ALGEBRAIC EMERGENCE

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Abstract. We define emergence algebraically in the context of discrete dynamical systems modeled as transformation semigroups. Emergence happens when a quotient structure (coarse-grained dynamics) is not a substructure of the original system. We survey small groups to show that algebraic emergence is neither ubiquitous nor rare. Then, we describe connections with hierarchical decompositions and explore some of the philosophical implications of the algebraic constraints.

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1. Introduction

Informally, the phenomenon of *emergence* can be summarized by the phrase “the whole is greater than the sum of its parts”. To be slightly more precise, we can distinguish between two levels of descriptions of a system. The *micro* level describes

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the parts, while the *macro* level is about the whole system. A simple example is gas molecules in a container: positions and momenta of the molecules comprise the micro level, while pressure is a macro level property. The intuitive idea of emergence becomes complicated when we apply it to tougher problems, e.g., how mental activity arises from the physical and biochemical level. The key question is *'How exactly the macro level properties emerge from the micro level dynamics?'*.

The paper titled *'What Emergence Can Possibly Mean'* [1] aims to clarify the possible meanings of emergence with a physics-based approach of describing systems with states and evolution rules. Here, we aim to bring even more algebraic precision for defining emergence in discrete dynamical systems. The alternative title for these notes could be *'And What Emergence Can(not) Mean According to Algebra'*.

Our line of reasoning starts with the observation that the physical system definition of emergence implies algebraic formulation, namely the use of morphisms. Once algebra is admitted, we are bound to investigate what constraints algebraic results put on emergence.

Effort is made to separate mathematical and the philosophical parts. The mathematics part is elementary, but its external meaning is more open to debate.

2. Transformation Semigroups as Discrete Dynamical Systems

We will use semigroup theory [2, 3], since *semigroups*, as a generalization of groups, allow to model irreversible dynamics as well. More precisely, we model discrete dynamical systems as *transformation semigroups* [4]. Compared to the generic models for physical systems (state set plus time evolution rules), they are both more specific and more general at the same. They are defined with discrete state sets (as opposed to continuous), and they allow interactive dynamics of states, not just time evolution. The arguments in this paper only apply to systems that can be modeled this way.

Definition 2.1. A *transformation semigroup* (X, S) is a finite nonempty set of *states* X and a set S of total transformations of the states closed under composition.

In other words we have functions of type $X \to X$. We compose on the right, so s_1s_2 reads as 's_1 then s_2'. When starting from state x, the resulting state is expressed as xs_1s_2 for this sequence of transformations. This notation is due to the connection to automata theory, where sequences of transformations are represented by words of input symbols.

We can define a semigroup by a set of *generators*. In practice, we prefer a limited number of transformations with the property that all transformations in the semigroup can be produced by some combination of these generators. The technical definition for the semigroup generated by A is the smallest semigroup containing A. The condition for being closed under composition connects the formal definition (being the smallest) to the informal (all distinct combinations).

The simplest case is when we have a single generator t and the semigroup elements are of the form $t, t^2, t^3, \ldots$. This is a good model for representing the passage of time, t acting as a clock-tick. While prevalent in physical models, monogenic (cyclic) transformation semigroups have an overly simple structure. A transformation has cycle components with basins of attractions, thus its complete dynamics can be captured by a convenient notation [5]. Monogenic transformation semigroups only produce a limited set of groups (reversible dynamics).

The transposition $\left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 1 & 3 & \cdots & n \end{smallmatrix}\right)$, the cycle $\left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \end{smallmatrix}\right)$, and the elementary state collapsing $\left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 3 & \cdots & n \end{smallmatrix}\right)$ generate the *full transformation semigroup*, the semigroup of all transformations of n states. The first two generate the *symmetric group*, the group of all permutations. Adding an elementary collapsing, the smallest possible information loss, generate all possible dynamics.

The generators of the semigroup can be interpreted as ways to *interact* with the system. For example, to a chemical mixture in a beaker we can add a new substrate as an action and define the next state as the resulting steady state [6].

3. Morphic Relations

The usual description of emergent phenomena tells the following story. We can take a micro state, and do time evolution on it, then use the emergence map to find the macro state corresponding to the result. Or, we can map the micro state first to the macro domain, and do the time evolution there. We get the same result. The diagram commutes. It is immediate that this is an instance of a morphism, where we establish compatibility of operations in two different domains.

Observation 3.1. *An emergence relation is an algebraic morphism.*

We define morphisms for transformation semigroups as they explicitly represent states. This is not a limitation, any abstract semigroup or any other representation of it can also be represented by a suitable transformation semigroup.

Definition 3.2 (Relational Morphism)**.** A *relational morphism* of transformation semigroups $(X, S) \xrightarrow{\theta,\varphi} (Y, T)$ is a pair of relations $(\theta : X \to Y, \varphi : S \to T)$ that are fully defined, i.e., $\theta(x) \neq \varnothing$ and $\varphi(s) \neq \varnothing$, and satisfy the condition of compatible actions for all $x \in X$ and $s \in S$:

$$y \in \theta(x), t \in \varphi(s) \implies yt \in \theta(xs),$$

or more succinctly: $\theta(x)\varphi(s) \subseteq \theta(xs)$, which can be depicted by a subcommutative diagram.

$$\begin{array}{ccc} X \times S & \longrightarrow & X \\ \big\downarrow{\scriptstyle \theta\times\varphi} & \subseteq & \big\downarrow{\scriptstyle \theta} \\ Y \times T & \longrightarrow & Y \end{array}$$

We map states to states and transformations to transformations in a way that the action of transformations on the states remains compatible. The morphism is defined as a relation, not as function. It is necessary for semigroup decompositions. Note that $\theta(x) \cap \theta(y)$ can be non-empty for $x \neq y$, i.e., the image sets can even overlap. Special cases of relational morphisms are exactly the usual homomorphisms including the bijective isomorphisms. We can also define morphisms for abstract semigroups. The condition of compatible operations is simpler. The relation $\varphi : S \to T$ is a morphism if $\varphi(s)\varphi(s') \subseteq \varphi(ss')$ for all $s, s' \in S$.

4. What could novelty mean?

How can an emergence map create something new? Morphisms are structure preserving maps. They can only lose information, or in the edge case of isomorphism, keep everything. Seemingly, there is no possibility for creating new dynamics. However, coarse-graining allows that, but not all coarse-grainings are creative. There is an example below where novelty does not appear, and another where it happens.

We will use groups instead of semigroups for the sake of simplicity and familiarity. Also, we represent groups abstractly. Here we assume some background knowledge on groups.

4.1. **No novelty: the morphic image is a part.** The simplest example that can accommodate two levels of description is $\mathbb{Z}_4$, the group of additions modulo 4. It consists of elements $+0, +1, +2, +3$. The identity is $+0$, and $+1$ is a generator.

We can do coarse-graining on $\mathbb{Z}_4$ by the partition $\{\{+0, +2\}, \{+1, +3\}\}$. Acting on these sets: $+0$ and $+2$ fix them (they only 'move' the elements inside), while $+1$ and $+3$ swap the two classes. We treat the sets in the partition as the macro states. Thus, we have a morphism with image $\mathbb{Z}_2$, giving us the macro dynamics.

This construction is the factor (quotient) group of $\mathbb{Z}_4$ by its normal subgroup $\mathbb{Z}_2$, denoted by $\mathbb{Z}_4/\mathbb{Z}_2$. The coarse-graining is done by the cosets. However, we can see that the macro dynamics is embedded in the micro dynamics. We got nothing new: $\mathbb{Z}_4/\mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_2 = \{+0, +2\}$.

4.2. **Novelty: the morphic image is distinct from all parts.** The quaternion group Q_8 has 8 elements $\{1, -1, i, -i, j, -j, k, -k\}$ and the following multiplication table.

Q_8	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

The order of an element is the size of the group generated by it. In other words, the order tells which power of the element is the identity. The identity has order 1, -1 has order 2, and all the other elements have order 4.

Factoring by $\{1, -1\}$ gives the partition $\{\{1, -1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}\}$. What is this factor group $Q_8/\{1, -1\}$? Let's denote the cosets by $\{1, I, J, K\}$. We get the following multiplication table by extending the multiplication to sets .

	1	I	J	K
1	1	I	J	K
I	I	1	K	J
J	J	K	1	I
K	K	J	I	1

This is the Klein-group K_4, the symmetry group of a non-square rectangle. It is $\mathbb{Z}_2 \times \mathbb{Z}_2$, which can be shown by the mapping $1 \mapsto (0, 0)$, $I \mapsto (0, 1)$, $J \mapsto (1, 0)$, and $K \mapsto (1, 1)$, where we do coordinate-wise addition modulo 2.

All non-identity elements in K_4 have order 2. Since isomorphisms are order preserving, K_4 is not isomorphic to any subgroup of Q_8, since the quaternion group has only element with order 2. Therefore, coarse-graining produced a group that cannot be found by observing the multiplication in Q_8. The point is that only coarse-graining can reveal K_4 in the dynamics of Q_8.

5. Algebraic Emergence

We observed that it is possible to give precise meaning to novelty in the algebraic context. We use the presence of such novelty as the defining property of emergence.

Definition 5.1. A transformation semigroup (Y,T) is *emergent* for (X,S) if there exists a surjective relational morphism $(X,S) \xrightarrow{\theta,\varphi} (Y,T)$ and there is no subsemigroup $(X',S') \leq (X,S)$ such that $(Y,T) \cong (X',S')$.

We can simply say that the morphic image (Y,T) is emergent if it is not a subsemigroup of (X,S), in case we consider isomorphic semigroups to be identical. Note that $(X,S) \ncong (Y,T)$ is implied by the definition, as the second condition rules out the possibility of an isomorphism.

How rare is algebraic emergence? If the quaternion group was the only one having emergent structures, then definition would not be an interesting one.

In algebraic emergence we have the base structure (the whole system) and the emergent structure (the macro level) in an asymmetric relationship. When looking for examples, we can ask two different questions. Given a structure S,

(1) what emergent structures can we find on top of S?,
(2) what base structures can give rise to S?

We look at the first question.

5.1. Non-examples.

First, we can think of cases where we know for sure that they cannot have emergent structures.

Simple groups have no normal subgroups (other than the group itself and the trivial group). Therefore, they do not have any non-trivial factor groups. They can have immense complexity, but they do not admit emergence.

Symmetric groups S_n of degree n have all the permutations of n states. Consequently, they have all degree n and smaller permutation groups as subgroups. Therefore, any morphic image is already contained.

Similar situation arises with cyclic groups. With elementary proofs we can show that all their subgroups and factor groups are cyclic. They both correspond to divisors of the size of the group. This implies that time evolution based (single generator) reversible discrete dynamical systems cannot exhibit algebraic emergence.

5.2. Computational Survey: Small Groups.

We can do a quick survey of small groups to see how many of them have emergent factor groups. We use the SmallGroups library [7] for the GAP computer algebra system [8]. Appendix A contains the source code for the function deciding emergence.

Table 1 lists the first few example of groups with emergence and their emergent groups. Table 2 shows the numbers of emergent groups. Our example Q_8 is indeed the smallest one, but algebraic emergence is not a rare property. It is also the first to be an emergent group that is also a base group. This shows that multiple levels of emergence exist, as we would expect from the informal idea of emergence. Another observation we can make is that algebraic emergence is not transitive: $\mathbb{Z}_2 \times \mathbb{Z}_2$ is emergent in Q_8 but not in $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$.

Order	Base Group	Emergent Groups
8	Q_8	$\mathbb{Z}_2 \times \mathbb{Z}_2$
12	$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$	S_3
16	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	D_8
16	$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	Q_8, D_8
16	Q_{16}	$D_8, \mathbb{Z}_2 \times \mathbb{Z}_2$
16	$\mathbb{Z}_2 \times Q_8$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
16	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
20	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	D_{10}

TABLE 1. The first few emergent groups. The order is the size of the base group. D_{2n} is the the dihedral group, $\rtimes$ denotes semidirect products, Q_n the dicyclic (quaternion) groups.

Order	8	12	16	20	24	28	32	36	...	63	64
#groups	5	5	14	5	15	4	51	14	...	4	267
#emergent	1	1	5	1	5	1	33	4	...	1	233

TABLE 2. Number of groups with emergence. Order 63 is the first odd size to have an emergent factor group.

6. Hierarchical Decompositions

According to a simplistic description of scientific understanding, we start with the observation of the macro level, then proceed to the micro level, finally getting a detailed understanding of the whole system.

macro level

system micro level

In the hierarchical decompositions of semigroups [9], we understand the complete system by creating a hierarchy of several levels of descriptions. In other words, we rebuild the system with the levels clearly separated. The hierarchy is defined by a *wreath product*, i.e., there is a linear order of component transformation semigroups and control information flows only one way, from top to bottom. What the top level does depends only on its input, but components below should consider the state(s) of the component(s) above too. In practice, we use substructures of the wreath product defined by generator sets.

In particular, the Covering Lemma method [10] works with transformation semigroups and produces a two level decomposition. This fits the above scheme of emergence well, but it works the other way around. We start with the complete system we do not understand, and try to simplify with a suitable surjective morphism.

system	macro level
	residual micro level

In other words, we are picking a suitable macro level description. Then, the decomposition algorithm extracts the relevant information from the system and generates the corresponding micro level. The micro level contains all the information that is lost in the surjective morphism.

How to pick the right macro level? That is an interesting research question for each problem domain. In general, we want to have a surjective map with low computational complexity. This condition implies the ability to work with the generators only (no need to compute the system completely). The other aspect is the usefulness of the macro level. Are the algebraically emergent macro levels good for understanding purposes? One special class of macro level descriptions has a central role in physics.

6.1. **Conserved Quantities.** If we partition the state set by θ, i.e., $x_1 \neq x_2 \implies \theta(x_1) \cap \theta(x_2) = \varnothing$, and the semigroup action respects this partition, i.e., classes are sent to classes, then we have a *congruence*. The action defined on congruence classes has another physical analogy, the *conserved quantities*. These serve as a top level understanding of a system [6]. For understanding a dynamical system we introduce some (formal) quantities with the following properties.

- They describe some important aspects of the system.
- The result of an action depends on the current value and the action only, and not on the precise current state of the system.

Any state in the equivalence class of x can be used for computing the next class, which is exactly the defining property of a congruence. The formal quantity is the same within equivalence class, so states in the class form a "closed system". This is an analogue for conserved quantities in physics, like energy level, or angular momentum.

6.2. **Locality.** There is nothing intrinsically local in the definition of the state space of transformation semigroups. However, when using the Covering Lemma method [10], locality appears as a consequence of the macro level description. There are 'islands of composability' on the micro level determined by the macro level component. Thus the micro level component is not a semigroup, but a *semigroupoid*. If we naively forced it to be semigroup, then we would get some phantom elements, byproducts of the encoding scheme of the micro level.

7. Philosophical Implications

Up to this point we only talked about elementary, proven, and thus noncontroversial mathematical facts, and created a definition of algebraic emergence. Here, we make philosophical arguments why the definition could be useful.

7.1. **The Two Pillars of Algebraic Emergence: Abstraction and Morphism.** The first important idea is that by *abstraction*, by forgetting details we create new information. In itself, this is not enough, since it allows to do unreasonable things. For instance, we can start with a text message, and by a specially crafted erasure of letters we can have an albeit shorter, but completely new message. This feels like cheating as there is no intrinsic relation between the two messages.

The second pillar is the map between compatible actions, the morphism. We need to erase information in a way that we keep compatibility with the original system. Emergence only makes sense if there is some compatibility between the base system and the emergent structures on top of it. Associating one system to another without morphism is just a made up arbitrary connection.

7.2. **No miracle morphisms.** Algebraically obvious, but it is worth stating explicitly that there are no miracle morphisms. There are information preserving maps (isomorphisms), and information forgetting ones (homomorphisms), but there are no information creating maps in algebra. One cannot map T_n to T_{n+1} isomorphically, as there are simply not enough transformations and not enough states in T_n. We can only *embed* the smaller into the bigger one. The simplest such map would be fixing a point.

7.3. **Limited Novelty.** Morphisms are generalizations of equality. If one structure is a morphic image of another one, then *to some extent* it is the same. In the space of all maps they are rare, and there is no guarantee that we will get a morphism between arbitrarily chosen objects.

In universal algebra [11], we have families of similar algebraic objects by equational identities. They are generated by the HSP operator, i.e., taking homomorphic images, substructures, and direct products. We identified emergence as morphic images that are not substructures. Arguably, nothing 'truly' novel is produced by the HSP operator, as we stay inside an equational class. For instance, an abelian group will never have non-abelian emergent macro level. An aperiodic semigroup (no non-trivial reversible dynamics) cannot have an emergent group structure.

7.4. **Weak versus Strong Emergence.** In philosophy, there is the distinct between strong emergence and weak emergence [12]. Without getting into the details of the traditional debate, with the algebraic emergence we have two choices.

(1) Algebraic emergence is strong emergence, or
(2) strong emergence cannot be interpreted in the algebraic framework.

Showing that Q_8 has emergent dynamics K_4, which is not contained in it already, is far from proving that mental processes emerge from the firings in a neural network. At least, the possibility is not ruled out mathematically. However, proponents of the strong emergence also talk about downward causation, and there are no inherent causal relations in the hierarchical model.

7.5. **Orthogonal to Causality.** The algebraic model of the hierarchical dependence (the wreath product) seems to include causal relationships: the lower levels 'listen to' the upper levels. The control flow is from top to bottom. However, this is just a representation. We can take an arbitrary semigroup, do a hierarchical decomposition even if it is defined by a recurrent mechanism.

Given a base system and a macro level description, we can find the missing micro level algorithmically. We could do this the other way around too. The algebra does not say that the macro level arises from the micro level, as the macro level can be used to find suitable micro levels too. Causal relationships are external to the mathematical theory.

The algebra deals only with three descriptions of a system, the complete, the macro level (coarse-grained), and the micro (localized). If any of them is missing, we can recover that from the other two.

7.6. **Emergent structures belong to their base.** Much of the confusion in philosophy about emergence may come from the assumption that the describing a base system is in some sense complete. As if, describing the pieces and the basic connections of a LEGO set would be sufficient. Then, of course, we get surprised by the 'emergent' things, vehicles, castles we can build using the pieces. Alternatively, we could include all the possible constructions in a complete description. In other words, the LEGO set can be defined by relations to all the real and imaginary objects it can build.

In category theory, the Yoneda Lemma (e.g., [13]) says that in order to know everything about a mathematical object it suffices to know how it relates to other objects. In a way, the essence is in the relationships. We already used one of its corollaries, Cayley's Theorem, when we switched to abstract groups from permutation groups.

We can apply this relational thinking for algebraic emergence. It is not enough to look inside the semigroups, and check only the subsemigroups, we need to check the morphic images as well. In physics, the understanding of matter (defined as elementary particles, atoms) will be complete once we enumerated all its possible emergent structures.

8. Conclusion

If a process can be modeled as a discrete dynamical system, then its algebraic representation has a clear condition for emergence. Emergence happens when a morphic image of the system is not already present in it as a substructure. In other words, emergent dynamics can only be derived by (generalized) coarse-graining.

Due to the importance of composition, the algebraic structure is fundamental for many systems. Therefore, any more detailed classification of types emergence should be preceded by the condition for algebraic emergence. It can serve as a necessary, if not as a sufficient condition.

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Appendix A. Finding Emergent Groups

The following GAP [8] function finds all emergent groups of a given group G. The algorithm is straightforward. We find all normal subgroups and construct all corresponding factor groups. We retain only those that are not isomorphic to any subgroups of G. No optimization is made, thus the function only works were enumerating all subgroups and constructing all factor groups are feasible.

```
EmergentGroups := function(G)
  local factor_groups, subgroups, emergent;
  factor_groups := List(NormalSubgroups(G),
                        N -> FactorGroup(G,N));
  subgroups := AllSubgroups(G);
  emergent := Filtered(factor_groups,
                       F -> ForAll(subgroups,
                                   S -> (fail = IsomorphismGroups(F,S))));
  return emergent;
end;
```

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