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The Premiss-Based Approach to Judgment Aggregation

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Abstract

In the framework of judgment aggregation, we assume that some formulas of the agenda are singled out as premisses, and that both Independence (formula-wise aggregation) and Unanimity Preservation hold for them. Whether premiss-based aggregation thus defined is compatible with conclusion-based aggregation, as defined by Unanimity Preservation on the non-premisses, depends on how the premisses are logically connected, both among themselves and with other formulas. We state necessary and sufficient conditions under which the combination of both approaches leads to dictatorship (resp. oligarchy), either just on the premisses or on the whole agenda. Our analysis is inspired by the doctrinal paradox of legal theory and is arguably relevant to this field as well as political science and political economy. When the set of premisses coincides with the whole agenda, a limiting case of our assumptions, we obtain several existing results in judgment aggregation theory.

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1 Introduction

The theory of judgment aggregation - or logical aggregation, as we call it here - originates in a famous conundrum of legal theory, i.e., the doctrinal paradox of Kornhauser [16] and Kornhauser and Sager [18].[1] This is the problem of how a multi-judge court should decide a case when its members disagree on some of the legal issues that bear on the case according to the prevailing legal doctrine; by the latter is meant a systematized version of the existing jurisprudence and statutes. *Prima facie*, there are two plausible ways for the court to reach a decision by taking majority votes. One - the *issue-based* method - is to have the judges vote separately on each issue underlying the case, and then draw the logical consequences that the legal doctrine entails in view of these results. The other - the *case-based* method - consists in collecting the judges' votes on the case alone, regardless of how they assess the issues. For some patterns of opinions, the two methods deliver opposite results. In Kornhauser and Sager's view, this discrepancy constitutes a *paradox* because, for one, it comes as a surprise, and for another, it leads to a hard choice; indeed, either method can recommend itself on some normative grounds. Further, the paradox is *doctrinal*, because it jeopardizes the conformity to the legal doctrine at the collective level: if the court takes polls on both the issues and the case, its overall position clashes with the legal doctrine any time the discrepancy occurs.

Here is the didactic, by now celebrated, example by which Kornhauser and Sager illustrate their paradox. Suppose that there are three judges 1, 2 and 3, who should collectively decide on a case of breach of contract. The legal doctrine, which each judge acknowledges, states that the defendant owes a compensation to the plaintiff ($= c$) if and only if the contract between them is valid ($= a$) and the defendant broke it ($= b$). Judges follow the issue-based method if they vote on a and b, and then apply the biimplicative doctrine to conclude, and they follow the case-based method if they directly vote on c. Specifically, suppose that they entertain the following opinions on a, b, c (observe that each individually obeys the legal doctrine):

	a 'contract valid'	b 'contract broken'	c 'compensation due'	the legal doctrine
Judge 1	Y	Y	Y	Y
Judge 2	N	Y	N	Y
Judge 3	Y	N	N	Y
Court, issue-based	Y	Y	Y	Y
Court, case-based			N	

As the table shows, the plaintiff would be compensated on one method but not on the other. Since this and related examples circulated among legal theorists, they have been arguing about the merits of the two ways, sometimes taking one to be superior to the other, sometimes taking both to be questionable and seeking a third way of escape.[2]

[1] These works are the proper references for the doctrinal paradox. The widely cited paper by Kornhauser and Sager [17] is the first to raise the decision problems of a multi-judge court, but without yet formulating the paradox.

[2] See the survey of proposed solutions in Nash [22]. Kornhauser and Sager [18] themselves favour a

Following the lead of Pettit [30] and List and Pettit [20], the theory of logical aggregation has encapsulated the doctrinal paradox into a complex net of impossibility theorems, roughly paralleling the move in social choice theory from the Condorcet paradox to Arrow's theorem. However, this work had less effect on legal theory than the earlier one did on political science. Neither Kornhauser and Sager nor their followers have paid much attention to the impossibility theorems. Though it contributes to the same technical corpus, the present paper is motivated by the thought that this communication failure should, if possible, be remedied.

Logical aggregation theorists handle the doctrinal paradox in terms of a logical language in which they express not only the propositions relative to the issues and the case, but also the legal doctrine itself, and this may be a cause of dissatisfaction for the legal theorists, given the complexity and elusiveness of the latter concept. We will attend to this objection in our logical treatment of the doctrine, but we are primarily concerned with another twist, which consists in treating all propositions, including the doctrine, on a par. The logical aggregation theorists' basic move is to gather all relevant propositions into a single set of logical formulas, called the *agenda*, and then attribute to each individual and the aggregate a *judgment set*, which is made out of accepted agenda formulas. Within this framework, the distinction between the issues and the case vanishes. The logical properties of judgment sets become the sole focus of attention, and the doctrinal paradox boils down to the inconsistency in the judgment set resulting from majority votes:

$$\{a, b, c \leftrightarrow a \wedge b, \neg c\}.$$

Legal theorists could complain that the doctrinal paradox was defined more specifically. For them, it means the conflict between two methods of decision, or in logical terms, the contradiction between those two formulas - c and $\neg c$ - which the methods deliver to resolve the case, a definition lost in the new framework.

We address this concern by making a limited technical modification to the existing theory. We keep the judgment sets and investigate their logical properties, but subdivide the agenda X into *premisses* and *non-premisses*, a distinction that abstractly generalizes Kornhauser and Sager's between the issues and the case. We allow *any* non-empty subset P of X, including $P = X$, to count as a set of premisses, provided it contains the negation of each of its member formulas. For the doctrinal paradox agenda, given by

$$X = \{a, b, c, c \leftrightarrow a \wedge b, \neg a, \neg b, \neg(c \leftrightarrow a \wedge b)\},$$

legal theory would suggest taking as premisses a, b, and possibly $c \leftrightarrow a \wedge b$ if a vote is taken on the legal doctrine, plus the negations; but we would allow for many other choices of P.

Similarly, we abstractly generalize the distinction between issue-based and case-based methods. Taking majority voting only as a special case, we define the *premiss-based approach* by the twofold condition that the aggregative rule satisfies Independence (i.e., formula-wise aggregation) on P and Unanimity Preservation on P. As

third way, consisting for the court in voting on the method to be applied (the "metavote" solution).

to the *conclusion-based approach*, we define it by the single condition that the aggregative rule satisfies Unanimity Preservation on $X \backslash P$. In this way, the doctrinal paradox reduces to the following question: to what extent are the premiss-based and conclusion-based approaches mutually compatible? This amounts to asking to what extent Independence on P is compatible with Unanimity Preservation on the whole of X.

As defined here, the conclusion-based approach is sufficiently modest to be *prima facie* agreeable to the premiss-based approach. Having left this way of escape, if we eventually find that our axiomatic conditions clash, we will have extended the doctrinal paradox the more significantly. This is indeed the outcome of the paper, and it should strike legal theorists as relevant to their own inquiry. Technically, we will prove necessary and sufficient conditions on P and X for any aggregative rule satisfying restricted Independence and global Unanimity Preservation to be dictatorial on the premisses (Theorem 1) or, more strongly, on all formulas (Theorem 2); an oligarchic variant accompanies these impossibility theorems. The conditions on P and X, which amount to connecting the premisses logically, can be met in a number of different ways.

Earlier works in logical aggregation theory have examined premiss-based and conclusion-based aggregative rules. But their definitions of premisses are more restrictive the present one. They superimpose one or several of the following features of premisses: (i) premisses obey a definite logical pattern, typically logical independence; (ii) premisses are fully determining, i.e., if decisions are made for or against all premisses, decisions result for or against all non-premisses; (iii) only one formula (and its negation) are not premisses, but 'conclusions'.[3] Surprisingly, none of these features is required for our impossibility results. Some of the earlier writers have impossibility results too, but use independence unrestrictedly, with the exceptions of Dietrich [2] and Mongin [21], and also implicitly of Nehring [23].[4]

In the limiting case $P = X$, Theorems 1 and 2 (and their accompanying variants) reduce to known impossibility theorems derived under unrestricted independence, including results in the present symposium. See the detailed literature review in section 5. Our proof techniques build on that earlier work. The initial version of this paper also strengthened the existing theory by allowing for *infinite populations*, but since this is an orthogonal direction of generality, it eventually seemed best to pursue it elsewhere.[5]

[3]Premisses have been conceptualized variously. List and Pettit [20] and Dietrich [2] assume (i); Nehring and Puppe [26], (ii); Mongin [21] and Dokow and Holzman [11], (i) and (ii); Nehring and Puppe [27], (i) and (iii); and Nehring [23], all the three. Dietrich's [3] relevance-based definition may be the only one without any such restrictions.

[4]Dietrich and List [7] also restrict independence in their positive characterization of strategy-proofness.

[5]Results for infinite populations are to be found in sections 4 and 5 of the preprint version (Dietrich and Mongin [10]).

2 A general logic framework

A *logic* consists of a logical language, which defines the permissible formulas, and of a formal statement of logical links between these formulas. In logical aggregation theory, the propositions on which the individuals and society make judgments are represented by formulas, and the acceptance of formulas should respect the logical links. Since Dietrich [4], it has become clear that the theory does not need to be specific about either side of the logic, and importantly, that it is not limited to the propositional calculus. Recall that the language of the propositional calculus consists of propositional variables (which stand for the elementary propositions, like a, b, c in the doctrinal paradox example) and Boolean connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, which stand for "not", "and", "or", "if-then", "if-and-only-if"). The general logic of this paper abstracts from this choice of language, as well as from the specific definition of logical links in the propositional calculus.

The logical language $\mathcal{L}$. Its formulas, designated by $\varphi, \psi, \chi...$, are typically constructed from elementary formulas, to be designated by $a, b, c...$, and logical symbols, among which $\neg$ must be present. Formally, all we impose on $\mathcal{L}$ is that for all $\varphi \in \mathcal{L}$, $\neg\varphi \in \mathcal{L}$. There may or may not be in $\mathcal{L}$ symbols of further Boolean connectives and of non-Boolean operators. In the legal context, two relevant examples of the latter group are the unary operators of deontic logic - symbolically, O, which stands for "it is obligatory that", and its dual P, which stands for "it is permissible that". Formally, O and P take any formula φ in $\mathcal{L}$ into another formula in $\mathcal{L}$, $O\varphi$ resp. $P\varphi$. Unlike $\neg$, O and P are non-Boolean, or equivalently, *not truth-functional*; that is to say, knowing the truth-value of φ is sufficient to fix the truth-value of $\neg\varphi$, but not that of $O\varphi$ or $P\varphi$ (think of "it is obligatory to pay one's taxes" as an example of this).

Another relevant, this time binary, non-Boolean operator is the implication of conditional logic, symbolically $\varphi \hookrightarrow \psi$. The difference with the so-called *material* or *classical* implication $\rightarrow$ is again that $\rightarrow$ is truth-functional but $\hookrightarrow$ is not, i.e., the truth-values of φ and ψ determine that of $\varphi \rightarrow \psi$ but not that of $\varphi \hookrightarrow \psi$. There are alternative ways to define $\hookrightarrow$, which are explored in conditional logic but not reviewed here. Arguably, the legal doctrine of Kornhauser and Sager should be paraphrased as "the defendant would owe a compensation to the plaintiff if and only if the contract between them were valid and the defendant had broken it", or "as a matter of legal obligation, the defendant owes a compensation to the plaintiff if and only if the contract between them is valid and the defendant broke it", both analyses leading again into the non-classical realm.[6]

The logical links within $\mathcal{L}$. They can be stated in two ways, either by defining an entailment relation $\vdash$, or by defining a set $\mathcal{I}$ of inconsistent subsets of $\mathcal{L}$. Since both notions are technically useful, it is immaterial which is chosen as a primitive, provided that the two are interdefinable. Section 7 makes the two axiomatic exercises in turn, expanding on Dietrich's [4] analysis. At this stage, we take inconsistency as being the primitive. Any subset $\mathcal{I} \subseteq 2^{\mathcal{L}}$ satisfying the conditions (I1)-(I5) in section

[6]The non-classical biimplication is of the *subjunctive* (or *counterfactual*) type in the first paraphrase, and of the *strict* (or *strong*) type in the second. See Lewis [33]. For an application to judgment aggregation, see Dietrich [5].

7 can be chosen to represent the set of inconsistent subsets. Then, derivatively, a set $S \subseteq \mathcal{L}$ *entails* $\varphi \in \mathcal{L}$ - written $S \vdash \varphi$ - if $S \cup \{\neg\varphi\} \in \mathcal{I}$; and φ is a *logical truth* if $\{\neg\varphi\} \in \mathcal{I}$, a *contradiction* if $\{\varphi\} \in \mathcal{I}$, and a *contingent* formula if neither holds. The entailment relation automatically satisfies the conditions (E1)-(E6) of section 7. For later technical purposes, we define $S \subseteq \mathcal{L}$ to be *minimally inconsistent* if $S \in \mathcal{I}$ and for all $T \subsetneq S$, $T \notin \mathcal{I}$.

The agenda and the judgment sets. The agenda represents the propositions on which judgments are passed, at both the individual and collective level. Formally, it is any non-empty subset $X \subseteq \mathcal{L}$ that contains only contingent formulas and takes the form of a union of pairs $\{\varphi, \neg\varphi\}$, where φ does not begin with $\neg$. From now on, when we write "$\neg\psi$" with $\psi \in X$, we mean *the other element* of the pair to which ψ belongs. For any $S \subseteq X$, we put $S^{\pm} = \{\varphi, \neg\varphi : \varphi \in S\}$ and define S to be *negation-closed* if $S = S^{\pm}$. A *subagenda* of X is any non-empty and negation-closed subset $P \subseteq X$. A *judgment set* is a set $B \subseteq X$ of formulas representing the propositions accepted by an individual or the collectivity. It is *complete with respect to* $S \subseteq X$ if for all $\varphi \in S$, $\varphi \in B$ or $\neg\varphi \in B$, and *deductively closed with respect to* S if for all $\varphi \in S$, it follows from $B \vdash \varphi$ that $\varphi \in B$. When $S = X$, we just call B complete (respectively, deductively closed).

If a consistent judgment set is complete, then from the conditions of section 7, it is also deductively closed, while the converse does not hold. Accordingly, we define D to be the set of all judgment sets that are consistent and complete, and $D^* \supset D$ to be the set of all judgment sets that are consistent and deductively closed.

We consider a set of individuals N of any cardinality $|N| \geq 2$ (the group in question), and define a *social judgment function* as a mapping

$$F : D^N \to 2^X,$$

with $D^N \to D$ and $D^N \to D^*$ as particular cases. Here, D^N is the (unrestricted) domain of all possible *profiles* $(A_i)_{i \in N}$ of judgment sets across the group. Our main theorem assumes N to be finite, but most of our preparatory lemmas are stated without this restriction.

3 The axioms on social judgment functions

We represent the propositions singled out as premisses in terms of any fixed subagenda $P \subseteq X$, with the letters p, q, r,... to denote its elements. Now, consider the following two axioms:

Independence on P. For all $p \in P$ and all $(A_i)_{i \in N}, (A_i^*)_{i \in N} \in D^N$, if for all $i \in N$, $p \in A_i \Leftrightarrow p \in A_i^*$, then

$$p \in F((A_i)_{i \in N}) \Leftrightarrow p \in F((A_i^*)_{i \in N}).$$

In words, aggregation takes place formula by formula on P.

Unanimity Preservation. For all $\varphi \in X$ and all $(A_i)_{i \in N} \in D^N$, if for all $i \in N$, $\varphi \in A_i$, then $\varphi \in F((A_i)_{i \in N})$.

By definition, any F satisfying both Independence on P and Unanimity Preservation also restricted to P belongs to the premiss-based approach, and any F satisfying Unanimity Preservation restricted to $X \backslash P$ belongs to the conclusion-based approach. To illustrate the two approaches, consider *premiss-based majority voting* F_{PBM} and *conclusion-based majority voting* F_{CBM}, respectively. Given N of finite cardinality, for every profile $(A_i)_{i \in N} \in D^N$, we define $F_{PBM}((A_i)_{i \in N})$ by first taking $P_{maj} = \{p \in P : |\{i : p \in A_i\}| > |N|/2\}$, and then putting

$$F_{PBM}((A_i)_{i \in N}) = P_{maj} \cup \{\varphi \in X \backslash P : P_{maj} \vdash \varphi\}.$$

The judgment sets generated by F_{PBM} are in D^* if P has no minimal inconsistent subset Y with $|Y| \geq 3$ and $|N|$ is odd. As to F_{CBM}, given N of finite cardinality, it is defined by putting

$$P_{CBM}((A_i)_{i \in N}) = \{\varphi \in X \backslash P : |\{i : \varphi \in A_i\}| > |N|/2\}$$

for every profile $(A_i)_{i \in N} \in D^N$.

The premiss-based approach excludes *constant* social judgment functions, but allows for oligarchy and dictatorship. Define F to be an *oligarchy on* P if there is a non-empty $M \subseteq N$ - the *oligarchs on* P - such that, for all $(A_i)_{i \in N} \in D^N$, $F((A_i)_{i \in N}) \cap P = \cap_{i \in M}(A_i \cap P)$, and to be a *dictatorship on* P if F is an oligarchy on P with $M = \{i\}$ for some i, the *dictator on* P. *Dictatorship* and *oligarchy* simpliciter refer to the whole of X. Generally, any condition or rule that does not mention a set is meant to refer to X.[7]

Whether a social judgment function for finite N degenerates into dictatorship or oligarchy on P when satisfying Independence on P and Unanimity Preservation depends on how the premisses are logically connected, both with each other and with formulas in $X \backslash P$. We introduce the three conditions that section 4 demonstrates to be pivotal. The last one requires an auxiliary notion: for any $\varphi, \psi \in X$, φ *conditionally entails* ψ - symbolically, $\varphi \vdash^* \psi$ - if there is a (possibly empty) set $Y \subseteq X$ such that $Y \cup \{\varphi\} \vdash \psi$ and both $Y \cup \{\varphi\}$ and $Y \cup \{\neg\psi\}$ are consistent.[8] The shorthand $Y_{\neg Z}$ denotes $(Y \backslash Z) \cup \{\neg\varphi : \varphi \in Z\}$, i.e., the set obtained from Y by negating the formulas of one of its subsets Z.

CONDITIONS ON PREMISSES:

(a) There is a minimal inconsistent set $Y \subseteq X$ such that $|Y \cap P| \geq 3$.

(b) There is a minimal inconsistent set $Y \subseteq X$ such that $Y_{\neg Z}$ is consistent for some set $Z \subseteq Y \cap P$ of even cardinality.

(c) For all $p, q \in P$, there is a sequence $p_1, ..., p_k \in P$ $(k \geq 2)$ such that $p = p_1 \vdash^* p_2 \vdash^* ... \vdash^* p_k = q$.

[7] Oligarchy here obeys the definition of Dietrich and List [8] or Dokow and Holzman [13]. Gärdenfors [14], Nehring [24], Nehring and Puppe [26] and Dokow and Holzman [11] have different notions.

[8] Using that our logic is compact (see section 7), the conditional entailment $\varphi \vdash^* \neg\psi$ can equivalently be defined by the property that $\varphi \neq \neg\psi$ and that there exists a minimal inconsistent set $Y' \subseteq X$ containing both φ and $\neg\psi$.

In the statement of (b), 'of even cardinality' can be changed into 'of cardinality two'; the equivalence is shown in the proof of Lemma 3 in section 8.

These conditions parallel existing ones (as discussed in section 5), but unconventionally refer to P. The connections stated by (a), (b), (c) can either take place inside or outside P. At one extreme, there can be none in terms of P alone, except for the trivial ones between p and $\neg p$; then, the inconsistent sets Y of (a) and (b) necessarily contain non-premisses, and similarly with the sets $Y_1, ..., Y_k$ supporting conditional entailments in (c). Such is the case when the premisses are logically independent, and in particular when they are the literals of the propositional calculus (the *literals* are the propositional variables and their negations). For another example, take $X = \{a, a \to b, b\}^{\pm}$ and $P = \{a, a \to b\}^{\pm}$. Although the members of P are not logically independent, they typically become so if a non-classical $a \hookrightarrow b$ replaces the material $a \to b$. Indeed, most axiomatizations in conditional logic make the truth-values of a and $a \hookrightarrow b$ independent of each other; emphatically, the truth of the implication cannot be concluded anymore from the falsity of its antecedent. At the other extreme, all relevant interconnections trivially take place within P if $P = X$.

Though substantial, the list does not seem exaggeratedly demanding. Condition (a) is flexible. If, e.g., $P = \{a, b, c\}^{\pm}$, it can be met by taking X to contain one of $a \vee b \vee c$, $(a \to (b \to c))$, $(a \leftrightarrow (b \leftrightarrow c))$, or still other formulas. Condition (c) can be met with highly roundabout connections, and (b) may be the easiest to satisfy. If, e.g., $X = \{a, b, a \wedge b\}^{\pm}$ and $P = \{a, b\}^{\pm}$, (b) holds with $Y = \{a, b, \neg(a \wedge b)\}$ and $Z = \{a, b\}$, whereas (a) and (c) fail.

4 Main results

To get an idea of how logical interconnections narrow down the set of possible social judgment functions, observe how F_{PBM} can violate Unanimity Preservation. Take the doctrinal paradox agenda with $X = \{a, b, c \leftrightarrow a \wedge b, c\}^{\pm}$, and unconventionally, let $P = \{a, b, c\}^{\pm}$. If $(A_i)_{i \in N}$ is the profile of the introduction, all individuals accept the non-premiss $c \leftrightarrow a \wedge b$, and yet

$$F_{PBM}((A_i)_{i \in N}) = \{a, b, \neg c, \neg(c \leftrightarrow a \wedge b)\}.$$

The following theorem encompasses this and many other troubling examples. Its conclusions vary in strength with the requirement placed on collective judgment sets.

Theorem 1. Suppose that N is finite with $|N| \geq 3$. Then, if (a), (b) and (c) hold, every social judgment function $F : D^N \to D^*$ (resp. D) that is independent on P and unanimity-preserving is an oligarchy (resp. a dictatorship) on P, and the converse implication also holds.

As the proof makes clear, the direct statement also holds for $|N| = 2$, but the counterexamples to establish the converse take at least $|N| = 3$. This proof is subdivided into nine lemmas reported in this section and proved in section 8. Some make use of two further properties of social judgment functions that need now introducing.

Systematicity on P. For all $p, p^* \in P$ and all $(A_i)_{i\in N}, (A_i^*)_{i\in N} \in D^N$, if for all $i \in N$, $p \in A_i \Leftrightarrow p^* \in A_i^*$, then

$$p \in F((A_i)_{i\in N}) \Leftrightarrow p^* \in F((A_i^*)_{i\in N}).$$

Monotonicity on P. For all $p \in P$ and all $(A_i)_{i\in N}, (A_i^*)_{i\in N} \in D^N$, if for all $i \in N$, $p \in A_i \Rightarrow p \in A_i^*$, and for some $j \in N$, $p \notin A_j$ and $p \in A_j^*$, then

$$p \in F((A_i)_{i\in N}) \Rightarrow p \in F((A_i^*)_{i\in N}).$$

By itself, the proof of Theorem 1 brings to light three more results holding under the previous cardinality restrictions on N. These are variants, not corollaries of the theorem, because they involve both weaker assumptions and weaker conclusions. Briefly put, Systematicity on P makes it possible to dispense with condition (c), and Monotonicity on P with condition (b). Technically:

(i) If and only if (a) and (b) hold, every social judgment function $F : D^N \to D^*$ (resp. D) that is systematic on P and unanimity-preserving is an oligarchy (resp. a dictatorship) on P;

(ii) If and only (a) holds, every social judgment function $F : D^N \to D$ (resp. D^*) that is systematic on P, monotonic on P, and unanimity-preserving, is an oligarchy (resp. a dictatorship) on P;

(iii) If and only if (a) and (c) hold, every social judgment function $F : D^N \to D^*$ (resp. D) that is independent on P, monotonic on P, and unanimity-preserving is an oligarchy (resp. a dictatorship) on P.

The lemmas use the classic set-theoretic language of filters and ultrafilters that logical aggregation theorists have borrowed from social choice theory. None of the involved notions require N to be finite, which is precisely the reason why they were first introduced. The finiteness assumption is made only in Lemma 5 to secure the familiar steps from filters to oligarchies, and ultrafilters to dictatorships.

For concreteness, we refer to subsets $C \subseteq N$ as *coalitions*. Now, a set of coalitions $\mathcal{D} \subseteq 2^N$ is *superset-closed* if for all coalitions C, C^*, $C \in \mathcal{D}$ and $C \subseteq C^* \subseteq N$ imply $C^* \in \mathcal{D}$; *intersection-closed* if for all coalitions C, C^*, $C, C^* \in \mathcal{D} \Rightarrow C \cap C^* \in \mathcal{D}$; *complete* if for all coalitions C, $C \notin \mathcal{D} \Rightarrow N \backslash C \in \mathcal{D}$; a *filter* if $\mathcal{D}$ is superset-closed and intersection-closed, with $\emptyset \notin \mathcal{D}$.[9] Finally, an *ultrafilter* is a filter that is complete.[10]

For any social judgment function $F : D^N \to 2^X$, let us say that $\mathcal{D} \subseteq 2^N$ *generates F on* $p \in P$ if

$$(*) \;\; \forall (A_i)_{i\in N} \in D^N,\; p \in F((A_i)_{i\in N}) \Leftrightarrow \{i : p \in A_i\} \in \mathcal{D}.$$

We then denote $\mathcal{D}$ by $\mathcal{C}_p^F$. This functional notation makes sense because there can be at most one such $\mathcal{D}$. If moreover the same $\mathcal{D}$ generates F on every $p \in P$, we

[9] Filters are sometimes defined without requiring $\emptyset \notin \mathcal{D}$, in which case our filters become the *proper* filters. See Chang and Keisler [1, p. 164].

[10] Or, equivalently, a filter that is maximal for set inclusion.

say that $\mathcal{D}$ *generates* F *on* P and denote it by $\mathcal{C}^F$. For concreteness again, we call the members of $\mathcal{C}^F$ *(F-)winning coalitions (on P)*. When there is no ambiguity, we may drop reference to F, writing $\mathcal{C}_p$ for $\mathcal{C}_p^F$ and $\mathcal{C}$ for $\mathcal{C}^F$. To illustrate, suppose that F is an oligarchy on P with a set of oligarchs $M \subseteq N$; then, F is generated by $\mathcal{C}^F = \{C \subseteq N : C \subseteq M\}$.

Lemma 1. A social judgment function $F : D^N \to 2^X$ is (i) independent on P if and only if for every $p \in P$, there is $\mathcal{C}_p^F \subseteq 2^N$ generating F on p, and (ii) systematic on P if and only if there is $\mathcal{C}^F \subseteq 2^N$ generating F on P.

An example of a social judgment function that is independent on P, but not systematic on P, is the constant rule $F((A_i)_{i\in N}) = \overline{A}$, for any fixed $\overline{A} \subseteq X$. Here, $\mathcal{C}_p^F = 2^N$ if $p \in \overline{A}$ and $\mathcal{C}_p^F = \emptyset$ if $p \notin \overline{A}$.

Lemma 2. Let a social judgment function $F : D^N \to D^*$ be independent on P. Then, for all $p \in P$ and all $C \subseteq N$, if $C \in \mathcal{C}_p^F$, then $N \backslash C \notin \mathcal{C}_{\neg p}^F$, and if moreover $F : D^N \to D$, the converse implication holds. Also, $N \in \mathcal{C}_p^F$ and $\emptyset \notin \mathcal{C}_p^F$ if F is unanimity-preserving.

Lemma 3. Assume (b). Then, if a social judgment function $F : D^N \to D^*$ is systematic on P and unanimity-preserving, $\mathcal{C}^F$ is superset-closed.

Lemma 4. Assume (a) and (b). Then, if a social judgment function $F : D^N \to D^*$ is systematic on P and unanimity-preserving, $\mathcal{C}^F$ is intersection-closed.

Lemma 5. Assume (a) and (b). Then, if a social judgment function $F : D^N \to D^*$ (resp. D) is systematic on P and unanimity-preserving, $\mathcal{C}^F$ is a filter (resp. ultrafilter), and for finite N, F is an oligarchy (resp. dictatorship) on P.

This implies one direction of Theorem 1 via a last lemma.

Lemma 6. If a social judgment function $F : D^N \to D^*$ is independent on P and unanimity-preserving, then for all $p, q \in P$, $p \vdash^* q \Rightarrow \mathcal{C}_p \subseteq \mathcal{C}_q$; and if (c) also holds, F is systematic on P.

As to the variants of Theorem 1, the sufficiency part in (i) is already proved at the stage of Lemma 5, while those in (ii) and (iii) follow from adapting Lemmas 4 and 5 slightly.[11]

The other direction of Theorem 1 follows from the next three lemmas, using the fact that an oligarchy (hence also a dictatorship) on P is necessarily generated on P by a filter $\mathcal{C}^F$.

[11] That F is monotonic on P should replace (b) in Lemmas 4 and 5. Inspection of the proofs shows that they carry through.

Lemma 7. If $|N| \geq 3$ and (a) is violated, a social judgment function $F : D^N \to D$ exists that is systematic (hence independent) on P and unanimity-preserving, and such that $\mathcal{C}^F$ is not a filter.

Lemma 8. If $|N| \geq 3$ and (b) is violated, a social judgment function $F : D^N \to D$ exists that is systematic (hence, independent) on P and unanimity-preserving, and such that $\mathcal{C}^F$ is not a filter.

Lemma 9. If (c) is violated, a social judgment function $F : D^N \to D$ exists that is independent on P and unanimity-preserving, and such that $\mathcal{C}_p^F$ is not the same for all $p \in P$.

5 Applications and connections with the literature

In this section, we consider special sets P for which the results of the previous section are simplified, and by this process obtain earlier theorems as corollaries; this leads us to discuss the extant literature. We also illustrate the premiss-based approach with examples of P that are relevant to legal theory and beyond.

First, suppose that $P = X$. Then, Independence on P becomes (standard) Independence; (c) becomes *total blockedness* (introduced by Nehring and Puppe [25], later adopted by Dokow and Holzman [12] and others); (b) becomes *even-number-negatability* (Dietrich [4]), which is equivalent to *non-affineness* for finite X (Dokow and Holzman [12]); finally, (a) can be dropped as it follows from total blockedness. As a corollary of Theorem 1, we obtain a by now classic result:

Corollary 1. Suppose N is finite and $|N| \geq 3$. If the agenda is even-number-negatable and totally blocked, every independent and unanimity-preserving social judgment function $F : D^N \to D^*$ (resp. D) is an oligarchy (resp. a dictatorship), and the converse implication also holds.

Concerning the dictatorial agenda, see Dokow and Holzman [12] for the full characterization, and Dietrich and List [6] for the sufficiency part. Concerning the oligarchic agenda, see the characterizations in Dietrich and List [8] and Dokow and Holzman [13]. The three variants of Theorem 1 based on Systematicity and/or Monotonicity imply corresponding variants of Corollary 1, which are also known; see Nehring and Puppe [25, 28] for the two Monotonicity-based dictatorship variants, which were the first to be discovered. Some of the results we recover were originally proved for a finite X, a restriction that we avoid by occasionally drawing on the compactness assumption of section 7.

By contrast, Theorem 1 has no direct bearing on those impossibility results which do not have Unanimity Preservation among their stated conditions.[12] Nor does it generalize oligarchy results based on other oligarchy notions, as referenced in fn. 7.

[12] As in List and Pettit [20], Dietrich [2, 4], Pauly and van Hees [29], van Hees [34], Dietrich and List [6, 9].

Theorem 1 provides representations of the social judgment function that are limited to P. When does the local dictatorship result translate into a global one? Here is the relevant condition:

(d) For all $S \subseteq P$ that are consistent and complete with respect to P, and all $\varphi \in X$, either $S \vdash \varphi$ or $S \vdash \neg\varphi$.

In view of the logical conditions of section 7, it is equivalent to require that for all $B \in D$,

$$B = \{\varphi \in X | B \cap P \vdash \varphi\}.$$

In words, any complete and consistent judgment set can be recovered from its premisses by entailment.[13] The following theorem gives necessary and sufficient conditions for global dictatorship.

Theorem 2. Suppose N is finite and $|N| \geq 3$. If (a), (b), (c) and (d) hold, every social judgment function $F : D^N \to D$ that is independent on P and unanimity-preserving is a dictatorship, and the converse implication also holds.

To derive the two directions of Theorem 2 from the corresponding ones in Theorem 1, we draw on two more lemmas.

Lemma 10. Assume (d) holds. If a social judgment function $F : D^N \to D$ is a dictatorship on P, it is a dictatorship.

Lemma 11. Let N be finite. If (d) is violated, a social judgment function $F : D^N \to D$ exists that is systematic (hence independent) on P and unanimity-preserving, but is not a dictatorship.

A *fully determining* set of premisses P, in the sense of (d), should be seen as exceptional, which reduces the impossibility flavour of Theorem 2 and its variants (they can be devised after those of Theorem 1). With the doctrinal paradox agenda $X = \{a, b, c \leftrightarrow a \wedge b, c\}^{\pm}$, (d) is met with $P = \{a, b, c \leftrightarrow a \wedge b\}^{\pm}$, but not with $P = \{a, b\}^{\pm}$. The latter choice of P is not implausible on legal grounds, because a court that would adopt it could adjust its position on the legal doctrine to the positions it takes on a, b, and c.[14] This facilitates compatibility with premiss-based majority voting, but at the cost of wrecking full determination. However, even the larger and more conventional P does not fully determine X if the material $\leftrightarrow$ gives way to a non-classical $\hookleftarrow\!\!\hookrightarrow$, as in our preferred reconstruction of the legal doctrine. (Take the set $S = \{\neg a, b, \neg(c \hookleftarrow\!\!\hookrightarrow a \wedge b)\}$; it entails neither c nor $\neg c$.)

For the sake of generality, we may consider a *partly determining* P, as defined by an existential variant (d') of (d) (i.e., with "some $S \subseteq P$" replacing "all $S \subseteq P$"). A *profile dictator* for some $(A_i)_{i \in N} \in D^N$ is an individual j such that $A_j =$

[13] Condition (d) or related ones are also considered by Dietrich [2], Nehring and Puppe [26, 27], and Dokow and Holzman [11].

[14] That judges revise their conception of the doctrine in view of the case is a possibility essential to the British and US common law.

$F((A_i)_{i\in N})$. Even for finite N, if (a)-(c) and (d') hold, F meeting the conditions need not be a dictatorship on the full domain D^N, though it is a profile dictatorship for all $(A_i)_{i\in N} \in D^N$ such that $A_j \cap P$ entails every non-premiss or its negation, where j is the dictator on P (who exists by Theorem 1). For instance, if $P = \{a, c, c \leftrightarrow a \wedge b\}^{\pm}$, a case where (d) fails but (d') holds, the individual j is a profile dictator for those $(A_i)_{i\in N} \in D^N$ with $A_j = \{a, c, c \leftrightarrow a \wedge b, ...\}$, but not for those $(A_i)_{i\in N} \in D^N$ with $A_j = \{\neg a, \neg c, c \leftrightarrow a \wedge b, ...\}$.

At the other extreme of the mainstream literature, Mongin [21] specializes in a very small set of premisses, defining P to be the set of literals of $X \subseteq \mathcal{L}$, where $\mathcal{L}$ is the language of standard propositional logic. Using properties of the inference rule of propositional logic, he shows that Independence on P, with P so defined, leads to dictatorship in the presence of Unanimity Preservation. This result can also be obtained from Theorem 2, because its stated agenda conditions entail the present conditions (a), (b), (c) and (d). Technically, Mongin assumes (a), a condition amounting to (b), given that even-number and binary negatability are equivalent (see section 3), (c), and *Closure under Propositional Variables*, which is essentially equivalent to (d) given his choice of logic. According to the last condition, P should contain the propositional variables $a \in \mathcal{L}$ that occur in any formula $\varphi \in X$.[15]

We end up this section with sets of premisses that are specially relevant to applications in legal theory and politics; they will also illustrate the possibility side of our results. Starting from a propositional conditional logic, take X and P such that $X \backslash P$ contains only literals; these will represent the decision- or policy-oriented propositions (such as "the defendant owes a compensation to the plaintiff", since this commits the court to implement an action, unlike "the contract was broken"). Suppose further that P contains only formulas of the following types:

1. Literals representing the factual and prescriptive reasons for or against the decisions.
2. Non-classical implications $p \hookrightarrow \varphi$, or negations thereof, with p belonging to type-1 premisses or the Boolean expressions built from them, and φ belonging to the non-premisses or the Boolean expressions built from them.

In many conditional logics, with *negated* type 2 premisses, nothing can be inferred on non-premisses, and thus some judgment sets will not state anything on the decisions to be taken. This already shows that (d) does not hold, hence from Theorem 2 that there exist non-dictatorial social judgment functions among those which are independent on P and unanimity-preserving. Condition (c) does not hold either, because - at least in various conditional logics - negated type-2 premisses entail no premisses other than negated type-2 premisses. Thus, from Theorem 1, there exist social judgment functions that are well-behaved even on P.

By contrast, conditions (a) and (b) can easily be met, so that the first two variants of Theorem 1 still apply. Suppose that $a, b, a \wedge \neg b \hookrightarrow \varphi \in P$, where $\{a, \neg b\}$ is consistent, and $\varphi \in X \backslash P$. Then, (a) and (b) hold with $Y = \{a, \neg b, a \wedge \neg b \hookrightarrow \varphi, \neg\varphi\}$ and $Z = \{a, \neg b\}$, so the first two variants of Theorem 1 predict that plausible rules

[15] By the propositional calculus, Closure under Propositional Variables entails (d), and if its definition is slightly weakend, with "occur" being replaced by "occur *essentially*", the two become equivalent (as an example, a does not essentially occur in $b \wedge (a \vee \neg a)$).

such as premiss-based majority voting will degenerate. As this argument shows, the doctrinal paradox is tenacious.

6 Conclusion

Starting afresh from Kornhauser and Sager's legal analysis, we have tried to remain more faithful to it than logical aggregation theory has usually been. Because they define the doctrinal paradox as a conflict between the issue-based and case-based methods of court decisions, the theory cannot claim to pursue it in the same sense unless it formalizes a related distinction. In responding to this problem, our definitions of the premiss-based and conclusion-based approaches led us to restate the doctrinal paradox as a tension between two axioms put on a social aggregation function, Independence on premisses and Unanimity Preservation on both premisses and non-premisses. Theorems 1 and 2, along with their variants, demonstrate that all functions satisfying the two axioms degenerate into dictatorships or oligarchies, either local or global, under moderately demanding conditions put on the agenda and the set of premisses. Viewed in this light, the doctrinal paradox appears to be a deep obstacle to the formation of collective judgments.

7 The general logic

This section states the general logic under which our results are proved.[16] Most actual logics, whether classical or not, satisfy the conditions below. The major exceptions are the recently developed non-monotonic logics, which mean to capture inductive, rather than deductive, reasoning, and are out of scope here, and the paraconsistent logics, which are deductive and call for an explanation below.

One way of capturing the general logic is by axiomatizing logical inconsistency, or more precisely, a set $\mathcal{I}$ of subsets $S \subseteq \mathcal{L}$ that are intended to represent the inconsistent sets of formulas:

(I1) $\emptyset \notin \mathcal{I}$ (non-triviality).

(I2) For all $\varphi \in \mathcal{L}$, $\{\varphi, \neg\varphi\} \in \mathcal{I}$ (reflexivity).

(I3) For all $S \subseteq \mathcal{L}$ and all $\varphi \in \mathcal{L}$, if $S \notin \mathcal{I}$, either $S \cup \{\varphi\} \notin \mathcal{I}$ or $S \cup \{\neg\varphi\} \notin \mathcal{I}$ (one-step completability).

(I4) For all $S \subseteq S' \subseteq \mathcal{L}$, if $S \in \mathcal{I}$, then $S' \in \mathcal{I}$ (monotonicity).

(I5) For all $S \subseteq \mathcal{L}$, if $S \in \mathcal{I}$, there is a finite $S_0 \subseteq S$ such that $S_0 \in \mathcal{I}$ (compactness).

Given (I4) and (I5), (I3) implies the following stronger property.

(I3$^+$) For all $S \subseteq \mathcal{L}$, if $S \notin \mathcal{I}$, there is $T \subseteq \mathcal{L}$ such that $S \subseteq T$, $T \notin \mathcal{I}$, and T contains a member of each pair $\varphi, \neg\varphi \in \mathcal{L}$ (completability).

[16] The general logic of this section is equivalent to Dietrich's [4] one, but improves on its axiomatization by formulating and making extensive use of the new condition of one-step completability - i.e., (I3) and (E3) below.

Proposition 1 *If (I3)-(I5) hold, so does (I3$^+$).*

Proof. Take a consistent set $S \subseteq \mathcal{L}$. Let $\mathcal{T}$ be the set of all consistent sets $T \subseteq \mathcal{L}$ which include S; it is partially ordered by set-inclusion $\subseteq$.

Claim 1. Every chain $\mathcal{T}^* \subseteq \mathcal{T}$ (i.e., every linearly ordered subset of $\mathcal{T}$) has an upper bound in $\mathcal{T}$.

Let $\mathcal{T}^* \subseteq \mathcal{T}$ be a chain. We show that $\cup_{T \in \mathcal{T}^*} T$ is an upper bound of $\mathcal{T}^*$ in $\mathcal{T}$ by establishing that this set is consistent. If not, by compactness, it has a finite inconsistent subset U. By finiteness, $U \subseteq T$ for some $T \in \mathcal{T}^*$, which contradicts monotonicity.

From Claim 1 and Zorn's Lemma, $\mathcal{T}$ has a maximal element; call it T.

Claim 2. T contains a member of each pair $\varphi, \neg\varphi \in \mathcal{L}$.

If not, there is $\varphi \in \mathcal{L}$ such that $\varphi, \neg\varphi \notin T$. Then, by one-step completability, either $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ is consistent. This contradicts the maximality of T. ■

This proof follows the style of existing ones to establish Lindenbaum's Lemma, a basic result in logic (see, e.g., Chiang and Keisler [1, p. 10]).

With inconsistency defined as the primitive notion, logical entailment becomes a derivative one. Formally, entailment is a binary relation $\vdash$ holding between sets $S \subseteq \mathcal{L}$ and formulas $\varphi \in \mathcal{L}$ and given by:

(*) $S \vdash \varphi$ if and only if $S \cup \{\neg\varphi\} \in \mathcal{I}$.

In the sequel, '$\psi \vdash \varphi$' is short for '$\{\psi\} \vdash \varphi$'.

By analogy with (I1)-(I6), the following conditions can be devised on logical entailment:

(E1) There is no $\varphi \in \mathcal{L}$ such that $\emptyset \vdash \varphi$ and $\emptyset \vdash \neg\varphi$ (non-triviality).

(E2) For all $\varphi \in \mathcal{L}$, $\varphi \vdash \varphi$ (reflexivity).

(E3) For all $S \subseteq \mathcal{L}$ and all $\varphi, \psi \in \mathcal{L}$, if $S \nvdash \psi$, then $S \cup \{\varphi\} \nvdash \psi$ or $S \cup \{\neg\varphi\} \nvdash \psi$ (one-step completability).

(E4) For all $S \subseteq S' \subseteq \mathcal{L}$ and all $\varphi \in \mathcal{L}$, if $S \vdash \varphi$, then $S' \vdash \varphi$ (monotonicity).

(E5) For all $S \subseteq \mathcal{L}$ and all $\varphi \in \mathcal{L}$, if $S \vdash \varphi$, there is a finite subset $S_0 \subseteq S$ such that $S_0 \vdash \varphi$ (compactness).

Here is another condition on $\vdash$, which has no previous analogue on the side of $\mathcal{I}$:

(E6) For all $S \subseteq \mathcal{L}$, if there is a $\varphi \in \mathcal{L}$ such that $S \vdash \varphi$ and $S \vdash \neg\varphi$, then for all $\varphi \in \mathcal{L}$, $S \vdash \varphi$ (non-paraconsistency).

As a matter of definition, the *paraconsistent* logics are those which do not satisfy (E6). What they accomplish in effect is to weaken the ordinary notion of a contradiction; this is further explained below.

Proposition 2 *If $\mathcal{I}$ satisfies (I1)-(I5) and $\vdash$ is defined by (*), then $\vdash$ satisfies (E1)-(E6).*

Sketch of the proof. (E1) derives from (I1), (I3) and (*); (E2) from (L2) and (*); (E3) from (I3) and (*); (E4) from (I4) and (*); and (E5) from (I4), (I5) and (*). As to (E6), suppose that there is ψ such that $S \vdash \psi$ and $S \vdash \neg\psi$. From (*), $S \cup \{\neg\psi\} \in \mathcal{I}$ and $S \cup \{\neg\neg\psi\} \in \mathcal{I}$, so that $S \in \mathcal{I}$ by (I3). For any φ, (I4) implies that $S \cup \{\neg\varphi\}$ $\in \mathcal{I}$, hence that $S \vdash \varphi$ by another application of (*). ■

To present framework is reversible, i.e., one can take $\vdash$ to be the primitive notion, and $\mathcal{I}$ to be the derivative one. *Prima facie*, there are (at least) two plausible ways to define $\mathcal{I}$ in terms of $\vdash$:

(**) $S \in \mathcal{I}$ if and only if for all $\varphi \in \mathcal{L}$, $S \vdash \varphi$,

(***) $S \in \mathcal{I}$ if and only if there is $\varphi \in \mathcal{L}$ such that $S \vdash \varphi$ and $S \vdash \neg\varphi$,

and either way has its problems. The former may be too weak in the right to left direction of the proposed equivalence, whereas the latter may be too weak in the left to right direction; this suggests that neither might be sufficiently assertive. We by-pass this problem by assuming (E6), i.e., non-paraconsistency, which ensures that (**) and (***) can be used interchangeably to characterize $\mathcal{I}$.

Proposition 3 *If $\vdash$ satisfies (E1)-(E6) and $\mathcal{I}$ is defined by (**) or, equivalently, (***), then $\mathcal{I}$ satisfies (I1)-(I5).*

Sketch of the proof. (I1) derives from (E1) and (***); (I2) from (E2), (E4) and (***); (I3) from (E3) and (**); (I4) and (I5) from (E4) and (E5), respectively, and either (**) or (***). By (E6), both definitions can be used. ■

Moreover, uniqueness holds in the following sense: the inconsistent sets constructed from an entailment relation lead back to the same relation, and the other way round. This is what makes the order of priority indifferent. Formally:

Proposition 4 *(a) If $\vdash$ is defined from $\mathcal{I}$ in the first place, with $\mathcal{I}$ satisfying (I3) and (I4), and $\mathcal{I}'$ is defined from the obtained $\vdash$, then $\mathcal{I} = \mathcal{I}'$.*

(b) If $\mathcal{I}$ is defined from $\vdash$ in the first place, with $\vdash$ satisfying (E2), (E3), (E4) and (E6), and $\vdash'$ is defined from the obtained $\mathcal{I}$, then $\vdash = \vdash'$.

Proof. (a) Starting from $\mathcal{I}$ satisfying (I3) and (I4), we define $\vdash$ by (*). To define $\mathcal{I}'$, we can choose either (**) or (***), given that (E6) obtains on $\vdash$, and we take the former. Now, suppose $S \in \mathcal{I}$. Then, (I4) entails that for all φ, $S \cup \{\neg\varphi\} \in \mathcal{I}$, and (*) that for all φ, $S \vdash \varphi$, which leads to $S \in \mathcal{I}'$ by (**). Conversely, suppose $S \in \mathcal{I}'$ and apply (**). Then, for all φ, $S \vdash \varphi$ and $S \vdash \neg\varphi$, whence by (*) for all φ, $S \cup \{\neg\varphi\} \in \mathcal{I}$ and $S \cup \{\neg\neg\varphi\} \in \mathcal{I}$, and $S \in \mathcal{I}$ follows by (I3).

(b) In the other direction, we start from $\vdash$ satisfying (E2), (E3), (E4) and (E6), and having the choice between (**) or (***) to define $\mathcal{I}$, we take the former. We define $\vdash'$ by (*). Now, suppose $S \vdash \varphi$; then, from (E4), $S \cup \{\neg\varphi\} \vdash \varphi$, and since (E2) and (E4) entail that $S \cup \{\neg\varphi\} \vdash \neg\varphi$, (E6) can be invoked to get $S \cup \{\neg\varphi\} \in \mathcal{I}$, hence $S \vdash' \varphi$ from (*). Conversely, suppose $S \vdash' \varphi$ and apply (*) to get $S \cup \{\neg\varphi\} \in \mathcal{I}$. By (**), $S \cup \{\neg\varphi\} \vdash \varphi$. Since $S \cup \{\varphi\} \vdash \varphi$ holds because of (E2) and (E4), $S \vdash \varphi$ holds because of (E3). ■

Carefully note the role of (E6) in the second half of this proof. By assuming it, we reject the paraconsistent claim that a set - in this paper, a belief set - can be *weakly* inconsistent, in the sense of entailing one contradiction, without at the same time being *strongly* inconsistent, in the sense of entailing any contradiction (see Priest et al. [32] and Priest [31]). The very existence of paraconsistent logics is proof that this rejection is necessary, as well as sufficient, to obtain a fully reversible framework. Indeed, starting from a paraconsistent $\vdash$, definitions (**) and (***) (or plausible alternatives) lead to one's violating the recovery equation $\vdash = \vdash'$ of Proposition 4.

The proofs given in the next section tacitly assume either of the two systems (I1)-(I5), plus (*), or (E1)-(E6), plus (**) or (***), also tacitly exploiting their proven equivalence. An exception to this rule of silence is made for (I5) and (E5), i.e., compactness, which the present groundwork has shown to be detachable from the remaining conditions. Its major role is to prove Proposition 1, which we will apply without saying when we extend consistent sets of formulas in X to judgment sets in the D^*- or D-sense, but it also occurs in Lemmas 3, 7, 8 and 9, and these applications will be mentioned.[17]

8 Proof of the lemmas

Notation. When a profile $(A_i)_{i\in N}$ is given, we often write A instead of $F((A_i)_{i\in N})$, and for $p \in P$, N_p instead of $\{i \in N : p \in A_i\}$. For $Z \subseteq X$, we denote $\{\neg p : p \in Z\}$by $\neg Z$.

Proof of Lemma 1. Associate with the social judgment function $F : D^N \to 2^X$ and possibly with $p \in P$ the following sets of coalitions:

$$\mathcal{D}_p = \left\{C \subseteq N : \exists (A_i)_{i\in N} \in D^N, \{i : p \in A_i\} = C \;\; \& \;\; p \in F((A_i)_{i\in N})\right\},$$
$$\overline{\mathcal{D}}_p = \left\{C \subseteq N : \forall (A_i)_{i\in N} \in D^N, \{i : p \in A_i\} = C \Rightarrow p \in F((A_i)_{i\in N})\right\},$$

and

$$\mathcal{D} = \left\{C \subseteq N : \forall p \in P, \forall (A_i)_{i\in N} \in D^N, \{i : p \in A_i\} = C \Rightarrow p \in F((A_i)_{i\in N})\right\}.$$

Clearly, $\mathcal{D} \subseteq \overline{\mathcal{D}}_p \subseteq \mathcal{D}_p$. It is easy to see that F is independent on P if and only if $\mathcal{D}_p = \overline{\mathcal{D}}_p$. Now, if this equality holds, $\overline{\mathcal{D}}_p$ satisfies condition $(*)$ of the text. Hence, if F is independent on P, there exists $\mathcal{C}_p$ generating F on p. The converse implication is trivial. By the same token, F is systematic on P if and only if $\mathcal{D} = \mathcal{D}_p$ for all $p \in P$. But if these equalities hold, condition $(*)$ holds for the same set regardless of p. Hence, if F is systematic on P, there exists $\mathcal{C}^F$ generating F on P. The converse is trivial. ■

Proof of Lemma 2. Left to the reader.

Proof of Lemma 3. Let (b) hold. We first derive a consequence of (b), and then proceed to the proof itself.

[17] Compactness has the effect of excluding some otherwise perfectly well-behaved logics, such as the probabilistic logics (see Heifetz and Mongin [15]) and some, but not all, of the logics of common belief and common knowledge (for a discussion, see Lismont and Mongin [19]).

Claim. There is a minimal inconsistent set $Y \subseteq X$ and distinct premises $p, q \in Y \cap P$ such that $Y_{\neg\{p,q\}}$ is consistent.

By (b), there is at least one pair (Y, Z) of a minimal inconsistent set $Y \subseteq X$ and an even-sized set of premisses $Z \subseteq Y \cap P$ such that $Y_{\neg Z}$ is consistent. Among these pairs, choose one (Y, Z) such that Z has *smallest* size. If $|Z| = 2$, we are done. Thus, we assume that $|Z| > 2$.

Choose any distinct $p, q \in Z$. By the minimality condition in the choice of Y and Z, the set $Y_{\neg\{p,q\}}$ is inconsistent. In fact, it is *minimal* inconsistent, by the following argument (made in a different framework by Dokow and Holzman [12]). There is a minimal inconsistent subset $W \subseteq Y_{\neg\{p,q\}}$ by compactness, and we want to show that $W = Y_{\neg\{p,q\}}$. Both $\neg p$ and $\neg q$ are in W as otherwise W would be included in the consistent sets $Y_{\neg\{q\}}$ and $Y_{\neg\{p\}}$. The set $W_{\neg\{\neg p,\neg q\}}$ is inconsistent by the minimality condition on the choice of Y and Z. As this inconsistent set $(= \{p, q, ...\})$ is included in the minimal inconsistent set Y, it follows that $Y = W_{\neg\{\neg p,\neg q\}}$, hence that $Y_{\neg\{p,q\}} = W$, as desired.

Now, in the minimal inconsistent set $Y_{\neg\{p,q\}}$, negate the members of $Z \backslash \{p, q\}$; this leads to $Y_{\neg Z}$, a consistent set by the choice of Y, Z. But this contradicts the minimality condition in this choice, because $Z \backslash \{p, q\} \subseteq Y_{\neg\{p,q\}} \cap P$ is even-sized and has smaller cardinality than Z.

Main proof. Let F be as specified. Let Y, p, q be as in the above claim. We consider any $C \subseteq C^* \subseteq N$ with $C \in \mathcal{C}$ and show that $C^* \in \mathcal{C}$. Now, by Lemma 1, it is enough for the proof to select a particular $(A_i)_{i \in N} \in D^N$ and a particular $r \in P$ s.t.

$$C = N_r \text{ and } r \in A.$$

We extend the consistent sets $Y_{\neg\{p\}}$, $Y_{\neg\{q\}}$ and $Y_{\neg\{p,q\}}$ to JS in D, resp. $A_{Y_{\neg\{p\}}}$, $A_{Y_{\neg\{q\}}}$ and $A_{Y_{\neg\{p,q\}}}$, and consider the profile $(A_i)_{i \in N} \in D^N$ defined by

$$A_i = \begin{cases} A_{Y_{\neg\{p\}}} & \text{if } i \in C \\ A_{Y_{\neg\{p,q\}}} & \text{if } i \in C^* \backslash C \\ A_{Y_{\neg\{q\}}} & \text{if } i \in N \backslash C^*. \end{cases}$$

Now, A contains q since $N_q = C \in \mathcal{C}$, and all $\varphi \in Y \backslash \{p, q\}$, since $N_\varphi = N \in \mathcal{C}$ by Lemma 2. So $Y \backslash \{p\} \subseteq A$. The inconsistency of Y ensures that $Y \backslash \{p\} \vdash \neg p$, whence $\neg p \in A$ by the assumption that $A \in D^*$. So $\{i : \neg p \in A_i\} \in \mathcal{C}$, which implies that $C^* \in \mathcal{C}$ as desired. ■

Proof of Lemma 4. Assume (a) and (b). Take F as specified, the associated $\mathcal{C}$, and any $C, C^* \in \mathcal{C}$. Take $Y \subseteq X$ as in (a). There are at least three pairwise distinct formulas $p, q, r \in Y \cap P$, and the sets $Y_{\neg\{p\}}, Y_{\neg\{q\}}$ and $Y_{\neg\{r\}}$ are consistent by the minimal inconsistency of Y. Hence, there is $(A_i)_{i \in N} \in D^N$ as follows:

- for all $i \in C \cap C^*$, A_i extends $Y_{\neg\{p\}}$,
- for all $i \in C^* \backslash C$, A_i extends $Y_{\neg\{r\}}$,
- for all $i \in N \backslash C^*$, A_i extends $Y_{\neg\{q\}}$.

Unanimity Preservation ensures that $Y \backslash \{p, q, r\} \subseteq A$. Further, $q \in A$ because $N_q = (C \cap C^*) \cup (C^* \backslash C) = C^* \in \mathcal{C}$, and $r \in A$ because $N_r = (C \cap C^*) \cup (N \backslash C^*) \supseteq$

$C \in \mathcal{C}$ and $\mathcal{C}$ is superset-closed by Lemma 3. Thus, $Y \backslash \{p\} \subseteq A$, and $\neg p \in A$ since Y is inconsistent and $A \in D^*$. This is sufficient for the conclusion that $C \cap C^* \in \mathcal{C}$, as was to be proved. ■

Proof of Lemma 5. Assume (a) and (b). Take F as specified and the associated $\mathcal{C}$. From Lemma 2, $\mathcal{C}$ does not contain $\emptyset$, and from Lemmas 3 and 4, $\mathcal{C}$ is superset- and intersection-closed. Hence $\mathcal{C}$ is a filter. If moreover $F : D^N \rightarrow D$, Lemma 2 implies the stronger conclusion that $\mathcal{C}$ is an ultrafilter. As is well-known, if N is finite, every filter is the set of supersets of some $M \subseteq N$, and every ultrafilter the set of supersets of $\{i\}$ for some $i \in N$; so that F is either an oligarchy or a dictatorship, respectively. ■

Proof of Lemma 6. For F as specified, consider $p, q \in P$ and the associated $\mathcal{C}_p, \mathcal{C}_q$. Take $p \vdash^* q$, and let $C \in \mathcal{C}_p$. By definition of $\vdash^*$, there is $Y \subseteq X$ s.t. $Y \cup \{p\}$ and $Y \cup \{\neg q\}$ are consistent, and $Y \cup \{p, \neg q\}$ is inconsistent. As the last claim implies, $Y \cup \{p, q\}$ and $Y \cup \{\neg p, \neg q\}$ are consistent, and there exists $(A_i)_{i \in N} \in D^N$ as follows:

- for all $i \in C$, A_i extends $Y \cup \{p, q\}$,
- for all $i \notin C$, A_i extends $Y \cup \{\neg p, \neg q\}$.

With this profile, $Y \subseteq A$ by Unanimity Preservation, and $p \in A$ because $\{i : p \in A_i\} \in \mathcal{C}_p$. So $q \in A$ since $A \in D^*$. By $N_q = C$ and $q \in A$, we have that $C \in \mathcal{C}_q$, as was to be proved.

Suppose now that (c) holds. Then, for all $p, q \in P$, the sequence of conditional entailments $p \vdash^* p_2, ..., p_{k-1} \vdash^* q$ made available by this condition leads to a corresponding sequence of inclusions $\mathcal{C}_p \subseteq \mathcal{C}_{p_2}, ..., \mathcal{C}_{p_{k-1}} \subseteq \mathcal{C}_q$, and then to $\mathcal{C}_p = \mathcal{C}_q$, so that by Lemma 1, F is systematic on P. ■

Proof of Lemma 7. Let $|N| \geq 3$, and let (a) be violated. Then there is an odd-sized coalition $M \subseteq N$ with $|M| \neq 1$. For any $(A_i)_{i \in N} \in D^N$, we define the set

$$B = (\cap_{i \in N} A_i) \cup \{p \in P : |\{i \in M : p \in A_i\}| > |M|/2\}.$$

(In words, B collects all formulas unanimously accepted and all premisses accepted by a majority within M.) We will show that B is consistent. If not, by compactness, B has a finite minimal inconsistent subset $Y \subseteq B$. As (a) does not hold, $|Y \cap P| \leq 2$, hence $|Y \cap P| = 2$ since X contains no contradictions. Say $Y \cap P = \{p, q\}$. Within M, a majority accepts p, and a majority accepts q. As two majorities must overlap, there is an $j \in M$ s.t. $\{p, q\} \subseteq A_j$. So $Y \cap P \subseteq A_j$. Hence, as also $Y \backslash P \subseteq A_j$, $Y \subseteq A_j$. So A_i is inconsistent, a contradiction.

Having just shown that B is consistent, B can be extended to a set in D; let $F((A_i)_{i \in N})$ be one such extension. Note that, as B is already complete w.r.t. P,

$$F((A_i)_{i \in N}) \cap P = B \cap P = \{p \in P : |\{i \in M : p \in A_i\}| > |M|/2\}.$$

So we have defined a social judgment function $F : D^N \rightarrow D$ that is generated on P by $\mathcal{C} = \{C \subseteq M : |C| > |M|/2\}$. Lemma 1 implies that F is systematic. Also, F is unanimity-preserving by construction. And $\mathcal{C}$ is not a filter, because $\mathcal{C}$ is not intersection-closed (take, e.g., two majorities of $\frac{|M|+1}{2}$ individuals that intersect on a singleton). ■

Proof of Lemma 8. Let $|N| \geq 3$; so N contains three distinct individuals, to be labelled 1,2,3. Let (b) be violated. For any $(A_i)_{i\in N} \in D^N$, we define $B = B_1 \cup B_2$, where

$$B_1 = A_1 \cap A_2 \cap A_3$$

and

$$B_2 = \{p \in P : p \text{ is in exactly one of } A_1, A_2, A_3\}.$$

We prove that B is consistent. Suppose not, then by compactness there is a finite minimal inconsistent subset $Y \subseteq B$. Define $Y^* = Y \cap P$ and $A_i^* = A_i \cap P$ for all $i \in N$. We have $Y^* \neq \emptyset$, as otherwise $Y \subseteq B_1 \subseteq A_1$, an impossibility since Y is inconsistent and A_1 is consistent. Now, Y^* can be expressed as the pairwise disjoint union of the following sets:

$$Z_0 = Y^* \cap A_1^* \cap A_2^* \cap A_3^*,\ Z_1 = Y^* \cap [A_1^* \backslash (A_2^* \cup A_3^*)]\,,$$
$$Z_2 = Y^* \cap [A_2^* \backslash (A_1^* \cup A_3^*)]\,,\ Z_3 = Y^* \cap [A_3^* \backslash (A_1^* \cup A_2^*)]\,.$$

There must exist two sets among Z_1, Z_2, Z_3, say w.l.g. Z_1, Z_2, such that $|Z_1 \cup Z_2|$ is even and $Z_1 \cup Z_2 \neq \emptyset$. (The first claim is simply combinatorial, and the second one follows by contradiction from the consistency of A_3, since $Z_1 \cup Z_2 = \emptyset$ leads to $Y^* \subseteq A_3$, hence to $Y \subseteq A_3$.) Put $Z = Z_1 \cup Z_2$. Since (b) does not hold, we will have derived a contradiction if we show that

$$Y_{\neg Z} = (Y \backslash Z) \cup \neg Z$$

is consistent. Now, $Y_{\neg Z}$ can be obtained as the union

$$Y = (Y \backslash Y^*) \cup (Y^* \backslash Z) \cup \neg Z,$$

where (i) $Y \backslash Y^* \subseteq B_1 \subseteq A_3$, (ii) $Y^* \backslash Z = Y^* \cap A_3^* \subseteq A_3^*$, and (iii) $\neg Z = \{\neg p : p \in Y^*, p \notin A_3^*\} \subseteq \{\neg p : p \in P, p \notin A_3^*\} = A_3^*$. The last equality holds as $A_3^* = A_3 \cap P$ contains exactly one member of each pair $p, \neg p \in P$. Putting (i), (ii) and (iii) together, we see that $Y_{\neg Z} \subseteq A_3$, hence that $Y_{\neg Z}$ is consistent, as we aimed at proving.

Having shown B to be consistent, we can extend B to a set in D; this set is our $F((A_i)_{i\in N})$. Note that, as B is already complete w.r.t. P,

$$F((A_i)_{i\in N}) \cap P = B \cap P = \{p \in P : |N_p| \text{ is odd}\}.$$

So the just-defined social judgment function $F : D^N \mapsto D$ is generated on P by

$$\mathcal{C}^F = \{C \subseteq N : |C \cap \{1,2,3\}| \text{ is odd}\}.$$

Hence, F is systematic on P from Lemma 1; it is also unanimity-preserving since $\cap_{i\in N} A_i \subseteq B_1 \subseteq F((A_i)_{i\in N})$. But $\mathcal{C}^F$ is not a filter, as it is not superset-closed. ■

Proof of Lemma 9. Let (c) be violated. As $|N| \geq 2$, N contains distinct individuals, to be labelled 1 and 2. For $p, q \in P$, define pRq if there is a sequence of conditional entailments from p to q as in the statement of (c). As (c) does not hold, there are $\bar{p}, \bar{q} \in P$ such that not $\bar{p}R\bar{q}$, and P can be partitioned into two non-empty sets

$$S_1 = \{p \in P : \bar{p}Rp\} \text{ and } S_2 = \{p \in P : \text{not } \bar{p}Rp\}.$$

Note that

$$p \nvdash^* q \text{ for all } p \in S_1 \text{ and all } q \in S_2. \tag{1}$$

We can further partition S_1 into the sets

$$S_{11} = \{p \in S_1 : \neg p \in S_1\} \text{ and } S_{12} = \{p \in S_1 : \neg p \in S_2\},$$

and similarly, S_2 into the sets

$$S_{21} = \{p \in S_2 : \neg p \in S_1\} \text{ and } S_{22} = \{p \in S_2 : \neg p \in S_2\}.$$

Now, consider any $(A_i)_{i \in N} \in D^N$. We first define $B \subseteq P$ as follows: for all $p \in P$,

$$p \in B \Leftrightarrow \begin{cases} p \in A_1 & \text{if } p \in S_{11} \\ p \in A_2 & \text{if } p \in S_{22} \\ p \in A_1 \cup A_2 & \text{if } p \in S_{12} \\ p \in A_1 \cap A_2 & \text{if } p \in S_{21}. \end{cases}$$

We set out to prove that $B \cup (A_1 \cap A_2)$ is a consistent set. Suppose not; then by compactness, there is a minimal inconsistent subset $Y \subseteq B \cup (A_1 \cap A_2)$. Hence,

$$p \vdash^* \neg q \text{ for all distinct } p, q \in Y. \tag{2}$$

We will prove six claims relative to $Y^* = Y \cap B$, leading eventually to a contradiction.

(i) $Y^* \nsubseteq S_{11} \cup S_{21}$. If not, the definition of B implies that $Y^* \subseteq A_1$, and $Y \subseteq A_1$, a consistent set.

(ii) $Y^* \nsubseteq S_{22} \cup S_{21}$ by a similar argument.

(iii) $Y^* \cap S_{12} \neq \emptyset$. If not, $Y^* \subseteq S_{11} \cup S_{22} \cup S_{21}$, and by (i) and (ii), there are $p, q \in Y^*$ with $p \in S_{11}$ and $q \in S_{22}$, hence also $\neg q \in S_{22}$. By (2), $p \vdash^* \neg q$, contradicting (1).

(iv) $Y^* \cap S_{12} = \{r\}$. If there were $r, s \in Y^* \cap S_{12}$, $r \neq s$, (2) would imply that $s \vdash^* \neg r$, in contradiction with (1).

(v) $Y^* \cap S_{11} = \emptyset$. If not, by (2) $p \vdash^* \neg r$, contradicting (1).

(vi) $Y^* \cap S_{22} = \emptyset$ by a similar argument.

From (iv), (v) and (vi), $Y^* \subseteq \{r\} \cup S_{21} \subseteq \{r\} \cup (A_1 \cap A_2)$, where the second inclusion follows from the definition of B. Since $Y \subseteq Y^* \cup (A_1 \cap A_2)$, it also holds that $Y \subseteq \{r\} \cup (A_1 \cap A_2)$. The definition of B implies that $r \in A_1$ or $r \in A_2$, whence either $Y \subseteq A_1$ or $Y \subseteq A_2$, a contradiction with the consistency of A_1 and A_2.

For all $(A_i)_{i \in N} \in D^N$, one can extend the consistent set $B \cup (A_1 \cap A_2)$ to one in D, so as to define a social judgment function $F : D^N \to D$. As B was already complete w.r.t. P, we have $F((A_i)_{i \in N}) \cap P = B \cap P$ for all $(A_i)_{i \in N} \in D^N$. It follows that, for every $p \in P$, F is generated on p by some $\mathcal{C}_p$, hence by Lemma 1 that F is independent on P. F is unanimity-preserving since $\cap_{i \in N} A_i \subseteq A_1 \cap A_2 \subseteq A$. Finally, $\mathcal{C}_p$ is not the same for all $p \in P$, because S_1 and S_2 are each non-empty, and if $p \in S_1$

then $\mathcal{C}_p$ is $\{C \subseteq N : 1 \in C\}$ or $2^N \backslash \{\emptyset\}$, whereas if $p \in S_2$ then $\mathcal{C}_p$ is $\{C \subseteq N : 2 \in C\}$ or $\{N\}$. ■

Proof of Lemma 10. Let N be finite. Suppose (d) holds and $F : D^N \to D$ is a dictatorship on P. Let j be the dictator on P. Let $G : D^N \to D$ be dictatorship on X by individual j. To show that $F = G$, we consider any $(A_i)_{i \in N} \in D^N$ and show that $F((A_i)_{i\in N}) = G((A_i)_{i\in N})$. As $F((A_i)_{i\in N})$ and $G((A_i)_{i\in N})$ and each in D, it suffices to show that $F((A_i)_{i\in N}) \subseteq G((A_i)_{i\in N})$. Consider any $\varphi \in F((A_i)_{i\in N})$. Let $S := F((A_i)_{i\in N}) \cap P$. By (d), either $S \vdash \varphi$ or $S \vdash \neg\varphi$. It cannot be that $S \vdash \neg\varphi$, since otherwise $F((A_i)_{i\in N})$ would contain $\neg\varphi$ by deductive closure, hence be inconsistent. So $S \vdash \varphi$. By definition of G, $G((A_i)_{i\in N}) \cap P = F((A_i)_{i\in N}) \cap P$, whence $G((A_i)_{i\in N}) \cap P = S$. So also $G((A_i)_{i\in N})$ entails φ. Hence, as $G((A_i)_{i\in N})$ is deductively closed, $\varphi \in G((A_i)_{i\in N})$, as desired. ■

Proof of Lemma 11. If (d) is violated, there is a set S that is complete w.r.t. P and s.t. for some $\varphi \in X \backslash P$, both $S \cup \{\varphi\}$ and $S \cup \{\neg\varphi\}$ are consistent. These two sets can be extended, so that there are $B, B' \in D$ with $B \cap P = B' \cap P$, but $B \neq B'$. Let 1 be any individual in N, and let $F : D^N \to D$ be defined by the condition that for all $(A_i)_{i\in N} \in D^N$,

$$A = \begin{cases} B & \text{if } A_1 = B' \text{ and } A_i = B \text{ for all } i \in N \backslash \{1\} \\ A_1 & \text{otherwise.} \end{cases}$$

This F is not dictatorial, and unanimity-preserving; and because 1 is a dictator on P it is also systematic, hence independent, on P. ■

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