Divine Proportions: Rational Trigonometry to Universal Geometry

by Norman Wildberger

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Unlike "lesser" disciplines, mathematics is not rent by disputes over what is true. What we have proved true has stayed true, give or take rare exceptions. Our argumentative energy has not gone to waste, however, and mathematicians debate vigorously questions on what topics are interesting, what conjectures credible, how classical fields can be better seen in the light of new results, and of course, how to teach.

A heated debate a hundred years ago – one with close parallels to the revolution in trigonometry that Wildberger urges in his new book – resulted in major changes to linear algebra. That is a branch of mathematics which the naïve student might expect to have developed smoothly, rationally and without controversy. Axioms, span, independence – what is there to become heated about in that? Yet the modern point of view is the end point of recovering from several false starts, notably Hamilton's inept attempt to do vector geometry and physics with quaternions and Grassmann's barely intelligible foundation of the subject on what we call flags of subspaces. There were also difficulties in moving beyond coordinates and matrices to the more abstract point of view of vectors and linear maps. It was only in the 1920s that British mathematicians and engineers swallowed their pride and admitted that the Germans had it right about vectors. (The story is told in Crowe's *History of Vector Analysis*.)

Trigonometry is a much older and more settled branch of mathematics than linear algebra. It comes much earlier in the syllabus and every becoming-numerate generation invests enormous effort in the painful calculation of the lengths and angles of complicated figures. Surveying, navigation and computer graphics are intensive users of the results. Much of that effort is wasted, Wildberger argues. The concentration on angles, especially, is a result of the historical accident that serious study of the subject began with spherical trigonometry for astronomy and long-range navigation, which meant there was altogether too much attention given to circles.

Wildberger's alternative is simple. We should avoid the concepts of length and angle as far as possible, and so do without their complicated formulas involving square roots and transcendental arcsins and the like. They should be replaced with two (algebraically) simpler concepts, "quadrance" and "spread". Quadrance is just the square of length, so its formula in terms of co-ordinates just involves the sum of squares of co-ordinates. Spread is a measure of separation of lines. It is (to slip into oldspeak for a moment, though the aim is to learn to think in the new language as fast as possible) the square of the sine of the angle(s) between the lines. The spread between the the lines $ax + by = 0$ and $cx + dy = 0$ is a simple rational expression in a,b,c,d.

Let us take one elementary and one more mathematical example to showcase the point of doing things this way. Consider the problem, useful in such fields as railway engineering, of the relation between slopes when climbing a hill "at an angle". For example, if a grade of one in fifty is the maximum a train can climb and the hill has a grade of one in thirty, in what direction across the hill must one build the railway? Standard trigonometry would attack this problem using angles and their tangents, but the problem and its answer do not mention angles. The solution in terms of spreads (p. 231) is very simple.

Mathematicians may be more excited by the way that the avoidance of square roots and transcendentals renders the results independent of the real field, and hence a true "universal geometry". For example, at first sight the result that the spread subtended by a chord of a circle is a constant (p. 178) seems much the same kind of result as the classical result which states that the angle subtended by a chord is constant. But there are subtle differences. With angles, one must consider which side of the chord the angle lies. That is awkward in itself and prevents generalization beyond the field of real numbers. For spreads, constant really means constant, and one may change the underlying field and retain the theorem.

Wildberger develops his universal geometry at length, dealing for example with the replacements of the sine and cosine rules, an alternative to spherical and polar co-ordinates with applications to moments and centers of inertia, and simplified treatments of classical surveying problems like the Snellius-Pothenot and Hansen's problems. Reform is intended not just of trigonometry but of the foundations of Euclidean geometry. The subject is developed from first principles over a general field – one cannot have "on this side of the line" in fields other than the reals, but almost all other Euclidean geometrical properties remain available (including the inside and outside of circles). Similarly conics are treated from a point of view that resembles algebraic geometry but includes a metric.

It is true that there is a need to retain the "circular" or "harmonic" functions to deal with circular motion, Fourier analysis and the like, but those wave-like functions with no natural zero would be better not called "trigonometric". They are not related to triangles.

It is certainly convincing that we would have been better off if trigonometry had developed this way instead of the way it did.

Now to the crunch. Is it feasible for the mathematical world to junk its immense investment in the old technology and move to a new one? It is a big ask, a very big ask, but there are a few reasons to think it might just be possible. The first is that despite the dead hand of conservatism, it has happened before. Replacing co-ordinates, matrices and quaternions with abstract vectors and linear transformations was an effort, but worthwhile in the end. The same was true of replacing sines and cosines in Fourier analysis by complex exponentials. Long before that, Arabic numerals replaced Roman simply because they were more rational. Revolutions are possible. One must regretfully call the author's attention to the fact that they usually take more than a single lifetime. It could be questioned also whether launching the project from a small independent publisher in Australia is a good idea, but in the world of the internet and blogs, perhaps that does not matter in the twenty-first century.

Secondly, a careful examination of 3D vector geometry will reveal that a certain amount of Wildberger's philosophy is implicit in it already, suggesting that he is on the right track at a more basic level. What makes geometry with vectors so successful is that all the information about lengths and angles is contained in the scalar product of vectors, which is algebraically very simple. The student soon learns that the way to approach typical problems, say on the closest distance between two non-intersecting lines, is to stay with vectors and their scalar products as long as possible and only extract any needed lengths and angles at the last moment. Wildberger simply goes one step further: he recommends we do the same in two dimensions, and suggests that we hardly ever have any real need for lengths and angles in any case.

There has been considerable debate by interested amateurs on internet forums about this book. There needs to be more mature consideration in better informed mathematics and mathematics education circles. Having things done better is one major payoff, but equally important would be a removal of a substantial blockage to the education of young mathematicians, the waterless badlands of traditional trigonometry that youth eager to reach the delights of higher mathematics must spend painful years crossing. Wildberger's book deserves very careful examination.