $\circ\alpha \notin K$. Suppose aditionally that $\circ\neg\alpha \notin K$ (in **Cbr** this is a consequence of the fact that $\circ\alpha \notin K$), $\neg\alpha \notin Cn(\emptyset)$ and $\neg\alpha \in K$. For instance, take α as a propositional variable p and $K = Cn(\{\neg p\})$. Then $\neg\alpha \in (K + \alpha) \setminus (K * \alpha)$ in AGMp or AGM$\circ$, where the internal revision $*$ can be taken as extensional or not. This constitutes a family of counterexamples to Tanaka's Lemma. A particular instance is the following:

Example 7.3 *Recall Example 7.2. It is easy to see that $\circ p \notin K$ if* **L** *is* **mbC**.[10] *Indeed, it is enough to consider the valuation v of Example 7.2. From this, $\neg p \in (K + p) \setminus (K * p)$ in AGMp and AGM$\circ$. This means that $K + p \neq K * p$ in AGMp and AGM$\circ$, despite $\circ p \notin K$. This holds for the internal revision $*$ being extensional or not.*

Thus, the very general approach to paraconsistent belief revision provided by AGMp and AGM$\circ$ allows us to overcome limiting results such as the ones found in [24]. This is justified by the fact that his construction presented for each logic (based on a special case of Grove's spheres systems, in which each theory is finitely axiomatizable) satisfies the AGM-like postulates, but the converse result is not proved. That is, a representation theorem for each operation is missing. Being so, the constructions are too specific, whence results as the collapse of revision with expansion mentioned above could be expected. In contrast, our proposal is based on AGM-like postulates, on the one hand, and a family of constructions based on selection functions, on the other, obtaining a representation theorem for each operation. With this general framework at hand, it is possible to analyze the relationship between paraconsistency and belief change from a broad perspective, avoiding such trivializing results.

[10] Note that in **mbCciw** (and its extensions) the sentences ${\sim}p$ and $\circ p \wedge \neg p$ are interderivable. Thus, given that ${\sim}p \in K$ it follows that $\circ p \in K$.

8 Final Remarks

This papers studies, from a very general perspective, AGM-like systems of belief change based on paraconsistent logics. Two basic proposals were developed with full technical details: AGMp and AGM∘. The first one is oriented to supraclassical paraconsistent logics, that its, expansions of the classical propositional logic **CPL** by adding (at least) a paraconsistent negation $\neg$ satisfying (at least) the law of excluded middle. It is shown that revision can be defined from contraction, as usual, by means of the Levi identity (namely, $K * \alpha =_{def} (K \div \neg\alpha) + \alpha$), and basically these operations are the same as in AGM, by changing the classical negation $\sim$ by the paraconsistent negation $\neg$; however, some minor adjustements are required. The real novelty of AGMp is that, capitalizing on the features of the paraconsistent negations, it is possible to define, for belief sets, revisions from contractions by means of the *reverse Levi identity* introduced by Hansson only for belief bases, namely: $K \circledast \alpha =_{def} (K + \alpha) \div \neg\alpha$.

The second paradigm proposed here, AGM∘, is specifically designed for the Logics of Formal Inconsistency (**LFI**s), in which a *consistency* operator $\circ$ allows us to recover all the classical inferences (including the *explosion law*) within the logic. The idea behind AGM∘ is that, if $\circ\alpha \in K$ then α cannot be retracted from K (similar to what happens when α is a theorem of the underlying logic). Thus, only 'unsecure' or 'unreliable' information is subject to change. Besides contraction, both internal and external revisions are defined in AGM∘ by means of (direct and reverse) Levi identity. Of course both the postulates and the concrete constructions by means of selection functions must be adapted to this setting.

Finally, Hansson's *consolidation* and *semi-revision* for belief bases are extended to belief sets, capitalizing once again on the non-explosiveness of the paraconsistent negation $\neg$. Additionally, the 'classical' Levi identity (that is, w.r.t. the classical negation) was also considered for AGMp and AGM∘ in order to define 'classical' internal revision (even though based on paraconsistent logics). A few toy examples show that paraconsistent revision do not necessarily coincides with plain expansion, as it is claimed by some authors.

Both systems AGMp and AGM∘ presented in this paper apprehend the dynamics of contradictory theories, particularly represented by the operators of *external revision* and *semi-revision.* Furthermore, AGM∘ provides to the Logics of Formal Inconsistency an intuitive interpretation for the formal consistency connective, as an epistemic attitude.

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Appendix: Proofs of the main results

Firstly, some general properties about tarskian logics and remainder sets will be stated in order to prove the representation theorems. Recall the definition of tarskian and standard propositional logics:

Definition (Tarskian and standard logics)
A logic **L** *defined over a language* $\mathbb{L}$ *and with a consequence relation* $\vdash$ *is* tarskian *if it satisfies the following properties, for every* $X \cup Y \cup \{\alpha\} \subseteq \mathbb{L}$*:*

(i) if $\alpha \in X$ *then* $X \vdash \alpha$*;*

(ii) if $X \vdash \alpha$ *and* $X \subseteq Y$ *then* $Y \vdash \alpha$*;*

(iii) if $X \vdash \alpha$ *and* $Y \vdash \beta$ *for every* $\beta \in X$ *then* $Y \vdash \alpha$.

A tarskian logic **L** *is* finitary *if it satisfies the following:*

(iv) if $X \vdash \alpha$ *then there exists a finite subset* X_0 *of* X *such that* $X_0 \vdash \alpha$.

A tarskian logic **L** *defined over a propositional language* $\mathbb{L}$ *generated by a signature from a set of propositional variables is called* structural *if it satisfies the following property:*

(v) if $X \vdash \alpha$ *then* $\sigma[X] \vdash \sigma(\alpha)$*, for every substitution* σ *of formulas for variables.*

A propositional logic is standard *if it is tarskian, finitary and structural (see [30]).*

All the logics considered in this paper are standard, and so 'logic' will stand for 'standard propositional logic'. The consequence operator $Cn : \wp(\mathbb{L}) \to \wp(\mathbb{L})$ associated to a logic **L** is defined as follows: $Cn(X) = \{\alpha \in \mathbb{L} \ : \ X \vdash \alpha\}$.

Lemma A (Distributivity)
Let **L** *be a logic with a classical disjunction* $\vee$ *and a classical conjunction* $\wedge$*. That is, for every set of formulas* $X \cup \{\alpha, \beta\}$*:* $Cn(X \cup \{\alpha\}) \cap Cn(X \cup \{\beta\}) = Cn(X \cup \{\alpha \vee \beta\})$*, and* $Cn(X \cup \{\alpha, \beta\}) = Cn(X \cup \{\alpha \wedge \beta\})$*. Then, the following* distributivity *law holds: if* $\emptyset \neq X_i = Cn(X_i)$ *(for* $i = 1, 2$*) then*

$$Cn(X_1 \cup \{\alpha\}) \cap Cn(X_2 \cup \{\alpha\}) = Cn((X_1 \cap X_2) \cup \{\alpha\})$$

for every formula α.

Proof. By monotonicity of Cn, $Cn((X_1 \cap X_2) \cup \{\alpha\}) \subseteq Cn(X_i \cup \{\alpha\})$ (for $i = 1, 2$). Thus $Cn((X_1 \cap X_2) \cup \{\alpha\}) \subseteq Cn(X_1 \cup \{\alpha\}) \cap Cn(X_2 \cup \{\alpha\})$. Now, let $\beta \in Cn(X_1 \cup \{\alpha\}) \cap Cn(X_2 \cup \{\alpha\})$. Then $\beta \in Cn(X_i \cup \{\alpha\})$ and so, by finitariness and monotonicity of Cn, there exists a finite set of formulas $F_i \subseteq X_i$ such that $\beta \in Cn(F_i \cup \{\alpha\})$, for $i = 1, 2$. By monotonicity of Cn, it can be assumed that each F_i has at least one element. Since **L** has a classical conjunction $\wedge$ then there exists a formula α_i (the conjunction of the elements of F_i) such that $\beta \in Cn(\{\alpha_i\} \cup \{\alpha\})$, for $i = 1, 2$. Since **L** has a classical disjunction $\vee$ it follows that $\beta \in Cn(\{\alpha_1 \vee \alpha_2\} \cup \{\alpha\})$. But $\alpha_1 \vee \alpha_2 \in Cn(X_1) \cap Cn(X_2) = X_1 \cap X_2$, hence $\beta \in Cn((X_1 \cap X_2) \cup \{\alpha\})$. □

Lemma B *Let **L** be a logic with a classical disjunction $\vee$ (see Lemma A) and with a negation $\neg$ such that $\vdash \alpha \vee \neg\alpha$ in **L**, for every formula α. Let $X \cup \{\alpha\} \subseteq \mathbb{L}$. Then,*

$$X, \alpha \vdash \neg\alpha \quad \textit{implies} \quad X \vdash \neg\alpha.$$

Proof. Suppose that $X, \alpha \vdash \neg\alpha$. Since **L** is a tarskian logic then $X, \neg\alpha \vdash \neg\alpha$. By the basic property of disjunction $\vee$ (see Lemma A) it follows that $X, \alpha \vee \neg\alpha \vdash \neg\alpha$. But $\vdash \alpha \vee \neg\alpha$ by hypothesis and then $X \vdash \neg\alpha$, since **L** is tarskian. □

Lemma C *If $X \in K \bot \alpha$, then $X \in Th(\mathbf{L})$.*

Proof. Let $X \in K \bot \alpha$. If $\beta \in Cn(X) \setminus X$ then $\alpha \in Cn(X \cup \{\beta\})$. Since **L** is tarskian, this implies that $\alpha \in Cn(X)$, a contradiction. Then $X = Cn(X)$ and so $X \in Th(\mathbf{L})$. □

Next result, a fundamental one, is an adaptation to the present framework of a well-known result due to Lindembaum-Łos.

Lemma D (Upper-bound)
*Let K be a belief set in **L** and $\alpha \in K$. If $X \subseteq K$ is such that $\alpha \notin Cn(X)$, then there is a set $X' \in K \bot \alpha$ such that $X \subseteq X'$.*

Proof. First, assuming that the language $\mathbb{L}$ is denumerable, let us arrange the sentences of K into a sequence $\beta_1, \beta_2, \ldots$ (if $\mathbb{L}$ is not denumerable, the proof above must be extended in order to use transfinite induction). Let $X = X_0$ and for each $n \geq 0$ we define X_{n+1} as follows:

$$X_{n+1} = \begin{cases} X_n & \text{if } \alpha \in Cn(X_n \cup \{\beta_{n+1}\}) \\ X_n \cup \{\beta_{n+1}\} & \text{otherwise} \end{cases} .$$

By construction, for every n, $\alpha \notin Cn(X_n)$. Let $X' = \bigcup_n X_n$. It is easy to verify that $X \subseteq X' \subseteq K$. By compactness, if $\alpha \in Cn(X')$ then $\alpha \in Cn(X'')$ for some finite $X'' \subseteq K$. It follows that $\alpha \in Cn(X_n)$ for some n, a contradiction. Then $\alpha \notin Cn(X')$. Moreover, if $\beta \in K$ and $\beta \notin X'$ then, in particular, $\beta \notin X_{n+1}$ where $n + 1$ is such that $\beta = \beta_{n+1}$. This means that $\alpha \in Cn(X_n \cup \{\beta\})$, by construction, and so $\alpha \in Cn(X' \cup \{\beta\})$, by monotonicity. Thus, $X' \in K \bot \alpha$. □

Corollary E

Let K be a belief set in $\mathbf{L}$*. Then* $\alpha \in K \setminus Cn(\emptyset)$ *if and only if* $K \perp \alpha \neq \emptyset$*.*

Proof. 'Only if' part: Let $\alpha \in K \setminus Cn(\emptyset)$ and take $X = \emptyset$ in Lemma D. It guarantees that $K \perp \alpha \neq \emptyset$.
'If' part: By Definition 2.3 of remainder, if $\alpha \notin K$ or $\alpha \in Cn(\emptyset)$ then $K \perp \alpha = \emptyset$.
□

Theorem 4.6

An operation $* : Th(\mathbf{L}) \times \mathbb{L} \to Th(\mathbf{L})$ *is an AGMp internal revision over* $\mathbf{L}$ *iff it is an internal partial meet revision operator over* $\mathbf{L}$*, that is: there is an AGMp selection function* γ *in* $\mathbf{L}$ *such that* $K * \alpha = \left(\bigcap \gamma(K, \neg\alpha)\right) + \alpha = (K \div_\gamma \neg\alpha) + \alpha$*, for every* K *and* α*.*

Proof.

(construction ⇒ postulates)

Let γ be an AGMp selection function, and define $K * \alpha = \left(\bigcap \gamma(K, \neg\alpha)\right) + \alpha$ for every $(K, \alpha) \in Th(\mathbf{L}) \times \mathbb{L}$. We have to prove that $*$ satisfies the postulates for internal AGMp partial meet revision of Definition 4.5.

The satisfaction of the $*$*closure*, $*$*success* and $*$*inclusion* postulates follows from the construction (and the definition of γ).

$*$*vacuity:* Suppose that $\neg\alpha \notin K$. Hence $\gamma(K, \neg\alpha) = \{K\}$ (by Corollary E and Definition 4.3). From this, $K * \alpha = \left(\bigcap \gamma(K, \neg\alpha)\right) + \alpha = K + \alpha$.

$*$*non-contradiction*: Suppose that $\neg\alpha \notin Cn(\emptyset)$. If $K \perp \neg\alpha \neq \emptyset$ then $\emptyset \neq \gamma(K, \neg\alpha) \subseteq K \perp \neg\alpha$ and so $\neg\alpha \notin K' = \bigcap \gamma(K, \neg\alpha)$. By Lemma B, $\neg\alpha \notin K' + \alpha = K * \alpha$. On the other hand, if $K \perp \neg\alpha = \emptyset$ then $\gamma(K, \neg\alpha) = \{K\}$ and so $K * \alpha = K + \alpha$. By Corollary E, $\neg\alpha \notin K$ or $\neg\alpha \in Cn(\emptyset)$. But $\neg\alpha \notin Cn(\emptyset)$, hence $\neg\alpha \notin K$. By Lemma B, $\neg\alpha \notin K + \alpha = K * \alpha$.

$*$*relevance*: Let $\beta \in K \setminus K * \alpha$. Then $\beta \notin \left(\bigcap \gamma(K, \neg\alpha)\right) + \alpha$ whence there exists $X \in K \perp \neg\alpha$ such that $\beta \notin X$. By definition of $*$, $K \cap (K * \alpha) \subseteq K \cap (X + \alpha)$. Observe that $X \subseteq K \cap (X + \alpha)$. Suppose that there exists $\psi \in (K \cap (X + \alpha)) \setminus X$. Then $X, \psi \vdash \neg\alpha$ (since $X \in K \perp \neg\alpha$). But $X, \alpha \vdash \psi$, therefore $X, \alpha \vdash \neg\alpha$. Then $X \vdash \neg\alpha$ by Lemma B, a contradiction. This means that $X = K \cap (X + \alpha)$ and so $K \cap (K * \alpha) \subseteq X = Cn(X) \subseteq K$. From the fact that $X \in K \perp \neg\alpha$, it follows that $\neg\alpha \notin Cn(X)$ and $\neg\alpha \in Cn(X) + \beta$, since $\beta \in K \setminus X$.

(postulates ⇒ construction)

Let $*$ be an operator satisfying the postulates of Definition 4.5 and consider a function $\gamma : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ defined as follows:
(i) Suppose that $(K, \beta) \in Th(\mathbf{L}) \times \mathbb{L}$ is such that $\beta = \neg\alpha$ for some α. Then

$$\gamma(K, \beta) = \begin{cases} \{X \in K \perp \neg\alpha \;:\; K \cap (K * \alpha) \subseteq X\} & \text{if } K \perp \neg\alpha \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

(ii) Otherwise (that is, if $\beta \neq \neg\alpha$), let

$$\gamma(K,\beta) = \begin{cases} K\bot\beta & \text{if } K\bot\beta \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

It will be proven that (1) γ is an AGMp selection function, and (2) $K * \alpha = \big(\bigcap \gamma(K, \neg\alpha)\big) + \alpha$ for every (K, α).

1. If $K\bot\neg\alpha = \emptyset$ then $\gamma(K, \neg\alpha) = \{K\}$. If $K\bot\neg\alpha \neq \emptyset$, it must be proven that $\gamma(K, \neg\alpha) \neq \emptyset$. By Corollary E, $\neg\alpha \in K \setminus Cn(\emptyset)$. Then, by $*$*non-contradiction*, $\neg\alpha \notin K * \alpha$. From this, $\neg\alpha \notin K \cap (K * \alpha) \subseteq K$. By Lemma D, there exists $X' \in K\bot\neg\alpha$ such that $K \cap (K * \alpha) \subseteq X'$, hence $X' \in \gamma(K, \neg\alpha)$ and then $\gamma(K, \neg\alpha) \neq \emptyset$.

2. Firstly it will be proven that $K * \alpha \subseteq \big(\bigcap \gamma(K, \neg\alpha)\big) + \alpha$. By construction, $K \cap (K * \alpha) \subseteq \bigcap \gamma(K, \neg\alpha)$. Hence, $\big(K \cap (K * \alpha)\big) + \alpha \subseteq \big(\bigcap \gamma(K, \neg\alpha)\big) + \alpha$. Since the logic **L** satisfies the hypothesis of Lemma A, and K and $K * \alpha$ are non-empty closed theories of **L**, it follows that $\big(K \cap (K * \alpha)\big) + \alpha = (K + \alpha) \cap ((K * \alpha) + \alpha)$. From this, $(K + \alpha) \cap ((K * \alpha) + \alpha) \subseteq \big(\bigcap \gamma(K, \neg\alpha)\big) + \alpha$. By $*$*success* and $*$*inclusion*, $(K * \alpha) + \alpha = K * \alpha \subseteq K + \alpha$. From this, $K * \alpha \subseteq \big(\bigcap \gamma(K, \neg\alpha)\big) + \alpha$. In order to prove the other inclusion, suppose by absurd that $\beta \in \big(\bigcap \gamma(K, \neg\alpha)\big) \setminus (K * \alpha)$. Since $\bigcap \gamma(K, \neg\alpha) \subseteq K$ then $\beta \in K \setminus (K * \alpha)$. By $*$*relevance*, there exists X such that $K \cap (K * \alpha) \subseteq Cn(X) \subseteq K$, $\neg\alpha \notin Cn(X)$, and $\neg\alpha \in Cn(X) + \beta$. From this, $\neg\alpha \in K \setminus Cn(\emptyset)$. By Corollary E, $K\bot\neg\alpha \neq \emptyset$ and so $\gamma(K, \neg\alpha) = \{X \in K\bot\neg\alpha \ : \ K \cap (K * \alpha) \subseteq X\}$. By Lemma D and Lemma C, there exists $X' \in K\bot\neg\alpha$ such that $K \cap (K * \alpha) \subseteq Cn(X) \subseteq X' = Cn(X')$. This means that $X' \in \gamma(K, \neg\alpha)$ and so $\bigcap \gamma(K, \neg\alpha) \subseteq X'$. Thus, $\beta \in X'$. But $\neg\alpha \in Cn(X) + \beta$, then $\neg\alpha \in Cn(X')$, a contradiction (since $X' \in K\bot\neg\alpha$). Therefore $\bigcap \gamma(K, \neg\alpha) \subseteq K * \alpha$. From this, $\big(\bigcap \gamma(K, \neg\alpha)\big) + \alpha \subseteq (K * \alpha) + \alpha = K * \alpha$, by $*$*success*.

□

THEOREM 4.8
An operation $\circledast : Th(\mathbf{L}) \times \mathbb{L} \to Th(\mathbf{L})$ *is an AGMp external revision over* **L** *iff it is an external partial meet revision operator over* **L**, *that is: there is an AGMp selection function* γ *in* **L** *such that* $K \circledast \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$, *for every* K *and* α.

Proof. (construction ⇒ postulates)

$\circledast$*closure:* By the definition of $\circledast$.

$\circledast$*success:* Suppose that $(K + \alpha)\bot\neg\alpha \neq \emptyset$ and let $X \in (K + \alpha)\bot\neg\alpha$ such that $\alpha \notin X$. Consider $X' = X \cup \{\alpha\}$. Since $X \subset X' \subseteq K + \alpha$ then $\neg\alpha \in Cn(X')$, by item (iii) of Definition 2.3, that is, $X, \alpha \vdash \neg\alpha$. Hence

$X \vdash \neg\alpha$ by Lemma B. But this contradicts the fact that $\neg\alpha \notin Cn(X)$, by item (ii) of Definition 2.3. Hence $\alpha \in X$ for every $X \in (K+\alpha)\bot\neg\alpha$. Thus, if $(K+\alpha)\bot\neg\alpha \neq \emptyset$ then $\alpha \in \bigcap\gamma(K+\alpha,\neg\alpha) = K \circledast \alpha$. Now, if $(K+\alpha)\bot\neg\alpha = \emptyset$ then $\alpha \in \bigcap\gamma(K+\alpha,\neg\alpha) = K \circledast \alpha$, since in this case $\gamma(K+\alpha,\neg\alpha) = \{K+\alpha\}$, by Definition 5.10 (and obviously $\alpha \in K+\alpha$).

$\circledast$*inclusion:* Clearly $K \circledast \alpha = \bigcap\gamma(K+\alpha,\neg\alpha) \subseteq K+\alpha$, by definitions 2.3 and 4.3.

$\circledast$*non-contradiction:* Supose that $\neg\alpha \in K \circledast \alpha = (K+\alpha) \div_\gamma \neg\alpha$. By Theorem 4.4 and $\div$*success* (see Definition 4.2), it follows that $\neg\alpha \in Cn(\emptyset)$.

$\circledast$*relevance:* Let $\beta \in K \setminus ((K+\alpha) \div_\gamma \neg\alpha)$. Hence $(K+\alpha)\bot\neg\alpha \neq \emptyset$ (otherwise $K \circledast \alpha = (K+\alpha) \div_\gamma \neg\alpha = K+\alpha$ and then $K \setminus ((K+\alpha) \div_\gamma \neg\alpha) = \emptyset$, a contradiction). Then there exists $X \in \gamma(K+\alpha,\neg\alpha) \subseteq (K+\alpha)\bot\neg\alpha$ such that $\beta \notin X$. By Lemma C, $X = Cn(X)$. By definition of $\circledast$, $K \circledast \alpha \subseteq X \subseteq K+\alpha$. Let $X' = Cn(X \cup \{\beta\})$. Hence $X \subset X' \subseteq K+\alpha$ (since $\beta \in K$). By Definition 2.3, $\neg\alpha \notin Cn(X)$ and $X' \vdash \neg\alpha$, that is, $X, \beta \vdash \neg\alpha$.

$\circledast$*pre-expansion:* $(K+\alpha) \circledast \alpha = ((K+\alpha)+\alpha) \div_\gamma \neg\alpha = (K+\alpha) \div_\gamma \neg\alpha = K \circledast \alpha$.

(postulates $\Rightarrow$ construction)
Let $\circledast$ be an operator satisfying the postulates of Definition 4.7 and consider a function $\gamma : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ defined as follows:
(i) Suppose that $(K,\beta) \in Th(\mathbf{L}) \times \mathbb{L}$ is such that $\beta = \neg\alpha$ for some α, where $\alpha \in K$. Then

$$\gamma(K,\beta) = \begin{cases} \{X \in K\bot\neg\alpha \ : \ K \circledast \alpha \subseteq X\} & \text{if } K\bot\neg\alpha \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

(ii) Otherwise (that is, if $\beta \neq \neg\alpha$, or $\beta = \neg\alpha$ but $\alpha \notin K$), let

$$\gamma(K,\beta) = \begin{cases} K\bot\beta & \text{if } K\bot\beta \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

We will prove that (1) it is an AGMp selection function (recall Definition 4.3), and (2) $K \circledast \alpha = \bigcap\gamma(K+\alpha,\neg\alpha)$.

1. It is obvious that item (ii) of the definition of γ characterizes a selection function in the sense of Definition 4.3. For item (i), suppose that $\alpha \in K$ (and so $K = K+\alpha$). Clearly $\gamma(K,\neg\alpha) \subseteq K\bot\neg\alpha$ when $K\bot\neg\alpha \neq \emptyset$, and $\gamma(K,\neg\alpha) = \{K\}$ otherwise. It remains to prove that $\gamma(K,\neg\alpha) \neq \emptyset$ if $K\bot\neg\alpha \neq \emptyset$. Then, suppose that $K\bot\neg\alpha \neq \emptyset$. Hence $\neg\alpha \notin Cn(\emptyset)$ by Corollary E. By $\circledast$*non-contradiction* it is the case that $\neg\alpha \notin K \circledast \alpha$. By $\circledast$*closure* and $\circledast$*inclusion*, $\neg\alpha \notin K \circledast \alpha = Cn(K \circledast \alpha) \subseteq K+\alpha = K$. Hence, by Lemma D, there exists $X \in K\bot\neg\alpha$ such that $K \circledast \alpha \subseteq X$ (observe that

$K\bot\neg\alpha \neq \emptyset$ implies, by Corollary E, that $\neg\alpha \in K$ and so Lemma D can be applied). Then $X \in \gamma(K, \neg\alpha)$ and so $\gamma(K, \neg\alpha) \neq \emptyset$ if $K\bot\neg\alpha \neq \emptyset$.

2. Now let us prove that $K \circledast \alpha = (K + \alpha) \div \neg\alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$.

   (a) Suppose that $(K + \alpha)\bot\neg\alpha \neq \emptyset$. Then $\neg\alpha \in K + \alpha$, by Corollary E. Clearly $K \circledast \alpha \subseteq \bigcap \gamma(K + \alpha, \neg\alpha)$ by definition of γ. Let $\beta \notin K \circledast \alpha$. We have to prove that there exists $X \in \gamma(K + \alpha, \neg\alpha)$ such that $\beta \notin X$. If $\beta \notin K + \alpha$ then $\beta \notin X$ for any $X \in \gamma(K + \alpha, \neg\alpha)$ (since every $X \in \gamma(K + \alpha, \neg\alpha)$ is contained in $K + \alpha$). Suppose now that $\beta \in K + \alpha$. By $\circledast$*pre-expansion* $\beta \notin (K + \alpha) \circledast \alpha$ and then, by $\circledast$*relevance*, there exists Z such that $K \circledast \alpha = (K + \alpha) \circledast \alpha \subseteq Z \subseteq (K + \alpha) + \alpha = K + \alpha$, $\neg\alpha \notin Cn(Z)$ and $\neg\alpha \in Cn(Z) + \beta$. By Lemma D there exists $X \in (K + \alpha)\bot\neg\alpha$ such that $K \circledast \alpha \subseteq Z \subseteq X$. Hence $X \in \gamma(K + \alpha, \neg\alpha)$. Since $\neg\alpha \in Cn(Z) + \beta$, then $X, \beta \vdash \neg\alpha$ and hence $X \nvdash \beta$ (otherwise $X \vdash \neg\alpha$). Then $\beta \notin X$ as required. It proves that $K \circledast \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$ if $(K + \alpha)\bot\neg\alpha \neq \emptyset$.

   (b) Finally suppose that $(K + \alpha)\bot\neg\alpha = \emptyset$. Then $\bigcap \gamma(K + \alpha, \neg\alpha) = K + \alpha$, by definition of γ, whence $K \circledast \alpha \subseteq K + \alpha$ by $\circledast$*inclusion*. On the other hand, if there exists $\beta \in (K + \alpha) \setminus (K \circledast \alpha)$ then, by $\circledast$*pre-expansion* and $\circledast$*relevance*, there is a set $X \subseteq K + \alpha$ such that $\neg\alpha \notin Cn(X)$ (hence $\neg\alpha \notin Cn(\emptyset)$) but $\neg\alpha \in Cn(X) + \beta$ (hence $\neg\alpha \in K + \alpha$). By Corollary E, $(K + \alpha)\bot\neg\alpha \neq \emptyset$, a contradiction. Then $K \circledast \alpha = K + \alpha = \bigcap \gamma(K + \alpha, \neg\alpha)$.

□

THEOREM 5.11
An operation $\div : Th(\mathbf{L}) \times \mathbf{L} \longrightarrow Th(\mathbf{L})$ *satisfies the postulates of AGM○ contraction iff there exists an AGM○ selection function* ς *in* $\mathbf{L}$ *such that* $K \div \alpha = \bigcap \varsigma(K, \alpha)$*, for every* K *and* α.

Proof. (construction ⇒ postulates)

$\div$*closure:* Since every $X \in K\bot\alpha$ is a closed theory (by Lemma C) and K itself is a closed theory, then $K \div_\varsigma \alpha = \bigcap \varsigma(K, \alpha)$ is a closed theory, since the intersection of closed theories is also closed.

$\div$*success:* Suppose that $\alpha \notin Cn(\emptyset)$ and $\circ\alpha \notin K$. If $K\bot\alpha = \emptyset$ then $\alpha \notin K$, by Corollary E. Then $\alpha \notin \bigcap \varsigma(K, \alpha)$ since, in this case, $\varsigma(K, \alpha) = \{K\}$. On the other hand, if $K\bot\alpha \neq \emptyset$ then $\emptyset \neq \varsigma(K, \alpha) \subseteq K\bot\alpha$. But $\alpha \notin X$ for every $X \in K\bot\alpha$ and so $\alpha \notin \bigcap \varsigma(K, \alpha)$.

$\div$*inclusion:* Follows directly from the construction.

$\div$*failure:* Follows directly from the construction.

$\div$*relevance:* If $\beta \in K \setminus (K \div_\varsigma \alpha)$ then there exists $X \in \varsigma(K, \alpha)$ such that $\beta \notin X$. By definition of $K\bot\alpha$ and by Lemma C, $K \div_\varsigma \alpha \subseteq X = Cn(X) \subseteq K$, $\alpha \notin Cn(X)$ and $\alpha \in Cn(X \cup \{\beta\}) = Cn(X) + \beta$.

(postulates ⇒ construction)

Let $\div$ be an operator satisfying the postulates for AGM$\circ$ contraction of Definition 5.9 and let ς be the following function:

$$\varsigma(K,\alpha) = \begin{cases} \{X \in K\perp\alpha \;:\; K \div \alpha \subseteq X\} & \text{if } \circ\alpha \notin K \text{ and } K\perp\alpha \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

We have to prove that 1) ς is an AGM$\circ$ selection function, and 2) $K \div \alpha = \bigcap\varsigma(K,\alpha)$.

1. The fact that $\varsigma(K,\alpha) \subseteq Th(\mathbf{L})$ follows directly by construction. If $\circ\alpha \notin K$ and $K\perp\alpha \neq \emptyset$ then $\alpha \in K$ and $\alpha \notin Cn(\emptyset)$, by Corollary E. Then, the $\div$*success* and $\div$*inclusion* postulates guarantee that $\alpha \notin K \div \alpha \subseteq K$. By Lemma D, there exists X such that $K \div \alpha \subseteq X \in K\perp\alpha$, whence $\varsigma(K,\alpha) \neq \emptyset$. On the other hand, if $\circ\alpha \in K$ or $K\perp\alpha = \emptyset$ then $\varsigma(K,\alpha) = \{K\}$.

2. Note that $K \div \alpha \subseteq \bigcap\varsigma(K,\alpha) = K \div_\varsigma \alpha$ by construction. Suppose that $\beta \in \bigcap\varsigma(K,\alpha) \setminus (K \div \alpha)$. Then, $\beta \in K \setminus (K \div \alpha)$, since $\bigcap\varsigma(K,\alpha) \subseteq K$ by definition. By $\div$*relevance*, there exists X such that $K \div \alpha \subseteq Cn(X) \subseteq K$ and $\alpha \notin Cn(X)$, but $\alpha \in Cn(X) + \beta$. Thus, $\alpha \in K \setminus (K \div \alpha)$. By Lemma D, there exists $X' \in K\perp\alpha$ such that $X \subseteq X'$, whence $K\perp\alpha \neq \emptyset$. By $\div$*failure*, $\circ\alpha \notin K$. Thus, by definition of $\varsigma(K,\alpha)$, $X' \in \varsigma(K,\alpha)$. From this, $\beta \in X'$. But $\alpha \in Cn(X) + \beta$, hence $\alpha \in Cn(X') + \beta$. This means that $\alpha \in Cn(X')$, a contradiction. Therefore $K \div \alpha = \bigcap\varsigma(K,\alpha)$.

□

Theorem 5.13

An operation $* : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ over $\mathbf{L}$ satisfies the postulates for internal partial meet AGM$\circ$ revision (see Definition 5.12) if and only if there exists an AGM$\circ$ selection function ς in $\mathbf{L}$ such that $K * \alpha = \left(\bigcap\varsigma(K,\neg\alpha)\right) + \alpha$, for every K and α.

Proof.

(construction ⇒ postulates)

Let ς be an AGM$\circ$ selection function (recall Definition 5.10), and define $K * \alpha = \left(\bigcap\varsigma(K,\neg\alpha)\right) + \alpha$ for every $(K,\alpha) \in Th(\mathbf{L}) \times \mathbb{L}$. We have to prove that $*$ satisfies the postulates for internal AGM$\circ$ partial meet revision of Definition 5.12.

The satisfaction of the $*$*closure*, $*$*success* and $*$*inclusion* postulates follows from the construction (and the definition of ς).

$*$*vacuity:* Suppose that $\neg\alpha \notin K$. This implies that $\varsigma(K,\neg\alpha) = \{K\}$, by Corollary E and Definition 5.10. Therefore $K * \alpha = \left(\bigcap\varsigma(K,\neg\alpha)\right) + \alpha = K + \alpha$.

$*$*non-contradiction*: Suppose that $\neg\alpha \notin Cn(\emptyset)$ and $\circ\neg\alpha \notin K$. If $K\perp\neg\alpha \neq \emptyset$ then $\emptyset \neq \varsigma(K,\neg\alpha) \subseteq K\perp\neg\alpha$ and so $\neg\alpha \notin K' = \bigcap\varsigma(K,\neg\alpha)$. By Lemma B,

$\neg\alpha \notin K' + \alpha = K * \alpha$. On the other hand, if $K\bot\neg\alpha = \emptyset$ then $\varsigma(K,\neg\alpha) = \{K\}$ and so $K*\alpha = K+\alpha$. By Corollary E, $\neg\alpha \notin K$ or $\neg\alpha \in Cn(\emptyset)$. But $\neg\alpha \notin Cn(\emptyset)$, hence $\neg\alpha \notin K$. By Lemma B, $\neg\alpha \notin K+\alpha = K*\alpha$.

$*$*failure*: If $\circ\neg\alpha \in K$ then $K' = \bigcap\varsigma(K,\neg\alpha)$ is K, by definition of ς. Hence $K*\alpha = K'+\alpha = K+\alpha$.

$*$*relevance*: It is proved as in the proof of Theorem 4.6 for the corresponding postulate.

(postulates $\Rightarrow$ construction)

Let $*$ be an operator satisfying the postulates of AGM∘ internal revision and consider a function $\varsigma : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ defined as follows:
(i) Suppose that $(K,\beta) \in Th(\mathbf{L}) \times \mathbb{L}$ is such that $\beta = \neg\alpha$ for some α. Then

$$\varsigma(K,\beta) = \begin{cases} \{X \in K\bot\neg\alpha \ : \ K\cap(K*\alpha) \subseteq X\} & \text{if } \circ\neg\alpha \notin K \text{ and } K\bot\neg\alpha \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

(ii) Otherwise (that is, if $\beta \neq \neg\alpha$), let

$$\varsigma(K,\beta) = \begin{cases} K\bot\beta & \text{if } K\bot\beta \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

It will be proven that (1) ς is an AGM∘ selection function (recall Definition 5.10), and (2) $K*\alpha = \big(\bigcap\varsigma(K,\neg\alpha)\big)+\alpha$ for every (K,α).

1. If $K\bot\neg\alpha = \emptyset$ then $\varsigma(K,\neg\alpha) = \{K\}$. If $K\bot\neg\alpha \neq \emptyset$, it must be proven that $\varsigma(K,\neg\alpha) \neq \emptyset$. By Corollary E, $\neg\alpha \in K \setminus Cn(\emptyset)$. If $\circ\neg\alpha \in K$ then $\varsigma(K,\neg\alpha) = \{K\} \neq \emptyset$. If $\circ\neg\alpha \notin K$ then, by $*$*non-contradiction*, $\neg\alpha \notin K*\alpha$. From this, $\neg\alpha \notin K\cap(K*\alpha) \subseteq K$. By Lemma D, there exists $X' \in K\bot\neg\alpha$ such that $K\cap(K*\alpha) \subseteq X'$, hence $X' \in \varsigma(K,\neg\alpha)$ and then $\varsigma(K,\neg\alpha) \neq \emptyset$.

2. Firstly it will be proven that $K*\alpha \subseteq \big(\bigcap\varsigma(K,\neg\alpha)\big)+\alpha$. By the very construction, $K\cap(K*\alpha) \subseteq \bigcap\varsigma(K,\neg\alpha)$. Hence, $\big(K\cap(K*\alpha)\big)+\alpha \subseteq \big(\bigcap\varsigma(K,\neg\alpha)\big)+\alpha$. Since the logic $\mathbf{L}$ satisfies the hypothesis of Lemma A, and K and $K*\alpha$ are non-empty closed theories of $\mathbf{L}$, it follows that $\big(K\cap(K*\alpha)\big)+\alpha = (K+\alpha)\cap((K*\alpha)+\alpha)$. From this, $(K+\alpha)\cap((K*\alpha)+\alpha) \subseteq \big(\bigcap\varsigma(K,\neg\alpha)\big)+\alpha$. By $*$*success* and $*$*inclusion*, $(K*\alpha)+\alpha = K*\alpha \subseteq K+\alpha$. From this, $K*\alpha \subseteq \big(\bigcap\varsigma(K,\neg\alpha)\big)+\alpha$. In order to prove the other inclusion, there are two cases to analyze:

   (a) If $\circ\neg\alpha \in K$ then, by $*$*failure*, $K*\alpha = K+\alpha$. Using Corollary E and definition of ς it follows that $\varsigma(K,\neg\alpha) = \{K\}$, hence $\big(\bigcap\varsigma(K,\neg\alpha)\big)+\alpha = K+\alpha = K*\alpha$.

(b) If $\circ\neg\alpha \notin K$, suppose by absurd that $\beta \in \left(\bigcap \varsigma(K, \neg\alpha)\right) \setminus (K * \alpha)$. Then $\beta \in K \setminus (K * \alpha)$, since $\bigcap \varsigma(K, \neg\alpha) \subseteq K$. Using $*$*relevance*, there exists X such that $K \cap (K * \alpha) \subseteq Cn(X) \subseteq K$, $\neg\alpha \notin Cn(X)$, and $\neg\alpha \in Cn(X) + \beta$. Thus, $\neg\alpha \in K \setminus Cn(\emptyset)$. Using Corollary E it follows that $K \bot \neg\alpha \neq \emptyset$, whence $\varsigma(K, \neg\alpha) = \{X \in K \bot \neg\alpha \ : K \cap (K * \alpha) \subseteq X\}$. By Lemma D and Lemma C, there exists $X' \in K \bot \neg\alpha$ such that $K \cap (K * \alpha) \subseteq Cn(X) \subseteq X' = Cn(X')$. But then $X' \in \varsigma(K, \neg\alpha)$ and so $\bigcap \varsigma(K, \neg\alpha) \subseteq X'$. As a consequence of this, $\beta \in X'$. But $\neg\alpha \in Cn(X) + \beta$, then $\neg\alpha \in Cn(X')$, a contradiction (since $X' \in K \bot \neg\alpha$). This means that $\bigcap \varsigma(K, \neg\alpha) \subseteq K * \alpha$. Therefore, $\left(\bigcap \varsigma(K, \neg\alpha)\right) + \alpha \subseteq (K * \alpha) + \alpha = K * \alpha$, by $*$*success*.

□

Theorem 5.15
An operation $\circledast : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow Th(\mathbf{L})$ over $\mathbf{L}$ satisfies the postulates for external partial meet AGM$\circ$ revision (see Definition 5.14) iff there is an AGM$\circ$ selection function ς in $\mathbf{L}$ such that $K \circledast \alpha = \bigcap \varsigma(K + \alpha, \neg\alpha)$, for every K and α.

Proof. (construction ⇒ postulates)

$\circledast$*closure*: It follows as in the proof of Theorem 5.11.

$\circledast$*success*: Suppose that $\circ\neg\alpha \in K$ or $(K + \alpha) \bot \neg\alpha = \emptyset$. Then, $\varsigma(K + \alpha, \neg\alpha) = \{K + \alpha\}$ by definition and so $\alpha \in K + \alpha = \bigcap \varsigma(K + \alpha, \neg\alpha)$. Now, suppose that $\circ\neg\alpha \notin K$ and $(K + \alpha) \bot \neg\alpha \neq \emptyset$. Let $X \in (K + \alpha) \bot \neg\alpha$, and suppose by absurd that $\alpha \notin X = Cn(X)$. Then $X \subset X \cup \{\alpha\} \subseteq K + \alpha$ and so $X, \alpha \vdash \neg\alpha$ (by item (iii) of Definition 2.3). But then $X \vdash \neg\alpha$, by Lemma B, a contradiction. Therefore $\alpha \in X$ for every $X \in (K + \alpha) \bot \neg\alpha$ and so $\alpha \in \bigcap \varsigma(K + \alpha, \neg\alpha)$.

$\circledast$*inclusion:* It follows by construction.

$\circledast$*non-contradiction*: Suppose that $\neg\alpha \notin Cn(\emptyset)$ and $\circ\neg\alpha \notin K + \alpha$. If $(K + \alpha) \bot \neg\alpha = \emptyset$ then $\varsigma(K + \alpha, \neg\alpha) = \{K + \alpha\}$ and $\neg\alpha \notin K + \alpha$, by Corollary E. This implies that $\bigcap \varsigma(K + \alpha, \neg\alpha) = K + \alpha$, whence $\neg\alpha \notin \bigcap \varsigma(K + \alpha, \neg\alpha)$. On the other hand, if $(K + \alpha) \bot \neg\alpha \neq \emptyset$ then $\emptyset \neq \varsigma(K + \alpha, \neg\alpha) \subseteq (K + \alpha) \bot \neg\alpha$. This implies that $\neg\alpha \notin \bigcap \varsigma(K + \alpha, \neg\alpha)$.

$\circledast$*failure*: Suppose that $\circ\neg\alpha \in K + \alpha$. By Definition 5.10, $\varsigma(K + \alpha, \neg\alpha) = \{K + \alpha\}$. Then $\bigcap \varsigma(K + \alpha, \neg\alpha) = K + \alpha$.

$\circledast$*relevance*: Let $\beta \in K \setminus \bigcap \varsigma(K + \alpha, \neg\alpha)$. Therefore $(K + \alpha) \bot \neg\alpha \neq \emptyset$ (otherwise, $\bigcap \varsigma(K + \alpha, \neg\alpha) = K + \alpha$ and so $K \setminus \bigcap \varsigma(K + \alpha, \neg\alpha) = K \setminus (K + \alpha) = \emptyset$, a contradiction). Hence, there exists $X \in \varsigma(K + \alpha, \neg\alpha) \subseteq (K + \alpha) \bot \neg\alpha$ such that $\beta \notin X$. By construction and Lemma C, $\bigcap \varsigma(K + \alpha, \neg\alpha) \subseteq X = Cn(X) \subseteq K + \alpha$. Let $X' = X \cup \{\beta\}$. Therefore $X \subset X' \subseteq K + \alpha$ by the fact that $\beta \in K$. Then $\neg\alpha \in Cn(X')$ and hence $\neg\alpha \in Cn(X) + \beta$, while $\neg\alpha \notin Cn(X)$.

$\circledast$*pre-expansion*: $\bigcap\varsigma((K+\alpha)+\alpha,\neg\alpha) = \bigcap\varsigma(K+\alpha,\neg\alpha)$.

(postulates ⇒ constructions)

Let $\circledast$ be an operator satisfying the postulates of AGM$\circ$ external revision and consider a function $\varsigma : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ defined as follows:
(i) Suppose that $(K,\beta) \in Th(\mathbf{L}) \times \mathbb{L}$ is such that $\beta = \neg\alpha$ for some α, where $\alpha \in K$. Then

$$\varsigma(K,\beta) = \begin{cases} \{X \in K\bot\neg\alpha \ : \ K \circledast \alpha \subseteq X\} & \text{if } \circ\neg\alpha \notin K \text{ and } K\bot\neg\alpha \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

(ii) Otherwise (that is, if $\beta \neq \neg\alpha$, or $\beta = \neg\alpha$ but $\alpha \notin K$), let

$$\varsigma(K,\beta) = \begin{cases} K\bot\beta & \text{if } K\bot\beta \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

We have to prove that (1) ς is an AGM$\circ$ selection function (see Definition 5.10), and (2) $K \circledast \alpha = \bigcap\varsigma(K+\alpha,\neg\alpha)$.

1. By considering the case (ii) of the construction of ς, it is clear that the conditions of Definition 5.10 are fullfilled. Now, suppose that $\alpha \in K$ (hence $K = K+\alpha$) and let us analyze the definition of $\varsigma(K,\neg\alpha)$. If $K\bot\neg\alpha = \emptyset$ then $\varsigma(K,\neg\alpha) = \{K\}$ as required. It remains to prove that $\emptyset \neq \varsigma(K,\neg\alpha)$ if $K\bot\neg\alpha \neq \emptyset$, and that $\varsigma(K,\neg\alpha) \subseteq K\bot\neg\alpha$ if, additionally, $\circ\neg\alpha \notin K$. The latter holds by the very definition of ς. Suppose then that $K\bot\neg\alpha \neq \emptyset$; by Corollary E, $\neg\alpha \notin Cn(\emptyset)$, and $\neg\alpha \in K$. If $\circ\neg\alpha \in K$ then $\varsigma(K,\neg\alpha) = \{K\} \neq \emptyset$. Finally, if $\circ\neg\alpha \notin K$ then $\varsigma(K,\neg\alpha) = \{X \in K\bot\neg\alpha \ : \ K \circledast \alpha \subseteq X\}$. By $\circledast$*non-contradiction*, $\neg\alpha \notin K \circledast \alpha$. By $\circledast$*closure* and $\circledast$*inclusion*, $\neg\alpha \notin K \circledast \alpha = Cn(K \circledast \alpha) \subseteq K+\alpha = K$. By Lemma D (recalling that $\neg\alpha \in K$), there exists $X \in K\bot\neg\alpha$ such that $K \circledast \alpha \subseteq X$. This means that $X \in \varsigma(K,\neg\alpha)$, whence $\varsigma(K,\neg\alpha) \neq \emptyset$.

2. Suppose firstly that $\circ\neg\alpha \notin K+\alpha$ and $(K+\alpha)\bot\neg\alpha \neq \emptyset$. Then $\neg\alpha \in K+\alpha$ and $\neg\alpha \notin Cn(\emptyset)$. By construction, $K \circledast \alpha \subseteq \bigcap\varsigma(K+\alpha,\neg\alpha)$. In order to prove the converse inclusion, let $\beta \notin K \circledast \alpha$. It is enough to prove that there exists $X \in \varsigma(K+\alpha,\neg\alpha)$ such that $\beta \notin X$. If $\beta \notin K+\alpha$ then $\beta \notin X$ for every $X \in \varsigma(K+\alpha,\neg\alpha)$ (since $X \subseteq K+\alpha$ for every $X \in \varsigma(K+\alpha,\neg\alpha)$). Suppose now that $\beta \in K+\alpha$. By $\circledast$*pre-expansion*, $\beta \notin (K+\alpha)\circledast\alpha$ and then, by $\circledast$*relevance*, there exists Z such that $K\circledast\alpha = (K+\alpha) \circledast \alpha \subseteq Cn(Z) \subseteq (K+\alpha)+\alpha = K+\alpha$, $\neg\alpha \notin Cn(Z)$ and $\neg\alpha \in Cn(Z)+\beta$. By Lemma D (taking into account that $\neg\alpha \in K+\alpha$), there exists $X \in (K+\alpha)\bot\neg\alpha$ such that $K \circledast \alpha \subseteq Cn(Z) \subseteq X$. Hence, $X \in \varsigma(K+\alpha,\neg\alpha)$. Since $\neg\alpha \in Cn(Z)+\beta$, then $\neg\alpha \in X+\beta$ and therefore $\beta \notin X = Cn(X)$ (otherwise, $\neg\alpha \in Cn(X)$). It follows that $\beta \notin \bigcap\varsigma(K+\alpha,\neg\alpha)$. From this it is concluded that $K\circledast\alpha = \bigcap\varsigma(K+\alpha,\neg\alpha)$.

Finally, suppose that $\circ\neg\alpha \in K+\alpha$ or $(K+\alpha)\perp\neg\alpha = \emptyset$. By construction, it follows that $\bigcap\varsigma(K+\alpha,\neg\alpha) = K+\alpha$. Hence $K \circledast \alpha \subseteq \bigcap\varsigma(K+\alpha,\neg\alpha)$, by $\circledast$*inclusion*. There are two case to analyze:

(a) If $\circ\neg\alpha \in K+\alpha$ then, by $\circledast$*failure*, $K \circledast \alpha = K+\alpha = \bigcap\varsigma(K+\alpha,\neg\alpha)$.

(b) If $(K+\alpha)\perp\neg\alpha = \emptyset$ then, by Corollary E, $\neg\alpha \notin K+\alpha$ or $\neg\alpha \in Cn(\emptyset)$. Suppose (by absurd) that there exists $\beta \in (K+\alpha) \setminus (K \circledast \alpha)$. By $\circledast$*pre-expansion*, $\beta \in (K+\alpha) \setminus ((K+\alpha) \circledast \alpha)$. By $\circledast$*relevance*, there exists $X \subseteq K+\alpha$ such that $\neg\alpha \notin Cn(X)$ and $\neg\alpha \in Cn(X)+\beta$. But then $\neg\alpha \in K+\alpha$ and $\neg\alpha \notin Cn(\emptyset)$, a contradiction. From this $K \circledast \alpha = K+\alpha = \bigcap\varsigma(K+\alpha,\neg\alpha)$.

□

THEOREM 4.14 (**Representation for extensional AGMp contraction**)

Proof. The proof is similar to the one for standard AGM, with minor differences with respect to $\div$*extensionality*, on the one hand, and the first property of the selection function, on the other. Let us concentrate on these differences:

(construction ⇒ postulates)

Let γ be a general AGMp selection function. In order to prove that $K \div_\gamma \alpha = \bigcap\gamma(K,\alpha)$ satisfies $\div$*extensionality*, suppose that $\alpha \equiv_{\mathbf{L}} \beta$ and $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$. By Definition 4.13, $\gamma(K,\alpha) = \gamma(K,\beta)$. Being so, $K \div_\gamma \alpha = K \div_\gamma \beta$.

(postulates ⇒ construction)

Suppose now that $\div$ is an extensional AGMp contraction over $\mathbf{L}$, and define a function γ as follows:

$$\gamma(K,\alpha) = \begin{cases} \{X \in K\perp\alpha \ : \ K \div \alpha \subseteq X\} & \text{if } K\perp\alpha \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

We have to prove that 1) γ is a general AGMp selection function, and 2) $K \div \alpha = \bigcap\gamma(K,\alpha)$. For 1), it is enough to prove that item 1 of Definition 4.13 is satisfied (since the other properties are proved as in classical AGM, using AGM-compliance). Thus, suppose that $\alpha \equiv_{\mathbf{L}} \beta$ and $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$. Then $K\perp\alpha = K\perp\beta$, and $K \div \alpha = K \div \beta$, by $\div$*extensionality*. This means that $\gamma(K,\alpha) = \gamma(K,\beta)$. The proof of 2) is as in classical AGM, using AGM-compliance (observe that $\div$*extensionality* is not used here). □

THEOREM 4.15 (**Representation for extensional AGMp internal revision**)

Proof. It is similar to the one given for Theorem 4.6, with the following changes:

(construction ⇒ postulates)

Given a general AGMp selection function γ, it is necessary to prove that $K * \alpha = \big(\bigcap\gamma(K,\neg\alpha)\big) + \alpha$ satisfies, additionally, $*$*extensionality*. Thus, suppose that $\alpha \equiv_{\mathbf{L}} \beta$ and $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$. Then $\neg\neg\alpha \equiv_{\mathbf{L}} \alpha \equiv_{\mathbf{L}} \beta \equiv_{\mathbf{L}} \neg\neg\beta$ and so $\gamma(K,\neg\alpha) = \gamma(K,\neg\beta)$, by Definition 4.13. From this, $K * \alpha = \big(\bigcap\gamma(K,\neg\alpha)\big) +$

$\alpha = \left(\bigcap \gamma(K, \neg\beta)\right) + \alpha = \left(\bigcap \gamma(K, \neg\beta)\right) + \beta = K * \beta$.

(postulates $\Rightarrow$ construction)

Let $*$ be an operator satisfying the postulates of an external AGMp internal revision and consider a function $\gamma : Th(\mathbf{L}) \times \mathbb{L} \longrightarrow \wp(Th(\mathbf{L})) \setminus \{\emptyset\}$ defined as follows:

$$\gamma(K, \alpha) = \begin{cases} \{X \in K\bot\alpha \ : \ K \cap (K * \neg\alpha) \subseteq X\} & \text{if } K\bot\alpha \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

With a proof similar to that for Theorem 4.6, it can be seen that 1) γ is an AGMp selection function, and 2) $K * \alpha = \left(\bigcap \gamma(K, \neg\alpha)\right) + \alpha$ for every (K, α). To see 1), the only detail to be taken into account is that $\alpha \equiv_{\mathbf{L}} \neg\neg\alpha$ (and so $\neg\alpha \equiv_{\mathbf{L}} \neg\neg\neg\alpha$), whence $K * \alpha = K * \neg\neg\alpha$ by $*$*extensionality.* Finally, using again $*$*extensionality*, it is immediate to prove that γ is, in fact, a general AGMp selection function.

□

THEOREM 4.16 (**Representation for extensional AGMp external revision**)

Proof. It is similar to the one given for Theorem 4.8, with minor changes. These changes are analogous to the ones given in the proof of Theorem 4.15. The details are left to the reader. □

THEOREM 5.20 (**Representation for extensional AGM$\circ$ contraction**)

Proof. It is similar to the one given for Theorem 5.11, with minor changes. These changes are analogous to the ones given in the respective proof of theorems 4.15 and 4.16. For instance, in order to see that, given an extensional AGM$\circ$ contraction $\div$, the induced mapping ς is a general AGM$\circ$ selection function, it is enough to observe the following: if $\alpha \equiv_{\mathbf{L}} \beta$, $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$ and $\circ\alpha \equiv_{\mathbf{L}} \circ\beta$ then $K \div \alpha = K \div \beta$, $K\bot\alpha = K\bot\beta$, and $\circ\alpha \in K$ iff $\circ\beta \in K$. From this $\varsigma(K, \alpha) = \varsigma(K, \beta)$. The details are left to the reader. □

THEOREM 5.21 (**Representation for extensional AGM$\circ$ internal revision**)

Proof. It is similar to the one given for Theorem 5.13, with minor changes. These changes are analogous to the ones given in the proof of Theorem 4.15. For instance, in order to see that, given an extensional AGM$\circ$ internal revision $*$, the induced mapping ς is a general AGM$\circ$ selection function, observe that ς must be defined as follows:

$$\varsigma(K, \alpha) = \begin{cases} \{X \in K\bot\alpha \ : \ K \cap (K * \neg\alpha) \subseteq X\} & \text{if } K\bot\alpha \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

Now, it is enough to observe the following: if $\alpha \equiv_{\mathbf{L}} \beta$, $\neg\alpha \equiv_{\mathbf{L}} \neg\beta$ and $\circ\alpha \equiv_{\mathbf{L}} \circ\beta$ then $\neg\neg\alpha \equiv_{\mathbf{L}} \neg\neg\beta$ and so $K * \neg\alpha = K * \neg\beta$. On the other hand,

$K \perp \alpha = K \perp \beta$, and $\circ\alpha \in K$ iff $\circ\beta \in K$. From this $\varsigma(K, \alpha) = \varsigma(K, \beta)$. The details are left to the reader. $\square$

THEOREM 5.22 (**Representation for extensional AGM∘ external revision**)

Proof. It is similar to the one given for Theorem 5.15, with minor changes. These changes are analogous to the ones given in the proof of theorem 5.21. $\square$

The upper-bound lemma (Lemma D) as well as its corollary (Corollary E) can be generalized for sets, taking into account Definition 6.4.

LEMMA F *If $X \in K \perp A$ then $X \in Th(\mathbf{L})$.* **Proof.** It is analogous to the proof of Lemma C. $\square$

LEMMA G (UPPER-BOUND FOR SETS)
Let K be a belief set in $\mathbf{L}$, and $\emptyset \neq A \subset \mathbb{L}$ such that $K \cap A \neq \emptyset$. Let $X \subseteq K$ such that $Cn(X) \cap A = \emptyset$. Then, there exists $X' \in K \perp_P A$ such that $X \subseteq X'$.

Proof. It is analogous to the proof of Lemma D, but now X_{n+1} is defined as follows:

$$X_{n+1} = \begin{cases} X_n & \text{if } Cn(X_n \cup \{\beta_{n+1}\}) \cap A \neq \emptyset \\ X_n \cup \{\beta_{n+1}\} & \text{otherwise} \end{cases}$$

$\square$

COROLLARY H *Let K be a belief set in $\mathbf{L}$, and $\emptyset \neq A \subset \mathbb{L}$. Then: $K \perp_P A \neq \emptyset$ if and only if $K \cap A \neq \emptyset$ and $Cn(\emptyset) \cap A = \emptyset$.* **Proof.** If $K \cap A \neq \emptyset$ and $Cn(\emptyset) \cap A = \emptyset$ take $X = \emptyset$ in Lemma G. The converse follows by the very definition of $K \perp_P A$. $\square$

Recall that a set X in a normal paraconsistent logic is contradictory if $\alpha \wedge \neg\alpha \in Cn(X)$ for some formula α. Then:

COROLLARY I *Let K be a belief set in a normal paraconsistent logic $\mathbf{L}$, and let Ω_K be the set of contradictory sentences of K (recall Definition 6.5). Then: $K \perp_P \Omega_K \neq \emptyset$ if and only if K is contradictory.* **Proof.** Immediate from Corollary H and the definitions. $\square$

THEOREM 6.7
An operation $! : Th(\mathbf{L}) \longrightarrow Th(\mathbf{L})$ over a normal paraconsistent logic $\mathbf{L}$ satisfies the postulates of Definition 6.3 iff there exists a consolidation function γ^f in $\mathbf{L}$ (in the sense of Definition 6.6) such that $K! = \bigcap \gamma^f(K)$ for every belief set K in $\mathbf{L}$.

Proof.

(construction ⇒ postulates)

closure: By Lemma F, every $X \in K\perp_P\Omega_K$ is a closed theory, and K itself is a closed theory. From this, $\bigcap\gamma^f(K)$ is a closed theory, since the intersection of closed theories is also closed.

inclusion: It follows by construction.

non-contradiction: Suppose that $K \neq \mathbb{L}$. If $K\perp_P\Omega_K \neq \emptyset$ then $\emptyset \neq \gamma^f(K) \subseteq K\perp_P\Omega_K$. Let $X \in K\perp_P\Omega_K$. Then, $\left(\bigcap\gamma^f(K)\right) \cap \Omega_K \subseteq X \cap \Omega_K = \emptyset$. On the other hand, if $K\perp_P\Omega_K = \emptyset$ then $\gamma^f(K) = \{K\}$. By Corollary I, $K \cap \Omega_K = \emptyset$. Thus $\left(\bigcap\gamma^f(K)\right) \cap \Omega_K = K \cap \Omega_K = \emptyset$.

failure: It follows from the definition of γ^f.

relevance: Let $\beta \in K \setminus \bigcap\gamma^f(K)$. Then, $\bigcap\gamma^f(K) \neq K$ and so, by construction, $K\perp_P\Omega_K \neq \emptyset$. Thus, there exists $X \in \gamma^f(K) \subseteq K\perp_P\Omega_K$ such that $\beta \notin X$. By construction and by Lemma F, $\bigcap\gamma^f(K) \subseteq X = Cn(X) \subseteq K$. Let $X' = X \cup \{\beta\}$. Then $X \subset Cn(X') \subseteq K$ by the fact that $\beta \in K$. By Definition 6.4, $\Omega_K \cap Cn(X') \neq \emptyset$, that is, $\Omega_K \cap (Cn(X) + \beta) \neq \emptyset$.

(postulates $\Rightarrow$ construction)
Consider the following function:

$$\gamma^f(K) = \begin{cases} \{X \in K\perp_P\Omega_K \ : \ K! \subseteq X\} & \text{if } K \neq \mathbb{L} \text{ and } K\perp_P\Omega_K \neq \emptyset, \\ \{K\} & \text{otherwise} \end{cases}.$$

We must prove that (1) γ^f is a consolidation function in the sense of Definition 6.6, and (2) $K! = \bigcap\gamma^f(K)$.

1. It follows by construction that $\gamma^f(K) \subseteq K\perp_P\Omega_K$ if $K \neq \mathbb{L}$ and $K\perp_P\Omega_K \neq \emptyset$, and $\gamma^f(K)\{K\}$ otherwise. It remains to prove that $\gamma^f(K) \neq \emptyset$ whenever $K \neq \mathbb{L}$ and $K\perp_P\Omega_K \neq \emptyset$. Observe that, if $K \neq \mathbb{L}$ and $K\perp_P\Omega_K \neq \emptyset$, then $\Omega_K \cap Cn(K!) = \emptyset$, by *non-contradiction* and *closure*. By *inclusion*, $K! \subseteq K$. Then, by Lemma G, there exists $X \in K\perp_P\Omega_K$ such that $K! \subseteq X$. It follows that $X \in \gamma^f(K)$ and then $\gamma^f(K) \neq \emptyset$.

2. It follows by construction and by *inclusion* that $K! \subseteq \gamma^f(K)$. We must show that $\gamma^f(K) \subseteq K!$. To this end, it is sufficient to show that, if $\beta \notin K!$ then $\beta \notin \bigcap\gamma^f(K)$. Thus, let $\beta \notin K!$. If $\beta \notin K$ then $\beta \notin \gamma^f(K)$ trivially. Now, suppose that $\beta \in K$. By *relevance*, there exists X such that $K! \subseteq Cn(X) \subseteq K$, $Cn(X) \cap \Omega_K = \emptyset$, but $Cn(X \cup \{\beta\}) \cap \Omega_K \neq \emptyset$. By Lemma G and Lemma F, there exists $X' \in K\perp_P\Omega_K$ such that $K! \subseteq Cn(X) \subseteq X' = Cn(X')$. Hence, $X' \in \gamma^f(K)$. Since $\Omega_K \cap Cn(X \cup \{\beta\}) \neq \emptyset$ it follows that $\beta \notin Cn(X') = X'$ (otherwise $\Omega_K \cap Cn(X') \neq \emptyset$). From this, $\beta \notin \bigcap\gamma^f(K)$.

□

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