The relation between degrees of belief and binary beliefs: A general impossibility theorem

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Abstract

Agents are often assumed to have degrees of belief ("credences") and also binary beliefs ("beliefs simpliciter"). How are these related to each other? A much-discussed answer asserts that it is rational to believe a proposition if and only if one has a high enough degree of belief in it. But this answer runs into the "lottery paradox": the set of believed propositions may violate the key rationality conditions of consistency and deductive closure. In earlier work, we showed that this problem generalizes: there exists no local function from degrees of belief to binary beliefs that satisfies some minimal conditions of rationality and non-triviality. "Locality" means that the binary belief in each proposition depends only on the degree of belief in that proposition, not on the degrees of belief in others. One might think that the impossibility can be avoided by dropping the assumption that binary beliefs are a function of degrees of belief. We prove that, even if we drop the "functionality" restriction, there still exists no local relation between degrees of belief and binary beliefs that satisfies some minimal conditions. Thus functionality is not the source of the impossibility; its source is the condition of locality. If there is any non-trivial relation between degrees of belief and binary beliefs at all, it must be a "holistic" one. We explore several concrete forms this "holistic" relation could take.

1 Introduction

We commonly take agents to have two kinds of belief: *degrees of belief* (also known as *credences*) and *binary beliefs* (also known as *beliefs* simpliciter). Degrees of belief take the form of subjective probability assignments to certain propositions, while binary beliefs take the form of the acceptance of some propositions and the non-acceptance of others. What is the relationship between these two kinds of belief? Can binary beliefs be "reduced" to degrees of belief? Or, even if no such reduction is possible, can we identify some other systematic relationship between the two? In particular, are there certain rationality constraints telling us what binary beliefs we should hold – or may permissibly hold – when we hold certain degrees of beliefs?[1]

[1]This question has recently received renewed attention. We omit a detailed literature review, but refer readers to contributions by, e.g., Hawthorne and Bovens (1999), Douven and Williamson (2006), Lin and Kelly (2012a,b), and Leitgeb (2014). For further references, see Dietrich and List (2018).

A widely studied answer to this question asserts that it is rational for an agent to believe a proposition in the binary sense if and only if he or she has a high enough degree of belief in it. "High enough" is a free parameter that can be specified in different ways. For example, we might consider a degree of belief of 0.9 or 0.95 "high enough" for full belief. Yet, this proposal – called a "threshold rule" – runs into some well-known difficulties. Suppose, for instance, that you believe of each lottery ticket among a million tickets that this particular ticket will not win, since your degree of belief in this proposition is extremely high, namely 0.999999. Your believed propositions will then entail that none of the tickets will win. But, of course, you know this to be false, and you have a degree of belief of 1 in the negation of this proposition: some ticket will win. This example shows that, under a threshold rule, binary beliefs may be neither deductively closed (some implications of believed propositions are not believed) nor consistent (some beliefs contradict others), a problem known as the "lottery paradox".[2]

In earlier work, we showed that the lottery paradox illustrates a more general impossibility theorem: there exists no function from degrees of belief to binary beliefs that satisfies certain minimal conditions of rationality and non-triviality.[3] A central such condition is *propositionwise independence* ("*locality*"): an agent's binary belief in each proposition should depend only on the agent's degree of belief in that proposition, not on the degrees of belief in others. Threshold rules satisfy propositionwise independence, but they are just one kind of example of propositionwise independent rules. The upshot of this theorem is that the lottery paradox is not restricted to threshold rules alone, but occurs for propositionwise independent belief-binarization rules much more generally. (We derived this result from an impossibility theorem in the different area of judgment-aggregation theory. This derivation was possible due to certain structural parallels between the lottery paradox and the paradoxes of social choice.)

A crucial presupposition of our earlier impossibility theorem, however, was that the relationship between degrees of belief and binary beliefs is a *functional* or *determinate* one: degrees of belief fully *determine* beliefs. Put slightly differently, in a rational agent, binary beliefs "supervene" on degrees of belief. The theorem leaves open whether the impossibility might go away if we lift this functionality constraint. Perhaps there are systematic constraints governing the relationship between degrees of belief and binary beliefs, but they do not take the form of a *function* from one to the other. One might think that the source of the lottery paradox and the more general impossibility result is the assumption that binary beliefs are fully determined by degrees of belief.

In this paper, we prove a more general impossibility theorem about the relationship between degrees of belief and binary beliefs. It shows that, even if we drop the constraint of functionality, there still exists no relation between degrees of belief and binary beliefs that satisfies certain conditions of rationality and non-triviality. Once again, a "locality" or "propositionwise independence" constraint is a central one of those conditions. The earlier theorem is a special case of our new theorem, applied to credence-belief relations that satisfy functionality. Our conclusion is that, contrary to what one might have expected, functionality is not the source of the impossibility;

[2]See Kyburg (1961). On the related "preface paradox", see Makinson (1965).

[3]See Dietrich and List (2018).

rather, its source is the condition of locality. If there is any non-trivial relation between degrees of belief and binary beliefs at all, it must be a "holistic" one.

2 The formalism

We begin with our basic formalism. Let Ω be some non-empty set of possible worlds. A *proposition* is a subset $p \subseteq \Omega$. As is standard, we assume that p consists of all those worlds at which the proposition is true. For each proposition p, we write $\overline{p}$ to denote its complement $\Omega \backslash p$ (*negation*). Moreover, for any two propositions p and q, the intersection $p \cap q$ represents their *conjunction*, and the union $p \cup q$ represents their *disjunction*. We further call any set S of propositions *consistent* if there is at least one world in Ω at which all the propositions in S are true, formally $\bigcap_{p \in S} p \neq \varnothing$; and we say that a set S of propositions *entails* another proposition q if q is true at every world in Ω at which all of the propositions in S are true, formally $\bigcap_{p \in S} p \subseteq q$. Let X be the set of propositions on which beliefs are held. For the moment, we assume that X is a *non-trivial algebra*, i.e., it is closed under both union and complement (i.e., for every $p, q \in X$, $p \cup q \in X$ and $\overline{p} \in X$), and it contains more than one contingent proposition-negation pair (where a proposition p is *contingent* if it is distinct from Ω and $\varnothing$).

Degrees of belief are formally represented by credence functions. A *credence function* is a function Cr that assigns to each proposition $p \in X$ a number $Cr(p)$ in the interval from 0 to 1, where this assignment is probabilistically coherent. When X is an algebra, as assumed so far, probabilistic coherence simply means that Cr is a well-defined probability function on X.[4]

Binary beliefs are formally represented by belief sets. A *belief set* is a subset $B \subseteq X$. It is

- *consistent* if it is a consistent set of propositions, as defined above;
- *deductively closed* if it contains every proposition in X that is entailed by B.

In our previous work, we studied belief-binarization rules. A *belief-binarization rule* is a function f that maps each credence function Cr on X (within some domain of admissible such functions) to a belief set $B = f(Cr)$. It thus specifies a relation by which degrees of belief *determine* binary beliefs. For example, under a *threshold rule* with threshold t, each credence function Cr is mapped to the belief set B defined as follows:

$$B = \{p \in X : Cr(p) \text{ exceeds } t\}.^5$$

The problem with threshold rules, as already noted, is that they do not generally produce consistent and deductively closed belief sets. Our earlier theorem showed that similar problems occur for a much broader class of "propositionwise independent"

[4] Formally, (i) $Cr(\Omega) = 1$, and (ii) for any $p, q \in X$ with $p \cap q = \varnothing$, $Cr(p \cup q) = Cr(p) + Cr(q)$. For a general X, which is not necessarily an algebra, probabilistic coherence of Cr means that Cr is extendable to a function with properties (i) and (ii) on an algebra including X.

[5] Here, "$Cr(p)$ exceeds t" could mean *either* $Cr(p) > t$ (in the case of a "strict" threshold) *or* $Cr(p) \geq t$ (in the case of a "weak" threshold).

belief-binarization rules. "Propositionwise independence" means that the binary belief in each proposition p depends only on the degree of belief in p, not on the degrees of belief in others; we give a formal definition later.

What we want to investigate is whether this picture changes if we give up the assumption of *functionality*, according to which binary beliefs are determined by degrees of belief via some function f. We will consider a less demanding relationship between degrees of belief and binary beliefs. Specifically, we will capture that relationship in terms of what we call a *credence-belief relation*. This is a binary relation, denoted $\sim$, between credence functions and belief sets which specifies which credence functions and belief sets are rationally co-tenable and which are not.

Formally, let us call any pair (Cr, B) consisting of a credence function and a belief set a *credence-belief pair*. The relation $\sim$ can now be defined as a set of such pairs. We write $Cr \sim B$ whenever (Cr, B) is contained in that set (technically, $(Cr, B) \in \sim$) and interpret $Cr \sim B$ to mean that the pair (Cr, B) is *coherent* or, equivalently, the belief set B is *co-tenable* with the credence function Cr. We can also interpret the pairs (Cr, B) that are deemed coherent by the relation $\sim$ as the "rationally permissible" credence-belief pairs. We further call a credence function Cr *rationally permissible* if it occurs in at least one coherent pair (Cr, B) (with $Cr \sim B$), and similarly, we call a belief set B *rationally permissible* if it occurs in at least one coherent pair (Cr, B).

The notion of a *credence-belief relation* generalizes the notion of a *belief-binarization rule*. A belief-binarization rule can be viewed as a special case of a credence-belief relation, where the following constraint is satisfied:

Functionality: No credence function Cr coheres with more than one belief set B.

Specifically, when this constraint is satisfied, the relation $\sim$ induces a belief-binarization rule f, defined as follows:

for each rationally permissible Cr, $f(Cr)$ is the belief set B such that $Cr \sim B$.

While a functional credence-belief relation is equivalent to belief-binarization rule, a general credence-belief relation is equivalent to a *belief-binarization correspondence*, i.e., a function F that assigns to each permissible credence function Cr the non-empty set $F(Cr)$ consisting of all belief sets B that cohere with Cr. In the special case of a functional relation, $F(Cr)$ is always singleton. Now our question is whether there exists a credence-belief relation that satisfies certain conditions of rationality and non-triviality.

3 A general impossibility result

We introduce five conditions on a credence-belief relation. The first condition says that the credence-belief relation should not rule out any possible credence functions from the outset. To understand this requirement, we should bear in mind that any credence function is by definition probabilistically coherent.

Universality: Every credence function is rationally permissible.

The second condition says that binary beliefs should always be consistent and deductively closed. This is a standard rationality requirement on beliefs, albeit a somewhat idealized one.

Belief consistency and deductive closure: Only consistent and deductively closed belief sets are rationally permissible.

Note that the demandingness of this requirement depends on how rich the algebra X of propositions is. If X is relatively small, then belief consistency and deductive closure may be relatively easily attainable. The requirement becomes more demanding as X gets larger.

To state the third condition, we need one further definition. We say that a credence function Cr *requires accepting proposition* p if p belongs to every belief set B that is coherent with Cr. Our third condition now says that whenever one assigns a credence of 1 ("certainty") to a particular proposition, then one is required to accept that proposition in the binary sense.

Propositionwise certainty preservation: For any rationally permissible credence function Cr and any proposition $p \in X$, if $Cr(p) = 1$, then Cr requires accepting p.

The fourth condition rules out "trivially loose" credence-belief relations, in which credences do not impose any non-trivial requirements on beliefs:

Non-looseness: There is at least one rationally permissible credence function Cr that requires accepting some proposition $p \in X$ with $Cr(p) < 1$.

While the first four conditions are relatively undemanding, the fifth is significantly more demanding. It imposes a "locality" constraint on the credence-belief relation, which rules out certain "holistic" relationships between degrees of belief and binary beliefs. Whether this is a plausible requirement is a debatable matter, as is clear from earlier work on this topic. In fact, we will later recommend relaxing locality.

Acceptance locality: Requirements to accept propositions depend only on the degrees of belief in those propositions themselves, not on the degrees of belief in others. Formally, for any rationally permissible credence functions Cr and Cr' and any proposition $p \in X$, if $Cr(p) = Cr'(p)$, then either both of Cr and Cr' require accepting p, or neither of them does so.

It is worth noting that this last condition also has a "dual", which concerns requirements to reject rather than to accept propositions. To state this "dual" condition, let us say that a credence function Cr *requires rejecting proposition* p if p belongs to no belief set B that is coherent with Cr.

Rejection locality: Requirements to reject propositions depend only on the degrees of belief in those propositions themselves, not on the degrees of belief in others. Formally, for any rationally permissible credence functions Cr and Cr' and any proposition $p \in X$, if $Cr(p) = Cr'(p)$, then either both of Cr and Cr' require rejecting p, or neither of them does so.

The conjunction of acceptance locality and rejection locality is what we call "full locality" or "propositionwise independence". Interestingly, our main impossibility theorem requires only the first of these two locality conditions.[6] Here is our result, proved in the appendix.

Theorem 1. Given any non-trivial algebra of propositions X, there exists no credence-belief relation satisfying universality, belief consistency and deductive closure, propositionwise certainty preservation, non-looseness, and acceptance locality.

In short, our five conditions are mutually inconsistent. Any credence-belief relation can satisfy at most four of them at once. Before turning to the question of which of the five conditions should be interpreted as the main "source" of the impossibility, and what possibilities open up if we relax some of them, we would like to look briefly at the special case of credence-belief relations satisfying functionality.

4 The special case of functionality

It should be evident that, insofar as Theorem 1 establishes the mutual inconsistency of its five conditions, the impossibility continues to hold if we add any further conditions, such as functionality. In particular, if the five existing conditions cannot be met by any credence-belief relation in general, then *a fortiori* they cannot be met by any credence-belief relation satisfying functionality. It is instructive to restate the conditions and our theorem in the functional case. As already noted, a functional credence-belief relation $\sim$ is equivalent to a belief-binarization rule f, defined as follows:

for each rationally permissible Cr, $f(Cr)$ is the belief set B such that $Cr \sim B$.

Given this translation, universality turns into the familiar universal domain condition:

Universal domain: The domain of admissible inputs to the belief-binarization rule consists of every possible credence function.

[6] In fact, just as acceptance locality has a dual in the form of rejection locality, so our non-looseness condition has a dual. As stated, it rules out relations which never require accepting a proposition except possibly when one's credence in it is 1. Such relations might be called *acceptance-loose*. The dual concept is that of a *rejection-loose* relation, which never requires rejecting a proposition except possibly when one's credence in it is 0. Fully loose relations, which are loose in both senses, are extremely non-functional: binary belief sets are totally unconstrained by credences (except possibly for propositions for which one's credence is 0 or 1). If a relation is just acceptance-loose or rejection-loose, it can still be functional. For instance, the functional relation under which each Cr coheres only with $B = \{p \in X : Cr(p) = 1\}$ is acceptance-loose (but it is far from rejection-loose, as almost all propositions must be rejected). Our theorem requires non-looseness only in the form of non-acceptance-looseness (just as it requires only acceptance locality).

The next two conditions remain essentially as before:

Belief consistency and deductive closure: The co-domain of admissible outputs of the belief-binarization rule consists of consistent and deductively closed belief sets.

Propositionwise certainty preservation: For any credence function Cr in the domin of f and any proposition $p \in X$, if $Cr(p) = 1$, then $p \in B$, where $B = f(Cr)$.

Non-looseness reduces to the requirement that belief simpliciter is not always restricted to propositions with credence one.

Non-triviality: There is at least one credence function Cr in the domain of f such that, for at least one proposition $p \in X$, we have $Cr(p) < 1$ and $p \in B$, where $B = f(Cr)$.

Finally, acceptance locality and rejection locality each become equivalent to the following "full" locality condition:

Propositionwise independence: The belief in each proposition depends only on the degree of belief in that proposition itself, not on the degrees of belief in others. Formally, for any credence functions Cr and Cr' in the domain of f and any proposition $p \in X$, if $Cr(p) = Cr'(p)$, then either both of B and B' contain p, or neither of them does so, where $B = f(Cr)$ and $B' = f(Cr')$.

Theorem 1 now immediately implies the mutual incompatibility of these five conditions:

Theorem 2. Given any non-trivial algebra of propositions X, there exists no belief-binarization rule satisfying universal domain, belief consistency and deductive closure, propositionwise certainty preservation, non-triviality, and propositionwise independence.

This theorem is equivalent to the second main theorem in our earlier work.[7] Instead of explicitly demanding non-triviality, we had stated the result as follows:

Theorem 2 (restated). Given any non-trivial algebra of propositions X, the only belief-binarization rule satisfying universal domain, belief consistency and deductive closure, propositionwise certainty preservation, and propositionwise independence is the threshold rule with threshold one, i.e., for every Cr, $f(Cr) = B$ where $B = \{p \in X : Cr(p) = 1\}$.

[7] See Dietrich and List (2018). Our earlier result uses a slightly different (namely, global rather than propositionwise) certainty preservation condition, which says that if the degrees of belief take only the extremal values 0 or 1 on all propositions in X, then these extremal degrees of belief should be preserved as the binary beliefs.

While one might have thought that this impossibility result is due to the restrictive nature of the belief-binarization framework – specifically, the requirement of functionality – our present analysis shows that the impossibility is much more general and occurs even when the functionality restriction is lifted.

5 Escape routes from the impossibility

Since our impossibility theorem has five conditions, there are in principle five logically possible escape routes: we could relax each of the five conditions. We will now briefly discuss all of these routes. The upshot, however, will be that there is only one genuinely compelling escape route from the impossibility, which is to relax locality.[8] We will illustrate that route by reviewing a number of possible credence-belief relations, including some that, as far as we know, have not previously been considered. Throughout this section, we assume that the algebra of propositions, X, is finite.

5.1 The first route: Relaxing universality

There seems little reason to rule out certain credence functions as rationally impermissible from the outset. Yet it is worth noting that if the rationally permissible credence functions were suitably restricted, then this would open up an escape route from the impossibility.

Suppose, for instance, we would like the relationship between degrees of belief and binary beliefs to be captured by a treshold rule with threshold t, i.e.,

$$B = \{p \in X : Cr(p) \text{ exceeds } t\}.$$

We can then distinguish between those credence functions Cr for which the associated belief set B is consistent and deductively closed, and those for which it is not. Call the former credence functions *t-permissible* and the latter *t-impermissible*. Let us introduce the following stipulative (though rather *ad hoc*) restriction as a replacement for universality:

t-permissibility: All (and only) t-permissible credence functions are rationally permissible.

Then we can define a credence-belief relation $\sim$ satisfying t-permissibility as well as the other four conditions. Specifically, the following relation will do the job: for all Cr and all B,

$$Cr \sim B \text{ if and only if } [Cr \text{ is } t\text{-permissible and } B = \{p \in X : Cr(p) \text{ exceeds } t\}].$$

In fact, this particular relation even satisfies functionality. The main point of the example, however, is to illustrate that the impossibility result ceases to hold if the

[8] Our discussion of these escape routes partly draws on our parallel discussion in the case of belief binarization (the case of a functional credence-belief relation). See Dietrich and List (2018), to which we refer readers for further details.

universality condition is suitably relaxed, though it is unclear how this relaxation would be justified.

5.2 The second route: Relaxing belief consistency and deductive closure

A second logically possible escape route from our impossibility theorem is to relax the rationality requirements on binary beliefs – consistency and deductive closure – and to permit their violation. While violations of consistency would seem very problematic from the perspective of rationality, violations of deductive closure seem less worrisome. In particular, when the set X of propositions on which beliefs are held is large and complex, then requiring deductive closure amounts to the implausible requirement of logical omniscience: knowledge of the logical implications of all of one's beliefs.

Suppose we weaken the requirement on binary beliefs as follows. Call a belief set B *closed under implication by singletons* if it contains every proposition p in X that is entailed by some individual proposition in B. Formally, this requires that, for any $q \in X$, if $p \subseteq q$ for some $p \in B$, then $q \in B$. Closure under implication by singletons is much less demanding than full deductive closure.

Belief consistency and closure under implication by singletons: Only belief sets that are consistent and closed under implication by singletons are rationally permissible.

It can be shown that there exist credence-belief relations satisfying this weaker requirement, together with the other four conditions. For example, given that X is finite, the following relation will always work: for all Cr and all B,

$$Cr \sim B \text{ if and only if } B = \{p \in X : Cr(p) > \tfrac{k-1}{k}\},$$

where k is the size of the largest minimal inconsistent subset of X. (A subset Y of X is *minimal inconsistent* if it is inconsistent but all its proper subsets are consistent.)

The size of the largest minimal inconsistent subset of X can be interpreted as a simple measure of the complexity of the logical interconnections in X. Clearly, the more complex the interconnections in X, the more demanding the acceptance threshold for any proposition must be, in order to ensure that the resulting binary beliefs are both consistent and closed under implication by singletons. In principle, one might also explore the possibility of relaxing not only the requirement of deductive closure but also that of consistency, but we set this possibility aside here.[9]

5.3 The third route: Relaxing propositionwise certainty preservation

The idea that a credence of 1 in proposition p should be sufficient to require accepting p seems very natural and hard to give up. Yet, from a logical perspective, the impossibility result ceases to hold if we drop that requirement. To illustrate this point, let Y be some fixed consistent subset of X. We can think of Y as the set of those propositions

[9] For a brief discussion of this route, see Dietrich and List (2018).

whose acceptance is somehow mandatory, irrespective of one's degree of belief in them. Now define a credence-belief relation $\sim$ as follows: for all Cr and all B,

$$Cr \sim B \text{ if and only if } B \text{ is a consistent and deductively closed extension of } Y.$$

This relation violates propositionwise certainty preservation, but satisfies our other four conditions (in case of non-looseness, provided Y contains at least one proposition distinct from Ω). Needless to say, the present credence-belief relation, which rigidly mandates the acceptance of all the propositions in Y, merely illustrates a technical point and has little independent plausibility.

5.4 The fourth route: Relaxing non-looseness

Recall that non-looseness is a non-triviality requirement on a credence-belief relation. It says that, at least in some cases, accepting a proposition is required even when one's credence in that proposition is less than one. Informally, a belief in some proposition may sometimes be rationally required even in the absence of complete certainty. If we are willing to give up this non-triviality requirement, then we can find credence-belief relations satisfying the other four conditions.

For example, the following relation will work: for all Cr and all B,

$$Cr \sim B \text{ if and only if } \begin{bmatrix} B \text{ is consistent and deductively closed} \\ \text{and, for every } p \in X, \text{ if } Cr(p) = 1, \text{ then } p \in B \end{bmatrix}.$$

Under this relation, *any* consistent and deductively closed belief set B that contains at least those propositions with $Cr(p) = 1$ is co-tenable with Cr. Clearly, this is an extremely "loose" coherence constraint between degrees of belief and binary beliefs.

We can also give an alternative but logically equivalent definition of the same relation $\sim$ in terms of some concepts that will become useful in our subsequent discussion. Let us say that a set of propositions $Y \subseteq X$ *Cr-entails* a proposition $p \in X$ if there is zero credence in p being false while all the propositions in Y are true, formally $Cr((\bigcap_{q \in Y} q) \backslash p) = 0$. We call a belief set B *Cr-closed* if it contains every proposition $p \in X$ that is Cr-entailed by B.[10] We can now define $\sim$ in these terms:

$$Cr \sim B \text{ if and only if } B \text{ is consistent and } Cr\text{-closed.}$$

Emphatically, this is not meant to be a particularly compelling credence-belief relation, insofar as it gives up a very plausible non-triviality requirement (namely, non-looseness). The notion of Cr-closure, however, will come up again at several points in what follows.

5.5 The fifth and most compelling route: Relaxing locality

The relative implausibility of the first four escape routes from our impossibility result suggests that the only genuinely plausible escape route is the fifth one: the relaxation

[10] It is easy to see that B is Cr-closed if and only if B is deductively closed and, for every $p \in X$, if $Cr(p) = 1$, then $p \in B$.

of locality. Unlike the other four conditions, which are all quite undemanding, the locality condition is arguably too strong. Since beliefs form interconnected webs, it is reasonable to think that requirements to accept propositions may depend not only on the degrees of belief in those propositions themselves, but also on the degrees of belief in other related propositions. In short, the relation between degrees of belief and binary beliefs may be holistic. Once we give up locality and permit such holism, there are many plausible credence-belief relations. They fall into two broad categories: functional and non-functional relations. In our previous work, we considered several examples in the functional category, in the form of belief-binarization rules giving up propositionwise independence, including "premise-based", "sequential", and "relevance-based" ones. Here, we focus on non-functional credence-belief relations. We consider four kinds of such relations. The first two do not seem to have received much attention in the existing literature; the last two are variants of existing proposals from the literature.

Holistic threshold relations

We can obtain a well-behaved holistic credence-belief relation by introducing a cut-off threshold for belief that is applied not to individual propositions, as in the case of standard threshold rules, but to sets of propositions in their entirety. Given any credence function Cr, we define the credence in a belief set B as the credence in the conjunction of all the propositions in B, formally

$$Cr(B) = Cr(\bigcap_{p \in B} p).$$

Note that this is always well-defined, since we have assumed that X is finite. We can now fix a threshold $\Delta \in [0, 1)$ and define the credence-belief relation as follows. For all Cr and all B,

$$Cr \sim B \text{ if and only if } Cr(B) > \Delta \text{ and } B \text{ is } Cr\text{-closed (*)}.$$

(Recall the definition of Cr-closure from the end of Section 5.4.) It is easy to see that, if $\Delta > 0$, the present relation satisfies all of our conditions except locality. If $\Delta = 0$, the relation can be shown to reduce to that defined in Section 5.4; so it then satisfies locality, but violates non-looseness.

The smaller the threshold Δ becomes, the more propositions can be believed in the binary sense. For instance, if $\Delta = \frac{1}{10}$, the credence in one's belief set can drop to (just above) $\frac{1}{10}$. In particular, if Δ is smaller than $Cr(p)$ for some *atom* p of X (where, for each $q \in X$, p entails either q or $\overline{q}$), then even *complete* belief sets become rationally permissible, i.e., belief sets that contain a member of each proposition-negation pair in X. On the other hand, even for a small threshold Δ, the relation $\sim$ permits credence-belief pairs (Cr, B) for which $Cr(B)$ is as high as 1 and B is very incomplete, perhaps containing only propositions $p \in X$ with $Cr(p) = 1$. To limit incompleteness in the belief set B, one might amend the definition of $\sim$ by imposing a maximality condition on B, in the following way:

$$Cr \sim B \text{ if and only if } B \text{ is a maximal belief set such that } Cr(B) > \Delta \text{ (**)},$$

where *maximality* means that no proper superset B' of B also has the property that $Cr(B') > \Delta$. Again, the relation $\sim$ satisfies all of our conditions except locality (unless $\Delta = 0$, in which case the relation becomes local but loose).

Note that any maximal belief set B such that $Cr(B) > \Delta$ automatically has the property of Cr-closure. This is why the belief-credence relation defined in (**) is a refinement of the relation previously defined in (*). The smaller the threshold Δ is, the less incompleteness in B will be permissible, where this can be measured by how rarely one has no opinion on a proposition-negation pair $p, \overline{p}$, in the sense of believing neither p nor $\overline{p}$. If $\Delta = 0$, the present definition renders only complete belief sets permissible. So, the relation takes this form:

$$Cr \sim B \text{ if and only if } B \text{ is complete and } Cr(B) > 0 \text{ (***)}.$$

Relations with partially complete binary beliefs

As should be evident, there is a tradeoff between the extent to which one's binary belief set B is complete and one's overall credence in that belief set, $Cr(B)$. The credence-belief relations just defined limit the amount of incompleteness, either by demanding that B be a *maximal* belief set for which $Cr(B)$ exceeds a given threshold (in the case of (**)), or by demanding completeness of B simpliciter (in the case of (***)). We now consider another way in which one might limit the amount of incompleteness in B.

Suppose there are certain propositions on which we wish to rule out the possibility of having no opinion. Let $Y \subseteq X$ be the set of those "non-abstention" propositions (where $p \in Y$ if and only if $\overline{p} \in Y$); a firm opinion on the propositions in Y is required. Let us call a belief set B *complete within* Y if, for all $p \in Y$, we have $p \in B$ or $\overline{p} \in B$. We can now define the following credence-belief relation: for all Cr and all B,

$$Cr \sim B \text{ if and only if } B \text{ is complete within } Y\text{, consistent, and } Cr\text{-closed.}$$

This relation violates locality but satisfies our other conditions except non-looseness, as we will explain in a moment.[11] In the special case $Y = X$, completeness within Y reduces to completeness simpliciter, and the conjunction of consistency and Cr-closure can be replaced by the simple clause that $Cr(B) > 0$. So, the relation $\sim$ then reduces to the one defined in (***) above.

The non-looseness violation of the present credence-belief relation is due to the fact that the relation never specifies which of two propositions $p, \overline{p} \in Y$ is to be believed in the binary sense, except when $Cr(p)$ is 0 or 1. This is so even if $Cr(p)$ is much greater than $Cr(\overline{p})$ or vice versa. In response, one might refine the relation $\sim$ by demanding that $Cr(B)$ be as high as possible. Formally, for all Cr and all B,

$$Cr \sim B \text{ if and only if } \left[\begin{array}{c} B \text{ is complete within } Y \text{ and } Cr\text{-closed, and} \\ \text{for any such } B', Cr(B') \leq Cr(B) \end{array} \right].$$

This relation strikes a balance in the aforementioned tradeoff between the completeness of the belief set B and the overall credence in it. Again, if $Y = X$, the definition of $\sim$

[11] We set aside the limiting case $Y = \emptyset$, where the relation $\sim$ reduces to the one defined in Section 5.4. In that case, it no longer violates locality.

can be simplified. Completeness within Y reduces to completeness simpliciter, and it is sufficient to require deductive closure rather than Cr-closure. So, the definition of $\sim$ becomes the following:

$$Cr \sim B \text{ if and only if } \left[\begin{array}{c} B \text{ is complete and deductively closed, and} \\ \text{for any such } B', Cr(B') \leq Cr(B) \end{array} \right].$$

Distance-based relations

Our next example is a variant of a proposal discussed in our earlier work.[12] It captures the idea that binary beliefs should "approximate" degrees of belief as closely as possible, subject to the constraint of consistency and deductive closure. We need one preliminary definition. For any belief set B, define a membership function $B : X \to \{0, 1\}$, where, for each p in X,

$$B(p) = \begin{cases} 1 & \text{if } p \in B \\ 0 & \text{if } p \notin B \end{cases}.$$

Now we stipulate that a belief set B is coherent with a credence function Cr if and only if B's membership function best approximates Cr, or, in different words, it minimizes the "distance" from Cr. Formally, for any Cr and any B, define

$$\text{distance}(Cr, B) = \textstyle\sum_{p \in X} |B(p) - Cr(p)|.$$

Then our credence-belief relation is defined as follows: for all Cr and all B,

$$Cr \sim B \text{ if and only if } \left[\begin{array}{c} B \text{ is consistent and deductively closed, and} \\ \text{for any such } B', \text{distance}(Cr, B) \leq \text{distance}(Cr, B') \end{array} \right].$$

In our earlier work, we called this proposal the "Hamming rule", in light of its structural similarity to an equally named proposal in judgment-aggregation theory. It satisfies all of our conditions except locality. The proposal does not satisfy functionality, but displays violations of functionality only in special cases, namely when several distinct belief sets equally minimize the distance from a given credence function. Note that, in the present construction, we could also use other definitions of distance, such as quadratic rather than absolute ones. We have chosen the current definition of distance(Cr, B) just for illustrative purposes.

An alternative variant of the present proposal replaces the idea of "optimizing" the distance between binary beliefs and degrees of belief with the idea of "satisficing". Here, a belief set B is deemed coherent with a credence function Cr just in case B is sufficiently (rather than maximally) close to Cr. Formally, fix a tolerance parameter $\delta > 1$. Then

$$Cr \sim B \text{ if and only if } \left[\begin{array}{c} B \text{ is consistent and deductively closed, and} \\ \text{for any such } B', \text{distance}(Cr, B) \leq \delta \times \text{distance}(Cr, B') \end{array} \right].$$

[12]See Dietrich and List (2018).

Stability-theoretic relations

Our final example of a holistic credence-belief relation builds on Hannes Leitgeb's work.[13] Leitgeb proposes a way of relating degrees of belief to binary beliefs. First, he defines a proposition $p \in X$ to be *stable* relative to a credence function Cr, for short *Cr-stable*, if $Cr(p|q) > \frac{1}{2}$ for any proposition $q \in X$ that is consistent with p and satisfies $Cr(q) > 0$.[14] It is easy to see that Cr-stable propositions always exist. For instance, any proposition $p \in X$ with $Cr(p) = 1$ is Cr-stable. The term "stable" refers to the fact that the credence in a non-empty Cr-stable proposition always exceeds 1/2 and continues to exceed 1/2 after Bayesian conditionalization on any other proposition that is consistent with it. Now, the credence-belief relation can be defined as follows: for all Cr and all B,

$$Cr \sim B \text{ if and only if } \left[\begin{array}{c} B = \{p \in X : Cr(p) \geq t\} \text{ for some threshold } t \text{ equal to} \\ \text{the credence in some } Cr\text{-stable proposition in } X \backslash \{\emptyset\} \end{array} \right].$$

As Leitgeb notes, on this proposal, belief does indeed correspond to "high enough" degree of belief, but what counts as "high enough" may depend on the credence function in question. The credence-belief relation we have just defined satisfies all of our conditions, except locality. In that sense, it captures a "holistic" relationship between degrees of belief and binary beliefs. Leitgeb recognizes this holism, which he describes as "a strong form of sensitivity of belief to context".[15] That there is such holism is not accidental at all. As our impossibility result shows, if there is to be any relation between degrees of belief and binary beliefs which satisfies universality, belief consistency and deductive closure, propositionwise certainty preservation, and non-looseness, then it must be holistic.

6 A further generalization

Up to this point, we have assumed that the set X of propositions on which beliefs are held has the structure of an algebra, i.e., it is a non-empty set of propositions that is closed under both union and complement (and thereby also intersection). We now want to extend our analysis to more general sets of propositions. The assumption that X is closed under union and intersection is somewhat restrictive. For instance, we might assign credences to propositions about the weather, and to propositions about inflation, without assigning credences to any conjunctions or disjunctions of those propositions. In such a case, the set X does not satisfy the given closure conditions.

We will show that our impossibility result does not depend on the assumption that X is an algebra, but applies to any set of propositions that satisfies some less demanding structural conditions. For a start, let us assume that X is a non-empty (and for present purposes again finite) set of propositions that is closed under complement (i.e., for every $p \in X$, we also have $\overline{p} \in X$), but not necessarily under conjunction (and disjunction). Recall that a *credence function* Cr assigns to each proposition p in X a number $Cr(p)$

[13] See Leitgeb (2014).

[14] Two propositions p and q are consistent with each other if $p \cap q \neq \emptyset$.

[15] See Leitgeb (2014, p. 168).

in the interval from 0 to 1, where this assignment is probabilistically coherent. In the present case, probabilistic coherence means that Cr is extendable to a probability function (with the standard properties) on an algebra including X. A *belief set*, as before, is a subset $B \subseteq X$. We will now ask what conditions on X are necessary and sufficient for our impossibility result to arise, and conversely, what conditions on X are necessary and sufficient for the avoidance of that result.

To state those conditions, we need to begin with some preliminary definitions. Let us say that a proposition $p \in X$ *conditionally entails* another proposition $q \in X$ if $\{p\} \cup Y$ entails q for some (possibly empty) set of propositions $Y \subseteq X$ which is consistent with p and consistent with $\overline{q}$ (i.e., where $(\underset{r \in Y}{\cap} r) \cap p \neq \emptyset$ and $(\underset{r \in Y}{\cap} r) \cap \overline{q} \neq \emptyset$). Now, for any two propositions $p, q \in X$, we say that there is a *path of conditional entailments* from p to q if there exist propositions $p_1, p_2, ..., p_k \in X$ with $p_1 = p$ and $p_k = q$ such that p_1 conditionally entails p_2, p_2 conditionally entails p_3, ..., and p_{k-1} conditionally entails p_k. We call the set X *negation-connected* if there is a path of conditional entailments from each contingent proposition $p \in X$ to its complement $\overline{p}$.

The following two results state the weakest conditions on X under which our earlier impossibility results hold.

Theorem 1*. If X is negation-connected, then there exists no credence-belief relation satisfying universality, belief consistency and deductive closure, propositionwise certainty preservation, non-looseness, and acceptance locality. Conversely, if X is not negation-connected, then there exists such a credence-belief relation.

Theorem 2*. If X is negation-connected, then there exists no belief-binarization rule satisfying universal domain, belief consistency and deductive closure, propositionwise certainty preservation, non-triviality, and propositionwise independence. Conversely, if X is not negation-connected, then there exists such a belief-binarization rule.

These generalized versions of our theorems – proved in the appendix – show two things. First, they show that the assumption that X is a non-trivial algebra is inessential for our impossibility result. The impossibility arises as soon as X is negation-connected. While all non-trivial algebras are negation-connected, the converse is not true. For an example of a negation-connected set of propositions that is not an algebra, consider the set X consisting of p, q, $p \cap q$, $p \cup q$, r, s, $r \cap s$, $r \cup s$, and the negations of these propositions, where p, q, r, and s are each contingent and logically independent from one another.[16] This set violates closure under conjunction. For instance, it fails to include $p \cap r$ and $q \cap s$.

Secondly, Theorems 1* and 2* show that if X is not negation-connected, then there does in fact exist a credence-belief relation (or a belief-binarization rule in the functional case) satisfying the specified conditions. Whether the possibilities that open up if X is not negation-connected are substantively interesting or rather degenerate depends on the set X in question. In general, if there are few or no logical interconnections between the propositions in X, many plausible credence-belief relations or belief-binarization

[16] For instance, we might have $\Omega = \{0,1\}^4$ with $p = \{(1, x, y, z) : x, y, z \in \{0,1\}\}$, $q = \{(x, 1, y, z) : x, y, z \in \{0,1\}\}$, $r = \{(x, y, 1, z) : x, y, z \in \{0,1\}\}$, and $s = \{(x, y, z, 1) : x, y, z \in \{0,1\}\}$.

rules satisfy all of our conditions. For instance, if there are no logical interconnections at all between different proposition-negation pairs $p, \overline{p}$ in X, we can use any threshold rule which accepts a proposition just in case the credence in it exceeds some threshold between $\frac{1}{2}$ and 1. By contrast, if X exhibits lots of logical interconnections despite not being negation-connected, then the resulting possibilities tend to be more degenerate. Let us sketch a solution which works in general.

Suppose that X is not negation-connected. Note that X can be partitioned into two subsets that are each closed under complement, labelled $X_{\vdash}$ and $X_{\nvdash}$: $X_{\vdash}$ contains all propositions $p \in X$ for which there are paths of conditional entailments from p to $\overline{p}$ and from $\overline{p}$ to p, and $X_{\nvdash}$ contains all other propositions, for which there is either no path from p to $\overline{p}$ or no path from $\overline{p}$ to p. We can now define a belief-binarization rule for X (and by implication, a credence-belief relation). For any credence function Cr, we specify the corresponding belief set B as follows. For each proposition $p \in X_{\vdash}$, we stipulate that $p \in B$ if and only if $Cr(p) = 1$. For all the propositions $p \in X_{\nvdash}$, we use a different construction. A technical lemma shows that $X_{\nvdash}$ has a subset S containing exactly one member of each proposition-negation pair $\{p, \overline{p}\} \subseteq X_{\nvdash}$ such that no proposition in S conditionally entails any proposition in $X \backslash S$ (intuitively, propositions in S can be accepted without any risk of creating conflicts with beliefs about propositions in $X \backslash S$). We pick such a subset S and make it the "default belief set" among the propositions in $X_{\nvdash}$, in the sense that, for any $p \in S$, we stipulate that $p \in B$ except when $Cr(p) = 0$, while for any $p \in X_{\nvdash} \backslash S$, we stipulate that $p \in B$ if and only if $Cr(p) = 1$. In the appendix, we show that the resulting belief set B is both consistent and deductively closed, and that the resulting belief-binarization rule (or more generally, credence-belief relation) satisfies the rest of our conditions. While this is indeed a formal possibility to which our impossibility theorem does not apply, it is obviously only of limited substantive interest.

7 Concluding remarks

We have seen that if

(i) agents have both degrees of belief and binary beliefs,

(ii) these satisfy standard rationality conditions (as captured by universality and belief consistency and deductive closure), and

(iii) the two kinds of belief stand in a systematic relationship to each other (as captured by propositionwise certainty preservation and non-looseness),

then

(iv) the resulting relationship cannot be a "local" or "propositionwise" one.

Rather, whether it is rationally required or permissible to accept a proposition in the binary sense will often depend not only on the degree of belief in that proposition itself, but also on the degrees of belief in others. The "unit of belief binarization", as we put it in our earlier work, is not each individual proposition in isolation, but rather a web of interconnected propositions. Our present results show that this point is not an artifact of the restrictive nature of *functional* relations between degrees of belief and binary beliefs. Rather, the point applies much more generally. Even if the relation between degrees of belief and binary beliefs is allowed to be many-to-many – meaning that more than one possible belief set is deemed coherent with each credence function, and vice versa – the relation still cannot be a local one.

The lesson is that if we insist that an agent's credal state involves both degrees of belief and binary beliefs, holism in the relationship between these two kinds of belief is unavoidable. We have given several examples of holistic credence-belief relations. Hopefully, future research will explore which of these and other holistic relations offer the best account of the relationship between degrees of belief and binary beliefs.

References

Dietrich, Franz, and Christian List (2007) "Arrow's Theorem in Judgment Aggregation", *Social Choice and Welfare* 29(1): 19-33.

Dietrich, Franz, and Christian List (2018) "From Degrees of Belief to Binary Beliefs: Lessons from Judgment-Aggregation Theory", *Journal of Philosophy* 115(5): 225–270.

Dokow, Elad, and Ron Holzman (2010) "Aggregation of binary evaluations", *Journal of Economic Theory* 145(2): 495–511.

Douven, Igor, and Timothy Williamson (2006) "Generalizing the Lottery Paradox", *British Journal for the Philosophy of Science* 57(4): 755–779.

Hawthorne, James, and Luc Bovens (1999) "The Preface, the Lottery, and the Logic of Belief", *Mind* 108(430): 241–264.

Kyburg, Henry E. (1961) *Probability and the Logic of Rational Belief*, Middletown: Wesleyan University Press.

Leitgeb, Hannes (2014) "The Stability Theory of Belief", *Philosophical Review* 123(2): 131–171.

Lin, Hanti, and Kevin T. Kelly (2012a) "Propositional Reasoning that Tracks Probabilistic Reasoning", *Journal of Philosophical Logic* 41(6): 957–981.

Lin, Hanti, and Kevin T. Kelly (2012b) "A Geo-logical Solution to the Lottery Paradox", *Synthese* 186(2): 531–575.

Makinson, David C. (1965) "The Paradox of the Preface", *Analysis* 25(6): 205–207.

Nehring, Klaus, and Clemens Puppe (2010) "Abstract Arrowian Aggregation", *Journal of Economic Theory* 145(2): 467-494.

A Negation-connected agendas

As shown in Section 6, our impossibility result – in both its relational and its functional form – is not dependent on the assumption that the set X of propositions on which beliefs are held – which we now call the *agenda* – is an algebra. Rather, the result continues to hold if X is not closed under disjunction (union) or conjunction (intersection), as long as it is negation-connected. Generally, an *agenda* – a term that comes from judgment-aggregation theory – is a non-empty set of propositions X that is closed under negation (i.e., if $p \in X$ then $\overline{p} \in X$). A *negation-connected* agenda, in turn, is one in which there is a path of conditional entailments from each contingent proposition $p \in X$ to its complement $\overline{p}$ (where a *path of conditional entailments* is as defined in Section 6).

In our earlier work on this topic, we discussed some structural parallels between the problem of judgment aggregation and the problem of belief binarization, and we showed that the same agendas for which certain impossibility results of judgment aggregation arise also lead to impossibility results in the area of belief-binarization. One of the most important classes of agendas in the area of judgment aggregation is the class of *path-connected* agendas. An agenda is *path-connected* if, for any pair of contingent propositions $p, q \in X$, there is a path of conditional entailments from p to q.[17] As is well known, path-connectedness is a necessary agenda condition for the judgment-aggregation variant of Arrow's classic impossibility theorem.[18] Manifestly, negation-connetedness, the condition introduced in the present paper, is less demanding than path-connectedness, insofar as it requires the existence of paths of conditional entailments only between propositions and their negations, not between any two propositions (assuming these propositions are contingent).

We can say more about the relationship between negation-connectedness and path-connectedness. Let us begin with two preliminary defitions. A *subagenda* of an agenda X is a subset $X' \subseteq X$ that is itself an agenda, i.e., it is non-empty and closed under complement. Subagendas $X_1, ..., X_k$ of X are *logically independent* if the propositions they contain are logically independent from each other, i.e., if, for all consistent belief sets $B_1 \subseteq X_1, ..., B_n \subseteq X_n$, the union $B_1 \cup \cdots \cup B_n$ is consistent.

Proposition 1 *For any finite agenda X, the following are equivalent:*

(a) X is negation-connected;

(b) X can be partitioned into (one or more) path-connected subagendas;

(c) X can be partitioned into (one or more) logically independent path-connected subagendas.

In this case, assuming $\Omega, \varnothing \notin X$, there is a unique partition as in (c) and a coarsest partition as in (b), both consisting of the maximal path-connected subagendas.[19]

[17]This condition was originally introduced by Nehring and Puppe (2010) under the name "total blockedness". See also Dietrich and List (2007) and Dokow and Holzman (2010).

[18]See Dietrich and List (2007) and Dokow and Holzman (2010).

[19]If X can be infinite, (a) is still equivalent to (b), in which case (assuming $\Omega, \varnothing \notin X$) there still is a coarsest partition as in (b) which consists of the maximal path-connected subagendas.

One might call the maximal path-connected subagendas of an agenda X the *path-connected components* of X. The simplest negation-connected agendas are those which have a single path-connected component, hence are themselves path-connected. The negation-connected agenda given as an example in Section 6 has two path-connected components: the subagenda containing $p, q, p \cap q, p \cup q$ (and their negations), and the subagenda containing $r, s, r \cap s, r \cup s$ (and their negations).

B Proofs

The proof architecture is as follows. We first establish that the relational theorem, in the version of Theorem 1 or 1*, is reducible to its functional counterpart, i.e., to Theorem 2 or 2*, respectively (Section B.1). Meanwhile note that Theorems 1 and 2 follow from Theorems 1* and 2*, respectively, once we restrict attention to the case of finite X, the case assumed in Theorems 1* and 2*. This is because any non-trivial algebra X is negation-connected.[20] After proving Proposition 1 on the structure of negation-connected agendas (Section B.2), we therefore only prove Theorem 2* (Sections B.3 and B.4). While Theorems 1 and 2 hold even for infinite X, we omit the proofs here.

B.1 Reducing the relational theorem to the functional theorem

The key idea behind the reduction is that we can go back and forth between credence-belief relations and belief-binarization rules, while preserving our conditions. As already noted, every belief-binarization rule is equivalent to a (very special) credence-belief relation: a functional one. This first translation by definition preserves our conditions:

Lemma 1 *If f is a belief-binarization rule and $\sim$ is the functional credence-belief relation equivalent to f, then*

(a) if f satisfies universal domain, then $\sim$ satisfies universality;

(b) if f satisfies belief consistency and deductive closure, then so does $\sim$;

(c) if f satisfies propositionwise independence, then $\sim$ satisfies full locality;

(d) if f satisfies propositionwise certainty preservation, then so does $\sim$;

(e) if f satisfies non-triviality, then $\sim$ satisfies non-looseness.

Conversely, an arbitrary credence-belief relation $\sim$, though of course usually not equivalent to a belief-binarization rule, *induces* a belief-binarization rule f, defined as follows: f permits the same credence functions Cr as $\sim$ (i.e., the credence functions coherent with some belief set), and for any permissible Cr, $f(Cr)$ consists of the

[20] Why? Let X be a non-trivial algebra. Take any contingent $p \in X$. We need to construct a path of conditional entailments from p to $\overline{p}$. *Case 1*: p is not an atom of the algebra. Then p has some strict subset $q \neq \varnothing$ in the algebra. Here is a path from p to $\overline{p}$: p entails q conditional on $q \cup \overline{p}$; q entails $q \cup \overline{p}$ conditional on no propositions; and $q \cup \overline{p}$ entails $\overline{p}$ conditional on $\overline{q}$. *Case 2*: p is an atom of the algebra. Then $\overline{p}$ is not an atom of the algebra (as the algebra is non-trivial), and so $\overline{p}$ has a strict subset $q \neq \varnothing$ in the algebra. Here is a path from p to $\overline{p}$: p entails $p \cup q$ conditional on no propositions; $p \cup q$ entails q conditional on $\overline{p}$; and q entails $\overline{p}$ conditional on no propositions. In fact, non-trivial algebras are not just negation-connected, but even path-connected.

propositions whose acceptance is required by Cr, i.e., which belong to *all* belief sets coherent with Cr:

$$f(Cr) = \{p \in X : Cr \text{ requires acceptance of } p\} = \cap\{B : Cr \sim B\}.$$

This converse translation also preserves our conditions:

Lemma 2 *If $\sim$ is any credence-belief relation and f is the induced belief-binarization rule as just defined, then*

(a) if $\sim$ satisfies universality, then f satisfies universal domain;

(b) if $\sim$ satisfies belief consistency and deductive closure, then so does f;

(c) if $\sim$ satisfies acceptance locality, then f satisfies propositionwise independence;

(d) if $\sim$ satisfies propositionwise certainty preservation, then so does f;

(e) if $\sim$ satisfies non-looseness, then f satisfies non-triviality.

Proof. Part (b) holds as the intersection of consistent and deductively closed belief sets is consistent and deductively closed. All other parts hold by definition of f. ■

On the basis of the two previous lemmas, we can derive our reduction result:

Lemma 3 *Theorem 2 implies Theorem 1, and Theorem 2* implies Theorem 1*.*

Proof. We assume that Theorem 2* holds, and prove both directions of Theorem 1* (analogously, Theorems 2 implies Theorem 1). If some credence-belief relation $\sim$ satisfies all the conditions of Theorem 1*, then by Lemma 2 the induced belief-binarization rule satisfies all the conditions of Theorem 2*, so that by Theorem 2* X is not negation-connected. Conversely, if X is not negation-connected, then by Theorem 2* some belief-binarization rule f satisfies all the conditions of Theorem 2*, so that by Lemma 1 the functional credence-belief relation equivalent to f satisfies all the conditions of Theorem 1* (in fact, even with *full* locality). ■

B.2 Proof of Proposition 1

We begin with two lemmas. Let $\vdash\vdash$ be the transitive closure of the conditional entailment relation.

Lemma 4 *If X is negation-connected, then the relation $\vdash\vdash$ is symmetric.*

Proof. Let X be negation-connected. Assume $p \vdash\vdash q$. Then

- $q \vdash\vdash \overline{q}$, as X is negation-connected (and as q is contingent, since $p \vdash\vdash q$);
- $\overline{q} \vdash\vdash \overline{p}$, as $p \vdash\vdash q$, and as conditional entailment (and so $\vdash\vdash$) satisfies contraposition;
- $\overline{p} \vdash\vdash p$, as X is negation-connected (and as $\overline{p}$ is contingent, since $p \vdash\vdash q$).

So, by transitivity of $\vdash\vdash$, $q \vdash\vdash p$. ■

Lemma 5 *If X is negation-connected, then on $X\backslash\{\Omega, \varnothing\}$ the relation $\vdash\vdash$ is an equivalence relation whose equivalence classes are path-connected subagendas.*

Proof. Let X be negation-connected. The relation $\vdash\vdash$ is transitive and symmetric by Lemma 4; and it is reflexive on $X\backslash\{\Omega, \varnothing\}$, since each contingent proposition conditionally (in fact even unconditionally) entails itself. So $\vdash\vdash$ is an equivalence relation on $X\backslash\{\Omega, \varnothing\}$. Each equivalence class is a subagenda, as it is non-empty and (by negation-connectedness of X) closed under complement; path-connectedness is obvious. ■

Proof of Proposition 1. Let X be any finite agenda.

Part 1 – equivalence of (a) and (b). Obviously, (b) implies (a). Now assume (a). By Lemma 5, $X\backslash\{\Omega, \varnothing\}$ can be partitioned into path-connected subagendas $X_1, X_2, ..., X_k$ ($k \geq 1$). Clearly, the subagendas $X_1 \cup \{\Omega, \varnothing\}, X_2, ..., X_k$ are still all path-connected, and they partition X. This establishes (b).

Part 2 – existence and nature of a coarsest partition in (b). Assume (b) and let $\Omega, \varnothing \notin X$. Let Π be the (non-empty) set of partitions of X into path-connected subagendas. We show that Π has a coarsest element, namely the meet of the partitions in Π, denoted $\wedge\Pi$ and defined as the finest partition of X that is at least as coarse as each $\mathcal{P} \in \Pi$ (it then follows that $\wedge\Pi$ consists of the maximal path-connected subagendas). So we must show that $\wedge\Pi$ belongs to Π, i.e., consists itself of path-connected subagendas. Take any $X' \in \wedge\Pi$. X' is a subagenda as it is the union of subagendas (from Π). To see why X' is path-connected, take any $p, q \in X'$. As $X' \in \wedge\Pi$, there are $X_1, ..., X_k \in \Pi$ such that $p \in X_1$, $q \in X_k$, and $X_t \cap X_{t+1} \neq \varnothing$ for all $t \in \{1, ..., k-1\}$. As each of $X_1, ..., X_k$ is path-connected, there are paths of conditional entailments from p to a proposition in $X_1 \cap X_2$, from there to a proposition in $X_2 \cap X_3$, and so on until we reach q. Concatenating these paths, we obtain a path from p to q.

Part 3 – equivalence of (b) and (c). Obviously, (c) implies (b). Now we suppose (b) and show (c). We may assume without loss of generality that $\Omega, \varnothing \notin X$. By Part 2, it suffices to show that the subagendas in $\wedge\Pi$ are logically independent. For a contradiction, let this be false. Then there is a minimal inconsistent set $B \subseteq X$ which intersects with more than one subagenda in $\wedge X$. We can coarsen $\wedge\Pi$ through replacing those subagendas which intersect with B by their union, where (as one easily checks) that union is again a path-connected subagenda – a contradiction.

Part 4 – uniqueness and nature of the partition in (c). Assume (c) and let $\Omega, \varnothing \notin X$. Let $\mathcal{P}$ be some partition of X of the type in (c). It remains to show that $\mathcal{P}$ is the coarsest of the partitions in (b). As shown in Part 2, there indeed exists a coarsest partition of type in (b), denoted $\wedge\Pi$. If $\mathcal{P}$ differed from $\wedge\Pi$, then $\mathcal{P}$ would be strictly finer than $\wedge\Pi$, so that some $X' \in \mathcal{P}$ would be the union of two or more subagendas from $\wedge\Pi$. Hence there would exist paths of conditional entailments between these subagendas (as X' is path-connected), contradicting the logical independence of the subagendas in $\wedge\Pi$ established under Part 3. ■

B.3 Proof of Theorem 2*: sufficiency

From now on, let the agenda X be finite, as assumed in Theorem 2*. In this subsection, we show that negation-connectedness of X is *sufficient* for non-existence of a belief-binarization rule satisfying Theorem 2*'s conditions. We proceed in several lemmas.

Lemma 6 *Given a propositionwise independent belief-binarization rule f,*

(a) f induces for each subagenda X' a propositionwise independent belief-binarization rule $f_{X'}$ defined by $f_{X'}(Cr|_{X'}) = f(Cr) \cap X'$ for all credence functions Cr permitted by f;

(b) if f satisfies universal domain, then so does $f_{X'}$ for each subagenda X';

(c) if f satisfies belief consistency and deductive closure, then so does $f_{X'}$ for each subagenda X';

(d) if f satisfies propositionwise certainty preservation, then so does $f_{X'}$ for each subagenda X';

(e) if f satisfies non-triviality and $\{X_1, ..., X_k\}$ is a partition of X into subagendas, then at least one of $f_{X_1}, ..., f_{X_k}$ satisfies non-triviality.

Proof. Details are left to the reader. We just note two things, relevant for (a) and (b), respectively. Let X' be any subagenda. First, $f_{X'}$ is well-defined by propositionwise independence of f. Second, $f_{X'}$ is defined on the domain $\{Cr|_{X'} : Cr$ is permitted by $f\}$, which contains *all* credence functions on X' whenever f permits *all* credence functions on X. ■

Lemma 7 *If no belief-binarization rule satisfies the conditions of Theorem 2* for any path-connected X, then no belief-binarization rule satisfies these conditions for any negation-connected X.*

Proof. Assume no belief-binarization rule satisfies Theorem 2*'s conditions for any path-connected X. For a contradiction, let X be negation-connected and f a belief-binarization rule satisfying those conditions. Using Proposition 1, we partition X into path-connected subagendas $X_1, ..., X_k$ (for some $k \geq 1$). By Lemma 6, at least one of the induced belief-binarization rules $f_{X_1}, ..., f_{X_k}$ satisfies all the conditions, a contradiction. ■

By Lemma 8, we can prove sufficiency in Theorem 2* by assuming that X is path-connected, not just negation-connected. We can also assume that $X \neq \{\Omega, \varnothing\}$, as otherwise there clearly is no non-trivial belief-binarization rule.

Assumptions for the rest of the section: X is path-connected and distinct from $\{\Omega, \varnothing\}$, and f is a belief-binarization rule satisfying Theorem 2*'s conditions except possibly non-triviality.

We must show that f is trivial. This will take several steps. We will borrow Lemmas 8 and 9 from our own earlier work on belief binarization,[21] and Lemmas 10 and 14 from

[21] See Dietrich and List (2018).

judgment-aggregation theory.[22] For completeness, we shall give proofs even of these four lemmas.

For any $p \in X$, let $\mathcal{W}_p$ be the set of *acceptance credences* for p, i.e., the set of values $w \in [0,1]$ such that p is believed in the binary sense whenever $Cr(p) = w$ (formally, for all credence functions Cr, $Cr(p) = w \Rightarrow p \in f(Cr)$). As f is propositionwise independent, it is fully determined by the acceptance credences:

$$f(Cr) = \{p \in X : Cr(p) \in \mathcal{W}_p\} \text{ for all credence functions } Cr.$$

For all $Y \subseteq X$ and $Z \subseteq Y$, we write $Y_{\neg Z}$ for the set arising from Y by negating Z's members: $Y_{\neg Z} = (Y \backslash Z) \cup \{\overline{z} : z \in Z\}$.

Lemma 8 *If a proposition $p \in X$ conditionally entails another proposition $q \in X$, then $\mathcal{W}_p \subseteq \mathcal{W}_q$. So (as X is path-connected) all contingent propositions $p \in X$ have the same set of acceptance credences $\mathcal{W}_p$, to be denoted $\mathcal{W}$.*

Proof. Let $p \in X$ conditionally entail $q \in X$, say $\{p\} \cup Y$ entails q where $Y \subseteq X$ is consistent with p and with $\overline{q}$. Then $\{p, q\} \cup Y$ is consistent, and also $\{\overline{p}, \overline{q}\} \cup Y$ is consistent (as $\{\overline{q}\} \cup Y$ is consistent while $\{p, \overline{q}\} \cup Y$ is inconsistent). Fix any $w \in \mathcal{W}_p$. We show that $w \in \mathcal{W}_q$. As $p \cap q \cap (\cap Y) \neq \varnothing$ and $\overline{p} \cap \overline{q} \cap (\cap Y) \neq \varnothing$, there is a credence function Cr such that $Cr(p \cap q \cap (\cap Y)) = w$ and $Cr(\overline{p} \cap \overline{q} \cap (\cap Y)) = 1 - w$. Now $f(Cr)$ contains p as $Cr(p) = w \in \mathcal{W}_p$, and contains all $y \in Y$ as $Cr(y) = 1$ where $1 \in \mathcal{W}_y$ by propositionwise certainty preservation. As $\{p\} \cup Y \subseteq f(Cr)$, $f(Cr)$ entails q, hence contains q by deductive closure. So $\mathcal{W}_q$ contains $Cr(q) = w$. ■

We must ultimately show that $\mathcal{W} = \{1\}$, as this means that f is trivial.

Lemma 9 *$1 \in \mathcal{W}$, and $w \in \mathcal{W} \Rightarrow 1 - w \notin \mathcal{W}$.*

Proof. By propositionwise certainty preservation, $1 \in \mathcal{W}$. Now let $w \in \mathcal{W}$. Take any $p \in X \backslash \{\Omega, \varnothing\}$, and any credence function Cr such that $Cr(p) = w$, i.e., $Cr(\overline{p}) = 1 - w$. We have $1 - w \notin \mathcal{W}$, as otherwise $Cr(p), Cr(\overline{p}) \in \mathcal{W}$, i.e., $p, \overline{p} \in f(Cr)$, violating consistency. ■

A set of propositions Y is *minimal inconsistent* if it is inconsistent, and all its proper subsets are consistent. X is *non-simple* if it has a minimal inconsistent subset Y such that $|Y| \geq 3$. For instance, if X contains two logically independent propositions p and q and contains $p \cap q$, then X is non-simple as $Y = \{p, q, \overline{p \cap q}\}$ is minimal inconsistent.

Lemma 10 *X is non-simple.*

Proof. Pick a contingent $p \in X$. Then $\overline{p}$ is also contingent. As X is path-connected, there is a path of conditional entailments in X from p to $\overline{p}$, say $p = p_1, p_2, ..., p_k = \overline{p} \in X$ where each p_t $(t < k)$ conditionally entails p_{t+1}. At least one of these conditional entailments is not an (ordinary) entailment, as otherwise p would entail $\overline{p}$ by transitivity

[22] See Nehring and Puppe (2010) and Dokow and Holzman (2010), respectively.

of entailment. Say p_t does not entail p_{t+1}. By conditional entailment, we may pick a set $Z \subseteq X$ such that $\{p_t\} \cup Z$ entails p_{t+1}, where $Z \cup \{p_t\}$ and $Z \cup \{\overline{p_{t+1}}\}$ are each consistent. As $\{p_t, \overline{p_{t+1}}\} \cup Z$ is inconsistent, it has a minimal inconsistent subset Y. Now $\{p_t, \overline{p_{t+1}}\} \subseteq Y$, as otherwise Y would be a subset of one of the consistent sets $Z \cup \{p_t\}$ and $Z \cup \{\overline{p_{t+1}}\}$. Further, $\{p_t, \overline{p_{t+1}}\} \neq Y$, as $\{p_t, \overline{p_{t+1}}\}$ is consistent. So, $|Y| \geq 3$. ■

Lemma 11 *If $\mathcal{W} \cap [0, \frac{1}{2}) = \varnothing$, then $\mathcal{W} = \{1\}$.*

Proof. For a contradiction, let $\mathcal{W} \cap [0, \frac{1}{2}) = \varnothing$ and $\mathcal{W} \neq \{1\}$. As $\inf \mathcal{W} < 1$ we have $\inf \mathcal{W} < \frac{1}{2} + \frac{\inf \mathcal{W}}{2}$. So we can pick a $t \in \mathcal{W}$ such that $t < \frac{1}{2} + \frac{\inf \mathcal{W}}{2}$. (If $\mathcal{W}$ contains $\inf \mathcal{W}$, we could let $t = \inf \mathcal{W}$.)

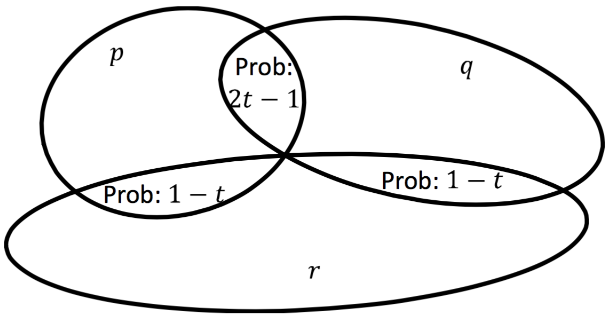

Figure 1: Venn diagram for the proof of Lemma 11, assuming $Y = \{p, q, r\}$

By Lemma 10, we may pick a minimal inconsistent set $Y \subseteq X$ with three distinct members p, q, r. As $Y_{\neg\{p\}}$, $Y_{\neg\{q\}}$ and $Y_{\neg\{r\}}$ are each consistent, there is a credence function Cr such that $Cr(\cap Y_{\neg\{p\}}) = Cr(\cap Y_{\neg\{q\}}) = 1-t$ and $Cr(\cap Y_{\neg\{r\}}) = 2t-1$. Note that Cr is probabilistically well-defined, since $Cr(\cap Y_{\neg\{p\}})$, $Cr(\cap Y_{\neg\{q\}})$ and $Cr(\cap Y_{\neg\{r\}})$ are non-negative (as $\frac{1}{2} \leq t < 1$) and of sum 1. Now $p, q \in f(Cr)$, as $Cr(p) = Cr(q) = t \in \mathcal{W}$. Also, for all $y \in Y \backslash \{p, q, r\}$, $y \in f(Cr)$, as $Cr(y) = 1 \in \mathcal{W}$. In sum, $Y \backslash \{r\} \subseteq f(Cr)$. So, as $Y \backslash \{r\}$ entails $\overline{r}$ and $f(Cr)$ is deductively closed, $\overline{r} \in f(Cr)$. Hence, $\mathcal{W}$ contains $Cr(\overline{r}) = 2t-1$. Yet $2t-1 < \inf \mathcal{W}$ (as $t < \frac{1}{2} + \frac{\inf \mathcal{W}}{2}$), a contradiction. ■

X is *pair-negatable* if it has a minimal inconsistent subset $Y \subseteq X$ with distinct elements p, q such that $Y_{\neg\{p,q\}}$ is consistent.

Lemma 12 *If X is pair-negatable, then $w \in \mathcal{W} \Rightarrow [w, 1] \subseteq \mathcal{W}$.*

Proof. Let X be pair-negatable. Suppose $w \in \mathcal{W}$. We fix a $w' \in [w, 1]$ and show that $w' \in \mathcal{W}$. Via pair-negatability, pick a minimal inconsistent set $Y \subseteq X$ and distinct $p, q \in Y$ such that $Y_{\neg\{p,q\}}$ is consistent. As $Y_{\neg\{p,q\}}$ is consistent, and as also $Y_{\neg\{p\}}$ and $Y_{\neg\{q\}}$ are consistent (by Y's minimal inconsistency), the intersections $\cap Y_{\neg\{p,q\}}$, $\cap Y_{\neg\{p\}}$ and $\cap Y_{\neg\{q\}}$ are non-empty. So there is a credence function Cr such that $Cr(\cap Y_{\neg\{q\}}) =$

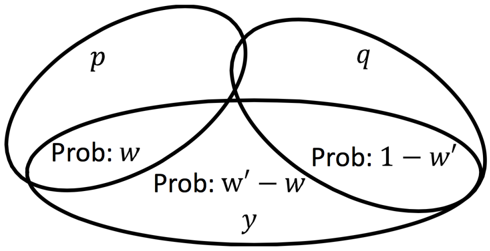

Figure 2: Venn diagrams for the proof of Lemma 12, assuming $Y = \{p, q, y\}$.

w, $Cr(\cap Y_{\neg\{p,q\}}) = w' - w$ and $Cr(\cap Y_{\neg\{p\}}) = 1 - w'$. As $Cr(p) = w \in \mathcal{W}$, we have $p \in f(Cr)$. For all $y \in Y \backslash \{p, q\}$, $Cr(y) = 1 \in \mathcal{W}$, whence $y \in f(Cr)$. In sum, $Y \backslash \{q\} \subseteq f(Cr)$. So, as $Y \backslash \{q\}$ entails $\overline{q}$ and $f(Cr)$ is deductively closed, $\overline{q} \in f(Cr)$. So $\mathcal{W}$ contains $Cr(\overline{q}) = w'$. ■

Lemma 13 *If X is pair-negatable, then $\mathcal{W} \subseteq [\frac{1}{2}, 1]$.*

Proof. Let X be pair-negatable. If $\mathcal{W}$ contained $w < \frac{1}{2}$, then $1 - w \in \mathcal{W}$ by Lemma 12 while $1 - w \notin \mathcal{W}$ by Lemma 5, a contradiction. ■

Lemma 14 *If X is not pair-negatable, then $Y_{\neg Z}$ is consistent for all minimal inconsistent sets $Y \subseteq X$ and all three-member subsets Z of Y.*

Proof. Assume $Y \subseteq X$ is a minimal inconsistent with a three-member subset Z such that the set $Y_{\neg Z}$ is inconsistent. We show pair-negatability. Pick a minimal inconsistent subset Y' of $Y_{\neg Z}$. Y' shares at least two propositions with $\{\overline{z} : z \in Z\}$; otherwise $Y' \subseteq Y_{\neg\{z\}}$ for some $z \in Z$, a contradiction since Y' is inconsistent although $Y_{\neg\{z\}}$ is consistent by Y's *minimal* inconsistency. So we can partition $\{\overline{z} : z \in Z\}$ into a two-member set $Z' \subseteq Y'$ and a singleton set $\{\overline{z}\}$. Whether or not $\overline{z}$ belongs to Y', the set $Y'_{\neg Z'}$ is a subset of $(Y \backslash \{z\}) \cup \{\overline{z}\} = Y_{\neg\{z\}}$, hence is consistent (by Y's *minimal* inconsistency). This proves pair-negatability. ■

Lemma 15 *If X is not pair-negatable, then $\mathcal{W} \subseteq (\frac{2}{3}, 1]$.*

Proof. Let X be not pair-negatable. For a contradiction, assume $w \in \mathcal{W}$ and $w \leq \frac{2}{3}$. By Lemma 10, we can pick a minimal inconsistent set $Y \subseteq X$ with a three-member subset $Z \subseteq Y$. By Lemma 14, $Y_{\neg Z}$ is consistent, i.e., $\cap Y_{\neg Z} \neq \varnothing$. By Y's minimal inconsistency, for each $z \in Z$ the set $Y_{\neg\{z\}}$ is consistent, i.e., $\cap Y_{\neg\{z\}} \neq \varnothing$. So we can pick a credence function Cr such that

$$Cr(\cap Y_{\neg\{z\}}) = \frac{w}{2} \text{ for all three } z \in Z,$$
$$Cr(\cap Y_{\neg Z}) = 1 - \left(\frac{w}{2} + \frac{w}{2} + \frac{w}{2}\right) = 1 - \frac{3w}{2}.$$

Here the inequality $w \leq \frac{2}{3}$ guarantees that $1 - \frac{3w}{2} \geq 0$. For each $z \in Z$, we have $Cr(z) = \frac{w}{2} + \frac{w}{2} = w \in \mathcal{W}$, and hence $z \in f(Cr)$. Moreover, for each $y \in Y \backslash Z$ we have $Cr(y) = 1 \in \mathcal{W}$, and hence $y \in f(Cr)$. In sum, $Y \subseteq f(Cr)$, a consistency violation. ■

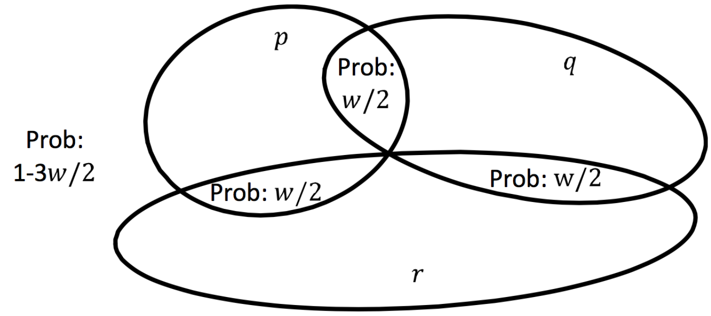

Figure 3: Venn diagram for the proof of Lemma 15, assuming $Y = Z = \{p, q, r\}$

Completing the proof of sufficiency in Theorem 2.* As noted, it remains to show that $\mathcal{W} = \{1\}$. By Lemma 9, $1 \in \mathcal{W}$. If $\mathcal{W}$ contained a $w < 1$, then $\mathcal{W}$ would even contain a $w < \frac{1}{2}$ by Lemma 11, in contradiction with Lemmas 13 and 15. ■

B.4 Proof of Theorem 2*: necessity

We finally prove that if X is not negation-connected, then there is a belief-binarization rule satisfying the conditions of Theorem 2*. We begin with a technical lemma. Consider the set $X_{\vdash} := \{p \in X : p \vdash\vdash \overline{p} \text{ and } \overline{p} \vdash\vdash p\}$ of propositions with paths of conditional entailments to and from their complement, and the set $X_{\nvdash} := X \backslash X_{\vdash}$ of all propositions without at least one of the paths. $X_{\vdash}$ and $X_{\nvdash}$ are each closed under complement and possibly empty.

Lemma 16 *There is a set $S \subseteq X_{\nvdash}$ containing exactly one member of each pair $\{p, \overline{p}\} \subseteq X_{\nvdash}$ such that no proposition in S conditionally entails any proposition in $X \backslash S$ (equivalently, each minimal inconsistent set $Y \subseteq X$ shares with S at most one proposition, and no proposition if $Y \cap X_{\vdash} \neq \varnothing$).*

Proof. The result follows from a lemma due to Klaus Nehring and Clemens Puppe (our sets X, $X_{\vdash}$, $X_{\nvdash}$ and S correspond to their sets $\mathcal{H}$, $\mathcal{H}_0$, $\mathcal{H}_1^+ \cup \mathcal{H}_1^- \cup \mathcal{H}_2$ and $\mathcal{H}_1^- \cup \mathcal{H}_2^-$, respectively).[23] ■

Proof of necessity in Theorem 2.* We give a constructive proof. Suppose X is not negation-connected, i.e., $X_{\nvdash} \neq \varnothing$. Choose S as in Lemma 16, and let f be the propositionwise independent belief-binarization rule with universal domain given by the following acceptance credences:

- any $p \in S$ has set of acceptance credences $\mathcal{W}_p = (0, 1]$,
- any $p \in X \backslash S$ has set of acceptance credences $\mathcal{W}_p = \{1\}$.

We show that f satisfies the conditions of Theorem 2* (in case of non-triviality because X is not negation-connected). Universal domain and propositionwise independence hold by definition. Propositionwise certainty preservation holds because $1 \in \mathcal{W}_p$ for all

[23]See Nehring and Puppe (2010), Lemma 3.

$p \in X$. Non-triviality holds because for $p \in S$ the set $\mathcal{W}_p$ contains values other than 1, and because $S \neq \varnothing$ (as $X_{\nvdash} \neq \varnothing$ by violation of negation-connectedness). Finally, let Cr be any credence function, and let us show that $f(Cr)$ is consistent and deductively closed.

Consistency. For a contradiction, let $f(Cr)$ be inconsistent. Then $f(Cr)$ has a minimal inconsistent subset Y. By Lemma 16, Y contains at most one proposition from S. So there is a $q \in Y$ such that $Y \backslash \{q\} \subseteq X \backslash S$ (q may or not belong to S). For all $p \in Y \backslash \{q\}$, $Cr(p) = 1$ as $p \in f(Cr)$ and $\mathcal{W}_p = \{1\}$. So, as $Y \backslash \{q\}$ entails $\overline{q}$, $Cr(\overline{q}) = 1$, i.e., $Cr(q) = 0$. Hence $q \notin f(Cr)$ as $0 \notin \mathcal{W}_q$. This is impossible as $q \in Y \subseteq f(Cr)$.

Deductive closure. Assume $f(Cr)$ entails $q \in X$. We must show that $q \in f(Cr)$. Let Z be a minimal subset of $f(Cr)$ that entails q. Note that $Z \cup \{\overline{q}\}$ is *minimal* inconsistent (Z is consistent as $f(Cr)$ is consistent).

- *Case 1*: $Z \cap S = \varnothing$. Then for all $p \in Z$, $Cr(p) = 1$, as $p \in f(Cr)$ and $\mathcal{W}_p = \{1\}$. So $Cr(q) = 1$, as Z entails q. Hence $q \in f(Cr)$, as $1 \in \mathcal{W}_q$.

- *Case 2*: $Z \cap S \neq \varnothing$. By Lemma 16 and the minimal inconsistency of $Z \cup \{\overline{q}\}$, we can infer three things. First, $Z \cap S$ is singleton, say $Z \cap S = \{z\}$. Second, $\overline{q} \notin S$. Third, $(Z \cup \{\overline{q}\}) \cap X_{\vdash} = \varnothing$, i.e., $Z \cup \{\overline{q}\} \subseteq X_{\nvdash}$. As $\overline{q} \in X_{\nvdash} \backslash S$, we have $q \in S$. For all $p \in Z \backslash \{z\}$, $Cr(p) = 1$, as $p \in f(Cr)$ and $\mathcal{W}_p = \{1\}$. So, as Z entails q, $Cr(q) \geq Cr(z)$. Meanwhile, $Cr(z) > 0$, as $z \in f(Cr)$ and $\mathcal{W}_z = (0, 1]$. So $Cr(q) > 0$. Hence, $q \in f(Cr)$, as $\mathcal{W}_q = (0, 1]$. ■