Swap structures semantics for Ivlev-like modal logics

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Abstract

In 1988 J. Ivlev propose some (non-normal) modal systems which are semantically characterized by four-valued non-deterministic matrices in the sense of A. Avron and I. Lev. Swap structures are multialgebras (a.k.a. hyperalgebras) of a special kind, which were introduced in 2016 by W. Carnielli and M. Coniglio in order to give a non-deterministic semantical account for several paraconsistent logics known as *logics of formal inconsistency*, which are not algebraizable by means of the standard techniques. Each swap structure induces naturally a non-deterministic matrix. The aim of this paper is to obtain a swap structures semantics for some Ivlev-like modal systems proposed in 2015 by M. Coniglio, L. Fariñas del Cerro and N. Peron. Completeness results will be stated by means of the notion of Lindenbaum-Tarski swap structures, which constitute a natural generalization to multialgebras of the concept of Lindenbaum-Tarski algebras.

1 Introduction

In 1981 and 1988 respectively, J. Kearns (see [13]) and J. Ivlev (see [12]) independently propose a new semantics for some modal systems defined by means of four-valued multivalued truth-functions. Both contributions anticipated the non-deterministic matrices (a.k.a. Nmatrices) introduced in 2001 by A. Avron and I. Lev. Kearns proposes a semantics of four-valued Nmatrices with a restriction on the valuations (which are called *level valuations*) for the systems **T**, **S4** and **S5**. On the other hand, Ivlev proposes a genuine semantics of four-valued Nmatrices (in the sense of Avron and his collaborators, see Definition 2.7 below) for several weak modal systems which do not have the necessitation rule. Afterwards, Kearns and Ivlev's approaches were revisited and extended in [7], [8] and [15].

Non-deterministic matrices were considered by Avron and his collaborators in order to semantically characterize logics. In particular, some paraconsistent logics known as *Logics of Formal Inconsistency* (**LFI**s for short, see for instance [6]) were characterized by this kind of semantics. In formal terms, Nmatrices are logical matrices in which the underlying algebra is replaced by a multialgebra, that is, an algebra in which the operations interpreting the connectives are multiple-valued: instead of returning a single value, a non-empty set of outputs is obtained from a single input. This change requires to consider a new notion of valuation, which choose, for any complex formula, a possible value generated by the multioperators from the values already given to its subformulas (see Definition 2.5 below).

Multialgebras (also known as *hyperalgebras*) where introduced in 1934 by F. Marty (see [14]), by means of the notion of *hypergroup*. Afterwards, several classes of multialgebras were proposed in the literature. This concept was studied from many points of view and applied to several areas of Mathematics and Computer Science. In the field of Logic, besides the already mentioned Nmatrix semantics obtained by Avron and his collaborators, Carnielli and Coniglio propose in [6, Chapter 6] a class of semantical structures based on multialgebras called *swap structures*, suitable for characterizing several **LFI**s. A formal study of swap structures by using tools from category theory and universal algebra was started in [10].

The present paper generalizes the finite Nmatrix semantics introduced in [7] and [8] to some Ivlev-like systems by considering swap structures semantics for them. After defining the class of swap structures for each of such modal logics, soundness and completeness theorems will be obtained for the Hilbert calculi associated to these logics with respect to the corresponding swap structures semantics. In order to obtain the completeness theorem, the notion of *Lindenbaum-Tarski swap structures* will be introduced, which generalizes in a quite natural way the classical Lindenbaum-Tarski technique to these logics which are not algebraizable in the usual sense.

2 Multialgebras and non-deterministic matrices

In this section the notion of multialgebra and submultialgebra adopted in this paper will be recalled, taken from [10]. The related notion of non-deterministic matrices will also be recalled, for the reader's convenience.

Notation 1 If $f : A \to B$ is a function and $X \subseteq A$ then $f[X]$ will stand for the set $\{f(x) \ : \ x \in X\}$. The powerset of any set A will be denoted by $\mathcal{P}(A)$.

Definition 2.1 (Signature) *Let Ξ be a denumerable set of propositional variables. A (propositional) signature is a family of sets $\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}}$, such that $\Xi \cap \Sigma_n = \emptyset$ for every n and, if $n \neq m$, then $\Sigma_n \cap \Sigma_m = \emptyset$. The free algebra of terms generated by Ξ over a signature Σ will be denoted by $For(\Sigma)$.*[1]

Elements in Σ_n are called *connectives of arity n*. In particular, elements in Σ_0 are called *constants*. Elements in $For(\Sigma)$ are called *formulas* over Σ. From now on, and whenever $|\Sigma| = \bigcup_{n\geq 0} \Sigma_n$ is finite, the signature Σ will be identified with $|\Sigma|$, assuming that the arity of each connective is clear from the context.

Definition 2.2 (Multialgebras) *Let Σ be a signature. A* multialgebra *(or* hyperalgebra*) over Σ is a pair $\mathcal{A} = \langle A, \sigma_{\mathcal{A}}\rangle$ such that A is a nonempty set (the* universe *or* support *of $\mathcal{A}$) and $\sigma_{\mathcal{A}}$ is a mapping assigning to each $c \in \Sigma_n$ a function (called* multioperation *or* hyperoperation*) $c^{\mathcal{A}} : A^n \to (\mathcal{P}(A) \setminus \{\emptyset\})$. In particular, $\emptyset \neq c^{\mathcal{A}} \subseteq A$ if $c \in \Sigma_0$.*

[1]This notation makes sense, provided that the set Ξ of generators will keep fixed along this paper.

Definition 2.3 (Submultialgebras, [10]) *Let $\mathcal{B} = \langle B, \sigma_{\mathcal{B}} \rangle$ and $\mathcal{A} = \langle A, \sigma_{\mathcal{A}} \rangle$ be two multialgebras over Σ. Then $\mathcal{B}$ is said to be a* submultialgebra *of $\mathcal{A}$ if the following conditions hold:*

(i) $B \subseteq A$,

(ii) *if $c \in \Sigma_n$ and $\vec{b} \in B^n$, then $c^{\mathcal{B}}(\vec{b}) \subseteq c^{\mathcal{A}}(\vec{b})$; in particular, $c^{\mathcal{B}} \subseteq c^{\mathcal{A}}$ if $c \in \Sigma_0$.*

The general notion of non-deterministic matrix (or Nmatrix) was formally introduced in [3] (see [5] for a survey on non-deterministic matrices). However, Nmatrices were already used in the literature for some specific cases: see, for instance, N. Rescher's non-deterministic implication proposed in [16] and the modal Nmatrices proposed by J. Kearns and J. Ivlev to be discussed in Section 3. In formal terms, Nmatrices are nothing else than multialgebras with a non-empty set of designated elements (generalizing the idea of logical matrices, which are algebras together with a set of designated values).

Definition 2.4 (Non-deterministic matrices, [3])
Let Σ be a signature. A non-deterministic matrix (or Nmatrix) over Σ is a pair $\mathcal{M} = \langle \mathcal{A}, D \rangle$ such that $\mathcal{A} = \langle A, \sigma_{\mathcal{A}} \rangle$ is a multialgebra over Σ with support A, and D is a non-empty subset of A. The elements in D are called designated elements.

Recall now the definition of valuations over non-deterministic matrices proposed in [3] (see also [4], [2]).

Definition 2.5 (Valuations, [3]) *Let $\mathcal{M} = \langle \mathcal{A}, D \rangle$ be a non-deterministic matrix over a signature Σ. A* valuation[2] *over $\mathcal{M}$ is a function $v : For(\Sigma) \to |\mathcal{A}|$ such that, for every $c \in \Sigma_n$ and every $\alpha_1, \ldots, \alpha_n \in For(\Sigma)$:*

$$v(c(\alpha_1, \ldots, \alpha_n)) \in c^{\mathcal{A}}(v(\alpha_1), \ldots, v(\alpha_n)).$$

In particular, $v(c) \in c^{\mathcal{A}}$, for every $c \in \Sigma_0$.

Remark 2.6 *It is worth noting that the same notion of valuation over Nmatrices was already considered (in a somewhat informal way) in the pioneering works [16], [13] and [12]. In the case of [13], a restricted subset of valuations was also considered, as it was mentioned in Section 1.*

Definition 2.7 (Consequence relation, [3])
Let $\mathcal{M} = \langle \mathcal{A}, D \rangle$ be a non-deterministic matrix over a signature Σ, and let $\Gamma \cup \{\alpha\} \subseteq For(\Sigma)$. We say that α is a consequence *of Γ in the non-deterministic matrix $\mathcal{M}$, denoted by $\Gamma \models_{\mathcal{M}} \alpha$, if the following holds: for every valuation v over $\mathcal{M}$, if $v[\Gamma] \subseteq D$ then $v(\alpha) \in D$. In particular, α is* valid *in $\mathcal{M}$, denoted by $\models_{\mathcal{M}} \alpha$, if $v(\alpha) \in D$ for every valuation v over $\mathcal{M}$.*

[2] Avron and his collaborators use in [3], [1], [5] the term *legal valuation* to refer to valuations over an Nmatrix.

3 Nondeterministic Ivlev-like modal logics

As mentioned in Section 1, J. Ivlev propose in [12] a family of (non-normal) modal logics (that is, modal logics in which the necessitation rule is no longer valid) which are characterized by four-valued Nmatrices. Some generalizations of Ivlev's logics were proposed in [7] (see also [8] and [15]).

In order to define four-valued non-deterministic matrices, Ivlev considers the set $\{T, t, f, F\}$ of truth-values with the following intended meaning:

T: Necessarily true

t: Possibly true

f: Possibly false

F: Necessarily false

The set of designated truth-values is $\{T, t\}$, while $\{F, f\}$ is the set of non-designated truth-values. Let $\Sigma' = \{\neg, \to, \Box\}$ be a propositional signature for modal systems. The following (four-valued) non-deterministic matrices where independently proposed by Ivlev (system S_a+, see [12]) and Kearns (system **T**, see [13]):[3]

$\to$	T	t	f	F
T	T	t	f	F
t	T	$\{T,t\}$	f	f
f	T	$\{T,t\}$	$\{T,t\}$	t
F	T	T	T	T

α	$\neg\alpha$	$\Box\alpha$
T	F	$\{T,t\}$
t	f	$\{f,F\}$
f	t	$\{f,F\}$
F	T	$\{f,F\}$

The connectives $\vee$ and $\Diamond$ can be defined, as in the classical case by $\alpha \vee \beta = \neg\alpha \to \beta$ and $\Diamond\alpha = \neg\Box\neg\alpha$, respectively, thus producing the following multioperators:

$\vee$	T	t	f	F
T	T	T	T	T
t	T	$\{T,t\}$	$\{T,t\}$	t
f	T	$\{T,t\}$	f	f
F	T	t	f	F

α	$\Diamond\alpha$
T	$\{T,t\}$
t	$\{T,t\}$
f	$\{T,t\}$
F	$\{f,F\}$

A Hilbert-style deductive system for S_a+ was introduced in [7], under the name of **Tm**. A slightly different version will be introduced here, by following the modifications suggested in [15] and [8].

Definition 3.1 *The system* ***Tm*** *over* $\Sigma' = \{\neg, \to, \Box\}$ *is given by the following axiom schemes:*

$$\alpha \to (\beta \to \alpha) \quad (\mathit{Ax1})$$

[3]As observed in Section 1, Kearns restricts the valuations over the Nmatrices in order to deal with the necessitation rule.

$$(\alpha \to (\beta \to \sigma)) \to ((\alpha \to \beta) \to (\alpha \to \sigma)) \quad (\mathit{Ax2})$$

$$(\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta) \quad (\mathit{Ax3})$$

$$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta) \quad (\mathit{K})$$

$$\Box(\alpha \to \beta) \to (\Box\neg\beta \to \Box\neg\alpha) \quad (\mathit{K1})$$

$$\neg\Box\neg(\alpha \to \beta) \to (\Box\alpha \to \neg\Box\neg\beta) \quad (\mathit{K2})$$

$$\Box\neg\alpha \to \Box(\alpha \to \beta) \quad (\mathit{M1})$$

$$\Box\beta \to \Box(\alpha \to \beta) \quad (\mathit{M2})$$

$$\Box\neg(\alpha \to \beta) \to \Box\neg\beta \quad (\mathit{M3})$$

$$\Box\neg(\alpha \to \beta) \to \Box\alpha \quad (\mathit{M4})$$

$$\Box\alpha \to \alpha \quad (\mathit{T})$$

$$\Box\alpha \to \Box\neg\neg\alpha \quad (\mathit{DN1})$$

$$\Box\neg\neg\alpha \to \Box\alpha \quad (\mathit{DN2})$$

together with the following inference rule:

$$\frac{\alpha, \alpha \to \beta}{\beta} \quad (\mathit{MP})$$

The systems **T4m**, **T45m**, **TBm** and **Dm** were introduced in [7] (see also [8]), obtained from **Tm** by considering the following axioms:

$$\Box\alpha \to \Box\Box\alpha \quad (4)$$

$$\neg\Box\neg\Box\alpha \to \Box\alpha \quad (5)$$

$$\neg\Box\neg\Box\alpha \to \alpha \quad (\mathsf{B})$$

$$\Box\alpha \to \neg\Box\neg\alpha \quad (\mathsf{D})$$

Let us consider firstly the systems **T4m**, **T45m** and **TBm**. A Hilbert calculus for each of them is given, respectively, by:

$$\mathbf{T4m} = \mathbf{Tm} \cup \{(4)\}$$

$$\mathbf{T45m} = \mathbf{T4m} \cup \{(5)\}$$

$$\mathbf{TBm} = \mathbf{Tm} \cup \{(\mathsf{B})\}$$

The non-deterministic matrices for the systems **T4m**, **T45m** and **TBm** are obtained from the one for **Tm** by respectively changing the multioperation $\Box$ to:

α	$\Box^{\mathbf{T4m}}\alpha$
T	T
t	$\{f, F\}$
f	$\{f, F\}$
F	$\{f, F\}$

α	$\Box^{\mathbf{T45m}}\alpha$
T	T
t	F
f	F
F	F

α	$\Box^{\mathbf{TBm}}\alpha$
T	$\{T, t\}$
t	$\{f, F\}$
f	F
F	F

Remark 3.2 *The operation $\Box^{T45m}$ coincides with the interpretation for $\Box$ proposed by Kearns in [13] for the system* **S5** *and by Ivlev for the system S_b+ in [12]. By taking $\Box(F) = \{F\}$ instead of $\Box(F) = \{f, F\}$ in $\Box^{T4m}$, the resulting multioperation coincides with the one proposed by Kearns [13] in order to interpret $\Box$ in the system* **S4**.

As usual, $\Diamond\alpha$ will be an abbreviation of the modal formula $\neg\Box\neg\alpha$. Recall that deontic logics are modal logics in which the operators $\Box$ and $\Diamond$ are intuitively interpreted as *it is obligatory that* and *it is permitted that*, respectively. Under this interpretation, it is natural to consider the principle $\Box\alpha \to \Diamond\alpha$ instead of the stronger principles $\Box\alpha \to \alpha$ and $\alpha \to \Diamond\alpha$.

Now, swap structures will be applied to a deontic-like modal system. The goal is to provide a slightly more general example. Let us consider the modal system **Dm** introduced in [7] and [8] which constitute a weaker version of the deontic logic KD. It is obtained from **Tm** by changing axiom (T) by (D). From the semantical point of view, it produces a six-valued characteristic non-deterministic matrix semantics. Given a proposition p, some usual deontic-like operators can be defined in the system **Dm**:

- p is *fulfilled* whenever it is obligatory and it is the case. In symbols: $(\Box p \wedge p)$.
- p is *infringed* whenever it is obligatory and it is not the case. In symbols: $(\Box p \wedge \neg p)$
- p is *optional* if it is neither obligatory nor forbidden. In symbols: $\neg(\Box p \vee \Box\neg p)$. Note that p is optional iff $\neg p$ is optional.
- p is *preferred* whenever it is optional and it is the case. In symbols: $(\neg(\Box p \vee \Box\neg p) \wedge p)$.

From this, a plausible interpretation for the six truth-values in the system **Dm** can be proposed:

T^+: p is fulfilled;

C^+: p is preferred;

F^+: $\neg p$ is infringed;

T^-: p is infringed;

C^-: $\neg p$ is preferred; and

F^-: $\neg p$ is fulfilled.

As mentioned above, a Hilbert calculus for **Dm** is given by:

$$\mathbf{Dm} = \big(\mathbf{Tm} \setminus \{(\mathsf{T})\}\big) \cup \{(\mathsf{D})\}$$

The Nmatrix semantics for the system **Dm** proposed in [7] is given by the multioperations $\to^{\mathbf{Dm}}$ and $\Box^{\mathbf{Dm}}$ and by the operation $\neg^{\mathbf{Dm}}$ displayed below:

$\to^{\mathbf{Dm}}$	T^+	T^-	C^+	C^-	F^+	F^-
T^+	T^+	T^-	C^+	C^-	F^+	F^-
T^-	T^+	T^+	C^+	C^+	F^+	F^+
C^+	T^+	T^-	$\{T^+, C^+\}$	$\{T^-, C^-\}$	C^+	C^-
C^-	T^+	T^+	$\{T^+, C^+\}$	$\{T^+, C^+\}$	C^+	C^+
F^+	T^+	T^-	T^+	T^-	T^+	T^-
F^-	T^+	T^+	T^+	T^+	T^+	T^+

α	$\neg^{\mathbf{Dm}}\alpha$
T^+	F^-
T^-	F^+
C^+	C^-
C^-	C^+
F^+	T^-
F^-	T^+

α	$\Box^{\mathbf{Dm}}\alpha$
T^+	$\{T^+, C^+, F^+\}$
T^-	$\{T^+, C^+, F^+\}$
C^+	$\{T^-, C^-, F^-\}$
C^-	$\{T^-, C^-, F^-\}$
F^+	$\{T^-, C^-, F^-\}$
F^-	$\{T^-, C^-, F^-\}$

in which $\{T^+, C^+, F^+\}$ is the set of designated values.

Remark 3.3 *It is worth noting that the six-valued Nmatrix for* **Dm** *extends the four-valued Nmatrix for* **Tm** *in the following sense: the truth-values T, t, f and F corresponds to T^+, C^+, C^- and F^-, respectively. Indeed, the multialgebra underlying the Nmatrix for* **Tm** *is a submultialgebra of the multialgebra underlying the Nmatrix for* **Dm** *(recall the notion of submultialgebra given in Definition 2.3 taken from [10]). Moreover, the respective sets $D_{\mathbf{Tm}}$ and $D_{\mathbf{Dm}}$ of designated values are related as follows: $D_{\mathbf{Tm}} = D_{\mathbf{Dm}} \cap \big\{T, t, f, F\big\}$.*

Notation 2 From now on, $\mathbb{L}$ will denote the set

$$\big\{\mathbf{Tm}, \mathbf{T4m}, \mathbf{T45m}, \mathbf{TBm}, \mathbf{Dm}\big\}$$

of modal logics.

In [7] and [8] it was obtained the following characterization result:

Theorem 3.4 *Let $\mathcal{L} \in \mathbb{L}$ and let $\mathcal{M}_{\mathcal{L}}$ be the Nmatrix for $\mathcal{L}$ as defined above. Then, for every $\alpha \in For(\Sigma')$,*

$$\vdash_{\mathcal{L}} \alpha \Leftrightarrow \ \vDash_{\mathcal{M}_{\mathcal{L}}} \alpha.$$

In the next section, swap structures semantics for the systems in $\mathbb{L}$ will be introduced, which generalizes the Nmatrix semantics analyzed above.

4 From Nmatrices to swap structures semantics

As it was discussed in Section 1, the notion of *swap structures* was introduced in [6, Chapter 6] in order to give a semantical account (with an “algebraic flavour”) to several paraconsistent logics in the class of *logics of formal inconsistency* (**LFI**s) which cannot be algebraizable by the standard methods. These structures are multialgebras formed by triples $z = (z_1, z_2, z_3)$ of elements over a given Boolean algebra $\mathcal{A}$, which are called *snapshots*.[4] The intended meaning of a snapshot z is that z_1 represents a given (algebraic) truth-value for a formula φ in $\mathcal{A}$, while z_2 and z_3 represent a (possible) truth-value for the formulas $\neg\varphi$ and $\circ\varphi$, respectively.[5] Each swap structure for a given **LFI** induces a non-deterministic matrix in a natural way, where the class of all such Nmatrices characterizes the given logic. In [10] a formal study of the category of swap structures for some **LFI**s was developed, by adapting notions and tools from universal algebra.

In this section, the original approach to swap structures semantics for **LFI**s will be adapted to the family of non-normal modal systems presented in Section 3. As in the case of **LFI**s, the swap structures proposed here are multialgebras whose elements are triples (snapshots) in a given Boolean algebra. In the present case, the intended meaning of a snapshot $z = (z_1, z_2, z_3)$ is that z_1 represents a given truth-value for a formula φ in $\mathcal{A}$, while z_2 and z_3 represent a (possible) truth-value for the formulas $\Box\varphi$ and $\Box\neg\varphi$, respectively.

Definition 4.1 (Swap structures for Tm) *Let*

$$\mathcal{A} = \langle A, \supset, \sqcup, \sqcap, \sim, 0, 1\rangle$$

be a Boolean algebra and let

$$\mathbb{B}_{\mathcal{A}}^{\boldsymbol{Tm}} = \{(a_1, a_2, a_3) \in A^3 \ : \ a_2 \leq a_1 \ \ \textit{and} \ \ a_1 \sqcap a_3 = 0\}.$$

A swap structure for ***Tm*** *over* $\mathcal{A}$ *is any multialgebra*

$$\mathcal{B} = \langle B, \tilde{\to}, \tilde{\neg}, \tilde{\Box}\rangle$$

over $\Sigma' = \{\to, \neg, \Box\}$ *such that* $B \subseteq \mathbb{B}_{\mathcal{A}}^{\boldsymbol{Tm}}$ *and the multioperations satisfy the following, for every* (a_1, a_2, a_3) *and* (b_1, b_2, b_3) *in* B*:*

(i) $(a_1, a_2, a_3) \mathrel{\tilde{\to}} (b_1, b_2, b_3) = \{(c_1, c_2, c_3) \in B \ : \ c_1 = a_1 \supset b_1, \ c_3 = a_2 \sqcap b_3 \ \ \textit{and} \ \ a_3 \sqcup b_2 \leq c_2 \leq (a_1 \supset b_1) \sqcap (a_2 \supset b_2) \sqcap (b_3 \supset a_3)\}$

(ii) $\tilde{\neg}(a_1, a_2, a_3) = \{(\sim a_1, a_3, a_2)\}$

(iii) $\tilde{\Box}(a_1, a_2, a_3) = \{(c_1, c_2, c_3) \in B \ : \ c_1 = a_2\}$

The unique swap structure for **Tm** with domain $\mathbb{B}_{\mathcal{A}}^{\mathbf{Tm}}$ will be denoted by $\mathcal{B}_{\mathcal{A}}^{\mathbf{Tm}}$. It will be called the *full swap structure for* ***Tm*** *over* $\mathcal{A}$.

[4] This terminology is inspired by its use in computer science to refer to states.

[5] Here $\neg$ represents the paraconsistent negation of a given **LFI**, while $\circ$ represents a *consistency operator* w.r.t. $\neg$. See for instance [6] for more information about **LFI**s.

Remark 4.2 *The intuition behind the universe $\mathbb{B}_{\mathcal{A}}^{\boldsymbol{Tm}}$ and the definition of the multioperations for swap structures for* $\boldsymbol{Tm}$ *is the following: suppose that $z = (a_1, a_2, a_3)$ is a snapshot in a given swap structure $\mathcal{B}$ for* $\boldsymbol{Tm}$. *Then, as explained above, a_1, a_2 and a_3 represent a possible truth-value (over the Boolean algebra $\mathcal{A}$) for φ, $\Box\varphi$ and $\Box\neg\varphi$, respectively, for a given formula φ. By axiom (T) it follows that $\Box\varphi \to \varphi$ is valid, which means that $a_2 \supset a_1 = 1$ under such interpretation. But the latter is equivalent to say that $a_2 \leq a_1$. By an analogous argument, $a_3 \leq {\sim}a_1$, where ${\sim}a_1$ represents the truth-value of $\neg\varphi$ under such interpretation (recall by Definition 4.1 that the Boolean complement in the Boolean algebra $\mathcal{A}$ is denoted by $\sim$). Hence, $a_1 \sqcap a_3 = 0$. With respect to the multioperations, the definition of the (standard) operation $\neg$ in a given swap structure $\mathcal{B}$ for* $\boldsymbol{Tm}$ *is natural under such interpretation: indeed, since $z = (a_1, a_2, a_3)$ represents the truth-values of φ, $\Box\varphi$ and $\Box\neg\varphi$, then $({\sim}a_1, a_3, a_2)$ represents the truth-values of $\neg\varphi$, $\Box\neg\varphi$ and $\Box\neg\neg\varphi$, given that $\Box\neg\neg\varphi$ is equivalent to $\Box\varphi$ in* $\boldsymbol{Tm}$. *Concerning the multioperation associated to $\Box$, it is clear that any snapshot $w = (c_1, c_2, c_3)$ in the set $\Box z$ must satisfy that $c_1 = a_2$, since a_2 represents $\Box\varphi$. The truth-values c_2 and c_3 for $\Box\Box\varphi$ and $\Box\neg\Box\varphi$ cannot be expressed in* $\boldsymbol{Tm}$ *in terms of a Boolean combination of the given truth-valued for φ, $\Box\varphi$ and $\Box\neg\varphi$, and this is why the second and third coordinates of w cannot be additionally restricted. As we shall see below, the axiomatic extensions of* $\boldsymbol{Tm}$ *anayzed here will impose some retrictions to the multioperation $\Box$. The definition of the multioperation associated to $\to$ is not so intuitive, and it is obtained by generalizing the definition of Kearns and Ivlev's four-valued Nmatrix for* $\boldsymbol{Tm}$ *to arbitrary Boolean algebras. Indeed, as it will be seen in Remark 4.6, such Nmatrix corresponds to the swap structure $\mathcal{B}_{\mathcal{A}_2}^{\boldsymbol{Tm}}$ defined over the two-element Boolean algebra $\mathcal{A}_2$. Finally, it is worth noting that, by the very definition, the class of swap structures for* $\boldsymbol{Tm}$ *is closed under submultialgebras (recall Definition 2.3). Moreover, any swap structure for* $\boldsymbol{Tm}$ *over $\mathcal{A}$ is a submultialgebra of $\mathcal{B}_{\mathcal{A}}^{\boldsymbol{Tm}}$.*

As it will be shown in Corollary 4.5, the implication in any swap structure for **Tm** is a well-defined multioperation. Namely: $(a_1, a_2, a_3) \tilde{\to} (b_1, b_2, b_3) \neq \emptyset$ for any (a_1, a_2, a_3) and (b_1, b_2, b_3) in $\mathbb{B}_{\mathcal{A}}^{\mathbf{Tm}}$.

Proposition 4.3 *: If $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{B}_{\mathcal{A}}^{\boldsymbol{Tm}}$ then*

$$a_3 \sqcup b_2 \leq (a_1 \supset b_1) \sqcap (a_2 \supset b_2) \sqcap (b_3 \supset a_3).$$

Proof: Let be $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{B}_{\mathcal{A}}^{\mathbf{Tm}}$.

As $a_1 \sqcap a_3 = 0$, then $a_3 \leq {\sim}a_1$. But, as $a_2 \leq a_1$, then ${\sim}a_1 \leq {\sim}a_2$ and, therefore, $a_3 \leq {\sim}a_2$. So,

$$a_3 \sqcup b_2 \leq {\sim}a_2 \sqcup b_2 = a_2 \supset b_2 \tag{1}$$

It also holds that $(a_3 \sqcup b_2) \sqcap a_1 = (a_3 \sqcap a_1) \sqcup (b_2 \sqcap a_1) = 0 \sqcup (b_2 \sqcap a_1) = b_2 \sqcap a_1 \leq b_2 \leq b_1$. So,

$$a_3 \sqcup b_2 \leq a_1 \supset b_1 \tag{2}$$

On the other hand, $(a_3 \sqcup b_2) \sqcap b_3 = (a_3 \sqcap b_3) \sqcup (b_2 \sqcap b_3) \leq (a_3 \sqcap b_3) \sqcup (b_1 \sqcap b_3) = (a_3 \sqcap b_3) \sqcup 0 = a_3 \sqcap b_3 \leq a_3$. So,

$$a_3 \sqcup b_2 \leq b_3 \supset a_3 \tag{3}$$

From (1), (2) and (3), we have $a_3 \sqcup b_2 \leq (a_1 \supset b_1) \sqcap (a_2 \supset b_2) \sqcap (b_3 \supset a_3)$. ■

Proposition 4.4 *: If $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{B}_{\mathcal{A}}^{\boldsymbol{Tm}}$, then $(a_1 \supset b_1) \sqcap (a_2 \sqcap b_3) = 0$.*

Proof: Straightforward. ■

Corollary 4.5 *If* $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{B}_{\mathcal{A}}^{\boldsymbol{Tm}}$ *then*

$$\emptyset \neq (a_1, a_2, a_3) \tilde{\to} (b_1, b_2, b_3) \subseteq \mathbb{B}_{\mathcal{A}}^{\boldsymbol{Tm}}.$$

Remark 4.6 *Observe that, in the case of the two-element Boolean algebra* $\mathcal{A}_2$ *with domain* $A_2 = \{0, 1\}$*, eight triples are possible:* $(1, 1, 1)$, $(1, 0, 1)$, $(0, 1, 0)$, $(0, 1, 1)$, $(1, 1, 0)$, $(1, 0, 0)$, $(0, 0, 0)$ *and* $(0, 0, 1)$. *However, in order to satisfy the requirements of Definition 4.1, the snapshots* $(0, 1, 0)$, $(0, 1, 1)$, $(1, 1, 1)$ *and* $(1, 0, 1)$ *must be discarded. So,* $\mathbb{B}_{\mathcal{A}_2}^{\boldsymbol{Tm}} = \{(1, 1, 0), (1, 0, 0), (0, 0, 0), (0, 0, 1)\}$. *By identifying the values* $(1, 1, 0)$, $(1, 0, 0)$, $(0, 0, 0)$ *and* $(0, 0, 1)$ *with* T, t, f *and* F, *respectively, it is immediate to see that the multioperations of the Nmatrix* $\mathcal{M}_{\mathbf{Tm}}$ *for* **Tm** *coincide with the ones for* $\mathcal{B}_{\mathcal{A}_2}^{\boldsymbol{Tm}}$.

Definition 4.7 (Swap structures for T4m) *A swap structure for* ***T4m*** *over* $\mathcal{A}$ *is a swap structure for* ***Tm*** *where the multioperation* $\Box$ *is given by* $\tilde{\Box}(a_1, a_2, a_3) = \{(c_1, c_2, c_3) \in B \ : \ c_1 = a_2 \text{ and } a_2 \leq c_2\}$ *for every* (a_1, a_2, a_3) *in* B.

Definition 4.8 (Swap structures for T45m) *A swap structure for* ***T45m*** *over* $\mathcal{A}$ *is a swap structure for* ***Tm*** *such that the multioperation* $\Box$ *is given by* $\tilde{\Box}(a_1, a_2, a_3) = \{(c_1, c_2, c_3) \in B \ : \ c_1 = a_2, \ a_2 \leq c_2 \text{ and } c_3 \sqcup c_1 = 1\}$ *for every* (a_1, a_2, a_3) *in* B.

Definition 4.9 (Swap structures for TBm) *A swap structure for* ***TBm*** *over* $\mathcal{A}$ *is a swap structure for* ***Tm*** *such that the multioperation* $\Box$ *is given by* $\tilde{\Box}(a_1, a_2, a_3) = \{(c_1, c_2, c_3) \in B \ : \ c_1 = a_2 \text{ and } a_1 \sqcup c_3 = 1\}$ *for every* (a_1, a_2, a_3) *in* B.

Remark 4.10 *As in the case of* ***Tm****, assume that a snapshot* $z = (a_1, a_2, a_3)$ *represents the truth-values of* φ, $\Box\varphi$ *and* $\Box\neg\varphi$. *Then, the multioperation associated to* $\Box$ *in the previous definitions is natural under such interpretation:*

In the case for ***T4m****, the additional axiom* (4) *imposes the condition* $a_2 \leq c_2$ *in Definition 4.7.*

In the case for ***T45m****, the additional axioms* (4) *and* (5) *impose the conditions* $a_2 \leq c_2$ *and* $c_3 \sqcup c_1 = 1$ *in Definition 4.8.*

Finally, in the case for ***TBm****, the axiom* (B) *imposes the condition* $a_1 \sqcup c_3 = 1$ *in Definition 4.9.*

Remark 4.11 *Let* $\mathcal{L} \in \{\boldsymbol{T4m}, \boldsymbol{T45m}, \boldsymbol{TBm}\}$ *and consider again the case of the two-element Boolean algebra* $\mathcal{A}_2$. *Thus,* $\mathbb{B}_{\mathcal{A}_2}^{\mathcal{L}} = \mathbb{B}_{\mathcal{A}_2}^{\mathbf{Tm}} = \{(1, 1, 0), (1, 0, 0), (0, 0, 0), (0, 0, 1)\}$. *The unique swap structure for* $\mathcal{L}$ *with domain* $\mathbb{B}_{\mathcal{A}}^{\boldsymbol{Tm}}$ *will be denoted by* $\mathcal{B}_{\mathcal{A}}^{\mathcal{L}}$. *It will be called the* full swap structure for $\mathcal{L}$ over $\mathcal{A}$. *As in the case of* **Tm***, its multioperations coincide with the ones for the Nmatrix* $\mathcal{M}_{\mathcal{L}}$ *for* $\mathcal{L}$.

Definition 4.12 (Swap structures for Dm) *Let*

$$\mathcal{A} = \langle A, \supset, \sqcup, \sqcap, \sim, 0, 1 \rangle$$

be a Boolean algebra and let

$$\mathbb{B}_{\mathcal{A}}^{\boldsymbol{Dm}} = \{(a_1, a_2, a_3) \in A^3 \ : \ a_2 \sqcap a_3 = 0\}.$$

A swap structure for ***Dm*** *over* $\mathcal{A}$ *is any multialgebra*

$$\mathcal{B} = \langle B, \tilde{\to}, \tilde{\neg}, \tilde{\Box} \rangle$$

over $\Sigma' = \{\to, \neg, \Box\}$ *such that* $B \subseteq \mathbb{B}_{\mathcal{A}}^{\boldsymbol{Dm}}$ *and the multioperations satisfy the following, for every* (a_1, a_2, a_3) *and* (b_1, b_2, b_3) *in* B:

(i) $(a_1, a_2, a_3) \tilde{\to} (b_1, b_2, b_3) = \{(c_1, c_2, c_3) \in B \ : \ c_1 = a_1 \supset b_1, \ c_3 = a_2 \sqcap b_3$ *and* $a_3 \sqcup b_2 \leq c_2 \leq (a_2 \supset b_2) \sqcap (b_3 \supset a_3)\}$

(ii) $\tilde{\neg}(a_1, a_2, a_3) = \{(\sim a_1, a_3, a_2)\}$

(iii) $\tilde{\Box}(a_1, a_2, a_3) = \{(c_1, c_2, c_3) \in B \ : \ c_1 = a_2\}$

Remark 4.13 *As in the case of* $\boldsymbol{Tm}$*, the intuition behind the universe* $\mathbb{B}_{\mathcal{A}}^{\boldsymbol{Dm}}$ *is the following: if* $z = (a_1, a_2, a_3)$ *is a snapshot in a given swap structure* $\mathcal{B}$ *for* $\boldsymbol{Dm}$ *then* a_1*,* a_2 *and* a_3 *represent a possible truth-value (over the Boolean algebra* $\mathcal{A}$*) for* φ*,* $\Box\varphi$ *and* $\Box\neg\varphi$*, respectively, for a given formula* φ*. By axiom* (D) *it follows that* $\Box\varphi \to \neg\Box\neg\varphi$ *is valid, which means that* $\neg(\Box\varphi \wedge \Box\neg\varphi)$ *is valid under such interpretation. But the latter is equivalent to say that* $a_2 \sqcap a_3 = 0$*. With respect to the multioperations in a swap structure for* $\boldsymbol{Dm}$*:* $\tilde{\neg}$ *and* $\tilde{\Box}$ *are defined as in Definition 4.1, and* $\tilde{\to}$ *is also defined as in Definition 4.1, but now without requiring that* $c_2 \leq (a_1 \supset b_1)$*.*

As in the case of **Tm**, it will be shown that the implication $(a_1, a_2, a_3) \tilde{\to} (b_1, b_2, b_3)$ in **Dm** returns a non-empty set, for $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{B}_{\mathcal{A}}^{\mathbf{Dm}}$:

Proposition 4.14 *: If* $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{B}_{\mathcal{A}}^{\boldsymbol{Dm}}$*, then*

$$a_3 \sqcup b_2 \leq (a_2 \supset b_2) \sqcap (b_3 \supset a_3).$$

Proof: Let be $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{B}_{\mathcal{A}}^{\mathbf{Dm}}$. We have $a_3 \sqcap \sim a_2 = 0 \sqcup (a_3 \sqcap \sim a_2) = (a_3 \sqcap a_2) \sqcup (a_3 \sqcap \sim a_2) = a_3 \sqcap (a_2 \sqcup \sim a_2) = a_3 \sqcap 1 = a_3$. But, $a_3 \sqcap \sim a_2 = a_3$ iff $a_3 \leq \sim a_2$. Then

$$a_3 \sqcup b_2 \leq \sim a_2 \sqcup b_2 = a_2 \supset b_2 \tag{4}$$

On the other hand, $b_2 \sqcap \sim b_3 = 0 \sqcup (b_2 \sqcap \sim b_3) = (b_2 \sqcap b_3) \sqcup (b_2 \sqcap \sim b_3) = b_2 \sqcap (b_3 \sqcup \sim b_3)$ $= b_2 \sqcap 1 = b_2$. But, $b_2 \sqcap \sim b_3 = b_2$ iff $b_2 \leq \sim b_3$. Then

$$a_3 \sqcup b_2 \leq a_3 \sqcup \sim b_3 = b_3 \supset a_3 \tag{5}$$

From (4) and (5), $a_3 \sqcup b_2 \leq (a_2 \supset b_2) \sqcap (b_3 \supset a_3)$. ∎

Proposition 4.15 *If* $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{B}_{\mathcal{A}}^{\boldsymbol{Dm}}$*, then* $(a_3 \sqcup b_2) \sqcap (a_2 \sqcap b_3) = 0$*.*

Proof: $(a_3 \sqcup b_2) \sqcap (a_2 \sqcap b_3) = (a_2 \sqcap b_3 \sqcap a_3) \sqcup (a_2 \sqcap b_3 \sqcap b_2) = (0 \sqcap b_3) \sqcup (a_2 \sqcap 0) = 0 \sqcup 0 = 0$. ∎

Corollary 4.16 *If* $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{B}_{\mathcal{A}}^{\boldsymbol{Dm}}$ *then*

$$\emptyset \neq (a_1, a_2, a_3) \tilde{\to} (b_1, b_2, b_3) \subseteq \mathbb{B}_{\mathcal{A}}^{\boldsymbol{Dm}}.$$

The unique swap structure for **Dm** with domain $\mathbb{B}_{\mathcal{A}}^{\mathbf{Dm}}$ will be denoted by $\mathcal{B}_{\mathcal{A}}^{\mathbf{Dm}}$. It will be called the *full swap structure for* ***Dm*** *over* $\mathcal{A}$.

Remark 4.17 *Recall that* $\mathbb{L}$ *denotes the set* $\{\boldsymbol{Tm}, \boldsymbol{T4m}, \boldsymbol{T45m}, \boldsymbol{TBm}, \boldsymbol{Dm}\}$ *(see Notation 2). In the case of the two-element Boolean algebra* $\mathcal{A}_2$*, it is clear that* $\mathbb{B}_{\mathcal{A}_2}^{\mathbf{Dm}} = \mathbb{B}_{\mathcal{A}_2}^{\mathbf{Tm}} \cup \{(1,0,1), (0,1,0)\}$ *such that* $(1,0,1)$ *and* $(0,1,0)$ *correspond to the "new" truth-values* F^+ *and* T^-*, respectively. As in the four-valued cases, the multioperations of the Nmatrix* $\mathcal{M}_{\mathbf{Dm}}$ *for* **Dm** *coincide with the ones for* $\mathcal{B}_{\mathcal{A}_2}^{\mathbf{Dm}}$*. It is possible to give an intuitive reading for the definition of the multioperations in the class of swap structures for any* $\mathcal{L} \in \mathbb{L} \setminus \{\mathbf{Tm}\}$*, analogous to the analysis for* **Tm** *found in Remark 4.2.*

Definition 4.18 *If $\mathcal{L} \in \mathbb{L}$, we denote by $\mathbb{K}_{\mathcal{L}}$ the class of swap structures for $\mathcal{L}$.*

Definition 4.19 *Let $\mathcal{L} \in \mathbb{L}$ and let $D_{\mathcal{B}} = \{(z_1, z_2, z_3) \in |\mathcal{B}| \ : \ z_1 = 1\}$ for each $\mathcal{B} \in \mathbb{K}_{\mathcal{L}}$. The* non-deterministic matrix associated to $\mathcal{B}$ *is $\mathcal{M}(\mathcal{B}) = \langle \mathcal{B}, D_{\mathcal{B}} \rangle$.*

Definition 4.20 *If $\mathcal{L} \in \mathbb{L}$, then we denote by $Mat(\mathbb{K}_{\mathcal{L}}) = \{\mathcal{M}(\mathcal{B}) \ : \ \mathcal{B} \in \mathbb{K}_{\mathcal{L}}\}$ the class of non-deterministic matrices associated to the class of swap structures for the system $\mathcal{L}$.*

Recall the semantics associated to non-deterministic matrices where the relation $\Delta \models_{\mathcal{M}} \alpha$ is the consequence relation defined over a non-deterministic matrix $\mathcal{M}$ (see Definition 2.7):

Definition 4.21 *For $\mathcal{L} \in \mathbb{L}$, let $\Delta \cup \{\alpha\} \subseteq For(\Sigma')$ be a set of formulas of $\mathcal{L}$. We say that α is a consequence of Δ in the class $Mat(\mathbb{K}_{\mathcal{L}})$ of non-deterministic matrices, and we denote it by $\Delta \models_{Mat(\mathbb{K}_{\mathcal{L}})} \alpha$, if $\Delta \models_{\mathcal{M}} \alpha$ for every $\mathcal{M} \in Mat(\mathbb{K}_{\mathcal{L}})$. In particular, α is valid in $Mat(\mathbb{K}_{\mathcal{L}})$, denoted by $\models_{Mat(\mathbb{K}_{\mathcal{L}})} \alpha$, if it is valid in every $\mathcal{M} \in Mat(\mathbb{K}_{\mathcal{L}})$.*

Theorem 4.22 (Deduction-detachment theorem) *Let $\mathcal{L} \in \mathbb{L}$. Then, for every set of formulas $\Delta \cup \{\alpha, \beta\}$: $\Delta \cup \{\alpha\} \vdash_{\mathcal{L}} \beta$ iff $\Delta \vdash_{\mathcal{L}} \alpha \to \beta$.*

Proof: Since every system in $\mathbb{L}$ is an axiomatic extension of classical propositional logic in which *Modus Ponens* is the only inference rule, the result follows easily. ■

Corollary 4.23 *Let $\mathcal{L} \in \mathbb{L}$ and let $\mathcal{M}_{\mathcal{L}}$ be the Nmatrix for $\mathcal{L}$. Then, for every $\Delta \cup \{\alpha\} \subseteq For(\Sigma')$,*

$$\Delta \vdash_{\mathcal{L}} \alpha \Leftrightarrow \ \Delta \vDash_{\mathcal{M}_{\mathcal{L}}} \alpha.$$

Proof: It follows from theorems 3.4 and 4.22, and the fact that the syntactical consequence relation $\vdash_{\mathcal{L}}$ for each $\mathcal{L}$ is finitary (since it is given by a standard Hilbert calculus). ■

Lemma 4.24 *Let $\pi_i : A^3 \to A$ be the canonical projection on the ith coordinate, namely $\pi_i((a_1, a_2, a_3)) = a_i$ for $i = 1, 2, 3$. Let $\alpha, \beta \in For(\Sigma')$ and let v be any valuation for a swap structure for $\mathcal{L} \in \mathbb{L}$ (in the sense of Definition 2.5). Then,*

i) $\pi_1(v(\alpha \to \beta)) = \pi_1(v(\alpha)) \supset \pi_1(v(\beta))$;

ii) $\pi_1(v(\neg\alpha)) = {\sim}\pi_1(v(\alpha))$.

Proof: Immediate from the definitions. ■

Theorem 4.25 (Soundness) *Let $\mathcal{L} \in \mathbb{L}$. For every $\Delta \cup \{\varphi\} \subseteq For(\Sigma')$: if $\Delta \vdash_{\mathcal{L}} \alpha$ then $\Delta \vDash_{Mat(\mathbb{K}_{\mathcal{L}})} \alpha$.*

Proof: Let v be a valuation for a swap structure $\mathcal{B}$ for $\mathcal{L}$ such that $v(\gamma) \in D_B$ for every $\gamma \in \Delta$. We will prove, by induction on the length of a given deduction of φ from Δ in $\mathcal{L}$, that $v(\varphi) \in D_B$. In order to do this, it is enough to verify that: 1) $v(\gamma) \in D_B$ for every instance of an axiom of $\mathcal{L}$; and 2) (MP) preserves trueness, that is: if $v(\gamma \to \delta) \in D_B$ and $v(\gamma) \in D_B$ then $v(\delta) \in D_B$. To simplify the proof, the following notation will be adopted: if γ is a formula and v is a valuation, we will write $|\gamma|_i$ instead of $\pi_i(v(\gamma))$, for $i = 1, 2, 3$.

Item 2) is obviously true, by the very definitions. Indeed, suppose that $v(\gamma \to \delta) \in D_B$ and $v(\gamma) \in D_B$. By definition of D_B and by Lemma 4.24, $|\gamma \to \delta|_1 = |\gamma|_1 \supset |\delta|_1 = 1$, and $|\gamma|_1 = 1$. Then $1 = |\gamma|_1 \leq |\delta|_1$ whence $|\delta|_1 = 1$ as well.[6]

In order to conclude the proof, item 1) must be verified. The task of checking the axioms will be divided into three parts:

Part 1: If α is an axiom in $\{(\mathsf{Ax1}), (\mathsf{Ax2}), (\mathsf{Ax3}), (\mathsf{K2}), (\mathsf{M1}), (\mathsf{M2}), (\mathsf{M3}), (\mathsf{M4}), (\mathsf{DN1}), (\mathsf{DN2})\}$, the proof is the same for any system $\mathcal{L}$. The proof for (Ax1), (Ax2) and (Ax3) is immediate, by definition of D_B and by Lemma 4.24. For the other axioms:

- If α is (K2) $\neg\Box\neg(\delta \to \beta) \to (\Box\delta \to \neg\Box\neg\beta)$: note that $|\neg\Box\neg(\delta \to \beta)|_1 = {\sim}|\Box\neg(\delta \to \beta)|_1 = {\sim}|\neg(\delta \to \beta)|_2 = {\sim}|(\delta \to \beta)|_3 = {\sim}(|\delta|_2 \sqcap |\beta|_3) = {\sim}|\delta|_2 \sqcup {\sim}|\beta|_3 = |\delta|_2 \supset {\sim}|\beta|_3$. And $|\Box\delta \to \neg\Box\neg\beta|_1 = |\Box\delta|_1 \supset |\neg\Box\neg\beta|_1 = |\delta|_2 \supset {\sim}|\Box\neg\beta|_1 = |\delta|_2 \supset {\sim}|\neg\beta|_2 = |\delta|_2 \supset {\sim}|\beta|_3$. In particular, $|\neg\Box\neg(\delta \to \beta)|_1 \leq |\Box\delta \to \neg\Box\neg\beta|_1$.

- If α is (M1) $\Box\neg\delta \to \Box(\delta \to \beta)$: then $|\Box\neg\delta|_1 = {\sim}|\delta|_2 = |\delta|_3$ and $|\Box(\delta \to \beta)|_1 = |\delta \to \beta|_2$. But $|\delta|_3 \leq |\delta|_3 \sqcup |\beta|_2 \leq |\delta \to \beta|_2$. Therefore, $|\Box\neg\delta|_1 \leq |\Box(\delta \to \beta)|_1$.

- If α is (M2) $\Box\beta \to \Box(\delta \to \beta)$: then $|\Box\beta|_1 = |\beta|_2$ and $|\Box(\delta \to \beta)|_1 = |\delta \to \beta|_2$. On the other hand, $|\beta|_2 \leq |\delta|_3 \sqcup |\beta|_2 \leq |\delta \to \beta|_2$. From this, $|\Box\beta|_1 \leq |\Box(\delta \to \beta)|_1$.

- If α is (M3) $\Box\neg(\delta \to \beta) \to \Box\neg\beta$: then $|\Box\neg\beta|_1 = |\neg\beta|_2 = |\beta|_3$ and $|\Box\neg(\delta \to \beta)|_1 = |\neg(\delta \to \beta)|_2 = |\delta \to \beta|_3 = |\delta|_2 \sqcap |\beta|_3$. So, $|\Box\neg(\delta \to \beta)|_1 = |\delta|_2 \sqcap |\beta|_3 \leq |\beta|_3 = |\Box\neg\beta|_1$.

- If α is (M4) $\Box\neg(\delta \to \beta) \to \Box\delta$: note that $|\Box\delta|_1 = |\delta|_2$ and $|\Box\neg(\delta \to \beta)|_1 = |\neg(\delta \to \beta)|_2 = |\delta \to \beta|_3 = |\delta|_2 \sqcap |\beta|_3$. Then, $|\Box\neg(\delta \to \beta)|_1 = |\delta|_2 \sqcap |\beta|_3 \leq |\delta|_2 = |\Box\delta|_1$.

- If α is (DN1) $\Box\delta \to \Box\neg\neg\delta$ or (DN2) $\Box\neg\neg\delta \to \Box\delta$: we have $|\Box\delta|_1 = |\delta|_2$ and $|\Box\neg\neg\delta|_1 = |\neg\neg\delta|_2 = |\neg\delta|_3 = |\delta|_2$. In particular, $|\Box\delta|_1 \leq |\Box\neg\neg\delta|_1$ and $|\Box\neg\neg\delta|_1 \leq |\Box\delta|_1$.

Part 2: If α is an axiom in $\{(\mathsf{K}), (\mathsf{K1})\}$ then the proof is the same for any system $\mathcal{L} \in \mathbb{L} \setminus \{\mathbf{Dm}\}$. So, the proof of each one will be divided into two parts.

- If α is (K) $\Box(\delta \to \beta) \to (\Box\delta \to \Box\beta)$ and if $\mathcal{L} \in \mathbb{L} \setminus \{\mathbf{Dm}\}$: we have $|\Box(\delta \to \beta)|_1 = |\delta \to \beta|_2$. So, $|\delta|_3 \sqcup |\beta|_2 \leq |\delta \to \beta|_2 \leq (|\delta|_1 \supset |\beta|_1) \sqcap (|\delta|_2 \supset |\beta|_2) \sqcap (|\beta|_3 \supset |\delta|_3)$. On the other hand, $|\Box\delta \to \Box\beta|_1 = |\Box\delta|_1 \supset |\Box\beta|_1 = |\delta|_2 \supset |\beta|_2$. So, $|\Box(\delta \to \beta)|_1 \leq (|\delta|_1 \supset |\beta|_1) \sqcap |\Box\delta \to \Box\beta|_1 \sqcap (|\beta|_3 \supset |\delta|_3)$. Therefore, $|\Box(\delta \to \beta)|_1 \leq |\Box\delta \to \Box\beta|_1$.

  If $\mathcal{L} = \mathbf{Dm}$, from $|\Box(\delta \to \beta)|_1 = |\delta \to \beta|_2$ we have $|\delta|_3 \sqcup |\beta|_2 \leq |\delta \to \beta|_2 \leq (|\delta|_2 \supset |\beta|_2) \sqcap (|\beta|_3 \supset |\delta|_3)$. On the other hand, $|\Box\delta \to \Box\beta|_1 = |\Box\delta|_1 \supset |\Box\beta|_1 = |\delta|_2 \supset |\beta|_2$. So, $|\Box(\delta \to \beta)|_1 \leq |\Box\delta \to \Box\beta|_1 \sqcap (|\beta|_3 \supset |\delta|_3)$. Therefore, $|\Box(\delta \to \beta)|_1 \leq |\Box\delta \to \Box\beta|_1$.

[6] Recall that, in any Boolean algebra, $a \supset b = 1$ iff $a \leq b$.

- If α is (K1) $\Box(\delta \to \beta) \to (\Box\neg\beta \to \Box\neg\delta)$: the proof is quite similar to the previous case.

Part 3: In this part, we will check the specific axioms of each system.

- If α is (T) $\Box\delta \to \delta$ and $\mathcal{L} \in \mathbb{L} \setminus \{\mathbf{Dm}\}$: as $|\Box\delta|_1 = |\delta|_2$ then, by definition of $\mathbb{B}_{\mathcal{A}}^{\mathbf{Tm}}$, we have $|\delta|_2 \leq |\delta|_1$. So, $|\Box\delta|_1 \leq |\delta|_1$.

- If α is (D) $\Box\delta \to \neg\Box\neg\delta$ and $\mathcal{L} = \mathbf{Dm}$: note that $|\Box\delta|_1 = |\delta|_2$ and $|\neg\Box\neg\delta|_1 = {\sim}|\Box\neg\delta|_1 = {\sim}|\neg\delta|_2 = {\sim}|\delta|_3$ and by definition of $\mathbb{B}_{\mathcal{A}}^{\mathbf{Dm}}$, we have $|\delta|_2 \sqcap |\delta|_3 = 0$. From this $|\delta|_2 \leq {\sim}|\delta|_3$. Therefore, $|\Box\delta|_1 \leq |\neg\Box\neg\delta|_1$.

- If α is (B) $\neg\Box\neg\Box\delta \to \delta$ and $\mathcal{L} = \mathbf{TBm}$: since $|\neg\Box\neg\Box\delta|_1 = {\sim}|\Box\neg\Box\delta|_1 = {\sim}|\neg\Box\delta|_2 = {\sim}|\Box\delta|_3$, such that $|\delta|_1 \sqcup |\Box\delta|_3 = 1$ (by Definition 4.9). But, then ${\sim}|\Box\delta|_3 \supset |\delta|_1 = 1$. And so, ${\sim}|\Box\delta|_3 \leq |\delta|_1$. Therefore, $|\neg\Box\neg\Box\delta|_1 \leq |\delta|_1$.

- If α is (5) $\neg\Box\neg\Box\delta \to \Box\delta$ and $\mathcal{L} = \mathbf{T45m}$: since $|\neg\Box\neg\Box\delta|_1 = {\sim}|\Box\neg\Box\delta|_1 = {\sim}|\neg\Box\delta|_2 = {\sim}|\Box\delta|_3$, such that $|\Box\delta|_1 \sqcup |\Box\delta|_3 = 1$ (by Definition 4.8). But, then ${\sim}|\Box\delta|_3 \supset |\Box\delta|_1 = 1$. And so, ${\sim}|\Box\delta|_3 \leq |\Box\delta|_1$. Therefore, $|\neg\Box\neg\Box\delta|_1 \leq |\Box\delta|_1$.

- If α is (4) $\Box\delta \to \Box\Box\delta$ and $\mathcal{L} \in \{\mathbf{T4m}, \mathbf{T45m}\}$: as $|\Box\delta|_1 = |\delta|_2$ such that (by Definitions 4.7 and 4.8) $|\delta|_2 \leq |\Box\delta|_2$ and $|\Box\Box\delta|_1 = |\Box\delta|_2$. So, $|\Box\delta|_1 \leq |\Box\Box\delta|_1$.

■

Definition 4.26 *Let $\mathcal{L} \in \mathbb{L}$ and let Δ be a theory in $\mathcal{L}$. The relation $\equiv_\Delta^{\mathcal{L}}$ between formulas of $\mathcal{L}$ is defined as follows:*

$$\alpha \equiv_\Delta^{\mathcal{L}} \beta \textit{ iff } \Delta \vdash_{\mathcal{L}} \alpha \to \beta \textit{ and } \Delta \vdash_{\mathcal{L}} \beta \to \alpha.$$

Proposition 4.27 *Let $\mathcal{L} \in \mathbb{L}$. Then, the relation $\equiv_\Delta^{\mathcal{L}}$ is a congruence w.r.t. connectives in $\Sigma'' = \{\to, \neg\}$.*

Proof: It follows from the fact that $\mathcal{L}$ contains classical logic over the signature Σ''. ■

Notation 5 If $\mathcal{L} \in \mathbb{L}$ and $\equiv_\Delta^{\mathcal{L}}$ is the congruence defined above, then $[\alpha]_\Delta = \alpha/_{\equiv_\Delta^{\mathcal{L}}} = \{\beta \in For(\Sigma') : \alpha \equiv_\Delta^{\mathcal{L}} \beta\}$ is the equivalence class of $\alpha \in For(\Sigma')$, while $For(\Sigma')/_{\equiv_\Delta^{\mathcal{L}}} = \{\alpha/_{\equiv_\Delta^{\mathcal{L}}} : \alpha \in For(\Sigma')\}$ denotes the set of all the equivalence classes.

By Proposition 4.27, $\equiv_\Delta^{\mathcal{L}}$ is a congruence w.r.t. the operations in $\Sigma'' = \{\to, \neg\}$. However, the connective $\Box$ is not congruential:

Proposition 4.28 *Let $\mathcal{L} \in \mathbb{L}$. Then, the connective $\Box$ is not congruential, that is: there are formulas α and β such that $\alpha \equiv_\Delta^{\mathcal{L}} \beta$ but $\Box\alpha \not\equiv_\Delta^{\mathcal{L}} \Box\beta$.*

Proof: The proof will be divided into two cases, the first for $\mathcal{L} \in \mathbb{L} \setminus \{\mathbf{Dm}\}$ and the second for $\mathcal{L} = \mathbf{Dm}$.

So, if $\mathcal{L} \in \mathbb{L} \setminus \{\mathbf{Dm}\}$, then we have $p \equiv_\Delta \neg p \to p$ but $\Box p \not\equiv_\Delta \Box(\neg p \to p)$, because there exist a valuation $v : For(\Sigma') \to \{T, t, f, F\}$ for $\mathcal{L}$, such that:

- If $\mathcal{L} \in \{\mathbf{Tm}, \mathbf{TBm}\}$, then $v(p) = t$ and for $v(\neg p \to p) = T$, we have $v(\Box p) \in \{F, f\}$, but $v(\Box(\neg p \to p)) \in \{T, t\}$;
- If $\mathcal{L} = \mathbf{T4m}$, then $v(p) = t$ and for $v(\neg p \to p) = T$, we have $v(\Box p) \in \{F, f\}$, but $v(\Box(\neg p \to p)) \in \{T\}$;
- If $\mathcal{L} = \mathbf{T45m}$, then $v(p) = t$ and for $v(\neg p \to p) = T$, we have $v(\Box p) \in \{F\}$, but $v(\Box(\neg p \to p)) \in \{T\}$.

If $\mathcal{L} = \mathbf{Dm}$, we have once again that $p \equiv_\Delta \neg p \to p$ but, $\Box p \not\equiv_\Delta \Box(\neg p \to p)$. Indeed, there exist a valuation $v : For(\Sigma') \to \{T^+, T^-, C^+, C^-, F^+, F^-\}$ for $\mathbf{Dm}$ such that if $v(p) = C^+$ and for $v(\neg p \to p) = T^+$, we have $v(\Box p) \in \{T^-, F^-, C^-\}$, but $v(\Box(\neg p \to p)) \in \{T^+, F^+, C^+\}$. ■

By means of this example it follows that the connective $\Box$ does not induce a well-defined operator on $For(\Sigma')/_{\equiv^{\mathcal{L}}_\Delta}$. As Carnielli and Coniglio did in [6, Chapter 6], in order to circumvent a similar problem with the operators $\circ$ and $\neg$ in the context of **LFI**s, the connective $\Box$ will be interpreted as a multioperation instead as an algebraic operator, by defining a swap structure over the Boolean algebra generated by the Lindenbaum-Tarski equivalence relation $\equiv^{\mathcal{L}}_\Delta$ (see Definition 4.31).

Proposition 4.29 *Let $\mathcal{L} \in \mathbb{L}$ and let Δ be a theory in $\mathcal{L}$. Let*

$$\mathcal{A}^{\mathcal{L}}_\Delta = \langle For(\Sigma')/_{\equiv^{\mathcal{L}}_\Delta}, \supset, \sqcup, \sqcap, {\sim}, 0_\Delta, 1_\Delta \rangle$$

be the structure given by

$$\begin{array}{lcl}
[\alpha]_\Delta \supset [\beta]_\Delta & = & [\alpha \to \beta]_\Delta \\
[\alpha]_\Delta \sqcup [\beta]_\Delta & = & [\alpha \vee \beta]_\Delta \\
[\alpha]_\Delta \sqcap [\beta]_\Delta & = & [\alpha \wedge \beta]_\Delta \\
{\sim}[\alpha]_\Delta & = & [\neg\alpha]_\Delta \\
0_\Delta & = & [\neg(\alpha \to \alpha)]_\Delta \\
1_\Delta & = & [\alpha \to \alpha]_\Delta.
\end{array}$$

Then, $\mathcal{A}^{\mathcal{L}}_\Delta$ is a Boolean algebra.

Proof: The operations and the constants are well defined, by Proposition 4.27. Since $\mathcal{L}$ contains classical logic, the rest of the proof is straightforward. ■

Remark 4.30 *It is worth noting that, if Δ is the trivial theory $For(\Sigma')$, then $\mathcal{A}^{\mathcal{L}}_\Delta$ is the trivial Boolean algebra with a single element.*

Definition 4.31 *Let $\mathcal{L} \in \mathbb{L}$ and let Δ be a theory in $\mathcal{L}$. Given the Boolean algebra $\mathcal{A}^{\mathcal{L}}_\Delta$, the unique swap structure over $\mathcal{A}^{\mathcal{L}}_\Delta$ with domain $\mathbb{B}^{\mathcal{L}}_{\mathcal{A}^{\mathcal{L}}_\Delta}$ is called* the Lindenbaum-Tarski swap structure for Δ in $\mathcal{L}$, *and it will denoted by $\mathcal{B}^{\mathcal{L}}_\Delta$.*

Definition 4.32 (Canonical Nmatrix) *Let $\mathcal{L} \in \mathbb{L}$ and let Δ be a theory in $\mathcal{L}$. Let $\mathcal{B}^{\mathcal{L}}_\Delta$ be the Lindenbaum-Tarski swap structure for Δ in $\mathcal{L}$. The* canonical Nmatrix *associated to Δ in $\mathcal{L}$ is $\mathcal{M}(\mathcal{B}^{\mathcal{L}}_\Delta)$, which will be denoted by $\mathcal{M}^{\mathcal{L}}_\Delta = \langle \mathcal{B}^{\mathcal{L}}_\Delta, D^{\mathcal{L}}_\Delta \rangle$.*

Lemma 4.33 *Let $\mathcal{L} \in \mathbb{L}$ and let $\mathcal{A}^{\mathcal{L}}_{\Delta}$ be as in Proposition 4.29. Then $[\alpha]_\Delta = 1_\Delta$ iff $\Delta \vdash_{\mathcal{L}} \alpha$.*

Proof: It is immediate. ■

Lemma 4.34 *Let $\mathcal{L} \in \mathbb{L}$ and let $\mathcal{A}^{\mathcal{L}}_{\Delta}$ be as in Proposition 4.29. Then, the following conditions are equivalent:*

(i) $[\alpha]_\Delta \leq [\beta]_\Delta$

(ii) $[\alpha]_\Delta \supset [\beta]_\Delta = 1_\Delta$

(iii) $[\alpha \to \beta]_\Delta = 1_\Delta$

(iv) $\Delta \vdash_{\mathcal{L}} \alpha \to \beta$

Proof: It is immediate from the definitions and from the properties of Boolean algebras. ■

Lemma 4.35 *Let $\mathcal{L} \in \mathbb{L}$ and let $\mathcal{A}^{\mathcal{L}}_{\Delta}$ be as in Proposition 4.29. The following holds in the indicated Boolean algebras:*

(i) $[\Box\neg\neg\alpha]_\Delta = [\Box\alpha]_\Delta$ *in* $\mathcal{A}^{\mathcal{L}}_{\Delta}$, *for* $\mathcal{L} \in \mathbb{L}$;

(ii) $[\Box\alpha]_\Delta \leq [\Box\Box\alpha]_\Delta$ *in* $\mathcal{A}^{\mathcal{L}}_{\Delta}$, *for* $\mathcal{L} \in \{\boldsymbol{T}4\boldsymbol{m},\ \boldsymbol{T}45\boldsymbol{m}\}$;

(iii) $[\Box\neg\Box\alpha]_\Delta \sqcup [\Box\alpha]_\Delta = 1_\Delta$ *in* $\mathcal{A}^{\boldsymbol{T45m}}_{\Delta}$;

(iv) $[\alpha]_\Delta \sqcup [\Box\neg\Box\alpha]_\Delta = 1_\Delta$*in* $\mathcal{A}^{\boldsymbol{TBm}}_{\Delta}$;

(v) $[\Box\neg(\alpha \to \beta)]_\Delta = [\Box\alpha]_\Delta \sqcap [\Box\neg\beta]_\Delta$ *in* $\mathcal{A}^{\mathcal{L}}_{\Delta}$, *for* $\mathcal{L} \in \mathbb{L}$;

(vi) $[\Box\neg\alpha]_\Delta \sqcup [\Box\beta]_\Delta \leq [\Box(\alpha \to \beta)]_\Delta$ *in* $\mathcal{A}^{\mathcal{L}}_{\Delta}$, *for* $\mathcal{L} \in \mathbb{L}$;

(vii) $[\Box(\alpha \to \beta)]_\Delta \leq [\alpha]_\Delta \supset [\beta]_\Delta$ *in* $\mathcal{A}^{\mathcal{L}}_{\Delta}$, *for* $\mathcal{L} \in \mathbb{L} \setminus \{\boldsymbol{Dm}\}$;

(viii) $[\Box(\alpha \to \beta)]_\Delta \leq [\Box\alpha]_\Delta \supset [\Box\beta]_\Delta$ *in* $\mathcal{A}^{\mathcal{L}}_{\Delta}$, *for* $\mathcal{L} \in \mathbb{L}$;

(ix) $[\Box(\alpha \to \beta)]_\Delta \leq [\Box\neg\beta]_\Delta \supset [\Box\neg\alpha]_\Delta$ *in* $\mathcal{A}^{\mathcal{L}}_{\Delta}$, *for* $\mathcal{L} \in \mathbb{L}$.

Proof:

(i) $[\Box\neg\neg\alpha]_\Delta = [\Box\alpha]_\Delta$ iff $\Delta \vdash_{\mathcal{L}} \Box\neg\neg\alpha \to \Box\alpha$ and $\Delta \vdash_{\mathcal{L}} \Box\alpha \to \Box\neg\neg\alpha$. But the latter holds by axioms (DN1) and (DN2).

(ii) $[\Box\alpha]_\Delta \leq [\Box\Box\alpha]_\Delta$ iff $\Delta \vdash_{\mathcal{L}} \Box\alpha \to \Box\Box\alpha$. But this holds by Axiom (4).

(iii) $[\Box\neg\Box\alpha]_\Delta \sqcup [\Box\alpha]_\Delta = 1_\Delta$ iff $[\Box\neg\Box\alpha \vee \Box\alpha]_\Delta = 1_\Delta$ iff $[\neg\Box\neg\Box\alpha \to \Box\alpha]_\Delta = 1_\Delta$ iff $\Delta \vdash_{\mathbf{T45m}} \neg\Box\neg\Box\alpha \to \Box\alpha$. But this holds by Axiom (5).

(iv) $[\alpha]_\Delta \sqcup [\Box\neg\Box\alpha]_\Delta = 1_\Delta$ iff $[\alpha \vee \Box\neg\Box\alpha]_\Delta = 1_\Delta$ iff $[\neg\Box\neg\Box\alpha \to \alpha]_\Delta = 1_\Delta$ iff $\Delta \vdash_{\mathbf{TBm}} \neg\Box\neg\Box\alpha \to \alpha$. But this holds by Axiom (B).

(v) $[\Box\neg(\alpha \to \beta)]_\Delta = [\Box\alpha]_\Delta \sqcap [\Box\neg\beta]_\Delta$ iff $[\Box\neg(\alpha \to \beta)]_\Delta = [\Box\alpha \wedge \Box\neg\beta]_\Delta$ iff $\Delta \vdash_{\mathcal{L}} \Box\neg(\alpha \to \beta) \to (\Box\alpha \wedge \Box\neg\beta)$ and $\Delta \vdash_{\mathcal{L}} (\Box\alpha \wedge \Box\neg\beta) \to \Box\neg(\alpha \to \beta)$. See the derivation below:

1. $\Delta \vdash_{\mathcal{L}} \neg\Box\neg(\alpha \to \beta) \to (\Box\alpha \to \neg\Box\neg\beta)$	(K2)
2. $\Delta \vdash_{\mathcal{L}} \neg(\Box\alpha \to \neg\Box\neg\beta) \to \neg\neg\Box\neg(\alpha \to \beta)$	1, CPC
3. $\Delta \vdash_{\mathcal{L}} \neg(\neg\Box\alpha \vee \neg\Box\neg\beta) \to \Box\neg(\alpha \to \beta)$	2, CPC
4. $\Delta \vdash_{\mathcal{L}} (\neg\neg\Box\alpha \wedge \neg\neg\Box\neg\beta) \to \Box\neg(\alpha \to \beta)$	3, CPC
5. $\Delta \vdash_{\mathcal{L}} (\Box\alpha \wedge \Box\neg\beta) \to \Box\neg(\alpha \to \beta)$	4, CPC

Now, consider the following derivation:

1. $\Delta \vdash_{\mathcal{L}} \Box\neg(\alpha \to \beta) \to \Box\neg\beta$	(M3)
2. $\Delta \vdash_{\mathcal{L}} \Box\neg(\alpha \to \beta) \to \Box\alpha$	(M4)
3. $\Delta \vdash_{\mathcal{L}} \Box\neg(\alpha \to \beta) \to (\Box\alpha \wedge \Box\neg\beta)$	1,2, CPC

(vi) $[\Box\neg\alpha]_\Delta \sqcup [\Box\beta]_\Delta \leq [\Box(\alpha \to \beta)]_\Delta$ iff $[\Box\neg\alpha \vee \Box\beta]_\Delta \leq [\Box(\alpha \to \beta)]_\Delta$ iff $\Delta \vdash_{\mathcal{L}} (\Box\neg\alpha \vee \Box\beta) \to \Box(\alpha \to \beta)$. See the derivation below:

1.	$\Delta \vdash_{\mathcal{L}} \Box\neg\alpha \to \Box(\alpha \to \beta)$	(M1)
2.	$\Delta \vdash_{\mathcal{L}} \Box\beta \to \Box(\alpha \to \beta)$	(M2)
3.	$\Delta \vdash_{\mathcal{L}} (\Box\neg\alpha \vee \Box\beta) \to \Box(\alpha \to \beta)$	1,2, CPC

(vii) $[\Box(\alpha \to \beta)]_\Delta \leq [\alpha]_\Delta \supset [\beta]_\Delta$ iff $[\Box(\alpha \to \beta)]_\Delta \leq [\alpha \to \beta]_\Delta$ iff $\Delta \vdash_{\mathcal{L}} \Box(\alpha \to \beta) \to (\alpha \to \beta)$. But this is a consequence of Axiom (T).

(viii) $[\Box(\alpha \to \beta)]_\Delta \leq [\Box\alpha]_\Delta \supset [\Box\beta]_\Delta$ iff $[\Box(\alpha \to \beta)]_\Delta \leq [\Box\alpha \to \Box\beta]_\Delta$ iff $\Delta \vdash_{\mathcal{L}} \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$. But this follows from Axiom (K).

(ix) $[\Box(\alpha \to \beta)]_\Delta \leq [\Box\neg\beta]_\Delta \supset [\Box\neg\alpha]_\Delta$ iff $[\Box(\alpha \to \beta)]_\Delta \leq [\Box\neg\beta \to \Box\neg\alpha]_\Delta$ iff $\Delta \vdash_{\mathcal{L}} \Box(\alpha \to \beta) \to (\Box\neg\beta \to \Box\neg\alpha)$. But this is a consequence of Axiom (K1).

$\blacksquare$

Proposition 4.36 *Let $\mathcal{L} \in \mathbb{L}$. Then, the function $v^{\mathcal{L}}_\Delta : For(\Sigma') \to \mathbb{B}^{\mathcal{L}}_{\mathcal{A}^{\mathcal{L}}_\Delta}$ given by $v^{\mathcal{L}}_\Delta(\alpha) = ([\alpha]_\Delta, [\Box\alpha]_\Delta, [\Box\neg\alpha]_\Delta)$ is a valuation in $\mathcal{M}^{\mathcal{L}}_\Delta$. That is, for every $\alpha, \beta \in For(\Sigma')$:*

(i) $v^{\mathcal{L}}_\Delta(\neg\alpha) \in \tilde{\neg} v^{\mathcal{L}}_\Delta(\alpha)$;

(ii) $v^{\mathcal{L}}_\Delta(\Box\alpha) \in \tilde{\Box} v^{\mathcal{L}}_\Delta(\alpha)$;

(iii) $v^{\mathcal{L}}_\Delta(\alpha \to \beta) \in v^{\mathcal{L}}_\Delta(\alpha) \tilde{\to} v^{\mathcal{L}}_\Delta(\beta)$.

Proof:

(i) $v^{\mathcal{L}}_\Delta(\neg\alpha) = ([\neg\alpha]_\Delta, [\Box\neg\alpha]_\Delta, [\Box\neg\neg\alpha]_\Delta)$. If $\mathcal{L} \in \mathbb{L}$ then, by Proposition 4.27,
$$\begin{aligned} \tilde{\neg} v^{\mathcal{L}}_\Delta(\alpha) &= \tilde{\neg}([\alpha]_\Delta, [\Box\alpha]_\Delta, [\Box\neg\alpha]_\Delta) \\ &= \{([\neg\alpha]_\Delta, [\Box\neg\alpha]_\Delta, [\Box\alpha]_\Delta)\} \end{aligned}$$
and by Lemma 4.35 (i), $[\Box\neg\neg\alpha]_\Delta = [\Box\alpha]_\Delta$. This implies that $v^{\mathcal{L}}_\Delta(\neg\alpha) \in \tilde{\neg} v^{\mathcal{L}}_\Delta(\alpha)$;

(ii) $v^{\mathcal{L}}_{\Delta}(\Box\alpha) = ([\Box\alpha]_\Delta, [\Box\Box\alpha]_\Delta, [\Box\neg\Box\alpha]_\Delta)$. If $\mathcal{L}$ belongs to $\{\mathbf{Tm}, \mathbf{Dm}\}$,

$$\begin{array}{rcl} \tilde{\Box}v^{\mathcal{L}}_{\Delta}(\alpha) & = & \tilde{\Box}([\alpha]_\Delta, [\Box\alpha]_\Delta, [\Box\neg\alpha]_\Delta) \\ & = & \{([\beta_1]_\Delta, [\beta_2]_\Delta, [\beta_3]_\Delta) \in \mathbb{B}^{\mathcal{L}}_{\Delta} : \\ & & [\beta_1]_\Delta = [\Box\alpha]_\Delta\}. \end{array}$$

If $\mathcal{L} = \mathbf{T4m}$, $\tilde{\Box}v^{\mathcal{L}}_{\Delta}(\alpha) = \tilde{\Box}([\alpha]_\Delta, [\Box\alpha]_\Delta, [\Box\neg\alpha]_\Delta) = \{([\beta_1]_\Delta, [\beta_2]_\Delta, [\beta_3]_\Delta) \in \mathbb{B}^{\mathcal{L}}_{\Delta} : [\beta_1]_\Delta = [\Box\alpha]_\Delta \text{ and } [\Box\alpha]_\Delta \leq [\beta_2]_\Delta\}$ and by Lemma 4.35 (ii) $[\Box\alpha]_\Delta \leq [\Box\Box\alpha]_\Delta$. If $\mathcal{L} = \mathbf{T45m}$,

$$\begin{array}{rcl} \tilde{\Box}v^{\mathcal{L}}_{\Delta}(\alpha) & = & \tilde{\Box}([\alpha]_\Delta, [\Box\alpha]_\Delta, [\Box\neg\alpha]_\Delta) \\ & = & \big\{([\beta_1]_\Delta, [\beta_2]_\Delta, [\beta_3]_\Delta) \in \mathbb{B}^{\mathcal{L}}_{\Delta} \; : \\ & & [\beta_1]_\Delta = [\Box\alpha]_\Delta, \\ & & [\Box\alpha]_\Delta \leq [\beta_2]_\Delta \;\text{ and } [\beta_3]_\Delta \sqcup [\beta_1]_\Delta = 1_\Delta\big\} \end{array}$$

and by Lemma 4.35 (ii) and (iii) $[\Box\alpha]_\Delta \leq [\Box\Box\alpha]_\Delta$ and $[\Box\neg\Box\alpha]_\Delta \sqcup [\Box\alpha]_\Delta = 1_\Delta$. If $\mathcal{L} = \mathbf{TBm}$, $\tilde{\Box}v^{\mathcal{L}}_{\Delta}(\alpha) = \tilde{\Box}([\alpha]_\Delta, [\Box\alpha]_\Delta, [\Box\neg\alpha]_\Delta) = \{([\beta_1]_\Delta, [\beta_2]_\Delta, [\beta_3]_\Delta) \in \mathbb{B}^{\mathcal{L}}_{\Delta} : [\beta_1]_\Delta = [\Box\alpha]_\Delta \text{ and } [\alpha]_\Delta \sqcup [\beta_3]_\Delta = 1_\Delta\}$ and by Lemma 4.35 (iv) $[\alpha]_\Delta \sqcup [\Box\neg\Box\alpha]_\Delta = 1_\Delta$.

(iii) $v^{\mathcal{L}}_{\Delta}(\alpha \to \beta) = ([\alpha \to \beta]_\Delta, [\Box(\alpha \to \beta)]_\Delta, [\Box\neg(\alpha \to \beta)]_\Delta)$.

If $\mathcal{L} \in \mathbb{L} \setminus \{\mathbf{Dm}\}$,

$$\begin{array}{l} v^{\mathcal{L}}_{\Delta}(\alpha) \tilde{\to} v^{\mathcal{L}}_{\Delta}(\beta) = \\ \quad ([\alpha]_\Delta, [\Box\alpha]_\Delta, [\Box\neg\alpha]_\Delta) \tilde{\to} ([\beta]_\Delta, [\Box\beta]_\Delta, [\Box\neg\beta]_\Delta) = \\ \quad \big\{([\gamma_1]_\Delta, [\gamma_2]_\Delta, [\gamma_3]_\Delta) \in \mathbb{B}^{\mathcal{L}}_{\mathcal{A}_\Delta} : [\gamma_1]_\Delta = [\alpha]_\Delta \supset [\beta]_\Delta, \\ \quad [\gamma_3]_\Delta = [\Box\alpha]_\Delta \sqcap [\Box\neg\beta]_\Delta \text{ and} \\ \quad [\Box\neg\alpha]_\Delta \sqcup [\Box\beta]_\Delta \leq [\gamma_2]_\Delta \leq \\ \quad ([\alpha]_\Delta \supset [\beta]_\Delta) \sqcap ([\Box\alpha]_\Delta \supset [\Box\beta]_\Delta) \sqcap \\ \quad ([\Box\neg\beta]_\Delta \supset [\Box\neg\alpha]_\Delta)\big\}. \end{array}$$

By Lemma 4.35 (v) and (vi), we have $[\Box\neg(\alpha \to \beta)]_\Delta = [\Box\alpha]_\Delta \sqcap [\Box\neg\beta]_\Delta$ and $[\Box\neg\alpha]_\Delta \sqcup [\Box\beta]_\Delta \leq [\Box(\alpha \to \beta)]_\Delta$. And by Lemma 4.35 (vii), (viii) and (ix), we have $[\Box(\alpha \to \beta)]_\Delta \leq ([\alpha]_\Delta \supset [\beta]_\Delta) \sqcap ([\Box\alpha]_\Delta \supset [\Box\beta]_\Delta) \sqcap ([\Box\neg\beta]_\Delta \supset [\Box\neg\alpha]_\Delta)$.

If $\mathcal{L} = \mathbf{Dm}$, the proof is quite similar to the previous cases.

■

Proposition 4.37 *Let $\mathcal{L} \in \mathbb{L}$. Then, the function $v^{\mathcal{L}}_{\Delta} : For(\Sigma') \to \mathbb{B}^{\mathcal{L}}_{\mathcal{A}^{\mathcal{L}}_{\Delta}}$ given by $v^{\mathcal{L}}_{\Delta}(\alpha) = ([\alpha]_\Delta, [\Box\alpha]_\Delta, [\Box\neg\alpha]_\Delta)$ is a canonical valuation in $\mathcal{M}^{\mathcal{L}}_{\Delta}$, that is: $v^{\mathcal{L}}_{\Delta}(\alpha) \in D^{\mathcal{L}}_{\Delta}$ iff $\Delta \vdash_{\mathcal{L}} \alpha$.*

Proof: By Proposition 4.36, $v^{\mathcal{L}}_{\Delta}(\alpha)$ is a valuation in $\mathcal{M}^{\mathcal{L}}_{\Delta}$. On the other hand, by the very definitions, $v^{\mathcal{L}}_{\Delta}(\alpha) \in D^{\mathcal{L}}_{\Delta}$ iff $[\alpha]_\Delta = 1_\Delta$ iff $\Delta \vdash_{\mathcal{L}} \alpha$, by Lemma 4.33. ■

Definition 4.38 (Canonical model) *Let $\mathcal{L} \in \mathbb{L}$ and let Δ be a theory in $\mathcal{L}$. The function $v^{\mathcal{L}}_{\Delta}$ is called the* canonical valuation *associated to Δ in $\mathcal{L}$. The pair $(\mathcal{M}^{\mathcal{L}}_{\Delta}, v^{\mathcal{L}}_{\Delta})$ is called the* canonical model *associated to Δ in $\mathcal{L}$.*

The canonical model allows to prove the completeness of $\mathcal{L} \in \mathbb{L}$ w.r.t. swap structures in a straightforward way:

Theorem 4.39 (Completeness) *For every $\mathcal{L} \in \mathbb{L}$ and for every $\Delta \cup \{\alpha\} \subseteq For(\Sigma')$,*

$$\textit{if } \; \Delta \models_{Mat(\mathbb{K}_{\mathcal{L}})} \alpha \;\; \textit{then } \; \Delta \vdash_{\mathcal{L}} \alpha.$$

Proof: Suppose that $\Delta \nvdash_{\mathcal{L}} \alpha$. Then, by Proposition 4.37, there exist a valuation $v^{\mathcal{L}}_{\Delta}$ over the canonical Nmatrix $\mathcal{M}^{\mathcal{L}}_{\Delta}$ for $\mathcal{L}$ such that $v^{\mathcal{L}}_{\Delta}(\beta) \in D^{\mathcal{L}}_{\Delta}$ for every $\beta \in \Delta$, but $v^{\mathcal{L}}_{\Delta}(\alpha) \notin D^{\mathcal{L}}_{\Delta}$. From this, $\Delta \not\models_{Mat(\mathbb{K}_{\mathcal{L}})} \alpha$. ■

Corollary 4.40 *Let $\mathcal{L} \in \mathbb{L}$ and let $\Delta \cup \{\alpha\} \subseteq For(\Sigma')$. Then, are equivalent:*

1. $\Delta \vdash_{\mathcal{L}} \alpha$;
2. $\Delta \models_{Mat(\mathbb{K}_{\mathcal{L}})} \alpha$;
3. $\Delta \models_{\mathcal{M}^{\mathcal{L}}_{\Delta}} \alpha$;
4. $\Delta \models_{\mathcal{M}(\mathcal{B}^{\mathcal{L}}_{\mathcal{A}_2})} \alpha$.

Proof:

(1) $\Leftrightarrow$ (2) By Theorems 4.25 and 4.39;

(1) $\Leftrightarrow$ (3) By Theorem 4.25 and the proof of Theorem 4.39;

(1) $\Leftrightarrow$ (4) By Corollary 4.23 and by remarks 4.6, 4.11 and 4.17.

■

Corollary 4.41 *For $\mathcal{L} \in \mathbb{L}$ and for every $\alpha \in For(\Sigma')$, are equivalent:*

1. $\vdash_{\mathcal{L}} \alpha$;
2. $\models_{Mat(\mathbb{K}_{\mathcal{L}})} \alpha$;
3. $\models_{\mathcal{M}^{\mathcal{L}}_{\Delta}} \alpha$;
4. $\models_{\mathcal{M}(\mathcal{B}^{\mathcal{L}}_{\mathcal{A}_2})} \alpha$.

The results stated in Corollary 4.40 show in a precise way that the class of Nmatrices associated to the swap structures for each modal logic $\mathcal{L} \in \mathbb{L}$ generalizes the original Nmatrix semantics presented in [7] and [8]. Moreover, it also provides an interesting family of examples, besides the ones obtained for **LFI**s in [10], that the usual Linbenbaum-Tarski construction can be generalized to non-algebraizable logics, by considering multialgebras instead of ordinary algebras.

5 Concluding remarks and future research

Non-deterministic semantic structures such as Nmatrices, swap structures and Fidel structures (see [6, Chapter 6]) can be considered as complementary to the algebraic structures associated to logics by means of the standard tools of abstract algebraic logic. Indeed, non-deterministic semantics can be applied to logics without non-trivial congruences, obtaining so algebraic-like interpretation for logics which lie outside the scope of the traditional approach to algebraic logic. It is worth observing that, as it was shown in [10], swap structures semantics for a given logic coincides with twist structures semantics, when the logic is Blok-Pigozzi algebraizable. This justifies the study of this kind of structures from the standpoint of universal algebra and category theory, thus broadening the horizons of abstract algebraic logic.

In this paper, new classes of swap structures were introduced as a suitable semantics for a family of non-normal modal systems. These classes of non-deterministic structures generalize the original semantical characterization of each of such logics by means of a single finite-valued non-deterministic matrix. Indeed, each finite characteristic Nmatrix is recovered from the respective class of models by considering the full swap structure over the two-element Boolean algebra. An analogous situation was observed in [6, Chapter 6] with respect to **LFI**s. Thus being so, the results obtained in the present paper suggest that a Birkhoff-like theorem for $\mathbb{K}_{\mathcal{L}}$ could be obtained for each $\mathcal{L} \in \mathbb{L}$, similar to the ones given in [10] for some **LFI**s. That is, to determine whether the finite-valued characteristic matrix $\mathcal{M}_{\mathcal{L}}$ generates the whole class $\mathcal{M}(\mathbb{K}_{\mathcal{L}})$ is an interesting question to be analyzed in future research.

In the present paper, just the deontic system **Dm** was analyzed under the perspective of swap structures. The extension of such structures to its axiomatic extensions **D4m**, **D45m** and **DBm** proposed in [7] and [8] should be obtained in a straightforward way. The 8-valued modal systems **Km**, **K4m** and **K45m** proposed in [9] can also be studied from the viewpoint of modal swap structures.

It should be observed that, in principle, the nature of the snapshots associated to a given logic does not seem to be *intrinsic* to that logic. In the cases studied in the present paper, the snapshots are conceived as triples (a_1, a_2, a_3) representing the truth values (over a given Boolean algebra $\mathcal{A}$) of given formulas α, $\Box\alpha$ and $\Box\neg\alpha$, respectively. But this choice seems to be rather arbitrary: it would be possible to change the interpretation of a_3 by $\neg\Box\neg\alpha$, that is, $\Diamond\alpha$. It would produce a finite-valued characteristic Nmatrix $\mathcal{M}'_{\mathcal{L}}$ different to $\mathcal{M}_{\mathcal{L}}$. It is an interesting topic of future research to relate the different classes of swap structures for a given logic which arise by modifying the nature of the corresponding snapshots.

The structures presented in this paper constitute the first examples of swap structures defined for logics outside the scope of the **LFI**s, the class of paraconsistent logics for which swap structures were originally proposed. Based on the ideas of the completeness proofs of **LFI**s with respect to Fidel structures semantics found in [6, Chapter 6], a quotient swap structure that we called *Lindenbaum-Tarski swap structure*, as well as a canonical valuation over it, were proposed. A similar technique was already used in [10] in the context of swap structures for **LFI**s. It is interesting to observe that these results show a way in which the standard Lindenbaum-Tarski construction can be generalized to non-congruential logics. The key is considering, for the non-congruential connectives, multioperations (or relations, in the case of Fidel structures) instead of operations. We conjecture that this generalized Lindenbaum-Tarski technique could be applied to a wide class of non-algebraizable logics, offering an interesting new paradigm for algebraizing logics by means of multialgebras and Fidel structures.

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