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6 Information-theoretic logic

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In memory of Alfred Tarski 1901-1983 and Alonzo Church 1903-1995 on the fortieth anniversary of their classic works: *Logic, Semantics, Metamathematics and Introduction to Mathematical Logic.*

6.1. Introduction

Information-theoretic approaches [114] to formal logic analyse the "common intuitive" concept of prepositional implication (or argumental validity) in terms of information content of propositions and sets of propositions: one given proposition implies a second if the former contains all of the information contained by the latter; an argument is valid if the conclusion contains no information beyond that of the premise-set. This paper locates information-theoretic approaches historically, philosophically and pragmatically. Advantages and disadvantages are identified by examining such approaches in themselves and by contrasting them with standard transformation-theoretic approaches. Transformation-theoretic approaches analyse validity (and thus implication) in terms of transformations that map one argument onto another: a given argument is valid if no transformation carries it onto an argument with all true premises and false conclusion. Model-theoretic, set-theoretic, and substitution-theoretic approaches, which dominate current literature, can be construed as transformation-theoretic, as can the so-called possible-worlds approaches. Ontic and epistemic presuppositions of both types of approaches are considered. Attention is given to the question of whether our historically cumulative experience applying logic is better explained from a purely information-theoretic perspective or from a purely transformation-theoretic perspective or whether apparent conflicts between the two types of approaches need to be reconciled in order to forge a new type of approach that recognizes their basic complementarity.

6.2. Preliminaries

The information-theoretic viewpoint dominated logic in the period during which the seeds of mathematical logic were being sown by Boole, De Morgan, Jevons, Venn and others. In fact, the writings of the logicians who succeeded and worked in the shadow of Boole and De Morgan show almost no trace of awareness of any other viewpoint. It is worthwhile to review some of the relevant passages in order to identify our topic and in order to confirm the pervasiveness of this mode of thought.

> The two ... logical inferences ... from the original [set of] propositions ... give us all the *information* which it *contains* respecting the class ...(George Boole, 1847, p.75).
>
> ... it is the office of a conclusion not to present us new truth, but only to bring into explicit form some *portion* of that truth which was implicitly involved in the premises ... [some portion of] the particular *information* conveyed in the premises ... (George Boole 1856?, p. 239).
>
> Every collective set of premises *contains* all its valid conclusions; ... speaking objectively, the assumption of them [the premises] is the assumption of the conclusion; though, ideally speaking, the presence of the premises in the mind is not necessarily the presence of the conclusion (Augustus De Morgan, 1847, p. 254).
>
> All the propositions of pure geometry, which multiply so fast that only a small ...class ... among mathematicians ... know all that has been done ... , are certainly *contained* in a very few notions ... .[The] consequences are virtually *contained* in the premises (Augustus De Morgan, 1847, p. 45).
>
> The very purpose of syllogism is to deduce a conclusion which will be true when the premises are true. The syllogism enables us to restate in a new form the *information* ... *contained* in the premises, just as a machine may deliver to us in a new form the material ... put into it (W. Stanley Jevons, 1870, p. 149).
>
> We extract out of the premises all the *information* ... useful for the purpose in view-and this is the whole which reasoning accomplishes (W. Stanley Jevons 1870,p.15).
>
> [To deduce is] ... to draw ... propositions as will necessarily be true when the premises are true. By deduction we investigate and unfold the *information contained* in the premises ... (W. Stanley Jevons, 1879, p. 49).
>
> These ... [consequences] ... *contain* every particle of *information* yielded by the original [premise] ... , or in any way deducible from it (John Venn, 1881, p. 296).
>
> That is, [in making this inference] we have had to let slip a part of the *information contained* in the data (John Venn, 1881, p. 362). [115]
>
> ... logicians in overwhelming majority maintain that every conclusion is implicitly *contained* in the premises (John Venn, 1889, p. 42).

Information-theoretic approaches to logic may be characterized to some extent by six remarks. All information and all propositions mentioned in these remarks are assumed to pertain to some one, limited, and coherent "domain of investigation" established in advance and remaining fixed throughout (Corcoran, 1995, §4.3). The purpose of this assumption is to limit the scope of the inquiry in order to avoid incoherent pseudo-questions, in order to circumvent extraneous issues, and in order to put a bound on "some" and "all" as applied to information and propositions. Thus, in particular, 'every proposition' is to be elliptical for 'every proposition pertaining to the domain of investigation' and 'all information' is to be elliptical for 'all pertinent information'. In view of the role of domains of investigation it is natural to refer to them also as "informational domains".

First, a given proposition follows from, is a consequence of, a given postulate set if all of the information contained in the proposition is contained within the set. Second, a given proposition is independent of, not a consequence of, a given postulate set if the proposition contains any information outside of the information content of the set. Third, a proposition is tautological if it is devoid of information; accordingly a sentence that expresses a tautology conveys no information. A tautology is thus implied by every pertinent proposition, and is thus useless as a postulate, whether postulate sets are intended as presentations of given information or whether they are intended as characterizations of a given subject matter. Fourth, a proposition is contradictory if it contains all information (pertaining to the domain of investigation); accordingly, a sentence that expresses a contradiction conveys all such information. A contradiction thus implies every pertinent proposition and is thus useless as a postulate if postulate sets are intended as characterizations of a given subject matter. No subject matter is accurately described or characterized by a contradictory proposition. Fifth, no proposition has any information in common with its own negation, although a proposition and its negation need not (but may) divide all pertinent information between them. Sixth, the disjunction of one given proposition with a second contains exactly the information that the first has in common with the second, i.e. the information that the two share.

As usual two propositions that imply each other are said to be logically equivalent (to each other). "No prime number exceeding two is even" is logically equivalent to "No even number exceeding two is prime" and to "No number exceeding two is both even and prime". The first of the above remarks characteristic of information-theoretic approaches to logic entails [116] that two propositions are logically equivalent if and only if they contain exactly the same information. In particular, every two tautologies, each being devoid of information, are logically equivalent and every two contradictions, each containing all pertinent information, are logically equivalent. For example, "Zero is zero" is logically equivalent to "One is one" and to "Every even number is either even or prime"; similarly "Zero isn't zero" is logically equivalent to "One isn't one" and to "Some even number is neither even nor prime". Having the same (information) content neither entails nor precludes having the same (logical) form (Corcoran, 1989, p. 27-31 and Cohen and Nagel, 1993, pp. XXI, XXXI-XXXVII).

Conversely, having the same form neither entails nor precludes having the same content: "Zero is odd" has the same form as "Zero is even"; "Some prime number is odd" has the same form as "Some prime number is even". Every proposition has a unique logical form and a unique information content; but a logical form *per se* does not have content and an information content *per se* does not have form. One might say that it is the amorphous character, the formlessness, of information content that enables it to take on various forms (Cohen and Nagel, 1993, pp. XXV-XXIX).

The amorphous character of every information content, in and of itself, neither entails nor precludes a kind of discreteness or atomicity. An earlier paper (Corcoran, 1995) pointed out the existence of informational atoms in one of the most important and best known informational domains, that of the 1931 Godel incompleteness paper. A proposition is an informational atom if it is *informative* (i.e. non-tautological) but it implies no weaker informative propositions. In other words, a proposition is an informational atom (of a given informational domain) if it is informative but there is no way to drop information from it without rendering it devoid of information. Besides tautologies, an informational atom implies only its own logical equivalents. The negation of the conjunction of the 1931 Godel axioms is informationally atomic.

As is well known, the Godel axiom set is semantically complete or complete with respect to consequences in the sense of Church (1956, p. 329). This means that the conjunction of the axioms implies every pertinent proposition that it does not contradict. Thus, this conjunction is a consistent proposition to which no pertinent information can be consistently added. Such propositions can be called (informational) *saturations*. Every informational atom is equivalent to the negation of an informational saturation and every informational saturation is equivalent to the negation of an informational atom.

The particulate, or atomic, character of the propositions noted above does not entail "an atomic theory of information"; even though the Godel domain [117] contains infinitely many informational atoms it is not the case that each of its propositions is logically equivalent to a set of informational atoms. In fact the Godel Axiom Set is not equivalent to a set of informational atoms (Corcoran, 1995, p. 75). Information seems to straddle "the continuous" and "the discrete", to share some aspects with "magnitudes" and some with "multitudes", to have some affinity with the category indicated by "mass nouns" and some affinity with the category indicated by "count nouns".

It is worth making explicit the fact that information-theoretic approaches to logic extend to propositions as abstract individuals a kind of "hylomorphism", or matter-and-form analysis, similar to that attributed by traditional Aristotelian ontology to concrete individuals. Just as an individual brass sphere involves brass as its matter and sphericity as its form, an individual arithmetic identity, say "One plus two is three", involves arithmetic information as its content and the logical form of the identity as its form. Just as the same brass admits of being contained in infinitely many geometrically dissimilar brass objects, as indicated above, the information content of the identity is contained in each of infinitely many formally dissimilar propositions. The hylomorphic analogy that naturally accompanies information-theoretic approaches helps to make logic accessible to beginning students, it helps to make logic more useful to those who apply it, and it helps to make logic more exploitable to researchers. Far from being a crude metaphor, as Morris Cohen once called it (Cohen 1944, p. 194) , the hylomorphic aspect of the information-theoretic viewpoint has pedagogical, practical, and heuristic benefits that can be enjoyed even by persons not ready to accept the viewpoint philosophically.

The expressions 'information content' and 'logical form' are far from self explanatory. Both are composed of notoriously ambiguous words and the ranges of senses of these words suitable for information-theoretic logic are severely restricted. In particular, the range of suitable senses of 'information' is limited by the formal properties of information content required by the six characteristic remarks given above. For example, every false proposition is informative. Thus, in the senses required here, "misinformation" is information, not all information is accurate. Moreover, the information content of a proposition is not to be measured by the number of its non-logical, or content, concepts: fewer non-logical concepts does not necessarily correlate with less information content. The proposition "Every number is inductive" contains much more information than "Every even number is [118] inductive". Of course, having the exact same content concepts does not necessarily correlate with having the exact same information content: in fact, "Every perfect number is even" has no information in common with "Some non-even number is perfect". In none of the senses of the expression 'information content' suitable for use with information-theoretic logic can a concept be said to have information content: information content belongs exclusively to propositions and sets of propositions.

As usual, (written) sentences, which are made up of conventional characters, or symbols, are distinguished from propositions, which are made up of concepts. One and the same proposition may be expressed by different sentences in different languages, or even in the same language. For example, "One plus two is three", the proposition that one plus two is three, is expressed by each of many sentences: '(1+2)=3', 'One plus two is three', 'One and two are three', 'Uno y dos son ires', and so on. This article follows the increasingly widespread convention of indicating sentences (and other expressions) by single quotes while indicating propositions (and other meanings) by double quotes. A sentence may be said to convey the information contained in a proposition that it expresses.

Our primary purpose is to locate information-theoretic conceptions of logic historically, pragmatically, and philosophically. Secondary purposes are to contribute to the dialogue concerning the nature of our "common intuitive" notion of consequence (or argument validity) and to explore the ontic and epistemic basis of logical judgements both of consequence (or argument validity) and of independence (or argument invalidity). Attention is given to the question of whether our cumulative experience in applying logic is better explained from an information-theoretic perspective or whether this perspective must be augmented, or entirely supplanted, by a transformation-theoretic approach. A background purpose is to contribute toward delimiting the formal properties that must be satisfied by any "information concept" adequate to serve as a foundation for an information-theoretic approach to logic.

6.2.1. Basic terminology

In the course of investigation of a given subject-matter or universe of discourse, it routinely occurs to the investigators to wonder whether or not a certain proposition is implied by or contradicted by a certain set of propositions. For example, and this by no means exhausts the types of situations in which such questions occur, the single proposition may be a hypothesis under investigation and the set may be composed of propositions which have actually been accepted as true, as in a branch of mathematics, or it may be composed of propositions which have been confirmed to a high degree or which have been adopted as working hypotheses, so to speak, as in one of the empirical [119] sciences or in commercial, medical, or criminal investigations. In this situation the investigators are wondering whether their problem, their hypothesis, can be settled affirmatively or negatively on the basis of information already "obtained", or whether settling of the hypothesis requires them to obtain more information. In some cases they will be asking whether an unsettled hypothesis is reducible to hypotheses already settled or whether further independent access to the subject-matter is necessary.

This type of question is typical, perhaps characteristic, of applied logic. My conception of logic can be seen as stemming from a focus on such practical questions. Accordingly, whenever propositions are compared below it is to go without saying that they are all pertinent to some one and the same investigation, in particular that they all concern a fixed universe of discourse and they involve concepts from a limited set of concepts fixed in advance and unchanged throughout an investigation. In the terminology introduced above, they are all presumed to belong to one *domain of investigation* or to one *investigational domain*.

Limitation of a given *logical* inquiry to propositions in a fixed domain of investigation corresponds to limitation of a *scientific* inquiry to a fixed universe of discourse. Moreover, just as the importance of limited universes of discourse became increasingly clear over the first century in which logic assumed an increasingly mathematical character, the importance of domains of investigation has become increasingly clear in the last 50 or 75 years. No longer do logicians consider *all* propositions without limitation but, in more and more instances, a logician will consider propositions from a limited class, e.g. only propositions about the natural numbers expressible in a certain interpreted formal language having a limited set of primitive concepts.

By an *argument* I mean a two-part system composed of a set of propositions, its premise-set, and a single proposition, its conclusion. An argument is *valid* if its conclusion is implied by (is a consequence of) its premise-set and an argument is *invalid* if its conclusion is not implied by (is independent of) its premise-set. A set of propositions *implies* a given proposition if the argument having the set as premise-set and the given proposition as conclusion is valid; accordingly, a set of propositions *does not imply* a given proposition if the corresponding argument is invalid. My last two sentences may well be reversed in order without changing the content. I do not define implication in terms of validity, or validity in terms of implication; I take them to be coordinate concepts.

It is important to notice that both concepts are ontic as opposed to epistemic, they are objective as opposed to subjective, they are impersonal as opposed to personal, and, in the sense of C.W. Morris, they are not pragmatic.[120] This point may perhaps be better stated by saying that a given argument is valid or invalid in and of itself without regard to whether anyone did or will or can determine its validity. The so-called Goldbach argument is either valid or invalid despite the fact that no one knows which and despite the possibility that its validity or invalidity is unknowable (Corcoran, 1973, p. 61 or Hughes, 1993, p. 88).

The Goldbach argument takes as its premise-set "the" basic premises of arithmetic, essentially "the" Peano Postulates and "the" definitions of addition, multiplication, and so on. Its conclusion is the so-called Goldbach Hypothesis, that every even number exceeding two is the sum of two primes.

Every argument is either valid or invalid. The fact that not every argument is known to be valid or known to be invalid is one of the things that makes applied *formal logic* interesting. The possibility that not every argument can be known to be valid or known to be invalid is one of the things that makes *formal epistemology* interesting. But with these two comments we are getting ahead of ourselves. These points are treated in the appendix (below).

An argument *per se* does not contain a sequence of propositions that may be taken to involve a "chain of reasoning" showing that the conclusion is an implication of the premise-set. A three-part system that results from adding a "chain of reasoning" to an argument I call an *argumentation* (Corcoran, 1989). An argumentation is said to be *cogent* or *fallacious* according as its chain of reasoning shows or does not show that its conclusion is implied by its premise-set. In contrast to validity and invalidity, cogency and fallaciousness are both epistemic and not purely ontic, they are both "subjective" in that both involve a cognitive "subject" in addition to their objective aspects, and they are both personal and pragmatic in that they involve thinkers.

I use the concepts of implication and validity in describing various historically given systems of logic even if the authors of these systems did not use them. For example, Aristotle has no implication relation but when he says that a conclusion follows of necessity from given premises we will feel free to report that he says that the given conclusion is implied by that premise-set. This kind of reporting could be misleading without a disclaimer. Likewise, Tarski has no argument concept and no validity concept. Of course, Tarski uses the words 'argument' and 'valid'. But when Tarski uses 'argument' it is often in reference to something involving a chain of reasoning. As a first approximation we may say that by 'argument' Tarski means something closer to "argumentation" (1941, pp. 26, 108, 126f., 158, 175 *passim*). Of course, Tarski also uses the word 'argument' in the mathematical sense to indicate "a value of an independent variable" (1941, pp. 98 ff.). Accordingly, Tarski never uses the word 'valid' in the above sense; he always speaks of the "validity" [121] of an "assumption", "statement", "theorem" or the like as opposed to the validity of an argument (1941, pp. 57, 118, 128, 135). Jan Tarski's new edition of his father's 1941 book contains a long footnote on this point (1994, p.52). Moreover, in the 1941 book Tarski does not use the verb 'imply' in the sense used here (1941, pp. 27-29).

Nevertheless, when Tarski has occasion to say that a certain proposition is a consequence of a certain set of propositions we will feel free to report him as having said that a certain argument is valid, and so on. Interpretation of a historically given text often involves a kind of "recategorization" of its subject-matter. We need to be alert to the dangers inherent in the interpretational enterprise.

There are several other locutions commonly used to say that a given argument is valid: were the premises all true then necessarily the conclusion would also be true; were the conclusion false then necessarily at least one of the premises would also be false; it is logically impossible for the premises to all be true without the conclusion being true, it is logically impossible for the conclusion to be false without at least one of the premises being false; were the premises all true it would be impossible for the conclusion to be false; it is logically impossible for the premises to all be true with the conclusion false (cf. Cohen and Nagel 1993, p. XXI). Although the above and similar modal expressions occur freely and unapologetically in certain parts of the logical literature, there are authors who tend to treat logical necessity and logical impossibility as psychological or subjective and there are others who conspicuously avoid modal expressions. For example, Tarski (1941, pp. 24-25) explains 'follows necessarily' in terms of compulsion to assume the consequences of propositions already assumed, a compulsion which he refers to as a psychological factor. Similarly, Quine (1952, pp. XIII-XIV) only uses the expression 'necessity' between quotation marks thus distancing himself from any literal usage. Moreover, he refers to the "necessity" of logical laws as something felt by people who are fully capable of feeling otherwise. Church (1956) avoids use of modal expressions altogether in his treatment of logical consequence.

6.3. The roles of logic in intellectual life

In order to prove that a given proposition is true it is sufficient to show that it is *implied* by propositions known to be true. In order to prove that a given proposition is false it is sufficient to show that it, alone or combined with propositions known to be true, *implies* a proposition known to be false. In order to assure ourselves that a given set of beliefs is consistent it is sufficient [122] to show that it *does not imply* the negation of any one of its own members. More generally, our methods of rational belief formation, whether directed at a proposition to be believed to be true, whether directed at a proposition to be disbelieved (believed to be false), or at a proposition to the effect that a given set of propositions is consistent, all involve understanding of implication. Rational belief formation regularly occurs in contexts that require us to make implicational judgements, affirmative judgements that a given proposition is an implication of a given set of propositions as well as negative judgements that a given proposition is not an implication of a given set of propositions (Quine, 1952/59, pp. XI-XVII and Hughes, 1993, pp. 1-5).

Implicational judgements are often required even before we are in a position to attempt to make a propositional judgement or even to form a prepositional belief or disbelief, however tentatively or however irresponsibly. The process of suspension of judgement, more properly suspension of prepositional belief and disbelief, is often predicated on not suspending, in fact on engaging in, implicational judgements.

In one sense of the verb 'to understand', it is necessary to understand a proposition *before* implicational judgements involving it can be made. Before determining whether one given proposition implies a second given proposition it is necessary to understand both, to *grasp* both.

6.3.1. Drawing the contained conclusions

For many years, even before the advent of mathematical logic, logicians have thought of the consequences of a set of propositions as being somehow contained in the set of propositions. It was clear that in some cases the containment was obvious, but that in many cases a set of propositions had hidden consequences (Scarre, 1984, p. 20). The process of determining that a given conclusion was implied by a given premise-set was spoken of as a drawing of the conclusion from the premise-set.

There is a parallelism between the above logical usage and the material usage whereby we say that a certain amount of water is *contained* in a certain well and that a certain amount of water has been *drawn* from the well. Treating the two usages along with many others that readily come to mind, we can say that a person *draws* an item from a source only if that item is *contained* in the source or that an item drawn from a source was necessarily contained in that source.

Some dictionaries treat the logical usage as a subordinate special case of a general usage which also includes the material usage (Niobey, 1986, p. 706). Some dictionaries treat the logical and the material usages as coordinate with others, such as drawing a card from a deck of cards (Mish, 1988, p. 381). And [123] some dictionaries treat the logical and material usages as coordinate while including as coordinate other usages that could easily be construed as including the logical and material uses as subordinate special cases (Brown, 1993, p. 745).

6.3.2. Sentences convey information; propositions contain information

An interpreted sentence is meaningful. Such a sentence may but need not convey information; it may or may not express a proposition. A proposition is either true or false, it can be entertained as a hypothesis, it can be believed, suspected, doubted or disbelieved, and it (in more or less rare cases) can be known to be true or known to be false. In some cases, two sentences expressing different propositions nevertheless convey the same information. The sentence 'One precedes two' and the sentence 'Two is preceded by one' occupy different positions in the alphabetical ordering of English sentences and the two propositions which they respectively express (under one obvious literal interpretation of their words) have different subjects and different predicates. These two propositions, the proposition that one precedes two and the proposition that two is preceded by one, contain the same information; the two have the same information content.

6.4. Conceptions of validity and invalidity

In this section I want to contrast two broad classes of approaches to logic: information-theoretic approaches and transformation-theoretic approaches. The latter class includes all of the possible-worlds approaches (cf. Bradley and Swartz, 1979), the model-theoretic approaches (including those taken by the early postulate theorists, Scanlan, 1991), the set-theoretic approaches perhaps epitomized in the 1987 Tarski-Givant book, the substitution-theoretic approaches favoured e.g. by Quine (1970/1986) and Suppes (1957), and even speech-act approaches as represented e.g. in Kearns (1995).

In order to avoid, on the one hand, verbal disagreements resulting from ambiguity of key terms and, on the other hand, begging of questions that should be regarded as issues rather than assumptions, I propose a perhaps unnecessarily restricted domain of investigation, viz. that of elementary number theory broadly conceived. In particular, our propositions all concern the class of natural numbers, they involve only such numerical concepts as occur in ordinary classical number theory, the class of their numerical (non-logical) concepts is presumed to be closed under definability but no distinction is made between "primitive" and "defined" concepts, and, finally, all propositions are assumed to be expressible in a standard formalized first-order language with identity and the other standard logical concepts. [124]

The main reasons for choosing this domain of investigation are the following. In the first place, it is well-known, accessible, rich, interesting and relatively uncontroversial. In the second place, it represents a kind of culmination or idealization of a series of investigations that include several high points of logic and foundations and that reach back to classical Greek thought, indeed to the mathematically-oriented presocratics. In the third place, there is a very wide agreement on which arguments in this domain are valid and which are invalid, i.e. on which propositions are implied by which sets of propositions and on which propositions are independent of which sets of propositions. This is *not* to say that each such argument is believed to be valid or believed to be invalid. On the contrary, and this only serves to emphasize the nature and extent of the agreement, in case after case there is wide agreement concerning where to withhold judgement. For example, first order versions of the Goldbach argument are widely regarded as open problems in formal logic. In the fourth place, there is wide agreement also on certain methods of validation and on certain methods of invalidation. In particular, without ruling out other methods, it is widely agreed that in this domain of investigation, a given conclusion is implied by given premises if and only if the conclusion is formally deducible from the premises in a standard deduction system. In effect, this stems from the 1929-1930 Godel completeness result (Godel, 1986). Moreover, again without ruling out other methods, it is widely agreed that a given conclusion is independent of given premises if and only if the premises can be transformed into truths and the conclusion into a falsehood by replacing some or all of their arithmetic concepts by new arithmetic concepts while perhaps also restricting the universe of discourse to a subset of the class of natural numbers. In effect, this stems from the Hilbert-Bernays strengthening of the Lowenheim-Skolem result (Quine, 1970/1986,p. 54).

It is important to realize that logicians can and do agree that an argument in this domain is valid if and only if it is formally deducible ... while at the same time disagreeing about whether validity *means* deducibility. Actually, most logicians since the 1930s think that it is a mistake to identify validity (or consequence) with its positive criterion, deducibility (Corcoran and Scanlan 1982, pp. 83-86). Likewise important is the fact that logicians can and do agree that an argument in this domain is invalid if and only if it admits of a countertransformation in a subuniverse of the natural numbers ... while at the same time disagreeing about whether invalidity *means* having such a countertransformation. Actually, I know of no support for the hypothesis that in this domain invalidity *means* countertransformability within a numerical [125] subuniverse. Compare Quine (1970). Moreover, the general question of whether in arbitrary domains invalidity means countertransformability has been raised; in fact, it has even been questioned whether invalidity is coextensive with countertransformability in arbitrary domains (Corcoran, 1972, p. 43 and Corcoran and Scanlan, 1982, pp. 83-85).

Thus, in the present domain, although there is widespread agreement on the extensions of the predicates 'valid' and 'invalid', logicians disagree about their intensions and thus also disagree about the correct analysis of our "common intuitive" concepts of validity and of invalidity.

6.4.1. Information-theoretic conceptions of validity and invalidity

As suggested in various passages above, information-theoretic approaches to logic take in a literal way common attributions of "information content" to propositions and sets of propositions. The proposition that every number precedes its own successor contains the information in the following: that seven precedes its own successor, that every even number precedes its own successor, that every non-even number precedes its own successor, that every number precedes a number that precedes its own successor, and so on. It is also obvious that "Some perfect number is not even" does not contain the information that some number is even, which is contained in "Some even number is not perfect". It is then clear that the existential negative proposition does not contain the information contained in its own converse, a point Aristotle made several times.

An information-theoretic approach to logic leads to a *premise-containment* conception of validity: an argument is valid if and only if its premise-set contains all of the information of its conclusion. Accordingly, it leads to a *conclusion-containment* conception of invalidity: an argument is invalid if and only if its conclusion contains information totally outside of the information content of its premise-set.

An information-theoretic approach to logic analyses our "common intuitive" conception of validity in terms of information content. It takes the validity of an argument to be an entirely intrinsic property of an argument; an argument is valid in virtue of an internal relation between the premise-set and the conclusion, a relation that makes no essential reference to anything outside of the argument. On an information-theoretic conception, being tautologous is an intrinsic property of a proposition that contrasts with the extrinsic property of being true. Roughly, truth is correspondence with external fact; tautologousness is being devoid of information. Moreover, an information-theoretic approach might be able to explain why it is that we are able to make affirmative and negative implicational judgements, judgements of validity on [126] the basis of our capacity to understand propositions. An information-theoretic approach suggests a way of answering the epistemic question of how judgements of validity and invalidity are possible.

The above paragraph highlights three aspects of an information-theoretic approach to logic: an analytical aspect, an ontological aspect and an epistemological aspect. Such an approach suggests an analysis of our "common intuitive" concept of validity; it attributes the ontological status of intrinsic property to validity; and it suggests an approach to explaining how implicational judgements are possible.

Once information content has been ascribed to propositions and to sets of propositions it is natural to define two arguments to have the same *argumental* content if their respective premise-sets have the same content and their respective conclusions have the same content. In other words, two arguments have the same content if and only if the premise-set of one is logically equivalent to the premise-set of the other and the conclusion of the one is logically equivalent to the conclusion of the other. Given this definition, the role of content in argument validity is reflected in the following *Principles of Content*: every two arguments having the same content are both valid or both invalid; every argument having the same content as a valid argument is valid; every argument having the same content as an invalid argument is invalid. From this point of view, it is a misleading half-truth to say that an argument is valid in virtue of its form; from this point of view, what "makes" an argument valid is its argumental content, specifically the relation of the premise set content to the conclusion content. The inadequacy of the statement that an argument is valid in virtue of its form was emphasized as early as 1934 by Cohen and Nagel (1993, pp. 8-12).

6.4.2. Transformation-theoretic conceptions of validity and invalidity

A transformation-theoretic approach to logic presupposes any one of a number of *ways* of transforming one argument into another. For example, the famous 1936 Tarski article 'On the concept of logical consequence' may be understood as involving transformations that replace the nonlogical concepts in a given argument by others of the same type. For example to see that the successor axiom, viz. "Distinct numbers have distinct successors", does not imply the zero axiom, viz. "No number's successor is zero" we can replace the successor concept by the square concept and replace the zero concept by the unit concept. The above argument is then transformed into the argument whose premise is "Distinct numbers have distinct squares", which is true of course, and whose conclusion is the following falsehood: "No number's square is one". The various possible-worlds approaches to logic obviously [127] admit of construal as transformation-theoretic approaches (Bradley and Swartz, 1979). The locution that a false proposition is true in some other possible world can not be taken literally because the proposition in question is about this world and hence is distinct from any proposition about some other possible world. Be that as it may, such locutions can readily be understood as involving transformations that carry propositions about a given existent subject-matter into propositions about a given non-existent but possible subject-matter.

The substitutional approaches of Quine (1970), Suppes (1957) and others can be construed as transformation-theoretic in nature as can the reinterpretational approaches represented classically in the 1932 Lewis and Langford work (1932, p. 342) and the speech-act approaches as represented in Kearns (1995). Incidentally, the appropriateness of collecting transformation-theoretic approaches under one rubric should not mask vast differences among them. For example, some are immanent, or this-world oriented, as is Quine's and the early Tarski's; some are transcendent, or other-world oriented, as are the possible-worlds approaches; and some postulate a realm of abstract entities that are the referents of the transformed propositions as do the set-theoretic approaches.

All of these approaches define an argument to be valid if and only if every transformation satisfying the premises satisfies the conclusion. This means of course that in order for a given argument to be invalid it is necessary and sufficient for there to exist a transformation that carries the given argument into one having all true premises and a false conclusion.

Thus a transformation-theoretic approach analyses validity as an extrinsic property of arguments; in order for a given argument to be valid it is necessary and sufficient for it to be related in a certain way to the class of all transformations, something that cannot be construed as internal to the given argument. The transformation theoretic-approach has the merit of taking account of the classical method of countermodels used to establish the standard independence results: the independence of the parallel postulate, the independence of the continuum hypothesis, the independence of the Peano axioms, and so on.

However, the transformation-theoretic approach seems to leave no room for judgements of invalidity not based on countertransformations. It would seem that the identification of invalidity with the existence of a countertransformation requires every judgement of invalidity to be preceded by the exhibition of a countertransformation. But, did we really need a countertransformation to judge that "Some perfect number is not even" does not entail "Some number is even"? Did we really need a countertransformation before [128] we could judge that "Every oblong number is even" doesn't imply "Some even number is not oblong"?

Thus one glaring difficulty with transformation-theoretic approaches concerns the epistemics of invalidation; such approaches entail that every judgement of independence, no matter how simple, requires construction of a countertransformation, be it a countermodel, a counterinterpretation, a countersubstitution, or whatever. Judgements of invalidity are all reduced to judgements of truth and falsehood. On these approaches invalidity is no longer a purely conceptual matter.

It would be an exaggeration to say that the transformation-theoretic approach makes judgements of invalidity, which had traditionally been considered *a priori*, into judgements that are *necessarily a posteriori*. But this exaggeration helps to reveal the nature of the difficulty.

An even more glaring difficulty comes to light when we consider the epistemics of validation. Traditionally validation was regarded as *a priori* judgement. It was deemed unnecessary to determine truth-values in this world or in any other in order to determine that a given valid argument is valid or, more particularly, to determine that a given tautological proposition is tautological.

But this is no longer the case when we come to the transformation-theoretic approach; for a person to judge correctly that a given argument is valid this approach requires knowledge of all transformations, viz. that each of them is a non-countertransformation. For example, on a possible-worlds approach knowledge that a given argument is valid requires not just the immanent knowledge that the argument is materially valid, that the conclusion is not false with the premises all being true, but it requires the transcendent knowledge that the same "factual" situation obtains in all possible worlds. In general, the transformation-theoretic approach makes knowledge of logical validity more onerous than knowledge of material validity, thus turning upside-down a view that has been almost universally accepted.

In order to emphasize this point, notice that the transformation-theoretic approach makes the task of gaining knowledge that a given proposition is tautological more onerous than that of gaining knowledge that a given proposition is true. In order to know that "Every even number is even" is tautological according to this approach, it is necessary to know that it is true in every model, or in every possible-world, or under every substitution, or whatever.

Such considerations seem to lead in the direction of rejection of transformation-theoretic approaches, and reaffirmation of the intrinsic nature of validity and tautologousness. This in turn points in the direction of information-theoretic logic.[129]

Appendix

Formal ontics, formal epistemics, and formal praxis (or practice). It has been said that a discipline is not determined by its subject-matter alone, but that identification of a discipline also requires an articulation of its problems and goals. From this perspective there are at least three disciplines that can be called 'logic'. Thus, it is possible to conceive of logic first as an autonomous science, as what has been called *formal ontics* (or *formal ontology*), or second as a study of certain aspects of cognitive processes as they take place in the autonomous sciences, what has been called *formal epistemics* (or *formal epistemology*), or third as series of applications of an organon, method, or instrument in each and every autonomous science, be it number theory, geometry, string theory, analysis, physics, or whatever. It is this third discipline, *formal praxis* (or *formal practice*), fundamentally a series of applications, that comes most readily to mind when the expression *formal logic* is used.

Formal praxis *per se* does not have a single most general problem; but for each field of application there is one general problem that serves to identify formal praxis *as it applies to that field*. For example, the most general *logical* problem in number theory can be considered to be the following: given an arbitrary argument whose premises and conclusion are exclusively number-theoretic propositions, to determine whether the argument is valid or invalid. In an alternative formulation the problem is the following: given an arbitrary set of arithmetic propositions and an arbitrary single arithmetic proposition, to determine whether the given proposition is a logical consequence of, or is independent of, the given set. (Cf. Boole, 1854, p. 140).

Subsumed by the most general logical problem in number theory we have the problem of determining whether or not a given set of arithmetic "postulates" is independent, since a set of propositions is independent if and only if each of them is independent of the rest. Likewise subsumed by the most general problem is the so-called arithmetic consistency problem: given an arbitrary set of arithmetic propositions, to determine whether or not it is consistent. The reason for this subsumption is of course the fact that a given set of propositions is consistent if and only if that given set does not imply the negation of one of its own members.

The same situation obtains in geometry, set theory, and the rest. For example, the most general logical problem in geometry can be taken to be the problem of determining, given an arbitrary set of geometrical propositions and an arbitrary geometrical proposition, whether the proposition is logically implied by the set of propositions. In order to restate this problem succinctly, let us say that an argument is geometrical if its propositions involve [130] non-logical concepts other than geometrical concepts such as "point", "line", "plane", "congruence", etc. The most general logical problem of geometry is then: to determine of a given geometrical argument whether it is valid or invalid. As in the analogous situation for number theory, this problem subsumes as special cases all of the geometrical independence and consistency problems, it includes for example, the ancient problem of whether the parallel postulate follows from the other basic premises of Euclidean geometry, or equivalently, the problem of whether the "corresponding non-Euclidean geometry" is consistent.

The overarching field, which includes ontics, epistemics and praxis, is called *formal logic* for several reasons, four of which are relevant here. In the first place formal praxis *per se* is not about the subject matter of any of its applications. The question of whether, e.g., the Goldbach Hypothesis implies the Fermat Hypothesis is not a question about numbers, the way the two hypotheses are, but it is about the propositions. This is the principle of subject-matter irrelevance of formal praxis. In the second place, and this might well be regarded as entailed by the first point, formal praxis *per se* is not concerned to determine the truth or falsity of any of the propositions in the arguments whose validity it seeks to determine. This is the principle of truth-value irrelevance of formal praxis, a principle very easy to misunderstand. In the third place, formal logic applies in exactly the same way to every science and indeed to every homogeneous body of scientific propositions. This is why persons who hone their logical skills in connection with one science can apply those improved and refined skills in other sciences. This is the principle of topic neutrality of formal praxis. In the fourth place, any result achieved in formal praxis automatically applies in indefinitely many other situations. Whether a given set of propositions implies or does not imply a given proposition, the same is true in every formally similar situation. In other words, every argument in the same form as a given valid argument is valid and every argument in the same form as a given invalid argument is invalid. More generally, every two arguments in the same form are both valid or both invalid. This is the principle of form for arguments. The principle of form is the source of the economy of thought much heralded by mathematicians, especially algebraists. In fact, it is probably no exaggeration to say the principle of form was one of the main engines in the early development of modern abstract algebra. (Cf. Bourbaki, 1950, pp. 223-6).

The principle of form reinforces the principle of topic neutrality in that a particular result that an argument in one field is valid, say in geometry, automatically provides the basis for a class of results that arguments in other [131] fields are valid. Likewise, for invalidity results. To take an example with an algebraic flavour: the fact that the law of commutativity of addition in number theory neither implies nor is implied by the law of associativity of addition can be taken as the basis for obtaining formally similar analogous independence results in other fields, e.g. that the law of commutativity of union in the theory of classes neither implies nor is implied by the law of associativity of union.

Even though this work focuses entirely on formal praxis and formal epistemics, it is still worthwhile to compare characteristic problems of these two subfields with those of formal ontics. The contrast with formal epistemics is clear: formal praxis studies arguments to determine whether they are valid or invalid; formal epistemics studies methods of determining validity and methods of determining invalidity. For example, where formal praxis looks at a given argument to determine whether it is valid or invalid, formal epistemics would look at an argumentation to determine whether it is cogent or fallacious. There is no room in formal praxis *per se* for a study of deductions or for a study of fallacies, but these two topics make up part of formal epistemics.

In order to succinctly state characteristic problems of formal epistemics let us use the verb *validate* for an act of determining that an argument is valid and the verb *invalidate* for an act of determining that an argument is invalid. One characteristic task of formal epistemics is the evaluation of specific methods intended as validations or invalidations. Thus one characteristic problem of formal epistemics is the following: given an arbitrary argument together with a given method (of validation or of invalidation) applied to that argument, to determine whether or not the method accomplishes its task. In addition to evaluating specific methods applied to specific arguments, formal epistemics also evaluates general methods. This gives rise to the following characteristic problem: given a method (of validation or of invalidation) to determine whether it accomplishes its task. In this connection we can recall the fact that a method of validation proposed by Copi in 1956 was found to be fallacious by Parry and by others (Anellis, 1991 and Corcoran and Rudnicki, 1994). Likewise a method of invalidation proposed by Hilbert in 1898-9 was alleged to be fallacious by Frege in 1907-9 (Kluge, 1971).

In order to fully grasp the focus of formal epistemics it is necessary to realize that it is not focusing merely on the "logical adequacy" of methods but rather on their "epistemic adequacy". In particular, completeness and soundness per se do not belong to formal epistemics. For example, Frege's charge was not that Hilbert's method of invalidation produced false judgements of invalidity (i.e. was unsound), but that it did not produce knowledge of [132] invalidity. Frege charged that Hilbert's method was flawed, but Frege did not claim that the method would result in a valid argument being misjudged invalid. Flipping a coin is an epistemically inadequate method of judging prime or composite for numbers exceeding two, even if by some strange quirk it would produce the right result. It would be fair to say that formal praxis concerns *what* to think about questions of validity and invalidity, whereas formal epistemics concerns *how* to think about such questions.

Where formal praxis asks whether a given argument is valid or invalid, formal epistemics asks how we can determine whether it is valid or invalid, what methods can be used to determine of a given valid argument that it is indeed valid and what methods can be used to determine of a given invalid argument that it is indeed invalid. The distinction between direct and indirect reasoning belongs to formal epistemics and it is entirely outside of the range of problems dealt with in formal praxis *per se*. There is no danger of confusing formal praxis with cognitive psychology, but the border between formal epistemics and cognitive psychology is not so easy to draw in detail and with complete accuracy. The question of whether indirect reasoning is genuinely cogent or whether it is merely sound and persuasive is a question on the border between formal epistemics and cognitive psychology, but one that is entirely outside of formal logic. Formal praxis *per se* does not critically investigate methods of validation and invalidation, not even those that it uses itself.

Roughly speaking, formal ontics (or formal ontology) may be described as the attempt to determine the truth or falsity of propositions involving formal concepts of categories of being and totally devoid of reference to "concrete" particulars and properties. The simplest examples of results in formal ontology are the so-called "laws of thought": "Excluded Middle", "Noncontradiction" and so on.

Excluded Middle: Given any individual and any property, either the property belongs to the individual or the property does not belong to the individual.

Noncontradiction: Given any individual and any property, it is not the case that the property both belongs and does not belong to the individual.

Identity: Given any individual and any property, if the property belongs to the individual then the individual has the property.

Identity of Indiscernibles: Given any two distinct individuals, there exists at least one property that belongs to one but not to the other.

Complementation: The complement of the converse of a given relation is coextensive with the converse of the complement of that relation.

Principia Mathematica can be regarded as an axiomatic presentation of logic as formal ontology. The characteristic problem of formal ontology can [133] be taken to be the following: given an arbitrary proposition totally devoid of non-logical concepts to determine whether it is true or false. Formal ontology is beyond the scope of this paper, which focuses on formal praxis (or applied logic) and formal epistemics (or epistemology of logic).

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