

# Computational reverse mathematics and foundational analysis

Benedict Eastaugh\*

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## Abstract

Reverse mathematics studies which subsystems of second order arithmetic are equivalent to key theorems of ordinary, non-set-theoretic mathematics. The main philosophical application of reverse mathematics proposed thus far is *foundational analysis*, which explores the limits of different foundations for mathematics in a formally precise manner. This paper gives a detailed account of the motivations and methodology of foundational analysis, which have heretofore been largely left implicit in the practice. It then shows how this account can be fruitfully applied in the evaluation of major foundational approaches by a careful examination of two case studies: a partial realization of Hilbert's program due to Simpson [1988], and predicativism in the extended form due to Feferman and Schütte.

Shore [2010, 2013] proposes that equivalences in reverse mathematics be proved in the same way as inequivalences, namely by considering only  $\omega$ -models of the systems in question. Shore refers to this approach as *computational reverse mathematics*. This paper shows that despite some attractive features, computational reverse mathematics is inappropriate for foundational analysis, for two major reasons. Firstly, the computable entailment relation employed in computational reverse mathematics does not preserve justification for the foundational programs above. Secondly, computable entailment is a  $\Pi_1^1$  complete relation, and hence employing it commits one to theoretical resources which outstrip those available within any foundational approach that is proof-theoretically weaker than  $\Pi_1^1\text{-CA}_0$ .

## 1 Introduction

In ordinary mathematical practice, mathematicians prove theorems, reasoning from a fixed set of axioms to a logically derivable conclusion. The axioms in play are usually implicit: mathematicians rarely assert at the beginning of their papers that they work in, for example, PA or ZFC. Given a particular proof we might ask which axioms were employed and thus make explicit the author's assumptions. Now that we have a set of axioms  $\Gamma$  which are sufficient to prove some theorem  $\varphi$ , we could further inquire whether they are necessary to prove the theorem, or whether a strictly weaker set of axioms would suffice. To a first approximation, *reverse mathematics* is the program of discovering precisely which axioms are both necessary and sufficient to prove any given theorem of ordinary mathematics.

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\*Munich Center for Mathematical Philosophy, LMU Munich, Geschwister-Scholl-Platz 1, 80539 Munich, Germany. Email: [benedict@eastough.net](mailto:benedict@eastough.net)

Reverse mathematics was initiated by Harvey Friedman [1975, 1976], and extensively developed in the work of Stephen Simpson and his students. It determines the proof-theoretic strength of theorems of ordinary mathematics, by proving equivalences between formalised versions of those theorems and axiom systems in a hierarchy of known strength. Roughly speaking, the term “ordinary mathematics” means non-set-theoretic mathematics, i.e. those parts of mathematics which do not depend on abstract set-theoretical concepts. Typical examples of ordinary mathematics include real and complex analysis, countable algebra, and the topology of complete separable metric spaces.

The axiom systems used in reverse mathematics are subsystems of second order arithmetic or  $Z_2$ . This is an extension of familiar first order systems of arithmetic such as Peano arithmetic. In the intended interpretation, variables in first order arithmetic range over the natural numbers  $\mathbb{N}$ . Second order arithmetic also has number variables ranging over the natural numbers, but in addition to these it has set variables which range over sets of numbers  $X \subseteq \mathbb{N}$ . For the full technical background on second order arithmetic the reader should consult Simpson [2009], the primary reference work on reverse mathematics. Here we restrict ourselves to sketching the basic features of the framework and explaining some salient details.

The language of second order arithmetic,  $L_2$ , is a two-sorted first order language with the following nonlogical symbols: constant symbols 0 and 1, binary function symbols  $+$  and  $\cdot$ , and the binary relation symbols  $<$  and  $\in$ .  $L_2$ -structures have two domains: a first order domain  $|M|$  over which the number variables  $x_0, x_1, \dots$  range, and a second order domain  $\mathcal{S} \subseteq \mathcal{P}(|M|)$  over which the set variables  $X_0, X_1, \dots$  range. An  $L_2$ -structure  $M$  is thus a tuple of the form

$$(1) \quad M = \langle |M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M \rangle,$$

where  $0_M$  and  $1_M$  are elements of  $|M|$ ,  $+_M$  and  $\cdot_M$  are functions from  $|M| \times |M|$  to  $|M|$ , and  $<_M$  is a binary relation on  $|M|$ .

The formal system  $Z_2$  of second order arithmetic has a long history in work on foundations of mathematics, which we can trace back to Dedekind. The most substantive classical developments are those of Hilbert and Bernays [1968, 1970]. The axioms of  $Z_2$  fall into three groups: the basic axioms; a comprehension scheme; and an induction axiom. The basic axioms are those of Peano arithmetic, minus the induction scheme. To these is added the comprehension scheme

$$(CA) \quad \exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

for all  $L_2$ -formulae  $\varphi$  (with parameters). Many subsystems of second order arithmetic are obtained by restricting this comprehension scheme to particular syntactically-defined subclasses. Finally there is the induction axiom

$$(I_0) \quad \forall X ((0 \in X \wedge \forall n (n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n (n \in X)).$$

This is a single axiom rather than an axiom scheme. As such its strength is tied directly to the associated comprehension scheme: we have induction only for those sets which we can prove to exist by comprehension. Because  $Z_2$  includes the comprehension scheme for all  $L_2$ -formulae  $\varphi$ , every instance of the second order induction scheme

$$(I) \quad (\varphi(0) \wedge \forall n (\varphi(n) \rightarrow \varphi(n + 1))) \rightarrow \forall n \varphi(n)$$

is a theorem of  $Z_2$ . By restricting  $\varphi$  to formulae in the language of first-order arithmetic  $L_1$  we obtain the induction scheme of first-order Peano arithmetic (PA). A stronger restriction, limiting  $\varphi$  to  $\Sigma_1^0$  formulae, gives us the  $\Sigma_1^0$  induction scheme.

A subsystem  $T$  of  $Z_2$  is a formal system in the language  $L_2$  such that each axiom  $\varphi$  of  $T$  is a theorem of  $Z_2$ . The central subsystems are colloquially known as the *Big Five*. Each of them includes the basic axioms, the induction axiom and some other set existence axioms. The weakest is  $RCA_0$ , the usual base theory for reverse mathematics. Its axioms consist of the basic axioms,  $\Delta_1^0$  (that is, recursive) comprehension and  $\Sigma_1^0$  induction. Induction in  $RCA_0$  thus outstrips comprehension, although it is still quite limited compared to PA or  $Z_2$ . The other members of the Big Five are:  $WKL_0$ , obtained by adding to  $RCA_0$  the assertion that every infinite subtree of  $2^{<\mathbb{N}}$  has an infinite path through it;  $ACA_0$ , which is given by the comprehension scheme for all arithmetically definable sets;  $ATR_0$ , which extends  $ACA_0$  with an axiom allowing the iteration of the arithmetical operations along any wellordering; and finally  $\Pi_1^1\text{-}CA_0$ , whose defining axiom is the  $\Pi_1^1$  comprehension scheme.

At first glance, second order arithmetic may seem somewhat limited, and unsuitable for the development of large portions of mathematics, even when we restrict our attention to ordinary, non-set-theoretic mathematics. Formally, second order arithmetic includes only two kinds of entities: natural numbers and sets of natural numbers. While clever coding schemes allow objects from many branches of mathematics to be represented within this formally austere framework, limitations abound, generally to do with cardinality: one cannot quantify over uncountable sets of real numbers, prove theorems about topological spaces of arbitrary cardinality, and so on. Most obviously, the set  $\mathbb{R}$  of all real numbers cannot be directly represented.

Mathematics within second order arithmetic is thus limited to countable or countably representable structures such as complete separable metric spaces and countable abelian groups. Nevertheless, this turns out to include a wide variety of mathematical objects including real numbers, continuous functions on the real line and complex plane, and Borel and analytic sets. Constructing representations of these objects and proving that they are well-behaved usually requires a certain minimum of theoretical strength. Definitions in reverse mathematics are therefore often given relative to a particular system, usually the weak base system  $RCA_0$ .

While we can do enough in  $RCA_0$  to get mathematics off the ground, many key theorems require stronger axioms. A typical example is that  $RCA_0$  does not prove the Bolzano–Weierstraß theorem, a fundamental theorem in analysis which states that every bounded sequence of real numbers has a convergent subsequence. The Bolzano–Weierstraß theorem can be formalised as a sentence  $BW$  in the language of second order arithmetic. In two papers which inaugurated the study of reverse mathematics, Friedman [1975, 1976] showed that  $BW$  is equivalent over  $RCA_0$  to the arithmetical comprehension scheme—the defining axiom of the system  $ACA_0$ . To prove the equivalence between  $BW$  and  $ACA_0$ , one first shows that  $ACA_0$  implies  $BW$ , by formalising the usual proof of the theorem within that system. The reversal is then accomplished by adding  $BW$  to the axioms of  $RCA_0$  and showing that any instance of arithmetical comprehension is provable from  $RCA_0 + BW$ .

One way to think about the epistemic value of reverse mathematics is that it uncovers the resources required in ordinary mathematical reasoning: for example, if a proof uses a compactness argument, then weak König’s lemma must be amongst the stock of axioms which the mathematician draws upon, whether explicitly or implicitly. That nonconstructive methods, in the form of compactness, are required to prove the completeness theorem for first-order logic tells us something important

about that theorem and its epistemic standing. This has many ramifications for philosophical issues. Feferman [1992] points out the application of reverse mathematical methods to indispensability arguments. Such arguments are intended to show that certain mathematical entities, being indispensable to science, must be accorded the same ontological rights as those whose existence is empirically confirmed. This leaves two critical questions unanswered: which mathematical entities does this argument show us to be committed to the existence of, and what principles concerning those entities must we endorse in order to carry out the mathematics that is indispensable to science? Real analysis is a natural starting point, since our current best physical theories model spacetime in terms of a geometrical continuum, as a type of differentiable manifold. By formalising our best physical theories in second order arithmetic, we could obtain far sharper answers, by showing that theorems of analysis required for physics are equivalent to particular systems studied in reverse mathematics.

This paper, however, is not concerned with applications of reverse mathematics to the indispensability argument. Instead, it addresses the relationship of reverse mathematics to the foundational views espoused by Hilbert and Weyl, and the finitist and predicativist programs developed in their work and that of their successors. More generally, it explores and defends the usefulness of reverse mathematics for determining the limits of what can be proved within foundational schemes that can be formalised in the setting of second order arithmetic, and for demonstrating to adherents of particular foundational approaches that they are unable to recover a given part of mathematics within their chosen foundation. It then turns to the task of determining whether one component of standard reverse mathematical practice, namely a proof-theoretically weak base theory over which equivalences between mathematical theorems and subsystems of second order arithmetic are proved, is essential.

§2 gives a detailed account of the motivations and methodology that underpin the reverse mathematical analysis of foundations. §3 and §4 are then devoted to case studies of particular foundational programs: a partial realization of Hilbert’s program due to Simpson [1988], and predicativism as initially developed by Weyl and then extended through the work of Kreisel, Feferman, Schütte, and others. The paper then examines a proposal of Shore [2010, 2013] to abandon the standard practice of proving reverse mathematical equivalences over the base theory  $\text{RCA}_0$ , and instead concern ourselves only with whether the principles involved are true in the same Turing ideals, that is to say,  $L_2$ -structures whose first order parts consist of the standard natural numbers  $\omega$  and whose second order parts are classes of sets  $\mathcal{C} \subseteq \mathcal{P}(\omega)$  closed under Turing reducibility and recursive joins. In §5 we introduce and motivate Shore’s proposal, and then in §6 we argue that in failing to respect the justificatory structure of the foundational programs mentioned above, Shore’s equivalence relation shows itself to be inappropriate for analysing foundations for mathematics in the way described above. Finally, in §7 we show that this equivalence relation is highly complex, and thus brings with it attendant theoretical commitments that exceed those acceptable to proponents of the type of foundational programs being analysed.

## 2 Reverse mathematical analysis of foundations

One of the main philosophical roles attributed to reverse mathematics in the current literature is what we shall call *foundational analysis*. This application has been strongly promoted by Stephen Simpson, born out of his view (stated amongst other places in Simpson [2009] and Simpson [2010]) that there is a correspondence between subsystems of second order arithmetic and foundational programs such as Weyl’s

predicativism and Hilbert’s finitistic reductionism. By providing a hierarchy of comparable systems, and proving the equivalence of theorems of ordinary mathematics to these systems, reverse mathematics demonstrates what resources a particular theorem requires, and what theorems a given system cannot prove. In other words, when committing to a foundational system, reverse mathematics lets us know precisely what we are giving up. Crucially, it also tells us when a proponent of such a system employs mathematical resources that she is not entitled to, as they go beyond what her preferred foundation can prove.

The following example should clarify the notion of foundational analysis. Suppose Sarah is a predicativist in the tradition of [Weyl \[1918\]](#). She believes that the natural numbers form a completed, infinite totality, and that sets which can be defined arithmetically—i.e. with quantifiers ranging over the natural numbers, but not over sets of them—also exist. This would lead her to accept the arithmetical comprehension scheme, and thus the subsystem of second order arithmetic  $\text{ACA}_0$ . She might even accept a somewhat stronger system; this possibility is explored in §4. But given Sarah’s predicativist outlook she would resist the thoroughly impredicative axiom scheme of  $\Pi_1^1$  comprehension, and its associated subsystem of second order arithmetic  $\Pi_1^1\text{-CA}_0$ .

Now suppose that her colleague Rebecca disagrees with Sarah’s predicativism and wants to persuade her that it is an inappropriate foundation for mathematics. She might argue as follows: Since Sarah wants her predicativist outlook to provide a foundation for all of mathematics, it would be strange if she failed to account for important theorems of ordinary mathematics—say, in abelian group theory. Consider the statement

- ( $\star$ ) Every countable abelian group can be expressed as a direct sum of a divisible group and a reduced group.

The group theorist in the street, Rebecca argues, believes this to be true. Sarah might tentatively agree, whereupon Rebecca would point out the following theorem from reverse mathematics: assuming that ( $\star$ ) holds, one can prove (in  $\text{RCA}_0$ , which Sarah clearly accepts) the  $\Pi_1^1$  comprehension scheme [[Friedman, Simpson, and Smith 1983](#), theorem 6.3, p. 178].

It appears that Sarah has some explaining to do. Either she must abandon her predicativism, or she must push back against the naturalistic line Rebecca is urging upon her. Neither course appears terribly palatable, while the fact that this theorem is drawn not from set theory or some other area of mathematics whose ontological commitments might be thought extravagant could be taken as evidence that the problem here is a pressing one. The contentious statement is an ordinary theorem from a core area of mathematics, which reverse mathematical analysis shows us to have substantial proof-theoretic strength.

The broadly naturalistic argument that Rebecca makes to Sarah can be generalised in a straightforward way. Let  $\mathcal{F}$  be a foundation for mathematics which accepts classical logic as leading to correct conclusions, and let  $S_{\mathcal{F}}$  be a subsystem of second order arithmetic containing  $\text{RCA}_0$  such that the  $\mathcal{F}$ -theorist accepts that  $S_{\mathcal{F}}$  is a faithful formalisation of the principles of  $\mathcal{F}$ . Then any participant in the foundational dialectic may (as Rebecca does in the example above) fill in the following schematic argument with her favourite examples, and make it to the  $\mathcal{F}$ -theorist:

Consider the ordinary mathematical theorem  $P$ , which may be faithfully formalised in the language of second order arithmetic as the sentence  $\varphi$ .

$\varphi$  is equivalent over  $\text{RCA}_0$  to the subsystem of second order arithmetic  $S(\varphi)$ . But  $S_{\mathcal{F}}$  cannot prove the axioms of  $S(\varphi)$ , and thus cannot prove  $\varphi$ . So  $\mathcal{F}$  cannot recover the ordinary mathematical theorem  $P$ , and is thus inadequate as a foundation for mathematics.

It is not necessary to suppose that all instances of this argument scheme will be persuasive to all  $\mathcal{F}$ -theorists: foundational analysis does not provide a knockdown argument against predicativism, or indeed any foundational view with limited mathematical resources. Rather, it makes arguments like the dispute between Rebecca and Sarah precise: we can see, within a common framework (namely the base theory  $\text{RCA}_0$ , and the coding required to represent ordinary mathematical concepts in it), just where the boundaries of these foundational systems lie. As a rational agent, Sarah surely formed her foundational views in the full understanding that they require her to give up on any mathematics that her view deems to be unfounded. The decision to give up on or stick with her foundation is not one to be taken lightly, and it is one that should be made by considering the relevant facts. These facts can, in large part, be supplied by foundational analysis, which allows Sarah and the rest of us to see precisely what is at stake.

For foundational analysis to play a useful philosophical role in mediating between disputants with different foundational stances, it must be possible to carry out this analysis on ground which is common between the disputants.<sup>1</sup> So while a predicativist like Sarah and a platonist like Rebecca might disagree about whether  $\Pi_1^1$  comprehension is a valid axiom, they both accept the laws of classical logic and at least the axioms of  $\text{RCA}_0$ , as well as the faithfulness of the representation of the theorem  $(\star)$  in second order arithmetic, and thus both will agree that  $(\star)$  above is not predicatively provable. In other words, foundational analysis makes it clear where the fault lines lie, and the existence of common ground makes the conclusion available not just to those who accept stronger axioms or rules of inference, but those who are committed to a more limited foundational framework and will only accept mathematical conclusions derived within that framework.

Notice that Sarah already accepted that  $\Pi_1^1$  comprehension was not a predicative principle, otherwise she would not have been able to deduce that theorem  $(\star)$  about abelian groups was not predicatively provable. In accepting this Sarah goes beyond what her foundation can formally prove. If she accepts  $\text{ACA}_0$  and no more, then it is difficult to see how she can separate  $\Pi_1^1\text{-CA}_0$  from  $\text{ACA}_0$ .  $\Pi_1^1\text{-CA}_0$  implies all instances of arithmetical comprehension, so  $\text{ACA}_0$  is a subsystem of  $\Pi_1^1\text{-CA}_0$ , but in order to show that it is a proper subsystem, one typically construct a model of  $\text{ACA}_0$  that is not a model of  $\Pi_1^1\text{-CA}_0$ . In doing so, however, one thereby proves the consistency of  $\text{ACA}_0$ , which is (assuming that  $\text{ACA}_0$  really is consistent) not something that  $\text{ACA}_0$  can prove. Any proof that  $\Pi_1^1\text{-CA}_0$  properly extends  $\text{ACA}_0$  therefore relies on theoretical resources not available within  $\text{ACA}_0$  itself. Needless to say, we cannot eliminate the assumption of the consistency of  $\text{ACA}_0$ , since if  $\text{ACA}_0$  is inconsistent, then it proves everything that  $\Pi_1^1\text{-CA}_0$  does, which is to say every sentence in the language of second order arithmetic.

The upshot of this is that Sarah cannot prove that  $\Pi_1^1$  comprehension is not a predicative principle merely on the basis of her acceptance of any fixed predicative formal theory, no matter how strong it is, since we can re-run the above argument for any system  $S$  such that  $\text{ACA}_0 \subseteq S \subsetneq \Pi_1^1\text{-CA}_0$ . For Sarah or any predicativist, the

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<sup>1</sup>For a closely related discussion, albeit one which treats much stronger logics and axiomatic principles than those which are the subject of this article, see [Koellner \[2010\]](#).

judgement of the impredicativity of  $\Pi_1^1$  comprehension must therefore be justified by some other means. One candidate justification might be Sarah's acceptance of the soundness of the predicative formal theory  $\text{ACA}_0$ , or a predicative extension thereof. Alternatively, the impredicativity of  $\Pi_1^1\text{-CA}_0$  might itself be taken as a basic (albeit defeasible) belief. The  $\Pi_1^1$  comprehension scheme quantifies over all sets of natural numbers, and appears to do so in an essential way. In the absence of evidence to the contrary, Sarah should assume that  $\Pi_1^1\text{-CA}_0$  is an impredicative axiom system, and thus unacceptable on the basis of her predicativist stance. A third option is quietism about the impredicativity of  $\Pi_1^1\text{-CA}_0$ : Sarah could suspend judgement about whether or not  $\Pi_1^1$  comprehension is justifiable on a predicative basis. Sarah's predicativity would thus be an entirely positive view, as Sarah would accept any statement that can be shown to be predicatively provable (in the Feferman–Schütte sense), but not deny that a statement is predicative (save those which are predicatively refutable). While this is a coherent position, and one which can be applied quite generally to many foundational views, at least prima facie it fails to do justice to the predicativist outlook. Predicativism was historically motivated by the apparent vicious circularity of impredicative definitions, and the positive epistemic view that those objects exist which can be defined by quantifying over only objects already shown to exist is linked to the negative view that objects that cannot be so defined do not exist. Without such a negative view in the background, the positive program seems to lose some of its bite: while predicative mathematics is well and good, there is little to recommend it as a stopping point when the mathematical fruits of impredicativity are just over the horizon.

In order to make the kind of naturalistic argument sketched above, it is not sufficient to formalise a foundational system  $\mathcal{F}$  as a subsystem of second order arithmetic  $S_{\mathcal{F}}$  in a way which is acceptable to the  $\mathcal{F}$ -theorist. One must also ensure that the mathematical theorems whose proof-theoretic strength is appealed to in the argument are formalised in a faithful way. For example, there are theorems of topology which are provably equivalent to  $\Pi_2^1$  comprehension [Mummert and Simpson 2005]. Since the formal language of second order arithmetic only allows one to quantify over countable objects, any representation of uncountable objects must be indirect and relies on the availability of suitable countable codes for those objects. This availability, and thus the faithfulness of formalisations of ordinary mathematical notions in second order arithmetic, must be proved in some suitable metatheory which can quantify directly over the uncountable objects in question, and prove the existence of the countable codes of those objects.<sup>2</sup>

If Rebecca wanted to invoke these theorems in her attempt to persuade Sarah that predicativism is inadequate to mathematical practice, and thus mathematical truth, then her argument would appear to rely on a suppressed premise, namely the faithfulness of the representation in second order arithmetic of these topological spaces. Sarah could therefore respond that the statements in the language of second order arithmetic which Rebecca takes to be formalisations of theorems of topology are not, from her predicative perspective, anything of the sort. Instead they are simply  $L_2$ -sentences that are not predicatively provable. For them to be formalisations of particular theorems of topology requires that they are faithful translations of those theorems, and the proof of that faithfulness requires theoretical resources that, being strongly impredicative, she is not willing to commit to.

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<sup>2</sup>In some cases the metatheoretic axioms required are strong: the existence of an uncountable topological space with a countable basis was shown by Hunter [2008] to imply the axioms of  $Z_2$ , the full system of second order arithmetic.

Foundational analysis in the form sketched above thus seems to impose a natural criterion on formalisations, namely that the faithfulness of the codings used must be provable in a conservative extension  $S$  of a theory accepted by proponents of the foundation being analysed.  $S$  would be a formal version of the metatheory discussed above, with higher type variables ranging over uncountable sets, allowing the direct formalisation of higher type objects such as uncountable topological spaces. This would ensure reverse mathematical results could be read as intended, i.e. as demonstrating the mathematical resources necessary to prove given theorems of ordinary mathematics. It would then allow the kind of naturalistic argument given by Rebecca to be evaluated by proponents of a given foundation, within the theoretical framework they already accept. In the ideal case, the faithfulness of the codings involved would be provable in a conservative extension of the base theory, thus allowing reverse mathematical results to be evaluated by anyone who accepts the axioms of that base theory.

In the next two sections we will study more closely two historical, philosophically-motivated foundational programs, and their connections to reverse mathematics and subsystems of second order arithmetic: finitism in the sense descending from Hilbert's program in §3, and predicativism in the spirit of Weyl in §4. Before doing so, it is worth remarking that the role played by foundational analysis in the historical development of reverse mathematics is somewhat ambiguous. While reverse mathematics has a broadly foundational aim, namely determining the axioms necessary to prove theorems of ordinary mathematics, it is unclear how much research in reverse mathematics itself has been directly motivated by foundational analysis in the sense discussed here.<sup>3</sup>

lines of research in the related field of reductive proof theory that are more explicitly motivated

In contrast, work in the related field of reductive proof theory is more explicitly motivated by goals related to foundational analysis, namely determining what fragment of ordinary mathematics can be recovered in subsystems of second order arithmetic that are proof-theoretically reducible to finitistic or constructive systems. This line of research is known as the relativized Hilbert program. Its inception is usually traced back to [Bernays \[1967\]](#), and its goals, methods and results have been articulated by, amongst others, [Sieg \[1988\]](#) and [Feferman \[1988b\]](#).

### 3 Finitistic reductionism

Hilbert's program was to reduce infinitary mathematics to finitary mathematics. He viewed finitism as a secure foundation for mathematics, free of the paradoxes which arose from seemingly natural assumptions and normal mathematical reasoning about infinite collections. This reduction was to be accomplished by giving a finitary consistency proof for infinitary mathematics, which for present purposes can be identified with ZFC. Hilbert thought that employing infinitary methods in mathematics, such as assuming the existence of infinite collections, could be viewed simply as a way to supplement our finitistic theories with ideal statements, analogous to ideal elements in algebra. Ideal statements are thus intended to be eliminable, at least in principle: the purpose of Hilbert's desired consistency proof was to show that we can use

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<sup>3</sup> Indeed, much of the current research in reverse mathematics is focused on other concerns, especially the use of tools from computability theory to explore the growing constellation of intermediate and incomparable subsystems between  $RCA_0$  and  $ACA_0$  known as the Reverse Mathematics Zoo [[Dzhafarov 2015](#)]. A summary of current research frontiers can be found in [Montalbán \[2011\]](#).



infinitary mathematics to get finitary results, and that those results are finitistically acceptable.

Gödel’s second incompleteness theorem shows that there can be no such consistency proof, and thus that Hilbert’s program cannot be carried out in its entirety. Many even consider Gödel’s theorems to have shown that Hilbert’s program is entirely bankrupt.<sup>4</sup> While it certainly seems to block the full realization of the enterprise, Simpson [1988] argues that the possibility of partial realizations remains. But since the consistency proof Hilbert sought is out of reach, the latter-day finitistic reductionist must find other ways to demonstrate that their uses of ideal statements are in principle eliminable. Instead of trying to prove the consistency of infinitary systems directly, finitistic reductions of infinitary systems can be carried out in a relativised way, following the template laid down by Kreisel [1968]. We now sketch how such reductions work.<sup>5</sup>

Suppose we have two theories  $T_1$  (in a language  $\mathcal{L}_1$ ) and  $T_2$  (in  $\mathcal{L}_2$ ), both of which contain primitive recursive arithmetic. Suppose also that we have a primitive recursive set of formulae  $\Phi \subseteq Fml_{\mathcal{L}_1} \cap Fml_{\mathcal{L}_2}$  containing every closed equation  $t_1 = t_2$ . A proof-theoretic reduction of  $T_1$  to  $T_2$  which conserves  $\Phi$  is a partial recursive function  $f$  which, given any proof from the axioms of  $T_1$  of a sentence  $\varphi \in \Phi$ , produces a proof of  $\varphi$  from the axioms of  $T_2$ . If the existence of  $f$  can be proved in  $T_2$ , it then follows that  $T_2$  proves (a formalisation of) the following conditional statement: “If  $T_2$  is consistent then  $T_1$  is consistent.” For if  $T_1$  proves that  $0 = 1$ , then  $f$  will transform any proof of  $0 = 1$  in  $T_1$  into a proof of  $0 = 1$  in  $T_2$ .

If the existence of a proof-theoretic reduction of this sort can be proved in a finitary system, then we call it a *finitary reduction*. In order for a proof-theoretic reduction  $f$  from an infinitary system to a finitary one to provide a partial realization of Hilbert’s program,  $f$  must be a finitary reduction. Otherwise the result has a circular character unacceptable within a reductionist program: it would amount to using ideal methods to show that ideal methods are acceptable. Similarly, an infinitary proof of a conservativity theorem is insufficient to demonstrate the reducibility of an infinitary system to a finitary one.

If Hilbert had succeeded in providing a finitary consistency proof for infinitary mathematics then there would have been no need to mark out the boundary between finitary and infinitary methods with any precision, as the proof would have made use of methods which were clearly finitary in nature. In order to obtain the conservation results that demonstrate that certain infinitary systems are finitistically reducible, and thereby partially realize Hilbert’s program, Simpson’s route to a partial realization of Hilbert’s program requires that we formalise our conception of a finitary system. The formal system which Simpson selects is primitive recursive arithmetic (PRA), following the thesis proposed by Tait [1981]. The rest of Simpson’s argument rests squarely on this identification of finitist provability with provability in PRA: he does not offer any new considerations in support of Tait’s thesis, instead simply accepting it and proceeding accordingly.

Fixing PRA as the finitary system to which infinitary systems must be reduced to, the next question is which infinitary systems are finitistically reducible to PRA. One such system is  $WKL_0$ , the system obtained by adding weak König’s lemma (“Every infinite subtree of  $2^{<\mathbb{N}}$  has an infinite path”) to  $RCA_0$ . Friedman [1976, unpublished] used model-theoretic techniques to show that  $WKL_0$  is  $\Pi_2^0$  conservative over PRA,

<sup>4</sup>For a contrary view, see Detlefsen [1979].

<sup>5</sup>An excellent survey of this topic which also details the foundational picture behind such relativised versions of Hilbert’s program is Feferman [1988b].

and thus consistent relative to PRA; the proof can be found in [Simpson \[2009, §IX.3\]](#). Subsequently [Sieg \[1985\]](#) gave a primitive recursive proof transformation which, given a proof of a  $\Pi_2^0$  theorem  $\varphi$  in  $\text{WKL}_0$ , generates a proof of  $\varphi$  in PRA. Unlike Friedman’s result, this proof-theoretic derivation of the conservativity theorem is itself a piece of finitistic mathematics: it is provable within a finitary system, thus making the reduction finitary. As the complexity of consistency statements is  $\Pi_1^0$ , if  $\text{WKL}_0$  proves the consistency of PRA then so does PRA itself. From this Simpson concludes that  $\text{WKL}_0$  is finitistically reducible to PRA, and so the fragment of mathematical reasoning which one can carry out in  $\text{WKL}_0$  is finitistically acceptable, in the following sense. Any  $\Pi_1^0$  sentence provable in  $\text{WKL}_0$  is finitistically meaningful, in virtue of its form, but it is also provable in PRA (by the conservativity theorem), and thus finitistically provable (by Tait’s thesis). Any theorem of  $\text{WKL}_0$ , such as the Heine–Borel covering theorem or the Hahn–Banach theorem for separable spaces, is thus legitimised as a lemma that can be invoked in order to prove a finitistic theorem.

By referring to the reductionist project he proposes as a partial realization of Hilbert’s program, Simpson opens himself up to the criticism that his interpretation of Hilbert is a misreading, as alleged by [Sieg \[1990, p. 874\]](#), who suggests that Simpson’s project might be better understood as a partial realization of Kronecker’s views on foundations of mathematics. While questions of historical interpretation are important, our present purpose is not Hilbert scholarship, but determining whether there is a defensible core to Simpson’s position, and thus whether the reverse mathematics of  $\text{WKL}_0$  make a contribution to foundational analysis. Starting from Tait’s thesis that finitist provability is provability in PRA, together with the Hilbertian contention that only  $\Pi_1^0$  sentences are finitistically meaningful, the finitistically provable conservativity theorem gives us strong prima facie reason to take Simpson’s finitistic reductionism seriously. Since the foundational analysis of finitistic reductionism can be carried out in a base theory that is itself finitistically reducible, its results are available to the finitist, who can thereby see that (for example) the Heine–Borel theorem is finitistically reducible, but the Bolzano–Weierstraß theorem is not.

With the positive case in hand, we turn to potential criticisms of Simpson’s view. The first is his reliance on Tait’s thesis, which has taken fire from many quarters. Broadly speaking, such complaints fall into two camps: that PRA is too weak to encompass all of finitistic reasoning, and that it is too strong. Those in the former camp include [Kreisel \[1958\]](#), who concluded that finitist provability coincides with provability in PA. [Detlefsen \[1979\]](#) argued that adding instances of a restricted version of the  $\omega$ -rule is also finitistically acceptable, although Detlefsen’s position has in turn been criticised, for example by [Ignjatović \[1994\]](#). Two proposals that fall into the latter camp are made by [Ganea \[2010\]](#). From the broad spread of conclusions reached it is clear that what finitistic reasoning consists in is, to say the least, disputed. However, Tait’s arguments provide a robust defence of the thesis that primitive recursive arithmetic demarcates the limits of finitistic reasoning, and moreover, one that has gained wide acceptance. We therefore conclude that on the one hand, extant arguments against Tait’s thesis entail that we should not consider Simpson’s identification of finitistic reducibility with proof-theoretic reducibility to PRA to be established; but on the other, since a strong case can be made in favour of the thesis, Simpson’s finitistic reductionism should be taken seriously as a foundation of mathematics.

[Burgess \[2010\]](#) criticises the finitistic reducibility of  $\text{WKL}_0$  from another direction, arguing that the analysis leading to the identification of finitistic provability with provability in the formal system PRA cannot be carried out from a finitistic point of view. This means that the conservativity theorem does not, by itself, justify the

finitist in believing any  $\Pi_1^0$  sentence provable in  $\text{WKL}_0$ . At best, it provides a recipe for producing PRA proofs from  $\text{WKL}_0$  proofs, which the finitist must then verify by assuring themselves that each of the axioms of PRA used in the proof is in fact finitistically acceptable. Burgess offers the following way out for the finitistic reductionist (p. 139): by limiting the induction principle in  $\text{RCA}_0$  and  $\text{WKL}_0$  we can define subtheories  $\text{RCA}_*$  and  $\text{WKL}_*$ , in which  $\Sigma_1^0$  induction is replaced by  $\Sigma_0^0$  induction plus a sentence asserting that the exponential function is total.<sup>6</sup>  $\text{WKL}_*$  is conservative over a proper subtheory of PRA, known as  $\text{I}\Delta_0 + \text{exp}$ , and since provability in this system does not press up against the bounds of finitist provability, the finitist can recognise that all proofs in this system are finitistically acceptable [Simpson and Smith 1986]. Many, albeit not all, of the theorems of ordinary mathematics that are provable in  $\text{WKL}_0$  are also provable in  $\text{WKL}_*$ . For  $\Pi_1^0$  sentences provable in  $\text{WKL}_*$  the finitist can therefore work in the infinitary system without needing to check that the resulting proof in PRA uses only finitistically acceptable principles, since this is already guaranteed by the conservativity theorem—which, by a result of Sieg [1991], is finitistically provable.

A further objection is that Simpson’s argument does not in any way pick out  $\text{WKL}_0$  as the unique formal counterpart of the program of finitistic reductionism. Brown and Simpson [1993] present a system they call  $\text{WKL}_0^+$ , which extends  $\text{WKL}_0$  with a strong formal version BCT of the Baire Category Theorem. They prove, using a forcing argument, that  $\text{WKL}_0^+$  is  $\Pi_1^1$  conservative over  $\text{RCA}_0$ . It follows from a result of Parsons [1970] that  $\text{WKL}_0^+$  is  $\Pi_2^0$  conservative over PRA, and that this conservativity theorem can be proved in PRA itself.<sup>7</sup> So while  $\text{WKL}_0$  is, modulo Tait’s thesis, a finitistically reducible system, it is but one partial realization of Hilbert’s program.  $\text{WKL}_0^+$  is demonstrably another, and indeed a stronger one, since it satisfies the same criteria of finitistic reducibility whilst properly extending  $\text{WKL}_0$ . One might think that this undermines Simpson’s claim that the Big Five subsystems of second order arithmetic correspond to existing foundational programs, but this is not a fair reading of Simpson’s position: he does not claim that these systems are the unique formal correlates of these foundational approaches. It is consistent with his position that there are a variety of infinitary yet finitistically reducible systems. Nevertheless, it is weak König’s lemma that has been found equivalent to many theorems of ordinary mathematics, not BCT. This is evidence for the (defeasible) claim that  $\text{WKL}_0$  is a mathematically natural stopping point in a way that  $\text{WKL}_0^+$  is not.  $\text{WKL}_0^+$  is finitistically reducible just as  $\text{WKL}_0$  is, while being a proper extension of it, so mathematically natural stopping points do not appear to always align cleanly with justificatory stopping points—or if they do, then we have not yet identified the sources of justification of these axiom systems in a sufficiently fine-grained way.

Patey and Yokoyama [2016] have shown that the statement known as Ramsey’s theorem for pairs and two colours,  $\text{RT}_2^2$ , is finitistically reducible. Moreover, by prov-

<sup>6</sup>The theories  $\text{RCA}_*$  and  $\text{WKL}_*$  are often referred to in the reverse mathematics literature as  $\text{RCA}_0^*$  and  $\text{WKL}_0^*$ , for example by Simpson and Smith [1986], who first isolated these systems. However, this notation is confusing because in most other cases in reverse mathematics, the superscript is used to refer to additional set existence axioms adjoined to the theory, as in the case of  $\text{WKL}_0^+$ ,  $\text{ACA}_0^+$ , and so on, while the subscript is used to indicate a restricted induction axiom (ACA versus  $\text{ACA}_0$ , for example). We therefore use Montalbán [2011]’s convention and refer to the system defined by the basic axioms, the recursive comprehension scheme, and induction for  $\Sigma_0^0$  formulas plus the totality of exponentiation, as  $\text{RCA}_*$ , and the system obtained by adding weak König’s lemma to  $\text{RCA}_*$  as  $\text{WKL}_*$ .

<sup>7</sup>Avigad [1996] showed how the forcing arguments of Brown and Simpson [1993] and Harrington could be formalised in the base theory  $\text{RCA}_0$ , thus giving a new effective proof of the  $\Pi_1^1$  conservativity of  $\text{WKL}_0^+$  over  $\text{RCA}_0$ , with only a polynomial increase in the length of proofs.

ing an amalgamation theorem, they also show that  $WKL_0 + RT_2^2$  is finitistically reducible. Since  $RT_2^2$  is incomparable with  $WKL_0$  [Jockusch 1972, Liu 2012], this is a substantial extension of the principles of proof known to be finitistically reducible, which now includes a large class of combinatorial and model-theoretic principles that have been the focus of much of recent research in reverse mathematics (for a survey of these principles, see §4–5 of Shore [2010]; a more comprehensive introduction to the study of combinatorial principles related to Ramsey’s theorem is Hirschfeldt [2014]).

Having considered the extent to which the reverse mathematics of systems such as  $WKL_0$  provides a foundational analysis of the program of finitistic reductionism, and the relationship of this reductive program to Hilbert’s, we now turn to the study of predicativism in the spirit of Weyl and its connections to subsystems of second order arithmetic such as  $ACA_0$  and  $ATR_0$ .

## 4 Predicativism and predicative reductionism

Predicativism is the view that only those sets that can be defined without reference to themselves are legitimate, existing objects. Predicativism given the natural numbers is the view that the natural numbers  $\mathbb{N}$  form a completed, infinite totality and thus quantification over the natural numbers is a legitimate way to define sets. For the rest of this article, whenever ‘predicativism’ and its cognates are invoked, it is predicativism given the natural numbers that is meant. Predicativism can be seen as a middle ground between finitism, in which only finite entities are accorded real existence, and forms of set-theoretic platonism, where defining objects by impredicative quantification is acceptable because the objects being quantified over are considered to have an existence independent of their definition. Although the predicativist only accepts definitions that quantify over objects that are not themselves the subjects of the definition, collections introduced by previous predicative definitions are legitimate objects that one may quantify over when defining new objects. Predicative definability is thus an iterable notion. Given the natural numbers  $\mathbb{N}$  we may define the collection  $R_1$  of sets definable using only the language of arithmetic and quantifiers ranging over the natural numbers (in other words, the arithmetical sets). We may then take the collection of sets definable by formulas in the second-order language of arithmetic, but where the second order quantifiers are relativized to range over  $R_1$ . Since the second order quantifiers do not range over the objects being defined (as they are restricted to ranging over objects we have previously guaranteed the existence of), this gives us an expanded collection of predicatively definable sets of natural numbers  $R_2$ . The natural next step is to form a ramified hierarchy of sets of natural numbers. Given a domain  $D \subseteq \mathcal{P}(\omega)$ , let  $D^*$  be the collection of sets of natural numbers defined by formulas in the language  $L_2$  of second order arithmetic,  $\varphi^D$ , where the second order quantifiers in  $\varphi$  are relativized to range over  $D$ . We then define the ramified analytical hierarchy by transfinite recursion on ordinals as

$$\begin{aligned} R_0 &= \emptyset \\ R_{\alpha+1} &= (R_\alpha)^* \\ R_\lambda &= \bigcup_{\beta < \lambda} R_\beta \text{ for limit } \lambda. \end{aligned}$$

The ramified theory of types, developed by Russell and Whitehead in *Principia Mathematica* [Whitehead and Russell 1925], proved too cumbersome for real use in

mathematics. Hermann Weyl’s predicative development of analysis in *Das Kontinuum* [Weyl 1918] showed that a theory of the arithmetical sets  $R_1$  is already sufficient to develop a substantial portion of classical analysis, including a sequential form of the least upper bound principle. A modern formal reconstruction of arithmetical analysis in the mode of Weyl is given by the system  $\text{ACA}_0$ , defined as  $\text{RCA}_0$  plus the arithmetical comprehension axiom: every set definable by an arithmetical formula exists.<sup>8</sup> Weyl showed that the amount of mathematics one could develop in his predicative framework was extensive, and allowed one to recover much of classical analysis. The reverse mathematics of  $\text{ACA}_0$  can be viewed as a continuation of this project, showing us not only what mathematics can be predicatively proved, but also which theorems cannot be proved in  $\text{RCA}_0$  or  $\text{WKL}_0$ , and actually require arithmetical comprehension. This includes theorems in analysis such as the Cauchy convergence theorem and the Ascoli lemma, but also theorems from algebra and combinatorics: that every countable vector space over  $\mathbb{Q}$  has a basis, and Ramsey’s theorem that for every  $k \in \mathbb{N}$ , every colouring of  $[\mathbb{N}]^k$  has a homogeneous set.

Unlike  $\text{WKL}_0$ , which is only reducible to a finitistic system, and not a finitistic system in and of itself,  $\text{ACA}_0$  is a formal system whose axioms can all be directly justified on predicative grounds. No general, limitative account of what principles are predicatively acceptable is required in order to show that  $\text{ACA}_0$  is a predicative system. As indicated above, however, predicative definability is an iterable notion, and consequently Weyl accepted a Principle of Iteration that, as Feferman [1988a]’s analysis demonstrates, goes beyond what his restriction to the arithmetically definable sets allows.<sup>9</sup> The strength of Weyl’s system with the Principle of Iteration therefore exceeds that of  $\text{ACA}_0$ , as sketched in §8 of Feferman [1988a], although Feferman and Jäger [1993, 1996] proved that it is still a predicative system, in a sense we will now explore.

The iterability of predicative definability suggests that there should be a correspondingly iterable notion of *predicative provability*, and it is this notion that is the subject of Feferman and Schütte’s influential analysis of the limits of predicativity. The ramified analytical hierarchy provides a standard model on which to base the development of predicative theories, starting with the language. Supplementing the usual first order language of arithmetic, the *language of ramified analysis* includes a stock of set variables  $X^\beta, Y^\beta, Z^\beta, \dots$  for each recursive ordinal  $\beta$ . Iterated predicative definability is formalised by a transfinite progression of formal systems of ramified analysis  $\mathbf{RA}_\alpha$ . Each such system has the ramified comprehension scheme

$$\exists X^\beta \forall n (n \in X^\beta \leftrightarrow \varphi(n))$$

for all  $\beta \leq \alpha$  and all formulas  $\varphi(n)$  in the language of ramified analysis such that the bound and free set variables in the formula all have ordinal indices smaller than  $\beta$ . It also has the following limit rule, where for all (codes for) limit ordinals  $\lambda \leq \alpha$ , and each formula of ramified analysis  $\psi(X^\lambda)$  with just  $X^\lambda$  free, if  $\psi(X^0), \dots, \psi(X^\beta), \dots$  for all  $\beta < \lambda$ , then  $\psi(X^\lambda)$  also holds.

This definition just leaves open how far the iteration process can go and still be considered predicative. Suspicion naturally attaches to the ordinals indexing the theories  $\mathbf{RA}_\alpha$ , for two reasons. Firstly, the question of whether a recursive linear order

<sup>8</sup>Here we brush over the details of the connection between Weyl’s system and subsystems of second order arithmetic, for which Feferman [1988a] is the authoritative source. An accessible summary of Weyl’s development of arithmetical analysis can be found in Feferman [2005].

<sup>9</sup>See §7 of Weyl [1918] for a definition of the Principle of Iteration, and the discussion in Feferman [1988a], especially that on pp. 264–5 of the revised version in Feferman [1998].

$\prec$  is a wellordering is, in general, impredicative: the statement that  $\prec$  is wellordered quantifies over all sets of natural numbers. Secondly, in the language of ramified analysis there is no single formula that expresses that  $\prec$  is wellordered. As far as this second issue is concerned, let  $\text{WO}^\beta(\prec)$  be the statement that no set  $X^\beta$  codes an infinite descending sequence in the linear order  $\prec$ . Then, by the limit rule, any proof in a system of ramified analysis that  $\text{WO}^0(\prec)$  can be lifted to a proof that  $\text{WO}^\beta(\prec)$  for each new  $\beta$  introduced. The first issue can then be resolved by the introduction of the notion of a *predicative ordinal*. The predicative ordinals are those recursive ordinals which can be predicatively proved to be wellordered, and the predicatively acceptable systems of ramified analysis are those indexed by predicative ordinals. This might seem circular, since we are using the notion of a predicative ordinal in order to characterise predicative provability, and predicative provability in order to determine which are the predicative ordinals. But as the following *autonomy condition* should make clear, this is not the case. 0 is a predicative ordinal. Suppose  $\prec$  is a recursive linear order and  $\alpha$  is a predicative ordinal. If  $\mathbf{RA}_\alpha \vdash \text{WO}^0(\prec)$ , then the ordinal  $\beta = \text{ot}(\prec)$  is predicative. This allows us to define predicative provability in terms of autonomous transfinite progressions of systems of ramified analysis, as follows: a sentence  $\varphi$  of ramified analysis is predicatively provable if there is a predicative ordinal  $\delta$  such that  $\mathbf{RA}_\delta \vdash \varphi$ . On the basis of these definitions, Feferman [1964] and Schütte [1964, 1965] determined the limit of predicativity, namely the least ordinal that cannot be proved to be wellordered within an autonomous progression of systems of ramified analysis. This is the ordinal  $\Gamma_0$ , also known as the Feferman–Schütte ordinal, or the ordinal of predicativity.<sup>10</sup>

With the limits of predicativity characterised in this manner,<sup>11</sup> we might now ask how much of second order arithmetic can be justified on predicative grounds. Reducing subsystems of second order arithmetic to predicative systems has practical benefits for the predicativist, since ramified systems are so intractable in terms of the actual pursuit of mathematics. It is also valuable from the point of view of foundational analysis, since it allows one to determine the amount of mathematics recoverable in a predicative framework.

A formal system  $T$  is *predicatively reducible* if there is an  $\alpha < \Gamma_0$  such that  $T$  is proof-theoretically reducible to  $\mathbf{RA}_\alpha$ , and *locally predicatively reducible* if it is proof-theoretically reducible to  $\mathbf{RA}_{\Gamma_0} = \bigcup_{\alpha < \Gamma_0} \mathbf{RA}_\alpha$ , where proof-theoretic reducibility is defined as in §3.<sup>12</sup> Predicative reductionism thus offers a reductionist program similar in spirit to the finitistic reductionism discussed in §3.<sup>13</sup> By a theorem of Friedman, McAloon, and Simpson [1982], the subsystem of second order arithmetic known as  $\text{ATR}_0$  is locally predicatively reducible, with proof-theoretic ordinal  $\Gamma_0$ . Moreover, it is conservative over  $\mathbf{RA}_{\Gamma_0}$  for arithmetical sentences, and even for  $\Pi_1^1$  sentences.<sup>14</sup>

<sup>10</sup>For a more detailed summary and historical background on predicativism, predicative provability, and predicative reductionism, we refer the reader to Feferman [2005]. Complete proofs of Feferman and Schütte’s key results can be found in Schütte [1977, ch. VIII] or Pohlers [2009, ch. 8].

<sup>11</sup>Notwithstanding Weaver [2009]’s claim that  $\Gamma_0$ , as well as larger recursive ordinals such as the small Veblen ordinal, can be proved to be wellfounded using only predicative means.

<sup>12</sup>In his survey of predicativity, Feferman [2005] prefers the terms *predicatively justifiable* and *locally predicatively justifiable* to *predicatively reducible* and *locally predicatively reducible*. This paper sticks to the older terminology.

<sup>13</sup>The comparison between finitistic reductionism and predicative reductionism has been considered explicitly in these terms by Simpson [1985, §5].

<sup>14</sup>Strictly speaking there is no common notion of a  $\Pi_1^1$  statement shared between the language of second order arithmetic and the language of ramified analysis. However, given a  $\Pi_1^1$  sentence  $\varphi$  in the language of second order arithmetic with bound set variables  $X_1, \dots, X_n$ , such that  $\varphi$  is provable in  $\text{ATR}_0$ , there exists  $\alpha < \Gamma_0$  such that the translation  $\varphi^*$  of  $\varphi$  into the language of ramified analysis

So not only does  $\text{ATR}_0$  agree with the predicative part of ramified analysis about arithmetical truth, it also proves the same theorems about the arithmetical properties of all real numbers.

The formal system  $\text{ATR}_0$  consists of  $\text{ACA}_0$  plus a scheme of *arithmetical transfinite recursion*. This states that the arithmetical operations can be iterated, starting from any set  $X \subseteq \mathbb{N}$ , along any countable wellordering; a full formal definition can be found in Simpson [2009, §V.2].  $\text{ATR}_0$  is a significant strengthening of  $\text{ACA}_0$ , taking us from classical analysis to parts of descriptive set theory: arithmetical transfinite recursion is equivalent over  $\text{RCA}_0$  to the perfect set theorem (every uncountable closed set has a perfect subset), Lusin’s separation theorem (any two disjoint analytic sets can be separated by a Borel set), and a number of statements concerning ordinals, for example that any two countable wellorderings are comparable.

Burgess [2010, p. 140]’s caution about conservativity applies to the case of  $\text{ATR}_0$  and predicative provability just as it does to the case of  $\text{WKL}_0$  and finitist provability. The conservativity theorem in this case will, for a proof  $p$  in  $\text{ATR}_0$  of a  $\Pi_1^1$  statement  $\varphi$ , provide a primitive recursive function  $f$  that transforms  $\text{ATR}_0$  proofs of  $\Pi_1^1$  sentences into proofs in  $\mathbf{RA}_{\Gamma_0}$ , so that  $f(p)$  is a proof in  $\mathbf{RA}_\alpha$  of  $\varphi^*$ , for some  $\alpha < \Gamma_0$ . Nevertheless, the predicative mathematician cannot immediately conclude that  $\varphi^*$  is predicatively provable, because the Feferman–Schütte analysis of the limits of predicativity is external to the predicativist standpoint, and thus not something the predicativist has access to: they cannot, from a predicative standpoint, prove that all ordinals below  $\Gamma_0$  are wellfounded, but can only verify of particular presentations of ordinals below  $\Gamma_0$  that they do indeed code ordinals. To recognise that  $\varphi^*$  is indeed predicatively provable, the predicativist must first verify that  $\alpha$  is a predicative ordinal, by carrying out the bootstrapped process of proving linear orderings to be wellfounded within predicative systems of ramified analysis described by the analysis of predicativity.<sup>15</sup>

Paralleling the response Burgess sketches on behalf of the proponent of finitistic reductionism (discussed in §3), one might think that (globally, not merely locally) predicatively reducible subsystems of second order arithmetic offer a way to gain the benefits of predicative reductionism without running into the problems faced by locally predicatively reducible systems such as  $\text{ATR}_0$ . Unfortunately the advantages that working within predicatively reducible subsystems of second order arithmetic offer over working with the predicative system  $\text{ACA}_0$  are minimal: as Simpson [1985, §5] stresses, there are few theorems of ordinary mathematics that are known to be true in the hyperarithmetical sets  $\text{HYP}$  (when viewed as an  $\omega$ -model) that are not already true in the arithmetical sets  $\text{ARITH}$ . This is salient for predicative provability because  $\text{HYP} = R_{\omega_1^{ck}}$ , the sets definable by iterating the ramified analytical hierarchy up to  $\omega_1^{ck}$ .  $\text{HYP}$  is therefore an extension of the standard model  $R_{\Gamma_0}$  of ramified analysis up to  $\Gamma_0$ , so if an  $L_2$ -sentence  $\varphi$  is false in  $\text{HYP}$ , then it is not predicatively provable.

Simpson argues that the many theorems which provable in  $\text{ATR}_0$  but which do not

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is provable in  $\mathbf{RA}_\alpha$ , where  $\varphi^*$  is obtained by replacing each  $X_k$  for  $1 \leq k \leq n$  with  $X_k^0$ .

<sup>15</sup>Burgess [2010, p. 140] writes that “If we move up to the level of predicativism, the result on the conservativeness of the system called  $\text{ATR}_0$  over the system called  $\mathbf{IR}$  has the same character as the result on the conservativeness of  $\text{WKL}_0$  over  $\text{PRA}$ .” In a similar vein, Simpson [1985, p. 154] writes that “Feferman [1964] has argued successfully that his formal system  $\mathbf{IR}$  and others like it constitute a precise explication of predicative provability.” The system  $\mathbf{IR}$  was introduced by Feferman [1964], and is characterised by the  $\Delta_1^1$  comprehension rule and the Bar Rule. Strictly speaking, the quoted remarks are incorrect, since  $\mathbf{IR}$  is not a predicative system, but instead a locally predicatively reducible one, just as  $\text{ATR}_0$  is. Burgess’s point about conservativity, properly reformulated in terms of conservativity over  $\mathbf{RA}_{\Gamma_0}$ , thus applies to  $\mathbf{IR}$  just as it applies to  $\text{ATR}_0$ .

hold in the  $\omega$ -model HYP demonstrate the benefits of predicative reductionism, but not being predicatively provable, these theorems have only instrumental value to the predicativist: statements such as the perfect set theorem, or the theorem that any two countable wellorderings are comparable can be used to prove predicative theorems, but are not themselves predicative. All this goes to show that the mathematics that can be recovered by reductionist programs such as finitistic reductionism and predicative reductionism cannot be simply read off from reverse mathematical results, based on identifications (such as that of  $WKL_0$  for finitistic reductionism, or  $ATR_0$  for predicative reductionism) of subsystems of second order arithmetic associated with that foundational framework.

## 5 Shore's program

However much it may borrow from other areas of mathematical logic, reverse mathematics is ultimately a proof-theoretic endeavour. Given a theorem of ordinary mathematics, the reverse mathematician seeks to find a subsystem of  $Z_2$  that is equivalent over a weak base theory to the theorem concerned. She thereby finds the proof-theoretic strength of the theorem. Rooted in niceties of formal systems such as axiom schemes and complexity hierarchies of formulae, this approach may seem awkward and even unnatural to mathematicians in more mainstream fields. As number theorist Barry Mazur says [Mazur 2008, p. 224, emphasis in original],

when it comes to a crisis of rigorous argument, the open secret is that, for the most part, mathematicians who are not focussed on the architecture of formal systems per se, mathematicians who are *consumers* rather than *providers*, somehow achieve a sense of utterly firm conviction in their mathematical doings, without actually going through the exercise of translating their particular argumentation into a brand-name formal system.

Turning to the specific case of the strength of mathematical theorems, Shore [2010, p. 381] contends that most mathematicians do not approach this task from the viewpoint of reverse mathematics:

While they may concern themselves with (or attempt to avoid) the axiom of choice or transfinite recursion, they certainly do not think about (nor care), for example, how much induction is used in any particular proof.

Shore goes on to argue that adopting a computational approach to reverse mathematics would solve this exegetical problem, providing a natural way for mathematicians to understand the motivations and results of reverse mathematics.

A computational account of reverse mathematics can be considered plausible only if mathematical principles have computational content. At least in the case of arithmetic it is clear that this is true, as demonstrated by the pioneering results of Gödel, Church, Turing, Post, Kleene and Rosser in the 1930s. Computability theory holds an important status in reverse mathematics, both in virtue of its relationship to subsystems of reverse mathematics and because it provides a battery of tools for proving reverse mathematical results. It is these principles and techniques which Shore appeals to when constructing his account of computational reverse mathematics.

In particular, the major subsystems of second order arithmetic correspond to principles from computability theory. As well as shedding light on the model theory of these systems, these connections give us the basis for Shore's computational reverse



mathematics. The foundation of these correspondences lies in the notion of an  $\omega$ -*model*. These are  $L_2$ -structures whose first order parts consists of the standard natural numbers  $\omega = \{0, 1, 2, \dots\}$ , and whose arithmetical vocabulary is interpreted in the standard way, with a second order part  $\mathcal{S} \subseteq \mathcal{P}(\omega)$ .  $\omega$ -models are thus uniquely distinguished by their second order parts, and which sentences of  $L_2$  a given  $\omega$ -model  $M$  satisfies is determined entirely by the sets in its second order part. In the rest of the paper we shall therefore allow ourselves some sloppiness and identify  $\omega$ -models with their second order parts whenever no ambiguity is possible.

Since  $\omega$ -models of  $\text{RCA}_0$  satisfy recursive comprehension, they are closed under Turing reducibility: given an  $\omega$ -model  $\mathcal{S}$  of  $\text{RCA}_0$ , if  $X \in \mathcal{S}$  and  $Y \leq_{\mathbf{T}} X$ , then  $X \in \mathcal{S}$ . They are also closed under recursive joins: if  $X, Y \in \mathcal{S}$ , then  $X \oplus Y \in \mathcal{S}$ , where the recursive join operation is defined as

$$(2) \quad X \oplus Y = \{2x \mid x \in X\} \cup \{2y + 1 \mid y \in Y\}.$$

Subsets of  $\mathcal{P}(\omega)$  which are closed under Turing reducibility and recursive joins are known as *Turing ideals*, and the  $\omega$ -models of  $\text{RCA}_0$  are precisely the Turing ideals.

Similar closure conditions apply to the  $\omega$ -models of the other main subsystems of second order arithmetic.  $\omega$ -models of  $\text{ACA}_0$  are Turing ideals, since  $\text{RCA}_0$  is a sub-theory of  $\text{ACA}_0$ , but these models are also closed under the Turing jump operator, while those of  $\Pi_1^1\text{-CA}_0$  are closed under the hyperjump. Computability-theoretic closure conditions also characterise the  $\omega$ -models of the intermediate systems  $\text{WKL}_0$  and  $\text{ATR}_0$ . The  $\omega$ -models of  $\text{WKL}_0$  are related to the Jockusch–Soare low basis theorem [Jockusch and Soare 1972]. The  $\omega$ -models of  $\text{ATR}_0$  are closed under hyperarithmetical reducibility, although there are some subtleties here; see §VIII.4 and §VIII.6 of Simpson [2009]. The Big Five thus correspond closely to a hierarchy of computational principles of increasing power.

Shore proposes a new approach to reverse mathematics based on taking these computability-theoretic characterisations of the  $\omega$ -models of subsystems of  $\mathbf{Z}_2$  at face value as measuring the complexity of the theorems equivalent to those systems. In place of the usual relations employed in reverse mathematics, namely provability and logical equivalence over a weak base theory, he offers the notions of *computable entailment* and *computable equivalence*.

**Definition 5.1.** Let  $\mathcal{C}$  be a Turing ideal, and let  $\varphi$  be a sentence of second order arithmetic.  $\mathcal{C}$  *computably satisfies*  $\varphi$  if  $\varphi$  is true in the  $\omega$ -model whose second order part consists of  $\mathcal{C}$ . A sentence  $\psi$  *computably entails*  $\varphi$ ,  $\psi \models_{\mathcal{C}} \varphi$ , if every Turing ideal  $\mathcal{C}$  computably satisfying  $\psi$  also computably satisfies  $\varphi$ . Two sentences  $\psi$  and  $\varphi$  are *computably equivalent*,  $\psi \equiv_{\mathcal{C}} \varphi$ , if each computably entails the other. These definitions extend to theories in the standard way.

Computable entailment removes any need for an explicit base theory: this role is instead played by the restriction of the class of models under consideration to  $\omega$ -models whose second order parts are Turing ideals. The  $\omega$ -models of  $\text{RCA}_0$  are precisely those models, so the base theory has not disappeared entirely, but manifested itself in a different way, by being baked into the definition of the computable entailment relation. Since not all  $L_2$ -structures are  $\omega$ -models, failures of computable entailment are stronger than failures of logical implication over  $\text{RCA}_0$ , since the former entails the latter, but not vice versa. Conversely, computable entailment is weaker than logical implication over  $\text{RCA}_0$ . By the Henkin–Orey completeness theorem for  $\omega$ -logic [Henkin 1954, Orey 1956], the computable entailment relation is extensionally equivalent to allowing unrestricted use of the  $\omega$ -rule in  $\text{RCA}_0$ . The  $\omega$ -rule is an infinitary rule

of inference that, from the infinite set of premises  $\varphi(0), \varphi(1), \dots, \varphi(\bar{n}), \dots$  for all numerals  $\bar{n}$ , one may infer the universal statement  $\forall n \varphi(n)$ . Proofs using the  $\omega$ -rule can be represented by wellfounded, countably branching trees. Second order arithmetic with the  $\omega$ -rule is complete for  $\Pi_1^1$  sentences, but not for  $\Sigma_1^1$  sentences [Rosser 1937].

Shore puts forward a number of considerations in support of his proposal.

- (1) Computational reverse mathematics unites proofs of implications (and hence equivalences) with proofs of nonimplications (and hence inequivalences).
- (2) Computational reverse mathematics offers a more direct route to the complexity measures we take to underpin the identification of the strength of theorems of ordinary mathematics.
- (3) Computational mathematics is a more natural framework for ordinary mathematicians.
- (4) Computational reverse mathematics provides a way to deal directly with uncountable structures and extend reverse mathematics to the study of theorems concerning essentially uncountable structures.

These considerations can be understood as arguments for two quite different conclusions. The first is that computable entailment is an important reducibility notion with intrinsic mathematical interest, one which merits and will reward further study.<sup>16</sup> The second is that it should be the primary tool we use to carry out the general task of reverse mathematics, namely to show what mathematical resources are needed to prove ordinary mathematical theorems. Shore does not explicitly endorse either of these options; neither does he propose computational reverse mathematics as a framework in which to carry out foundational analysis. However, by reviewing the reasons underlying his proposal, it becomes clear that arguments based on these considerations make adopting computational reverse mathematics as a first-class replacement for classical reverse mathematics a serious option. This being the case, it seems reasonable to wonder whether his framework can contribute to the analysis of foundational programs just as classical reverse mathematics does. Before attempting to answer this question, we first analyse in detail Shore's considerations in favour of computational reverse mathematics.

## 5.1 Unity of reverse mathematics

The first consideration in favour of computational reverse mathematics is that (1) computable entailment unites proofs of implications with the existing practice of using computability-theoretic tools to construct Turing ideals witnessing the failures of implications, bringing a unity to the methods of proof in reverse mathematics. The procedure is particularly straightforward when the sentences in question are  $\Pi_2^1$ , where the following template applies. Given  $\Pi_2^1$  statements  $\Phi \equiv \forall X \exists Y \varphi(X, Y)$  and  $\Psi \equiv \forall X \exists Y \psi(X, Y)$ , one constructs a Turing ideal  $\mathcal{C}$  where for every  $X \in \mathcal{C}$ , there exists an  $Y \in \mathcal{C}$  such that  $\varphi(X, Y)$ , but there is no  $Y \in \mathcal{C}$  such that  $\psi(X, Y)$ .  $\mathcal{C}$  is therefore the second-order part of an  $\omega$ -model that satisfies  $\text{RCA}_0$  and  $\Phi$ , but not  $\Psi$ ,

<sup>16</sup>A comparable notion that provides a finer-grained degree structure is Weihrauch reducibility, recently taken up in the reverse mathematics context by Dorais, Dzhafarov, Hirst, Mileti, and Shafer [2016], but a comparison of Weihrauch reducibility to computational reverse mathematics is beyond the scope of this paper.

and so  $\text{RCA}_0$  does not prove that  $\Phi$  implies  $\Psi$ . Since many important mathematical theorems are  $\Pi_2^1$ , the technique is widely applicable. For instance, to show that weak König’s lemma does not imply arithmetical comprehension, we use the Jockusch–Soare low basis theorem to prove the existence of an  $\omega$ -model  $M$  of  $\text{WKL}_0$  in which all sets are low. Such a model will not contain  $0'$ , and thus  $M \not\models \text{ACA}_0$ , since  $\text{ACA}_0$  proves the existence of the Turing jump.

It is clear how Shore’s observation leads to his definition of the computable entailment relation, but to use it to support the adoption of computational reverse mathematics as the preferred general methodology in reverse mathematics, we must formulate a more explicit argument. The following reconstruction seems reasonable: that (1a) unity in methods of proof in reverse mathematics is a theoretical virtue, and that (1b) computational reverse mathematics possesses this unity in greater degree than classical reverse mathematics. Granting (1a) is compatible with there being other theoretical virtues, such as tractability, not requiring strong background assumptions, and so on. We stipulate for the sake of argument that (1c) the other theoretical virtues of computational reverse mathematics are at least as great as those of classical reverse mathematics. From (1a–c), it follows that computational reverse mathematics is a more theoretically virtuous framework than classical reverse mathematics in which to carry out reverse mathematics.

In the next two sections we shall discover that we have reason to doubt (1c), but for now let us grant it and concentrate on the more immediately problematic premise (1b). By the soundness theorem for first order logic, from exhibiting a model of  $\text{RCA}_0 + \Phi + \neg\Psi$  we can infer that there is no proof in  $\text{RCA}_0$  of  $\Phi \rightarrow \Psi$ . This shows that there is a unity in the methods of proof of classical reverse mathematics: to demonstrate an implication we prove that the formal system  $\text{RCA}_0$  proves it, and to demonstrate a nonimplication we prove that the formal system  $\text{RCA}_0$  does not prove the relevant implication. The soundness and completeness theorems for first order logic unite proof-theoretic and model-theoretic methods. If computational reverse mathematics has a greater unity of methods of proof, it can only be in a methodological sense, since as Shore points out, in practice the vast majority of proofs of nonimplications are carried out by constructing Turing ideals. But if the unity of methods of proof we are concerned with is merely methodological, concerning how practitioners happen to prove nonimplications in classical reverse mathematics, then premise (1a) starts to look shaky, since it seems entirely reasonable to take the unity of methods of proof (in this methodological sense) to be a purely instrumental virtue of a framework for reverse mathematics, and not a theoretical one. Even if we grant that computational reverse mathematics is better, from an instrumental point of view, than classical reverse mathematics as a framework for reverse mathematics, this seems like a minor reason to adopt it when weightier theoretical considerations are on the table.

## 5.2 Complexity and difficulty of proof

A more substantial motivation can be found in Shore’s suggestive remarks about the relationship between degrees of computability and methods of proof. The essential point is as follows. Proving a theorem that can be stated in the form “For all  $X$  such that  $\dots$ , there exists a  $Y$  such that  $\dots$ ”—that is to say, as a  $\Pi_n^1$  sentence—can be understood as providing, given as input a countably representable mathematical structure  $\mathcal{A}$ , a function or relation  $\mathcal{F}$  on  $\mathcal{A}$ . As we have seen, given some  $\mathcal{A}$ , a corresponding  $\mathcal{F}$  can be more or less complex, depending on the theorem. In classical reverse mathematics, the proof-theoretic strength of the theorem depends on where

such  $\mathcal{F}$  can be found in the hierarchy of Turing degrees relativized to  $\mathcal{A}$ . Shore proposes that we give a direct formulation of this complexity measure—the difficulty of computing a solution (the function or relation  $\mathcal{F}$ ) given a problem (the structure  $\mathcal{A}$ )—rather than mediating it through first order logical theories. Doing so “formalizes the intuition that ‘being harder to prove’ means ‘harder to compute’” [Shore 2013, p, 153]. Taking this underlying intuition into account, the appeal of Shore’s approach becomes more clear: since computation across nonstandard models of arithmetic is highly non-absolute, restricting the interpretation of arithmetical vocabulary to the standard natural numbers  $\omega$  allows us to fix a single underlying notion of computation, and thus the Turing reducibility relation  $\leq_{\mathbf{T}}$  that allows us to compare the complexity of solutions provided by theorems directly. This is not available within the classical reverse mathematics framework where the base theory  $\text{RCA}_0$  provides at best a weak constraint on the models of the arithmetical part of the theory.

We can regiment this argument as follows. (2a) The difficulty of proving a  $\Pi_n^1$  sentence  $\forall X \exists Y \varphi(X, Y)$  is just the difficulty of, given a problem  $X \subseteq \omega$ , computing a solution  $Y \subseteq \omega$  such that  $\varphi(X, Y)$ . (2b) Computable entailment captures the difficulty of—given a problem  $X \subseteq \omega$ —computing a solution  $Y \subseteq \omega$  such that  $\varphi(X, Y)$  better than  $\text{RCA}_0$ -provability does. Therefore, computable entailment captures the difficulty of proving  $\Pi_n^1$  sentences better than  $\text{RCA}_0$ -provability.

The content of the intuition (2a) that Shore takes his view to be formalising is, at least on a naïve reading, somewhat problematic. Firstly, the mathematician’s standard understanding of difficulty of proof does not square with Shore’s account: theorems in number theory are not thought less difficult to prove just because the solutions are not highly uncomputable. Secondly, while the notion of degrees of uncomputability is clear, it is arguable whether this measures difficulty of computation. Since the Turing jump and True Arithmetic are both uncomputable sets, and thus both intractable problems for finite computers, even idealised ones with unbounded time and space such as Turing machines, there is good reason to consider both of these sets equally difficult to compute, viz., impossible. We shall have more to say on this point in §7, but for now let us try to sidestep these issues by reformulating the argument above.

To do so, we must reject the naïve reading of the two notions that figure in Shore’s equivalence between difficulty of proof and difficulty of computation. Instead of the mathematician’s standard understanding of difficulty of proof, let us take “being harder to prove” to mean “requiring stronger axioms to prove”, since this is the standard of difficulty salient to reverse mathematics. To resolve the second difficulty, we replace the vague statement “harder to compute” with an explicit reference to the notion of relative computability. The reformulated argument then runs as follows. (2a\*) The strength of axioms which entail a  $\Pi_n^1$  sentence  $\forall X \exists Y \varphi(X, Y)$  is precisely captured by—given a problem  $X \subseteq \omega$ —the degrees of uncomputability of solutions  $Y \subseteq \omega$  such that  $\varphi(X, Y)$ . (2b\*) Computable entailment captures the degrees of uncomputability—given a problem  $X \subseteq \omega$ —of solutions  $Y \subseteq \omega$  such that  $\varphi(X, Y)$  better than  $\text{RCA}_0$ -provability does. Therefore, computable entailment captures the strength of assumptions which entail  $\Pi_n^1$  sentences better than  $\text{RCA}_0$ -provability.

Premise (2b\*) is well supported, given the non-absoluteness of Turing computability across nonstandard models (modulo the usual questions about the existence of the unique standard natural number structure  $\omega$ , and the background theory needed to establish it). We therefore focus on (2a\*), which is distinctly more plausible than (2a), since we already know that the Big Five subsystems of second order arithmetic are based on computability-theoretic principles, and that the hierarchy of proof-theoretic

strength of these systems correlates with the Turing degree of solutions whose existence is asserted in theorems equivalent to these systems.

A contentious point in assessing (2a\*) is the status of induction axioms. Neeman [2011] proves that  $\Sigma_1^1$  induction is required to prove that Jullien’s indecomposability theorem implies weak  $\Sigma_1^1$  choice: neither the  $\Sigma_1^0$  induction axiom of  $\text{RCA}_0$ , nor even  $\Delta_1^1$  induction, suffices. If we take the classical reverse mathematics framework to privilege  $\text{RCA}_0$  as a base theory then this is *prima facie* problematic. Since computational reverse mathematics fixes the first order part of the model as the standard natural numbers  $\omega$ , it does not have this difficulty. One problem with this view is that induction axioms are also a form set of existence principle, namely bounded comprehension schemes for finite sets. The  $\Sigma_1^0$  induction scheme is equivalent over  $\text{RCA}_*$  to bounded  $\Sigma_1^0$  comprehension: the scheme that for every  $n \in \mathbb{N}$  the set  $X = \{k < n \mid \varphi(k)\}$  exists, where  $\varphi(x)$  is any  $\Sigma_1^0$  formula. Stronger induction schemes are likewise equivalent to stronger bounded comprehension schemes. Simpson and Smith [1986] show that a number of theorems from algebra are equivalent over  $\text{RCA}_*$  to  $\Sigma_1^0$  induction. This undermines the idea that classical reverse mathematics should be identified with reverse mathematics in  $\text{RCA}_0$ , but perhaps this is no bad thing. Computational reverse mathematics, on the other hand, appears to trivialise bounded comprehension schemes, since they are all computably entailed.  $\Pi_n^1$  theorems, central to all of reverse mathematics, thus become almost the only subject countenanced (further reverse mathematical equivalences trivialised by the computable entailment relation are considered in §6).

### 5.3 Accessibility of reverse mathematics

Another of Shore’s motivations in introducing computational reverse mathematics is expository: making the tools and results of reverse mathematics more accessible to ordinary mathematicians who do not think, as logicians do, in terms of formal theories and proof systems. Although it still involves formalisation, computational reverse mathematics does allow us to sidestep some formal aspects of reverse mathematics. Instead of proving equivalences to syntactically defined subsystems of second order arithmetic, we can work directly with computability-theoretic and combinatorial closure conditions. Moreover, it allows us to use induction in the standard way, for any property, not just those definable in the language of arithmetic with a single first order quantifier.

Making reverse mathematics more accessible to ordinary mathematicians is clearly a valuable goal, but since computable entailment is not extensionally equivalent to provability in the standard base theory  $\text{RCA}_0$ , this is not by itself sufficient to motivate our adoption of computational reverse mathematics over classical reverse mathematics. In order to provide the relevant motivation, we could think of this as an instance of naturalistic deferral to mathematical practice, that is, (3) computable entailment is the right way to measure the strength of theorems because it better captures the way ordinary mathematicians work with the objects these theorems concern. In particular, ordinary mathematicians “do not think about (nor care) [...] how much induction is used in any particular proof” [Shore 2010, p. 381]. Moreover, definitions and inferences in informal mathematical practice are not carried out within a fixed formal framework. Working only with  $\omega$ -models thus reflects ordinary mathematical practice as, in practice, mathematicians consider the natural numbers to be categorical and to satisfy induction for all predicates.

## 5.4 Uncountable reverse mathematics

Since second order arithmetic does not allow quantification over uncountable sets, classical reverse mathematics is necessarily limited to treating theorems concerning objects that are either countable, or can be represented by countable codes. Computational reverse mathematics offers a way to overcome this barrier and allow for the development of a reverse mathematics of uncountable mathematics, by using one of the existing definitions of computation on uncountable mathematical structures.<sup>17</sup> This is done by altering the definition of computable entailment to quantify not over Turing ideals, but over over classes of sets that are closed under a different notion of relative computability appropriate to the particular uncountable setting. Shore [2013] takes some initial steps in this direction by developing a variation of computable entailment that uses  $\alpha$ -recursion theory, along with analogues of ACA and WKL in this setting, and proving some reverse mathematics-style results for them.

This suggests the following argument. (4a) A framework for reverse mathematics that allows us to analyse the strength of theorems throughout ordinary mathematics (i.e. not just countable and countably-representable mathematics) is superior to one that does not. (4b) Computational reverse mathematics allows us to carry out such an analysis. (4c) Classical reverse mathematics does not. Therefore, computational reverse mathematics is a superior framework in which to carry out reverse mathematics.

Little work has been done in this setting beyond Shore’s initial papers, so its full promise remains as yet unfulfilled. Nevertheless, the adaptability of Shore’s approach to different settings, with the underlying notion of computation allowed to vary so as to provide an appropriate measure of the computational strength of theorems in those settings, suggests a highly promising route to a reverse mathematics of the uncountable. That being said, if a computational reverse mathematics of the uncountable turns out to be a fruitful approach, there is no obvious barrier to developing an axiomatic counterpart that stands in a similar relation to, for example,  $\alpha$ -recursion on uncountable ordinals as  $\text{RCA}_0$  does to Turing reducibility. So even accepting (4a), the status of (4b) and (4c) remains unclear. It is therefore difficult to evaluate the extent to which its extensibility to uncountable structures weighs in favour of computational reverse mathematics.

Intriguingly, Shore [2010] suggests that computational reverse mathematics of the uncountable will also provide a testing ground for notions of computability on uncountable sets, and that (p. 387) “if a theory of computability for uncountable domains provides a satisfying analysis of mathematical theorems and constructions in the reverse mathematical sense based on the approach of [definition 5.1], then it has a strong claim to being a good notion of computation in the uncountable.” This would result in a virtuous circle of justification. On the one hand, the success of a notion of computation on uncountable sets in providing a reverse mathematics of the uncountable would vindicate it as the correct notion of computation in the uncountable setting. On the other, its status as the correct notion of computation in the uncountable setting would support its use as the notion of computation underlying the reverse mathematics of the uncountable. Alternatively, it may turn out that “there is no single ‘right’ [notion of computation on uncountable sets] but that certain ones may be better than others for different branches of mathematics” [Shore 2010, p. 387]. If pursuing computational reverse mathematics could help answer these questions it

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<sup>17</sup>Others have also proposed ways to extend reverse mathematics to the uncountable, most notably Kohlenbach [2002, 2005] who has developed an approach in higher-order arithmetic.

would certainly strengthen the case for doing so, but that it might do so does not provide a prima facie reason for preferring it over classical reverse mathematics as our general framework for reverse mathematics.

## 5.5 Closure conditions and the standard view of reverse mathematics

The standard view in reverse mathematics holds that the significance of reversals lies in the set existence principles they show to be necessary to prove ordinary mathematical theorems.<sup>18</sup> However, the relevant concept of a set existence principle, as used in reverse mathematics, has not been explicated in any detail. [Dean and Walsh \[2017\]](#) have argued that such a concept cannot be exhausted by the notion of a comprehension principle, since this would leave out weak König’s lemma and arithmetical transfinite recursion. In [\[Eastaugh 2018\]](#) I argue that even if one includes both comprehension principles and separation principles in the concept of a set existence principle, thereby incorporating all of the Big Five, there are other mathematically natural set existence principles that are left out, such as weak König’s lemma.<sup>19</sup>

[Shore \[2010\]](#)’s emphasis on the characterisation of the Big Five in terms of computability-theoretic closure conditions suggests that we could understand the concept of a set existence principle in terms of that of a closure condition on the powerset of the natural numbers. I advance a view along these lines in [\[Eastaugh 2018\]](#), where I take closure conditions as being axiomatized by those  $\Pi_n^1$  sentences ( $n \geq 2$ ) that are not equivalent to any less complex sentence.<sup>20</sup> This account of set existence principles as closure conditions has a number of advantages, including its generality, since it not only includes all of the Big Five and weak König’s lemma, but also other statements such as choice principles and Ramsey-like combinatorial statements. It also has an intuitive appeal, as attested by a number of authors who have identified set existence principles with closure conditions, albeit without providing a precise account of which sentences of second order arithmetic axiomatize closure conditions.<sup>21</sup>

Despite these appealing characteristics, the details of the account demonstrate an oddity, namely the status of induction axioms. Instances of the arithmetical induction scheme  $\Pi_\infty^0$ -IND are at most of  $\Pi_1^1$  complexity, and thus on this account are not considered to be set existence principles.<sup>22</sup> More complex fragments of the full induction scheme, however, such as  $\Sigma_1^1$  induction, will be axiomatized by  $\Pi_n^1$  sentences,  $n \geq 2$ , and will not be equivalent to less complex sentences except by a base theory that proves them outright. They should thus, according to the view put forward in [\[Eastaugh 2018\]](#), be considered closure conditions, and thus set existence principles. This is counterintuitive, since instances of induction are typically taken to concern the structure of the natural numbers, not the structure of its powerset.

This entanglement between the first and second order parts of the theory arises because of the existence of non-standard models of arithmetic: the standard natural

<sup>18</sup>The locus classicus of this view is [Simpson \[2009, p. 2\]](#).

<sup>19</sup>The restriction of weak König’s lemma to trees of positive measure.

<sup>20</sup>Here “equivalent” is to be understood as meaning provably equivalent in any suitable base theory  $B$  that does not prove the sentence in question, but which can otherwise be as strong as possible. Sentences of this sort are referred to in [\[Eastaugh 2018\]](#) as *essentially*  $\Pi_{n \geq 2}^1$  sentences; see §5 of that paper for a detailed account of these notions.

<sup>21</sup>For instance [Feferman \[1964, p. 8\]](#), [Feferman \[1992, p. 451\]](#), [Dorais, Dzhafarov, Hirst, Mileti, and Shafer \[2016, p. 2\]](#), and [Chong, Slaman, and Yang \[2014, p. 864\]](#).

<sup>22</sup>Cf. [Simpson \[2009, pp. 71–2\]](#), who argues that “despite appearances, the  $\Sigma_1^0$  induction axiom of  $\text{RCA}_0$  can be considered to be a set existence axiom”, due to its equivalence to the scheme of bounded  $\Sigma_1^0$  comprehension.

numbers  $\omega$  satisfy not just the full induction scheme  $\Pi_\infty^1$ -IND, but the induction scheme formulated in any language, no matter how expressively powerful. Moreover, all finite sets are already present in any model satisfying recursive comprehension, since finite sets are by definition recursive. It is only when we consider non-standard models that bounded comprehension schemes are required to ensure the existence of “finite” sets of size  $k$ , where  $k$  is a non-standard number.

Shore’s view offers us a natural response to this problem. Since in computational reverse mathematics we take the first order part of the model to be fixed as the standard natural numbers  $\omega$ , we can rule out instances of induction as axiomatizing closure conditions, for they will already be satisfied by any base theory  $B$ , as the first order variables of sentences of  $B$  will always range over standard natural numbers. The view that set existence principles are exactly the closure conditions axiomatized by essentially  $\Pi_{n \geq 2}^1$  sentences works smoothly in Shore’s framework, without counterintuitive instances of induction masquerading as closure conditions on  $\mathcal{P}(\mathbb{N})$ . Computational reverse mathematics thereby seems to offer us a better account of the concept of a set existence principle than classical reverse mathematics, and thus a more satisfactory way of vindicating the standard view that reversals are significant because they demonstrate the set existence principles necessary to prove theorems of ordinary mathematics.

## 6 Preservation of justification under computable entailment

Computable entailment collapses many distinctions present under the usual classical entailment relation, and thus the equivalence classes obtained under the computable equivalence relation are significantly different from those given by provable equivalence over  $\text{RCA}_0$ . For instance, the standard natural numbers satisfy the induction scheme for all predicates in the language of second order arithmetic. As a result, systems with only restricted induction and their counterparts with the full induction scheme are computably equivalent. The presence of full induction is indicated by the absence of the ‘0’ subscript in the system’s name:  $\text{RCA}$  is  $\text{RCA}_0$  but with full induction,  $\text{WKL}$  is  $\text{WKL}_0$  with full induction, and so on. In all cases, the system with full induction has precisely the same  $\omega$ -models as its counterpart with restricted induction, and thus they are computably equivalent.

This presents a problem given the connections between the Big Five and existing philosophically-motivated programs in the foundations of mathematics. At least in some cases these subsystems are formalisations in second order arithmetic of those foundational programs, but it is by no means obvious that the same is true for other axiom systems which are computably equivalent to them.  $\text{ACA}_0$  is a predicative system, but the mere fact that  $\text{ACA}$  is computably equivalent to it should not compel us to believe that  $\text{ACA}$  is similarly predicatively acceptable.

Another way to understand this point is by considering that a key property of any entailment relation is preserving justification: if we are justified in accepting the antecedent then we are justified in accepting the consequent. For computational reverse mathematics to be capable of the foundational analysis outlined earlier, computable entailment must preserve justification, just as deductive entailment does. Given any foundational program that we wish to analyse by proving reverse mathematical results, those results must be justified on the conception of justification internal to the foundational program itself. If computable entailment fails to satisfy this requirement then proponents of such foundational programs will be unmoved by any arguments drawn from computational reverse mathematics, as they will reject the underlying



assumption necessary to proving the results involved. In other words, the crux of the issue is not whether computable entailment preserves justification on some particular account of the epistemology of mathematics, but whether it respects the justificatory structure of the foundational programs being analysed.

In §3 we examined [Simpson \[1988\]](#)'s claim that the proof-theoretic reducibility of  $\text{WKL}_0$  to PRA constitutes a partial realization of Hilbert's program. There are reasons to question whether Simpson's interpretation of Hilbert is correct, and plenty of debate to be had over whether this is in fact a good foundation for mathematics. Nevertheless, the finitistic reductionism that Simpson proposes is nonetheless a foundational enterprise worthy of consideration. One part of such an assessment consists of the use of reverse mathematical methods to determine the parts of ordinary mathematics that can be developed within this foundational framework, that is, foundational analysis as studied in the preceding sections. In order to apply it in this way, our system of reverse mathematics should therefore be able to analyse Simpson's finitistic reductionism, and as argued above, that analysis should respect the justificatory structure of finitistic reductionism. With this concern in mind, the crucial question is whether or not finitistic reductionism can be extended from  $\text{WKL}_0$  to include all systems  $T$  that are computably equivalent to  $\text{WKL}_0$ . Only if this is the case can we conclude that Shore's computational reverse mathematics respects its justificatory structure.

One system that is computably equivalent to  $\text{WKL}_0$  is the system WKL. As mentioned earlier, this system augments  $\text{WKL}_0$  with the full induction scheme. If computable entailment is to preserve justification for the Tait-style finitist, then WKL must also be finitistically reducible. But the presence of the full induction scheme means that, as we shall see below, WKL proves the consistency of PRA. Therefore, it is not finitistically reducible to PRA, since the canonical formal consistency statement  $\text{Con}(\text{PRA})$  is a  $\Pi_1^0$  statement that PRA does not (if it is, in fact, consistent) prove. In other words, it rules out the possibility of a finitistic reduction of the sort delivered by Sieg for  $\text{WKL}_0$ , and thus rules out the possibility that WKL is a finitistically reducible system.

Recall that  $\text{I}\Sigma_n$  is the fragment of Peano arithmetic obtained by restricting the induction scheme to  $\Sigma_n^0$  formulae. The following is a standard result in the literature on first-order arithmetic. A full proof can be found in [Hájek and Pudlák \[1993, §I.4\]](#).

**Fact 6.1.**  $\text{I}\Sigma_{n+1}$  proves the consistency of  $\text{I}\Sigma_n$ .

**Corollary 6.2.**

1. PRA,  $\text{I}\Sigma_1$ ,  $\text{RCA}_0$  and  $\text{WKL}_0$  are equiconsistent.
2. WKL proves the consistency of the systems given in (1).
3. WKL is not  $\Pi_1^0$  conservative over the systems given in (1).

*Proof.*  $\text{I}\Sigma_1$  is  $\Pi_2^0$  conservative over PRA [[Parsons 1970](#)]; the first order part of  $\text{RCA}_0$  is  $\text{I}\Sigma_1$  (that is, they prove the same sentences in the language  $L_1$  of first order arithmetic); and  $\text{WKL}_0$  is  $\Pi_1^1$  conservative over  $\text{RCA}_0$  (this is a result of Leo Harrington; a proof appears in [Simpson \[2009, §IX.2\]](#)). Consequently any  $\Pi_2^0$  statement provable in  $\text{WKL}_0$  (or  $\text{RCA}_0$  or  $\text{I}\Sigma_1$ ) is also provable in PRA. Since the canonical consistency statements for PRA,  $\text{I}\Sigma_1$  and  $\text{WKL}_0$  are  $\Pi_1^0$ , any system proving the consistency of one of these systems proves the consistency of all the others.

By fact 6.1,  $\text{I}\Sigma_2$  proves the consistency of  $\text{I}\Sigma_1$  and hence the consistency of all the systems listed in (1). WKL extends  $\text{I}\Sigma_2$  and thus proves all the theorems it does.

Finally, by the complexity of consistency statements,  $WKL$  cannot be  $\Pi_1^0$  conservative over any of the systems listed in (1).  $\square$

The methods of infinitary mathematics are justified, according to Simpson’s finitistic reductionism, only to the extent that they are reducible to finitary ones. This seems to rule out  $WKL$  as a partial realization of Hilbert’s program quite straightforwardly. But if computable entailment preserves justification, then we are justified in accepting  $WKL$  if and only if we accept  $WKL_0$ , as they are computably equivalent. If this is not the case then computable equivalence seems to have failed as a way to analyse the mathematical resources required to derive theorems of ordinary mathematics, since it leads to underdetermination: we are no longer certain, given some theorem  $\varphi$ , whether it is acceptable to the finitistic reductionist if we know only that it is computably entailed by  $WKL_0$ . To resolve this underdetermination one could prove that  $\varphi$  follows from  $WKL_0$  using only resources acceptable to the finitistic reductionist—but since these resources are simply the axioms of a finitistically reducible system and the laws of classical logic, this amounts to simply proving the result in  $WKL_0$ , and we are no longer working in Shore’s framework, where all that is necessary to show that one principle follows from another is to demonstrate that the former is true in every  $\omega$ -model of the latter.

This being the case, we have at least one situation in which computational reverse mathematics is not sufficient to carry out a task in reverse mathematics of significant philosophical interest and importance. The computable entailment relation does not always preserve the justificatory structure of foundational theories, and hence Shore’s framework thus cannot be used to conduct the kind of foundational analysis articulated in the previous chapter—at least for one important example, namely Simpson’s finitistic reductionism.<sup>23</sup>

In fact, we can show more than this: that computable entailment does not preserve justification for the two other foundational programs we examined above: Weyl’s predicativism and the program of predicative reductionism that follows from the Feferman–Schütte analysis of predicative provability. Here we rely on a fact that will be demonstrated in lemma 7.5: if  $\varphi$  is a  $\Pi_1^1$  sentence in the language of second order arithmetic, then  $\varphi$  is true if and only if  $\models_c \varphi$ . We therefore have that for any code  $X$  for a recursive linear order  $<_X$ ,  $<_X$  is wellordered (i.e.  $X$  codes a recursive ordinal  $\alpha$ ) if and only if  $\models_c <_X$  is wellordered. Consider the ordinals  $\varepsilon_0$  and  $\Gamma_0$ , which are respectively the proof-theoretic ordinals of  $ACA_0$  and  $ATR_0$ . From an external point of view validating that the recursive codes for these ordinals really do code wellorderings, we have that  $WO(\varepsilon_0)$  and  $WO(\Gamma_0)$ . But then by the above fact,  $\models_c WO(\varepsilon_0)$  and  $\models_c WO(\Gamma_0)$ . This means that  $ACA_0 \equiv_c ACA_0 + WO(\varepsilon_0)$  and  $ATR_0 \equiv_c ATR_0 + WO(\Gamma_0)$ .

$ACA_0$  does not prove that  $\varepsilon_0$  is wellordered, but one might reasonably consider  $WO(\varepsilon_0)$  to be a predicative principle nonetheless, since stronger predicative theories, justified on the Feferman–Schütte analysis of predicativity, do prove it. However, on that understanding of predicativity, it is not predicatively provable that  $\Gamma_0$  is wellordered. There are therefore theories which are computably equivalent to predicative and predicative reducible ones but which are not themselves either predicative

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<sup>23</sup>This argument also shows that any modification of reverse mathematics to strengthen the induction principle of the base theory to include even  $\Sigma_2^0$  induction renders it inappropriate for the foundational analysis of finitistic reductionism. Friedman’s switch from subsystems of  $Z_2$  with full induction in [Friedman 1975] to systems with restricted induction in [Friedman 1976] is therefore a crucial one for the foundational analysis of finitistic reductionism.

or predicatively reducible. As we have argued above for the case of finitistic reductionism, this shows that computational reverse mathematics is not an appropriate setting in which to carry out the foundational analysis of predicativism and predicative reductionism.

One might reasonably wonder whether these latter examples are somehow artificial and do not constitute substantial counterexamples to the preservation of justification by the computable entailment relation. Since all true  $\Pi_1^1$  statements are computably entailed, this question reduces to the question of whether there are ordinary mathematical theorems that are  $\Pi_1^1$  and not justifiable on the basis of the foundational programs we are considering. The answer to this is positive, and indeed there are ordinary mathematical theorems equivalent to statements of the form we have just been considering, namely  $\text{WO}(\alpha)$  for  $\alpha < \omega_1^{ck}$ . Simpson [1988] showed that Hilbert's basis theorem is equivalent over  $\text{RCA}_0$  to the wellordering of  $\omega^\omega$ . This is the proof-theoretic ordinal of  $\text{WKL}_0$ , so  $\text{WO}(\omega^\omega)$  is computably entailed by  $\text{WKL}_0$  but not finitistically reducible, since over a weak base theory it implies the consistency of  $\text{WKL}_0$  and thus that of PRA. A much stronger example is Kruskal's theorem, a famous result in graph theory that is equivalent over  $\text{ACA}_0$  to the wellordering of the small Veblen ordinal  $\vartheta\Omega^\omega$  [Rathjen and Weiermann 1993, p. 62]. This is a recursive ordinal greater than  $\Gamma_0$ , and thus  $\text{WO}(\vartheta\Omega^\omega)$  is computably entailed by all predicative and predicatively reducible subsystems of second order arithmetic, but not predicatively provable.

## 7 The complexity of computable entailment

We now turn to a different but related issue with the computable entailment relation: its completeness-theoretic complexity. As we know from Church and Turing's negative answer to the Entscheidungsproblem, the classical provability relation is uncomputable. Indeed, the set of provable consequences of a theory like Peano arithmetic is a quintessential example of a recursively enumerable set that is not recursive. Consequently, while there is no general method for determining whether or not a sentence  $\varphi$  in the language of arithmetic is provable in  $\text{RCA}_0$ , there is a Turing machine which enumerates the provable consequences of  $\text{RCA}_0$ , amongst which are the equivalences of classical reverse mathematics.

Semantic relations such as truth tend to be far more complex than syntactic relations such as provability, since they are—usually ineliminably—infinitary in nature. The completeness theorem for classical first order logic gives us an important counterexample: since  $T \models \varphi \Leftrightarrow T \vdash \varphi$  for theories  $T$  and sentences  $\varphi$ , we can enumerate the model-theoretic consequences of a theory by enumerating its provable consequences, reducing a complex semantic relation to a finitary one. The same does not hold for computable entailment. Not only is it not recursive, but it is not even arithmetical. As a prelude to demonstrating this, we give a revised definition of computable entailment, generalised to accommodate parameters. For the rest of this paper we use the symbol  $\mathbb{N}$  to refer to the internal natural numbers of subsystems of second order arithmetic, and reserve the symbol  $\omega$  for the external natural numbers of the metatheory.

**Definition 7.1.** For any set  $X \subseteq \omega$ , and sentence  $\varphi$  in the language  $L_2$  expanded with a constant symbol for  $X$ , we say that  $\varphi$  is *X-computably entailed*, in symbols  $\models_c^X \varphi$ , iff for all Turing ideals  $M$  such that  $X \in M$ ,  $M \models \varphi$ .

At first glance this may appear less general than the earlier definition, but by the definition of the satisfaction relation,  $(\varphi \models_c^X \psi)$  iff  $\models_c^X (\varphi \rightarrow \psi)$ , and the new

definition is simpler to work with in the current context. Fixing a recursive, bijective Gödel coding of sentences of second order arithmetic, we represent the computable entailment relation by the set of Gödel codes for sentences which are computably entailed. For any  $X \subseteq \omega$ , let

$$(3) \quad C(X) = \{ \ulcorner \varphi \urcorner \mid \models_c^X \varphi \}$$

where  $\varphi$  is an  $L_2$ -sentence which may contain a constant  $\bar{X}$  denoting  $X$ . The parameter-free version of  $C(X)$  we denote simply  $C$ . Observing that the definition of computable entailment quantifies over  $\omega$ -models, we can see that  $C$  contains all the sentences of True Arithmetic, the first order theory of the natural numbers. True Arithmetic is not arithmetically definable, as this would contradict Tarski's theorem. So computable entailment cannot be arithmetical either.

A stronger lower bound for the complexity of computable entailment can be found by noting that arithmetical properties of reals are absolute to all  $\omega$ -models, and thus that all  $\Pi_1^1$  sets of natural numbers are 1-reducible to  $C$ . We can thus precisely characterise its complexity as  $\Pi_1^1$  complete, by showing that  $C$  can be captured by a  $\Pi_1^1$  definition. This theorem is essentially a classical one due to [Grzegorzcyk, Mostowski, and Ryll-Nardzewski \[1958, §3.4, pp. 386–7\]](#). Their result was proved for the second order functional calculus with the  $\omega$ -rule, which they refer to as  $A_\omega$ . We can understand this in the terminology of the present work as the following result: the set of Gödel numbers of  $L_2$ -sentences true in every  $\omega$ -model of second order arithmetic  $Z_2$  is a  $\Pi_1^1$  complete set. The proof presented below is due to [Mummert \[2012\]](#), who proves it for  $\omega$ -models of  $RCA_0$  rather than full  $Z_2$ . Relativizing computable entailment to a set parameter  $X \subseteq \omega$  we have the following.

**Theorem 7.2.** *For any set parameter  $X \subseteq \omega$ , the  $X$ -computable entailment relation  $C(X)$  is  $\Pi_1^1(X)$  complete.*

We shall need some standard definitions from computability theory. For more background the reader should consult a reference work such as [Rogers \[1967\]](#) or [Soare \[1987\]](#).

**Definition 7.3.** For sets  $X, Y \subseteq \omega$ ,  $X$  is many-one reducible to  $Y$ ,  $X \leq_m Y$ , just in case there is a total recursive function  $f$  such that for all  $m \in \omega$ ,

$$(4) \quad m \in X \Leftrightarrow f(m) \in Y.$$

If  $f$  is injective then  $X$  is 1-reducible to  $Y$ ,  $X \leq_1 Y$ , and if  $f$  is a bijection then  $X$  and  $Y$  are 1-equivalent.

**Definition 7.4.** Let  $\mathcal{X} \subseteq \mathcal{P}(\omega)$ . A set  $X \subseteq \omega$  is complete for  $\mathcal{X}$  iff  $X \in \mathcal{X}$  and  $Y \leq_1 X$  for every  $Y \in \mathcal{X}$ .

**Lemma 7.5.** *For any set parameter  $X \subseteq \omega$ , every  $\Pi_1^1(X)$  set  $A$  is 1-reducible to  $C(X)$ .*

*Proof.* Let  $\varphi(m_1, X_1)$  be a  $\Pi_1^1$  formula. We refer to  $(\omega, \mathcal{P}(\omega))$  as the full model.

*Claim:* For any  $n \in \omega$  and  $X \subseteq \omega$ ,  $\varphi(n, X)$  is true in the full model iff it is true in all Turing ideals containing  $X$ .

( $\Leftarrow$ ) The full model is a Turing ideal containing  $X$ , so if  $\varphi(n, X)$  is false in the full model then it is false in that ideal.

( $\Rightarrow$ ) Assume without loss of generality that  $\varphi(n, X) \equiv \forall Y \psi(n, X, Y)$  where  $\psi$  is arithmetical. Suppose there is a Turing ideal  $\mathcal{C}$  containing  $X$  such that  $\mathcal{C} \not\models \varphi(X)$ .

Then there is some counterexample  $B \in \mathcal{C}$  such that  $\mathcal{C} \not\models \psi(X, B)$ . Since the interpretation of the first order quantifiers and nonlogical symbols are the same in all  $\omega$ -models, such a  $B$  will remain a counterexample in the full model.

This completes the proof of the claim.

Given  $\varphi(m_1, X_1)$  as above, let  $A = \{n \in \omega \mid \varphi(n, X)\}$ . Define the function  $f_A : \omega \rightarrow \omega$  as  $f_A(n) = \ulcorner \varphi(\bar{n}, \bar{X}) \urcorner$ . This function is recursive and injective, since if  $a \neq b$  then  $\ulcorner \varphi(\bar{a}, \bar{X}) \urcorner \neq \ulcorner \varphi(\bar{b}, \bar{X}) \urcorner$  by the properties of the Gödel coding. Finally by the claim above and the fact that  $\varphi(m_1, X_1)$  is  $\Pi_1^1$ ,  $n \in A \leftrightarrow \varphi(n, X) \leftrightarrow \ulcorner \varphi(\bar{n}, \bar{X}) \urcorner = f_A(n) \in C(X)$ .  $\square$

Having shown that  $C$  is  $\Pi_1^1$ -hard, i.e. that all sets  $A \in \Pi_1^1$  are 1-reducible to it, we shall show that  $C$  is itself  $\Pi_1^1$  and thus is  $\Pi_1^1$  complete. In doing so we shall lean on the following definition which shows how a set can code a countable Turing ideal. A countable coded  $\omega$ -model is a set  $W$  which codes countable sequence of sets  $\langle (W)_n \mid n \in \mathbb{N} \rangle$  where  $(W)_n = \{i \mid (i, n) \in W\}$ . For a full definition of countable coded  $\omega$ -models see Simpson [2009, §VII.2].

**Definition 7.6.** Suppose  $W \subseteq \mathbb{N}$  is a set coding the countable model  $M$  and  $X \subseteq \mathbb{N}$ .  $W$  codes a countable Turing ideal containing  $X$  iff

- (i) For every  $m, n$ , there exists a  $k$  such that  $(W)_k = (W)_m \oplus (W)_n$ ;
- (ii) For every  $m$ , if  $Y \leq_{\mathbf{T}} (W)_m$  then there exists a  $k$  such that  $(W)_k = Y$ ;
- (iii) There exists some  $k$  such that  $(W)_k = X$ .

**Lemma 7.7.** Let  $X, W \subseteq \mathbb{N}$ . The predicate “ $W$  codes a countable Turing ideal containing  $X$ ” is arithmetical.

*Proof.* Throughout we use the countable coded  $\omega$ -model  $W$  as a parameter. The following formula is an analogue of condition (i) of definition 7.6.

$$(5) \quad \forall m \forall n \exists k \forall x \forall y [x \in (W)_m \wedge y \in (W)_n \leftrightarrow 2x \in (W)_k \wedge 2y + 1 \in (W)_k].$$

For (ii), let  $\pi(e, n, Y)$  be a universal lightface  $\Pi_1^0$  formula with the given free variables. The existence of such formulae is provable in  $\text{RCA}_0$ ; a definition is provided in Simpson [2009, definition VII.1.3, p. 244]. They play the role of universal Turing machines.

$$(6) \quad \forall m \forall e_0 \forall e_1 [\forall n (\pi(e_0, n, (W)_m) \leftrightarrow \neg \pi(e_1, n, (W)_m)) \rightarrow \exists k \forall n (n \in (W)_k \leftrightarrow \pi(e_0, n, (W)_m))].$$

Finally we add condition (iii) that  $X$  is an element of the Turing ideal coded by  $W$ ,

$$(7) \quad \exists k \forall n (n \in X \leftrightarrow n \in (W)_k).$$

One can (tediously) verify that these conditions hold of  $W$  if and only if the  $\omega$ -model coded by  $W$  is a Turing ideal containing  $X$ .  $\square$

**Lemma 7.8.** For any set parameter  $X \subseteq \omega$ , if an  $L_2(X)$ -sentence  $\varphi$  is false in any Turing ideal containing  $X$ , then it is false in a countable Turing ideal containing  $X$ .

*Proof.* Let  $M$  be a Turing ideal containing  $X$ , and assume that  $M \models \neg\varphi$ . By the downwards Löwenheim–Skolem theorem,  $M$  has a countable  $\omega$ -submodel  $M' \subseteq_\omega M$  such that  $X \in M'$ .  $M'$  is a Turing ideal, as this property is definable by an  $L_2(X)$  sentence which is true in  $M$ , and thus in  $M'$  by elementarity. Finally,  $\varphi$  is false in  $M'$ , again by elementarity.  $\square$

*Proof of theorem 7.2.* Fix a set parameter  $X$ . By lemma 7.5,  $C(X)$  1-reduces every  $\Pi_1^1(X)$  set. It only remains to show that  $C(X)$  is itself a  $\Pi_1^1(X)$  set.

Let  $C^\dagger(X)$  be the set of Gödel codes of  $L_2$ -sentences  $\varphi$  such that every countable Turing ideal containing  $X$  satisfies  $\varphi$ . Lemma 7.8 shows that any sentence  $\varphi$  of second order arithmetic is satisfied by every Turing ideal containing  $X$  iff it is satisfied by every countable Turing ideal containing  $X$ . So  $\ulcorner\varphi\urcorner \in C(X) \Leftrightarrow \ulcorner\varphi\urcorner \in C^\dagger(X)$ . Thus by proving that  $C^\dagger(X)$  is a  $\Pi_1^1(X)$  set, we show that  $C(X)$  is also  $\Pi_1^1(X)$ .

The relation  $\ulcorner\varphi\urcorner \in C^\dagger(X)$  can be defined in second order arithmetic as:

$$(8) \quad (\forall \text{ countable Turing ideals } M)(X \in M \rightarrow M \models \varphi)$$

By lemma 7.7, the predicate “ $W$  codes a countable Turing ideal  $M$ ” is arithmetical.  $M \models \varphi$  means “There exists a satisfaction function  $f$  for  $M$  such that  $f(\ulcorner\varphi\urcorner) = 1$ .” Although this is  $\Sigma_1^1$ , every such  $f$  is provably unique, and thus  $M \models \varphi$  is equivalent to a  $\Pi_1^1$  formula.  $\square$

Computable entailment thus transcends arithmetical truth, being recursively isomorphic to the  $\Pi_1^1$  theory of the natural numbers, and also to membership in Kleene’s  $\mathcal{O}$ , the set of notations for recursive ordinals. Nevertheless its complexity is towards the lower end of the logics considered by Väänänen [2001] and Koellner [2010], being for instance far less complex than the full second-order consequence relation. But as we shall soon see, such complexity is incompatible with the requirements of foundational analysis.

The Entscheidungsproblem was considered by Hilbert and others to be of such importance because a positive solution would have meant we could obtain, by finite means, knowledge of the provability or unprovability of all mathematical statements. The computational intractability of the classical provability relation constitutes an epistemic difficulty for mathematics. From this perspective, we should be troubled by an entailment relation such as Shore’s with a far greater degree of uncomputability.

It’s well known that truth definitions are not simple: Kripke’s fixed-point construction of a truth predicate over the natural numbers is also  $\Pi_1^1$  complete [Kripke 1975]. Provability, at least for classical first-order logic, is comparatively uncomplicated. If  $\text{RCA}_0 \models \varphi$  then we can produce a finitary proof witness by an exhaustive search. We have no such assurance when  $\models_c \varphi$ : computable entailment does not satisfy Gödel’s completeness theorem, so we are unable to reduce this complex semantic relation to the more finitistically acceptable provability relation.

$\omega$ -logic does have a completeness theorem of sorts, namely the  $\omega$ -completeness theorem of Henkin and Orey, as mentioned in §5. By this theorem, restricting to  $\omega$ -models is equivalent to closing one’s consequence set under the  $\omega$ -rule. This is typically formalised in terms of an infinitary proof calculus, where proofs are well-founded trees which branch infinitely on uses of the  $\omega$ -rule. However, this completeness theorem does not induce a reduction in the complexity of the computable entailment relation: computable entailment is irredeemably infinitary. Computable entailment is also impredicative. Shore’s definition quantifies over all Turing ideals, and while theorem 7.2 shows that a definition quantifying only over countable Turing ideals is

in fact equivalent to Shore's, computable entailment is still  $\Pi_1^1$  complete, and thus an archetypal impredicative relation.

While determining the complexity of the computable entailment relation in a traditional, computability-theoretic way as we have above is a useful classification exercise, it comes with some disadvantages. Principally, it does not make clear what proof-theoretic resources are required in order to prove the result. This means that it is unclear whether the result is epistemically accessible to the convinced  $\mathcal{F}$ -theorist, where  $\mathcal{F}$  is a given foundational approach such as those studied in §3 and §4. From an external viewpoint we can see that the computable entailment relation is definable in the language of second order arithmetic, by quantifying over all countable Turing ideals. We therefore turn to the resources of reverse mathematics and show that an analogue of theorem 7.2 can be proved within a predicative subsystem of second order arithmetic.

To do so we must select the correct base theory, and then formulate the principle that truth sets for the  $X$ -computable entailment relations exist with some care. The first barrier is that  $\text{RCA}_0$  does not prove that codes for countable Turing ideals exist. Nor, given a countable coded Turing ideal  $M$ , does it prove that the satisfaction function  $f$  for  $M$  exists. We therefore work in the stronger theory  $\text{ACA}_0$ . However,  $\text{ACA}_0$  does not prove that the full satisfaction function for any countable coded Turing ideal (considered as an  $\omega$ -model) exists, since such a satisfaction function is essentially a truth predicate for the first-order language of arithmetic, and thus not arithmetically definable. We therefore formulate the definition of the truth set for  $X$ -computable entailment  $C(X)$  in a slightly modified form, using not full satisfaction functions but valuation functions for single sentences.<sup>24</sup> For details of the notion of a valuation function see definition VII.2.1 of Simpson [2009].

**Definition 7.9.** The following definition is made in  $\text{ACA}_0$ . Let  $X \subseteq \mathbb{N}$  be any set and let  $\ulcorner \varphi \urcorner$  be a Gödel code for a sentence  $\varphi$  in the language of second order arithmetic  $L_2$  extended with a constant symbol for  $X$ . We say that  $\varphi$  is  *$X$ -computably entailed*,  $\ulcorner \varphi \urcorner \in C(X)$ , if for every code  $W$  for a countable Turing ideal  $M$  such that  $X \in M$ , and for every valuation function  $f : \text{Sub}_M(\varphi) \rightarrow \{0, 1\}$ , we have that  $f(\ulcorner \varphi \urcorner) = 1$ .

**Lemma 7.10.** *Suppose  $\varphi(m, X)$  is a  $\Pi_1^1$  formula with exactly the displayed free variables. Then the following is provable in  $\text{ACA}_0$ . For all  $X \subseteq \mathbb{N}$ , if  $C(X)$  exists, then  $Y = \{m \mid \varphi(m, X)\}$  exists and  $Y \leq_{\mathbf{T}} C(X)$ .*

*Proof.* Let  $\varphi(m, X)$  be as above. By the Kleene normal form theorem for  $\Pi_1^1$  formulas, there is a  $\Sigma_1^0$  formula  $\sigma(m, f, X)$  with exactly the displayed free variables such that  $\text{ACA}_0$  proves

$$\forall m \forall X (\varphi(m, X) \leftrightarrow \forall f \sigma(m, f, X)).$$

Given  $m \in \mathbb{N}$  and  $X \subseteq \mathbb{N}$ , we reason in  $\text{ACA}_0$  and show that

$$\forall f \sigma(m, f, X) \leftrightarrow \ulcorner \forall f \sigma(m, f, X) \urcorner \in C(X).$$

( $\Rightarrow$ ) Suppose  $\ulcorner \forall f \sigma(m, f, X) \urcorner \notin C(X)$ . Then there exists a code  $W_1$  for a countable Turing ideal  $M_1$  containing  $X$ , and a valuation function  $g_1 : \text{Sub}_{M_1}(\psi_1) \rightarrow \{0, 1\}$  (where  $\psi_1 \equiv \forall f \sigma(m, f, X)$ ), such that  $g_1(\psi_1) = 0$ . By the definition of the valuation

<sup>24</sup>This definition is extensionally equivalent to the previous one using full satisfaction functions, as can be seen from the viewpoint of proof-theoretically stronger but still predicatively reducible theories such as  $\text{ACA}_0^+$ , which prove that satisfaction functions for countable coded  $\omega$ -models exist. Dorais [2012] explains many of the subtleties involved.

function, there exists  $k \in \mathbb{N}$  which is the index of a function  $f_1 = (W)_k$ , such that  $g_1(\ulcorner \sigma(m, f_1, X) \urcorner) = 0$ .  $f_1$  exists by recursive comprehension in the parameter  $W$ , and since arithmetical formulas are absolute between the ambient model and any countable coded  $\omega$ -model, we have that  $\neg\sigma(m, f_1, X)$ , and so  $\neg\forall f\sigma(m, f, X)$ .

( $\Leftarrow$ ) Suppose there exists  $f_2$  such that  $\neg\sigma(m, f_2, X)$ . By arithmetical comprehension there exists a code  $W_2$  for a countable Turing ideal  $M_2$  such that  $X, f_2 \in M_2$  (for example, just take the code for the ideal consisting of the sets recursive in  $X \oplus f_2$ ). Arithmetical comprehension also proves the existence of a valuation function  $g_2 : \text{Sub}_{M_2}(\psi_2) \rightarrow \{0, 1\}$ , where  $\psi_2 \equiv \neg\sigma(m, f_2, X) \wedge \forall f\sigma(m, f, X)$ . By absoluteness,  $g_2(\ulcorner \neg\sigma(m, f_2, X) \urcorner) = 1$ , so  $g_2(\ulcorner \forall f\sigma(m, f, X) \urcorner) = 0$ , and hence  $\ulcorner \forall f\sigma(m, f, X) \urcorner \notin C(X)$ .

Given a set  $X \subseteq \mathbb{N}$ , assume that  $C(X)$  exists. Let  $Y$  be the set of all  $m$  such that  $\ulcorner \forall f\sigma(m, f, X) \urcorner \in C(X)$ .  $Y$  is recursive in  $C(X)$ , and thus exists by recursive comprehension. So by the equivalence just proved,  $\forall m(m \in Y \leftrightarrow \varphi(m, X))$ .  $\square$

While this result demonstrates that the complexity of the computable entailment relationship is in some sense accessible to the predicativist, if not the finitistic reductionist, it is not clear what the philosophical moral should be. A natural interpretation might be that a  $\Pi_1^1$  complete entailment relation such as computable entailment is much more uncomputable than the relation of provability in classical logic, and that this degree of uncomputability can be comprehended from within a predicativist formal theory. This conclusion goes hand in hand with a more general view that the Turing degrees track a hierarchy of relative difficulty of problem solving: problems with higher Turing degree are harder to solve than those with lower Turing degree.

This is related to the naïve reading of the intuition discussed in §5.2 that “being harder to prove” means “harder to compute”, and is an unsatisfying interpretation for similar reasons. The first concerns the use of the computable entailment relation. We are not seeking a general method that for any  $\varphi, \psi$  in the language of second order arithmetic tells us whether or not  $\varphi \models_c \psi$ . Rather, given specific statements of mathematical interest, we try to prove (or disprove) that one computably entails the other. The proofs involved here are typical mathematical proofs, carried out in the usual way, not infinitary inferences: they quantify over Turing ideals, but they do not require that we are able, as mathematicians, to solve the halting problem or determine membership in Kleene’s  $\mathcal{O}$ . Secondly, it is unclear why—given that they are both uncomputable—that the computable entailment relation is epistemically any more intractable than the standard first order provability relation of classical logic. Absent the ability to carry out supertasks, we cannot solve the halting problem, so even in principle, determining for arbitrary  $\varphi, \psi$  whether or not  $\text{RCA}_0 \vdash \varphi \rightarrow \psi$  seems as out of reach as determining whether  $\varphi \models_c \psi$ .

Given this, a more plausible reconstruction of mathematical practice when we prove computable entailments or failures of computable entailment is that we work in a way that can be formalised in a standard deductive calculus, but that in doing so we assume that quantifying over all Turing ideals (or equivalently, over all countable Turing ideals) is well-defined. One way of ensuring this well-definedness is to work in a background theory that proves that the extension of the computable entailment relation exists. In this context, knowing the precise Turing degree of the computable entailment relation takes on greater significance, since it allows us to determine what axioms are both necessary and sufficient to prove its well-definedness.

**Corollary 7.11.** *The following are equivalent over  $\text{ACA}_0$ .*

1.  $\Pi_1^1$  comprehension.



2. For every  $X \subseteq \mathbb{N}$ , the truth set  $C(X)$  of the  $X$ -computable entailment relation exists.

*Proof.* The definition of  $C(X)$  is  $\Pi_1^1$ , as can be seen from definition 7.9, so  $\Pi_1^1$  comprehension proves that for any  $X \subseteq \mathbb{N}$ ,  $C(X)$  exists.

For the reversal we work in  $\text{ACA}_0$ . Let  $\varphi(m)$  be a  $\Pi_1^1$  formula. Using pairing, join all  $n$  free set variables in  $\varphi(m)$ , to produce an equivalent formula  $\varphi'(m, Y)$  with a single free set variable  $Y$ , such that  $\text{ACA}_0$  proves

$$\forall X_1 \dots \forall X_n \forall Y \forall m (\varphi(m) \leftrightarrow \varphi'(m, Y)).$$

Given  $Y$ , assume that  $C(X)$  exists. Lemma 7.10 applied to  $\varphi'(m, Y)$ , together with the above equivalence, implies the existence of a set  $Z$  such that  $\forall m (m \in Z \leftrightarrow \varphi(m))$ . This proves  $\Pi_1^1$  comprehension.  $\square$

$\Pi_1^1\text{-CA}_0$  is the strongest of the subsystems of second order arithmetic usually studied in reverse mathematics. Computational reverse mathematics therefore draws on resources which are unavailable in the four members of the Big Five that are proof-theoretically weaker than  $\Pi_1^1\text{-CA}_0$ . Moreover, since 7.11 is provable within a predicative system, the predicativist is in a position to calibrate the strength of the commitment involved in accepting computable entailment. Doing so, she will see that not only is it stronger than predicative systems like  $\text{ACA}_0$ , but also predicatively reducible ones like  $\text{ATR}_0$ . So not only does the existence of the truth set for the computable entailment relation exceed the strength of the predicativist and the predicative reductionist's theoretical resources, but they are in a position to see that it does. Since they reject impredicative mathematics, and thus reject  $\Pi_1^1$  comprehension, they must therefore reject the equivalent statement that the truth set for computable entailment exists.

For foundational analysis to be a useful and worthwhile endeavour within the philosophy of mathematics, the fruits of its analysis must be epistemically available to disputants. Recall our example of Sarah the predicativist from §2. Since she accepts  $\text{ACA}_0$ , she believes that the equivalence between  $\Pi_1^1$  comprehension and the statement  $(\star)$ , “Every countable abelian group can be expressed as a direct sum of a divisible group and a reduced group” is true, since it is provable in a system contained in  $\text{ACA}_0$  (namely  $\text{RCA}_0$ ). How she responds to Rebecca's challenge that Sarah's predicativism is misguided, since it does not allow her to prove the ordinary mathematical theorem  $(\star)$ , will depend on the details of her views about the foundations of mathematics, but she cannot dismiss the equivalence as question-begging. On the other hand, suppose Rebecca were instead to present Sarah with the following argument:  $\Pi_1^1\text{-CA}_0$  and  $(\star)$  are computably equivalent, that is to say they are true in exactly the same Turing ideals. Sarah should therefore accept  $\Pi_1^1\text{-CA}_0$ , since  $(\star)$  is an ordinary mathematical theorem that any reasonable foundational system should prove. In this case Sarah can resist the conclusion by refusing to accept the antecedent: computable equivalence is not a well-defined notion, since it presupposes theoretical resources which predicativism denies. Any argument presupposing that computable equivalence is a well-defined notion therefore begs the question against her position.

We argued in §2 that philosophical arguments that attempts to invoke reverse mathematical results in foundational analysis should, if they are to have any force, appeal only to principles that targets of these argument already accept. In other words, its presuppositions must not exceed their theoretical commitments. But the argument above shows that the theoretical commitments which accompany the use

of computable entailment outstrip those acceptable to partisans of most of the foundational programs associated with subsystems of second order arithmetic. Computational reverse mathematics does not allow one, in general, to persuasively demonstrate the mathematical limits of these foundational programs to those who accept them.

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