A Structural Tonk

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Abstract

When logicians work with multiple-conclusion systems, they use a metalinguistic comma ‘,’ to aggregate premises and/or conclusions. In this note, I present an analogy between this comma and Prior’s infamous connective TONK. The analogy reveals that these expressions have much in common. I argue that, indeed, the comma can be seen as a structural incarnation of TONK. The upshot is that, whatever story one has to tell about TONK, there are good reasons to tell a similar story about the comma in typical multiple-conclusion systems, and vice versa.

Key words: multiple conclusions; tonk; logical inferentialism; harmony

1 Introduction

After Gentzen’s (1934) seminal work on sequent calculi, it has slowly become widespread to work with multiple-conclusion logical systems. In the literature on philosophical logic, pretty much all systems of this sort induce a reading of validity roughly as follows: Γ entails Δ just in case the conjunction of the things in Γ entails the disjunction of the things in Δ—where Γ and Δ are collections of the appropriate kind. Vindications of multiple conclusions so understood can be found, e.g. in Carnap (1943), Scott (1971), Shoesmith and Smiley (1978), Restall (2005) and Dicher (2020).

The philosophical standpoint known as *logical inferentialism* maintains that the meaning of logical constants is determined by the rules that govern their behaviour.[1] As is well-known, Prior (1960) challenged this standpoint by presenting his now-infamous connective TONK, whose rules trivialise transitive consequence relations under fairly weak conditions.

[1] See Murzi and Steinberger (2017) for a nice overview of inferentialism in its various forms.

When logicians work with multiple-conclusion systems, they use a metalinguistic comma ‘,’ to aggregate premises and/and or conclusions. This comma abbreviates a union-like operation between collections of the appropriate kind. Thus, for instance, if $\vdash$ stands for entailment, the relata of $\vdash$ are sets and A, B, C and D are formulas, then a claim of the form $A, B \vdash C, D$ abbreviates $\{A\} \cup \{B\} \vdash \{C\} \cup \{D\}$.

In this note, I present what strikes me as an illuminating analogy between the behaviour of the comma in typical multiple-conclusion systems, on the one hand, and the behaviour of TONK on the other. The analogy reveals that these expressions have much in common. I argue that, indeed, the comma can be seen as a structural incarnation of TONK. In my view, this is surprising, because TONK and multiple conclusions are in general subject to very different assessments. TONK is seen as a completely useless and potentially harmful aberration. Multiple conclusions, in contrast, are seen as a largely innocuous and often useful technical artifice. One would not expect that so different beasts turn out being of the same blood.

The upshot of the discussion is that, whatever story one has to tell about TONK, there are good reasons to tell a similar story about the comma in typical multiple-conclusion systems, and vice versa. In particular, inferentialists who think that TONK is meaningless had better feel dubious about multiple conclusions. By the same token, those who sympathise with multiple conclusions should think twice before rejecting TONK as meaningless.

2 Preliminaries

We identify languages with the sets of their formulas. Let $\mathcal{L}$ be a propositional language with parameters $p, q, r, ...$, and without any logical constants, and let $\mathcal{L}^{\mathrm{T}}$ be the result of expanding $\mathcal{L}$ with a dyadic connective TONK. We use $A, B, C, ...$ for arbitrary formulas of a given language, and $\Gamma, \Delta, \Sigma, ...$ for sets thereof. We use ‘iff’ for ‘if and only if’.

Let R be a dyadic relation on a set X. R is *reflexive* iff for every $a \in \mathrm{X}$, aRa. R is *transitive* iff for every $a, b, c \in \mathrm{X}$, if aRb and bRc then aRc. We shall frequently appeal to the following sequent principles encoding reflexivity and transitivity:

$$\text{Fla-ID}\ \frac{}{A \Rightarrow A} \qquad\qquad \text{Fla-TR}\ \frac{A \Rightarrow B \qquad B \Rightarrow C}{A \Rightarrow C}$$

$$\text{Set-ID}\ \frac{}{\Gamma \Rightarrow \Gamma} \qquad\qquad \text{Set-TR}\ \frac{\Gamma \Rightarrow \Delta \qquad \Delta \Rightarrow \Sigma}{\Gamma \Rightarrow \Sigma}$$

We say that a consequence relation $\vdash$ is *formula-reflexive* (*set-reflexive*) iff it satisfies Fla-ID (Set-ID), and *formula-transitive* (*set-transitive*) iff it satisfies Fla-TR (Set-TR). Keep in mind that, if $\vdash$ is single-conclusion and single-premise (viz. it goes from formulas to formulas) then it is formula-reflexive (formula-transitive) iff it is reflexive (transitive) *simpliciter*. Similarly, if $\vdash$ is multiple-conclusion and multiple-premise, then it is set-reflexive (set-transitive) iff it is reflexive (transitive) *simpliciter*.

3 The Analogy

My comparison between TONK and the comma will revolve around three closely related issues. First, how these expressions behave, and in what sense their behaviour can be regarded as bad. Second, what we can do to handle them without triviality. Third, why they behave as they do—or what's going on with them. Allowing myself a medical metaphor, I shall call these the *symptoms*, the *cures* and the *diagnoses*, respectively, of TONK and the comma. I tackle each of the issues in turn.

The symptoms. Let's start with TONK. We want to compare it with the comma of multiple-conclusion systems, and systems of this latter kind are typically presented by means of sequent calculi. Thus, we characterise TONK with the sequent rules

$$\textsc{tonk}\text{-R}\ \frac{A \Rightarrow B}{A \Rightarrow B \,\textsc{tonk}\, C} \qquad\qquad \textsc{tonk}\text{-L}\ \frac{A \Rightarrow B}{C \,\textsc{tonk}\, A \Rightarrow B}$$

These are the usual sequent rules for TONK (cf. Dicher, 2020; Ripley, 2015) with the only difference that we restrict them to a single-conclusion and single-premise framework. This allows a more neat comparison between TONK and the comma, for it precludes the latter from appearing in the rules of the former—thus keeping both expressions apart.

The sense in which rules TONK-L and TONK-R are *prima facie* pathological is quite straightforward. Consider any consequence relation $\vdash \subseteq \mathcal{L}^{\mathrm{T}} \times \mathcal{L}^{\mathrm{T}}$ induced by a system $\mathcal{S}$ that contains Fla-ID and Fla-TR. The result of extending $\mathcal{S}$ with TONK-L and TONK-R delivers a trivial consequence relation, as witnessed by the simple derivation

$$\dfrac{\dfrac{\overline{A \Rightarrow A}}{A \Rightarrow A \,\textsc{tonk}\, B} \qquad \dfrac{\overline{B \Rightarrow B}}{A \,\textsc{tonk}\, B \Rightarrow B}}{A \Rightarrow B}$$

So, at the very least, TONK-L and TONK-R do not get along with consequence relations that are formula-reflexive and formula-transitive.

Now, let's tackle the comma. For our present purposes, we stipulate that a multiple-conclusion system is *typical* just in case, when it is restricted to a language with no logical constants (viz. a language like $\mathcal{L}$), it is soundly and completely axiomatised by Fla-ID together with rules

$$\text{Set-R}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \Sigma} \qquad\qquad \text{Set-L}\ \frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta}$$

This stipulation singles out a wide class of systems—which includes classical logic and many non-classical logics as well.[2] Rule Fla-ID does not feature the comma in its formulation. Hence, we can say that rules SET-R and SET-L characterise the behaviour of the comma in all these systems.

It is immediate to see that rules SET-R and SET-L are formally identical to TONK-R and TONK-L, respectively: in each case, the rule for the comma results by taking the rule for TONK and uniformly replacing arbitrary formulas with sets and TONK with the comma. Both expressions are introduced as conjunctions on the left-hand side of the turnstile, and as disjunctions on the right-hand side. Of course, it would be a categorical mistake to say that they are just notational variants of one another since they do not even pertain to the same language. However, it is transparent that their respective behaviours can be encoded in similar patterns of inference.

More importantly, SET-R and SET-L display quite similar *prima facie* pathological properties. Consider any consequence relation $\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ induced by a system $\mathcal{S}$ that contains Fla-ID and Set-TR. Let $A \in \Gamma$ and $B \in \Delta$. The result of extending $\mathcal{S}$ with rules Set-L and Set-R enables the following derivation:

[2] In this paper we focus on consequence relations defined on sets. But this is only for simplicity. With minor adjustments, our analogy carries over to many systems defined by means of other kinds of formula aggregation (e.g. multisets, sequences).

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}}{\Gamma \Rightarrow A}}{\Gamma \Rightarrow \Gamma, \Delta} \qquad \dfrac{\dfrac{\dfrac{}{B \Rightarrow B}}{B \Rightarrow \Delta}}{\Gamma, \Delta \Rightarrow \Delta}$$
$$\overline{\qquad\qquad \Gamma \Rightarrow \Delta \qquad\qquad}$$

That is, we prove $\Gamma \Rightarrow \Delta$ for any non-empty Γ and Δ. Thus, at the very least, multiple conclusions do not get along with consequence relations that are formula-reflexive and set-transitive.

At this point, the reader may perhaps object that there are some differences between TONK and the comma. Since we are assuming that Γ, Δ, etc. are sets, the comma satisfies by definition certain properties as, e.g. commutativity: A, B and B, A are everywhere intersubstitutable without loss of validity.[3] However, TONK does not satisfy these properties by virtue of its rules alone.

But I think that the objection has not much philosophical significance. First, the point of my analogy is not that the inferential behaviours of TONK and the comma are similar in all possible respects. The point is that they are similar in certain *relevant* aspects, which concern mainly the pathological character that they seem to exhibit. And I think that *this* point is not undermined by the objection. Secondly, if e.g. commutativity turned out to be relevant for some reason, we could easily define a sibling connective, say STONK (for 'symmetric TONK'), governed by rules

$$\frac{A \Rightarrow B}{A \Rightarrow B \,\textsc{stonk}\, C} \qquad \frac{A \Rightarrow B}{A \Rightarrow C \,\textsc{stonk}\, B} \qquad \frac{A \Rightarrow B}{C \,\textsc{stonk}\, A \Rightarrow B} \qquad \frac{A \Rightarrow B}{A \,\textsc{stonk}\, C \Rightarrow B}$$

This connective would allow us to rerun our analogy avoiding the objection. (A similar point applies to other properties of the comma, as e.g. contraction: A, A can be everywhere replaced by A without loss of validity.) For simplicity, I shall keep talking just about TONK, and leave the reference to its potential relatives implicit.

The cures. Few attempts have been made to design logical systems where TONK is admissible without triviality. Cook, for instance (2005), presents a non-transitive logic

[3] One could perhaps wonder whether we are entitled to *assume* that Γ, Δ, etc. are sets. Shouldn't we *guarantee* this by adding, e.g. contraction and exchange to the rules for the comma? I submit a negative answer. The role of the structural rules is to describe the *logical* aspects of the behaviour of sets. In doing such a description, we are entitled to use the non-logical facts about sets that our underlying set theory gives us.

such that a connective satisfying TONK-L and TONK-R can be defined and conservatively added to the language. Fjellstad, however (2015), argues that a proper logic for TONK should be both non-transitive and non-reflexive. The main reason is that in a sequent calculus constituted by the rules of TONK and a principle of reflexivity, TONK in not uniquely defined.[4] Thus, we could not say that the system admits *the* connective TONK; at best, we can argue that it admits a whole family of connectives, each characterised by the same pair of rules. The author provides a non-reflexive and non-transitive sequent calculus where TONK *is* uniquely defined, and a semantics with respect to which the calculus in question is sound and complete.

So, we have good reasons to think that a logic that is suitable for tonk is non-transitive and probably also non-reflexive. The similarity with the comma becomes apparent when we observe that typical multiple-conclusion systems are formula-reflexive and formula-transitive, but *neither* set-reflexive nor set-transitive (so, they are non-reflexive and non-transitive *simpliciter*). On the one hand, we typically have $\varnothing \nvdash \varnothing$, violating Set-ID. On the other hand, $p \vdash p, q$ and $p, q \vdash q$ but $p \nvdash q$, violating Set-TR. Thus, we can say that, since Gentzen, the strategy to admit the comma without triviality is analogous to our best available strategies to admit TONK.

An immediate objection could be that, even if typical multiple-conclusion systems are not transitive *tout court*, they often satisfy certain variants of transitivity, as for instance the rule of Cut:

$$\text{Set-Cut}\ \frac{\Gamma \Rightarrow \Delta, B \qquad B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

In contrast, logics for TONK do not admit cut. Hence—the objection goes—the comma and TONK are not incompatible with transitivity in the same sense.

However, I think that the objection can be resisted. TONK is an object-language connective, and as such, it is attached to *formulas*. As we have seen, it is incompatible with *formula*-transitivity. On the other hand, the comma is an expression of the metalanguage, and as such, it is attached to *sets* of formulas. Accordingly, it is incompatible with *set*-transitivity. Hence, both TONK and the comma are incompatible with

[4]Informally, we say that a connective $\star$ is *uniquely defined by its rules* just in case any connective $\circ$ whose rules are formally identical to those of $\star$ is such that formulas $\star(A_1, ..., A_n)$ and $\circ(A_1, ..., A_n)$ follow from and entail the same things. See Belnap (1962) for the technical definition.

the transitivity of the things to which they are attached. That's quite a similar sense of being incompatible with transitivity, in my view. To allay any remaining suspicions, I note that Fjellstad's system for TONK validates the principle

$$\text{T-Cut}\ \frac{A \Rightarrow C \text{ TONK } B \qquad B \text{ TONK } A \Rightarrow C}{A \Rightarrow C}$$

which results by taking Set-Cut and uniformly replacing sets by formulas and commas by TONKs. Thus, TONK also satisfies some *variant* of transitivity.[5]

The diagnosis. Roughly, there are two families of explanations of what's going on with TONK. One of them appeals to the idea that the rules of TONK clash with our background assumptions about the behaviour of logical consequence. For a connective not to clash with those assumptions, it must produce a *conservative extension* when we add it to our system. The idea was put forward by Belnap (1962), and it is viewed with approval by various authors (e.g. Cook, 2005; Dicher, 2016; Ripley, 2015). Let's call it the *Belnap explanation*. According to this view, there need not be anything wrong with TONK. It is just that TONK is incompatible with the transitivity of logical consequence. If we had good reasons for abandoning transitivity, we could happily accept TONK as legitimate. In this approach, then, the question of whether a given connective is legitimate is a *relative* one: it can receive different answers depending on how we assume that logical consequence behaves. The other family of explanations appeals to the idea that the rules of TONK are not in *harmony*. This means, roughly, that one of them is too weak or too strong with respect to the other. The idea was contained *in nuce* in some remarks by Gentzen, and was then developed by Prawitz (1974) and Dummett (1991), among many others (e.g. Read, 2010; Tennant, 2007). Let's call it the *Gentzen-Prawitz-Dummett explanation*. According to this view, TONK is inherently illegitimate, and the question of whether a connective is legitimate or not is an *absolute* one: it receives the same answer, irrespective of our assumptions on how logical consequence behaves.

[5]Perhaps, another possible objection is that TONK is compatible with formula-reflexivity while the comma is not compatible with set-reflexivity. But first, the reason for this asymmetry does not stem from the rules of TONK and the coma, but rather from the non-logical properties of these expressions: the operation of union has a neutral element (namely $\varnothing$) while the operation of TONK-ing sentences together does not. Secondly, we could easily make the comma compatible with set-reflexivity by restricting Set-R to non-empty Δs, and Set-L to non-empty Γs.

Both kinds of explanations can be transposed to the case of the comma. The Belnap explanation appeals to our earlier trivialisation result (p. 4). Let $\vdash$ be a non-trivial, single-premise and single-conclusion consequence relation on $\mathcal{L}$, induced by a calculus $\mathcal{S}$ containing Fla-ID and Set-TR. In the metalanguage of $\vdash$, things of the form $A \vdash C$ are always well-formed, but things of the form $\Gamma \vdash \Delta$ are well-formed only when Γ and Δ are singletons.[6] Consider, now, a multiple-premise and multiple-conclusion relation $\vdash^\star$. The metalanguage of $\vdash^\star$ is richer, since $\Gamma \vdash^\star \Delta$ is well-formed for any $\Gamma, \Delta \subseteq \mathcal{L}$. But suppose that $\vdash^\star$ is induced by the system $\mathcal{S}^\star$ that results by extending $\mathcal{S}$ with Set-L and Set-R. Then, we obtain triviality: $A \vdash^\star B$ for any A and B. Thus, we can say (in a non-orthodox but intuitively clear sense) that $\vdash^\star$ is a *non-conservative extension* of $\vdash$. This would imply that rules Set-L and Set-R clash with our prior assumptions on the behaviour of consequence—in particular, they clash with Set-TR and Fla-ID.

As for the Gentzen-Prawitz-Dummett explanation, the story would go like this. There have been many proposals as to how to make precise the idea of harmony.[7] But most of them (if not all) entail that a connective is harmonious only if it satisfies what Prawitz (1965) called an *inversion principle*. For our purposes, we can put it as follows: *a connective is legitimate only if the direct grounds for introducing it allow us to infer whatever follows by eliminating it.* We align with standard practice, and interpret Set-R as the rule that introduces the comma (it says what things entail it) and Set-L as the rule that eliminates the comma (it says what things follow from it). The latter tells us that we can infer Δ from the set Σ, Γ and the assumption that the sequent $\Gamma \Rightarrow \Delta$ is valid. But the former tells us that the grounds for inferring Σ, Γ are given by the set Σ and the assumption that the sequent $\Sigma \Rightarrow \Sigma$ is valid. Of course, Σ and $\Sigma \Rightarrow \Sigma$ are not in general enough to infer Δ. Thus, we conclude that the comma violates a (non-formalised but intuitively clear) structural variant of the inversion principle.

I would like to consider one last objection against the analogy between TONK and the comma. It concerns rules Fla-TR and Set-TR. There seems to be a relevant difference between these rules: Set-TR involves the concept at issue in the rules of the comma, whereas Fla-TR does not involve the concept at issue in the rules of TONK. Thus, we

[6] We are assuming that a formula A and the singleton $\{A\}$ are for all our purposes equivalent; this makes expressions of the form $\{A\} \vdash \{C\}$ well-formed.

[7] See Steinberger (2011a) for a critical analysis of various such proposals.

can blame Set-TR for being incoherent with the rules of the comma, whereas we cannot plausibly blame Fla-TR for being incoherent with the rules of TONK—a connective that the rule does not even mention. This allows us to claim that the comma is meaningful, while TONK is not.

For starters, I deny that Set-TR involves the concept at issue in the rules of the comma. Let me make a comparison with rules Set-R and Set-L. The comma abbreviates the operation of set union. The rules in question mention the comma explicitly, and moreover, any reformulation of them that avoids mentioning the comma will appeal, in a more or less disguised way, to the union of some of the sets displayed. Thus, I read Set-L and Set-R as talking about the logical behaviour of set union—or, in other words, about what unions of sets follow from what. In contrast, Set-TR does not mention the comma, and it does not appeal to any union of the sets displayed in it. Hence, I read Set-TR as talking about the logical behaviour of sets in general—that is, making abstraction of any particular operation that may be applied to them.

Secondly, even if there was a sense in which Set-TR *does* involve the concept at issue in the rules of the comma, the objection would fail anyway. The fact that Set-TR, Set-R, and Set-L share their subject matter only indicates that these three rules are not *jointly* coherent; it tells us nothing about which of them we should accept and which of them we should reject. Pending an additional argument, claiming that Set-TR is invalid while Set-R and Set-L are valid is just to beg the question in favour of the comma. Notice also that, while Fla-TR and the rules of TONK do not share their subject matter, they are also jointly untenable, so we must choose among them. And if we arbitrarily chose the comma over Set-TR, we could just as arbitrarily choose TONK over Fla-TR. In a nutshell: the fact that Set-TR involves the comma is no argument in favour of the comma.

4 Philosophical Import

I have argued that TONK and the comma of typical multiple-conclusion systems are strikingly akin to one another. They can be characterised by formally identical rules, they trivialise transitive systems under evenly weak conditions, they have been handled in a similar way in the literature (namely, by abandoning transitivity and reflexivity of consequence) and alike explanations can be given of what is going on with them. We can conclude, I submit, that *the comma is nothing more (or less) than a structural* TONK.

As I anticipated, the philosophical upshot is that, whatever story we choose to believe about TONK, we have good reasons to believe a similar story about the comma, and vice versa. The comma is just as aberrant as TONK, or TONK is just as innocuous as the comma—depending on one's background commitments.

In view of this, I see at least three possible stances avoid triviality while doing justice to our analogy. The first is to accept both TONK and the comma as legitimate expressions, but reject both formula- and set- transitivity as properties of logical consequence—hopefully, giving some good reasons for this along the way. This route is conspicuously taken by Ripley (2015).

The second stance is to reject both TONK and the comma as legitimate expressions. Various authors have already complained that multiple conclusions are artificial in that they cannot be found in our everyday argumentative practices.[8] Moreover, Steinberger (2011b) has recently given independent reasons to think that multiple conclusions are not acceptable by inferentialist standards. So, this note could be read as providing further evidence for Steinberger's claim.

The third stance is to claim that Fla-TR is valid while Set-TR is invalid; as a consequence, the comma can be accepted as meaningful but TONK cannot. This position certainly fits better with standard practice. It abides by our analogy as long as it concedes that neither TONK nor the comma is inherently meaningful (in the sense of harmonious); they are only meaningful (in the sense of conservative) relative to the absence of certain principles. In my view, the challenge faced by this position is to give a (non-question-begging) motivation for its non-uniform policy towards transitivity; in other words, to explain what is the difference between formulas and sets that makes transitivity plausible for the former but unacceptable for the latter.

I have to emphasise that we have focused only on the comma as it works in typ-

[8]For some remarks in this direction, see, e.g. Beall and Restall (2006), Cintula and Paoli (2021), Rumfitt (2008) and Tennant (2002).

ical multiple-conclusion systems in philosophical logic. In algebraic logic we often see multiple-conclusion systems where the comma on the right-hand side of the turnstile does not behave as a disjunction, but as a conjunction.[9] Recently, even in philosophical logic some authors considered this type of systems.[10] It is obvious that my analogy with TONK does not extend to them.

We can close by paraphrasing Shoesmith and Smiley (1978, p. 4) and say that supporters of multiple conclusions, like so many Monsieur Jourdains, have been speaking TONK-ish all their lives without even knowing it. The question that arises is what they will do when they find that out.[11]

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[9]Cf. Galatos and Tsinakis (2009).

[10]Cf. Cintula and Paoli (2021).

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