Modal logic S4 as a paraconsistent logic with a topological semantics

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Dedicated to Amilcar Sernadas,
a brilliant mind

ABSTRACT. In this paper the propositional logic **LTop** is introduced, as an extension of classical propositional logic by adding a paraconsistent negation. This logic has a very natural interpretation in terms of topological models. The logic **LTop** is nothing more than an alternative presentation of modal logic **S4**, but in the language of a paraconsistent logic. Moreover, **LTop** is a logic of formal inconsistency in which the consistency and inconsistency operators have a nice topological interpretation. This constitutes a new proof of **S4** as being "the logic of topological spaces", but now under the perspective of paraconsistency.

1 Topology, Modal Logic and Paraconsistency

The studies on the relationship between modal logic, topology and paraconsistency, have a relatively long history.

By extending the Stone representation theorem for Boolean algebras, McKinsey and Tarski (see [14]) proved in 1944 that it is possible to characterize modal logic **S4** by means of a topological semantics. Within this semantics, the necessity operator $\Box$ and the possibility operator $\Diamond$ are interpreted as the interior and the closure topological operators, respectively, and so this result states that **S4** is, in a certain sense, "the logic of topological spaces". Moreover, they prove that **S4** is semanticaly characterized by the real line (with the usual topology) or, in general, by any dense-in-itself separable metrizable space. Several variants and generalizations of McKinsey and Tarski's result have been proposed in the literature (see, for instance, [18] and [12]).

Semantics for Paraconsistent logics (that is, logics having a negation which produces some non-trivial contradictory theories) have been defined in topological terms by several authors. For instance, Mortensen studies in [16] some topological properties by means of paraconsistent and paracomplete logics. Goodman already proposes in [10] an "anti-intuitionistic logic" which is paraconsistent and it is endowed with a topological semantics. Along the same lines, Priest analyzes a paraconsistent negation obtained by dualizing the intuitionistic negation, defining a topological semantics for such negation (see [17]). From a broader perspective, Başkent proposes in [1] an interesting study of topological models for paraconsistency and paracompleteness.

By its turn, the relationship between paraconsistency and modal logic is also very close. Already in 1948, Jaśkowski presented in [11] his "discussive logic", which is considered the first formal system for a paraconsistent logic, and it was formalized in terms of modalities. Beziau observes in [2] (see also [3]) that the operator $\neg\alpha \stackrel{\text{def}}{=} {\sim}\Box\alpha$ defines in modal logic **S5** a paraconsistent negation (here, $\sim$ denotes the classical negation). This relation between modalities and paraconsistent negation was already observed by Beziau in 1998 (despite the paper was published only in 2006, see [4]), from the perspective of Kripke semantics, when defining the logic **Z**. However, already in 1987, de Araújo et al. observed in [9] that a Kripke-style semantics can be given for Sette's 3-valued paraconsistent logic **P1**, based on Kripke frames for the modal logic **T**. In that semantics, the formula $\neg\alpha$ (for the paraconsistent negation $\neg$ of **P1**) is interpreted exactly as the modal formula ${\sim}\Box\alpha$. Beziau's approach was generalized by Marcos in [13], showing that there is a close correspondence between non-degenerate modal logics and the paraconsistent logics known as *logics of formal inconsistency* (see Section 6).

This paper contributes to this discussion by introducing a propositional logic, called **LTop**, which extends classical propositional logic with a paraconsistent negation. This logic has a very natural interpretation in terms of topological models which associate to each formula a set (not necessarily open or close). The classical connectives are interpreted as usual, and the paraconsistent negation is interpreted as the topological closure of the complement, in a dual form to the usual interpretation of the intuitionistic negation (namely, the interior of the complement). This topological interpretation of a paraconsistent negation is very natural, and it was already proposed in [10], [16] and [1]. Modalities $\Box$ and $\Diamond$ can be defined in the language of **LTop**, which are interpreted as the interior and closure operator, respectively. As expected, the logic **LTop** is nothing more than an alternative presentation of modal logic **S4**, but in a language (and through a Hilbert calculus) corresponding to an extension of classical logic by means of a paraconsistent negation. It is worth noting that the logic **Z** in [4] introduces an axiomatization of **S5** as an extension of classical logic with a paraconsistent negation, and so the present result is a kind of generalization of such result, provided that **S5** can be obtained from **S4** by adding an additional axiom. Additionally, it is proved that **LTop** is a logic of formal inconsistency (see Section 6) in which the consistency and inconsistency operators have a nice topological interpretation. This constitutes a new proof of **S4** as being "the logic of topological spaces", but now under the perspective of paraconsistent logics. It is finally shown that intuitionistic propositional logic can be interpreted in **LTop** through a very natural conservative translation.

2 The propositional logic LTop

In this section, the propositional logic **LTop** will be introduced by means of a Hilbert calculus, with a modal-like notion of derivations.

DEFINITION 1. Let $\mathcal{V} = \{p_n \ : n \geq 1\}$ be a denumerable set of *propositional variables*. Given the propositional signature $\Sigma = \{\sim, \neg, \to\}$ let $\mathbb{L}$ be the language generated by the set $\mathcal{V}$ over the signature Σ.

As suggested by the notation, $\sim$ and $\neg$ are unary connectives representing two different negations, while $\to$ is a binary connective which represents an implication (in the logic **LTop** to be defined below). The connective $\sim$ will represent a classical negation, while $\neg$ will represent a paraconsistent negation. The implication will be also classical. The following usual abbreviations can be introduced in the language $\mathbb{L}$:

(conj) $\alpha \wedge \beta \stackrel{\text{def}}{=} \sim(\alpha \to \sim\beta)$

(disj) $\alpha \vee \beta \stackrel{\text{def}}{=} \sim\alpha \to \beta$

DEFINITION 2 (Propositional Logic **LTop**). The logic **LTop** is given by the Hilbert calculus over the language $\mathbb{L}$ defined by following the axioms and inference rules:

Axiom schemas:

$$\alpha \to (\beta \to \alpha) \tag{\textbf{Ax1}}$$
$$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)) \tag{\textbf{Ax2}}$$
$$(\sim\alpha \to \beta) \to ((\sim\alpha \to \sim\beta) \to \alpha) \tag{\textbf{Ax3}}$$
$$\alpha \to \neg\sim\alpha \tag{\textbf{Ax4}}$$
$$\neg\sim\neg\sim\alpha \to \neg\sim\alpha \tag{\textbf{Ax5}}$$
$$\neg(\alpha \wedge \beta) \to \neg\alpha \vee \neg\beta \tag{\textbf{Ax6}}$$
$$\sim\neg(\alpha \to \alpha) \tag{\textbf{Ax7}}$$

Inference rules:

$$\frac{\alpha \quad \alpha \to \beta}{\beta}\ (\textbf{MP}) \qquad \frac{\alpha \to \beta}{\neg\beta \to \neg\alpha}\ (\textbf{CR}) \qquad \frac{\alpha \quad \beta}{\alpha \wedge \beta}\ (\textbf{DR})$$

The Logic **LTop** has the following notion of derivation:

DEFINITION 3 (Derivations in **LTop**).
(1) A *derivation* of a formula α in **LTop** is a finite sequence of formulas $\alpha_1 \dots \alpha_n$ such that α_n is α and every α_i is either an instance of an axiom or it is the consequence of some inference rule whose premises appear in the sequence $\alpha_1 \dots \alpha_{i-1}$. We say that α is *derivable in* **LTop**, and we write $\vdash_{\mathbf{LTop}} \alpha$, if there exists a derivation of it in **LTop**.
(2) Let $\Gamma \cup \{\alpha\}$ be a set of formulas. We say that α is *derivable in* **LTop** *from* Γ, and we write $\Gamma \vdash_{\mathbf{LTop}} \alpha$, if either $\vdash_{\mathbf{LTop}} \alpha$ or there exists a finite, non-empty subset $\{\gamma_1, \dots, \gamma_n\}$ of Γ such that $(\gamma_1 \wedge (\gamma_2 \wedge (\dots \wedge (\gamma_{n-1} \wedge \gamma_n) \dots))) \to \alpha$ is derivable in **LTop**.

By the very definition, $\emptyset \vdash_{\mathbf{LTop}} \alpha$ iff $\vdash_{\mathbf{LTop}} \alpha$.

REMARK 4. The consequence relation $\vdash_{\mathbf{LTop}}$ for **LTop** given in the previous definition is Tarskian and finitary, that is, it satisfies the following properties:

(i) if $\alpha \in \Gamma$ then $\Gamma \vdash_{\mathbf{LTop}} \alpha$; (ii) if $\Gamma \vdash_{\mathbf{LTop}} \alpha$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash \alpha$; (iii) if $\Gamma \vdash_{\mathbf{LTop}} \Delta$ and $\Delta \vdash_{\mathbf{LTop}} \alpha$ then $\Gamma \vdash_{\mathbf{LTop}} \alpha$, where $\Gamma \vdash \Delta$ means that $\Gamma \vdash \delta$ for every $\delta \in \Delta$; and (iv) if $\Gamma \vdash_{\mathbf{LTop}} \alpha$ then $\Gamma_0 \vdash_{\mathbf{LTop}} \alpha$ for some finite Γ_0 contained in Γ. This can be proven by adapting to the logic **LTop**, together with the pair of connectives $(\rightarrow, \wedge)$, a general result (Theorem 2.10.2) concerning entailment systems found in [20].

REMARK 5. *Note that **(Ax1)**, **(Ax2)** and **(Ax3)** plus **(MP)** contitute an axiomatization of propositional classical logic* **CPL** *over the signature* $\{\sim, \rightarrow\}$.

3 Basic Propositions derivable in LTop

In this section some basic propositions will be derived in **LTop**.

DEFINITION 6. Define the relation $\equiv$ in $\mathbb{L}$ as follows:

$$\alpha \equiv \beta \quad \text{iff} \quad \vdash_{\mathbf{LTop}} \alpha \rightarrow \beta \ \text{ and } \ \vdash_{\mathbf{LTop}} \beta \rightarrow \alpha$$

An immediate consequence of the rules **CR** and **MP** is the following:

PROPOSITION 7. *If* $\alpha \equiv \beta$ *then* $\neg\alpha \equiv \neg\beta$.

From Proposition 7 and Remark 5 it follows:

COROLLARY 8 (Weak Replacement). *If* $\alpha_i \equiv \beta_i$ *for* $i = 1, \ldots, n$ *then, for every formula* $\varphi(p_1, \ldots, p_n)$*, it holds:* $\varphi[p_1/\alpha_1 \cdots p_n/\alpha_n] \equiv \varphi[p_1/\beta_1 \cdots p_n/\beta_n]$.

In the terminology introduced by Wójcicki, the latter result shows that **LTop** is a *selfextensional* logic. On the other hand, the following meta-property of **LTop** can be easily proved:

PROPOSITION 9 (Deduction Theorem). *For every set of formulas* $\Gamma \cup \{\alpha, \beta\}$*:* $\Gamma, \alpha \vdash_{\mathbf{LTop}} \beta$ *iff* $\Gamma \vdash_{\mathbf{LTop}} (\alpha \rightarrow \beta)$.

Proof. It is an immediate consequence of the notion of derivations in **LTop** and the properties of **CPL**. The details are left to the reader. ■

The relation between the two primitive negations is as follows:

PROPOSITION 10. $\vdash_{\mathbf{LTop}} \sim\alpha \rightarrow \neg\alpha$.

Proof. Consider the following (meta)derivation in **LTop**:

1. $\vdash_{\mathbf{LTop}} \sim\alpha \rightarrow \neg\sim\sim\alpha$ (by **Ax4**)
2. $\vdash_{\mathbf{LTop}} \sim\alpha \rightarrow \neg\alpha$ (Replacement)

■

The converse does not hold in general (see Proposition 16 below). The weak negation $\neg$ satisfies the following basic properties:

PROPOSITION 11. *The following holds in* **LTop***:*

(i) $\vdash_{\mathbf{LTop}} \alpha \vee \neg\alpha$.

(ii) $\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$.

(iii) $\vdash_{\mathbf{LTop}} \neg(\alpha \vee \beta) \to (\neg\alpha \wedge \neg\beta)$.

Proof. (i) Consider the following (meta)derivation in **LTop**:

1. $\vdash_{\mathbf{LTop}} {\sim}\alpha \to \neg\alpha$ (by Proposition 10)
2. $\vdash_{\mathbf{LTop}} (\alpha \vee {\sim}\alpha) \to (\alpha \vee \neg\alpha)$ (1, **CPL**)
3. $\vdash_{\mathbf{LTop}} \alpha \vee {\sim}\alpha$ (**CPL**)
4. $\vdash_{\mathbf{LTop}} \alpha \vee \neg\alpha$ (2,3 **MP**)

(ii) By (**Ax6**) it is enough to show that $\vdash_{\mathbf{LTop}} (\neg\alpha \vee \neg\beta) \to \neg(\alpha \wedge \beta)$. Thus, consider the following (meta)derivation in **LTop**:

1. $\vdash_{\mathbf{LTop}} (\alpha \wedge \beta) \to \alpha$ (**CPL**)
2. $\vdash_{\mathbf{LTop}} \neg\alpha \to \neg(\alpha \wedge \beta)$ (**CR**, 1)
3. $\vdash_{\mathbf{LTop}} (\alpha \wedge \beta) \to \beta$ (**CPL**)
4. $\vdash_{\mathbf{LTop}} \neg\beta \to \neg(\alpha \wedge \beta)$ (**CR**, 3)
5. $\vdash_{\mathbf{LTop}} (\neg\alpha \vee \neg\beta) \to \neg(\alpha \wedge \beta)$ (2,4, **CPL**)

(iii) By **CPL**, $\vdash_{\mathbf{LTop}} \alpha \to (\alpha \vee \beta)$ whence $\vdash_{\mathbf{LTop}} \neg(\alpha \vee \beta) \to \neg\alpha$, by (**CR**). Analogously, $\vdash_{\mathbf{LTop}} \neg(\alpha \vee \beta) \to \neg\beta$. The result follows ny **CPL**. ∎

The converse of item (iii) of the latter proposition does not hold in general. It will be proven in the next section by using the topological semantics for **LTop** (see Proposition 16).

4 Topological Semantics for LTop

In this section, an intuitive semantics for **LTop** over topological spaces will be given. Given a a topological space (X, τ) and $A \subseteq X$, the interior and the closure of A in the given topology, as well as its complement (relative to X), will be denoted by $Int(A)$, $\overline{A}$ and A^c, respectively. The powerset of X will be denoted by $\wp(X)$.

DEFINITION 12. A *Topological structure* for **LTop** is a topological space $\mathcal{T} = \langle X, \tau\rangle$. A *Topological model* for **LTop** is a pair $\mathcal{M} = \langle \mathcal{T}, v\rangle$ such that $\mathcal{T}$ is a topological structure $\langle X, \tau\rangle$ for **LTop** and $v : \mathbb{L} \to \wp(X)$ is a function, called *valuation*, satisfying the following conditions:

1. $v(\alpha \to \beta) = v(\alpha)^c \cup v(\beta)$;
2. $v({\sim}\alpha) = v(\alpha)^c$;

3. $v(\neg\alpha) = \overline{v(\alpha)^c}$.

Note that, because of (conj) and (disj):

(v-conj) $v(\alpha \wedge \beta) = v({\sim}(\alpha \to {\sim}\beta)) = (v(\alpha)^c \cup v(\beta)^c)^c = v(\alpha) \cap v(\beta)$;

(v-disj) $v(\alpha \vee \beta) = v({\sim}\alpha \to \beta) = (v(\alpha)^c)^c \cup v(\beta) = v(\alpha) \cup v(\beta)$.

DEFINITION 13 (Semantical consequence in **LTop**).
(1) A formula α in $\mathbb{L}$ is *true* in a topological model $\mathcal{M} = \langle\langle X, \tau\rangle, v\rangle$, written as $\mathcal{M} \models \alpha$, if $v(\alpha) = X$.
(2) A formula α in $\mathbb{L}$ is *valid* in **LTop**, denoted by $\models_{\mathbf{LTop}} \alpha$, if $\mathcal{M} \models \alpha$ for every topological model $\mathcal{M}$.
(3) Let $\Gamma \cup \{\alpha\}$ be a set of formulas. We say that α is *a semantical consequence of* Γ *in* **LTop**, denoted by $\Gamma \models_{\mathbf{LTop}} \alpha$, if either $\models_{\mathbf{LTop}} \alpha$ or there exists a finite non-empty set $\Gamma_0 \subseteq \Gamma$ such that $\bigcap_{\gamma \in \Gamma_0} v(\gamma) \subseteq v(\alpha)$, for every topological model $\langle\langle X, \tau\rangle, v\rangle$.

Note that, by the very definition, $\emptyset \models_{\mathbf{LTop}} \alpha$ iff $\models_{\mathbf{LTop}} \alpha$.

PROPOSITION 14. *[Soundness] The logic* **LTop** *is sound with respect to the topological semantics, that is:* $\Gamma \vdash_{\mathbf{LTop}} \alpha$ *implies* $\Gamma \models_{\mathbf{LTop}} \alpha$*, for every set of formulas* $\Gamma \cup \{\alpha\}$*. In particular,* $\vdash_{\mathbf{LTop}} \alpha$ *implies that* $\models_{\mathbf{LTop}} \alpha$.

Proof. We first prove that $\vdash_{\mathbf{LTop}} \alpha$ implies $\models_{\mathbf{LTop}} \alpha$, since **LTop** is finitary and by Proposition 9. But the latter is easily proved by observing that every axiom is valid in every topological space, and that inference rules preserve validity. The case for $\Gamma \neq \emptyset$ follows from the very definitions. ■

COROLLARY 15. *If* $\alpha \equiv \beta$ *then* $v(\alpha) = v(\beta)$ *for every topological model* $\langle \mathcal{T}, v\rangle$.

Proof. Suppose that $\alpha \equiv \beta$, and let $\langle\langle X, \tau\rangle, v\rangle$ be a topological model for **LTop**. The result follows by observing that $v(\alpha \to \beta) = X$ iff $v(\alpha) \subseteq v(\beta)$. ■

The converse of the last corollary will follow from the completeness theorem (see Theorem 36 below). Thanks to that, and by the definition of the topological semantics, it is easy to see that Proposition 11(ii) corresponds to one of the Kuratowski axioms for the closure operator, namely: $\overline{A} \cup \overline{B} = \overline{A \cup B}$ for every $A, B \subseteq X$.

By using again the soundness theorem of **LTop** w.r.t. topological semantics, it can be seen that some properties of the classical negation $\sim$ fail for the negation $\neg$:

PROPOSITION 16. *Let* p *and* q *be two different propositional variables. Then:*

(i) $\nvdash_{\mathbf{LTop}} (\neg p \wedge \neg q) \to \neg(p \vee q)$.

(ii) $\nvdash_{\mathbf{LTop}} p \to \neg\neg p$.

(iii) $\nvdash_{\mathbf{LTop}} \neg\neg p \to p$.

(iv) $\nvdash_{\mathbf{LTop}} \neg p \to {\sim} p$.

Proof. Consider $\mathcal{M} = \langle\langle \mathbb{R}, \tau\rangle, v\rangle$ such that τ is the usual topology on the set $\mathbb{R}$ of real numbers.

(i) Let $v(p) = (-\infty, 0] \cup [1, +\infty)$ and $v(q) = (-\infty, 1] \cup [2, +\infty)$. Then $v(\neg p) = \overline{(0,1)} = [0,1]$ and $v(\neg q) = \overline{(1,2)} = [1,2]$ and so $v(\neg p \wedge \neg q) = v(\neg p) \cap v(\neg q) = \{1\}$. On the other hand, $v(p \vee q) = \mathbb{R}$ and so $v(\neg(p \vee q)) = \overline{\emptyset} = \emptyset$. Since $v(\neg p \wedge \neg q) \nsubseteq v(\neg(p \vee q))$ it follows that $v((\neg p \wedge \neg q) \to \neg(p \vee q)) \neq \mathbb{R}$. By soundness of **LTop** w.r.t. topological models, $\nvdash_{\mathbf{LTop}} (\neg p \wedge \neg q) \to \neg(p \vee q)$.

(ii), (iii) and (iv) Let $v(p) = (0,1) \cup \{2\}$. Then $v(\neg\neg p) = [0,1]$, which is incomparable with $v(p)$. On the other hand, $v(\neg p) \nsubseteq v({\sim} p)$. ■

5 LTop as a modal logic

Consider the following abbreviations:

1. $\Box\alpha \stackrel{\text{def}}{=} {\sim}\neg\alpha$
2. $\Diamond\alpha \stackrel{\text{def}}{=} \neg{\sim}\alpha$

Semantically, it means the following:

1. $v(\Box\alpha) = v({\sim}\neg\alpha) = (\overline{v(\alpha)^c})^c = Int(v(\alpha))$
2. $v(\Diamond\alpha) = v(\neg{\sim}\alpha) = \overline{(v(\alpha)^c)^c} = \overline{v(\alpha)}$

The relationship between $\Box$ and $\Diamond$ is as expected.

PROPOSITION 17. *The following holds in* **LTop***:*

(i) $\Box\alpha \equiv {\sim}\Diamond{\sim}\alpha$.

(ii) $\Diamond\alpha \equiv {\sim}\Box{\sim}\alpha$.

Proof. It is immediate from the axioms and rules of **LTop**. ■

The following properties of $\Diamond$ can be proven in **LTop**:

PROPOSITION 18. *The operator* $\Diamond$ *satisfies the following properties in* **LTop***:*

(i) $\vdash_{\mathbf{LTop}} \alpha \to \Diamond\alpha$.

(ii) $\Diamond\alpha \equiv \Diamond\Diamond\alpha$.

(iii) $\Diamond(\alpha \vee \beta) \equiv \Diamond\alpha \vee \Diamond\beta$.

(iv) If $\vdash_{\mathbf{LTop}} (\alpha \to \beta)$ *then* $\vdash_{\mathbf{LTop}} (\Diamond\alpha \to \Diamond\beta)$.

Proof. (i) and (ii): It follows easily from the axioms and rules of **LTop**.
(iii) Since, for any α and β,

$$(\neg\alpha \vee \neg\beta) \equiv \neg(\alpha \wedge \beta)$$

then, in particular,

$$(\neg{\sim}\alpha \vee \neg{\sim}\beta) \equiv \neg({\sim}\alpha \wedge {\sim}\beta).$$

But ${\sim}\alpha \wedge {\sim}\beta \equiv {\sim}(\alpha \vee \beta)$ (by **CPL**) and then, using Replacement, we have that

$$(\neg{\sim}\alpha \vee \neg{\sim}\beta) \equiv \neg{\sim}(\alpha \vee \beta)$$

namely $\Diamond\alpha \vee \Diamond\beta \equiv \Diamond(\alpha \vee \beta)$.
(iv) Consider the following (meta)derivation in **LTop**:

1. $\vdash_{\mathbf{LTop}} (\alpha \to \beta)$ (by Hypothesis)
2. $\vdash_{\mathbf{LTop}} ({\sim}\beta \to {\sim}\alpha)$ (**CPL**)
3. $\vdash_{\mathbf{LTop}} (\neg{\sim}\alpha \to \neg{\sim}\beta)$ (**CR**)

■

REMARK 19. Observe that the logical properties of the connective $\Diamond$ in **LTop** stated in Proposition 18 reflect, when interpreted in topological structures, the basic properties of a closure operator. Indeed, they represent the following properties, for all subsets A, B of a topological space X: (i) $A \subseteq \overline{A}$; (ii) $\overline{A} = \overline{\overline{A}}$; (iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$; and (iv) if $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.

Dually, the following properties of $\Box$, seeing as an interior operator, can be proved in **LTop**:

PROPOSITION 20. *The following holds in* **LTop***:*

(i) $\vdash_{\mathbf{LTop}} \Box(\alpha \to \alpha)$

(ii) $\vdash_{\mathbf{LTop}} \Box\alpha \to \alpha$.

(iii) $\Box\alpha \equiv \Box\Box\alpha$.

(iv) $\Box(\alpha \wedge \beta) \equiv \Box\alpha \wedge \Box\beta$.

(v) If $\vdash_{\mathbf{LTop}} (\alpha \to \beta)$ *then* $\vdash_{\mathbf{LTop}} (\Box\alpha \to \Box\beta)$.

(vi) $\vdash_{\mathbf{LTop}} \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$.

Proof. Items (i)-(iv) are left to the reader.
(v) Consider the following (meta) derivation in **LTop**:

1. $\vdash_{\mathbf{LTop}} (\alpha \to \beta)$ (Hypothesis)

2. $\vdash_{\mathbf{LTop}} (\neg\beta \to \neg\alpha)$ (**CR**)

3. $\vdash_{\mathbf{LTop}} (\neg\beta \to \neg\alpha) \to (\sim\neg\alpha \to \sim\neg\beta)$ (**CPL**)

4. $\vdash_{\mathbf{LTop}} (\sim\neg\alpha \to \sim\neg\beta)$ (2,3 **MP**)

(vi) Since $\vdash_{\mathbf{LTop}} ((\alpha \to \beta) \wedge \alpha) \to \beta$ then $\vdash_{\mathbf{LTop}} \Box((\alpha \to \beta) \wedge \alpha) \to \Box\beta$, by (v). The result is an immediate consequence of $\Box((\alpha \to \beta) \wedge \alpha) \equiv \Box((\alpha \to \beta)) \wedge \Box\alpha$ (by item (iv)), Replacement and the properties of **CPL**. ■

REMARK 21. Observe that properties (ii), (iii) and (vi) of the last proposition correspond to the well-known modal axioms (**T**), (**4**) and (**K**), respectively. By its turn, property (v), together with (i), produce the modal necessitation rule. In fact, as we shall see in Section 9, **LTop** coincides with modal logic **S4** up to language. Additionally, the logical properties (i)-(v) of the connective $\Box$ stated in the last proposition reflect, when interpreted in topological structures, the basic properties of an interior operator. Specifically, they state the following properties, for all subsets A, B of a topological space X: (i) $Int(X) = X$; (ii) $Int(A) \subseteq A$; (iii) $Int(A) = Int(Int(A))$; (iv) $Int(A \cap B) = Int(A) \cap Int(B)$; and (v) if $A \subseteq B$ then $Int(A) \subseteq Int(B)$.

6 LTop as a logic of formal inconsistency

The *Logics of Formal Inconsistency* (**LFI**s) where introduced by W. Carnielli and J. Marcos in [8], and additionally studied in [7; 6] (among others). They are paraconsistent logics (that is, logics with a non-explosive negation $\neg$) and with a *consistency* operator which alows to recover the explosion law w.r.t. $\neg$ in a non-trivial way. In formal terms:[1]

DEFINITION 22. Let $\mathbf{L} = \langle \Theta, \vdash \rangle$ be a Tarskian, finitary and structural logic defined over a propositional signature Θ, which contains a negation $\neg$, and let $\circ$ be a (primitive or defined) unary connective. Then, $\mathbf{L}$ is said to be a *Logic of Formal Inconsistency* (an **LFI**, for short) with respect to $\neg$ and $\circ$ if the following holds:

(i) $\varphi, \neg\varphi \nvdash \psi$ for some φ and ψ;

(ii) there are two formulas α and β such that

  (ii.a) $\circ(\alpha), \alpha \nvdash \beta$;

  (ii.b) $\circ(\alpha), \neg\alpha \nvdash \beta$;

(iii) $\circ(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ.

[1]This is a simplified version of the definition of **LFI**s. In the general definition, a non-empty set $\bigcirc(p)$ of formulas depending exactly on the propositional variable p is considered, instead of a connective $\circ(p)$.

Condition (ii) of the definition of **LFI**s is required in order to satisfy condition (iii) (called *gentle explosion law*) in a non-trivial way. Examples of consistency operators defined in **LTop** violating condition (ii) will be given in Remark 30.

Consider the following proposal for a consistency operator in **LTop** w.r.t. the negation $\neg$:

(cons) $\circ\alpha \stackrel{\text{def}}{=} \Diamond\alpha \to \Box\alpha \equiv \Box{\sim}\alpha \vee \Box\alpha = {\sim}\neg{\sim}\alpha \vee {\sim}\neg\alpha$.

Let **L** be an **LFI**. An *inconsistency* operator $\bullet$ can also be defined, which satisfies the following: $\vdash (\alpha \wedge \neg\alpha) \to \bullet\alpha$. In general, $\bullet$ can be defined in **L** in two ways: $\bullet\alpha \stackrel{\text{def}}{=} \neg\circ\alpha$, or $\bullet\alpha \stackrel{\text{def}}{=} {\sim}\circ\alpha$, where $\sim$ is a classical negation definable in **L**. In the case of **LTop**, these alternatives produce the following:

(incons) $\bullet\alpha \stackrel{\text{def}}{=} {\sim}\circ\alpha = {\sim}(\Diamond\alpha \to \Box\alpha) = {\sim}(\Diamond\alpha \to {\sim}\neg\alpha) = \Diamond\alpha \wedge \neg\alpha$;

(incons') $\bullet'\alpha \stackrel{\text{def}}{=} \neg\circ\alpha \equiv \neg({\sim}\neg{\sim}\alpha \vee {\sim}\neg\alpha) \equiv \neg{\sim}(\neg{\sim}\alpha \wedge \neg\alpha) = \Diamond(\Diamond\alpha \wedge \neg\alpha)$.

Let $\langle\langle X, \tau\rangle, v\rangle$ be a topological model for **LTop**. Recall that $\partial A = \overline{A} \cap \overline{A^c}$ and $Ext(A) = Int(A^c)$ denote the *boundary* and the *exterior* of a set $A \subseteq X$. Thus, the operators $\circ$, $\bullet$ and $\bullet'$ for consistency and inconsistency in **LTop** are semantically characterized as follows (by using Corollary 15):

(v-cons) $v(\circ\alpha) = v(\Box{\sim}\alpha \vee \Box\alpha) = Ext(v(\alpha)) \cup Int(v(\alpha))$;

(v-incons) $v(\bullet\alpha) = v(\Diamond\alpha \wedge \neg\alpha) = \overline{v(\alpha)} \cap \overline{v(\alpha)^c} = \partial v(\alpha)$;

(v-incons') $v(\bullet'\alpha) = v(\Diamond(\Diamond\alpha \wedge \neg\alpha)) = \overline{\overline{v(\alpha)} \cap \overline{v(\alpha)^c}}$.

Given that the intersection of closed sets is a closed set, it follows that $\overline{v(\alpha)} \cap \overline{v(\alpha)^c} = \overline{\overline{v(\alpha)} \cap \overline{v(\alpha)^c}}$ and so $v(\bullet\alpha) = v(\bullet'\alpha)$. This means that, given the consistency operator proposed for **LTop**, there is just one inconsistency operator generated from it by means of the usual definitions. It is worth noting that $\circ$ and $\bullet$, as well as the negation $\neg$, have a nice interpretation in topological terms.

REMARK 23. Note that, for every topological model $\mathcal{M}$,

$$v(\alpha \wedge \neg\alpha) = v(\alpha) \cap \overline{v(\alpha)^c} \subseteq \overline{v(\alpha)} \cap \overline{v(\alpha)^c} = v(\bullet\alpha)$$

and so $\bullet$ satisfies the basic requirement for an inconsistency operator. The inclusion above is strict in general: let $\mathcal{M} = \langle\langle\mathbb{R}, \tau\rangle, v\rangle$ such that τ is the usual topology on the set $\mathbb{R}$ of real numbers, and $v(p) = (0, 1)$, where p is a propositional variable. Then $v(\neg p) = \overline{(-\infty, 0] \cup [1, +\infty)}$ and so $v(p \wedge \neg p) = v(p) \cap v(\neg p) = \emptyset$. On the other hand $\overline{v(p)} = [0, 1]$ and so $v(\bullet p) = \{0, 1\} \not\subseteq \emptyset = v(p \wedge \neg p)$. This means that **LTop** separates the notions of $\neg$-contradiction and inconsistency: every $\neg$-contradiction is an inconsistency but the converse is not always true. In logical terms:[2]

$$\vdash_{\mathbf{LTop}} (\alpha \wedge \neg\alpha) \to \bullet\alpha \quad \text{but} \quad \nvdash_{\mathbf{LTop}} \bullet\alpha \to (\alpha \wedge \neg\alpha).$$

In order to prove that **LTop** is an **LFI** w.r.t. $\neg$ and $\circ$, we begin by proving the following:

LEMMA 24. *Let p and q be two different propositional variables. Then:*

(i) $p, \neg p \nvdash_{\mathbf{LTop}} q$;

(ii) $\circ p, p \nvdash_{\mathbf{LTop}} q$;

(iii) $\circ p, \neg p \nvdash_{\mathbf{LTop}} q$.

Proof. Let $\mathcal{M} = \langle\langle \mathbb{R}, \tau\rangle, v\rangle$ such that τ is the usual topology on the set $\mathbb{R}$ of real numbers, $v(p) = [0, 1)$ and $v(q) = (2, 3)$.

(i) Since $v(p)^c = (-\infty, 0) \cup [1, +\infty)$ it follows that $v(\neg p) = (-\infty, 0] \cup [1, +\infty)$. Hence $v(p \wedge \neg p) = v(p) \cap v(\neg p) = \{0\} \not\subseteq v(q)$.

(ii) Note that $Ext(v(p)) = (-\infty, 0) \cup (1, +\infty)$. Given that $Int(v(p)) = (0, 1)$ it follows that $v(\circ p) = Ext(v(p)) \cup Int(v(p)) = \mathbb{R} \setminus \{0, 1\}$. Hence, $v(\circ p) \cap v(p) = (0, 1) \not\subseteq v(q)$.

(iii) From (i) and (ii) it follows that $v(\circ p) \cap v(\neg p) = \big(\mathbb{R} \setminus \{0, 1\}\big) \cap \big((-\infty, 0] \cup [1, +\infty)\big) = (-\infty, 0) \cup (1, +\infty) \not\subseteq v(q)$. ∎

The previous lemma shows that $\neg$ and $\circ$ satisfy in **LTop** properties (i) and (ii) of Definition 22. In order to prove that condition (iii) of such definition is also satisfied, it is necessary to prove some previous results in **LTop**.

LEMMA 25. *If* $\vdash_{\mathbf{LTop}} \alpha \to \gamma$ *and* $\vdash_{\mathbf{LTop}} \beta \to \gamma$ *then* $\vdash_{\mathbf{LTop}} (\alpha \vee \beta) \to \gamma$.

Proof. Assume that $\vdash_{\mathbf{LTop}} \alpha \to \gamma$ and $\vdash_{\mathbf{LTop}} \beta \to \gamma$. By using **CPL**, we also have that $\vdash_{\mathbf{LTop}} (\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma)$. The result follows by (**MP**). ∎

LEMMA 26. $\vdash_{\mathbf{LTop}} \Box\alpha \to (\alpha \to (\neg\alpha \to \beta))$.

Proof. By **CPL**, it is enough to shown that $\vdash_{\mathbf{LTop}} (\Box\alpha \wedge \alpha \wedge \neg\alpha) \to \beta$. Thus, it will shown that $\vdash_{\mathbf{LTop}} ({\sim}\neg\alpha \wedge \alpha \wedge \neg\alpha) \to \beta$. Consider the following (meta)derivation in **LTop**:

1. $\vdash_{\mathbf{LTop}} ({\sim}\neg\alpha \wedge \alpha \wedge \neg\alpha) \to ({\sim}\neg\alpha \wedge \neg\alpha)$ (**CPL**)
2. $\vdash_{\mathbf{LTop}} ({\sim}\neg\alpha \wedge \neg\alpha) \to \beta$ (**CPL**)

[2] Of course this holds after proving the completeness of **LTop** w.r.t. topological models, see Theorem 36 below.

3. $\vdash_{\mathbf{LTop}} (\sim\neg\alpha \wedge \alpha \wedge \neg\alpha) \to \beta$ (1, 2, **CPL**)

$\blacksquare$

Similarly, it can be proven the following:

LEMMA 27. $\vdash_{\mathbf{LTop}} \Box{\sim}\alpha \to (\alpha \to (\neg\alpha \to \beta))$.

Finally:

LEMMA 28. $\vdash_{\mathbf{LTop}} \circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$.

Proof. Recall that $\circ\alpha \equiv \Box{\sim}\alpha \vee \Box\alpha$. The result follows from lemmas 26, 27 and 25, and by Replacement. $\blacksquare$

PROPOSITION 29. **LTop** *is an* **LFI** *w.r.t.* $\neg$ *and* $\circ$.

Proof. It follows from lemmas 24 and 28. $\blacksquare$

REMARK 30. Let $\circ'\alpha \stackrel{\text{def}}{=} \Box\alpha$ and $\circ''\alpha \stackrel{\text{def}}{=} \Box{\sim}\alpha$. Then, lemmas 26 and 27 state that these unary operators satisfy item (iii) of Definition 22 of **LFI**s. Since item (i) of that definition is also satisfied (because **LTop** is $\neg$-paraconsistent) one wonders if $\circ'$ and $\circ'$ could be considered as alternative consistency operators in **LTop** w.r.t. $\neg$. However, it is easy to see that they are trivial, in the sense that property (ii) of Definition 22 fails for both of them. Indeed, $v(\circ'\alpha\wedge\neg\alpha) = \emptyset$ in every topological model and so condition (ii.b) fails for $\circ'$. On the other hand, $v(\circ''\alpha \wedge \alpha) = \emptyset$ in every topological model, hence condition (ii.a) fails for $\circ''$. Being so, the gentle exlosion law is satisfied by both operators in a trivial way.

It is worth noting that Marcos suggested in [13] the following definition of a consistency operator inside a modal logic: $\oslash\alpha \stackrel{\text{def}}{=} \alpha \to \Box\alpha$. Considered in **LTop**, this formula defines a consistency operator in the sense of Definition 22. The inconsistency operators naturally associated to it are $\bullet\alpha = {\sim}\oslash\alpha = \alpha \wedge \neg\alpha$ (where inconsistency is identified with contradiction) and $\bullet'\alpha = \neg\oslash\alpha = \overline{\alpha \wedge \neg\alpha}$ (where inconsistency and contradiction are diferent notions).

7 *Intermezzo*: A problem on Kuratowski operators

In this section a technical result (Proposition 31) concerning the definition of a closure operator from a given colletion of sets enjoying some properties will be given. This result will be used in the proof of completeness of **LTop** w.r.t. topological models, in Section 8.

Let $X \neq \emptyset$ be a set. Let $\mathcal{B} \subseteq \wp(X)$ be a collection of subsets of X such that: (i) $\emptyset \in \mathcal{B}$ and $X \in \mathcal{B}$; and (ii) if $F, G \in \mathcal{B}$ then $F \cup G \in \mathcal{B}$. Now, let $\widehat{(\cdot)} : \mathcal{B} \to \mathcal{B}$ be a mapping such that:

1. $\widehat{\emptyset} = \emptyset$;

2. $F \subseteq \widehat{F}$ for every $F \in \mathcal{B}$;

3. $\widehat{F \cup G} = \widehat{F} \cup \widehat{G}$ for every $F, G \in \mathcal{B}$;

4. $\widehat{\widehat{F}} = \widehat{F}$ for every $F \in \mathcal{B}$.

Note that $\widehat{X} = X$, from 2. From 3 it follows that $\widehat{(\cdot)}$ is monotonic: $F \subseteq G$ implies $\widehat{F} \subseteq \widehat{G}$, for every $F, G \in \mathcal{B}$. We ask wether it is possible to extend $\widehat{(\cdot)}$ to a Kuratowski closure operator $\overline{(\cdot)} : \wp(X) \to \wp(X)$ over X (that is, satisfying properties 1-4 above for every $F, G \in \wp(X)$). Next result give us a positive answer.

PROPOSITION 31. *Let $\mathcal{B}$ and $\widehat{(\cdot)}$ as above. The operator $\overline{(\cdot)} : \wp(X) \to \wp(X)$ given by $\overline{A} = \bigcap\{\widehat{F} \ : \ F \in \mathcal{B}$ and $A \subseteq \widehat{F}\}$, is a Kuratowski closure operator over X such that $\overline{F} = \widehat{F}$ if $F \in \mathcal{B}$.*

Proof. First, it is easy to see that $\overline{F} = \widehat{F}$ if $F \in \mathcal{B}$. In fact, given $F \in \mathcal{B}$ then $F \subseteq \widehat{F}$ and so $\overline{F} = \bigcap\{\widehat{G} \ : \ G \in \mathcal{B}$ and $F \subseteq \widehat{G}\} \subseteq \widehat{F}$. On the other hand, if $G \in \mathcal{B}$ and $F \subseteq \widehat{G}$ then $\widehat{F} \subseteq \widehat{\widehat{G}} = \widehat{G}$, and so $\widehat{F} \subseteq \overline{F}$.

This means that $\overline{(\cdot)}$ extends the operator $\widehat{(\cdot)}$ to $\wp(X)$. We will prove now that $\overline{(\cdot)}$ is a Kuratowski closure operator.

(1) From the observation above, $\overline{\emptyset} = \widehat{\emptyset} = \emptyset$.

(2) Clearly $A \subseteq \overline{A}$.

(3) Let $A, B \subseteq X$. Then

$$\overline{A} \cup \overline{B} = \left(\bigcap_{A \subseteq \widehat{F}} \widehat{F}\right) \cup \left(\bigcap_{B \subseteq \widehat{G}} \widehat{G}\right) = \bigcap \mathcal{F}$$

such that $\mathcal{F} = \{\widehat{F} \cup \widehat{G} \ : \ F, G \in \mathcal{B};\ A \subseteq \widehat{F}$ and $B \subseteq \widehat{G}\}$. On the other hand, $\overline{A \cup B} = \bigcap \mathcal{G}$ such that $\mathcal{G} = \{\widehat{H} \ : \ H \in \mathcal{B}$ and $A \cup B \subseteq \widehat{H}\}$. Since $\widehat{F \cup G} = \widehat{F} \cup \widehat{G}$ then $\mathcal{F} \subseteq \mathcal{G}$. On the other hand, let $\widehat{H} \in \mathcal{G}$. Then $\widehat{H} \in \mathcal{B}$ is such that $A \subseteq \widehat{H}$ and $B \subseteq \widehat{H}$ and so $\widehat{H} = \widehat{H} \cup \widehat{H} \in \mathcal{F}$. From this, $\mathcal{G} \subseteq \mathcal{F}$. Therefore, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(4) By definition, $\overline{\overline{A}} = \bigcap \mathcal{F}$ such that $\mathcal{F} = \{\widehat{F} \ : \ F \in \mathcal{B}$ and $\overline{A} \subseteq \widehat{F}\}$. By its turn, $\overline{A} = \bigcap \mathcal{G}$ such that $\mathcal{G} = \{\widehat{G} \ : \ G \in \mathcal{B}$ and $A \subseteq \widehat{G}\}$. Let $\widehat{G} \in \mathcal{G}$. Then $G \in \mathcal{B}$ such that $\overline{A} \subseteq \widehat{G}$ and so $\widehat{G} \in \mathcal{F}$. This means that $\mathcal{G} \subseteq \mathcal{F}$ and so $\overline{\overline{A}} = \bigcap \mathcal{F} \subseteq \bigcap \mathcal{G} = \overline{A}$. ■

8 Completeness Theorem for the logic LTop

Recall that, given a logic $\mathbf{L}$ and a formula α in the language $\mathbb{L}$ of $\mathbf{L}$, a set $\Delta \subseteq \mathbb{L}$ is α-saturated in $\mathbf{L}$ if: (1) $\Delta \nvdash_{\mathbf{L}} \alpha$; and (2) if $\beta \notin \Delta$ then $\Delta, \beta \vdash_{\mathbf{L}} \alpha$. It follows that α-saturated sets are closed non-trivial theories of $\mathbf{L}$, that is: $\Delta \neq \mathbb{L}$, and $\beta \in \Delta$ iff $\Delta \vdash_{\mathbf{L}} \beta$. The following classical result will be useful:

THEOREM 32 (Lindenbaum-Łos). *Let* **L** *be a Tarskian and finitary logic over a language* $\mathbb{L}$. *Let* $\Gamma \cup \{\alpha\}$ *be a set of formulas of* $\mathbb{L}$ *such that* $\Gamma \nvdash_{\mathbf{L}} \alpha$. *Then, it is possible to extend* Γ *to an* α*-saturated set* Δ *in* **L**.

Proof. See [19, Theorem 22.2]. ■

Since the logic **LTop** is Tarskian and finitary (see Remark 4) then the last theorem holds for it. Additionally, α-saturated sets in **LTop** satisfy the following properties:

PROPOSITION 33. *Let* Δ *be an* α*-saturated set in* **LTop**. *Then:*
(i) $(\beta \to \beta) \in \Delta$ *and* $\sim\neg(\beta \to \beta) \in \Delta$ *for every* β*;*
(ii) $(\beta \to \gamma) \in \Delta$ *iff* $\beta \notin \Delta$ *or* $\gamma \in \Delta$*;*
(iii) $\sim\beta \in \Delta$ *iff* $\beta \notin \Delta$*;*
(iv) $(\beta \wedge \gamma) \in \Delta$ *iff* $\beta \in \Delta$ *and* $\gamma \in \Delta$*;*
(v) $(\beta \vee \gamma) \in \Delta$ *iff* $\beta \in \Delta$ *or* $\gamma \in \Delta$*;*
(vi) $\beta \notin \Delta$ *implies* $\neg\beta \in \Delta$.

Now, let $X_c = \{\Delta \subseteq \mathbb{L} \ : \ \Delta$ is an α-saturated set in **LTop** for some $\alpha \in \mathbb{L}\}$. For every $\varphi \in \mathbb{L}$ let $F_\varphi = \{\Delta \in X_c \ : \ \varphi \notin \Delta\}$. Observe that $F_\varphi = F_\psi$ whenever $\varphi \equiv \psi$. Let $\mathcal{B} = \{F_\varphi \ : \ \varphi \in \mathbb{L}\}$. Clearly:

(i) $\emptyset \in \mathcal{B}$ since $\emptyset = F_{(\varphi\to\varphi)}$, by Proposition 33(i); additionally, $X_c \in \mathcal{B}$ since $X_c = F_{\sim(\varphi\to\varphi)}$ (recalling that α-saturated sets are non-trivial theories);

(ii) $F_\varphi \cup F_\psi = F_{(\varphi\wedge\psi)} \in \mathcal{B}$, by Proposition 33(iv).

Notice that $X_c \setminus F_\varphi = F_{\sim\varphi}$, by Proposition 33(iii). Consider now a mapping $\widehat{(\cdot)} : \mathcal{B} \to \mathcal{B}$ defined as follows:

$$\widehat{F_\varphi} = F_{\sim\neg\varphi} = \{\Delta \in X_c \ : \ \neg\varphi \in \Delta\}.$$

Observe the following:

(i) $\widehat{\emptyset} = \emptyset$. In fact: $\widehat{\emptyset} = \widehat{F_{(\varphi\to\varphi)}} = F_{\sim\neg(\varphi\to\varphi)} = \emptyset$, by Proposition 33(i).

(ii) $F_\varphi \subseteq \widehat{F_\varphi}$. In fact: if $\Delta \in F_\varphi$ then $\varphi \notin \Delta$ and so $\neg\varphi \in \Delta$, by Proposition 33(vi). Thus $\Delta \in \widehat{F_\varphi}$.

(iii) $\widehat{F_\varphi \cup F_\psi} = \widehat{F_\varphi} \cup \widehat{F_\psi}$. In fact: $\widehat{F_\varphi} \cup \widehat{F_\psi} = F_{\sim\neg\varphi} \cup F_{\sim\neg\psi} = F_{(\sim\neg\varphi\wedge\sim\neg\psi)} = F_{\sim\neg(\varphi\wedge\psi)} = \widehat{F_{(\varphi\wedge\psi)}} = \widehat{F_\varphi \cup F_\psi}$, by Proposition 20(iv).

(iv) $\widehat{\widehat{F_\varphi}} = \widehat{F_\varphi}$. In fact: $\widehat{\widehat{F_\varphi}} = \widehat{F_{\sim\neg\varphi}} = F_{\sim\neg\sim\neg\varphi} = F_{\sim\neg\varphi} = \widehat{F_\varphi}$, by Proposition 20(iii).

Then, by Proposition 31, the operator $\overline{(\cdot)} : \wp(X_c) \to \wp(X_c)$ given by

$$\overline{A} = \bigcap\{F_{\sim\neg\varphi} \ : \ A \subseteq F_{\sim\neg\varphi}\}$$

is a Kuratowski closure operator over X_c such that $\overline{F_\varphi} = \widehat{F_\varphi} = F_{\sim\neg\varphi}$.

DEFINITION 34. The *canonical structure* for **LTop** is the topological space $\mathcal{M}_c = \langle X_c, \tau_c\rangle$ such that τ_c is the topology over X_c generated by the Kuratowski closure $\overline{(\cdot)}$ defined above. The *canonical model* for **LTop** is the model $\langle \mathcal{M}_c, v_c\rangle$ such that $v_c(p) = F_{\sim p} = \{\Delta \in X_c \ : \ p \in \Delta\}$, for every $p \in \mathcal{V}$.

Notice that, since $X_c \setminus F_\varphi = F_{\sim\varphi}$, then each F_φ is clopen in $\mathcal{M}_c$.

LEMMA 35 (Truth Lemma). *For every $\varphi \in \mathbb{L}$ it holds:*

$$v_c(\varphi) = F_{\sim\varphi} = \{\Delta \in X_c \ : \ \varphi \in \Delta\}.$$

Proof. Since it was observed, $F_{\sim\varphi} = \{\Delta \in X_c \ : \ \varphi \in \Delta\}$. By induction on the complexity of φ it will be proven that $v_c(\varphi) = F_{\sim\varphi}$.
If $\varphi = p \in \mathcal{V}$ then the result holds by definition of v_c.
If $\varphi = (\psi \to \gamma)$ then $v_c(\varphi) = v_c(\psi \to \gamma) = (X_c \setminus v_c(\psi)) \cup v_c(\gamma)$, by definition of v_c. By induction hypothesis and Proposition 33, the latter is equal to $(X_c \setminus F_{\sim\psi}) \cup F_{\sim\gamma} = F_{\sim\sim\psi} \cup F_{\sim\gamma} = F_\psi \cup F_{\sim\gamma} = F_{(\psi\wedge\sim\gamma)} = F_{\sim(\psi\to\gamma)} = F_{\sim\varphi}$.
If $\varphi = {\sim}\psi$ then $v_c(\varphi) = v_c({\sim}\psi) = (X_c \setminus v_c(\psi))$, by definition of v_c. By induction hypothesis it is equal to $(X_c \setminus F_{\sim\psi}) = F_{\sim\sim\psi} = F_{\sim\varphi}$.
If $\varphi = \neg\psi$ then $v_c(\varphi) = v_c(\neg\psi) = \overline{(X_c \setminus v_c(\psi))}$, by definition of v_c. By induction hypothesis and the definition of closure it is equal to $\overline{(X_c \setminus F_{\sim\psi})} = \overline{F_{\sim\sim\psi}} = \overline{F_\psi} = \widehat{F_\psi} = F_{\sim\neg\psi} = F_{\sim\varphi}$. ■

THEOREM 36 (Completeness of **LTop**). *For every set of formulas $\Gamma \cup \{\varphi\}$ it holds: $\Gamma \models_{\mathbf{LTop}} \varphi$ implies $\Gamma \vdash_{\mathbf{LTop}} \varphi$.*

Proof. Suppose that $\Gamma \nvdash_{\mathbf{LTop}} \varphi$, and assume that $\Gamma \neq \emptyset$. By Theorem 32, there exists a φ-saturated set Δ in **LTop** such that $\Gamma \subseteq \Delta$. Consider the canonical model $\langle \mathcal{M}_c, v_c\rangle$. Since $\Gamma \subseteq \Delta$ then $\Delta \in v_c(\gamma)$ for every $\gamma \in \Gamma$, by Lemma 35, and so $\Delta \in \bigcap_{\gamma\in\Gamma} v_c(\gamma)$. On the other hand $\varphi \notin \Delta$ and so $\Delta \notin v_c(\varphi)$, by Lemma 35. Thus $\bigcap_{\gamma\in\Gamma} v_c(\gamma) \nsubseteq v_c(\varphi)$. This shows that $\Gamma \not\models_{\mathbf{LTop}} \varphi$. The case $\Gamma = \emptyset$ is proved analogously. ■

The latter result can be applied to the question of defining unary connectives in **LTop**, which can be solved semantically in terms of the Kuratowski's closure-complement problem. Indeed, a theorem due to Kuratowski states that there are at most 14 distinct sets obtainable by iterations of closure (k) and complement (c) to a given subset of X. As a consequence of this, and by soundness and completeness of **LTop** w.r.t. topological models, there are only 14 unary conectives definable in **LTop** by iterations of $\neg$ and $\sim$. They are listed in the table below (as usual, the composition of operators must be read from right to left, and so, for instance, kc stands for c followed by k).

Operator	Connective
id	$\sim\sim$
c	$\sim$
k	$\neg\sim$
ck	$\sim\neg\sim$
kc	$\neg$
ckc	$\sim\neg$
kck	$\neg\neg\sim$
ckck	$\sim\neg\neg\sim$
kckc	$\neg\neg$
ckckc	$\sim\neg\neg$
kckck	$\neg\neg\neg\sim$
ckckck	$\sim\neg\neg\neg\sim$
kckckc	$\neg\neg\neg$
ckckckc	$\sim\neg\neg\neg$

9 LTop is S4 up to language

In this section it will be shown that **LTop** is nothing more than the well-known modal logic **S4**, presented in a different (non-modal) language. In order to do this, two translation mappings $* : \mathbb{L}_4 \to \mathbb{L}$ and $\otimes : \mathbb{L} \to \mathbb{L}_4$ will be defined, where $\mathbb{L}_4$ is the set of formulas of **S4**, satisfying the following: (1) $\Theta \vdash_{\mathbf{S4}} \beta$ iff $\Theta^* \vdash_{\mathbf{LTop}} \beta^*$ for every $\Theta \cup \{\beta\} \subseteq \mathbb{L}_4$; and (2) $\Gamma \vdash_{\mathbf{LTop}} \alpha$ iff $\Gamma^{\otimes} \vdash_{\mathbf{S4}} \alpha^{\otimes}$ for every $\Gamma \cup \{\alpha\} \subseteq \mathbb{L}$, where $\vdash_{\mathbf{S4}}$ denotes the consequence relation of **S4** (observe that derivations in **S4** from non-empty sets of premisses are defined in a similar way as in **LTop**). Here, $\Theta^* = \{\gamma^* \ : \ \gamma \in \Theta\}$ and γ^* denotes $*(\gamma)$, for every $\gamma \in \mathbb{L}_4$. A similar notation is adopted for the translation mapping $\otimes$. Moreover, $\beta \equiv_4 (\beta^*)^{\otimes}$ and $\alpha \equiv (\alpha^{\otimes})^*$ for every $\beta \in \mathbb{L}_4$ and $\alpha \in \mathbb{L}$, where $\beta \equiv_4 \gamma$ means that $\vdash_{\mathbf{S4}} \beta \to \gamma$ and $\vdash_{\mathbf{S4}} \gamma \to \beta$. From this result, it can be said that **LTop** coincides with **S4** 'up to translations' or 'up to language'.

Previous to prove this result, it will be shown that the modal *Necessitation rule* ('if α is a theorem then $\Box\alpha$ is a theorem') is admissible in **LTop** (where $\Box\varphi$ denotes, as stated above, the formula $\sim\neg\varphi$ of $\mathbb{L}$). This means that adding this rule to **LTop** does not add any new theorem to the resulting logic.

THEOREM 37 (Admissibility in **LTop** of the Necessitation rule). *Consider, in the language $\mathbb{L}$ of **LTop**, the Necessitation rule:*

$$(\mathbf{NEC})\ \frac{\alpha}{\Box\alpha}$$

(where, as stated above, $\Box\alpha$ denotes $\sim\neg\alpha$). Then, (**NEC**) *is admissible in* **LTop**, *that is: if $\vdash_{\mathbf{LTop}} \varphi$ then $\vdash_{\mathbf{LTop}} \Box\varphi$.*

Proof. Suppose that $\vdash_{\mathbf{LTop}} \varphi$. By soundness of **LTop** with respect to topological structures, it follows that $\models_{\mathbf{LTop}} \varphi$. This means that $v(\varphi) = X$ for every model $\langle\langle X, \tau\rangle, v\rangle$. But then $v(\neg\varphi) = \overline{v(\varphi)^c} = \overline{\emptyset} = \emptyset$ and so $v(\Box\varphi) = v(\sim\neg\varphi) = v(\neg\varphi)^c = X$, for every model $\mathcal{M}$. Thus $\models_{\mathbf{LTop}} \Box\varphi$ and so $\vdash_{\mathbf{LTop}} \Box\varphi$, by the completeness theorem. ■

Now, the logic **S4** will be formally analyzed. Recall from Remark 5 that the Hilbert calculus formed by axioms (**Ax1**), (**Ax2**) and (**Ax3**) of **LTop** plus the rule (**MP**) constitutes a Hilbert calculus $\mathbf{H_{CPL}}$ for **CPL** over the signature $\{\sim, \to\}$.

DEFINITION 38 (Modal logic **S4**). Consider the signature $\Sigma_\Box = \{\Box, \sim, \to\}$ and let $\mathbb{L}_4$ be the language generated by the set $\mathcal{V}$ of propositional variables (recall Definition 1) over the signature $\Sigma_\Box$. The modal logic **S4** is defined over the language $\mathbb{L}_4$ by adding to $\mathbf{H_{CPL}}$ the following axioms and rules:

$$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta) \qquad (\mathbf{K})$$

$$\Box\alpha \to \alpha \qquad (\mathbf{T})$$

$$\Box\alpha \to \Box\Box\alpha \qquad (\mathbf{4})$$

$$\frac{\alpha}{\Box\alpha} \ (\mathbf{NEC})$$

Derivations (without and with premisses) are defined in **S4** in a similar way as in **LTop** (recall Definition 3), as usual in modal systems. It is well-known (consult, for instance, [5]) that the Hilbert calculus presented in Definition 38, together with its notion of derivations, is adequate for the modal logic **S4**.

If $\beta, \gamma \in \mathbb{L}_4$ then $\beta \equiv_4 \gamma$ will mean that $\vdash_{\mathbf{S4}} \beta \to \gamma$ and $\vdash_{\mathbf{S4}} \gamma \to \beta$.

Now, the two translation mappings will be defined.

DEFINITION 39. Let $* : \mathbb{L}_4 \to \mathbb{L}$ be a mapping defined recursively as follows (here, $*(\gamma)$ will be denoted by γ^*, for every $\gamma \in \mathbb{L}_4$):

(i) $p^* = p$ if p is a propositional variable;

(ii) $(\Box\gamma)^* = \sim\neg(\gamma^*)$;

(iii) $(\sim\gamma)^* = \sim(\gamma^*)$;

(iv) $(\gamma \to \delta)^* = (\gamma^*) \to (\delta^*)$.

DEFINITION 40. Let $\otimes : \mathbb{L} \to \mathbb{L}_4$ be a mapping defined recursively as follows (here, $\otimes(\gamma)$ will be denoted by $\gamma^\otimes$, for every $\gamma \in \mathbb{L}$):

(i) $p^\otimes = p$ if p is a propositional variable;

(ii) $(\neg\gamma)^\otimes = \sim\Box(\gamma^\otimes)$;

(iii) $(\sim\gamma)^\otimes = \sim(\gamma^\otimes)$;

(iv) $(\gamma \to \delta)^\otimes = (\gamma^\otimes) \to (\delta^\otimes)$.

PROPOSITION 41. *For every* $\beta \in \mathbb{L}_4$, $\beta \equiv_4 (\beta^*)^\otimes$.

Proof. The proof is done by induction on the complexity of $\beta \in \mathbb{L}_4$ (which is defined as usual). By definition of $*$, it is enough to analyze the induction step when $\beta = \Box\gamma$. Then, we have that $\beta^* = {\sim}\neg(\gamma^*)$. From this, by definition of $\otimes$, by Replacement of **S4** and by induction hypothesis (from which $\gamma \equiv_4 (\gamma^*)^\otimes$),

$$(\beta^*)^\otimes = {\sim}{\sim}\Box((\gamma^*)^\otimes) \equiv_4 \Box((\gamma^*)^\otimes) \equiv_4 \Box\gamma = \beta.$$

That is, $\beta \equiv_4 (\beta^*)^\otimes$. ■

PROPOSITION 42. *For every $\alpha \in \mathbb{L}$, $\alpha \equiv (\alpha^\otimes)^*$.*

Proof. The proof is analogous to that of Proposition 41, but now the only case to be analyzed is when $\alpha = \neg\delta$. The details are left to the reader ■

PROPOSITION 43. *For every $\beta \in \mathbb{L}_4$, $\vdash_{\mathbf{S4}} \beta$ implies that $\vdash_{\mathbf{LTop}} \beta^*$.*

Proof. Consider the Hilbert calculus for **S4** presented above. It is immediate to see, by the results about **LTop** proved in the previous sections, that any instance β of an axiom of **S4** is such that β^* is a theorem of **LTop**. On the other hand, by definition of $*$, it is immediate that (**MP**) is translated as itself by $*$. Finally, the translation of (**NEC**) by $*$ is an admissible rule in **LTop**, by Theorem 37. From this, it is easy to prove, by induction on the length of a derivation of β in **S4**, that $\vdash_{\mathbf{S4}} \beta$ implies that $\vdash_{\mathbf{LTop}} \beta^*$. The details are left to the reader. ■

PROPOSITION 44. *For every $\alpha \in \mathbb{L}$, $\vdash_{\mathbf{LTop}} \alpha$ implies that $\vdash_{\mathbf{S4}} \alpha^\otimes$.*

Proof. The proof uses an argument analogous to that of Proposition 43. Let us begin by observing that the translation by $\otimes$ of axioms (**Ax1**), (**Ax2**) and (**Ax3**) of **LTop** are instances of the same axioms in **S4**. The same happens with the rule (**MP**). Now, observe that the translation by $\otimes$ of axioms (**Ax4**) and (**Ax5**) of **LTop** corresponds to axioms (**T**) and (**4**) of **S4**, but written in terms of the possibility operator. Then, they are derivable in **S4**. The translation by $\otimes$ of any instance axiom (**Ax6**) of **LTop** has the form ${\sim}\Box(\gamma \wedge \delta) \to ({\sim}\Box\gamma \vee {\sim}\Box\delta)$. The latter is equivalent, by **CPL**, to $(\Box\gamma \wedge \Box\delta) \to \Box(\gamma \wedge \delta)$, which is a theorem of **S4**. The translation by $\otimes$ of any instance of axiom (**Ax7**) of **LTop** has the form $\Box(\gamma \to \gamma)$, which is derivable in **S4**. The rule (**DR**) is translated by $\otimes$ as itself, being clearly a derived rule in the Hilbert calculus of **S4**. Finally, the translation by $\otimes$ of the rule (**CR**) has the form $\gamma \to \delta / {\sim}\Box\delta \to {\sim}\Box\gamma$. Suppose that $\vdash_{\mathbf{S4}} \gamma \to \delta$. Then $\vdash_{\mathbf{S4}} \Box\gamma \to \Box\delta$ (this is a well known fact of **S4**) and so, by **CPL**, $\vdash_{\mathbf{S4}} {\sim}\Box\delta \to {\sim}\Box\gamma$. This proves that the translation by $\otimes$ of the rule (**CR**) is an admissible rule in **S4**, concluding the proof. ■

COROLLARY 45. *Let $\alpha \in \mathbb{L}$ and $\beta \in \mathbb{L}_4$. Then:*

(i) $\vdash_{\mathbf{LTop}} \alpha$ *iff* $\vdash_{\mathbf{S4}} \alpha^\otimes$.

(ii)) $\vdash_{\mathbf{S4}} \beta$ *iff* $\vdash_{\mathbf{LTop}} \beta^*$.

Proof. (i) If $\vdash_{\mathbf{LTop}} \alpha$ then $\vdash_{\mathbf{S4}} \alpha^{\otimes}$, by Proposition 44. Conversely, if $\vdash_{\mathbf{S4}} \alpha^{\otimes}$ then $\vdash_{\mathbf{LTop}} (\alpha^{\otimes})^*$, by Proposition 43. The result follows by Proposition 42.
(ii) It is proved analogously, by using Proposition 41 in the last step. ■

COROLLARY 46. *Let* $\Gamma \cup \{\alpha\} \subseteq \mathbb{L}$ *and* $\Theta \cup \{\beta\} \subseteq \mathbb{L}_4$. *Then:*

(i) $\Gamma \vdash_{\mathbf{LTop}} \alpha$ *iff* $\Gamma^{\otimes} \vdash_{\mathbf{S4}} \alpha^{\otimes}$.

(ii) $\Theta \vdash_{\mathbf{S4}} \beta$ *iff* $\Theta^* \vdash_{\mathbf{LTop}} \beta^*$.

Proof. It is immediate from Corollary 45, the definition of the respective consequence relation in both logics, and the fact that both translations preserve the connectives of **CPL**. The details are left to the reader. ■

The last result, together with propositions 41 and 42, justify the claim that **LTop** and **S4** define the same logic, up to language.

10 Encoding Intuitionistic Logic inside LTop

Finally, it will be shown that **LTop** has the expressive power to encode Intuitionistic Propositional Logic **IPL** by means of a conservative translation. This is not surprising, since **LTop** is **S4** up to translations, as it was proven in the previous section. On the other and, it was proved by McKinsey-Tarski in [15] that Gödel's translation T of **IPL** into **S4** is conservative for valid formulas, that is: α is intuitionistically valid iff $T(\alpha)$ is derivable in **S4**.

In order to define a conservative translation from **IPL** to **LTop**, consider the signature $\Sigma_{int} = \{\sim, \to, \wedge, \vee\}$ for **IPL** and let $\mathbb{L}_{\mathbf{IPL}}$ be the language generated by the set $\mathcal{V}$ of propositional variables (recall Definition 1) over the signature Σ_{int}. The consequence relation of **IPL**, denoted by $\vdash_{\mathbf{IPL}}$, can be axiomatized by a Hilbert calculi over $\mathbb{L}_{\mathbf{IPL}}$ (see, for instance, [15]). It is well known that $\vdash_{\mathbf{IPL}}$ is semantically characterized by a topological semantics given by topological models $\mathcal{M} = \langle \mathcal{T}, v_0 \rangle$ such that $\mathcal{T} = \langle X, \tau \rangle$ is a topological space and $v_0 \colon \mathbb{L}_{\mathbf{IPL}} \to \tau$ is a valuation satisfying the following conditions:

1. $v_0({\sim}\alpha) = Int(v_0(\alpha)^c)$;
2. $v_0(\alpha \to \beta) = Int(v_0(\alpha)^c \cup v_0(\beta))$;
3. $v_0(\alpha \wedge \beta) = v_0(\alpha) \cap v_0(\beta)$;
4. $v_0(\alpha \vee \beta) = v_0(\alpha) \cup v_0(\beta)$.

It generates a semantical consequence as follows: $\models_{\mathbf{IPL}} \alpha$ iff $v_0(\alpha) = X$ for every topological model $\langle\langle X, \tau\rangle, v_0\rangle$. Moreover, $\Gamma \models_{\mathbf{IPL}} \alpha$ iff either $\models_{\mathbf{IPL}} \alpha$ or there exists a finite non-empty set $\Gamma_0 \subseteq \Gamma$ such that $\bigcap_{\gamma\in\Gamma_0} v_0(\gamma) \subseteq v_0(\alpha)$, for every topological model $\langle\langle X, \tau\rangle, v_0\rangle$. By definition, $\emptyset \models_{\mathbf{IPL}} \alpha$ iff $\models_{\mathbf{IPL}} \alpha$.

Consider now the following translation mapping:

DEFINITION 47. Let $\odot : \mathbb{L}_{\mathbf{IPL}} \to \mathbb{L}$ be a mapping defined recursively as follows:

(i) $p^{\odot} = \Box p = \sim\neg p$ if p is a propositional variable;

(ii) $(\sim\gamma)^{\odot} = \Box\sim(\gamma^{\odot}) = \sim\neg\sim(\gamma^{\odot})$;

(iii) $(\gamma \to \delta)^{\odot} = \Box((\gamma^{\odot}) \to (\delta^{\odot})) = \sim\neg((\gamma^{\odot}) \to (\delta^{\odot}))$;

(iv) $(\gamma \wedge \delta)^{\odot} = (\gamma^{\odot}) \wedge (\delta^{\odot})$;

(v) $(\gamma \vee \delta)^{\odot} = (\gamma^{\odot}) \vee (\delta^{\odot})$.

Now, the desired result is the following:

THEOREM 48. *Let* $\Gamma \cup \{\varphi\} \subseteq \mathbb{L}_{\mathbf{IPL}}$. *Then:*

$$\Gamma \vdash_{\mathbf{IPL}} \varphi \quad \textit{iff} \quad \Gamma^{\odot} \vdash_{\mathbf{LTop}} \varphi^{\odot}.$$

Proof. ('Only if' part) Suppose that $\Gamma \vdash_{\mathbf{IPL}} \varphi$ such that $\Gamma \neq \emptyset$, and let $\langle\langle X, \tau\rangle, v\rangle$ be a topological model for **LTop**. It is easy to prove that, for every $\alpha \in \mathbb{L}_{\mathbf{IPL}}$, $v(\alpha^{\odot}) \in \tau$. Thus, there is a mapping $v_0 : \mathbb{L}_{\mathbf{IPL}} \to \tau$ given by $v_0(\alpha) \stackrel{\text{def}}{=} v(\alpha^{\odot})$ for every α. It can be seen that v_0 is a valuation for **IPL** whence $\langle\langle X, \tau\rangle, v_0\rangle$ is a topological model for **IPL**. By soundness of **IPL** w.r.t. topological models it follows that $\bigcap_{\gamma\in\Gamma} v_0(\gamma) \subseteq v_0(\varphi)$. That is, $\bigcap_{\gamma\in\Gamma} v(\gamma^{\odot}) \subseteq v(\varphi^{\odot})$. This means that $\Gamma^{\odot} \models_{\mathbf{LTop}} \varphi^{\odot}$ and so $\Gamma^{\odot} \vdash_{\mathbf{LTop}} \varphi^{\odot}$, by completeness of **LTop** w.r.t. topological models. The case when $\vdash_{\mathbf{IPL}} \varphi$ is proved analogously.
('If' part) Suppose that $\Gamma^{\odot} \vdash_{\mathbf{LTop}} \varphi^{\odot}$ such that $\Gamma \neq \emptyset$, and let $\langle\langle X, \tau\rangle, v_0\rangle$ be a topological model for **IPL**. Define a mapping $v_1 : \mathcal{V} \to \wp(X)$ such that $v_1(p) = v_0(p)$ for every propositional variable p. Extend v_1 to a valuation $v : \mathbb{L} \to \wp(X)$ for **LTop** by using the clauses of Definition 12. It is easy to prove that $v_0(\alpha) = v(\alpha^{\odot})$ for every $\alpha \in \mathbb{L}_{\mathbf{IPL}}$. Let $\langle\langle X, \tau\rangle, v\rangle$ be the resulting topological model for **LTop**. By hypothesis and soundness of **LTop** w.r.t. topological models, $\bigcap_{\gamma\in\Gamma} v(\gamma^{\odot}) \subseteq v(\varphi^{\odot})$ and so $\bigcap_{\gamma\in\Gamma} v_0(\gamma) \subseteq v_0(\varphi)$. This shows that $\Gamma \models_{\mathbf{IPL}} \varphi$, and then $\Gamma \vdash_{\mathbf{IPL}} \varphi$ by completeness of **IPL** w.r.t. topological models. The case when $\vdash_{\mathbf{LTop}} \varphi^{\odot}$ is proved analogously. ■

The latter result shows that $\odot$ is a conservative translation from **IPL** to **LTop**.

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