THE JOURNAL OF SYMBOLIC LOGIC
Volume 37, Number 4, Dec. 1972

# COMPLETENESS OF AN ANCIENT LOGIC

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In previous articles ([4], [5]) it has been shown that the deductive system developed by Aristotle in his "second logic" (cf. Bochenski [2, p. 43]) is a natural deduction system and not an axiomatic system as previously had been thought [6]. It was also pointed out that Aristotle's logic is self-sufficient in two senses: First, that it presupposed no other logical concepts, not even those of propositional logic; second, that it is (strongly) complete in the sense that every valid argument formable in the language of the system is demonstrable by means of a formal deduction in the system. Review of the system makes the first point obvious. The purpose of the present article is to prove the second. Strong completeness is demonstrated for the Aristotelian system.

**§1. The language.** The logic in question was developed by Aristotle as an underlying logic (in the sense of Church [3, p. 317]) for an axiomatized science. Because the question of whether Aristotle recognized the possibility of a science with an infinite vocabulary of *nonlogical constants* is irrelevant to present concerns, we simply assume a set $V$ containing at least two characters to play the role of the vocabulary of "categorical terms." For *logical constants* we take four characters, $A$, $N$, $S$, and $\$$ (not in $V$). The *language* L contains all strings consisting in a logical constant followed by *two distinct* nonlogical constants in $V$. Members of L are called *sentences*. If $x$ and $y$ are in $V$ and $X$ and $Y$ are "corresponding" categorical terms (e.g., man, animal) the following are heuristic correspondence results: $Axy$ (*A*ll $X$'s are $Y$'s), $Nxy$ (*N*o $X$'s are $Y$'s), $Sxy$ (*S*ome $X$'s are $Y$'s), $\$xy$ (*S*ome $X$'s are *not* $Y$'s).

The fact that each sentence contains two distinct nonlogical constants reflects a systematic avoidance by Aristotle of "sentences" such as $Axx$ and $Sxx$. Extension of the language to accommodate such sentences would have rather trivial mathematical consequences but would entail rather more deviation from the Aristotelian text than the present framework requires.

In terms of the grammar of L we make two further definitions which make contact with traditional terminology and which are useful below. An *argument* is an ordered pair $(P, d)$ where $P$ is a set of sentences (called the premises) and $d$ is a single sentence (called the conclusion). $Axy$ and $Nxy$ are defined to be *contradictories of* $\$xy$ and $Sxy$ respectively (and vice versa) and $C(d)$ indicates the contradictory of $d$.

**§2. The semantics.** The semantic system S is defined as follows. An *interpretation* $i$ of L is a function defined on $V$ and having as values nonempty sets (cf.

Received January 7, 1972.

§5 below). In order to characterize the association of truth-values with sentences under an interpretation $i$ we extend the domain of definition of $i$ to include all of **L** so that sentences in **L** get their expected truth values. Explicitly, $iAxy = t$ if $ix$ is included in $iy$ and $iAxy = f$ otherwise; $iNxy = t$ if $ix$ is disjoint with $iy$ and $iNxy = f$ otherwise; similarly for $iSxy$ and $i\$xy$.

As usual, if $id = t$ for some sentence $d$ then $i$ is said to be a *true interpretation of* $d$ and if $i$ is a true interpretation of every sentence in a set $P$ then $i$ is a *true interpretation of* $P$. The term "false interpretation" is not used. A sentence $d$ is said to be a *logical consequence* of a set $P$ of sentences if every true interpretation of $P$ is a true interpretation of $d$. To indicate that this relation holds we write $P \models d$. It is also convenient to make further contact with "traditional" terminology by defining an *argument* $(P, d)$ to be *valid* when $P \models d$, otherwise *invalid*. $P + Q$ is the union of $P$ and $Q$ and we always drop the brackets in the notation for unit sets.

The following obvious facts concerning the semantics will play a role in developments below.

2.1. *Semantic principles.* Let $x$, $y$ and $z$ be different members of $V$. Let $P$ be a set of sentences and let $d$ and $e$ be sentences.

*Law of Contradiction.* For all $i$, $id \neq iC(d)$.

*Conversion Laws.* (C1) $Nxy \models Nyx$, (C2) $Axy \models Syx$, (C3) $Sxy \models Syx$.

*Laws of Perfect Syllogisms.* (PS1) $Azy + Axz \models Axy$, (PS2) $Nzy + Axz \models Nxy$, (PS3) $Azy + Sxz \models Sxy$, (PS4) $Nzy + Sxz \models \$xy$.

*Reductio Law.* $P \models d$ *if* $P + C(d) \models e$ and $P + C(d) \models C(e)$.

## §3. Aristotle's deductive system.

The system of deductions treated in Aristotle's second logic seems to be a natural deductive system (i.e., has several rules but no axioms) which consists in two distinct classes of deductions—the direct deductions and the indirect deductions. Generally speaking, a direct deduction is a finite list beginning with the premises, after which each new line is obtained by applying a rule to previous lines and, of course, ending with the conclusion. An indirect deduction, on the other hand, does not contain its conclusion but rather it is, in effect, a direct deduction containing the contradictory of the conclusion as an added assumption and having a pair of contradictories for its last two lines. For Aristotle, an indirect proof of a conclusion from premises was obtained by deducing contradictory sentences from the premises together with the contradictory of the conclusion (for detailed scholarship see [5]).

We proceed with an exact definition of the system **D** of deductions. First, restate the laws of conversion and perfect syllogisms as rules of inference. Use the terms 'a **D**-conversion of a sentence' to indicate the result of applying one of the three conversion rules to it. Use the terms '**D**-inference from two sentences' to indicate the result of applying one of the perfect syllogism rules to the two sentences.

A *direct deduction in* **D** *of* $d$ *from* $P$ is defined to be a finite list of sentences ending with $d$, beginning with all or some of the sentences in $P$ and such that each subsequent line (after those in $P$) is either (a) a repetition of a previous line, (b) a **D**-conversion of a previous line or (c) a **D**-inference from two previous lines.

An *indirect deduction in* **D** *of* $d$ *from* $P$ is defined to be a finite list of sentences ending in a pair of contradictions [$e$ and $C(e)$], beginning with a list of all or some of the sentences in $P$ followed by the contradictory of $d$, and such that each subsequent additional line (after the contradictory of $d$) is either (a) a repetition of a previous line, (b) a **D**-conversion of a previous line or (c) a **D**-inference from two previous lines.

All examples of deductions will be annotated according to the following scheme.

(1) Premises will be prefixed by '+' so that '$+Axy$' can be read "assume $Axy$ as a premise."

(2) After the premises are put down, we interject the conclusion prefixed by '?' so that '$?Axy$' can be read "we want to show why $Axy$ follows."

(3) The hypothesis of an indirect (reductio) deduction is prefixed by '$h$' so that '$hAxy$' can be read "suppose $Axy$ for purposes of reasoning."

(4) A line entered by repetition is prefixed by '$a$' so that '$aAxy$' can be read "we have already accepted $Axy$."

(5) Lines entered by conversion and syllogistic inference are prefixed by '$c$' and '$s$' respectively.

(6) Finally, the last line of an indirect deduction has '$B$' prefixed to its other annotation so that '$BaAxy$' can be read "but we have already accepted $Axy$," etc. We define an *annotated deduction in* **D** to be a deduction in **D** annotated according to the above scheme.

**Examples.**

(1) Let $M$ be predicated of no $N$
and of all $X$.
(*conclusion omitted in text*)
Then, since the negative premise converts, $N$ belongs to no $M$.
But it was supposed that $M$ belongs to all $X$.
Therefore $N$ will belong to no $X$.

$+Nnm$
$+Axm$
$(?Nxn)$
$cNmn$
$aAxm$
$sNxn$

(2) Again, if $M$ belongs to all $N$
and to no $X$,
$X$ will belong to no $N$.
For if $M$ belongs to no $X$,
$X$ belongs to no $M$.
But $M$ belonged to all $N$.
Therefore, $X$ will belong to no $N$.

$+Anm$
$+Nxm$
$?Nnx$
$aNxm$
$cNmx$
$aAnm$
$sNnx$

To exemplify an indirect deduction we do the same for [1, 28b, 18].

(3)

For if *R* belongs to all *S*,
but *P* does not belong to some *S*,
it is necessary that *P* does not belong to some *R*.
For if *P* belongs to all *R*,
and *R* belongs to all *S*,
then *P* will belong to all *S*;
but we assumed that it did not.

$+Asr$
$+\not{S}sp$
$?\not{S}rp$
$hArp$
$aAsr$
$sAsp$
$Ba\not{S}sp$

We give three examples above; two of direct deductions and one of an indirect deduction. The others raise no problems. First we reproduce two of Aristotle's deductions ([1, 27a, 5–15]; [7, p. 34]), each followed by the corresponding annotated deductions in **D**.

Readers can verify (by "translating" Aristotle's proofs of the syllogisms he proved, using ingenuity in the other cases) that all valid arguments in any of the four traditional figures are deducible in **D**.

3.1. *The reduced deductive system.* The system **D** above, in all essential respects due to Aristotle, is unusual from a modern point of view because it lacks a *reductio* rule and instead has a special class of deductions, viz. the indirect deductions. The essential points are two: First, the conclusion of an indirect deduction does not occur as a subsequently usable line in the indirect deduction; and (consequently) second, there are no deductions employing a nested (or even iterated) *reductio* strategy. One key point in the proof of strong completeness shows in effect that multiple *reductio* strategies are not necessary. This is Lemma M2 below. Because of its logical form, it is not surprising that it is easier to prove Lemma M2 for a *weaker* system than it is for **D** itself. The weaker system **RD** is obtained from **D** by deleting the rules corresponding to C3, PS3 and PS4. Aristotle himself had considered a system very close to **RD** and had observed (but not proved, evidently) that it was equivalent to **D** [5, §4.2.1].

$P \vdash d$ means that there is a deduction in **RD** of $d$ from $P$. The balance of the paper proves that *if* $P \models d$ then $P \vdash d$.

3.2. *Some properties of the deductive system* **RD**. The first thing to notice is that the property of being a deduction is unaffected by permutation of premises. Next notice that the operands of *all* of the rules (except repetition) are "universal" sentences (*Axy* or *Nxy*) so that once a "particular" sentence (*Sxy* or $\not{S}xy$) gets into a deduction thereafter it can only be repeated. In particular, one may delete from a direct deduction all occurrences of all particular sentences (except the conclusion if it is particular) and obtain thereby another direct deduction of the same conclusion from the same premises always (and in case a premise was particular, from a smaller set of premises). Consideration of the implications of having a particular sentence as the hypothesis of an indirect proof leads to Lemma M1.

LEMMA M1. *Let e be universal and let P be an indirect deduction of e from $S + d$. Then* ***either*** *there is an indirect deduction of $C(d)$ from S* ***or else*** *there is a direct deduction of e from $S + d$.*

Assume the hypothesis. Without loss assume that in $P$ the premises from $S$ come first, then $d$, then $C(e)$, then the intermediate lines, finally $f$ and $C(f)$. Since $e$ is universal $C(e)$ is particular. If $C(e)$ is neither $f$ nor $C(f)$, then every occurrence of $C(e)$ can be deleted producing a direct deduction from $S + d$ which ends with $f$ and $C(f)$. But this is an indirect deduction of $C(d)$ from $S$. Now suppose that $C(e)$ is either $f$ or $C(f)$. In this case $P$ ends with $e$ and $C(e)$, perhaps not in that order. In any case every occurrence of $C(e)$ can be deleted, producing a direct deduction of $e$ from $S + d$. Q.E.D.

LEMMA M2. *If $S + d \vdash e$ and $S + d \vdash C(e)$ then $S \vdash C(d)$.*

Assume the hypothesis and let $P$ and $PC$ be deductions of $e$ and $C(e)$, respectively, both from $S + d$. Without loss of generality, assume that $d$ is the last premise in both and that both contain the same sentences from $S$. There are three cases according as both, only one, or neither of $P$ and $PC$ are direct. The first two cases are obvious and the third uses Lemma M1. Q.E.D.

A set of sentences is *inconsistent* if there are two deductions having all premises in the set and having contradictory conclusions. Otherwise, a set is *consistent*. A consistent set having no consistent supersets is *maximally consistent*. These definitions yield Lemma A using the previous lemma.

LEMMA A. *Let S be maximally consistent. Then the following hold*:

(0) *$d \in S$ iff $S \vdash d$;*
(1) *exactly one of $Axy$, $\$xy \in S$;*
(2) *exactly one of $Sxy$, $Nxy \in S$;*
(3) *at least one of $Sxy$, $\$xy \in S$;*
(4) *at most one of $Axy$, $Nxy \in S$.*

**§4. The completeness proof.** By a variant of a familiar argument, completeness is proved once we see how to construct a true interpretation for an arbitrary maximally consistent set.

There is a rather "natural" class of interpretations, constructible using subsets of $V$ as follows. Let $U$ be any class of subsets of $V$. For each such $U$ there is a unique *natural function f* from $V$ into the power set of $U$ such that, for each $x$ in $V$, $fx$ is the class of sets in $U$ containing $x$. The idea, of course, is that $U$ is the "universe of discourse" whose "objects" are subsets of $V$ and that the property associated with the "term" $x$ is the property of having $x$ as a member. In case $U$ contains, for each $x$ in $V$, at least one set containing $x$ *then* the natural function $f$ is actually an interpretation. We call such interpretations *natural.*

Under a natural interpretation a universal sentence ($Axy$ or $Nxy$) says that certain objects are *not* in $U$. In particular, $Axy$ says that all objects containing $x$ but lacking $y$ are excluded from $U$ and $Nxy$ says that all objects containing both $x$ and $y$ are excluded from $U$. It will be shown that if one starts with $PV$, the power set of $V$, and then, for a given maximally consistent $S$, one deletes from $PV$ exactly those objects "excluded by" *universal* sentences in $S$, the result is a set whose natural function $f$ is a true interpretation of $S$. Let the result of deleting from $PV$ the objects excluded by $S$ be called $U(S)$. In particular we have the following theorem which is immediate from the Lemma B (stated just after).

THEOREM. *If $S$ is maximally consistent then the natural function based on $U(S)$ is a true interpretation of $S$.*

LEMMA B. *Let $S$ be maximally consistent. Then the following hold:*

(0) *for each $x$ in $V$, $U(S)$ contains at least one set containing $x$ (i.e., the natural function is an interpretation);*

(1) *$Axy \in S$ iff $U(S)$ contains no sets containing $x$ but lacking $y$;*

(2) *$Nxy \in S$ iff $U(S)$ contains no sets containing $x$ and $y$;*

(3) *$Sxy \in S$ iff $U(S)$ contains a set containing $x$ and $y$;*

(4) *$\$xy \in S$ iff $U(S)$ contains a set containing $x$ but lacking $y$.*

The lemma is established as follows. Clause (0) is proved below and similar reasoning shows the 'if' parts of clauses (1) and (2). The 'only if' parts of (1) and (2) are by definition of $U(S)$. Clauses (3) and (4) follow from the previous clauses by Lemma A.

In order to express proofs of the clauses succinctly, some notation is needed. For $x$ in $V$, $[x]$ is any subset of $V$ containing $x$. For $x, y$ in $V$, $[xy]$ is any subset of $V$ containing both $x$ and $y$ while $[x\bar{y}]$ is any subset of $V$ containing $x$ but lacking $y$. If $Y$ is a subset of $V$ then $x + Y$ is the union of unit set $x$ and $Y$.

Notice that a positive sentence, $Axy$, cannot exclude $V$ and that a negative sentence, $Nxy$, cannot exclude a unit set. Another useful fact is that any set containing *all* sentences $Axy$, for $y$ in a set $Y$, and *any* sentence $Nuv$, $u$ and $v$ in $x + Y$, is inconsistent.

To see clause (0) let $S$ be maximally consistent and suppose that $x$ is in $V$ but that no $[x]$ is in $U(S)$. Thus $\{x\}$ is not in $U(S)$. Since $\{x\}$ is excluded only by sentences $Axy$, for some $y$, $S$ must contain such a sentence. Let $Axy'$ be one such sentence and let $Y$ be the set of $y$ in $V$ for which $Axy$ is in $S$. $x + Y$ must be all of $V$. [Otherwise since $x + Y$ is excluded and consistency precludes any $Nuv$ ($u$, $v$ in $x + Y$) from being in $S$ we must have $Ayz$ in $S$ for $y$ in $Y$ and $z$ *not* in $x + Y$. But since $Axy$ and $Ayz$ are both in $S$, maximality requires that $Axz$ is in $S$. So $z$ is in $Y$. A contradiction.] Thus $S$ contains all sentences $Axy$ ($y$ in $V$ and distinct from $x$). Since $V$ itself is excluded and since $V$ cannot be excluded by any sentence $Auv$, $S$ must contain a sentence $Nuv$. This contradicts consistency. Q.E.D.

**§5. Comments and corollaries.** It is interesting to notice that Aristotle himself had given some thought to the problem of proving strong completeness of his own system [1, Book I, Chapter 23]. There is no question that he thought that he had shown the deductive equivalence of the system **D** to the reduced system **RD** and that he used the reduction in his deliberation concerning strong completeness of **D**. Unfortunately, he does not seem to have been clear enough about his own semantics to formulate the problem precisely and it is certain that he had not demonstrated the result.

The fact that Aristotle's metaphysics required that each universal term hold of at least one particular provides the motivation for assigning nonempty sets to "terms" in addition to providing the key to a theoretical account of why Aristotle's logic had "existential import." Incidentally, in view of the above systematic account, it seems wrong to attribute "existential import" to sentences—the existential import clearly belongs to "terms" relative to the semantics.

Note that there are no logical truths in the above system; i.e., that for all $c$, if $P$ is empty then $c$ is not a logical consequence of $P$. Aristotle systematically avoided sentences having two occurrences of a single term. This may explain why there is no doctrine of logical truth in the Aristotelian corpus.

The referee pointed out that decision procedures for monadic logic are easily adapted for this system.

**Acknowledgements.** A paper containing the same result for a stronger system was presented in June 1971 to the Mathematics Department of Laval University, Quebec, Canada. Peter Malcolmson (Department of Mathematics, University of California, Berkeley) discovered and proved Lemmas M1 and M2, making it possible to obtain the present result.

ADDED IN PROOF. In February of 1972 it came to my attention that Timothy Smiley (Clare College, Cambridge) had developed an "interpretation" of Aristotle's logic which agrees in all substantial points with mine. In addition, he had demonstrated strong completeness for his system (which is slightly stronger than mine) and he had obtained certain other results which go beyond the present study. His results were then to be published in ***Journal of Philosophical Logic.*** Needless to say, prior to February 1972 Smiley and I had worked in complete independence of each other.

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