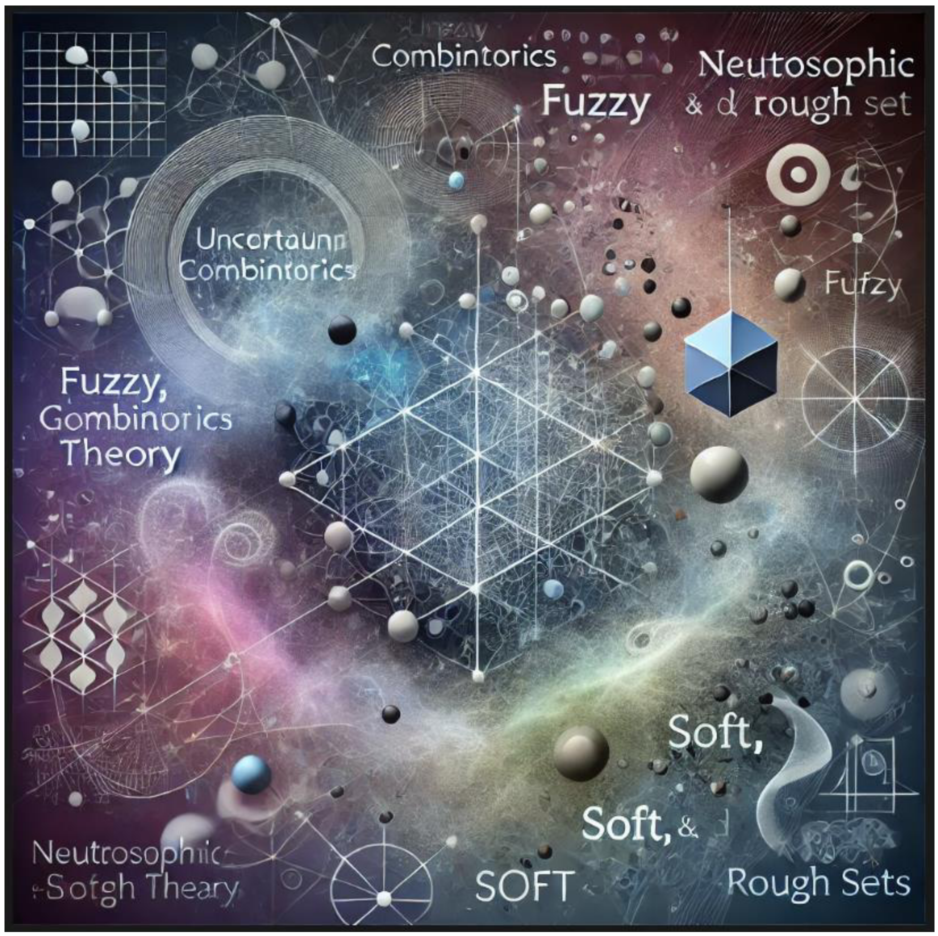
Takaaki Fujita
Florentin Smarandache

Advancing Uncertain Combinatorics

through Graphization, Hyperization, and Uncertainization:
Fuzzy, Neutrosophic, Soft, Rough, and Beyond

Second Volume



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Second Volume

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Foreword

The second volume of “Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond” presents a deep exploration of the progress in uncertain combinatorics through innovative methodologies like graphization, hyperization, and uncertainization. This volume integrates foundational concepts from fuzzy, neutrosophic, soft, and rough set theory, among others, to further advance the field. Combinatorics and set theory, two central pillars of mathematics, focus on counting, arrangement, and the study of collections under defined rules. Combinatorics excels in handling uncertainty, while set theory has evolved with concepts such as fuzzy and neutrosophic sets, which enable the modeling of complex real-world uncertainties by addressing truth, indeterminacy, and falsehood. These advancements, when combined with graph theory, give rise to novel forms of uncertain sets in "graphized" structures, including hypergraphs and superhypergraphs. Innovations such as Neutrosophic Oversets, Undersets, and Offsets, as well as the Nonstandard Real Set, build upon traditional graph concepts, pushing both theoretical and practical boundaries. The synthesis of combinatorics, set theory, and graph theory in this volume provides a robust framework for addressing the complexities and uncertainties inherent in both mathematical and real-world systems, paving the way for future research and application.

In the first chapter, “A Review of the Hierarchy of Plithogenic, Neutrosophic, and Fuzzy Graphs: Survey and Applications”, the authors investigate the interrelationships among various graph classes, including Plithogenic graphs, and explore other related structures. Graph theory, a fundamental branch of mathematics, focuses on networks of nodes and edges, studying their paths, structures, and properties. A Fuzzy Graph extends this concept by assigning a membership degree between 0 and 1 to each edge and vertex, representing the level of uncertainty. The Turiyam Neutrosophic Graph is introduced as an extension of both Neutrosophic and Fuzzy Graphs, while Plithogenic graphs offer a potent method for managing uncertainty.

The second chapter, “Review of Some Superhypergraph Classes: Directed, Bidirected, Soft, and Rough”, examines advanced graph structures such as directed superhypergraphs, bidirected hypergraphs, soft superhypergraphs, and rough superhypergraphs. Classical graph classes include undirected graphs, where edges lack orientation, and directed graphs, where edges have specific directions. Recent innovations, including bidirected graphs, have sparked ongoing research and significant advancements in the field. Soft Sets and their extension to Soft Graphs provide a flexible framework for managing uncertainty, while Rough Sets and Rough Graphs address uncertainty by using lower and upper approximations to handle imprecise data. Hypergraphs generalize traditional graphs by allowing edges, or hyperedges, to connect more than two vertices. Superhypergraphs further extend this by allowing both vertices and edges to represent subsets, facilitating the modeling of hierarchical and group-based relationships.

The third chapter, “Survey of Intersection Graphs, Fuzzy Graphs, and Neutrosophic Graphs”, explores the intersection graph models within the realms of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs. The chapter highlights their mathematical properties and interrelationships, reflecting the growing number of graph classes being developed in these areas. Intersection graphs, such as Unit Square Graphs, Circle Graphs, and Ray Intersection Graphs, are crucial for understanding complex graph structures in uncertain environments.

The fourth chapter, "Fundamental Computational Problems and Algorithms for SuperHyperGraphs", addresses optimization problems within the SuperHypergraph framework, such as the SuperHypergraph Partition Problem, Reachability, and Minimum Spanning SuperHypertree. The chapter also adapts classical problems like the Traveling Salesman Problem and the Chinese Postman Problem to the SuperHypergraph context, exploring how hypergraphs, which allow hyperedges to connect more than two vertices, can be used to solve complex hierarchical and relational problems.

The fifth chapter, "A Short Note on the Basic Graph Construction Algorithm for Plithogenic Graphs", delves into algorithms designed for Plithogenic Graphs and Intuitionistic Plithogenic Graphs, analyzing their complexity and validity. Plithogenic Graphs model multi-valued attributes by incorporating membership and contradiction functions, offering a nuanced representation of complex relationships.

The sixth chapter, "Short Note of Bunch Graph in Fuzzy, Neutrosophic, and Plithogenic Graphs", generalizes traditional graph theory by representing nodes as groups (bunches) rather than individual entities. This approach enables the modeling of both competition and collaboration within a network. The chapter explores various uncertain models of bunch graphs, including Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs.

In the seventh chapter, "A Reconsideration of Advanced Concepts in Neutrosophic Graphs: Smart, Zero Divisor, Layered, Weak, Semi, and Chemical Graphs", the authors extend several fuzzy graph classes to Neutrosophic graphs and analyze their properties. Neutrosophic Graphs, a generalization of fuzzy graphs, incorporate degrees of truth, indeterminacy, and falsity to model uncertainty more effectively.

The eighth chapter, "Short Note of Even-Hole-Graph for Uncertain Graph", focuses on Even-Hole-Free and Meyniel Graphs analyzed within the frameworks of Fuzzy, Neutrosophic, Turiyam Neutrosophic, and Plithogenic Graphs. The study investigates the structure of these graphs, with an emphasis on their implications for uncertainty modeling.

The ninth chapter, "Survey of Planar and Outerplanar Graphs in Fuzzy and Neutrosophic Graphs", explores planar and outerplanar graphs, as well as apex graphs, within the contexts of fuzzy, neutrosophic, Turiyam Neutrosophic, and plithogenic graphs. The chapter examines how these types of graphs are used to model uncertain parameters and relationships in mathematical and real-world systems.

The tenth chapter, "General Plithogenic Soft Rough Graphs and Some Related Graph Classes", introduces and explores new concepts such as Turiyam Neutrosophic Soft Graphs and General Plithogenic Soft Graphs. The chapter also examines models of uncertain graphs, including Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Graphs, all designed to handle uncertainty in diverse contexts.

The eleventh chapter, "Survey of Trees, Forests, and Paths in Fuzzy and Neutrosophic Graphs", provides a comprehensive study of Trees, Forests, and Paths within the framework of Fuzzy and Neutrosophic Graphs. This chapter focuses on classifying and analyzing graph structures like trees and paths in uncertain environments, contributing to the ongoing development of graph theory in the context of uncertainty.

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Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond

Second Volume

This series explores the advancement of uncertain combinatorics through innovative methods such as graphization, hyperization, and uncertainization, incorporating concepts from fuzzy, neutrosophic, soft, and rough set theory, among others. Combinatorics and set theory are fundamental mathematical disciplines that focus on counting, arrangement, and the study of collections under specified rules. While combinatorics excels at solving problems involving uncertainty, set theory has expanded to include advanced concepts like fuzzy and neutrosophic sets, which are capable of modeling complex real-world uncertainties by accounting for truth, indeterminacy, and falsehood. These developments intersect with graph theory, leading to novel forms of uncertain sets in "graphized" structures, such as hypergraphs and superhypergraphs. Innovations like Neutrosophic Oversets, Undersets, and Offsets, as well as the Nonstandard Real Set, build upon traditional graph concepts, pushing the boundaries of theoretical and practical advancements. This synthesis of combinatorics, set theory, and graph theory provides a strong foundation for addressing the complexities and uncertainties present in mathematical and real-world systems, paving the way for future research and application.

A Review of the Hierarchy of Plithogenic, Neutrosophic, and Fuzzy Graphs: Survey and Applications

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Abstract: As many readers may know, graph theory is a fundamental branch of mathematics that examines networks consisting of nodes and edges, with a focus on their paths, structures, and properties [157]. A Fuzzy Graph extends this concept by assigning a membership degree between 0 and 1 to each edge and vertex, capturing the level of uncertainty. Expanding on this idea, the Turiyam Neutrosophic Graph was introduced as an extension of both Neutrosophic and Fuzzy Graphs. Plithogenic graphs, in turn, offer a powerful approach for managing uncertainty. In this paper, we explore the relationships among various graph classes, including Plithogenic graphs, and investigate other related graph structures.

Keywords: Neutrosophic graph, plithogenic graphs, Turiyam Neutrosophic graph, Fuzzy graph

1. Introduction

1.1 Graph Theory

As the readers are aware, graph theory, a fundamental branch of mathematics, examines networks composed of nodes and edges, focusing on their paths, structures, and properties [157]. It is well-known that graph theory has been extensively studied for a wide range of applications[70, 98, 148, 172, 197, 350, 461]. In graph theory, various graph classes, structures, and algorithms have been explored, such as those related to tree structures[133, 163], path structures[107], and linear layouts[142, 164], each tailored to specific objectives. It is also known that focusing on specific structures rather than arbitrary ones allows for the development of more efficient algorithms, illustrating the practical benefits of using these graph classes[105].

1.2 Fuzzy, Neutrosophic, Turiyam neutrosophic, and Plithogenic Graphs

To address the uncertain parameters and concepts in the world, graphs such as Fuzzy, Neutrosophic, Turiyam Neutrosophic graphs, and Plithogenic Graphs are being studied.

A Fuzzy Graph is a graph in which each edge and vertex is assigned a membership degree between 0 and 1, reflecting the level of uncertainty. To put it simply, a Fuzzy Graph is a graphical representation of a fuzzy set (cf. [162,278,477,478]). In real-life applications, fuzzy graphs are used in areas like social networks, decision-making, and transportation systems to model imprecise relationships or uncertain connections[318,377]. Due to their applicability in various fields, they have been the subject of extensive research. Within fuzzy graph theory, various graph classes have been proposed to generalize fuzzy graphs or adapt them for real-world applications. These include Intuitionistic Fuzzy Graphs [341], Bipolar Fuzzy Graphs [15], Fuzzy Planar Graphs [392], Irregular Bipolar Fuzzy Graphs [391], General Fuzzy graph[173, 336], Semi Fuzzy Graph[52], and Complex Hesitant Fuzzy Graphs [6], among others. Studying these classes helps researchers uncover common properties, develop specialized algorithms, and apply findings to practical problems.

To address uncertainty and the relationships between concepts, several graph classes have been introduced, including Fuzzy Graphs [318,377], Vague Graphs [102,369,393], Plithogenic graphs [183,411,425,426], weighted graphs[313, 459, 466], probabilistic graphs[171, 264, 303], Vague hypergraphs [25], Graph entropy (cf.Fuzzy entropy[265]) [144], N-graphs[14], N-hypergraphs[19], Markov graphs[266], Hyperfuzzy Graph (Hyperfuzzy set)[177, 193, 236, 442], HyperNeutrosophic Graph[177], Soft Graph (Soft Set)[34, 222, 294, 314, 462], Hypersoft Graph[178, 182, 382, 383], and Rough Graphs (Rough set) [9, 152, 351–354]. Among these, this paper mainly focuses on Neutrosophic Graphs and Turiyam Neutrosophic Graphs [185–187, 436]. Each graph class has been studied with respect to its specific objectives.

Recently, Neutrosophic Graphs[33, 41, 115, 174, 177, 180, 200, 215, 239, 385, 429, 436] and Neutrosophic Hypergraphs[31, 45, 153, 287] have gained attention within the framework of Neutrosophic Set Theory[61, 443].

”Neutrosophic” refers to a mathematical approach that extends classical and fuzzy logic by incorporating degrees of truth, indeterminacy, and falsity. As generalizations of Fuzzy Graphs[318, 377], Neutrosophic Graphs have garnered significant interest due to their versatility and broad potential for applications, much like Fuzzy Graphs. Various related classes of Neutrosophic Graphs have also been developed and studied, including Bipolar Neutrosophic Graphs[45], Neutrosophic Incidence Graphs[440], Single valued neutrosophic signed graphs[306], Strong Neutrosophic Graphs[240], m-polar neutrosophic graphs[312], Complex neutrosophic hypergraphs[287], Bipolar neutrosophic hypergraphs[45].

Subsequently, the Turiyam Neutrosophic Graph was introduced as an extension of Neutrosophic and Fuzzy Graphs. A Turiyam Neutrosophic Graph is distinguished by each vertex and edge having four attributes: truth, indeterminacy, falsity, and a liberal state, thus expanding the framework of Neutrosophic and Fuzzy Graphs [185–187]. Moreover, a further generalization of these graphs, known as Plithogenic Graphs, has been developed and is also actively studied [241, 412, 449, 451].

Given the extensive and continuously expanding body of literature on fuzzy mathematics, it is inevitable that similar concepts may emerge independently across different journals and periods. Nevertheless, we believe that efforts to unify these concepts are essential and will greatly contribute to advancing the field. Additionally, when conducting research or exploring applications, it is important to consider and investigate a wide range of graph classes to identify the most suitable options. This comprehensive approach is crucial for thorough analysis.

1.3 Our Contribution

Based on the above, research on graphs and graph classes that address uncertainty is of great importance. The study of the Turiyam Neutrosophic Graph class has only just begun and remains far less known compared to Fuzzy Graphs, Intuitionistic Fuzzy Graphs, and Neutrosophic Graphs. This paper explores a new graph class for Intuitionistic Fuzzy Graphs and Turiyam Neutrosophic Graphs: General Intuitionistic Fuzzy Graphs, General Turiyam Neutrosophic Graphs and Turiyam Neutrosophic Hypergraphs. Additionally, we consider Extended Turiyam Neutrosophic Graphs, which are graphs designed to handle five parameters from the perspective of uncertainty. We also examine whether most of the graphs mentioned above can be generalized within the framework of Plithogenic Graphs.

The main result of this paper is presented in the following theorem. In addition to this theorem, we also examined the hierarchy of each graph class. Note that the Turiyam Neutrosophic Set is, in fact, a specific case of the Quadripartitioned Neutrosophic Set, achieved by replacing ”Contradiction” with ”Liberal.” (cf.[417, 434])

Theorem 1.1. *In each graph class, the following relationships hold.*

- *An empty graph and a null graph can be represented as 2-valued graphs and 3-valued graphs.*
- *Every edge-fuzzy graph can be transformed into a 2-valued graph by thresholding the edge membership values.*
- *Every fuzzy graph can be transformed into a 3-valued graph by mapping the fuzzy membership values of vertices and edges to the values {-1, 0, 1}.*
- *Every Intuitionistic Fuzzy Graph can be transformed into a Fuzzy Graph by restricting the non-membership function* v_A *to 0 for all vertices.*
- *Every Neutrosophic Graph can be transformed into an Intuitionistic Fuzzy Graph by setting the indeterminacy value to zero.*
- *Every Extended Turiyam Neutrosophic Graph is a generalization of the Turiyam Neutrosophic Graph.*
- *A plithogenic Graphs generalize Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, Extended Turiyam Neutrosophic Graphs.*
- *Every general plithogenic Graphs can be transformed into General Turiyam Neutrosophic Graph, General Fuzzy Graph, General Intuitionistic Fuzzy Graph, Four-Valued Fuzzy graph, Ambiguous graph, Picture Fuzzy Graph, Hesitant Fuzzy Graph, Intuitionistic Hesitant Fuzzy Graph, Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Quadripartitioned Neutrosophic graph, Pentapartitioned Neutrosophic graph, Turiyam Neutrosophic Graphs, Extended Turiyam Neutrosophic Graphs, and Spherical Fuzzy Graphs.*

Additionally, when considering the relationships between different graph classes, another goal is to organize the characteristics of this field by proving several other theorems related to each graph class as a byproduct, as well as investigating application examples.

In Section 2, we outline the definitions and characteristics used throughout the paper. This section not only introduces various graph classes, but also provides examples and discussions of other graph classes (some of which are already well-known), along with explanations of their applications. Section 3 defines General Intuitionistic Fuzzy Graphs, Turiyam Neutrosophic Graphs, and Turiyam Neutrosophic Hypergraphs, and explores their relationships with other graph types. In Section 4, we examine Plithogenic Graphs, explaining how Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Extended Turiyam Neutrosophic Graphs can be generalized within the framework of Plithogenic Graphs. Finally, Section 5 briefly discusses the future challenges related to graph parameters in Turiyam Neutrosophic Graphs. Graph parameters, simply put, are measures that indicate how closely a given graph matches the desired structure. As the study of graph classes and structures is essential, so too is the investigation of graph parameters [203, 396].

1.4 The Structure of the Paper

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2. Preliminaries and definitions

In this section, we provide a brief overview of the definitions and notations used throughout this paper. Specifically, we cover fundamental concepts of graphs, hypergraphs, fuzzy graphs, Intuitionistic Fuzzy Graphs, Turiyam Neutrosophic graphs/hypergraphs, Neutrosophic graphs, Plithogenic graphs, and single-valued Neutrosophic graphs. Furthermore, we introduce some key properties associated with each type of graph. Additionally, we highlight relevant applications for each graph class where applicable.

Note that this paper may also address set theory concepts in addition to graph theory. Please refer to relevant surveys or notes on set theory as needed [214, 234, 276].

2.1 Basic Graph Concepts

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [155, 156, 156, 157, 197, 473].

Definition 2.1 (Graph)**.** [157] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Example 2.2. Consider the graph $G = (V, E)$, where the vertex set V is defined as $V = \{A, B, C, D\}$, and the edge set E is defined as $E = \{(A, B), (B, C), (C, D), (A, D)\}$.

Definition 2.3 (Subgraph)**.** [157] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Example 2.4. A subgraph H of the graph $G = (V, E)$ from the previous example can be formed by selecting the vertex set $V_H = \{A, B, D\}$ and the edge set $E_H = \{(A, B), (A, D)\}$.

Definition 2.5 (Degree)**.** [157] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Example 2.6 (Degree)**.** Consider an undirected graph $G = (V, E)$ where

$$V = \{a, b, c, d\}$$

and

$$E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}\}$$

. The degree of each vertex is as follows:

- $\deg(a) = 3$ because a is connected to b, c, and d.
- $\deg(b) = 2$ as it connects to a and d.
- $\deg(c) = 1$ since it's only connected to a.
- $\deg(d) = 2$ due to connections with a and b.

For a directed graph, let $G' = (V', E')$ with $V' = \{x, y, z\}$ and $E' = \{(x, y), (y, z), (z, x), (y, x)\}$. The in-degrees and out-degrees are:

- $\deg^-(x) = 1, \deg^+(x) = 1$
- $\deg^-(y) = 2, \deg^+(y) = 2$
- $\deg^-(z) = 1, \deg^+(z) = 1$

Definition 2.7 (Path)**.** (cf.[483]) A path in a graph $G = (V, E)$ is a sequence of distinct vertices $v_1, v_2, \dots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \dots, k-1$. A path is represented as:

$$P = (v_1, v_2, \dots, v_k),$$

where no vertex is repeated. The length of a path is the number of edges it contains, i.e., $k - 1$.

Example 2.8 (Path)**.** Consider an undirected graph $G = (V, E)$ with $V = \{p, q, r, s\}$ and $E = \{\{p, q\}, \{q, r\}, \{r, s\}\}$. A path from vertex p to vertex s is represented as:

$$P = (p, q, r, s)$$

The length of this path is 3, corresponding to the 3 edges traversed: $\{p, q\}, \{q, r\}, \{r, s\}$.

Definition 2.9 (Cycle)**.** (cf.[483]) A cycle is a closed path, meaning it is a path where $v_1 = v_k$, forming a loop with no other repeated vertices. Formally, a cycle is a sequence:

$$C = (v_1, v_2, \dots, v_k, v_1),$$

where $v_1 = v_k$, and $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \dots, k-1$.

Example 2.10 (Cycle)**.** In the same graph $G = (V, E)$ as above, add an edge $\{s, p\}$ to form a new set of edges $E = \{\{p, q\}, \{q, r\}, \{r, s\}, \{s, p\}\}$. Now there exists a cycle:

$$C = (p, q, r, s, p)$$

This sequence starts and ends at p, creating a closed loop, and no other vertex is repeated within the cycle.

Definition 2.11 (Acyclicity)**.** A graph $G = (V, E)$ is called *acyclic* if it contains no cycles. That is, for any sequence of distinct vertices $v_1, v_2, \dots, v_k$ with $k \geq 3$, there does not exist a set of edges

$$\{(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k), (v_k, v_1)\}$$

that form a cycle.

Definition 2.12 (Tree)**.** (cf.[483]) A tree is a connected, acyclic graph. In other words, a tree is a graph where there is exactly one path between any two vertices, and no cycles exist.

Example 2.13. Here is an example of a Tree based on the provided definitions:

$$T = (V, E), \quad V = \{v_1, v_2, v_3, v_4, v_5\}, \quad E = \{(v_1, v_2), (v_1, v_3), (v_3, v_4), (v_3, v_5)\}$$

This graph T has the following properties:

- The set of vertices $V = \{v_1, v_2, v_3, v_4, v_5\}$ represents five distinct vertices.
- The set of edges $E = \{(v_1, v_2), (v_1, v_3), (v_3, v_4), (v_3, v_5)\}$ connects these vertices.
- T is connected, meaning there is a path between any pair of vertices.
- T is acyclic, meaning it contains no cycles.

Thus, T satisfies the definition of a *Tree*, as it is connected and acyclic. For example, there is exactly one path from v_1 to v_5 through v_3, and no cycles are formed.

2.1.1 Useful graph class

We introduce some representative classical graph classes. These graph classes have also been studied in the contexts of fuzzy, neutrosophic, and plithogenic settings.

Definition 2.14. (cf.[128, 275, 481]) The **line graph** $L(G)$ of a given graph $G = (V, E)$ is a graph constructed as follows:

- Each vertex in $L(G)$ represents an edge in the original graph G.
- Two vertices in $L(G)$ are connected by an edge if and only if their corresponding edges in G share a common vertex.

Definition 2.15. (cf.[143, 376, 460]) A graph $G = (V, E)$ is called a **planar graph** if it can be drawn on a plane in such a way that no two edges intersect except at their endpoints. In other words, there exists a way to represent the graph on a two-dimensional plane such that the edges only meet at shared vertices, without any of the edges crossing each other.

Definition 2.16. (cf.[198,329]) An **outer-planar graph** $G = (V, E)$ is an undirected graph that can be embedded in the plane such that all vertices lie on the outer face of the embedding, meaning no vertex is entirely enclosed by edges. In other words, it is possible to draw the graph on a plane without any edge crossings, with all vertices positioned on the unbounded (outer) face of the graph.

1. **Edge Crossing Condition**: The graph G must be drawn in such a way that no two edges intersect except at their endpoints.
2. **Outer Face Requirement**: All vertices of G must lie on the boundary of a single, unbounded face of the graph.

2.1.2 Empty Graph and Null graph

The definitions of an Empty Graph and a Null Graph are provided below.

Definition 2.17 (Empty Graph and Null graph)**.** (cf.[208, 400]) An *empty graph* is a graph that contains a set of vertices V but has no edges. Formally, an empty graph $G = (V, E)$ is defined as a graph where $E = \emptyset$, meaning the edge set is empty. This graph has no connections between its vertices. A *null graph* is a graph with no vertices and no edges. Formally, a null graph is a graph $G = (V, E)$ where both the vertex set $V = \emptyset$ and the edge set $E = \emptyset$. It is the simplest possible graph structure, containing no elements.

I will describe the operations of graph vertex addition and graph edge addition. These are commonly used operations in graph algorithms (ex.[104, 145, 211]).

Definition 2.18 (Graph Vertex Addition)**.** Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. **Graph vertex addition** is the operation of adding a new vertex v_{new} to the graph G, resulting in a new graph $G' = (V', E')$, where:

$$V' = V \cup \{v_{\text{new}}\}, \quad E' = E.$$

No new edges are introduced in this operation. This operation is commonly used in graph algorithms to expand the graph by introducing additional vertices.

Definition 2.19 (Graph Edge Addition)**.** Let $G = (V, E)$ be a graph. **Graph edge addition** is the operation of adding a new edge $e_{\text{new}} = (u, v)$, where $u, v \in V$, to the edge set E of the graph G, resulting in a new graph $G' = (V', E')$, where:

$$V' = V, \quad E' = E \cup \{e_{\text{new}}\}.$$

This operation is commonly used in graph algorithms to create connections between existing vertices.

Proposition 2.20. *If a vertex addition is performed on a null graph, the result is an empty graph.*

Proof. A *null graph* is defined as a graph with no vertices and no edges, formally represented as $G = (V, E)$, where $V = \emptyset$ and $E = \emptyset$.

Let us perform a *vertex addition* on this null graph by adding a single vertex v_{new}. After the addition of v_{new}, the updated vertex set V' becomes $V' = \{v_{\text{new}}\}$. The edge set remains unchanged because no edges are introduced during vertex addition, so $E' = \emptyset$.

Thus, the resulting graph $G' = (V', E')$ is a graph with one vertex and no edges. By definition, a graph with one or more vertices and no edges is called an *empty graph*. Therefore, the result of adding a vertex to a null graph is an empty graph. □

Proposition 2.21. *If an edge addition is performed on an empty graph, the result is a graph.*

Proof. An *empty graph* is defined as a graph with a set of vertices V, but no edges, i.e., $E = \emptyset$. Formally, an empty graph is represented as $G = (V, E)$, where $|V| \geq 1$ and $E = \emptyset$.

Let us perform an *edge addition* on this empty graph by adding an edge $e_{\text{new}} = (u, v)$, where $u, v \in V$. After the addition of the edge e_{new}, the updated edge set E' becomes $E' = \{e_{\text{new}}\}$, while the vertex set V' remains the same as $V' = V$.

The resulting graph $G' = (V', E')$ has the same set of vertices V, and now it has at least one edge $E' \neq \emptyset$. By definition, any graph with vertices and at least one edge is simply called a *graph*. Therefore, the result of adding an edge to an empty graph is a graph. □

2.1.3 2-Valued Graph and 3-Valued Graph

In this subsubsection, we provide an explanation of 2-Valued Graphs and 3-Valued Graphs. In graph theory, vertices and edges are often represented as either existing (denoted by 1) or not existing (denoted by 0). It is worth noting that the concept of a 2-Valued Graph is highly similar to the definition of a Crisp Graph[97, 307, 335, 358] commonly used in the field of Fuzzy Graphs.

When expressed as a 2-valued function, it takes the following form:

Definition 2.22. In a graph $G = (V, E)$, where V is the set of vertices and E is the set of edges, we can define a 2-valued function for both vertices and edges as follows:

- The function $f_V : V \to \{0, 1\}$ represents the state of each vertex $v \in V$:

$$f_V(v) = \begin{cases} 1 & \text{if vertex } v \text{ exists,} \\ 0 & \text{if vertex } v \text{ does not exist.} \end{cases}$$

- Similarly, the function $f_E : E \to \{0, 1\}$ represents the state of each edge $e \in E$:

$$f_E(e) = \begin{cases} 1 & \text{if edge } e \text{ exists,} \\ 0 & \text{if edge } e \text{ does not exist.} \end{cases}$$

To put it bluntly, the idea of extending the functions assigned to vertices and edges—such as by using membership functions or other types of functions—is a fundamental approach to handling real-world concepts and uncertainty. This makes the concept easier to understand.

Proposition 2.23. *An empty graph and a null graph can be represented as 2-valued graphs.*

Proof. Empty graph: Consider a graph $G = (V, E)$ where the vertex set V exists, but the edge set $E = \emptyset$. This is an empty graph. For each vertex $v \in V$, we define the following 2-valued function f_V:

$$f_V(v) = \begin{cases} 1 & \text{if vertex } v \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

For the edge set E, we define a 2-valued function $f_E(e) = 0$ (since no edges exist in the empty graph).

Null graph: In a null graph $G = (V = \emptyset, E = \emptyset)$, there are no vertices or edges. As such, the 2-valued functions f_V and f_E do not apply, effectively representing an empty graph structure.

Thus, both the empty graph and the null graph can be represented using 2-valued functions. □

Here is a rough example, but let's consider a 3-valued function (cf.[213, 282, 299]). We can define a 3-valued function for both vertices and edges in a graph $G = (V, E)$ as follows:

Definition 2.24. In a graph $G = (V, E)$, where V is the set of vertices and E is the set of edges, a 3-valued function can be defined for both vertices and edges as follows:

- **For vertices**: Let $f_V : V \to \{-1, 0, 1\}$ represent the state of each vertex $v \in V$. The function assigns values as follows:

$$f_V(v) = \begin{cases} 1 & \text{if vertex } v \text{ is in a positive state (exists),} \\ 0 & \text{if vertex } v \text{ is neutral,} \\ -1 & \text{if vertex } v \text{ is in a negative state (non-existent or unfavorable).} \end{cases}$$

- **For edges**: Similarly, define $f_E : E \to \{-1, 0, 1\}$ to represent the state of each edge $e \in E$:

$$f_E(e) = \begin{cases} 1 & \text{if edge } e \text{ is in a positive state (exists),} \\ 0 & \text{if edge } e \text{ is neutral,} \\ -1 & \text{if edge } e \text{ is in a negative state (non-existent or unfavorable).} \end{cases}$$

Theorem 2.25. *A 3-valued graph is a generalization of a 2-valued graph.*

Proof. This is evident. □

In this paper, we consider graphs with uncertainty incorporated into them.

2.2 Hypergraph Concepts

A hypergraph is a generalization of a graph where edges, called hyperedges, can connect any number of vertices, not just two. This structure is useful for modeling complex relationships in various fields like computer science and biology[170, 195, 196, 355]. The definition is provided below.

Definition 2.26. [106] A hypergraph is a pair $H = (V(H), E(H))$, consisting of a nonempty set $V(H)$ of vertices and a set $E(H)$ of subsets of $V(H)$, called the hyperedges of H. In this paper, we consider only finite hypergraphs.

Example 2.27. Let H be a hypergraph with vertex set $V(H) = \{A, B, C, D, E\}$ and hyperedge set $E(H) = \{e_1, e_2, e_3\}$, where:

$$e_1 = \{A, D\}, \quad e_2 = \{D, E\}, \quad e_3 = \{A, B, C\}.$$

Thus, H is represented by the pair $H(V, E) = (\{A, B, C, D, E\}, \{\{A, D\}, \{D, E\}, \{A, B, C\}\})$.

Theorem 2.28. *Any classical (binary) graph $G = (V, E)$, where V is the set of vertices and E is the set of edges, can be represented as a hypergraph $H = (V(H), E(H))$, where $V(H) = V$ and each edge $e \in E$ corresponds to a hyperedge in $E(H)$ consisting of two vertices.*

Proof. Let $G = (V, E)$ be a classical graph, where:

- $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices, and
- $E = \{e_1, e_2, \ldots, e_m\}$ is the set of edges, where each edge $e_i = (u_i, v_i)$ connects two vertices $u_i, v_i \in V$.

We define a corresponding hypergraph $H = (V(H), E(H))$, where:

- $V(H) = V$, meaning the vertex set of the hypergraph is the same as the vertex set of the graph.
- $E(H) = \{e_1, e_2, \ldots, e_m\}$, where each hyperedge $e_i \in E(H)$ is a 2-element subset of $V(H)$, corresponding to the edge $e_i = (u_i, v_i)$ in the graph.

For each edge $e_i = (u_i, v_i) \in E$ in the graph, we define a hyperedge $e_i' = \{u_i, v_i\} \in E(H)$ in the hypergraph. Thus, the hypergraph H is constructed such that each edge in the original graph is represented by a hyperedge containing exactly two vertices.

Since each edge in a graph connects exactly two vertices, and each hyperedge in the constructed hypergraph contains exactly two vertices, this hypergraph representation faithfully captures the structure of the original graph.

The classical graph $G = (V, E)$ is equivalent to a hypergraph $H = (V(H), E(H))$ where every edge in the graph corresponds to a 2-element hyperedge in the hypergraph. Therefore, any graph can be represented as a hypergraph where each hyperedge contains exactly two vertices. □

Definition 2.29. [106] For a hypergraph H and a subset $X \subseteq V(H)$, the subhypergraph induced by X is defined as $H[X] = (X, \{e \cap X \mid e \in E(H)\})$. We denote the hypergraph obtained by removing X from H as $H \setminus X := H[V(H) \setminus X]$.

This paper does not delve into detailed discussions, but Supergraphs [98] and Superhypergraph [73, 175, 181, 192, 229, 323, 360, 428, 429, 435, 439] are well-known graph classes that are closely related to both graphs and hypergraphs. The following diagram illustrates their relationship with these graph classes. The same applies to directed graphs as well.

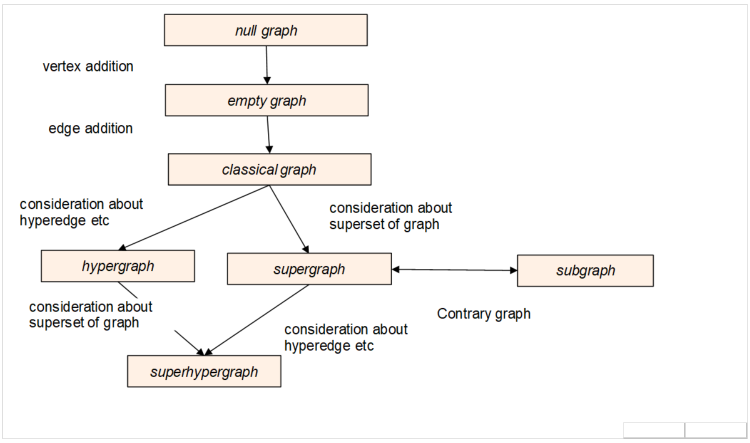

Fig. 1. Graph Hierarchy for Hypergraph

2.3 Fuzzy Graph

A Fuzzy Graph captures relationships involving uncertainty by assigning membership degrees to both vertices and edges, enabling flexible and nuanced analysis. It can also be viewed as a graphical representation of a fuzzy set (cf. [1, 161, 281, 357, 477]). Due to its significance, Fuzzy Graphs have been the subject of extensive research [32, 37, 48, 51, 54, 56, 93, 95, 237, 283, 316, 322, 447, 470].

The Fuzzy framework encompasses a wide range of mathematical and logical structures that have been extensively studied and developed in recent years. The primary concepts include:

1. Fuzzy Set [261, 293, 365, 477, 484]
2. Fuzzy Topological Spaces [285, 444]
3. Fuzzy Logics [90, 204]
4. Fuzzy Algebraic Structures [243]
5. Fuzzy Environment [89, 330]
6. Fuzzy Geometry [378]
7. Fuzzy Statistics [243]
8. Fuzzy Physics [86]
9. Fuzzy Control [160, 342, 356]
10. Fuzzy system [124, 132]

2.3.1 Basic concepts for fuzzy graph

In this subsubsection, we explain the basic concepts of fuzzy graphs. A Fuzzy Graph is often discussed in relation to a crisp graph [97, 307, 335, 358]. First, we provide the definition of a crisp graph below.

Definition 2.30. A crisp graph $G = (V, E)$ consists of V, a non-empty set of vertices (or nodes), and E, a set of edges. Each edge is associated with either one or two vertices, which are referred to as its endpoints. An edge is said to connect its endpoints.

To put it simply, a fuzzy graph can be seen as a crisp graph with the concept of fuzzy relation applied to it. The definition of a fuzzy relation is presented below.

Definition 2.31. Let S be a non-empty set. A **fuzzy relation** on S is defined as a fuzzy subset $\mu : S \times S \to [0, 1]$, which assigns a membership value to each pair of elements in S. The membership value $\mu(x, y)$ represents the degree of relationship between elements x and y in S.

Let $\sigma : S \to [0, 1]$ be a fuzzy set on S, where $\sigma(x)$ represents the degree of membership of element x in the fuzzy set. The fuzzy relation μ is said to be a relation on σ if, for all $x, y \in S$, the following condition holds:

$$\mu(x, y) \leq \sigma(x) \wedge \sigma(y),$$

where $\wedge$ denotes the minimum operation, i.e.,

$$\sigma(x) \wedge \sigma(y) = \min\{\sigma(x), \sigma(y)\}.$$

In this context, $\mu(x, y)$ indicates the strength of the relationship between x and y, and the condition ensures that the relationship between x and y does not exceed the membership values of x and y in the fuzzy set σ.

The formal definition of a Fuzzy Graph is as follows [317, 377].

Definition 2.32. [377] A fuzzy graph $G = (\sigma, \mu)$ with V as the underlying set is defined as follows:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of vertices, where $\sigma(x)$ represents the membership degree of vertex $x \in V$.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation on σ, such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ denotes the minimum operation.

The underlying crisp graph of G is denoted by $G^* = (\sigma^*, \mu^*)$, where:

- $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$
- $\mu^* = \sup p(\mu) = \{(x, y) \in V \times V : \mu(x, y) > 0\}$

Definition 2.33. [377] A fuzzy subgraph $H = (\sigma', \mu')$ of G is defined as follows:

- There exists $X \subseteq V$ such that $\sigma' : X \to [0, 1]$ is a fuzzy subset.
- $\mu' : X \times X \to [0, 1]$ is a fuzzy relation on σ', satisfying $\mu'(x, y) \leq \sigma'(x) \wedge \sigma'(y)$ for all $x, y \in X$.

Example 2.34. (cf.[116]) Consider a fuzzy graph $G = (\sigma, \mu)$ with four vertices $V = \{v_1, v_2, v_3, v_4\}$, as depicted in the diagram.

The membership degrees of the vertices are as follows:

$$\sigma(v_1) = 0.1, \quad \sigma(v_2) = 0.3, \quad \sigma(v_3) = 0.2, \quad \sigma(v_4) = 0.4$$

The fuzzy relation on the edges is defined by the values of μ, where $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. The fuzzy membership degrees of the edges are as follows:

$$\mu(v_1, v_2) = 0.1, \quad \mu(v_2, v_3) = 0.1, \quad \mu(v_3, v_4) = 0.1$$

$$\mu(v_4, v_1) = 0.1, \quad \mu(v_2, v_4) = 0.3$$

In this case, the fuzzy graph G has the following properties:

- Vertices v_1, v_2, v_3, v_4 are connected by edges with varying membership degrees.
- The fuzzy relations ensure that $\mu(x, y)$ for any edge (x, y) does not exceed the minimum membership of the corresponding vertices.

The definitions of "Complete" and "Strong," which are frequently used properties of fuzzy graphs, are provided below.

Definition 2.35. [189] A fuzzy graph $G = (\sigma, \mu)$ is called *complete* if for all $u, v \in V$, the following condition holds:

$$\mu(u, v) = \sigma(u) \wedge \sigma(v),$$

where $\wedge$ denotes the minimum operation.

Definition 2.36. [189] A fuzzy graph $G = (\sigma, \mu)$ is called *strong* if for all $u, v \in E$, the same condition holds:

$$\mu(u, v) = \sigma(u) \wedge \sigma(v).$$

Theorem 2.37. *Every complete fuzzy graph is a strong fuzzy graph, but not every strong fuzzy graph is a complete fuzzy graph.*

Proof. By definition, a complete fuzzy graph $G = (\sigma, \mu)$ satisfies:

$$\mu(u, v) = \sigma(u) \wedge \sigma(v) \quad \forall u, v \in V,$$

where $\wedge$ is the minimum operation. This condition holds for all pairs of vertices, ensuring that every complete fuzzy graph is a strong fuzzy graph, as a strong fuzzy graph also satisfies:

$$\mu(u, v) = \sigma(u) \wedge \sigma(v) \quad \forall u, v \in E.$$

Thus, every complete fuzzy graph is strong.

A strong fuzzy graph only requires the condition $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for edges $(u, v) \in E$, but does not require edges between every pair of vertices. In contrast, a complete fuzzy graph requires edges between all pairs of vertices. Therefore, not every strong fuzzy graph is complete. □

2.3.2 Other graph classes related to fuzzy graph

Next, we will examine the relationships with other classes of graphs.

Proposition 2.38. *Every Fuzzy Graph can be transformed into an null graph by restricting the membership values of vertices and edges to* 0.

Proof. Let $G = (V, E, \sigma, \mu)$ be a fuzzy graph, where $\sigma : V \to [0, 1]$ represents the membership function of the vertices, and $\mu : V \times V \to [0, 1]$ represents the membership function of the edges.

To transform G into a null graph (a graph with no vertices or edges), we need to set the membership values of all vertices and edges to 0.

Formally, for each vertex $v \in V$ and each edge $(u, v) \in E$, we redefine:

$$\sigma(v) = 0 \quad \text{and} \quad \mu(u, v) = 0.$$

This operation results in no vertices or edges having a non-zero membership value, thereby reducing the graph to a null graph. □

Proposition 2.39. *Every Fuzzy Graph can be transformed into an empty graph by restricting the membership values of edges to* 0.

Proof. Let $G = (V, E, \mu)$ be a fuzzy graph, where μ is the membership function of the edges. By definition, $\mu : V \times V \to [0, 1]$ assigns a membership value to each edge.

To transform G into an empty graph, we simply set the membership values of all edges to 0. Formally, redefine $\mu(u, v) = 0$ for all $(u, v) \in E$.

Since an empty graph has no edges, this operation effectively removes all connections, making the resulting graph an empty graph with only isolated vertices. □

Proposition 2.40. *Every Fuzzy Graph can be transformed into a Crisp Graph by restricting the membership values of vertices and edges to* 0 *or* 1.

Proof. Let $G_F = (V, E, \mu_V, \mu_E)$ be a Fuzzy Graph, where:

- $\mu_V : V \to [0, 1]$ assigns a membership value to each vertex $v \in V$,
- $\mu_E : E \to [0, 1]$ assigns a membership value to each edge $(u, v) \in E$.

We want to transform G_F into a Crisp Graph $G_C = (V_C, E_C)$, where:

- $V_C \subseteq V$ is the set of vertices in the Crisp Graph,
- $E_C \subseteq V_C \times V_C$ is the set of edges in the Crisp Graph.

The transformation is performed as follows:

1. Vertices:
$$V_C = \{v \in V : \mu_V(v) > 0\}.$$
If a vertex $v \in V$ has a positive membership value $\mu_V(v) > 0$, then v is included in the vertex set V_C of the Crisp Graph. If $\mu_V(v) = 0$, then v is excluded.

2. Edges:
$$E_C = \{(u, v) \in E : \mu_E(u, v) > 0\}.$$
Similarly, if an edge $(u, v) \in E$ has a positive membership value $\mu_E(u, v) > 0$, then (u, v) is included in the edge set E_C of the Crisp Graph. If $\mu_E(u, v) = 0$, then the edge is excluded.

Since the membership values in G_C are restricted to 0 or 1, this transformation converts the Fuzzy Graph into a Crisp Graph. □

Proposition 2.41. *Every fuzzy graph can be transformed into a 3-valued graph by mapping the fuzzy membership values of vertices and edges to the values {-1, 0, 1}.*

Proof. Let $G_F = (V, E, \sigma, \mu)$ be a fuzzy graph, where:

- $\sigma : V \to [0, 1]$ assigns membership values to vertices, and
- $\mu : E \to [0, 1]$ assigns membership values to edges.

To transform G_F into a 3-valued graph $G_T = (V_T, E_T, f_V, f_E)$, we define the following mappings for vertices and edges:

Define a function $f_V : V \to \{-1, 0, 1\}$ for each vertex $v \in V$ as follows:

$$f_V(v) = \begin{cases} 1 & \text{if } \sigma(v) > 0.5 \text{ (positive state)}, \\ 0 & \text{if } \sigma(v) = 0.5 \text{ (neutral state)}, \\ -1 & \text{if } \sigma(v) < 0.5 \text{ (negative state)}. \end{cases}$$

For edges: Similarly, define a function $f_E : E \to \{-1, 0, 1\}$ for each edge $e \in E$ as follows:

$$f_E(e) = \begin{cases} 1 & \text{if } \mu(e) > 0.5 \text{ (positive state)}, \\ 0 & \text{if } \mu(e) = 0.5 \text{ (neutral state)}, \\ -1 & \text{if } \mu(e) < 0.5 \text{ (negative state)}. \end{cases}$$

By applying these mappings, the fuzzy membership values $\sigma(v)$ and $\mu(e)$, which lie in the interval $[0, 1]$, are transformed into discrete values $\{-1, 0, 1\}$ corresponding to negative, neutral, and positive states. This preserves the uncertainty modeled by the fuzzy graph while allowing for a simpler 3-valued representation.

Thus, every fuzzy graph can be transformed into a 3-valued graph by appropriately mapping its membership values. □

2.3.3 Edge-fuzzy Graph

The above definition of a fuzzy graph considers membership values for both vertices and edges. However, there are also definitions that consider membership values solely for vertices or solely for edges. These are known as vertex fuzzy graphs and edge fuzzy graphs, respectively (cf.[159]).

Definition 2.42. An **Edge-fuzzy Graph** $G = (V, E, \mu)$ is a graph where:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- $\mu : E \to \mathbb{R}$ is a function that assigns a real value to each edge. The value $\mu(e) \in \mathbb{R}$ represents the weight, cost, or strength associated with the edge e.

Proposition 2.43. *Let $G = (\sigma, \mu)$ be a fuzzy graph. If the vertex membership function σ is fixed to a constant $c \in [0, 1]$, then the graph can be transformed into an edge-fuzzy graph $G' = (V, \mu')$.*

Proof. In a fuzzy graph $G = (\sigma, \mu)$, the membership value of an edge (u, v) is constrained by the vertex membership values as follows:
$$\mu(u, v) \leq \sigma(u) \wedge \sigma(v).$$
If we fix the membership values of all vertices to a constant $c \in [0, 1]$, then for all vertices $u, v \in V$, we have $\sigma(u) = \sigma(v) = c$. Therefore, the constraint simplifies to:
$$\mu(u, v) \leq c.$$

In the resulting edge-fuzzy graph $G' = (V, \mu')$, we define the edge membership function μ' as:

$$\mu'(u, v) = \frac{\mu(u, v)}{c}.$$

This definition ensures that $\mu'(u, v) \in [0, 1]$, which satisfies the requirements of an edge-fuzzy graph.

Hence, by fixing the vertex membership function σ to a constant c, the fuzzy graph can be transformed into an edge-fuzzy graph. □

Theorem 2.44. *Every edge-fuzzy graph can be transformed into a 2-valued graph by thresholding the edge membership values.*

Proof. Let $G = (V, E, \mu)$ be an edge-fuzzy graph, where $\mu(e) \in [0, 1]$ represents the membership value of edge $e \in E$. To transform this edge-fuzzy graph into a 2-valued graph, we can apply a threshold to the edge membership values, setting a threshold $\tau \in [0, 1]$ to decide whether an edge exists or not in the resulting 2-valued graph.

Define the edge set E_2 of the 2-valued graph G_2 as:

$$E_2 = \{e \in E \mid \mu(e) \geq \tau\}.$$

The function $f_E : E_2 \to \{0, 1\}$ is then defined as:

$$f_E(e) = \begin{cases} 1 & \text{if } \mu(e) \geq \tau, \\ 0 & \text{if } \mu(e) < \tau. \end{cases}$$

This mapping ensures that any edge $e \in E$ with a membership value $\mu(e) \geq \tau$ is included in the 2-valued graph as an existing edge, while edges with $\mu(e) < \tau$ are considered non-existent (i.e., their value is 0).

Thus, the edge-fuzzy graph G has been successfully transformed into a 2-valued graph G_2 based on the threshold τ. □

Based on the previous discussion, it is crucial to determine where to apply the fuzzy concept within the graph and which parts to keep fixed. In the literature [97], the graph types were defined as follows.

Definition 2.45 (Fuzzy Graph Type)**.** [97] A fuzzy graph GF is a graph that satisfies one of the following types of fuzziness (referred to as GF of the i-th type) or any combination thereof:

(i) $GF_1 = \{G_1, G_2, G_3, \ldots, G_F\}$ where fuzziness exists in each graph G_i.

(ii) $GF_2 = \{V, E_F\}$ where the edge set E_F is fuzzy.

(iii) $GF_3 = \{V, E(t_F, h_F)\}$ where both the vertex set V and edge set E are crisp, but the edges have fuzzy heads $h(e_i)$ and fuzzy tails $t(e_i)$.

(iv) $GF_4 = \{V_F, E\}$ where the vertex set V_F is fuzzy.

(v) $GF_5 = \{V, E(w_F)\}$ where both the vertex set V and edge set E are crisp, but the edges have fuzzy weights w_F.

2.3.4 N-graph and general fuzzy graph

In this subsubsection, we explain N-graphs and general fuzzy graphs. Various variants of fuzzy graphs have been defined due to their importance. The N-graph and general fuzzy graph are well-known concepts related to fuzzy graphs. Their definitions are presented below [14, 336].

Definition 2.46. [14] By an N-graph G of a graph, we mean a pair $G = (V, E)$, where V is an N-function on the vertex set V and E is an N-relation on the edge set E, such that for all $u, v \in V$, the relation $(u, v) \leq \max\{N(u), N(v)\}$.

Theorem 2.47. *(cf.[14]) Every N-graph can be transformed into a 2-valued graph by setting the N-function values to* 1 *for all vertices and edges where the membership is non-zero.*

Proof. Let $G = (V, E)$ be an N-graph with the function $N : V \to [0, 1]$ on vertices and the relation $N : E \to [0, 1]$ on edges. To transform G into a 2-valued graph, we need to modify the function values such that the new graph $G_2 = (V, E_2)$ only contains edges and vertices with function values 1 or 0.

Define the vertex function $f_V : V \to \{0, 1\}$ as follows:

$$f_V(v) = \begin{cases} 1 & \text{if } N(v) > 0, \\ 0 & \text{if } N(v) = 0. \end{cases}$$

This ensures that every vertex with a non-zero membership in the N-graph is mapped to 1 in the 2-valued graph.

For the edges, define the edge function $f_E : E \to \{0, 1\}$ as:

$$f_E(u, v) = \begin{cases} 1 & \text{if } N(u, v) > 0, \\ 0 & \text{if } N(u, v) = 0. \end{cases}$$

This transformation ensures that any edge with a non-zero relation in the N-graph is mapped to 1, and edges with zero relation are mapped to 0.

The relation $N(u, v) \leq \max\{N(u), N(v)\}$ still holds after transformation, as the maximum value of $N(u)$ and $N(v)$ is now either 1 or 0. Hence, the transformed edge function $f_E(u, v)$ is consistent with the original relation in the N-graph.

Thus, by setting all non-zero function values to 1, the N-graph has been successfully transformed into a 2-valued graph. □

Definition 2.48. [336] A *general fuzzy graph* with V as the underlying set is defined as a pair of functions $G = (\sigma, \mu)$, where:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of V,
- $\mu : V \times V \to [0, 1]$ is a fuzzy subset of $V \times V$.

This definition generalizes the traditional fuzzy graph by allowing the membership values of edges to be independent of the membership values of their incident vertices.

Definition 2.49. [336] A *general weak fuzzy graph* with V as the underlying set is defined as a pair of functions $G = (\sigma, \mu)$, where:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of V,
- $\mu : V \times V \to [0, 1]$ is a fuzzy subset of $V \times V$,

such that for all $(u, v) \in E$, the membership degree of the edge $\mu(u, v)$ satisfies:

$$\mu(u, v) \neq \sigma(u) \wedge \sigma(v),$$

where $\wedge$ denotes the minimum operation.

Proposition 2.50. *(cf.[336]) Every Weak General fuzzy Graph is a General fuzzy Graph.*

Proof. By the definition of a Weak General fuzzy Graph, for all vertices $u, v \in V$ and edges $(u, v) \in E$, the following condition holds:

$$\mu(u, v) \neq \sigma(u) \wedge \sigma(v),$$

where $\wedge$ denotes the minimum operation.

Now, consider the definition of a *General fuzzy Graph*. A General fuzzy Graph $G = (\sigma, \mu)$ consists of the same functions:

- $\sigma : V \to [0, 1]$,
- $\mu : V \times V \to [0, 1]$,

but without the restriction that $\mu(u, v)$ must equal $\sigma(u) \wedge \sigma(v)$. In other words, $\mu(u, v)$ can take any value independent of the membership degrees of the vertices.

Since the condition $\mu(u, v) \neq \sigma(u) \wedge \sigma(v)$ is less restrictive than requiring $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, any Weak General fuzzy Graph satisfies the conditions of a General fuzzy Graph.

Thus, every Weak General fuzzy Graph is a General fuzzy Graph. □

Proposition 2.51. *(cf.[336]) Every fuzzy Graph is a Weak General fuzzy Graph.*

Proof. For a fuzzy Graph, the membership degree of each edge $\mu(u, v)$ is required to satisfy:

$$\mu(u, v) = \sigma(u) \wedge \sigma(v),$$

where $\wedge$ is the minimum operation.

Now, consider the definition of a *Weak General fuzzy Graph*. In a Weak General fuzzy Graph, there is no requirement for $\mu(u, v)$ to be exactly $\sigma(u) \wedge \sigma(v)$, but instead, it only requires that $\mu(u, v) \neq \sigma(u) \wedge \sigma(v)$.

Since every fuzzy Graph satisfies $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, which is a specific case of $\mu(u, v) \neq \sigma(u) \wedge \sigma(v)$ (in the sense that it does not violate the condition), every fuzzy Graph also satisfies the conditions of a Weak General fuzzy Graph.

Thus, every fuzzy Graph is a Weak General fuzzy Graph. □

2.3.5 Related graph class for fuzzy graph

In this subsubsection, we explain Related graph classes for fuzzy graph. In the field of Fuzzy Graphs, numerous graph classes have also been proposed. Here, we briefly introduce the related graph classes for fuzzy graphs.

Notation 2.52. In this paper, we define the term "Related graph class" as a graph class that either extends or restricts a corresponding graph class in some way.

Theorem 2.53. *The following are examples of related graph classes, including but not limited to:*

- *Bipolar Fuzzy Graphs [15]*
- *Fuzzy Planar Graphs [392]*
- *Irregular Bipolar Fuzzy Graphs [391]*
- *Regular Bipolar Fuzzy Graphs [23]*
- *Picture Fuzzy Tolerance Graphs [135]*
- *Complex Hesitant Fuzzy Graphs [6]*
- *Strong Intuitionistic Fuzzy Graphs [21]*
- *Product Fuzzy Graphs [53]*
- *Partially Total Fuzzy Graphs [8]*
- *Fuzzy Influence Graphs [304]*
- *Picture Fuzzy Directed Hypergraphs [255]*
- *Radio Fuzzy Graphs [290]*
- *Line Regular Fuzzy Semigraphs [74]*
- *Fuzzy Incidence Graphs [158]*
- *Balanced Picture Fuzzy Graphs [66]*
- *Oscillating Polar Fuzzy Graphs [65]*
- *Cayley Fuzzy Graphs [457]*
- *Rough Fuzzy Digraphs [9]*
- *T-Spherical Fuzzy Graphs [199]*
- *Mixed Fuzzy Graphs [387]*
- *Einstein Fuzzy Graphs [228]*
- *Edge-Regular Fuzzy Graphs [55, 118, 339]*
- *Robust Fuzzy Graphs [448]*
- *Anti-Product Fuzzy Graphs [49]*
- *Valued Fuzzy Superhypergraphs [433]*
- *Inverse Fuzzy Graphs [100]*
- *Inverse Eccentric Fuzzy Graphs [305]*
- *Cubic Pythagorean Fuzzy Graphs [321]*

- *Complete Fuzzy Graphs [50]*
- *Mixed Picture Fuzzy Graphs [327]*
- *Extended Total Fuzzy Graphs [7]*
- *Pseudo Regular Fuzzy Graphs [292]*
- *Best Fuzzy Graphs [452]*
- *Intuitionistic Felicitous Fuzzy Graphs [94]*
- *Middle Fuzzy Graphs [309, 450]*
- *Bipolar Fuzzy P-Competition Graphs [337]*
- *Fuzzy Intersection Graphs [372]*
- *Fuzzy Semigraphs [380]*
- *Intuitionistic Fuzzy Soft Expert Graphs [464]*
- *m-Polar Fuzzy Graphs [17]*
- *Balanced Interval-Valued Fuzzy Graphs [370]*
- *Double Layered Fuzzy graph[260]*
- *Triple Layered Fuzzy Graph[379]*
- *Fuzzy Outerplanar Graphs [176, 231]*
- *Inverse fuzzy multigraphs[99]*
- *Bipolar inverse fuzzy graphs[253]*
- *Fuzzy zero divisor graphs [269, 270]*

Considering these fuzzy graph classes enables the identification of shared properties, which can lead to the development of efficient algorithms, deeper analysis, and practical applications across various fields.

Proof. Refer to each reference as needed. □

2.3.6 Application for fuzzy graph

In this subsubsection, we explain Applications for fuzzy graph. Fuzzy graphs have a wide range of applications. Here, we introduce a few practical examples.

- Neural Network: A neural network is a computational model inspired by the human brain, using interconnected nodes (neurons) to process data and learn patterns(cf.[5,202]). Various studies have been conducted on the application of fuzzy graphs to neural networks [267, 283, 472, 478].
- Decision-Making: Decision-making using graphs involves representing problems visually with nodes and edges, allowing for structured analysis of options, dependencies, and outcomes to reach optimal solutions (cf.[259, 278, 445]). Fuzzy graphs and decision-making are highly compatible, and numerous studies have been conducted on this topic[10, 123, 262, 402].
- COVID-19: COVID-19 is a contagious disease caused by the SARS-CoV-2 virus. It spreads through respiratory droplets and can cause symptoms such as fever, cough, and breathing difficulties. Severe cases may require hospitalization (cf. [127, 359]). Several papers have been published on the effective use of fuzzy graphs in relation to COVID-19 [212, 367, 387, 458].
- Communication networks: Communication networks are systems that allow the transfer of data or information between devices through interconnected nodes, using wired or wireless connections(cf.[315]). They include internet, phone, and satellite networks. Several papers have been published on the effective use of fuzzy graphs in relation to Communication networks[96, 232, 288, 289].
- SNS: SNS (Social Networking Service) is an online platform where users create profiles, share content, and connect with others (cf.[11, 131, 373]). Several papers have been published on the effective use of fuzzy graphs in relation to SNS[159, 230, 286].

2.4 Intuitionistic fuzzy graphs

An Intuitionistic Fuzzy Graph (IFG) extends fuzzy graphs by incorporating membership, non-membership, and hesitancy degrees for vertices and edges. An Intuitionistic Fuzzy Graphs have been the subject of extensive research [24,141,341,371]. To put it simply, it is a graph that represents the concept of Intuitionistic Fuzzy Sets (cf.[80–83,165]) in graph form.

The Intuitionistic Fuzzy framework encompasses a wide range of mathematical and logical structures that have been extensively studied and developed in recent years. The primary concepts include:

1. Intuitionistic Fuzzy Set [455]
2. Intuitionistic Fuzzy Topological Spaces [130,381]
3. Intuitionistic Fuzzy Logics [84]
4. Intuitionistic Fuzzy Algebra [22,233]
5. Intuitionistic Fuzzy Environment [69,277]

2.4.1 Definition of Intuitionistic Fuzzy Graph

The definition is provided below.

Definition 2.54 (Intuitionistic Fuzzy Graph (IFG))**.** [341] Let $G = (V, E)$ be a classical graph where V denotes the set of vertices and E denotes the set of edges. An *Intuitionistic Fuzzy Graph* (IFG) on G, denoted $G_{IF} = (A, B)$, is defined as follows:

1. (μ_A, ν_A) is an *Intuitionistic Fuzzy Set (IFS)* on the vertex set V. For each vertex $x \in V$, the degree of membership $\mu_A(x) \in [0, 1]$ and the degree of non-membership $\nu_A(x) \in [0, 1]$ satisfy:

$$\mu_A(x) + \nu_A(x) \leq 1$$

   The value $1 - \mu_A(x) - \nu_A(x)$ represents the hesitancy or uncertainty regarding the membership of x in the set.

2. (μ_B, ν_B) is an *Intuitionistic Fuzzy Relation (IFR)* on the edge set E. For each edge $(x, y) \in E$, the degree of membership $\mu_B(x, y) \in [0, 1]$ and the degree of non-membership $\nu_B(x, y) \in [0, 1]$ satisfy:

$$\mu_B(x, y) + \nu_B(x, y) \leq 1$$

   Additionally, the following constraints must hold for all $x, y \in V$:

$$\mu_B(x, y) \leq \mu_A(x) \wedge \mu_A(y)$$

$$\nu_B(x, y) \leq \nu_A(x) \vee \nu_A(y)$$

In this definition:

- $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and non-membership of the vertex x, respectively.
- $\mu_B(x, y)$ and $\nu_B(x, y)$ represent the degree of membership and non-membership of the edge (x, y), respectively.
- If $\nu_A(x) = 0$ and $\nu_B(x, y) = 0$ for all $x \in V$ and $(x, y) \in E$, then the Intuitionistic Fuzzy Graph reduces to a Fuzzy Graph.

Example 2.55 (Intuitionistic Fuzzy Graph)**.** Consider the Intuitionistic Fuzzy Graph $G = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$, where $V = \{v_1, v_2, v_3, v_4\}$ and the edges $E = \{(v_1, v_2), (v_1, v_4), (v_2, v_3), (v_3, v_4), (v_2, v_4)\}$. The membership and non-membership degrees for the vertices are given as:

$$\mu_A(v_1) = 0.1, \quad \nu_A(v_1) = 0.4$$

$$\mu_A(v_2) = 0.3, \quad \nu_A(v_2) = 0.3$$

$$\mu_A(v_3) = 0.2, \quad \nu_A(v_3) = 0.4$$

$$\mu_A(v_4) = 0.4, \quad \nu_A(v_4) = 0.6$$

For the edges, the membership and non-membership degrees are given as follows:

- $\mu_B(v_1, v_2) = 0.1, \quad \nu_B(v_1, v_2) = 0.4$
- $\mu_B(v_1, v_4) = 0.1, \quad \nu_B(v_1, v_4) = 0.6$
- $\mu_B(v_2, v_3) = 0.1, \quad \nu_B(v_2, v_3) = 0.4$
- $\mu_B(v_3, v_4) = 0.1, \quad \nu_B(v_3, v_4) = 0.6$
- $\mu_B(v_2, v_4) = 0.3, \quad \nu_B(v_2, v_4) = 0.6$

This provides an example of how an Intuitionistic Fuzzy Graph can be structured, representing uncertainty in both vertex and edge membership using membership and non-membership degrees.

The following types of graphs are known for Intuitionistic Fuzzy Graphs.

Definition 2.56 (Strong Intuitionistic Fuzzy Graph)**.** [21] An Intuitionistic Fuzzy Graph $G = (A, B)$ is called a *strong intuitionistic fuzzy graph* if for all $xy \in E$:

$$\mu_B(xy) = \min(\mu_A(x), \mu_A(y)) \quad \text{and} \quad \nu_B(xy) = \max(\nu_A(x), \nu_A(y)).$$

Definition 2.57 (Complete Intuitionistic Fuzzy Graph)**.** (cf.[468]) An Intuitionistic Fuzzy Graph $G = (A, B)$ is called *complete* if for all $xy \in E$:

$$\mu_B(xy) = \min(\mu_A(x), \mu_A(y)) \quad \text{and} \quad \nu_B(xy) = \min(\nu_A(x), \nu_A(y)).$$

Proposition 2.58. *Every Intuitionistic Fuzzy Graph can be transformed into a Fuzzy Graph by restricting the non-membership function ν_A to 0 for all vertices.*

Proof. To transform G_{IF} into a fuzzy graph, we set the non-membership functions $\nu_A(x) = 0$ for all vertices $x \in V$, and $\nu_B(x, y) = 0$ for all edges $(x, y) \in E$.

With this restriction, the conditions simplify as follows:

$$\mu_A(x) + \nu_A(x) = \mu_A(x) \leq 1 \quad \text{for all } x \in V,$$

$$\mu_B(x, y) + \nu_B(x, y) = \mu_B(x, y) \leq 1 \quad \text{for all } (x, y) \in E.$$

Thus, G_{IF} reduces to a fuzzy graph $G_F = (\sigma, \mu)$, where:

- $\sigma : V \to [0, 1]$ is the membership function for vertices, corresponding to μ_A,
- $\mu : E \to [0, 1]$ is the membership function for edges, corresponding to μ_B,

and the constraints:

$$\mu(x, y) \leq \sigma(x) \wedge \sigma(y) \quad \text{for all } (x, y) \in E,$$

are satisfied as in the definition of a fuzzy graph. Therefore, by restricting the non-membership function ν_A to 0, we transform the Intuitionistic Fuzzy Graph into a Fuzzy Graph. □

2.4.2 Graph class for Intuitionistic Fuzzy Graph

We consider about Graph class for Intuitionistic Fuzzy Graph.

Theorem 2.59. *The following are examples of related graph classes, including but not limited to:*

- *Strong Intuitionistic Fuzzy Graphs [21]*
- *Perfect intuitionistic fuzzy graphs [188]*
- *Intuitionistic fuzzy competition graphs[388]*
- *Intuitionistic fuzzy threshold graphs[475]*
- *Balanced Intuitionistic Fuzzy Graphs[246]*
- *Bipolar Intuitionistic Fuzzy Competition Graphs[150]*
- *Intuitionistic Felicitous Fuzzy Graphs [94]*
- *Intuitionistic Fuzzy Soft Expert Graphs [464]*
- *Intuitionistic fuzzy tolerance graphs[389]*
- *Intuitionistic fuzzy planar graphs[63]*

- *Edge regular intuitionistic fuzzy graph[247]*
- *Edge Irregular Intuitionistic Fuzzy Graphs[331]*
- *m-Neighbourly Irregular Instuitionistic Fuzzy Graphs[291]*
- *R-edge regular intuitionistic fuzzy graphs[57]*
- *Perfectly Edge-Regular Intuitionistic Fuzzy Graphs[3]*
- *Edge Regular Intuitionistic Fuzzy M-Polargraphs[191]*
- *Intuitionistic fuzzy labeling graphs[386]*
- *Regular Interval-Valued Intuitionistic Fuzzy Graphs[2]*
- *Anti intuitionistic fuzzy graph[227]*
- *Interval-valued intuitionistic fuzzy graphs [268]*
- *Intuitionistic fuzzy directed hypergraphs[328]*
- *Complex intuitionistic fuzzy graphs[72]*
- *Complex t-Intuitionistic Fuzzy Graph[250]*
- *Intuitionistic fuzzy incidence graphs[334]*
- *Intuitionistic fuzzy k-partite hypergraphs[325, 326]*
- *Irregular Intuitionistic Fuzzy Graphs[308]*
- *intuitionistic L-fuzzy graph[446, 453]*
- *Bipolar intuitionistic anti fuzzy graphs[149]*
- *Intuitionistic anti-fuzzy graphs[324]*
- *intuitionistic product fuzzy graphs[456]*
- *intuitionistic k-partitioned fuzzy graph[345]*
- *intutionistic fuzzy multigraphs[85]*

Proof. Refer to each reference as needed. □

2.4.3 Application of intuitionistic fuzzy graph

We introduce the applications of intuitionistic fuzzy Graphs. Intuitionistic fuzzy Graphs hold potential for a wide range of applications. Below, we present an example of such applications.

- water supply systems: Water supply systems are networks that collect, treat, and distribute water from sources like rivers or reservoirs to homes and businesses(cf.[129, 273]). Several papers have explored the use of Intuitionistic fuzzy Graphs in water supply systems[251, 404].
- cellular network: A cellular network is a communication system that uses multiple cell towers to provide wireless connectivity for mobile devices(cf.[482]). Several papers have explored the use of Intuitionistic fuzzy Graphs in cellular network[476].
- COVID-19: Several papers have explored the use of Intuitionistic fuzzy Graphs in COVID-19[451]

2.5 Neutrosophic Graph

We present the concept of a neutrosophic graph [41, 115, 200, 215, 385, 436], which builds upon and expands the theory of fuzzy graphs [237]. Neutrosophic graphs offer a powerful tool for addressing uncertainty, making them particularly useful in diverse domains such as social networks and industrial systems. While fuzzy graph theory has already proven its relevance in modern scientific and technological applications, including operations research, neural networks [4, 283], artificial intelligence [4, 283], and decision-making [237], neutrosophic graphs provide an enhanced framework for more complex scenarios.yaqoob2019complex

The Neutrosophic framework encompasses a wide range of mathematical and logical structures that have been extensively studied and developed in recent years. The primary concepts include:

1. Neutrosophic Set [110, 166, 419, 421, 438, 471]
2. Neutrosophic Offset[179, 418, 423, 431, 432]
3. Neutrosophic Topological Spaces [76, 390]
4. Neutrosophic Logics [78, 375, 419]

5. Neutrosophic Algebraic Structures [47, 242]
6. Neutrosophic Environment [384, 437]
7. NeutroGeometry[430]
8. Neutrosophic Statistics[64, 423]
9. Neutrosophic Physics[420]
10. Neutrosophic Sociology[427]
11. Neutrosophic Automata[184]

The definitions are provided below.

Definition 2.60. [116,436] A neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ is defined as a graph where $\sigma_i : V \to [0, 1]$, $\mu_i : E \to [0, 1]$, and for every $v_i v_j \in E$, the following condition holds: $\mu(v_i v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$.

1. σ is called the neutrosophic vertex set.
2. μ is called the neutrosophic edge set.
3. $|V|$ is called the order of NTG, and it is denoted by $O(NTG)$.
4. $\sum_{v \in V} \sigma(v)$ is called the neutrosophic order of NTG, and it is denoted by $On(NTG)$.
5. $|E|$ is called the size of NTG, and it is denoted by $S(NTG)$.
6. $\sum_{e \in E} \mu(e)$ is called the neutrosophic size of NTG, and it is denoted by $Sn(NTG)$.

The Examples of neutrosophic graph is following.

Example 2.61. (cf.[116]) Consider a neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ with four vertices $V = \{v_1, v_2, v_3, v_4\}$, as shown in the diagram.

The neutrosophic membership degrees of the vertices are as follows:

$$\sigma(v_1) = (0.5, 0.1, 0.4), \quad \sigma(v_2) = (0.6, 0.3, 0.2),$$
$$\sigma(v_3) = (0.2, 0.3, 0.4), \quad \sigma(v_4) = (0.4, 0.2, 0.5)$$

The neutrosophic membership degrees of the edges are as follows:

$$\mu(v_1 v_2) = (0.2, 0.3, 0.4), \quad \mu(v_2 v_3) = (0.3, 0.3, 0.4),$$
$$\mu(v_3 v_4) = (0.2, 0.3, 0.4), \quad \mu(v_4 v_1) = (0.1, 0.2, 0.5)$$

In this case, the neutrosophic graph NTG has the following properties:

- Vertices v_1, v_2, v_3, v_4 are connected by edges with varying neutrosophic membership degrees.
- The neutrosophic relations ensure that for every edge $v_i v_j \in E$, $\mu(v_i v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$, where $\wedge$ denotes the minimum operation.

Proposition 2.62. *(cf.[436]) Every neutrosophic Graph can be transformed into a fuzzy graph.*

Proof. To transform a neutrosophic graph $G_N = (V, E, T, I, F)$ into a fuzzy graph, we reduce the multiple components (truth T, indeterminacy I, and falsity F) into a single membership function.

Define the fuzzy graph $G_F = (V, E, \mu)$ where $\mu : V \times V \to [0, 1]$ is the membership function, given by:

$$\mu(u, v) = T(u, v) - I(u, v) - F(u, v),$$

subject to $\mu(u, v) \in [0, 1]$. This transformation subtracts the indeterminacy and falsity values from the truth value to produce a single fuzzy membership value.

- If $I(u, v) = F(u, v) = 0$, then $\mu(u, v) = T(u, v)$, reflecting a pure truth-based fuzzy graph.

- If $T(u, v) = 0$, then $\mu(u, v) = 0$, meaning no edge exists.
- If $I(u, v) + F(u, v) \geq T(u, v)$, then $\mu(u, v) = 0$, indicating that the edge is too uncertain or false.

Thus, the transformation from a neutrosophic graph to a fuzzy graph is valid. □

Proposition 2.63. *Every Neutrosophic Graph can be transformed into an Intuitionistic Fuzzy Graph by setting the indeterminacy value to zero.*

Proof. Let $G_N = (V, E, T, I, F)$ be a neutrosophic graph, where:

- $T : V \to [0, 1]$ is the truth-membership function for vertices,
- $I : V \to [0, 1]$ is the indeterminacy-membership function for vertices,
- $F : V \to [0, 1]$ is the falsity-membership function for vertices,
- Similarly, $T : E \to [0, 1]$, $I : E \to [0, 1]$, and $F : E \to [0, 1]$ are the truth, indeterminacy, and falsity functions for edges.

To transform G_N into an Intuitionistic Fuzzy Graph $G_{IF} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$, we make the following assignments:

- For vertices:
$$\mu_A(v) = T(v), \quad \nu_A(v) = F(v),$$
where $\mu_A(v)$ represents the membership degree of vertex v, and $\nu_A(v)$ represents the non-membership degree. The indeterminacy value $I(v)$ in the neutrosophic graph is set to zero, implying no uncertainty.
- For edges:
$$\mu_B(e) = T(e), \quad \nu_B(e) = F(e),$$
where $\mu_B(e)$ represents the membership degree of edge e, and $\nu_B(e)$ represents the non-membership degree.

Since the indeterminacy function $I(v)$ and $I(e)$ is set to zero, the hesitation degree (indeterminacy) vanishes, reducing the neutrosophic graph to an Intuitionistic Fuzzy Graph. □

2.5.1 Graph class related to neutrosophic graph

We describe the graph classes related to neutrosophic graphs. Due to the applicability and significance of neutrosophic graphs, many related graph classes have been extensively studied. The definitions are provided below.

Theorem 2.64. *The following are examples of related Neutrosophic graph classes, including but not limited to:*

- *Bipolar Neutrosophic Graphs [39]*
- *Interval-Valued Neutrosophic Graphs [409]*
- *Single-Valued Neutrosophic Graphs [43, 116]*
- *Interval Complex Neutrosophic Graphs [109, 220]*
- *Neutrosophic Hypergraphs [190]*
- *Neutrosophic Vague Line Graphs [220]*
- *Neutrosophic Vague Graphs [223]*
- *Single-Valued Neutrosophic Signed Graphs [306]*
- *Neutrosophic Soft Rough Graphs [33]*
- *Neutrosophic Incidence Graphs [40]*
- *Fermatean Neutrosophic Dombi Fuzzy Graphs [397]*
- *Interval valued pentapartitioned neutrosophic graphs [109]*
- *Single valued pentapartitioned neutrosophic graphs [140]*
- *pentapartitioned neutrosophic graphs[363]*
- *t-Neutrosophic Fuzzy graph[248]*

- *Complex t-Neutrosophic Graph[252]*
- *Regular neutrosophic graphs[215]*
- *Q-neutrosophic soft graphs[462]*
- *Balanced Neutrosophic Graphs[416]*
- *Neutrosophic minimum spanning tree graph [279]*
- *HyperNeutrosophic Graph[177]*
- *SuperHyperNeutrosophic Graph[177]*

Proof. Refer to each reference as needed. □

2.5.2 Application of neutrosophic graph

We introduce the applications of Neutrosophic Graphs. Neutrosophic Graphs, which can generalize fuzzy graphs, hold potential for a wide range of applications. Below, we present an example of such applications.

- Decision-making: Several papers have explored the use of Neutrosophic Graphs in Decision-making[38, 113, 465, 480].
- COVID-19: Several papers have explored the use of Neutrosophic Graphs in COVID-19 countermeasures[363]
- Hospital Infrastructure Design: A paper has explored the use of Neutrosophic Graphs in Hospital Infrastructure Design[62].
- Internet Streaming Services: Internet streaming services deliver digital content like video, music, or live broadcasts to users via the internet in real-time (cf.[474]). The study is conducted in [249].
- Natural disaster: A natural disaster is a catastrophic event caused by natural processes, such as earthquakes, tsunamis, hurricanes, or floods, leading to widespread damage and loss of life (cf.[60, 338]). In the study [58], the potential application of Neutrosophic Graphs in Japan's earthquake response is being explored. Additionally, [407] examines response methods for tsunami management in Japan.
- Networks: Several papers have explored the use of Neutrosophic Graphs in mobile network[117] and wireless networks[114, 205].

2.6 Turiyam Neutrosophic Graph

Research on Turiyam Neutrosophic Graphs, which incorporate parameters into Neutrosophic Graphs, is currently being conducted [185–187]. These graphs are a graphical representation of the Turiyam Neutrosophic Set [410]. Similar concepts include four-valued logic [91, 125]. The definition is provided below.

Definition 2.65 (Turiyam Neutrosophic Graph)**.** [185–187] Let $G = (V, E)$ be a classical graph with a finite set of vertices $V = \{v_i : i = 1, 2, \dots, n\}$ and edges $E = \{(v_i, v_j) : i, j = 1, 2, \dots, n\}$. A *Turiyam Neutrosophic Graph* of G, denoted $G^T = (V^T, E^T)$, is defined as follows:

1. **Turiyam Neutrosophic Vertex Set**: For each vertex $v_i \in V$, the Turiyam Neutrosophic graph assigns the following mappings:
$$t(v_i), iv(v_i), fv(v_i), lv(v_i) : V \to [0, 1],$$
where:
    - $t(v_i)$ is the truth value (tv) of the vertex v_i,
    - $iv(v_i)$ is the indeterminacy value (iv) of v_i,
    - $fv(v_i)$ is the falsity value (fv) of v_i,
    - $lv(v_i)$ is the Turiyam Neutrosophic state (or liberal value) (lv) of v_i,

    for all $v_i \in V$, such that the following condition holds for each vertex:
$$0 \le t(v_i) + iv(v_i) + fv(v_i) + lv(v_i) \le 4.$$

2. **Turiyam Neutrosophic Edge Set**: For each edge $(v_i, v_j) \in E$, the Turiyam Neutrosophic graph assigns the following mappings:

$$t(v_i, v_j), iv(v_i, v_j), fv(v_i, v_j), lv(v_i, v_j) : E \to [0, 1],$$

where:

   - $t(v_i, v_j)$ is the truth value of the edge (v_i, v_j),
   - $iv(v_i, v_j)$ is the indeterminacy value of (v_i, v_j),
   - $fv(v_i, v_j)$ is the falsity value of (v_i, v_j),
   - $lv(v_i, v_j)$ is the Turiyam Neutrosophic state (or liberal value) of (v_i, v_j),

for all $(v_i, v_j) \in E$, such that the following condition holds for each edge:

$$0 \le t(v_i, v_j) + iv(v_i, v_j) + fv(v_i, v_j) + lv(v_i, v_j) \le 4.$$

In this case, V^T represents the Turiyam Neutrosophic vertex set of the graph G^T, and E^T represents the Turiyam Neutrosophic edge set of G^T.

Proposition 2.66. *A Turiyam Neutrosophic graph is a generalization of the fuzzy graph.*

Proof. Let $G = (V, E)$ be a Turiyam Neutrosophic graph, where each vertex and edge is associated with four membership values: truth-membership (t), indeterminacy-membership (i), falsity-membership (f), and Turiyam Neutrosophic state-membership (l).

To reduce a Turiyam Neutrosophic graph to a fuzzy graph, we set the values of the indeterminacy-membership, falsity-membership, and Turiyam Neutrosophic state-membership to zero, keeping only the truth-membership value. This corresponds to the membership function used in a fuzzy graph.

For each vertex $v \in V$ and edge $e \in E$, we set the following:

- Indeterminacy-membership $i(v), i(e) = 0$,
- Falsity-membership $f(v), f(e) = 0$,
- Turiyam Neutrosophic state-membership $l(v), l(e) = 0$.

As a result, only the truth-membership value $t(v)$ for each vertex v and $t(e)$ for each edge e remains. Since a fuzzy graph is characterized by a single membership function μ that assigns values between 0 and 1 to each vertex and edge, this reduced form of the Turiyam Neutrosophic graph is equivalent to a fuzzy graph where:

$$\mu(v) = t(v) \quad \text{and} \quad \mu(e) = t(e).$$

Thus, by eliminating the indeterminacy, falsity, and Turiyam Neutrosophic state values, a Turiyam Neutrosophic graph reduces to a fuzzy graph. Therefore, a Turiyam Neutrosophic graph is a generalization of a fuzzy graph. □

Proposition 2.67. *A Turiyam Neutrosophic graph is a generalization of the neutrosophic graph.*

Proof. Let $G_N = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ be a neutrosophic graph, where:

- σ_1, σ_2, and σ_3 are the truth, indeterminacy, and falsity membership functions on the vertices, respectively, with $\sigma_i : V \to [0, 1]$.
- μ_1, μ_2, and μ_3 are the truth, indeterminacy, and falsity membership functions on the edges, respectively, with $\mu_i : E \to [0, 1]$.

The neutrosophic graph assigns three membership degrees to each vertex and edge, namely truth, indeterminacy, and falsity.

A Turiyam Neutrosophic graph $G^T = (V^T, E^T)$ extends the neutrosophic graph by adding a fourth component, the "Turiyam Neutrosophic state" or liberal value, to both vertices and edges. In a Turiyam Neutrosophic graph, for each vertex $v \in V$ and each edge $e \in E$, the following values are assigned:

$$t(v), iv(v), fv(v), lv(v) \quad \text{for each vertex } v \in V,$$

$$t(e), iv(e), fv(e), lv(e) \quad \text{for each edge } e \in E,$$

where:

- $t(v), iv(v), fv(v)$ correspond to the truth, indeterminacy, and falsity values of the vertex v, matching the components of a neutrosophic graph.
- $lv(v)$ is an additional component, representing the Turiyam Neutrosophic state (liberal value) of the vertex.
- Similarly, $t(e), iv(e), fv(e)$ correspond to the truth, indeterminacy, and falsity values of the edge e, and $lv(e)$ is the Turiyam Neutrosophic state of the edge.

Thus, the Turiyam Neutrosophic graph includes the structure of the neutrosophic graph but adds an extra dimension of information with the lv component, making it a more general structure. The conditions for the vertex set V^T and edge set E^T in the Turiyam Neutrosophic graph are:

$$0 \leq t(v) + iv(v) + fv(v) + lv(v) \leq 4 \quad \text{for each vertex } v,$$

$$0 \leq t(e) + iv(e) + fv(e) + lv(e) \leq 4 \quad \text{for each edge } e.$$

By setting $lv(v) = 0$ and $lv(e) = 0$ for all vertices and edges, the Turiyam Neutrosophic graph reduces to a neutrosophic graph where only the truth, indeterminacy, and falsity components are retained. Therefore, a neutrosophic graph can be viewed as a special case of a Turiyam Neutrosophic graph, where the liberal value is not considered.

This demonstrates that a Turiyam Neutrosophic graph is a generalization of the neutrosophic graph, as it extends the latter by adding an additional component, lv, while retaining the structure of truth, indeterminacy, and falsity. □

2.7 Single-Valued Neutrosophic Graph

A *Single-Valued Neutrosophic Graph* (SVNG) is a generalization of classical graphs, where each vertex and edge is characterized by degrees of truth, indeterminacy, and falsity, taking values from the interval $[0,1]$ [284]. The definition is provided below.

Definition 2.68. [46,244,284] A *Single-Valued Neutrosophic Graph G* is defined as a pair $G = (A, B)$, where:

- $A : V \to [0,1]$ is a single-valued neutrosophic set on the vertex set V,
- $B : V \times V \to [0,1]$ is a single-valued neutrosophic relation on the edge set $E \subseteq V \times V$.

For each pair of vertices $x, y \in V$, the following conditions hold:

$$T_B(xy) \leq \min\{T_A(x), T_A(y)\},$$

$$I_B(xy) \leq \min\{I_A(x), I_A(y)\},$$

$$F_B(xy) \leq \max\{F_A(x), F_A(y)\},$$

where:

- $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the truth-membership, indeterminacy-membership, and falsity-membership values of vertex x,
- $T_B(xy)$, $I_B(xy)$, and $F_B(xy)$ represent the truth-membership, indeterminacy-membership, and falsity-membership values of edge xy.
- **Single-Valued Neutrosophic Vertex Set** A: This defines the neutrosophic membership functions for the vertices. Each vertex $x \in V$ is associated with three membership degrees: truth ($T_A(x)$), indeterminacy ($I_A(x)$), and falsity ($F_A(x)$).
- **Single-Valued Neutrosophic Edge Set** B: This defines the neutrosophic membership functions for the edges. Each edge $xy \in E$ has corresponding neutrosophic membership degrees: truth ($T_B(xy)$), indeterminacy ($I_B(xy)$), and falsity ($F_B(xy)$).
- If the neutrosophic relation B is symmetric, the graph $G = (A, B)$ is considered an undirected single-valued neutrosophic graph.
- If the relation B is not symmetric, the graph $G = (A, B)$ is a directed single-valued neutrosophic graph.

2.7.1 Graph class related to single-valued neutrosophic graph

We introduce related graph classes of the single-valued neutrosophic graph. The following properties hold.

Theorem 2.69. *The following are examples of related Neutrosophic graph classes, including but not limited to:*

- *Single-Valued Neutrosophic Graphs [43, 116]*
- *Single-Valued Neutrosophic Signed Graphs [306]*
- *Single valued neutrosophic signed digraph[415]*
- *Single valued pentapartitioned neutrosophic graphs[140]*
- *Isolated single valued neutrosophic graphs[111]*
- *Bipolar Single Valued Neutrosophic Isolated Graphs[108]*
- *Uniform single valued neutrosophic graphs[112]*
- *Accessible single-valued neutrosophic graphs[206]*
- *single-valued co-neutrosophic graphs[154]*
- *Single-valued neutrosophic line graphs[332]*
- *Regular single valued neutrosophic hypergraphs[298]*
- *KM-single valued neutrosophic metric graphs[207]*
- *antipodal single valued neutrosophic graph[295]*
- *Regular bipolar single valued neutrosophic hypergraphs[297]*
- *Self-Centered Single Valued Neutrosophic Graphs[469]*

2.8 Single-Valued Neutrosophic Hypergraph

A *Single-Valued Neutrosophic Hypergraph* (SVNH) is a generalization of a classical hypergraph, incorporating neutrosophic sets to account for degrees of truth, indeterminacy, and falsity between vertices and edges[30, 31, 43, 296].

Definition 2.70. [43] Let $V = \{v_1, v_2, \ldots, v_n\}$ be a finite set of vertices, and let $E = \{E_1, E_2, \ldots, E_m\}$ be a finite family of single-valued neutrosophic subsets of V. Each hyperedge $E_i \in E$ is represented as:

$$E_i = \{(v_j, T_{E_i}(v_j), I_{E_i}(v_j), F_{E_i}(v_j)) \mid v_j \in V\}$$

where:

- $T_{E_i}(v_j)$ is the truth-membership degree of vertex v_j in hyperedge E_i,
- $I_{E_i}(v_j)$ is the indeterminacy-membership degree of vertex v_j in hyperedge E_i,
- $F_{E_i}(v_j)$ is the falsity-membership degree of vertex v_j in hyperedge E_i.

The pair $H = (V, E)$ is called a *single-valued neutrosophic hypergraph*, where:

- V is the set of vertices, and
- E is the family of single-valued neutrosophic hyperedges.

- **Adjacency of Vertices:** Two vertices u and v are adjacent if they belong to the same hyperedge E_i, i.e., $u, v \in \text{supp}(E_i)$.
- **Connectivity:** A single-valued neutrosophic hypergraph is connected if every pair of vertices can be connected by a sequence of adjacent vertices.
- **Adjacency of Edges:** Two hyperedges E_i and E_j are adjacent if they share at least one common vertex, i.e., $\text{supp}(E_i) \cap \text{supp}(E_j) \neq \emptyset$.
- **Order and Size:**
  - The *order* of the hypergraph is $|V|$, the number of vertices.
  - The *size* of the hypergraph is $|E|$, the number of hyperedges.

- **Cardinality of a Hyperedge:** The degree of a hyperedge E_i is the sum of the truth-membership, indeterminacy-membership, and falsity-membership values of the vertices in that hyperedge:

$$d_H(E_i) = \sum_{v_j \in E_i} \left(T_{E_i}(v_j) + I_{E_i}(v_j) + F_{E_i}(v_j)\right)$$

- **Rank and Anti-Rank:**
  - The *rank* of the hypergraph is the maximum cardinality of any hyperedge: $\max_{E_i \in E} d_H(E_i)$.
  - The *anti-rank* of the hypergraph is the minimum cardinality of any hyperedge: $\min_{E_i \in E} d_H(E_i)$.
- **Linear Single-Valued Neutrosophic Hypergraph:** The hypergraph is called *linear* if any two distinct vertices belong to at most one common hyperedge, i.e.,

$$|\text{supp}(E_i) \cap \text{supp}(E_j)| \leq 1 \quad \text{for all } E_i, E_j \in E.$$

- **Membership Constraint:** For each vertex v_j in any hyperedge E_i, the following constraint must hold:

$$0 \leq T_{E_i}(v_j) + I_{E_i}(v_j) + F_{E_i}(v_j) \leq 3$$

  This ensures that the sum of the truth, indeterminacy, and falsity values for any vertex in a hyperedge is valid.
- **Linear Hypergraphs:** The linear property ensures that any pair of vertices belongs to at most one common hyperedge, which preserves the structure of the hypergraph.

Example 2.71. Consider a single-valued neutrosophic hypergraph $H = (V, E)$ such that:

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}, \quad E = \{E_1, E_2, E_3, E_4, E_5, E_6\},$$

where:

$$E_1 = \{(v_1, 0.3, 0.4, 0.6), (v_3, 0.7, 0.4, 0.4)\},$$
$$E_2 = \{(v_1, 0.3, 0.4, 0.6), (v_2, 0.5, 0.7, 0.6)\},$$
$$E_3 = \{(v_2, 0.5, 0.7, 0.6), (v_4, 0.6, 0.4, 0.8)\},$$
$$E_4 = \{(v_3, 0.7, 0.4, 0.4), (v_6, 0.4, 0.2, 0.7)\},$$
$$E_5 = \{(v_3, 0.7, 0.4, 0.4), (v_5, 0.6, 0.7, 0.5)\},$$
$$E_6 = \{(v_5, 0.6, 0.7, 0.5), (v_6, 0.4, 0.2, 0.7)\},$$

and:

$$E_7 = \{(v_4, 0.6, 0.4, 0.8), (v_6, 0.4, 0.2, 0.7)\}.$$

2.9 Known Result

Based on the previous discussions, the following theorem can be understood.

Theorem 2.72. *In each graph class, the following relationships hold.*

- *An empty graph and a null graph can be represented as 2-valued graphs and 3-valued graphs.*
- *Every edge-fuzzy graph can be transformed into a 2-valued graph by thresholding the edge membership values.*
- *Every fuzzy graph can be transformed into a 3-valued graph by mapping the fuzzy membership values of vertices and edges to the values {-1, 0, 1}.*
- *Every Intuitionistic Fuzzy Graph can be transformed into a Fuzzy Graph by restricting the non-membership function ν_A to 0 for all vertices.*
- *Every Neutrosophic Graph can be transformed into an Intuitionistic Fuzzy Graph by setting the indeterminacy value to zero.*

3. New graph class definition and Result of this paper

The results in this paper are presented as follows.

3.1 General intuitionic fuzzy graph

Following the examples of the General Fuzzy Graph and General Neutrosophic Graph [173, 336], we introduce the definition of the General Intuitionistic Fuzzy Graph. The definition is provided below.

Definition 3.1 (General Intuitionistic Fuzzy Graph)**.** A *general intuitionistic fuzzy graph* with V as the underlying set is defined as a pair of functions $G_{IF} = (\mu_A, \nu_A, \mu_B, \nu_B)$, where:

- $\mu_A : V \to [0, 1]$ is an intuitionistic fuzzy subset of V representing the membership degree of vertices, and $\nu_A : V \to [0, 1]$ represents the non-membership degree of vertices.
- $\mu_B : V \times V \to [0, 1]$ is an intuitionistic fuzzy relation on $V \times V$ representing the membership degree of edges, and $\nu_B : V \times V \to [0, 1]$ represents the non-membership degree of edges.

This definition generalizes the traditional intuitionistic fuzzy graph by allowing the membership and non-membership values of edges to be independent of the membership and non-membership values of their incident vertices.

Definition 3.2 (Weak General Intuitionistic Fuzzy Graph)**.** A *weak general intuitionistic fuzzy graph* with V as the underlying set is defined as a pair of functions $G_{IF} = (\mu_A, \nu_A, \mu_B, \nu_B)$, where:

- $\mu_A : V \to [0, 1]$ is an intuitionistic fuzzy subset of V representing the membership degree of vertices, and $\nu_A : V \to [0, 1]$ represents the non-membership degree of vertices.
- $\mu_B : V \times V \to [0, 1]$ is an intuitionistic fuzzy relation on $V \times V$ representing the membership degree of edges, and $\nu_B : V \times V \to [0, 1]$ represents the non-membership degree of edges.

Additionally, for all $(u, v) \in E$, the membership degree of the edge $\mu_B(u, v)$ satisfies:

$$\mu_B(u, v) \neq \mu_A(u) \wedge \mu_A(v),$$

where $\wedge$ denotes the minimum operation.

This condition implies that the membership degree of an edge is not strictly determined by the membership degrees of its vertices.

The relationships with other graph classes are described below.

Theorem 3.3. *Every intuitionistic fuzzy graph is a weak general intuitionistic fuzzy graph.*

Proof. Let $G_{IF} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$ be an intuitionistic fuzzy graph. In an intuitionistic fuzzy graph, for each edge $(u, v) \in E$, the membership degree $\mu_B(u, v)$ is given by:

$$\mu_B(u, v) = \mu_A(u) \wedge \mu_A(v),$$

where $\wedge$ denotes the minimum operation.

In a weak general intuitionistic fuzzy graph, the condition required is $\mu_B(u, v) \neq \mu_A(u) \wedge \mu_A(v)$, but this does not prevent $\mu_B(u, v)$ from satisfying $\mu_B(u, v) = \mu_A(u) \wedge \mu_A(v)$ in specific cases. Thus, every intuitionistic fuzzy graph meets the weaker condition required by a weak general intuitionistic fuzzy graph.

Hence, every intuitionistic fuzzy graph is a weak general intuitionistic fuzzy graph. □

Theorem 3.4. *Every weak general intuitionistic fuzzy graph is a general intuitionistic fuzzy graph.*

Proof. By the definition of a weak general intuitionistic fuzzy graph, for all vertices $u, v \in V$ and edges $(u, v) \in E$, the following condition holds:

$$\mu_B(u, v) \neq \mu_A(u) \wedge \mu_A(v),$$

where $\wedge$ denotes the minimum operation.

Now, consider the definition of a general intuitionistic fuzzy graph. A general intuitionistic fuzzy graph $G = (\mu_A, \nu_A, \mu_B, \nu_B)$ allows the membership values of the edges $\mu_B(u, v)$ to take any value independently of the membership values of the vertices.

Since the condition $\mu_B(u, v) \neq \mu_A(u) \wedge \mu_A(v)$ is less restrictive than requiring $\mu_B(u, v) = \mu_A(u) \wedge \mu_A(v)$, any weak general intuitionistic fuzzy graph satisfies the conditions of a general intuitionistic fuzzy graph.

Thus, every weak general intuitionistic fuzzy graph is a general intuitionistic fuzzy graph. □

Theorem 3.5. *Every weak general intuitionistic fuzzy graph can be transformed into a weak general fuzzy graph by setting the non-membership functions to zero.*

Proof. Let $G_{IF} = (\mu_A, \nu_A, \mu_B, \nu_B)$ be a weak general intuitionistic fuzzy graph. To transform G_{IF} into a weak general fuzzy graph, set $\nu_A(v) = 0$ and $\nu_B(u, v) = 0$ for all vertices $v \in V$ and edges $(u, v) \in E$.

This results in a weak general fuzzy graph $G_F = (\sigma, \mu)$ where:

$$\sigma(v) = \mu_A(v) \quad \text{and} \quad \mu(u, v) = \mu_B(u, v).$$

The condition $\mu_B(u, v) \neq \mu_A(u) \wedge \mu_A(v)$ in the weak general intuitionistic fuzzy graph is preserved as $\mu(u, v) \neq \sigma(u) \wedge \sigma(v)$ in the weak general fuzzy graph.

Thus, every weak general intuitionistic fuzzy graph can be transformed into a weak general fuzzy graph. □

Corollary 3.6. *Every general intuitionistic fuzzy graph can be transformed into a general fuzzy graph by setting the non-membership functions to zero.*

Proof. It can be proven in the same way as above. □

3.2 Single-Valued Turiyam Neutrosophic Graph

A *Single-Valued Turiyam Neutrosophic Graph* (SVTG) is a generalization of classical graphs and Single-Valued Neutrosophic Graphs, where each vertex and edge is characterized by four degrees: truth, indeterminacy, falsity, and Turiyam Neutrosophic (liberal) state, taking values from the interval $[0, 1]$. Note that a Single-Valued Turiyam Neutrosophic Graph can be generalized within the framework of a Single-Valued Quadripartitioned Neutrosophic Graph (cf. [417])

Definition 3.7. A *Single-Valued Turiyam Neutrosophic Graph* G is defined as a pair $G = (A, B)$, where:

- $A : V \to [0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$ is a single-valued Turiyam Neutrosophic set on the vertex set V, assigning four degrees to each vertex: truth $t(v)$, indeterminacy $iv(v)$, falsity $f(v)$, and Turiyam Neutrosophic state $lv(v)$.
- $B : V \times V \to [0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$ is a single-valued Turiyam Neutrosophic relation on the edge set $E \subseteq V \times V$, assigning four degrees to each edge: truth $t(v_i, v_j)$, indeterminacy $iv(v_i, v_j)$, falsity $f(v_i, v_j)$, and Turiyam Neutrosophic state $lv(v_i, v_j)$.

  For each pair of vertices $v_i, v_j \in V$, the following conditions must hold:

$$t(v_i, v_j) \leq \min\{t(v_i), t(v_j)\},$$

$$iv(v_i, v_j) \leq \min\{iv(v_i), iv(v_j)\},$$

$$f(v_i, v_j) \leq \max\{f(v_i), f(v_j)\},$$

$$lv(v_i, v_j) \leq \min\{lv(v_i), lv(v_j)\}.$$

In addition, for any vertex $v \in V$, the following constraint applies:

$$0 \leq t(v) + iv(v) + f(v) + lv(v) \leq 4.$$

For any edge $(v_i, v_j) \in E$, the corresponding constraint is:

$$0 \leq t(v_i, v_j) + iv(v_i, v_j) + f(v_i, v_j) + lv(v_i, v_j) \leq 4.$$

Example 3.8. Consider a graph G with three vertices V_1, V_2, V_3 and three edges connecting them. Each vertex and edge is assigned four values representing the truth (t), indeterminacy (iv), falsity (f), and Turiyam Neutrosophic state (lv) values.

Let the vertex set $V = \{V_1, V_2, V_3\}$ be defined as follows:

$$V_1 = (t(V_1), iv(V_1), f(V_1), lv(V_1)) = (0.3, 0.4, 0.2, 0.5)$$

$$V_2 = (t(V_2), iv(V_2), f(V_2), lv(V_2)) = (0.2, 0.3, 0.3, 0.4)$$

$$V_3 = (t(V_3), iv(V_3), f(V_3), lv(V_3)) = (0.4, 0.4, 0.3, 0.4)$$

The edges connecting these vertices are given as:

$$(V_1, V_2) = (t(V_1, V_2), iv(V_1, V_2), f(V_1, V_2), lv(V_1, V_2)) = (0.2, 0.2, 0.3, 0.3)$$

$$(V_2, V_3) = (t(V_2, V_3), iv(V_2, V_3), f(V_2, V_3), lv(V_2, V_3)) = (0.3, 0.2, 0.2, 0.3)$$

$$(V_1, V_3) = (t(V_1, V_3), iv(V_1, V_3), f(V_1, V_3), lv(V_1, V_3)) = (0.3, 0.3, 0.2, 0.2)$$

Each vertex has been assigned values for truth, indeterminacy, falsity, and Turiyam Neutrosophic states, which satisfy the constraint:

$$0 \leq t(v) + iv(v) + f(v) + lv(v) \leq 4 \quad \forall v \in \{V_1, V_2, V_3\}.$$

- For V_1, we have $0.3 + 0.4 + 0.2 + 0.5 = 1.4$.
- For V_2, $0.2 + 0.3 + 0.3 + 0.4 = 1.2$.
- For V_3, $0.4 + 0.4 + 0.3 + 0.4 = 1.5$.

These satisfy the constraint for the vertex membership condition.

Similarly, each edge has four values assigned:

$$0 \leq t(v_i, v_j) + iv(v_i, v_j) + f(v_i, v_j) + lv(v_i, v_j) \leq 4 \quad \forall (v_i, v_j) \in \{(V_1, V_2), (V_2, V_3), (V_1, V_3)\}.$$

- For (V_1, V_2), we have $0.2 + 0.2 + 0.3 + 0.3 = 1.0$.
- For (V_2, V_3), $0.3 + 0.2 + 0.2 + 0.3 = 1.0$.
- For (V_1, V_3), $0.3 + 0.3 + 0.2 + 0.2 = 1.0$.

These satisfy the edge membership condition.

Theorem 3.9. *Every Single-Valued Turiyam Neutrosophic Graph $G^T = (V^T, E^T)$, where the vertices and edges have associated truth (t), indeterminacy (iv), falsity (f), and Turiyam Neutrosophic (lv) values, can be transformed into a Single-Valued Neutrosophic Graph $G^N = (V^N, E^N)$ by merging the Turiyam Neutrosophic value lv into the truth, indeterminacy, and falsity components of the graph, while maintaining the membership constraints $t + iv + f \leq 3$.*

Proof. For each vertex $v \in V^T$, and each edge $e \in E^T$, redistribute the Turiyam Neutrosophic value lv as follows:

$$t'(v) = t(v) + \alpha \cdot lv(v),$$
$$iv'(v) = iv(v) + \beta \cdot lv(v),$$
$$f'(v) = f(v) + \gamma \cdot lv(v),$$

where $\alpha, \beta, \gamma \in [0, 1]$ are constants such that $\alpha + \beta + \gamma = 1$. These constants determine how the Turiyam Neutrosophic value is distributed between the truth, indeterminacy, and falsity components.

After redistributing the Turiyam Neutrosophic value, the new membership values for vertices and edges must satisfy the Single-Valued Neutrosophic constraint:

$$0 \leq t'(v) + iv'(v) + f'(v) \leq 3.$$

By construction, the redistribution satisfies this condition because:

$$t'(v) + iv'(v) + f'(v) = (t(v) + \alpha \cdot lv(v)) + (iv(v) + \beta \cdot lv(v)) + (f(v) + \gamma \cdot lv(v)).$$

Simplifying:

$$t'(v) + iv'(v) + f'(v) = t(v) + iv(v) + f(v) + (\alpha + \beta + \gamma) \cdot lv(v).$$

Since $\alpha + \beta + \gamma = 1$, we have:

$$t'(v) + iv'(v) + f'(v) = t(v) + iv(v) + f(v) + lv(v).$$

From the original Turiyam Neutrosophic condition, we know that:

$$0 \leq t(v) + iv(v) + f(v) + lv(v) \leq 4.$$

Thus, after removing the Turiyam Neutrosophic component $lv(v)$, we have:

$$0 \leq t'(v) + iv'(v) + f'(v) \leq 3.$$

The same process applies to edges, ensuring that for each edge $e \in E^T$:

$$0 \leq t'(e) + iv'(e) + f'(e) \leq 3.$$

The result of the redistribution of the Turiyam Neutrosophic value is that the modified graph $G^N = (V^N, E^N)$ satisfies the conditions for a Single-Valued Neutrosophic Graph, with no Turiyam Neutrosophic values remaining.

Thus, the Single-Valued Turiyam Neutrosophic Graph G^T has been successfully transformed into a Single-Valued Neutrosophic Graph G^N, where:

- $t'(v), iv'(v), f'(v)$ are the new truth, indeterminacy, and falsity values for each vertex v,
- $t'(e), iv'(e), f'(e)$ are the new truth, indeterminacy, and falsity values for each edge e, and the constraints $t' + iv' + f' \leq 3$ are satisfied for both vertices and edges.

□

3.3 General Turiyam Neutrosophic Graph

Next, we consider the General Turiyam Neutrosophic Graph. Similar to how a General Fuzzy Graph is a generalization of a Fuzzy Graph, a General Turiyam Neutrosophic Graph is a generalization of a Turiyam Neutrosophic Graph. The definition is provided below.

Definition 3.10 (General Turiyam Neutrosophic Graph)**.** Let $G = (V, E)$ be a classical graph with vertex set V and edge set E. A *General Turiyam Neutrosophic Graph* $G^T = (V^T, E^T)$ is defined as follows:

- **Turiyam Neutrosophic Vertex Set**: For each vertex $v \in V$, assign four membership degrees:

  $$t(v), \quad iv(v), \quad f(v), \quad lv(v) \in [0, 1],$$

  representing the truth, indeterminacy, falsity, and Turiyam Neutrosophic (liberal) membership degrees of the vertex v, respectively, satisfying:

  $$0 \leq t(v) + iv(v) + f(v) + lv(v) \leq 4.$$

- **Turiyam Neutrosophic Edge Set**: For each edge $e = (u, v) \in E$, assign four membership degrees:

  $$t(e), \quad iv(e), \quad f(e), \quad lv(e) \in [0, 1],$$

  representing the truth, indeterminacy, falsity, and Turiyam Neutrosophic (liberal) membership degrees of the edge e, respectively, satisfying:

  $$0 \leq t(e) + iv(e) + f(e) + lv(e) \leq 4.$$

In a General Turiyam Neutrosophic Graph, the edge membership degrees are independent of the vertex membership degrees. That is, there is no requirement relating $t(e), iv(e), f(e), lv(e)$ to $t(u), iv(u), f(u), lv(u)$ and $t(v), iv(v), f(v), lv(v)$.

Definition 3.11 (Weak General Turiyam Neutrosophic Graph)**.** A *Weak General Turiyam Neutrosophic Graph* $G^T = (V^T, E^T)$ is defined similarly to the General Turiyam Neutrosophic Graph, with the following additional condition:

For each edge $e = (u, v) \in E$, the edge membership degrees satisfy:

$$t(e) \leq t(u) \wedge t(v), \quad iv(e) \leq iv(u) \wedge iv(v), \quad f(e) \geq f(u) \vee f(v), \quad lv(e) \leq lv(u) \wedge lv(v),$$

where $\wedge$ denotes the minimum operation and $\vee$ denotes the maximum operation.

This condition implies that the edge membership degrees are related to the vertex membership degrees, but not necessarily equal to any specific function of them.

Based on the above definition, we will explore the relationships with other graph classes. The following theorem holds.

Theorem 3.12. *Every Turiyam Neutrosophic Graph is a Weak General Turiyam Neutrosophic Graph.*

Proof. In a Turiyam Neutrosophic Graph, the edge membership degrees satisfy the following conditions for each edge $e = (u, v) \in E$:

$$t(e) = t(u) \wedge t(v), \quad iv(e) = iv(u) \wedge iv(v), \quad f(e) = f(u) \vee f(v), \quad lv(e) = lv(u) \wedge lv(v).$$

These conditions are stronger than those required in a Weak General Turiyam Neutrosophic Graph, where the edge membership degrees only need to satisfy:

$$t(e) \leq t(u) \wedge t(v), \quad iv(e) \leq iv(u) \wedge iv(v), \quad f(e) \geq f(u) \vee f(v), \quad lv(e) \leq lv(u) \wedge lv(v).$$

Therefore, every Turiyam Neutrosophic Graph satisfies the properties of a Weak General Turiyam Neutrosophic Graph. □

Theorem 3.13. *Every Weak General Turiyam Neutrosophic Graph is a General Turiyam Neutrosophic Graph.*

Proof. By definition, a Weak General Turiyam Neutrosophic Graph assigns membership degrees to vertices and edges, satisfying the constraints of the General Turiyam Neutrosophic Graph. The additional conditions relating edge and vertex membership degrees in a Weak General Turiyam Neutrosophic Graph do not contradict the General Turiyam Neutrosophic Graph's definition, which allows for edge membership degrees independent of vertex degrees. Therefore, every Weak General Turiyam Neutrosophic Graph is also a General Turiyam Neutrosophic Graph. □

Theorem 3.14. *A Turiyam Neutrosophic Graph is a special case of a General Turiyam Neutrosophic Graph where the edge membership degrees are constrained by the vertex membership degrees.*

Proof. In a Turiyam Neutrosophic Graph, the edge membership degrees are directly related to the vertex membership degrees through specific equalities:

$$t(e) = t(u) \wedge t(v), \quad iv(e) = iv(u) \wedge iv(v), \quad f(e) = f(u) \vee f(v), \quad lv(e) = lv(u) \wedge lv(v).$$

In a General Turiyam Neutrosophic Graph, there are no such constraints; the edge membership degrees are independent of the vertex membership degrees. Therefore, a Turiyam Neutrosophic Graph can be viewed as a General Turiyam Neutrosophic Graph with additional constraints, making it a special case. □

Definition 3.15. Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be two General Turiyam Neutrosophic Graphs with underlying crisp graphs $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$. The union $G = (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ is defined as follows:

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) & \text{if } u \in V_1 - V_2, \\ \sigma_2(u) & \text{if } u \in V_2 - V_1, \\ \max\{\sigma_1(u), \sigma_2(u)\} & \text{if } u \in V_1 \cap V_2. \end{cases}$$

$$(\mu_1 \cup \mu_2)(u, v) = \begin{cases} \mu_1(u, v) & \text{if } (u, v) \in X_1 - X_2, \\ \mu_2(u, v) & \text{if } (u, v) \in X_2 - X_1, \\ \max\{\mu_1(u, v), \mu_2(u, v)\} & \text{if } (u, v) \in X_1 \cap X_2. \end{cases}$$

The c-complement of a General Turiyam Neutrosophic Graph $G = (\sigma, \mu)$ is defined as:

$$\sigma^c = \sigma, \quad \mu^c(u, v) = \begin{cases} 0 & \text{if } \mu(u, v) = 0, \\ 1 - \mu(u, v) & \text{if } \mu(u, v) > 0. \end{cases}$$

The $|\mu|$-complement of a General Turiyam Neutrosophic Graph $G = (\sigma, \mu)$ is defined as:

$$\mu^{|\mu|}(u, v) = \begin{cases} 0 & \text{if } \mu(u, v) = 0, \\ |\sigma(u) \wedge \sigma(v) - \mu(u, v)| & \text{if } \mu(u, v) > 0. \end{cases}$$

Theorem 3.16. *General Turiyam Neutrosophic Graphs are closed under c-complement and $|\mu|$-complement.*

Proof. Based on the definitions of c-complement and $|\mu|$-complement, both operations preserve the conditions that the sum of the truth, indeterminacy, falsity, and liberal values for each vertex and edge does not exceed 4. Therefore, General Turiyam Neutrosophic Graphs remain closed under these operations. □

Theorem 3.17. *The union of two Weak General Turiyam Neutrosophic Graphs is not necessarily a Weak General Turiyam Neutrosophic Graph.*

Proof. Consider two Weak General Turiyam Neutrosophic Graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$. When taking the union $G = (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$, the relationships between edge membership values and vertex membership values may not be preserved, breaking the conditions required for a Weak General Turiyam Neutrosophic Graph. This can occur in cases where the edge values do not correspond to the minimum or maximum of the associated vertex values. □

3.4 Single-Valued Turiyam Neutrosophic Hypergraph

A *Single-Valued Turiyam Neutrosophic Hypergraph* (SVTH) is a generalization of classical hypergraphs and Single-Valued Neutrosophic Hypergraphs, where each vertex and hyperedge is characterized by four degrees: truth, indeterminacy, falsity, and Turiyam Neutrosophic (liberal) state, taking values from the interval $[0, 1]$.

Definition 3.18. A *Single-Valued Turiyam Neutrosophic Hypergraph $H = (V, E)$* is defined as follows:

- $V = \{v_1, v_2, \ldots, v_n\}$ is a finite set of vertices.
- $E = \{E_1, E_2, \ldots, E_m\}$ is a finite family of Turiyam Neutrosophic subsets of V, where each hyperedge E_i is represented as:
$$E_i = \left\{(v_j, t_{E_i}(v_j), iv_{E_i}(v_j), f_{E_i}(v_j), lv_{E_i}(v_j)) \mid v_j \in V\right\}.$$
  For each vertex $v_j \in V$ in a hyperedge E_i, the following four values are assigned:
  - $t_{E_i}(v_j)$ is the truth-membership degree of vertex v_j in hyperedge E_i,
  - $iv_{E_i}(v_j)$ is the indeterminacy-membership degree of vertex v_j in hyperedge E_i,
  - $f_{E_i}(v_j)$ is the falsity-membership degree of vertex v_j in hyperedge E_i,
  - $lv_{E_i}(v_j)$ is the Turiyam Neutrosophic state (liberal value) of vertex v_j in hyperedge E_i.

  The following condition holds for each vertex $v_j \in V$ within a hyperedge E_i:
$$0 \leq t_{E_i}(v_j) + iv_{E_i}(v_j) + f_{E_i}(v_j) + lv_{E_i}(v_j) \leq 4.$$
- **Turiyam Neutrosophic Vertex Set:** The set V represents the vertices of the hypergraph. Each vertex $v_j \in V$ has associated degrees of truth, indeterminacy, falsity, and Turiyam Neutrosophic state within each hyperedge it belongs to.
- **Turiyam Neutrosophic Hyperedge Set:** The set E represents the hyperedges of the hypergraph. Each hyperedge E_i is composed of vertices and their corresponding Turiyam Neutrosophic values: truth $t_{E_i}(v_j)$, indeterminacy $iv_{E_i}(v_j)$, falsity $f_{E_i}(v_j)$, and Turiyam Neutrosophic state $lv_{E_i}(v_j)$.
- **Adjacency of Vertices:** Two vertices u and v are adjacent if they belong to the same hyperedge E_i, i.e., $u, v \in E_i$.
- **Connectivity:** A Single-Valued Turiyam Neutrosophic Hypergraph is connected if every pair of vertices can be connected by a sequence of adjacent vertices.
- **Adjacency of Hyperedges:** Two hyperedges E_i and E_j are adjacent if they share at least one common vertex, i.e., $E_i \cap E_j \neq \emptyset$.
- **Order and Size:**
  - The *order* of the hypergraph is $|V|$, the number of vertices.
  - The *size* of the hypergraph is $|E|$, the number of hyperedges.

- **Cardinality of a Hyperedge:** The degree of a hyperedge E_i is the sum of the truth-membership, indeterminacy-membership, falsity-membership, and Turiyam Neutrosophic state values of the vertices in that hyperedge:

$$d_H(E_i) = \sum_{v_j \in E_i} \left(t_{E_i}(v_j) + iv_{E_i}(v_j) + f_{E_i}(v_j) + lv_{E_i}(v_j) \right).$$

- **Rank and Anti-Rank:**
  - The *rank* of the hypergraph is the maximum cardinality of any hyperedge: $\max_{E_i \in E} d_H(E_i)$.
  - The *anti-rank* of the hypergraph is the minimum cardinality of any hyperedge: $\min_{E_i \in E} d_H(E_i)$.
- **Linear Single-Valued Turiyam Neutrosophic Hypergraph:** A Turiyam Neutrosophic hypergraph is called *linear* if any two distinct vertices belong to at most one common hyperedge, i.e.,

$$|E_i \cap E_j| \leq 1 \quad \text{for all } E_i, E_j \in E.$$

  For each vertex v_j in any hyperedge E_i, the following constraint must hold:

$$0 \leq t_{E_i}(v_j) + iv_{E_i}(v_j) + f_{E_i}(v_j) + lv_{E_i}(v_j) \leq 4.$$

This ensures that the sum of the truth, indeterminacy, falsity, and Turiyam Neutrosophic values for any vertex in a hyperedge is valid.

Example 3.19. Consider a Single-Valued Turiyam Neutrosophic Hypergraph $H = (V, E)$ where the vertex set is:

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and the hyperedge set is:

$$E = \{E_1, E_2, E_3, E_4, E_5, E_6\}$$

The hyperedges are defined as follows:

$$E_1 = \{(v_1, 0.3, 0.4, 0.2, 0.3), (v_3, 0.4, 0.3, 0.2, 0.4)\}$$
$$E_2 = \{(v_1, 0.3, 0.4, 0.2, 0.3), (v_2, 0.5, 0.6, 0.3, 0.2)\}$$
$$E_3 = \{(v_2, 0.5, 0.6, 0.3, 0.2), (v_4, 0.6, 0.4, 0.3, 0.5)\}$$
$$E_4 = \{(v_3, 0.4, 0.3, 0.2, 0.4), (v_6, 0.3, 0.4, 0.2, 0.5)\}$$
$$E_5 = \{(v_3, 0.4, 0.3, 0.2, 0.4), (v_5, 0.6, 0.5, 0.3, 0.2)\}$$
$$E_6 = \{(v_5, 0.6, 0.5, 0.3, 0.2), (v_6, 0.3, 0.4, 0.2, 0.5)\}$$

Each vertex is associated with four membership values: truth (t), indeterminacy (iv), falsity (f), and Turiyam Neutrosophic state (lv). The following conditions for each vertex in every hyperedge are satisfied:

$$0 \leq t(v) + iv(v) + f(v) + lv(v) \leq 4$$

Let's verify this for each vertex:

For hyperedge E_1:

- For v_1, we have $0.3 + 0.4 + 0.2 + 0.3 = 1.2$ (valid).
- For v_3, we have $0.4 + 0.3 + 0.2 + 0.4 = 1.3$ (valid).

For hyperedge E_2:

- For v_1, we have $0.3 + 0.4 + 0.2 + 0.3 = 1.2$ (valid).
- For v_2, we have $0.5 + 0.6 + 0.3 + 0.2 = 1.6$ (valid).

For hyperedge E_3:

- For v_2, we have $0.5 + 0.6 + 0.3 + 0.2 = 1.6$ (valid).
- For v_4, we have $0.6 + 0.4 + 0.3 + 0.5 = 1.8$ (valid).

For hyperedge E_4:

- For v_3, we have $0.4 + 0.3 + 0.2 + 0.4 = 1.3$ (valid).
- For v_6, we have $0.3 + 0.4 + 0.2 + 0.5 = 1.4$ (valid).

For hyperedge E_5:

- For v_3, we have $0.4 + 0.3 + 0.2 + 0.4 = 1.3$ (valid).
- For v_5, we have $0.6 + 0.5 + 0.3 + 0.2 = 1.6$ (valid).

For hyperedge E_6:

- For v_5, we have $0.6 + 0.5 + 0.3 + 0.2 = 1.6$ (valid).
- For v_6, we have $0.3 + 0.4 + 0.2 + 0.5 = 1.4$ (valid).

Thus, all conditions for a Single-Valued Turiyam Neutrosophic Hypergraph are satisfied.

Theorem 3.20. *Every Single-Valued Turiyam Neutrosophic Hypergraph (SVTH) can be transformed into a Single-Valued Neutrosophic Hypergraph (SVNH) by merging the Turiyam Neutrosophic state into the truth, indeterminacy, and falsity values, while maintaining the neutrosophic membership constraints.*

Proof. The proof can be conducted in the same manner as for Single-Valued Turiyam Neutrosophic graphs. □

Corollary 3.21. *Every Linear Single-Valued Turiyam Neutrosophic Hypergraph (SVTH) can be transformed into a Linear Single-Valued Neutrosophic Hypergraph (SVNH) by merging the Turiyam Neutrosophic state into the truth, indeterminacy, and falsity values, while maintaining the neutrosophic membership constraints.*

Proof. The proof can be conducted in the same manner as for Single-Valued Turiyam Neutrosophic graphs. □

3.5 Ambiguous Graph

The concept of Ambiguous sets, as outlined below, has been a subject of research in recent years, with several papers exploring this topic [408, 413, 414]. We now define an Ambiguous Graph by extending this concept to graphs.

It is worth noting that both Ambiguous Graphs and Turiyam Neutrosophic Graphs can be generalized within the framework of Quadripartitioned Neutrosophic Graphs (cf. [417]).

Definition 3.22 (Ambiguous set)**.** [408, 413, 414] Let $H = \{h\}$ be a fixed finite non-empty set of any event h. An AS X for $h \in H$ is expressed as:

$$X = \{\langle h, t_X(h), f_X(h), ta_X(h), fa_X(h)\rangle \mid h \in H\},$$

where t_X, f_X, ta_X, and fa_X denote the true membership function, false membership function, true-ambiguous membership function, and false-ambiguous membership function, respectively.

Here,

$$t_X(h)\colon H \to]-0,1[, \quad f_X(h)\colon H \to]-0,1[, \quad ta_X(h)\colon H \to]-0,1[, \quad \text{and} \quad fa_X(h)\colon H \to]-0,1[$$

are called the True Membership Degree (TMD), False Membership Degree (FMD), True-Ambiguous Membership Degree (TAMD), and False-Ambiguous Membership Degree (FAMD) of $h \in H$ in X, respectively.

These membership functions must satisfy the condition:

$$-0 \leq t_X(h) + f_X(h) + ta_X(h) + fa_X(h) \leq 2^+$$

for all $h \in H$. Here, t_X, f_X, ta_X, and fa_X are real standard values or non-standard subsets of $]-0,1[$.

Definition 3.23 (Ambiguous Graph)**.** Let $G = (V, E)$ be a classical graph with a finite set of vertices $V = \{v_i : i = 1, 2, \ldots, n\}$ and edges $E \subseteq V \times V$. An **Ambiguous Graph** $G^A = (V^A, E^A)$ is defined as follows:

1. **Ambiguous Vertex Set**: For each vertex $v_i \in V$, the ambiguous graph assigns the following membership functions:
$$t_V(v_i), \quad f_V(v_i), \quad ta_V(v_i), \quad fa_V(v_i) : V \to]-0,1[,$$
where:

- $t_V(v_i)$ is the true membership degree (TMD) of vertex v_i,
- $f_V(v_i)$ is the false membership degree (FMD) of v_i,
- $ta_V(v_i)$ is the true-ambiguous membership degree (TAMD) of v_i,
- $fa_V(v_i)$ is the false-ambiguous membership degree (FAMD) of v_i,

for all $v_i \in V$, satisfying the condition:

$$-0 \le t_V(v_i) + f_V(v_i) + ta_V(v_i) + fa_V(v_i) \le 2^+.$$

2. **Ambiguous Edge Set**: For each edge $e_{ij} = (v_i, v_j) \in E$, the ambiguous graph assigns the following membership functions:

$$t_E(e_{ij}), \quad f_E(e_{ij}), \quad ta_E(e_{ij}), \quad fa_E(e_{ij}) : E \to]-0, 1[,$$

where:

- $t_E(e_{ij})$ is the true membership degree of edge e_{ij},
- $f_E(e_{ij})$ is the false membership degree of e_{ij},
- $ta_E(e_{ij})$ is the true-ambiguous membership degree of e_{ij},
- $fa_E(e_{ij})$ is the false-ambiguous membership degree of e_{ij},

for all $e_{ij} \in E$, satisfying the condition:

$$-0 \le t_E(e_{ij}) + f_E(e_{ij}) + ta_E(e_{ij}) + fa_E(e_{ij}) \le 2^+.$$

Here, V^A represents the ambiguous vertex set of the graph G^A, and E^A represents the ambiguous edge set of G^A.

Based on the above definitions, the properties of the Ambiguous Graph are outlined below.

Theorem 3.24. *Every Ambiguous Graph can be transformed into an Intuitionistic Fuzzy Graph by restricting the true-ambiguous membership degree and false-ambiguous membership degree to zero for all vertices and edges.*

Proof. Let $G^A = (V^A, E^A)$ be an Ambiguous Graph as defined. For each vertex $v_i \in V$ and edge $e_{ij} \in E$, the ambiguous graph assigns the membership degrees:

$$\begin{cases} t_V(v_i), \quad f_V(v_i), \quad ta_V(v_i), \quad fa_V(v_i) \\ t_E(e_{ij}), \quad f_E(e_{ij}), \quad ta_E(e_{ij}), \quad fa_E(e_{ij}) \end{cases}$$

satisfying:

$$-0 \le t_V(v_i) + f_V(v_i) + ta_V(v_i) + fa_V(v_i) \le 2^+$$

and

$$-0 \le t_E(e_{ij}) + f_E(e_{ij}) + ta_E(e_{ij}) + fa_E(e_{ij}) \le 2^+$$

To transform G^A into an Intuitionistic Fuzzy Graph $G^{IF} = (A, B)$, we restrict the true-ambiguous and false-ambiguous membership degrees to zero:

$$ta_V(v_i) = fa_V(v_i) = 0, \quad \forall v_i \in V$$

$$ta_E(e_{ij}) = fa_E(e_{ij}) = 0, \quad \forall e_{ij} \in E$$

Then, define the membership and non-membership functions for G^{IF} as:

$$\mu_A(v_i) = t_V(v_i), \quad \nu_A(v_i) = f_V(v_i)$$

$$\mu_B(e_{ij}) = t_E(e_{ij}), \quad \nu_B(e_{ij}) = f_E(e_{ij})$$

Since $t_V(v_i), f_V(v_i) \in [0, 1]$ and satisfy:

$$-0 \le t_V(v_i) + f_V(v_i) \le 2^+$$

which simplifies to:

$$0 \leq \mu_A(v_i) + \nu_A(v_i) \leq 1$$

because $t_V(v_i) + f_V(v_i) \leq 1$ in standard intervals. Similarly for edges:

$$0 \leq \mu_B(e_{ij}) + \nu_B(e_{ij}) \leq 1$$

These conditions match those of an Intuitionistic Fuzzy Graph. Therefore, by setting $ta_V(v_i) = fa_V(v_i) = 0$ and $ta_E(e_{ij}) = fa_E(e_{ij}) = 0$, every Ambiguous Graph G^A can be transformed into an Intuitionistic Fuzzy Graph G^{IF}. □

Theorem 3.25. *If in an Ambiguous Graph G^A, the sum of the true membership degree and the true-ambiguous membership degree equals 1 for all vertices and edges, then the graph represents maximum truthfulness under ambiguity.*

Proof. For any vertex $v_i \in V$, suppose:

$$t_V(v_i) + ta_V(v_i) = 1$$

Since all membership degrees are in $[0, 1]$, and the sum of all degrees does not exceed 2, it follows that:

$$f_V(v_i) + fa_V(v_i) = t_V(v_i) + f_V(v_i) + ta_V(v_i) + fa_V(v_i) - (t_V(v_i) + ta_V(v_i)) \leq 2 - 1 = 1$$

Thus, the total falsity (including ambiguity) does not exceed 1, indicating that the vertex has maximum truthfulness under the given ambiguity.

Similarly for edges $e_{ij} \in E$.

This proposition highlights a scenario where the graph exhibits the highest possible degree of truthfulness within the ambiguous framework. □

3.6 Extended Turiyam Neutrosophic Graph

The Extended Turiyam Neutrosophic Graph, which is an extension of the Turiyam Neutrosophic Graph, is described below. This graph handles five types of uncertainties. The definition is provided as follows. Note that a Extended Turiyam Neutrosophic Graph can be generalized within the framework of a pentapartitioned Neutrosophic Graph (cf. [140]).

Definition 3.26. Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices and $E \subseteq V \times V$ is the set of edges.

An *Extended Turiyam Neutrosophic Graph* $G^E = (V^E, E^E)$ is defined as follows:

- **Extended Turiyam Neutrosophic Vertex Set V^E:**
  For each vertex $v \in V$, the following mappings are assigned:

  $$t(v),\ iv(v),\ fv(v),\ lv(v),\ hv(v) : V \to [0, 1],$$

  where:

  - $t(v)$: Truth-membership degree of vertex v,
  - $iv(v)$: Indeterminacy-membership degree of vertex v,
  - $fv(v)$: Falsity-membership degree of vertex v,
  - $lv(v)$: Turiyam Neutrosophic state (liberal value) of vertex v,
  - $hv(v)$: Hesitancy-membership degree of vertex v.

  These degrees represent the five uncertainties associated with each vertex. The following condition holds for each vertex $v \in V$:

  $$0 \leq t(v) + iv(v) + fv(v) + lv(v) + hv(v) \leq 5.$$

- **Extended Turiyam Neutrosophic Edge Set E^E:**
  For each edge $e = (u, v) \in E$, the following mappings are assigned:

  $$t(e),\ iv(e),\ fv(e),\ lv(e),\ hv(e) : E \to [0, 1],$$

  where each degree corresponds to the edge e similarly as for vertices. The following condition holds for each edge $e \in E$:

  $$0 \leq t(e) + iv(e) + fv(e) + lv(e) + hv(e) \leq 5.$$

Theorem 3.27. *Every Extended Turiyam Neutrosophic Graph is a generalization of the Turiyam Neutrosophic Graph.*

Proof. In the Extended Turiyam Neutrosophic Graph $G^E = (V^E, E^E)$, each vertex $v \in V$ has five associated membership degrees: $t(v), iv(v), fv(v), lv(v), hv(v)$.

To show that G^E is a generalization of the Turiyam Neutrosophic Graph G^T, consider the special case where the hesitancy-membership degree $hv(v)$ is zero for all vertices and edges. That is, set $hv(v) = 0$ for all $v \in V$ and $hv(e) = 0$ for all $e \in E$.

With $hv(v) = 0$, the condition for vertices in G^E reduces to:

$$0 \leq t(v) + iv(v) + fv(v) + lv(v) \leq 4,$$

which is exactly the condition in the original Turiyam Neutrosophic Graph definition.

Therefore, the Extended Turiyam Neutrosophic Graph includes the Turiyam Neutrosophic Graph as a special case, making it a generalization. □

Theorem 3.28. *Every Extended Turiyam Neutrosophic Graph can be transformed into a Turiyam Neutrosophic Graph by aggregating the hesitancy-membership degree into the other uncertainties.*

Proof. Given an Extended Turiyam Neutrosophic Graph $G^E = (V^E, E^E)$, for each vertex $v \in V$, we can redistribute the hesitancy-membership degree $hv(v)$ into the other four uncertainties. Define a transformed Turiyam Neutrosophic Graph G^T where:

$$t'(v) = t(v) + \alpha hv(v),$$
$$iv'(v) = iv(v) + \beta hv(v),$$
$$fv'(v) = fv(v) + \gamma hv(v),$$
$$lv'(v) = lv(v) + \delta hv(v),$$

with coefficients $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + \beta + \gamma + \delta = 1$.

Similarly for edges.

This ensures that the total of the four uncertainties remains within the original bounds:

$$0 \leq t'(v) + iv'(v) + fv'(v) + lv'(v) \leq 4.$$

Thus, the Extended Turiyam Neutrosophic Graph can be transformed into a Turiyam Neutrosophic Graph by appropriately distributing the hesitancy-membership degree. □

Theorem 3.29. *In an Extended Turiyam Neutrosophic Graph, for any edge $e = (u, v)$, the truth-membership degree of the edge satisfies:*

$$t(e) \leq \min\{t(u), t(v)\}.$$

Proof. By the nature of membership degrees, the edge's truth-membership degree cannot exceed the truth-membership degrees of its incident vertices. Suppose, for contradiction, that $t(e) > \min\{t(u), t(v)\}$.

Without loss of generality, assume $t(u) \leq t(v)$, so $\min\{t(u), t(v)\} = t(u)$.

Then $t(e) > t(u)$, which is inconsistent because an edge cannot be more "true" than its least truthful vertex. Therefore, the inequality must hold:

$$t(e) \leq \min\{t(u), t(v)\}.$$

□

Theorem 3.30. *In an Extended Turiyam Neutrosophic Graph, the total uncertainty associated with any vertex or edge does not exceed 5.*

Proof. This follows directly from the definition of the Extended Turiyam Neutrosophic Graph. For any vertex v or edge e, the sum of the uncertainties satisfies:

$$0 \leq t(x) + iv(x) + fv(x) + lv(x) + hv(x) \leq 5,$$

where $x \in V \cup E$.

Since each membership degree $t(x), iv(x), fv(x), lv(x), hv(x)$ lies in the interval $[0, 1]$, the maximum possible sum is when all degrees are at their maximum value of 1:

$$t(x) = iv(x) = fv(x) = lv(x) = hv(x) = 1 \implies \text{Sum} = 5.$$

Therefore, the total uncertainty does not exceed 5. □

3.7 Refined Graph for uncertainty

In recent years, refined sets have been introduced to handle various types of uncertainty [77, 146, 238, 394,422]. Refined sets adopt the concept of splitting values. By splitting truth values, they generalize fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets. Based on these sets, we define the Refined Graph as follows.

Definition 3.31 (Refined Fuzzy Graph)**.** A *Refined Fuzzy Graph (RFG)* is an extension of a classical fuzzy graph where each vertex and edge is assigned a refined fuzzy value, represented by a Refined Fuzzy Set (RFS).

Let $G = (V, E)$ be a simple graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. The RFG assigns refined fuzzy sets to both vertices and edges, defined as follows:

- **RFS for Vertices**:
  For each vertex $v \in V$, a Refined Fuzzy Set (RFS) is assigned:
  $$\mu_v = (T_1(v), T_2(v), \dots, T_p(v)),$$
  where $p \geq 2$ and $T_j(v) \in [0, 1]$ for all $j = 1, 2, \dots, p$. Here, $T_j(v)$ represents the sub-membership degree of vertex v, and the sum of these values is allowed to be adjusted based on the number of values p:
  $$\sum_{j=1}^{p} T_j(v) \leq p.$$
  This allows for greater flexibility in modeling the membership degrees of vertices.
- **RFS for Edges**:
  For each edge $e = (u, v) \in E$, a Refined Fuzzy Set (RFS) is assigned:
  $$\mu_e = (T_1(e), T_2(e), \dots, T_p(e)),$$
  where $p \geq 2$ and $T_j(e) \in [0, 1]$ for all $j = 1, 2, \dots, p$. Here, $T_j(e)$ represents the sub-membership degree of edge e, and the sum of these values is similarly allowed to be adjusted:
  $$\sum_{j=1}^{p} T_j(e) \leq p.$$

Theorem 3.32. *If for every vertex $v \in V$ and edge $e \in E$ in a Refined Fuzzy Graph (RFG), all refined membership degrees $T_j(v)$ and $T_j(e)$ are equal and sum to 1, then the RFG reduces to a classical fuzzy graph.*

Proof. In a classical fuzzy graph, each vertex v and edge e has a single membership degree $\mu(v) \in [0, 1]$ and $\mu(e) \in [0, 1]$.

In the RFG, each vertex v has refined membership degrees:

$$\mu_v = (T_1(v), T_2(v), \dots, T_p(v)),$$

where $T_j(v) \in [0, 1]$ and $\sum_{j=1}^{p} T_j(v) = 1$.

Assuming all $T_j(v)$ are equal, we have:

$$T_j(v) = \frac{1}{p}, \quad \forall j = 1, 2, \dots, p.$$

Thus:

$$\sum_{j=1}^{p} T_j(v) = p \times \frac{1}{p} = 1.$$

Define the classical membership degree as:

$$\mu(v) = \sum_{j=1}^{p} T_j(v) = 1.$$

Similarly for edges:

$$\mu(e) = \sum_{j=1}^{p} T_j(e) = 1.$$

Therefore, the RFG reduces to a classical fuzzy graph where each vertex and edge has a membership degree of 1. □

Definition 3.33 (Refined Intuitionistic Fuzzy Graph)**.** A *Refined Intuitionistic Fuzzy Graph (RIFG)* extends a classical intuitionistic fuzzy graph by assigning refined membership and non-membership values to both vertices and edges. Each vertex and edge in the graph is associated with a Refined Intuitionistic Fuzzy Set (RIFS), defined as follows:

- **RIFS for Vertices**:
  For each vertex $v \in V$, the Refined Intuitionistic Fuzzy Set (RIFS) is given by:
  $$(\mu_v, \nu_v) = \left((T_1(v), T_2(v), \dots, T_p(v)); (F_1(v), F_2(v), \dots, F_s(v))\right),$$
  where $p \geq 1$, $s \geq 1$, and $T_j(v), F_l(v) \in [0, 1]$ for all $j = 1, 2, \dots, p$ and $l = 1, 2, \dots, s$. The sum of membership values $\sum_{j=1}^{p} T_j(v)$ and non-membership values $\sum_{l=1}^{s} F_l(v)$ is adjusted based on the number of truth and falsity values:
  $$\sum_{j=1}^{p} T_j(v) + \sum_{l=1}^{s} F_l(v) \leq p + s.$$

- **RIFS for Edges**:
  For each edge $e = (u, v) \in E$, a Refined Intuitionistic Fuzzy Set (RIFS) is given by:
  $$(\mu_e, \nu_e) = \left((T_1(e), T_2(e), \dots, T_p(e)); (F_1(e), F_2(e), \dots, F_s(e))\right),$$
  where the same conditions apply as for the vertices, and the sum is similarly adjusted:
  $$\sum_{j=1}^{p} T_j(e) + \sum_{l=1}^{s} F_l(e) \leq p + s.$$

Theorem 3.34. *If the sum of the refined membership degrees and non-membership degrees for each vertex and edge in a Refined Intuitionistic Fuzzy Graph (RIFG) is 1, and there is only one refined value for each, then the RIFG reduces to a classical intuitionistic fuzzy graph.*

Proof. In RIFG, for each vertex v, we have:
$$\mu_v = (T_1(v)), \quad \nu_v = (F_1(v)),$$
with $T_1(v), F_1(v) \in [0, 1]$ and $T_1(v) + F_1(v) = 1$.

Similarly for edges.

This aligns with the definition of a classical intuitionistic fuzzy graph, where each vertex and edge has a membership degree $\mu(v)$ and a non-membership degree $\nu(v)$ such that $\mu(v) + \nu(v) = 1$. □

Definition 3.35. (cf.[424]) Let $G = (V, E)$ be a simple graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Refined Neutrosophic Graph* assigns neutrosophic values to both vertices and edges, where each vertex and each edge has:

- A membership degree split into r values $\mu_1, \mu_2, \dots, \mu_r$,
- An indeterminacy degree split into s values $\sigma_1, \sigma_2, \dots, \sigma_s$,
- A non-membership degree split into t values $\nu_1, \nu_2, \dots, \nu_t$,

such that for each vertex or edge x, the following condition holds:
$$0 \leq \sum_{i=1}^{r} \mu_i(x) + \sum_{i=1}^{s} \sigma_i(x) + \sum_{i=1}^{t} \nu_i(x) \leq n,$$
where $n = r + s + t$.

For each vertex $v \in V$, a neutrosophic n-valued refined set (μ_v, σ_v, ν_v) is assigned, where
$$\mu_v = (\mu_1(v), \dots, \mu_r(v)), \quad \sigma_v = (\sigma_1(v), \dots, \sigma_s(v)), \quad \nu_v = (\nu_1(v), \dots, \nu_t(v)).$$

Similarly, for each edge $e = (u, v) \in E$, a neutrosophic n-valued refined set (μ_e, σ_e, ν_e) is assigned, where
$$\mu_e = (\mu_1(e), \dots, \mu_r(e)), \quad \sigma_e = (\sigma_1(e), \dots, \sigma_s(e)), \quad \nu_e = (\nu_1(e), \dots, \nu_t(e)).$$

Theorem 3.36. *In a Refined Neutrosophic Graph, the total of the refined membership, indeterminacy, and non-membership degrees for any vertex or edge does not exceed $n = r + s + t$.*

Proof. By definition, for each vertex or edge x:

$$0 \leq \sum_{i=1}^{r} \mu_i(x) + \sum_{i=1}^{s} \sigma_i(x) + \sum_{i=1}^{t} \nu_i(x) \leq n.$$

This ensures that the combined degrees do not exceed the total number of refined components n. □

Theorem 3.37. *If $r = s = t = 1$ in a Refined Neutrosophic Graph, it reduces to a classical neutrosophic graph.*

Proof. With $r = s = t = 1$, each vertex and edge has single degrees:

$$\mu(x) = \mu_1(x), \quad \sigma(x) = \sigma_1(x), \quad \nu(x) = \nu_1(x),$$

and the condition becomes:

$$0 \leq \mu(x) + \sigma(x) + \nu(x) \leq 3.$$

After normalizing by dividing each degree by 3, we obtain degrees in $[0, 1]$, matching the classical neutrosophic graph definition. □

Definition 3.38 (Refined Turiyam Neutrosophic Graph)**.** A *Refined Turiyam Neutrosophic Graph* is an extension of a Turiyam Neutrosophic Graph where the truth value, indeterminacy value, falsity value, and liberal value of each vertex and edge are refined into sub-values.

Let $G = (V, E)$ be a classical graph with a finite set of vertices $V = \{v_i : i = 1, 2, \ldots, n\}$ and edges $E = \{(v_i, v_j) : i, j = 1, 2, \ldots, n\}$. The *Refined Turiyam Neutrosophic Graph*, denoted $G^T = (V^T, E^T)$, is defined as follows:

For each vertex $v_i \in V$, the Refined Turiyam Neutrosophic Graph assigns the following mappings:

$$t(v_i) = (\mu_1^t(v_i), \ldots, \mu_r^t(v_i)), \quad iv(v_i) = (\sigma_1^i(v_i), \ldots, \sigma_s^i(v_i)),$$
$$fv(v_i) = (\nu_1^f(v_i), \ldots, \nu_t^f(v_i)), \quad lv(v_i) = (\lambda_1^l(v_i), \ldots, \lambda_q^l(v_i)), \tag{3.1}$$

where:

- $t(v_i)$ is the refined truth value of the vertex v_i, split into r sub-membership values $\mu_1^t(v_i), \ldots, \mu_r^t(v_i)$,
- $iv(v_i)$ is the refined indeterminacy value, split into s values $\sigma_1^i(v_i), \ldots, \sigma_s^i(v_i)$,
- $fv(v_i)$ is the refined falsity value, split into t sub-non-membership values $\nu_1^f(v_i), \ldots, \nu_t^f(v_i)$,
- $lv(v_i)$ is the refined liberal value, split into q values $\lambda_1^l(v_i), \ldots, \lambda_q^l(v_i)$.

These values satisfy the following condition for each vertex v_i:

$$0 \leq \sum_{j=1}^{r} \mu_j^t(v_i) + \sum_{j=1}^{s} \sigma_j^i(v_i) + \sum_{j=1}^{t} \nu_j^f(v_i) + \sum_{j=1}^{q} \lambda_j^l(v_i) \leq n,$$

where $n = r + s + t + q$ is the total number of refined values.

For each edge $(v_i, v_j) \in E$, the Refined Turiyam Neutrosophic Graph assigns the following mappings:

$$t(v_i, v_j) = (\mu_1^t(v_i, v_j), \ldots, \mu_r^t(v_i, v_j)), \quad iv(v_i, v_j) = (\sigma_1^i(v_i, v_j), \ldots, \sigma_s^i(v_i, v_j))$$
$$fv(v_i, v_j) = (\nu_1^f(v_i, v_j), \ldots, \nu_t^f(v_i, v_j)), \quad lv(v_i, v_j) = (\lambda_1^l(v_i, v_j), \ldots, \lambda_q^l(v_i, v_j)),$$

where:

- $t(v_i, v_j)$ is the refined truth value of the edge (v_i, v_j),
- $iv(v_i, v_j)$ is the refined indeterminacy value of the edge,
- $fv(v_i, v_j)$ is the refined falsity value of the edge,
- $lv(v_i, v_j)$ is the refined liberal value of the edge.

These values satisfy the following condition for each edge (v_i, v_j):

$$0 \leq \sum_{j=1}^{r} \mu_j^t(v_i, v_j) + \sum_{j=1}^{s} \sigma_j^i(v_i, v_j) + \sum_{j=1}^{t} \nu_j^f(v_i, v_j) + \sum_{j=1}^{q} \lambda_j^l(v_i, v_j) \leq n.$$

Theorem 3.39. *In a Refined Turiyam Neutrosophic Graph, the sum of the refined truth, indeterminacy, falsity, and liberal degrees for any vertex or edge does not exceed $n = r + s + t + q$.*

Proof. By definition, for each vertex or edge x:

$$0 \leq \sum_{j=1}^{r} \mu_j^t(x) + \sum_{j=1}^{s} \sigma_j^i(x) + \sum_{j=1}^{t} \nu_j^f(x) + \sum_{j=1}^{q} \lambda_j^l(x) \leq n.$$

This ensures the total degrees are within the allowable range. □

Theorem 3.40. *If $r = s = t = q = 1$ in a Refined Turiyam Neutrosophic Graph, it reduces to a classical Turiyam Neutrosophic Graph.*

Proof. Each vertex and edge then has single degrees:

$$t(x) = \mu_1^t(x), \quad iv(x) = \sigma_1^i(x), \quad fv(x) = \nu_1^f(x), \quad lv(x) = \lambda_1^l(x),$$

with the condition:

$$0 \leq t(x) + iv(x) + fv(x) + lv(x) \leq 4.$$

After normalization, this aligns with the classical Turiyam Neutrosophic Graph. □

Definition 3.41 (Refined Extended Turiyam Neutrosophic Graph)**.** A *Refined Extended Turiyam Neutrosophic Graph* is an extension of the Extended Turiyam Neutrosophic Graph that refines the truth, indeterminacy, falsity, liberal, and hesitancy values of both vertices and edges into multiple sub-values. This refined graph allows for more detailed and precise modeling of various types of uncertainties.

Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices and $E \subseteq V \times V$ is the set of edges. The *Refined Extended Turiyam Neutrosophic Graph* $G^R = (V^R, E^R)$ is defined as follows:

For each vertex $v \in V$, the following mappings are assigned:

$$t(v) = (\mu_1^t(v), \ldots, \mu_r^t(v)), \quad iv(v) = (\sigma_1^i(v), \ldots, \sigma_s^i(v)), \quad fv(v) = (\nu_1^f(v), \ldots, \nu_t^f(v)),$$

$$lv(v) = (\lambda_1^l(v), \ldots, \lambda_q^l(v)), \quad hv(v) = (\eta_1^h(v), \ldots, \eta_p^h(v)),$$

where:

- $t(v)$ is the refined truth-membership degree of vertex v, split into r sub-values $\mu_1^t(v), \ldots, \mu_r^t(v)$,
- $iv(v)$ is the refined indeterminacy-membership degree, split into s sub-values $\sigma_1^i(v), \ldots, \sigma_s^i(v)$,
- $fv(v)$ is the refined falsity-membership degree, split into t sub-values $\nu_1^f(v), \ldots, \nu_t^f(v)$,
- $lv(v)$ is the refined liberal value, split into q sub-values $\lambda_1^l(v), \ldots, \lambda_q^l(v)$,
- $hv(v)$ is the refined hesitancy-membership degree, split into p sub-values $\eta_1^h(v), \ldots, \eta_p^h(v)$.

These values represent the five refined uncertainties associated with each vertex, and they satisfy the following condition:

$$0 \leq \sum_{j=1}^{r} \mu_j^t(v) + \sum_{j=1}^{s} \sigma_j^i(v) + \sum_{j=1}^{t} \nu_j^f(v) + \sum_{j=1}^{q} \lambda_j^l(v) + \sum_{j=1}^{p} \eta_j^h(v) \leq n,$$

where $n = r + s + t + q + p$ is the total number of sub-values associated with each vertex.

For each edge $e = (u, v) \in E$, the following mappings are assigned:

$$t(e) = (\mu_1^t(e), \ldots, \mu_r^t(e)), \quad iv(e) = (\sigma_1^i(e), \ldots, \sigma_s^i(e)), \quad fv(e) = (\nu_1^f(e), \ldots, \nu_t^f(e)),$$

$$lv(e) = (\lambda_1^l(e), \ldots, \lambda_q^l(e)), \quad hv(e) = (\eta_1^h(e), \ldots, \eta_p^h(e)),$$

where:

- $t(e)$ is the refined truth-membership degree of edge e,
- $iv(e)$ is the refined indeterminacy-membership degree of edge e,
- $fv(e)$ is the refined falsity-membership degree of edge e,
- $lv(e)$ is the refined liberal value of edge e,
- $hv(e)$ is the refined hesitancy-membership degree of edge e.

These values satisfy the following condition for each edge $e \in E$:

$$0 \leq \sum_{j=1}^{r} \mu_j^t(e) + \sum_{j=1}^{s} \sigma_j^i(e) + \sum_{j=1}^{t} \nu_j^f(e) + \sum_{j=1}^{q} \lambda_j^l(e) + \sum_{j=1}^{p} \eta_j^h(e) \leq n,$$

where $n = r + s + t + q + p$ is the total number of sub-values associated with each edge.

Theorem 3.42. *If $r = s = t = q = p = 1$ in a Refined Extended Turiyam Neutrosophic Graph, it reduces to a classical Extended Turiyam Neutrosophic Graph.*

Proof. With single degrees:

$$t(x) = \mu_1^t(x), \quad iv(x) = \sigma_1^i(x), \quad fv(x) = \nu_1^f(x), \quad lv(x) = \lambda_1^l(x), \quad hv(x) = \eta_1^h(x),$$

and the condition:

$$0 \leq t(x) + iv(x) + fv(x) + lv(x) + hv(x) \leq 5.$$

This matches the classical definition after appropriate normalization. □

Hereafter, this paper will consider the restricted version of the refined graph. As a concrete example, in the case of a Refined Extended Turiyam Neutrosophic Graph, we assume $r = s = t = q = p = c$ (where c is a given constant).

4. Result:Property of Plithogenic graphs

In this section, we explore the properties of Plithogenic graphs. Specifically, we define Plithogenic graphs and General Plithogenic graphs, and then examine their relationships.

4.1 Plithogenic graphs

Recently, Plithogenic Graphs have been proposed as a generalization of Fuzzy Graphs and Turiyam Neutrosophic Graphs, as well as a graphical representation of Plithogenic Sets [426]. Plithogenic Graphs have been developed and are currently being actively studied [241, 412, 449, 451] The definition is provided below.

Definition 4.1. [425, 426] Let S be a universal set, and $P \subseteq S$. A **Plithogenic Set** PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0, 1]^s$ is the **Degree of Appurtenance Function (DAF)**.
- $pCF : Pv \times Pv \to [0, 1]^t$ is the **Degree of Contradiction Function (DCF)**.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. **Reflexivity of Contradiction Function**:

$$pCF(a, a) = 0$$

2. **Symmetry of Contradiction Function**:

$$pCF(a, b) = pCF(b, a)$$

Example 4.2. Here, $s, t \in \{1, 2, 3, 4, 5\}$.

- When $s = t = 1$, PS is called a **Plithogenic Fuzzy Set** and is denoted by PFS.
- When $s = 2, t = 1$, PS is called a **Plithogenic Intuitionistic Fuzzy Set** and is denoted by $PIFS$.
- When $s = 3, t = 1$, PS is called a **Plithogenic Neutrosophic Set** and is denoted by PNS.
- When $s = 4, t = 1$, PS is called a **Plithogenic Turiyam Neutrosophic Set** and is denoted by $PTuS$.
- When $s = 5, t = 1$, PS is called a **Plithogenic Extended Turiyam Neutrosophic Set** and is denoted by $PTuS$.

Definition 4.3. [451] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A **Plithogenic Graph** PG is defined as:

$$PG = (PM, PN)$$

where:

1. **Plithogenic Vertex Set** $PM = (M, l, Ml, adf, aCf)$:
   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the range of possible attribute values.
   - $adf : M \times Ml \to [0, 1]^s$ is the **Degree of Appurtenance Function (DAF)** for vertices.
   - $aCf : Ml \times Ml \to [0, 1]^t$ is the **Degree of Contradiction Function (DCF)** for vertices.
2. **Plithogenic Edge Set** $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \to [0, 1]^s$ is the **Degree of Appurtenance Function (DAF)** for edges.
   - $bCf : Nm \times Nm \to [0, 1]^t$ is the **Degree of Contradiction Function (DCF)** for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. **Edge Appurtenance Constraint**: For all $(x, a), (y, b) \in M \times Ml$:

$$bdf\ ((xy), (a, b)) \leq \min\{adf(x, a), adf(y, b)\}$$

   where $xy \in N$ is an edge between vertices x and y, and $(a, b) \in Nm \times Nm$ are the corresponding attribute values.

2. **Contradiction Function Constraint**: For all $(a, b), (c, d) \in Nm \times Nm$:

$$bCf\ ((a, b), (c, d)) \leq \min\{aCf(a, c), aCf(b, d)\}$$

3. **Reflexivity and Symmetry of Contradiction Functions**:

$$\begin{aligned}
aCf(a, a) &= 0, && \forall a \in Ml \\
aCf(a, b) &= aCf(b, a), && \forall a, b \in Ml \\
bCf(a, a) &= 0, && \forall a \in Nm \\
bCf(a, b) &= bCf(b, a), && \forall a, b \in Nm
\end{aligned}$$

Example 4.4.

- When $s = t = 1$, PG is called a **Plithogenic Fuzzy Graph**.
- When $s = 2, t = 1$, PG is called a **Plithogenic Intuitionistic Fuzzy Graph**.
- When $s = 3, t = 1$, PG is called a **Plithogenic Neutrosophic Graph**.

- When $s = 4, t = 1$, PG is called a **Plithogenic Turiyam Neutrosophic Graph**.
- When $s = 5, t = 1$, PG is called a **Plithogenic Extended Turiyam Neutrosophic Graph**.

Theorem 4.5. *Plithogenic Turiyam Neutrosophic Graph is Turiyam Neutrosophic Graph.*

Proof. In a Plithogenic Graph, each vertex and edge can have multiple attributes. By restricting the attributes in the Plithogenic Graph to the four attributes that define a Turiyam Neutrosophic Graph—truth, indeterminacy, falsity, and liberal state—and setting the *Degree of Contradiction Function (DCF)* to zero for all attributes (i.e., no contradictions between attributes), we obtain the structure of a Turiyam Neutrosophic Graph.

Let $v \in V$ be a vertex in the Plithogenic Turiyam Neutrosophic Graph, and $Ml = \{t(v), iv(v), fv(v), lv(v)\}$ be the set of attributes for the vertex. The *Degree of Appurtenance Function (DAF)*, adf, assigns values between 0 and 1 to each attribute. By setting the *Degree of Contradiction Function (DCF)*, aCf, to zero, we eliminate contradictions between these attributes, which is equivalent to the condition in the Turiyam Neutrosophic Graph:

$$0 \leq t(v) + iv(v) + fv(v) + lv(v) \leq 4.$$

Let $(v_i, v_j) \in E$ be an edge in the Plithogenic Turiyam Neutrosophic Graph, and $Nm = \{t(v_i, v_j), iv(v_i, v_j), fv(v_i, v$ be the set of attributes for the edge. The *Degree of Appurtenance Function (DAF)*, bdf, assigns values between 0 and 1 to each edge attribute. By setting the *Degree of Contradiction Function (DCF)*, bCf, to zero, we eliminate contradictions between edge attributes, ensuring that the sum of the edge membership degrees does not exceed 4:

$$0 \leq t(v_i, v_j) + iv(v_i, v_j) + fv(v_i, v_j) + lv(v_i, v_j) \leq 4.$$

By restricting the attributes in a Plithogenic Turiyam Neutrosophic Graph to those of a Turiyam Neutrosophic Graph (truth, indeterminacy, falsity, and liberal state) and setting the Degree of Contradiction Function (DCF) to zero, the Plithogenic Turiyam Neutrosophic Graph reduces to a Turiyam Neutrosophic Graph.

Thus, Plithogenic Turiyam Neutrosophic Graph is a generalization of a Turiyam Neutrosophic Graph. □

Theorem 4.6. *Plithogenic Graphs generalize Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Extended Turiyam Neutrosophic Graphs.*

Proof. As per the definitions:

- When $s = t = 1$, PG reduces to a Plithogenic Fuzzy Graph.
- When $s = 2, t = 1$, PG becomes a Plithogenic Intuitionistic Fuzzy Graph.
- When $s = 3, t = 1$, PG becomes a Plithogenic Neutrosophic Graph.
- When $s = 4, t = 1$, PG becomes a Plithogenic Turiyam Neutrosophic Graph.
- When $s = 5, t = 1$, PG becomes a Plithogenic Extended Turiyam Neutrosophic Graph.

Each of these specialized graphs is a particular case of the Plithogenic Graph with specific values of s and t. Therefore, the Plithogenic Graph encompasses all these types as special cases, thereby generalizing them. □

4.2 Other Graph class related to Plithogenic graphs

These Plithogenic graphs are highly manageable, and the following generalizations are also possible. In this subsection, we will explore their relationships with those graph classes.

First, we consider the General Plithogenic Graph, which is a relaxed version of the Plithogenic Graph. Simply put, this definition is designed to allow for modifications to each graph by incorporating conditions that are suitable for each specific graph, enabling their application in a more flexible manner.

Definition 4.7 (General Plithogenic Graph)**.** A **General Plithogenic Graph** G^{GP} is an extension of the classical Plithogenic Graph that allows for a more flexible and independent treatment of vertices and edges. It is defined as follows:

Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges.

A General Plithogenic Graph $G^{GP} = (PM, PN)$ consists of:

1. **General Plithogenic Vertex Set** PM:

$$PM = (M, l, Ml, adf, aCf)$$

where:

- $M \subseteq V$: Set of vertices.
- l: Attribute associated with the vertices.
- Ml: Range of possible attribute values.
- $adf : M \times Ml \to [0,1]^s$: Degree of Appurtenance Function (DAF) for vertices.
- $aCf : Ml \times Ml \to [0,1]^t$: Degree of Contradiction Function (DCF) for vertices.

2. **General Plithogenic Edge Set** PN:

$$PN = (N, m, Nm, bdf, bCf)$$

where:

- $N \subseteq E$: Set of edges.
- m: Attribute associated with the edges.
- Nm: Range of possible attribute values.
- $bdf : N \times Nm \to [0,1]^s$: Degree of Appurtenance Function (DAF) for edges.
- $bCf : Nm \times Nm \to [0,1]^t$: Degree of Contradiction Function (DCF) for edges.

The General Plithogenic Graph G^{GP} only needs to satisfy the following **Reflexivity and Symmetry** properties of the Contradiction Functions:

- Reflexivity and Symmetry of Contradiction Functions:

$$\begin{aligned} aCf(a,a) &= 0, & \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), & \forall a,b \in Ml \\ bCf(a,a) &= 0, & \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), & \forall a,b \in Nm \end{aligned}$$

Theorem 4.8. *Plithogenic Graph is General Plithogenic Graph.*

Proof. Obviously holds. □

We consider transforming this graph into various other graphs. As mentioned earlier, this definition is intended to facilitate modifications to each graph by incorporating conditions tailored to each specific graph, allowing for more flexible application.

4.2.1 Relation for spherical fuzzy graphs

For example, we can refer to spherical fuzzy graphs [37]. In the field of fuzzy graphs, numerous studies have been conducted on spherical fuzzy graphs [121, 199, 209, 245, 301].

Theorem 4.9. *The following are examples of related graph classes for spherical fuzzy graphs, including but not limited to:*

- *T-Spherical Fuzzy Graphs[199]*
- *Spherical Fuzzy Labelling Graphs[122]*
- *spherical fuzzy digraphs[302]*
- *T-spherical fuzzy Hamacher graphs[79]*
- *Regular spherical fuzzy graph[310]*
- *Spherical Fuzzy Cycle graph[274]*
- *Spherical Fuzzy Tree graph[274]*
- *Pseudo regular spherical fuzzy graphs[210]*
- *Spherical neutrosophic graph[13]*

Proof. Refer to each reference as needed. □

The definition is provided below[37].

Definition 4.10. [37] A *spherical fuzzy graph (SFG)* on an underlying set V is a pair $G = (M, N)$, where M is a spherical fuzzy set in V and N is a spherical fuzzy relation on $V \times V$, such that:

$$\alpha_N(a, b) \leq \min\{\alpha_M(a), \alpha_M(b)\},$$

$$\gamma_N(a, b) \leq \min\{\gamma_M(a), \gamma_M(b)\},$$

$$\beta_N(a, b) \leq \max\{\beta_M(a), \beta_M(b)\},$$

where α is known as the degree of truthness, γ is known as the degree of abstinence, and β is known as the degree of falseness. These satisfy the following condition for all $a, b \in V$:

$$0 \leq \alpha_N^2(a, b) + \gamma_N^2(a, b) + \beta_N^2(a, b) \leq 1,$$

where M is the spherical fuzzy vertex set and N is the spherical fuzzy edge set of G.

Theorem 4.11. *General Plithogenic graphs with $s = 3$ and $t = 1$ can be transformed into spherical fuzzy graphs.*

Proof. We want to show that a General Plithogenic Graph G^{GP} with parameters $s = 3$ and $t = 1$ can be transformed into a spherical fuzzy graph (SFG).

By definition, the General Plithogenic Graph $G^{GP} = (PM, PN)$ is composed of a vertex set PM and an edge set PN. These sets contain Degree of Appurtenance Functions (DAF) and Degree of Contradiction Functions (DCF) for vertices and edges:

For the vertex set PM:

$$adf : M \times Ml \to [0, 1]^s$$

Since $s = 3$, the DAF for vertices will have three components for each vertex $v \in M$, say $adf(v) = (\mu(v), \eta(v), \nu(v))$, representing membership, abstinence, and non-membership degrees, respectively.

For the edge set PN:

$$bdf : N \times Nm \to [0, 1]^s$$

Similarly, the DAF for edges will have three components $bdf(e) = (\mu(e), \eta(e), \nu(e))$, representing membership, abstinence, and non-membership degrees for each edge $e \in N$.

The DCF aCf has $t = 1$, meaning it is a single-valued function measuring contradiction between attributes. This will not affect the transformation into a spherical fuzzy graph since a spherical fuzzy graph's main components involve membership, abstinence, and non-membership values.

A spherical fuzzy graph $G = (M, N)$ is characterized by the following conditions:

$$\alpha_N(a, b) \leq \min\{\alpha_M(a), \alpha_M(b)\},$$

$$\gamma_N(a, b) \leq \min\{\gamma_M(a), \gamma_M(b)\},$$

$$\beta_N(a, b) \leq \max\{\beta_M(a), \beta_M(b)\},$$

with the constraint

$$0 \leq \alpha_N^2(a, b) + \gamma_N^2(a, b) + \beta_N^2(a, b) \leq 1.$$

Since the General Plithogenic Graph has DAF values for both vertices and edges in the form (μ, η, ν), where μ is the degree of membership, η is the degree of abstinence, and ν is the degree of non-membership, these can directly correspond to α, γ, and β in a spherical fuzzy graph. □

4.2.2 Relation for General graphs

General (Intuitionistic) fuzzy graphs, weak General (Intuitionistic) fuzzy graphs, General Turiyam Neutrosophic graphs, and weak General Turiyam Neutrosophic graphs can all be generalized using Plithogenic graphs. The following properties hold.

Theorem 4.12. *A Plithogenic graph with $s = 1$ and $t = 1$ satisfies all the conditions required to transform it into general fuzzy graph.*

Proof. Obviously holds. □

Theorem 4.13. *A General Plithogenic Graph with $s = 1$ and $t = 1$ satisfies all the conditions required to transform it into weak general fuzzy graph.*

Proof. Obviously holds. □

Theorem 4.14. *A General Plithogenic Graph with $s = 2$ and $t = 1$ satisfies all the conditions required to transform it into general Intuitionistic fuzzy graph.*

Proof. Obviously holds. □

Theorem 4.15. *A General Plithogenic Graph with $s = 2$ and $t = 1$ satisfies all the conditions required to transform it into weak general Intuitionistic fuzzy graph.*

Proof. Obviously holds. □

Theorem 4.16. *A General Plithogenic Graph with $s = 4$ and $t = 1$ satisfies all the conditions required to transform it into general Turiyam Neutrosophic graph.*

Proof. Obviously holds. □

Theorem 4.17. *A General Plithogenic Graph with $s = 4$ and $t = 1$ satisfies all the conditions required to transform it into weak general Turiyam Neutrosophic graph.*

Proof. Obviously holds. □

4.2.3 Relation for Pythagorean fuzzy graphs

Next, we consider the Pythagorean fuzzy graph. Pythagorean fuzzy graphs have also been extensively studied in the field of fuzzy theory [27, 28, 35, 401, 467].

Theorem 4.18. *The following are examples of related graph classes for Pythagorean fuzzy graphs, including but not limited to:*

- *Cubic Pythagorean fuzzy graphs [321]*
- *Complex Pythagorean Dombi fuzzy graphs [29]*
- *Interval-valued Pythagorean fuzzy graphs[36]*
- *Interval-valued complex Pythagorean fuzzy graph[441]*
- *Pythagorean Dombi fuzzy graphs[20]*
- *Pythagorean fuzzy soft graphs[401]*
- *Pythagorean neutrosophic fuzzy graphs[12]*
- *Complex pythagorean fuzzy planar graphs[18]*
- *Pythagorean neutrosophic Dombi fuzzy graphs [120]*
- *Complex Pythagorean fuzzy threshold graphs[16]*
- *Pythagorean Dombi fuzzy soft graphs[42]*
- *Pythagorean fuzzy incidence graphs[44, 405]*

Proof. Refer to each reference as needed. □

The following theorem holds.

Definition 4.19. [35] A *Pythagorean fuzzy graph (PFG)* on a nonempty set V is a pair $G = (P, Q)$, where:

- P is a Pythagorean fuzzy set (PFS) on V,
- Q is a Pythagorean fuzzy relation (PFR) on $V \times V$,

satisfying the following conditions for all $u, v \in V$:

$$Q(uv) \leq P(u) \wedge P(v),$$

$$Q(uv) \geq P(u) \vee P(v),$$

where $Q : V \times V \to [0,1]$ and $\overline{Q} : V \times V \to [0,1]$ represent the membership and non-membership functions of Q, respectively. These functions satisfy the Pythagorean fuzzy condition:

$$0 \leq Q^2(uv) + \overline{Q}^2(uv) \leq 1 \quad \forall uv \in E.$$

Theorem 4.20. *A General Plithogenic Graph with $s = 2$ and $t = 1$, representing membership and non-membership degrees, can be transformed into a Pythagorean fuzzy graph (PFG).*

Proof. To prove that a General Plithogenic Graph with $s = 2$ and $t = 1$ can be transformed into a Pythagorean fuzzy graph (PFG), we start by recalling the definition of a PFG. A PFG on a nonempty set V is a pair $G = (P, Q)$, where:

- P is a Pythagorean fuzzy set (PFS) on V,
- Q is a Pythagorean fuzzy relation (PFR) on $V \times V$.

In a General Plithogenic Graph with $s = 2$, the two uncertainty components μ_1 and μ_2 represent membership and non-membership degrees, respectively. We can map these directly to the PFG structure by defining:

$$Q(uv) = \mu_1(uv) \quad \text{and} \quad \overline{Q}(uv) = \mu_2(uv),$$

for each edge $uv \in E$.

The Pythagorean condition requires that:

$$0 \leq Q^2(uv) + \overline{Q}^2(uv) \leq 1.$$

Since μ_1 and μ_2 in the Plithogenic Graph satisfy this condition by definition (i.e., $\mu_1^2 + \mu_2^2 \leq 1$), they conform to the Pythagorean fuzzy requirements.

Furthermore, the Degree of Appurtenance Function (DAF) in the Plithogenic Graph ensures that:

$$\mu_1(uv) \leq \min(\mu_1(u), \mu_1(v)),$$

which aligns with the requirement that $Q(uv) \leq \min(P(u), P(v))$ in a PFG. □

4.2.4 Relation for Hesitancy Fuzzy Graphs

Next, we consider the Hesitancy Fuzzy Graph. Like other classes of fuzzy graphs, the Hesitancy Fuzzy Graph has been the subject of extensive research [201, 216, 344, 362, 366]. This type of graph introduces the concept of the hesitancy of an element $v_i \in V$ into the fuzzy graph structure.

Theorem 4.21. *The following are examples of related graph classes for Hesitancy fuzzy graph, including but not limited to:*

- *Complex Hesitant Fuzzy Graph [6]*
- *Constant hesitancy fuzzy graphs[343]*
- *Bipolar hesitancy fuzzy graph[217, 340]*
- *Hesitancy fuzzy magic labeling graph[374]*
- *Dual hesitant fuzzy graphs[87]*
- *Regular Hesitancy Fuzzy Soft Graphs[272]*
- *Hesitant fuzzy hypergraphs[194]*

Proof. Refer to each reference as needed. □

The definition is provided below.

Definition 4.22. [344] A Hesitancy Fuzzy Graph is of the form $G = (V, E)$, where

- $V = \{v_1, v_2, v_3, \dots, v_n\}$ such that $\mu_1 : V \to [0, 1]$, $\gamma_1 : V \to [0, 1]$, and $\beta_1 : V \to [0, 1]$ denote the degree of membership, non-membership, and hesitancy of the element $v_i \in V$, respectively, and

$$\mu_1(v_i) + \gamma_1(v_i) + \beta_1(v_i) = 1$$

for every $v_i \in V$, where

$$\beta_1(v_i) = 1 - [\mu_1(v_i) + \gamma_1(v_i)] \tag{1}$$

- $E \subseteq V \times V$ where $\mu_2 : V \times V \to [0, 1]$, $\gamma_2 : V \times V \to [0, 1]$, and $\beta_2 : V \times V \to [0, 1]$ are such that

$$\mu_2(v_i, v_j) \le \min[\mu_1(v_i), \mu_1(v_j)] \tag{2}$$

$$\gamma_2(v_i, v_j) \le \max[\gamma_1(v_i), \gamma_1(v_j)] \tag{3}$$

$$\beta_2(v_i, v_j) \le \min[\beta_1(v_i), \beta_1(v_j)] \tag{4}$$

and

$$0 \le \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) + \beta_2(v_i, v_j) \le 1$$

for every $(v_i, v_j) \in E$.

Theorem 4.23. *A General Plithogenic Graph with $s = 3$ and $t = 1$ can be transformed into a Hesitancy Fuzzy Graph.*

Proof. To prove that a General Plithogenic Graph with $s = 3$ and $t = 1$ can be transformed into a Hesitancy Fuzzy Graph, we start by examining the definitions of both graphs and establishing the required mappings.

A General Plithogenic Graph is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- P is a set of vertices.
- v is an attribute, and Pv is the set of possible values for v.
- **Degree of Appurtenance Function (DAF)** $pdf : P \times Pv \to [0, 1]^s$, where $s = 3$ implies three components representing uncertainty.
- **Degree of Contradiction Function (DCF)** $pCF : Pv \times Pv \to [0, 1]^t$, where $t = 1$.

The three components of the DAF in this case represent:

1. $\mu_1(v)$: Membership degree.
2. $\gamma_1(v)$: Non-membership degree.
3. $\beta_1(v)$: Hesitancy degree, such that $\mu_1(v) + \gamma_1(v) + \beta_1(v) = 1$.

In a Hesitancy Fuzzy Graph $G = (V, E)$, each vertex $v_i \in V$ has associated values $\mu_1(v_i)$, $\gamma_1(v_i)$, and $\beta_1(v_i)$ satisfying the same condition $\mu_1(v_i) + \gamma_1(v_i) + \beta_1(v_i) = 1$. For edges $(v_i, v_j) \in E$, the conditions:

$$\mu_2(v_i, v_j) \le \min[\mu_1(v_i), \mu_1(v_j)],$$
$$\gamma_2(v_i, v_j) \le \max[\gamma_1(v_i), \gamma_1(v_j)],$$
$$\beta_2(v_i, v_j) \le \min[\beta_1(v_i), \beta_1(v_j)]$$

must hold.

To transform a Plithogenic Graph into a Hesitancy Fuzzy Graph:

1. Map the DAF components directly:

$$\mu_1(v) = pdf_1(v), \quad \gamma_1(v) = pdf_2(v), \quad \beta_1(v) = pdf_3(v).$$

2. This satisfies $\mu_1(v) + \gamma_1(v) + \beta_1(v) = 1$ as required by both graph definitions.
3. For edges, map:

$$\mu_2(v_i, v_j) = pdf_1(v_i, v_j), \quad \gamma_2(v_i, v_j) = pdf_2(v_i, v_j), \quad \beta_2(v_i, v_j) = pdf_3(v_i, v_j).$$

These values naturally satisfy the conditions of the Hesitancy Fuzzy Graph due to the properties of DAF and DCF.

Finally, since $t = 1$, the single contradiction value in the Plithogenic Graph ensures consistent relationships, and thus, the transformation is valid. This completes the proof. ◻

4.2.5 Relation for Intuitionistic Hesitancy Fuzzy Graphs

Next, we consider the Intuitionistic Hesitancy Fuzzy Graph, a newly proposed class of graphs in recent years. It is known to be a generalized class of Hesitancy Fuzzy Graphs. The definition is provided below [319, 320].

Definition 4.24. [320] An Intuitionistic Hesitancy Fuzzy Graph is $G = (V, E, \sigma, \mu)$, where:

- V is the vertex set.
- $\lambda_1, \delta_1, \rho_1 : V \to [0, 1]$ represent the degree of membership (MS), non-membership (NMS), and hesitancy of $v \in V$, respectively, satisfying:

$$0 \le \lambda_1(v) + \delta_1(v) + \rho_1(v) \le 1$$

- $\lambda_2, \delta_2, \rho_2 : V \times V \to [0, 1]$ represent the degree of membership (MS), non-membership (NMS), and hesitancy of $x = (u, v) \in V \times V$, respectively, satisfying:

$$\lambda_2(x) \le \min\{\lambda_1(u), \lambda_1(v)\}$$

$$\delta_2(x) \le \max\{\delta_1(u), \delta_1(v)\}$$

$$\rho_2(x) \le \min\{\rho_1(u), \rho_1(v)\}$$

  and

$$0 \le \lambda_2(x) + \delta_2(x) + \rho_2(x) \le 1, \quad \forall x \in V \times V$$

Theorem 4.25. *A General Plithogenic Graph with $s = 3$ and $t = 1$, representing membership and non-membership degrees, can be transformed into a Intuitionistic Hesitancy Fuzzy Graphs.*

Proof. It can be proven in the same way as the previous theorem. □

4.2.6 Relation for picture fuzzy graph

Next, we consider the picture fuzzy graph. The picture fuzzy graph is a generalization of fuzzy graphs and has been extensively studied, with many applications also being explored [68, 263].

Theorem 4.26. *The following are examples of related graph classes for picture fuzzy graph, including but not limited to:*

- *Picture Dombi Fuzzy Graph [311]*
- *Picture fuzzy line graphs[126]*
- *Picture fuzzy planar graphs[92]*
- *Picture fuzzy tolerance graphs[135]*
- *Picture fuzzy ϕ-tolerance competition graphs[136]*
- *Interval-Valued Picture Fuzzy Graph [368]*
- *Picture Fuzzy Incidence Graph [333]*
- *Picture fuzzy threshold graphs[135]*
- *Picture Fuzzy Soft Graph [123]*
- *Cayley Picture Fuzzy Graph [256]*
- *m-Polar Picture Fuzzy Graphs[257]*
- *Interval-Valued Picture (S, T)-Fuzzy Graph [75]*
- *q-Rung Picture Fuzzy Graph [26]*
- *Mixed Picture Fuzzy Graph [327]*
- *Picture Fuzzy Directed Hypergraphs[255]*

- *Picture fuzzy cubic graphs[258]*
- *Picture Fuzzy Digraph [302]*
- *Balanced Picture Fuzzy Graph [67]*

Proof. Refer to each reference as needed. □

The definition is provided below [320].

Definition 4.27. [320] Let $G^* = (V, E)$ be a graph. A pair $G = (A, B)$ is called a *picture fuzzy graph* on G^*, where:

- $A = (\mu_A, \eta_A, \nu_A)$ is a picture fuzzy set on V,
- $B = (\mu_B, \eta_B, \nu_B)$ is a picture fuzzy set on $E \subseteq V \times V$,

such that for each arc $uv \in E$, the following conditions hold:

$$\mu_B(u, v) \leq \min(\mu_A(u), \mu_A(v)),$$

$$\eta_B(u, v) \leq \min(\eta_A(u), \eta_A(v)),$$

$$\nu_B(u, v) \geq \max(\nu_A(u), \nu_A(v)).$$

Theorem 4.28. *A Plithogenic graph with $s = 2$ and $t = 1$, representing membership and non-membership degrees, can be transformed into a picture fuzzy graph.*

Proof. To prove that a Plithogenic graph with $s = 2$ and $t = 1$ can be transformed into a picture fuzzy graph, we start by recalling the definition of a picture fuzzy graph $G = (A, B)$ on a graph $G^* = (V, E)$. It consists of:

- $A = (\mu_A, \eta_A, \nu_A)$, a picture fuzzy set on V,
- $B = (\mu_B, \eta_B, \nu_B)$, a picture fuzzy set on $E \subseteq V \times V$,

such that for each edge $uv \in E$, the conditions hold:

$$\mu_B(u, v) \leq \min(\mu_A(u), \mu_A(v)),$$

$$\eta_B(u, v) \leq \min(\eta_A(u), \eta_A(v)),$$

$$\nu_B(u, v) \geq \max(\nu_A(u), \nu_A(v)).$$

In a Plithogenic graph with $s = 2$ and $t = 1$, we have two components representing uncertainty:

- μ_1: Membership degree,
- μ_2: Non-membership degree.

To transform this into a picture fuzzy graph, we map these components as follows:

$$\mu_A(u) = \mu_1(u), \quad \eta_A(u) = \mu_2(u), \quad \nu_A(u) = 1 - (\mu_1(u) + \mu_2(u)),$$

for each vertex $u \in V$. Similarly, for edges $uv \in E$, we define:

$$\mu_B(u, v) = \mu_1(uv), \quad \eta_B(u, v) = \mu_2(uv), \quad \nu_B(u, v) = 1 - (\mu_1(uv) + \mu_2(uv)).$$

These definitions ensure that the conditions of the picture fuzzy graph are satisfied because:

$$\mu_B(u, v) \leq \min(\mu_A(u), \mu_A(v)),$$

$$\eta_B(u, v) \leq \min(\eta_A(u), \eta_A(v)),$$

$$\nu_B(u, v) \geq \max(\nu_A(u), \nu_A(v)),$$

naturally follow from the properties of the membership, non-membership, and hesitancy degrees.

Therefore, a Plithogenic graph with $s = 2$ and $t = 1$ can be effectively transformed into a picture fuzzy graph by interpreting its components accordingly. □

4.2.7 Relation for Ambiguous Graph

Next, we consider the relations for the Ambiguous Graph. The following holds:

Theorem 4.29. *A General Plithogenic Graph with $s = 4$ and $t = 1$ can be transformed into an Ambiguous Graph.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.8 Relation for Quadripartitioned Neutrosophic Set and Graph

The Quadripartitioned Neutrosophic Set and Graph are recently defined sets and graphs, and several studies have been conducted on them. Related concepts include Quadripartitioned Neutrosophic Soft Graphs [226], Quadripartitioned Neutrosophic Competition Graph [218], Quadripartitioned Single-Valued Neutrosophic Graph [219], Quadripartitioned Neutrosophic Pythagorean Soft Set [364], and Interval Quadripartitioned Neutrosophic Sets [361].

Definition 4.30 (Quadripartitioned Neutrosophic Set (QPNS))**.** (cf.[361]) A **quadripartitioned neutrosophic set** A on a universal set Y is defined as:

$$A = \{(y, t_A(y), c_A(y), u_A(y), f_A(y)) : y \in Y\},$$

where $t_A(y), c_A(y), u_A(y), f_A(y) \in [0, 1]$ represent the truth, contradiction, unknown, and falsity membership functions of each $y \in Y$, respectively. The membership functions $t_A(y), c_A(y), u_A(y), f_A(y)$ must satisfy the condition:

$$0 \leq t_A(y) + c_A(y) + u_A(y) + f_A(y) \leq 4.$$

Definition 4.31 (Quadripartitioned Neutrosophic Graph (QPNG))**.** [225] Let V be a finite set of vertices, and let $E \subseteq \{\{u, v\} : u, v \in V \text{ with } u \neq v\}$ be a set of edges. Let A be a quadripartitioned neutrosophic set (QPNS) on V, and B be a QPNS on E, such that for each $\{u, v\} \in E$, the following conditions hold:

$$t_B(u, v) \leq \min\{t_A(u), t_A(v)\},$$

$$c_B(u, v) \leq \min\{c_A(u), c_A(v)\},$$

$$u_B(u, v) \leq \max\{u_A(u), u_A(v)\},$$

$$f_B(u, v) \leq \max\{f_A(u), f_A(v)\}.$$

Then, $G = (A, B, V, E)$ is called a **quadripartitioned neutrosophic graph (QPNG)**.

1. Each $v \in V$ is called a vertex of G.
2. Each $\{u, v\} \in E$ is called an edge of G.

Theorem 4.32. *A General Plithogenic Graph with $s = 4$ and $t = 1$ can be transformed into an Quadripartitioned Neutrosophic Graph.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.9 Relation for Pentapartitioned Neutrosophic Set and Graph

The Pentapartitioned Neutrosophic Set and Graph are recently defined sets and graphs, and several studies have been conducted on them [109, 119, 139, 140, 363].

Definition 4.33 (Pentapartitioned Neutrosophic Set (PPNS) [300])**.** A **pentapartitioned neutrosophic set** A on a universal set Y is defined as:

$$A = \{(y, t_A(y), c_A(y), g_A(y), u_A(y), f_A(y)) : y \in Y\},$$

where $t_A(y), c_A(y), g_A(y), u_A(y), f_A(y) \in [0, 1]$ represent the truth, contradiction, ignorance, unknown, and falsity membership functions of each $y \in Y$, respectively. The membership functions $t_A(y), c_A(y), g_A(y), u_A(y), f_A(y)$ must satisfy the condition:

$$0 \leq t_A(y) + c_A(y) + g_A(y) + u_A(y) + f_A(y) \leq 5.$$

Definition 4.34 (Pentapartitioned Neutrosophic Graph (PPNG) [363])**.** Let V be a finite set of vertices, and let $E \subseteq \{\{u, v\} : u, v \in V \text{ with } u \neq v\}$ be a set of edges. Let A be a pentapartitioned neutrosophic set (PPNS) on V, and B be a PPNS on E, such that for each $\{u, v\} \in E$, the following conditions hold:

$$t_B(u, v) \leq \min\{t_A(u), t_A(v)\},$$

$$c_B(u, v) \geq \max\{c_A(u), c_A(v)\},$$
$$g_B(u, v) \geq \max\{g_A(u), g_A(v)\},$$
$$u_B(u, v) \geq \max\{u_A(u), u_A(v)\},$$
$$f_B(u, v) \geq \max\{f_A(u), f_A(v)\}.$$

Then, $G = (A, B, V, E)$ is called a **pentapartitioned neutrosophic graph (PPNG)**.

1. Each $v \in V$ is called a vertex of G.
2. Each $\{u, v\} \in E$ is called an edge of G.

Theorem 4.35. *A General Plithogenic Graph with* $s = 5$ *and* $t = 1$ *can be transformed into an Pentapartitioned Neutrosophic Graph.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.10 Relation for Penta-Valued Fuzzy Set and graph

The Penta-Valued Fuzzy Set has been previously proposed [347,349]. The graph-based definition of this concept is provided below.

Definition 4.36 (Penta-Valued Fuzzy Graph)**.** (cf.[347, 349]) Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges. A **penta-valued fuzzy graph** G^{PVF} is defined by associating a penta-valued fuzzy set to both the vertices and the edges of the graph.

1. **Penta-Valued Fuzzy Vertex Set:** For each vertex $v \in V$, the penta-valued fuzzy graph assigns the following membership functions:

   $$\mu_V(v), \quad \lambda_V(v), \quad \pi_V(v), \quad \omega_V(v), \quad \nu_V(v) : V \to [0, 1],$$

   where:

   - $\mu_V(v)$ represents the strong membership degree of the vertex v,
   - $\lambda_V(v)$ represents the weak membership degree of v,
   - $\pi_V(v)$ represents the uncertainty degree of v,
   - $\omega_V(v)$ represents the weak non-membership degree of v,
   - $\nu_V(v)$ represents the strong non-membership degree of v.

   These membership functions must satisfy the condition:

   $$\mu_V(v) + \lambda_V(v) + \pi_V(v) + \omega_V(v) + \nu_V(v) = 1 \quad \forall v \in V.$$

2. **Penta-Valued Fuzzy Edge Set:** For each edge $e = (u, v) \in E$, the penta-valued fuzzy graph assigns the following membership functions:

   $$\mu_E(e), \quad \lambda_E(e), \quad \pi_E(e), \quad \omega_E(e), \quad \nu_E(e) : E \to [0, 1],$$

   where:

   - $\mu_E(e)$ represents the strong membership degree of the edge e,
   - $\lambda_E(e)$ represents the weak membership degree of e,
   - $\pi_E(e)$ represents the uncertainty degree of e,
   - $\omega_E(e)$ represents the weak non-membership degree of e,
   - $\nu_E(e)$ represents the strong non-membership degree of e.

These membership functions must satisfy the condition:

$$\mu_E(e) + \lambda_E(e) + \pi_E(e) + \omega_E(e) + \nu_E(e) = 1 \quad \forall e \in E.$$

The penta-valued fuzzy graph G^{PVF} is represented as:

$$G^{PVF} = (V^{PVF}, E^{PVF}),$$

where:

- $V^{PVF} = \{(v, \mu_V(v), \lambda_V(v), \pi_V(v), \omega_V(v), \nu_V(v)) : v \in V\}$ is the penta-valued fuzzy vertex set.
- $E^{PVF} = \{(e, \mu_E(e), \lambda_E(e), \pi_E(e), \omega_E(e), \nu_E(e)) : e \in E\}$ is the penta-valued fuzzy edge set.

Theorem 4.37. *A General Plithogenic Graph with $s = 5$ and $t = 1$ satisfies all the conditions required to transform it into a Penta-Valued Fuzzy Graph.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.11 Relation for Four-Valued Fuzzy Set and graph

Four-Valued Fuzzy Sets and graphs are extensions of Tree-Valued Fuzzy Sets and Fuzzy Graphs. Several studies have been published on this topic [346, 348]. The definition is provided below. It is worth noting that a Four-Valued Fuzzy Set can be generalized within the framework of a Quadripartitioned Neutrosophic Set (cf. [417]).

Definition 4.38 (Four-Valued Fuzzy Set)**.** [346] Let X be a non-empty set (the universe of discourse). A **four-valued fuzzy set** W on X is defined as:

$$W = \{(x, \mu(x), \lambda(x), \omega(x), \nu(x)) : x \in X\},$$

where

$$\mu, \lambda, \omega, \nu : X \to [0, 1]$$

are the membership functions representing the following components:

- $\mu(x)$: True membership function indicating the degree to which x is strongly true.
- $\lambda(x)$: Contingent true membership function indicating the degree to which x is weakly true.
- $\omega(x)$: Contingent false membership function indicating the degree to which x is weakly false.
- $\nu(x)$: False membership function indicating the degree to which x is strongly false.

These membership functions must satisfy the condition:

$$\mu(x) + \lambda(x) + \omega(x) + \nu(x) = 1 \quad \forall x \in X.$$

The overall membership degree $w(x)$ of an element x in W is determined by the weighted sum:

$$w(x) = 1 \cdot \mu(x) + \frac{2}{3} \cdot \lambda(x) + \frac{1}{3} \cdot \omega(x) + 0 \cdot \nu(x).$$

Definition 4.39 (Four-Valued Fuzzy Graph)**.** Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges. A **four-valued fuzzy graph** G^{FVF} is defined by associating a four-valued fuzzy set to both the vertices and the edges of the graph.

For each vertex $v \in V$, the four-valued fuzzy graph assigns the following membership functions:

$$\mu_V(v), \lambda_V(v), \omega_V(v), \nu_V(v) : V \to [0, 1],$$

where:

- $\mu_V(v)$: Strong true membership degree of the vertex v,
- $\lambda_V(v)$: Contingent true membership degree of v,
- $\omega_V(v)$: Contingent false membership degree of v,

- $\nu_V(v)$: Strong false membership degree of v.

These membership functions must satisfy the condition:

$$\mu_V(v) + \lambda_V(v) + \omega_V(v) + \nu_V(v) = 1 \quad \forall v \in V.$$

For each edge $e = (u, v) \in E$, the four-valued fuzzy graph assigns the following membership functions:

$$\mu_E(e), \lambda_E(e), \omega_E(e), \nu_E(e) : E \to [0, 1],$$

where:

- $\mu_E(e)$: Strong true membership degree of the edge e,
- $\lambda_E(e)$: Contingent true membership degree of e,
- $\omega_E(e)$: Contingent false membership degree of e,
- $\nu_E(e)$: Strong false membership degree of e.

These membership functions must satisfy the condition:

$$\mu_E(e) + \lambda_E(e) + \omega_E(e) + \nu_E(e) = 1 \quad \forall e \in E.$$

The four-valued fuzzy graph G^{FVF} is represented as:

$$G^{FVF} = (V^{FVF}, E^{FVF}),$$

where:

- $V^{FVF} = \{(v, \mu_V(v), \lambda_V(v), \omega_V(v), \nu_V(v)) : v \in V\}$ is the four-valued fuzzy vertex set.
- $E^{FVF} = \{(e, \mu_E(e), \lambda_E(e), \omega_E(e), \nu_E(e)) : e \in E\}$ is the four-valued fuzzy edge set.

Theorem 4.40. *A General Plithogenic Graph with $s = 4$ and $t = 1$ satisfies all the conditions required to transform it into a Four-Valued Fuzzy Graph.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.12 Relation for Vague graphs

Vague graphs are a concept that graphically represent vague sets, and various studies have been conducted on them[25, 101, 103, 369, 393]. They are particularly closely related to intuitionistic fuzzy graphs.

We consider about Graph class for Vague graphs.

Theorem 4.41. *The following are examples of related graph classes, including but not limited to:*

- *Edge irregular product vague graphs[271]*
- *Vague incidence graphs[169]*
- *Complex vague graphs[479]*
- *Irregular vague graphs[88]*
- *Neutrosophic vague graphs[224]*
- *Neutrosophic vague line graphs[221]*
- *Cubic Vague Graphs[454]*
- *Vague Influence Graphs[406]*
- *Neutrosophic Bipolar Vague Line Graph[151]*
- *Neutrosophic vague incidence graph[221]*
- *Neutrosophic Bipolar Vague Incidence Graph[71]*
- *Neutrosophic Vague Soft Graphs[222]*

Proof. Refer to each reference as needed. □

The following is clearly satisfied.

Theorem 4.42. *A General Plithogenic Graph with $s = 2$ and $t = 1$ satisfies all the conditions required to transform it into a Vague graphs.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.13 Relation for Multi-fuzzy graphs

We explore the relationships within multi-fuzzy graphs, which are defined as an extension of the concept of multi-fuzzy sets [59, 398, 398, 399]. Related concepts include n-dimensional fuzzy sets [403], multidimensional fuzzy sets [167, 280, 395], and probabilistic hesitant fuzzy sets [168, 235, 254]. The definition is provided as follows.

Definition 4.43. Let $G = (V, E)$ be an undirected graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A **multi-fuzzy graph** G^M is a graph in which each vertex and each edge has multiple membership degrees associated with it.

Formally, a multi-fuzzy graph $G^M = (V^M, E^M)$ is defined as follows:

- **Multi-Fuzzy Vertex Set** V^M:
$$V^M = \{(v, \mu_1(v), \mu_2(v), \ldots, \mu_k(v)) \mid v \in V\}$$
where $\mu_i(v) \in [0, 1]$ represents the i-th membership degree of vertex v for $i = 1, 2, \ldots, k$. The integer k denotes the dimension of the multi-fuzzy graph.
- **Multi-Fuzzy Edge Set** E^M:
$$E^M = \{(e, \nu_1(e), \nu_2(e), \ldots, \nu_k(e)) \mid e \in E\}$$
where $\nu_i(e) \in [0, 1]$ represents the i-th membership degree of edge e for $i = 1, 2, \ldots, k$.

Theorem 4.44. *Let integer k the dimension of the multi-fuzzy graph. A General Plithogenic Graph with $s = k$ and $t = 1$ satisfies all the conditions required to transform it into a Multi-Fuzzy graphs.*

Proof. A General Plithogenic Graph G^{GP} is defined as $G^{GP} = (PM, PN)$, where:

1. **Plithogenic Vertex Set** $PM = (M, l, Ml, adf, aCf)$:
   - $M \subseteq V$: Set of vertices.
   - $adf : M \times Ml \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for vertices.
   - $aCf : Ml \times Ml \to [0, 1]^t$: Degree of Contradiction Function (DCF) for vertices.
2. **Plithogenic Edge Set** $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$: Set of edges.
   - $bdf : N \times Nm \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for edges.
   - $bCf : Nm \times Nm \to [0, 1]^t$: Degree of Contradiction Function (DCF) for edges.

Since $t = 1$, the Degree of Contradiction Functions aCf and bCf output a single value, thus $aCf : Ml \times Ml \to [0, 1]$ and $bCf : Nm \times Nm \to [0, 1]$.

A multi-fuzzy graph $G^M = (V^M, E^M)$ is defined with:

- **Multi-Fuzzy Vertex Set** V^M:
$$V^M = \{(v, \mu_1(v), \mu_2(v), \ldots, \mu_k(v)) \mid v \in V\}$$
where $\mu_i(v) \in [0, 1]$ represents the i-th membership degree of vertex v for $i = 1, 2, \ldots, k$.
- **Multi-Fuzzy Edge Set** E^M:
$$E^M = \{(e, \nu_1(e), \nu_2(e), \ldots, \nu_k(e)) \mid e \in E\}$$
where $\nu_i(e) \in [0, 1]$ represents the i-th membership degree of edge e for $i = 1, 2, \ldots, k$.

Given that $s = k$ in the General Plithogenic Graph G^{GP}:

- For each vertex $v \in M$, the Degree of Appurtenance Function $adf(v, a) = (\mu_1(v), \mu_2(v), \ldots, \mu_k(v))$ corresponds directly to the multi-membership degrees of v in the multi-fuzzy graph G^M. Thus, we can map each $adf(v, a)$ to the multi-fuzzy membership values $\mu_1(v), \mu_2(v), \ldots, \mu_k(v)$ for the vertex v.
- Similarly, for each edge $e \in N$, the Degree of Appurtenance Function $bdf(e, b) = (\nu_1(e), \nu_2(e), \ldots, \nu_k(e))$ corresponds to the multi-membership degrees of e in the multi-fuzzy graph G^M. We map $bdf(e, b)$ to $\nu_1(e), \nu_2(e), \ldots, \nu_k(e)$ for edge e.

Since aCf and bCf are now single values due to $t = 1$, they do not affect the multiple membership values in the transformation process. Therefore, we have proven that a General Plithogenic Graph with $s = k$ and $t = 1$ can be transformed into a Multi-Fuzzy Graph. □

4.2.14 Relation for intuitionistic multi-fuzzy graph

We consider the relationships for intuitionistic multi-fuzzy graphs. T hey are defined as graphs that extend the concept of intuitionistic multi-fuzzy sets [134, 137, 138]. The definition is as follows.

Definition 4.45. Let $G = (V, E)$ be a graph where V is a finite set of vertices and $E \subseteq V \times V$ is a set of edges. An intuitionistic multi-fuzzy graph G_{IMF} is defined as a pair $G_{IMF} = (V_{IMF}, E_{IMF})$ where:

1. **Vertices Membership Function**: For each vertex $v \in V$, there exists an associated intuitionistic multi-fuzzy set
$$\mu_G(v) = \{(\mu_1(v), \nu_1(v)), (\mu_2(v), \nu_2(v)), \ldots, (\mu_k(v), \nu_k(v))\},$$
where $0 \leq \mu_i(v) + \nu_i(v) \leq 1$ for each $i = 1, 2, \ldots, k$. Here, $\mu_i(v)$ represents the degree of membership of the vertex v at level i, and $\nu_i(v)$ represents the degree of non-membership of the vertex v at level i.

2. **Edges Membership Function**: Similarly, for each edge $e = (u, v) \in E$, there exists an associated intuitionistic multi-fuzzy set
$$\mu_G(e) = \{(\mu_1(e), \nu_1(e)), (\mu_2(e), \nu_2(e)), \ldots, (\mu_k(e), \nu_k(e))\},$$
where $0 \leq \mu_i(e) + \nu_i(e) \leq 1$ for each $i = 1, 2, \ldots, k$. Here, $\mu_i(e)$ represents the degree of membership of the edge e at level i, and $\nu_i(e)$ represents the degree of non-membership of the edge e at level i.

3. **Intuitionistic Multi-Fuzzy Value**: For each level i, the intuitionistic multi-fuzzy value (IMFV) or intuitionistic multi-fuzzy number (IMFN) is given by
$$\pi_i(v) = 1 - (\mu_i(v) + \nu_i(v))$$
for vertices, and
$$\pi_i(e) = 1 - (\mu_i(e) + \nu_i(e))$$
for edges. This value represents the degree of indeterminacy or hesitation at each level i.

The parameter k represents the number of different levels of membership and non-membership, allowing for a multi-faceted evaluation of each vertex and edge within the graph.

Theorem 4.46. *Let integer k the dimension of the multi-fuzzy graph. A General Plithogenic Graph with $s = 2 \times k$ and $t = 1$ satisfies all the conditions required to transform it into a intuitionistic Multi-Fuzzy graphs.*

Proof. The proof can be constructed in a manner similar to the previous theorem. □

4.2.15 Relation for other concepts

Additionally, concepts such as the Hexapartitioned Neutrosophic Graph, which includes an additional membership function for hesitancy alongside truth, contradiction, ignorance, unknown, and falsity, can also be considered. These can similarly be generalized as Plithogenic graphs. Moreover, this generalization can be extended to Plithogenic sets rather than just Plithogenic graphs.

Question 4.47. Is it possible to define a Layered Plithogenic Graph? If so, how can it be defined?

4.3 Restricted Refined Plithogenic graphs

As previously mentioned, research on refined sets and graphs has advanced in recent years (cf.[146, 147, 238, 463, 463]). The definition of Restricted Refined Plithogenic Graphs is provided as follows.

Definition 4.48. Let $G = (V, E)$ be a crisp graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges.

We define a **Restricted Refined Plithogenic Graph (RRPG)** as:
$$RRPG = (RPM, RPN)$$
where:

- **Restricted Refined Plithogenic Vertex Set** $RPM = (M, l, Ml, adf_r, aCf_r, s, r)$:
  - $M \subseteq V$ is the set of vertices.
  - l is an attribute associated with the vertices.

- Ml is the range of possible attribute values.
- $adf_r : M \times Ml \to [0,1]^{s \times r}$ is the **Restricted Refined Degree of Appurtenance Function (RDAF)** for vertices.
- $aCf_r : Ml \times Ml \to [0,1]^{t \times r}$ is the **Restricted Refined Degree of Contradiction Function (RDCF)** for vertices.
- s is the number of uncertainty components (e.g., truth, indeterminacy, falsity).
- r is the number of splits per uncertainty component.

• **Restricted Refined Plithogenic Edge Set** $RPN = (N, m, Nm, bdf_r, bCf_r, s, r)$:
  - $N \subseteq E$ is the set of edges.
  - m is an attribute associated with the edges.
  - Nm is the range of possible attribute values.
  - $bdf_r : N \times Nm \to [0,1]^{s \times r}$ is the **Restricted Refined Degree of Appurtenance Function (RDAF)** for edges.
  - $bCf_r : Nm \times Nm \to [0,1]^{t \times r}$ is the **Restricted Refined Degree of Contradiction Function (RDCF)** for edges.
  - s and r are as defined above.

Theorem 4.49. *If $r = 1$ (i.e., no refinement is applied), the Restricted Refined Plithogenic Graph reduces to the classical Plithogenic Graph.*

Proof. When $r = 1$, each uncertainty component has only one degree, so the refined functions become:

$$adf_r : M \times Ml \to [0,1]^s$$
$$aCf_r : Ml \times Ml \to [0,1]^t$$

This matches the definitions of the classical Plithogenic Graph, where the Degree of Appurtenance Function and Degree of Contradiction Function output vectors of length s and t, respectively. All constraints and properties remain valid, thus confirming that the Restricted Refined Plithogenic Graph reduces to the classical case when $r = 1$. □

Theorem 4.50. *Under appropriate choices of s and t, the Restricted Refined Plithogenic Graph reduces to the corresponding restricted refined fuzzy, intuitionistic fuzzy, neutrosophic, Turiyam neutrosophic, or extended Turiyam Neutrosophic graphs.*

Proof. By selecting specific values for s and t, we can align the uncertainty components with those of the desired restricted refined graph:

- **Restricted Refined Fuzzy Graph**: Set $s = 1$, $t = 1$, $r \geq 1$.
- **Restricted Refined Intuitionistic Fuzzy Graph**: Set $s = 2$, $t = 1$, $r \geq 1$.
- **Restricted Refined Neutrosophic Graph**: Set $s = 3$, $t = 1$, $r \geq 1$.
- **Restricted Refined Turiyam Neutrosophic Graph**: Set $s = 4$, $t = 1$, $r \geq 1$.
- **Restricted Refined Extended Turiyam Neutrosophic Graph**: Set $s = 5$, $t = 1$, $r \geq 1$.

With these settings, the Restricted Refined Plithogenic Graph's definitions and constraints match those of the corresponding restricted refined graphs, confirming the reduction. □

Based on the above, the following relationship holds as illustrated in the diagram below. Note that in the diagram, ”Turiyam Graph” refers to the ”Turiyam Neutrosophic Graph.”

Question 4.51. How is the definition of the Refined Plithogenic Graph related to other classes of Refined Graphs?

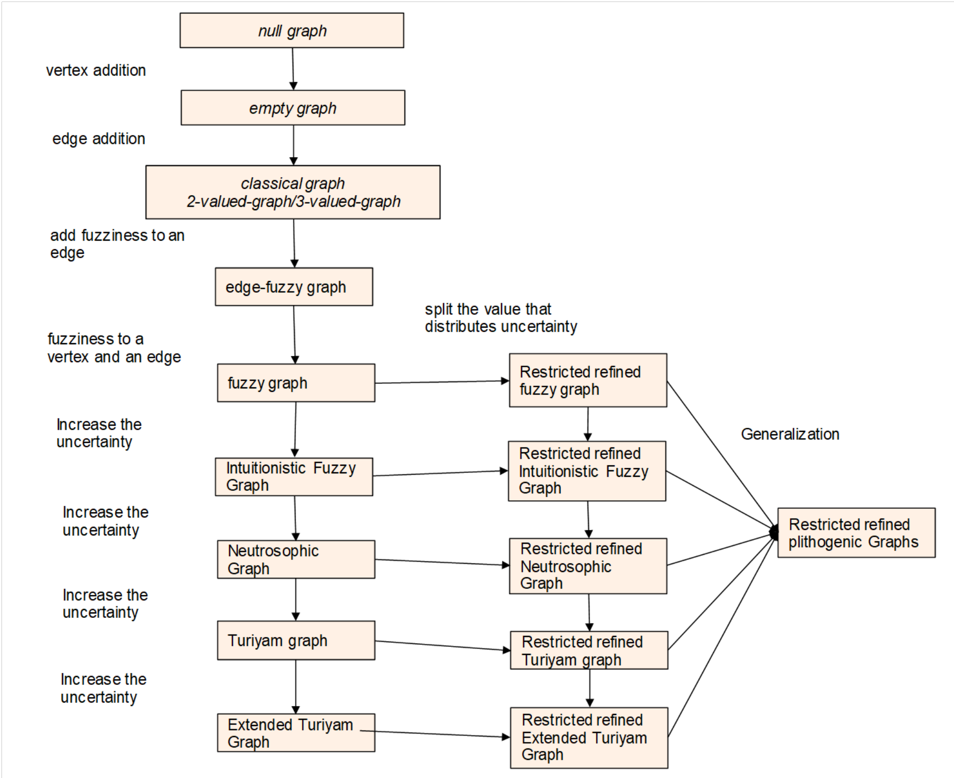

Fig. 2. Graph Hierarchy for Refined graph

4.4 Plithogenic Graph Type

Regarding the Plithogenic Graph type, specifically, we extend and examine the fuzzy graph type [97] into the Plithogenic Graph Type. As previously mentioned, I believe that determining where to incorporate uncertainty and where to maintain fixed elements is crucial for practical applications.

Definition 4.52 (Plithogenic Graph Type)**.** A plithogenic graph PG is a graph that satisfies one of the following types of plithogenic characteristics (referred to as PG of the i-th type) or any combination thereof:

(i) $PG_1 = \{G_1, G_2, G_3, \ldots, G_P\}$ where plithogenic characteristics exist in each graph G_i, incorporating different attributes and degrees of appurtenance and contradiction for vertices and edges.

(ii) $PG_2 = \{V, E_P\}$ where the edge set E_P is plithogenic, meaning that each edge is associated with a range of possible attributes and corresponding degrees of appurtenance and contradiction.

(iii) $PG_3 = \{V, E(t_P, h_P)\}$ where both the vertex set V and edge set E are crisp, but the edges have plithogenic heads $h(e_i)$ and plithogenic tails $t(e_i)$ with respect to certain attributes.

(iv) $PG_4 = \{V_P, E\}$ where the vertex set V_P is plithogenic, meaning each vertex has attributes with varying degrees of appurtenance and contradiction.

(v) $PG_5 = \{V, E(w_P)\}$ where both the vertex set V and edge set E are crisp, but the edges have plithogenic weights w_P, indicating the attribute-based degrees of appurtenance and contradiction.

The above corresponds to each graph type, specifically the fuzzy graph type, intuitionistic fuzzy graph type, neutrosophic graph type, Turiyam Neutrosophic graph type, and Extended Turiyam Neutrosophic graph type. It may be important to consider, apply, and study the appropriate graph type depending on the data being handled or the field of application.

Example 4.53. For the plithogenic graph PG, the values s and t correspond to the following graph types, each representing a specific plithogenic graph type:

- When $s = t = 1$, PG is classified as a **Plithogenic Fuzzy Graph**, denoted by PFG. This type corresponds to a fuzzy graph type where the degrees of appurtenance and contradiction for vertices and edges are governed by fuzzy logic, extending the concept of a traditional fuzzy graph into the plithogenic domain.
- When $s = 2$ and $t = 1$, PG becomes a **Plithogenic Intuitionistic Fuzzy Graph**, denoted by $PIFG$. As an intuitionistic fuzzy graph type, this form includes both membership and non-membership degrees for vertices and edges, thereby enhancing the traditional fuzzy graph with an intuitionistic dimension.
- When $s = 3$ and $t = 1$, PG is identified as a **Plithogenic Neutrosophic Graph**, denoted by PNG. This type, as a neutrosophic graph type, integrates three distinct attributes—degrees of truth, indeterminacy, and falsity—providing a more detailed and flexible representation within the plithogenic framework.
- When $s = 4$ and $t = 1$, PG is termed a **Plithogenic Turiyam Neutrosophic Graph**, denoted by $PTuG$. This graph type introduces four levels of evaluation for each attribute, adding an extra state beyond the neutrosophic type, thereby forming a more intricate type within the plithogenic classification.
- When $s = 5$ and $t = 1$, PG is known as a **Plithogenic Extended Turiyam Neutrosophic Graph**, denoted by $PETuG$. This represents the most comprehensive graph type, encompassing five different levels of evaluation for each attribute, and stands as the most generalized and inclusive plithogenic graph type.

5. Conclusions

In this paper, we introduced and explored a new class of graphs for Intuitionistic Fuzzy Graphs and Turiyam Neutrosophic Graphs, namely General Intuitionistic Fuzzy Graphs, General Turiyam Neutrosophic Graphs, and Turiyam Neutrosophic Hypergraphs. We also considered the concept of Extended Turiyam Neutrosophic Graphs. Furthermore, we investigated the potential for generalizing these graph types within the broader framework of Plithogenic Graphs. These concepts are not limited to graph theory alone; they can also be applied and hold true in other areas, such as set theory.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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Review of Some Superhypergraph classes: Directed, Bidirected, Soft, and Rough

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Abstract

Classical graph classes include undirected graphs, where edges lack orientation, and directed graphs, where edges have specific directions. Recently, concepts such as bidirected graphs have emerged, sparking ongoing research and significant advancements in the field. A Soft Set, or its extension Soft Graph, provides a flexible framework for managing uncertainty by associating elements with specific parameters, facilitating adaptable decision-making. Similarly, Rough Sets and Rough Graphs address uncertainty by employing lower and upper approximations to approximate imprecise data. A hypergraph generalizes the concept of a graph by allowing edges, called hyperedges, to connect any number of vertices, not just two. Superhypergraphs further extend this concept by permitting both vertices and edges to represent subsets, enabling the modeling of hierarchical and group-based relationships. In this paper, we explore and analyze various advanced graph structures, including directed superhypergraphs, bidirected hypergraphs, bidirected superhypergraphs, soft superhypergraphs, and rough superhypergraphs.

Keywords: Superhypergraph, hypergraph, soft graph, roudh graph, bidirected graph

1 Introduction

1.1 Graph Classes and Uncertain Graphs

Graph theory, the mathematical study of networks consisting of nodes (vertices) and connections (edges), is a well-established and widely explored field due to its extensive real-world applications [50]. Over the years, a variety of graph classes have been introduced to capture specific structural properties and characteristics [37].

Classical graph classes include undirected graphs, where edges have no orientation, and directed graphs, where edges have specific directions [32, 38, 72, 92, 93, 122]. Mixed graphs, which integrate both directed and undirected edges, further extend the classical framework [59, 126, 127]. Recently, the concept of bidirected graphs has emerged, offering greater flexibility in edge directionality and inspiring ongoing research [26, 52, 74].

To address uncertainties inherent in many real-world problems, uncertain graph models have been developed as extensions of classical graph theory. These include fuzzy graphs, intuitionistic fuzzy graphs, neutrosophic graphs, soft graphs, rough graphs, and plithogenic graphs—collectively referred to as ”uncertain graphs” [3–5, 8, 10–12, 60, 61, 63, 64, 66, 67, 69, 70, 124, 125, 131, 134, 139, 140]. For instance, fuzzy graphs model uncertain relationships by assigning membership degrees (ranging from 0 to 1) to edges and vertices, providing a flexible framework for network modeling [125]. Neutrosophic graphs, on the other hand, incorporate truth, indeterminacy, and falsehood degrees into edge and vertex definitions, enhancing their utility in analyzing complex systems. These uncertain graph models enable the representation of real-world networks with inherent ambiguity.

Among these models, this paper emphasizes soft graphs and rough graphs. A soft set [19,20,103,108,146], and its extension, the soft graph [15, 16, 87], provide a versatile framework for managing uncertainty by associating elements with specific parameters, thereby facilitating adaptable decision-making. Similarly, rough sets [81, 118–121] and rough graphs [40,47,88,114,143] employ upper and lower approximations to represent imprecise data. These models are widely studied due to their practical applicability and the mathematical elegance of their underlying structures [6, 7, 17, 21, 46, 55, 56, 65, 94, 99, 115, 129, 144, 147, 148].

1.2 Hypergraphs and Superhypergraphs

Hypergraphs generalize traditional graphs by allowing edges (referred to as hyperedges) to connect any number of vertices rather than just pairs. These structures have garnered attention due to their wide-ranging applications, including in graph databases and neural networks [18, 42, 57, 73, 82, 141].

Superhypergraphs further extend hypergraphs by permitting both vertices and edges to represent subsets, enabling the modeling of hierarchical and group-based relationships [83–85, 134–136]. These generalized structures have been further explored using fuzzy and neutrosophic frameworks, fostering active research into their theoretical and practical applications [9, 14, 31, 84, 101, 104, 105, 112].

1.3 Contributions of This Paper

Building upon these discussions, the study of superhypergraphs, soft graphs, and rough graphs holds significant importance in both theoretical and applied mathematics. This paper revisits and explores the concepts of directed superhypergraphs, bidirected hypergraphs (cf. [68]), bidirected superhypergraphs (cf. [68]), soft superhypergraphs, and rough superhypergraphs, offering new insights into their mathematical structures and potential applications. Note that some aspects involve revisiting and extending ideas from prior works, such as those presented in [68].

1.4 The Structure of the Paper

The structure of this paper is as follows.

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2 Preliminaries and Definitions

In this section, we present the Preliminaries and Definitions. For foundational concepts and notations in graph theory, readers are encouraged to consult standard texts, surveys, or lecture notes, such as [48–50, 145]. Basic principles from set theory are also utilized throughout this work. For detailed discussions on these topics, references such as [58, 86, 89, 90, 98] are recommended. For specific operations and related topics discussed in this paper, readers may refer to the respective references for further insights as needed.

2.1 hypergraph and superhypergraph

A hypergraph is a generalized graph concepts that extends traditional graph concepts by allowing hyperedges, which connect multiple vertices rather than just pairs, enabling more complex relationships between elements [28–30, 78–80]. The basic definitions of graphs and hypergraphs are provided below.

Definition 2.1 (Graph)**.** [50] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2.2 (Subgraph)**.** [50] Let $G = (V, E)$ be a graph. A *subgraph* $H = (V_H, E_H)$ of G is a graph such that:

- $V_H \subseteq V$, i.e., the vertex set of H is a subset of the vertex set of G.
- $E_H \subseteq E$, i.e., the edge set of H is a subset of the edge set of G.
- Each edge in E_H connects vertices in V_H.

Definition 2.3 (Hypergraph)**.** [30] A *hypergraph* $H = (V, E)$ consists of a set V of vertices and a set E of hyperedges. Each hyperedge $e \in E$ is defined as a subset of V, thus $e \subseteq V$, and $E \subseteq \mathcal{P}(V)$, where $\mathcal{P}(V)$ denotes the power set of V.

The SuperHyperGraph is a generalized graph concept that extends the Hypergraph by incorporating supervertices and superedges [62, 76, 83–85, 133–136, 138]. Definitions of this and related concepts are introduced below.

Definition 2.4 (SuperHyperGraph)**.** [133] Let V be a finite set of vertices. A *superhypergraph* is an ordered pair $H = (V, E)$, where:

- $V \subseteq P(V)$ (the power set of V), meaning that each element of V can be either a single vertex or a subset of vertices (called a *supervertex*).
- $E \subseteq P(V)$ represents the set of edges, called *superedges*, where each $e \in E$ can connect multiple supervertices.

In this framework, a superhypergraph can accommodate complex relationships among groups of vertices, including single edges, hyperedges, superedges, and multi-edges. Superhypergraphs provide a flexible structure to represent high-order and hierarchical relationships.

2.2 directed hypergraph

A directed hypergraph is a hypergraph generalization of a directed graph. Similar to undirected hypergraphs, directed hypergraphs have been extensively studied for their various derivatives and applications (cf. [13, 39, 54, 95, 97, 109–111, 116, 117]). Its definition is provided below.

Definition 2.5 (Directed Graph)**.** [149] A *directed graph* (digraph) $G = (V, E)$ consists of:

- V: A finite set of vertices.
- $E \subseteq V \times V$: A set of directed edges, where each edge is an ordered pair (u, v) with $u, v \in V$.

The edge (u, v) indicates a directed connection from vertex u (source) to vertex v (target).

Definition 2.6 (Directed Hypergraph)**.** [27, 71] A *Directed Hypergraph* H is a pair $H = (V, E)$, where:

- V is a finite set of vertices (or nodes).
- E is a finite set of hyperarcs. Each hyperarc $e \in E$ is an ordered pair $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$, where:
  - $\mathrm{Tail}(e) \subseteq V$ is a non-empty subset of vertices, called the *tail* of the hyperarc.
  - $\mathrm{Head}(e) \in V$ is a single vertex, called the *head* of the hyperarc.

Properties

- A hyperarc $e = (\mathrm{Tail}(e), \mathrm{Head}(e))$ connects all vertices in $\mathrm{Tail}(e)$ to the vertex $\mathrm{Head}(e)$.
- When $|\mathrm{Tail}(e)| = 1$ for all $e \in E$, the directed hypergraph reduces to a standard directed graph.

2.3 bidirected hypergraph

A bidirected graph is a graph that offers greater flexibility in assigning directions to edges compared to a directed graph. Due to its advantages over directed graphs in certain applications, bidirected graphs have been extensively studied, similar to other graph classes [33, 34, 36, 77, 91, 96, 123]. The definition is provided below [26, 52, 74].

Definition 2.7. (cf. [26, 52, 74]) A *bidirected graph* is a directed graph with independent orientations at each endpoint of an edge. Formally, a bidirected graph is defined as:

$$G = (V, E, \tau),$$

where:

- V is the set of vertices.
- E is the set of edges, each connecting two vertices.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function:
  - $\tau(v, e) = 1$: Edge e is directed toward vertex v.
  - $\tau(v, e) = -1$: Edge e is directed away from vertex v.
  - $\tau(v, e) = 0$: Vertex v is not incident to edge e.

Based on the above definition of a bidirected graph, we first define a bidirected hypergraph as described below. Subsequently, we examine its relationships with other graph classes.

Definition 2.8. A *bidirected hypergraph* extends the concept of bidirected graphs to hypergraphs. Formally:

$$H = (V, E, \tau),$$

where:

- V is the set of vertices.
- E is the set of hyperedges, where each $e \in E$ is a subset of V.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function.

Theorem 2.9. *A bidirected hypergraph $H = (V, E, \tau)$ can be reduced to a standard hypergraph $H' = (V, E')$ by removing the bidirection function τ, while preserving the vertex memberships in each hyperedge.*

Proof. Consider a bidirected hypergraph $H = (V, E, \tau)$, where:

- V is the set of vertices.
- E is the set of hyperedges.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function.

To transform H into a standard hypergraph, define:

$$H' = (V, E'),$$

where $E' = \{e \in E \mid e \subseteq V\}$. Since hyperedges in E in a bidirected hypergraph are subsets of V, we can preserve all hyperedge memberships while discarding the orientation information τ. The resulting structure satisfies the definition of a standard hypergraph.

Removing the bidirection function τ does not alter the membership structure of the hyperedges, resulting in a standard hypergraph. □

Theorem 2.10. *A bidirected hypergraph $H = (V, E, \tau)$ can be transformed into a directed hypergraph $H' = (V, E')$ by defining the tail and head sets of each hyperedge based on the bidirection function τ.*

Proof. Let $H = (V, E, \tau)$ be a bidirected hypergraph. For each hyperedge $e \in E$, define:

$$T(e) = \{v \in V \mid \tau(v, e) = -1\}, \quad H(e) = \{v \in V \mid \tau(v, e) = 1\}.$$

The tail set $T(e)$ represents vertices pointing away from the hyperedge, and the head set $H(e)$ represents vertices pointing toward the hyperedge. Construct the directed hypergraph:

$$H' = (V, E'),$$

where $E' = \{(T(e), H(e)) \mid e \in E\}$. Each directed hyperedge in E' satisfies the definition of a directed hypergraph edge, connecting a set of tail vertices to a set of head vertices.

By splitting each hyperedge in H into a tail set $T(e)$ and a head set $H(e)$, the bidirected hypergraph transforms into a directed hypergraph. □

Theorem 2.11. *A bidirected hypergraph $H = (V, E, \tau)$ can be reduced to a bidirected graph $G = (V, E', \tau')$ by restricting the hyperedges to pairs of vertices and preserving the bidirection function for these edges.*

Proof. Let $H = (V, E, \tau)$ be a bidirected hypergraph. For each hyperedge $e \in E$, if $|e| = 2$, let $e = \{u, v\}$. Define:

$$E' = \{\{u, v\} \mid e \in E, |e| = 2\}.$$

The bidirection function τ for the original hyperedges is inherited by the edges in E', i.e.,

$$\tau'(v, e) = \tau(v, e), \quad \forall v \in V,\ e \in E'.$$

Construct the bidirected graph:

$$G = (V, E', \tau').$$

The resulting graph G satisfies the properties of a bidirected graph since each edge in E' connects exactly two vertices, and the bidirection function τ' provides independent orientations for each vertex in e.

By restricting hyperedges to pairs of vertices and preserving the bidirection function, the bidirected hypergraph transforms into a bidirected graph. □

2.4 soft hypergraph

A *Soft Hypergraph* is a mathematical structure that combines the concepts of soft sets and hypergraphs to model relationships in a parameterized and flexible manner.

Definition 2.12. [15, 87] Let $G = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges. Let A be a non-empty set of parameters and $R \subseteq A \times V$ be a relation between elements of A and elements of V. Define a set-valued function $F : A \to P(V)$ by

$$F(x) = \{y \in V \mid xRy\}.$$

The pair (F, A) is a soft set over V.

A *Soft Graph* of G is defined as follows:

A soft set (F, A) over V is said to be a soft graph of G if the subgraph $F(x)$
is a connected subgraph of G for all $x \in A$. The set of all soft graphs of G is denoted by $SG(G)$.

Definition 2.13. (cf. [22, 75]) Let $H^* = (V, E)$ be a simple hypergraph, where:

- V is the vertex set, and
- $E \subseteq \mathcal{P}(V)$ is the set of hyperedges (subsets of V).

Additionally, let C be a non-empty set of parameters. A *Soft Hypergraph* is defined as a 4-tuple $H = (H^*, A, B, C)$, where:

1. $H^* = (V, E)$ is a simple hypergraph.
2. C is a non-empty set of parameters.
3. (A, C) is a soft set over V, where $A : C \to \mathcal{P}(V)$ is a mapping defined by:
$$A(c) = \{v \in V \mid (c, v) \in R\},$$
and $R \subseteq C \times V$ is a relation.
4. (B, C) is a soft set over Es (the set of all subhyperedges of H^*), where $B : C \to \mathcal{P}(\text{Es})$ is defined by:
$$B(c) = \{\text{m-subhyperedges } \langle A(c) \rangle\},$$
and m-subhyperedges $\langle A(c) \rangle$ is the set of all maximum subhyperedges formed from $A(c)$.

A soft hypergraph $H = (H^*, A, B, C)$ must satisfy the following conditions:

1. $H^* = (V, E)$ is a simple hypergraph.
2. C is a non-empty set of parameters.
3. (A, C) is a soft set over V.
4. (B, C) is a soft set over Es, the set of all subhyperedges of H^*.
5. For each $c \in C$, $(A(c), B(c))$ forms a semisubhypergraph of H^*, meaning:
$$A(c) \subseteq V \quad \text{and} \quad B(c) \subseteq \text{subhyperedges of } H^*.$$

The soft hypergraph H can also be represented as:

$$H = \{F(c) : c \in C\},$$

where $F(c) = (A(c), B(c))$ is referred to as a *hyperpart* (or *h-part*) of the soft hypergraph corresponding to the parameter c.

- **Soft Vertex Set** (A, C)**:** A parameterized view of the vertex set V, where each parameter $c \in C$ maps to a subset $A(c) \subseteq V$.
- **Soft Hyperedge Set** (B, C)**:** A parameterized view of subhyperedges of H^*, where each parameter $c \in C$ maps to a subset $B(c) \subseteq$ Es consisting of maximum subhyperedges derived from $A(c)$.

- **Hyperpart** $F(c) = (A(c), B(c))$**:** Describes a specific semisubhypergraph of H^* associated with the parameter c.

Theorem 2.14. *A* Soft Hypergraph *is a generalization of both* Soft Graphs *and* Hypergraphs*.*

Proof. Let $H = (H^*, A, B, C)$ be a soft hypergraph, where:

- $H^* = (V, E)$ is a simple hypergraph, with V as the vertex set and $E \subseteq \mathcal{P}(V)$ as the set of hyperedges.
- (A, C) is a soft set over V, providing a parameterized view of the vertex set.
- (B, C) is a soft set over Es, the set of all subhyperedges of H^*, parameterizing the hyperedges.
- C is a non-empty set of parameters.

Reduction to a Hypergraph If we ignore the parameterization induced by C, the soft hypergraph $H = (H^*, A, B, C)$ reduces to the underlying hypergraph $H^* = (V, E)$. Specifically:

- The vertex set V remains the same.
- The edge set E, which is a subset of the power set $\mathcal{P}(V)$, represents the hyperedges of the hypergraph.

Thus, H^* retains the structure of a standard hypergraph, demonstrating that soft hypergraphs generalize hypergraphs by adding a parameterization framework.

Reduction to a Soft Graph If the edge set E of the underlying hypergraph H^* is restricted such that all hyperedges $e \in E$ are pairs of vertices (i.e., $e = \{u, v\}$ with $|e| = 2$), the hypergraph $H^* = (V, E)$ reduces to a graph $G = (V, E)$. In this case:

- The soft set (A, C) over V provides a parameterized view of the vertex set.
- The soft set (B, C) over Es parameterizes the edges, where Es $= E$.

This results in the structure of a soft graph, as defined in the literature, showing that soft hypergraphs generalize soft graphs by allowing hyperedges of arbitrary size.

A soft hypergraph incorporates the concepts of both soft sets and hypergraphs, parameterizing vertices and hyperedges. By constraining its structure:

- Removing parameterization reduces the soft hypergraph to a standard hypergraph.
- Restricting hyperedges to pairs of vertices reduces the soft hypergraph to a soft graph.

Hence, a soft hypergraph is a unifying generalization of soft graphs and hypergraphs. □

2.5 rough hypergraph

A *rough hypergraph* extends the concept of hypergraphs using rough set theory. The definitions, including related concepts, are provided below [128].

Definition 2.15 (Approximation Space)**.** (cf. [102, 150]) Let Q be a non-empty set, and let φ be an equivalence relation (EQ) on Q. The pair (Q, φ) is called an *approximation space*.

For any subset $A \subseteq Q$, the *upper approximation* $\varphi(A)$ and *lower approximation* $\varphi(A)$ are defined as:

$$\varphi(A) = \{d \in Q \mid [d]_\varphi \cap A \neq \emptyset\},$$

$$\varphi(A) = \{d \in Q \mid [d]_\varphi \subseteq A\},$$

where $[d]_\varphi = \{g \in Q \mid (d, g) \in \varphi\}$ is the equivalence class of d under φ.

The pair $(\varphi(A), \varphi(A))$ is referred to as a *rough set* on Q.

Definition 2.16 (Rough Set)**.** [81, 118–121] Let U be a universe of discourse and R a relation on U that induces a partition of U into equivalence classes. For a subset $X \subseteq U$, the *lower approximation* of X with respect to R (denoted $R(X)$) is the set of elements that are certainly in X given the information provided by R:

$$R(X) = \{x \in U \mid [x]_R \subseteq X\}.$$

The *upper approximation* of X (denoted $R(X)$) is the set of elements that possibly belong to X:

$$R(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}.$$

The pair $(R(X), R(X))$ is called the *Rough Set* approximation of X with respect to R.

Definition 2.17 (Rough Graph)**.** [40, 47, 88, 114, 143] Let $G = (V, E)$ be a graph where V is the set of vertices and E is the set of edges. Let R be an attribute set on E, inducing an equivalence relation on the edges. For any edge set $X \subseteq E$, the *lower approximation* of X with respect to R (denoted $R(X)$) is defined as:

$$R(X) = \{e \in E \mid [e]_R \subseteq X\},$$

where $[e]_R$ denotes the equivalence class of e under R. The *upper approximation* of X (denoted $R(X)$) is defined as:

$$R(X) = \{e \in E \mid [e]_R \cap X \neq \emptyset\}.$$

A graph $G = (V, E)$ is called an *R-rough graph* if X is not exactly definable under R, and it is characterized by the pair $(R(X), R(X))$, where $R(X)$ is the *lower approximation graph* and $R(X)$ is the *upper approximation graph*.

Definition 2.18 (Rough Hypergraph)**.** [128] Let Q be a non-empty set, and let φ be an equivalence relation on Q. For $A \subseteq Q$, let $(\varphi(A), \varphi(A))$ represent a rough set on Q.

Let ψ be an equivalence relation on $M \subseteq \mathcal{P}(Q) \setminus \{\emptyset\}$ (the power set of Q excluding the empty set). For $D \subseteq \mathcal{P}(A) \setminus \{\emptyset\}$, the *upper* and *lower approximations* of D are defined as:

$$\psi(D) = \{V \in M \mid [V]_\psi \cap D \neq \emptyset\},$$

$$\psi(D) = \{V \in M \mid [V]_\psi \subseteq D\}.$$

The pair $(\psi(D), \psi(D))$ is referred to as a *rough relation* on Q. If $\psi(D) \subseteq \mathcal{P}(\varphi(A))$, the pair $(\psi(D), \psi(D))$ is a *rough relation* on $(\varphi(A), \varphi(A))$.

A *rough hypergraph* on Q is defined as a triplet $(Q, \varphi(A), \psi(D))$, where:

1. φ is an equivalence relation on the vertex set Q,
2. $(\varphi(A), \varphi(A))$ is a rough set on Q for $A \subseteq Q$,

3. ψ is an equivalence relation on $M \subseteq \mathcal{P}(Q) \setminus \{\emptyset\}$,
4. $(\psi(D), \psi(D))$ is a rough relation on $(\varphi(A), \varphi(A))$, where $\psi(D) \subseteq \mathcal{P}(\varphi(A))$.

The rough hypergraph $R = (Q, \varphi(A), \psi(D))$ can also be represented as:

$$R = (R, R),$$

where $R = (\varphi(A), \psi(D))$ and $R = (\varphi(A), \psi(D))$ are hypergraphs.

Theorem 2.19. *A rough hypergraph $R = (Q, \varphi_A, \psi_D)$, where φ_A represents the rough set approximations on the vertex set Q and ψ_D represents rough approximations on subsets of hyperedges, generalizes both hypergraphs and rough graphs.*

Proof. To show that a rough hypergraph generalizes a standard hypergraph, consider the following:

1. Let $R = (Q, \varphi_A, \psi_D)$ be a rough hypergraph, where:
   - Q is the set of vertices.
   - $\varphi_A = (\varphi A, \varphi A)$ is the rough approximation of a vertex set $A \subseteq Q$.
   - $\psi_D = (\psi D, \psi D)$ is the rough approximation of subsets of hyperedges $D \subseteq \mathcal{P}(Q)$.
2. Remove the rough approximations φ_A and ψ_D, and consider only the underlying sets:
   - Replace φ_A with Q, the vertex set.
   - Replace ψ_D with D, the collection of hyperedges.
3. The resulting structure is a hypergraph $H = (Q, D)$, where:
   - Q is the vertex set.
   - $D \subseteq \mathcal{P}(Q)$ is the set of hyperedges.

Thus, a rough hypergraph reduces to a standard hypergraph when rough approximations are ignored.

To show that a rough hypergraph generalizes a rough graph, consider the following:

1. Let $R = (Q, \varphi_A, \psi_D)$ be a rough hypergraph, where:
   - $\varphi_A = (\varphi A, \varphi A)$ represents rough approximations on the vertex set Q.
   - $\psi_D = (\psi D, \psi D)$ represents rough approximations on subsets of hyperedges.
2. Restrict the hyperedges $D \subseteq \mathcal{P}(Q)$ such that all hyperedges contain exactly two vertices:

$$D' = \{e \in D \mid |e| = 2\}.$$

3. The resulting structure is a rough graph $G = (Q, \varphi_A, \psi'_D)$, where:
   - Q is the set of vertices.
   - $\varphi_A = (\varphi A, \varphi A)$ represents the rough set approximations on the vertex set.
   - $\psi'_D = (\psi D', \psi D')$ represents rough approximations on the edges.

Thus, a rough hypergraph reduces to a rough graph when hyperedges are restricted to pairs of vertices.

By demonstrating that a rough hypergraph $R = (Q, \varphi_A, \psi_D)$ reduces to a hypergraph $H = (Q, D)$ or a rough graph $G = (Q, \varphi_A, \psi'_D)$ under appropriate transformations, it is proven that the concept of rough hypergraphs generalizes both hypergraphs and rough graphs. $\square$

2.6 k-zero-divisor hypergraph

The k-zero-divisor hypergraph is a well-established concept [41, 53, 130]. The definition, along with related concepts, is provided below.

Definition 2.20 (Commutative Ring)**.** (cf. [24, 106, 107]) A *commutative ring* is a set R equipped with two binary operations, addition + and multiplication $\cdot$, such that:

- $(R, +)$ is an abelian group.
- $(R, \cdot)$ is a monoid with an identity element $1 \in R$ (i.e., multiplication is associative, and there exists a multiplicative identity 1).
- Multiplication is commutative, i.e., $a \cdot b = b \cdot a$ for all $a, b \in R$.
- Multiplication is distributive over addition, i.e., $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$.

Definition 2.21 (Zero-Divisor)**.** (cf. [2,113]) In a commutative ring R, an element $a \in R$ is called a *zero-divisor* if there exists a non-zero element $b \in R$ such that $a \cdot b = 0$.

Definition 2.22 (Zero-divisor Graph)**.** (cf. [1, 23, 25, 100]) Let R be a commutative ring with unity, and let $Z(R)$ denote the set of zero-divisors of R. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is an undirected graph defined as follows:

- The vertex set of $\Gamma(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, i.e., the set of nonzero zero-divisors of R.
- For distinct $x, y \in Z(R)^*$, there is an edge between x and y if and only if $xy = 0$ in R.

Thus, the graph $\Gamma(R)$ captures the relationships between the nonzero zero-divisors of R. If R is an integral domain, $\Gamma(R)$ is the empty graph.

Definition 2.23 (k-zero-divisor hypergraph)**.** [41,53,130] Let R be a commutative ring with identity, and let $Z(R)$ denote the set of zero-divisors in R. A k-zero-divisor hypergraph, denoted $H_k(R) = (V_k(R), E_k(R))$, is defined as follows:

- **Vertex Set**:
$$V_k(R) = \{(x, i) \mid x \in Z_k(R), 1 \le i \le n_k(x)\} \subseteq Z_k(R) \times \{1, 2, \dots, k\},$$
where $Z_k(R)$ is the set of k-zero-divisors, and $n_k(x)$ is the appearance number of x.
- **Hyperedge Set**:
$$E_k(R) = \{\{(x_1, i_1), (x_2, i_2), \dots, (x_k, i_k)\} \mid x_1 x_2 \cdots x_k = 0 \text{ and } \prod_{x \in S} x \ne 0, \forall S \subset \{x_1, \dots, x_k\}\}.$$

Note that For any $k \ge 2$,
$$Z_{k+1}(R) \subseteq Z_k(R) \subseteq \cdots \subseteq Z_2(R) = Z(R) \setminus \{0\}.$$

Example 2.24. Let $R = \mathbb{Z}_6$. The zero-divisors are $Z(\mathbb{Z}_6) = \{2, 3, 4\}$. For $k = 3$, the hypergraph $H_3(\mathbb{Z}_6)$ has:
$$V_3(\mathbb{Z}_6) = \{2, 3, 4\}, \quad E_3(\mathbb{Z}_6) = \{\{2, 3, 4\}\}.$$

3 Result: Some Concepts

In this section, we examine Directed SuperHypergraphs, BiDirected SuperHypergraphs, Soft SuperHypergraphs, and Rough SuperHypergraphs.

3.1 Directed SuperHypergraph

Define the Directed SuperHypergraph and examine its relationships with other graph classes. The definition is provided below.

Definition 3.1 (Directed SuperHypergraph)**.** A *Directed SuperHypergraph* DSH is a tuple:

$$\mathrm{DSH} = (V, E),$$

where:

- V is a finite set of vertices.
- $E \subseteq 2^V \times 2^V$ is a set of directed superhyperedges. Each directed superhyperedge $e = (T, H) \in E$ satisfies:
  - $T \subseteq V$: the **tail set**, representing the source vertices.
  - $H \subseteq V$: the **head set**, representing the target vertices.

Properties

- A directed superhyperedge $e = (T, H)$ connects multiple source vertices in T to multiple target vertices in H.
- If $T = \{v_1\}$ and $H = \{v_2\}$, then e behaves like a directed edge in a standard directed graph.
- Directed superhyperedges extend the flexibility of traditional directed graphs and directed hypergraphs.

Theorem 3.2. *A Directed Hypergraph DH is a special case of a Directed SuperHypergraph DSH, where the head of every edge consists of a single vertex (i.e., $|H| = 1$ for all $(T, H) \in E$).*

Proof. Let $\mathrm{DH} = (V, E_{\mathrm{DH}})$ be a directed hypergraph, where:

$$E_{\mathrm{DH}} = \{(T, h) \mid T \subseteq V,\ h \in V\}.$$

Define a corresponding Directed SuperHypergraph $\mathrm{DSH} = (V, E_{\mathrm{DSH}})$ such that:

$$E_{\mathrm{DSH}} = \{(T, H) \mid (T, h) \in E_{\mathrm{DH}},\ H = \{h\}\}.$$

- For each directed hyperedge $(T, h) \in E_{\mathrm{DH}}$, construct a directed superhyperedge $(T, H) \in E_{\mathrm{DSH}}$ with $H = \{h\}$.
- Since $H \subseteq V$ and $|H| = 1$, DSH satisfies the definition of a Directed Hypergraph.

Thus, any Directed Hypergraph DH can be represented as a Directed SuperHypergraph DSH, proving that DSH generalizes DH. □

Theorem 3.3. *A Directed SuperHypergraph DSH $= (V, E)$ becomes a SuperHypergraph when all directed superhyperedges $(T, H) \in E$ are treated as undirected edges by merging their tail and head sets $e' = T \cup H$.*

Proof. Let $\mathrm{DSH} = (V, E)$, where each directed superhyperedge $e = (T, H) \in E$ satisfies $T, H \subseteq V$. To transform DSH into an undirected superhypergraph SH, define:

$$E' = \{T \cup H \mid (T, H) \in E\}.$$

The resulting graph $\mathrm{SH} = (V, E')$ satisfies the definition of a SuperHypergraph:

- V is the set of vertices, unchanged from DSH.
- $E' \subseteq 2^V$, where each edge $e' = T \cup H$ is a subset of V.

Since no directionality is preserved in E', SH represents an undirected SuperHypergraph. Thus, DSH reduces to a SuperHypergraph when all directed edges are treated as undirected by merging their tail and head sets. □

Theorem 3.4. *A Directed Graph $G = (V, A)$ is a special case of a Directed SuperHypergraph DSH $= (V, E)$ where each directed edge $a = (u, v) \in A$ is represented as a directed superhyperedge $e = (\{u\}, \{v\}) \in E$.*

Proof. Let $G = (V, A)$ be a Directed Graph, where $A \subseteq V \times V$. Define a Directed SuperHypergraph DSH $= (V, E)$ such that:

$$E = \{(\{u\}, \{v\}) \mid (u, v) \in A\}.$$

- V: The vertex set of DSH is the same as that of G.
- E: The set of directed superhyperedges in DSH corresponds exactly to the set of directed edges in G, with each edge $(u, v) \in A$ represented as $(\{u\}, \{v\}) \in E$.

Each directed edge (u, v) in G is uniquely represented in DSH as a directed superhyperedge $(\{u\}, \{v\})$. Hence, the structure of G is preserved within DSH, making G a special case of DSH where each superhyperedge connects exactly one vertex in the tail set to one vertex in the head set. □

3.2 Bidirected Superhypergraph

A **Bidirected Superhypergraph** is a mathematical structure that combines the concepts of superhypergraphs and bidirected hypergraphs, allowing for greater flexibility in the assignment of directions to edges and accommodating hierarchical relationships among vertices.

Definition 3.5 (Bidirected Superhypergraph)**.** Let U be a finite set called the universe. A *bidirected superhypergraph* is a triple $H = (V, E, \tau)$ where:

- $V \subseteq \mathcal{P}(U)$ is the set of *supervertices*, where each supervertex is a subset of U.
- $E \subseteq \mathcal{P}(V)$ is the set of *superedges*, where each superedge is a subset of V.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the *bidirection function*, assigning orientations to the incidence of supervertices and superedges:
  - $\tau(v, e) = 1$: Superedge e is directed *toward* supervertex v.
  - $\tau(v, e) = -1$: Superedge e is directed *away from* supervertex v.
  - $\tau(v, e) = 0$: Supervertex v is not incident to superedge e.

The bidirection function τ allows each supervertex to have an independent orientation with respect to each incident superedge.

Theorem 3.6. *A bidirected superhypergraph $H = (V, E, \tau)$ generalizes both* superhypergraphs *and* bidirected hypergraphs.

Proof. To prove this theorem, we show that by appropriate simplifications, a bidirected superhypergraph reduces to a superhypergraph or a bidirected hypergraph.

If we disregard the bidirection function τ, the bidirected superhypergraph $H = (V, E, \tau)$ reduces to the superhypergraph $H' = (V, E)$, where:

- $V \subseteq \mathcal{P}(U)$ is the set of supervertices.
- $E \subseteq \mathcal{P}(V)$ is the set of superedges.

This structure satisfies the definition of a superhypergraph, where the vertices are subsets of U (supervertices), and edges are subsets of V (superedges). By removing τ, we eliminate the directional information, thus obtaining a standard superhypergraph.

If we consider that all supervertices are singletons, i.e., $V = \{\{v\} \mid v \in U\}$, then V can be identified with U. In this case:

- Each supervertex $\{v\}$ corresponds to a vertex v in U.
- Each superedge $e \in E$ becomes a subset of V, which corresponds to a subset of U.
- The bidirection function τ becomes $\tau : U \times E \to \{-1, 0, 1\}$.

Thus, the bidirected superhypergraph $H = (V, E, \tau)$ reduces to a bidirected hypergraph $H'' = (U, E', \tau')$, where:

- U is the set of vertices.
- $E' = \{e' \subseteq U \mid e' = \{v \mid \{v\} \in e\} \text{ for } e \in E\}$.
- $\tau' : U \times E' \to \{-1, 0, 1\}$ is defined by $\tau'(v, e') = \tau(\{v\}, e)$.

This structure satisfies the definition of a bidirected hypergraph, where vertices are elements of U, edges are subsets of U, and τ' assigns independent orientations at each vertex-edge incidence.

Since a bidirected superhypergraph can be simplified to yield both a superhypergraph and a bidirected hypergraph, it generalizes both concepts. □

Theorem 3.7. *A Bidirected SuperHypergraph $H = (V, E, \tau)$ generalizes a Directed SuperHypergraph DSH $= (V, E')$, where $E' \subseteq 2^V \times 2^V$ is the set of directed superhyperedges.*

Proof. To prove this, we show that a Bidirected SuperHypergraph $H = (V, E, \tau)$ reduces to a Directed SuperHypergraph DSH $= (V, E')$ when appropriate restrictions are applied to the bidirection function τ.

Construction of DSH from H: Let $H = (V, E, \tau)$, where:

- $V \subseteq \mathcal{P}(U)$ is the set of supervertices.
- $E \subseteq \mathcal{P}(V)$ is the set of superedges.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function.

Define the directed superhyperedge set E' of DSH as follows:

$$E' = \{(T(e), H(e)) \mid e \in E\},$$

where the **tail set** $T(e)$ and **head set** $H(e)$ are determined from the bidirection function τ:

$$T(e) = \{v \in V \mid \tau(v, e) = -1\}, \quad H(e) = \{v \in V \mid \tau(v, e) = 1\}.$$

The directed superhypergraph DSH $= (V, E')$ satisfies the following:

- Each superhyperedge $e \in E$ in H corresponds to a directed superhyperedge $(T(e), H(e)) \in E'$.
- The bidirection function τ is used to assign vertices to either the tail set $T(e)$ or the head set $H(e)$, consistent with the definition of directed superhyperedges.

Reduction Conditions: A Bidirected SuperHypergraph $H = (V, E, \tau)$ reduces to a Directed SuperHypergraph DSH = (V, E') under the following conditions:

- The bidirection function τ is such that every $v \in V$ satisfies:

$$\tau(v, e) \in \{-1, 0, 1\}, \quad \text{and} \quad \tau(v, e) \neq 0 \text{ for } v \in T(e) \cup H(e).$$

- No vertex v belongs to both $T(e)$ and $H(e)$, i.e., $T(e) \cap H(e) = \emptyset$.

Under the above conditions, the Bidirected SuperHypergraph $H = (V, E, \tau)$ can be uniquely mapped to the Directed SuperHypergraph DSH = (V, E'). Since the bidirection function τ can specify distinct tail and head sets for each superedge e, Bidirected SuperHypergraphs naturally generalize Directed SuperHypergraphs. □

Theorem 3.8. *Let $H = (V, E, \tau)$ be a bidirected superhypergraph. For any subsets $V' \subseteq V$ and $E' \subseteq E$ such that $E' \subseteq \mathcal{P}(V')$, the structure $H' = (V', E', \tau')$ with $\tau' = \tau \restriction_{V' \times E'}$ is a* sub-bidirected superhypergraph *of H.*

Proof. Since $V' \subseteq V$ and $E' \subseteq E$, and τ' is the restriction of τ to $V' \times E'$, all definitions and properties required for H' to be a bidirected superhypergraph are satisfied. The incidence structure and bidirection function are preserved within V' and E'. □

Theorem 3.9. *A bidirected graph $G = (V, E, \tau)$ is a special case of a bidirected superhypergraph $H = (V, E, \tau)$ where all supervertices are singleton sets and all superedges are pairs of vertices.*

Proof. Let $G = (V, E, \tau)$ be a bidirected graph, where:

- V is the set of vertices.
- E is the set of edges, where each $e \in E$ connects exactly two vertices $u, v \in V$.
- $\tau : V \times E \to \{-1, 0, 1\}$ is the bidirection function.

Define a bidirected superhypergraph $H = (V', E', \tau')$ as follows:

- $V' = \{\{v\} \mid v \in V\}$, i.e., each vertex in V corresponds to a singleton supervertex in V'.
- $E' = \{\{\{u\}, \{v\}\} \mid e = (u, v) \in E\}$, i.e., each edge in E corresponds to a superedge connecting the respective supervertices in E'.
- $\tau'(\{v\}, e') = \tau(v, e)$, i.e., the orientation of each vertex with respect to an edge in G is preserved in H.

Since V' consists of singleton sets and E' consists of pairs of vertices, H satisfies the structure of a bidirected superhypergraph and retains all properties of G. Thus, G is a special case of H. □

Theorem 3.10. *A directed hypergraph $H = (V, E)$ is a special case of a bidirected superhypergraph $H' = (V, E', \tau)$ where all supervertices are singleton sets and all superedges have one subset of vertices directed into another.*

Proof. Let $H = (V, E)$ be a directed hypergraph, where:

- V is the set of vertices.
- E is the set of directed hyperedges, where each hyperedge $e = (\text{Tail}(e), \text{Head}(e))$ satisfies:
  - $\text{Tail}(e) \subseteq V$: the tail set.
  - $\text{Head}(e) \subseteq V$: the head set.

Define a bidirected superhypergraph $H' = (V', E', \tau)$ as follows:

- $V' = \{\{v\} \mid v \in V\}$, i.e., each vertex in V corresponds to a singleton supervertex in V'.
- $E' = \{\mathrm{Tail}(e) \cup \mathrm{Head}(e) \mid e \in E\}$, i.e., each hyperedge in E corresponds to a superedge connecting the union of tail and head vertices.
- $\tau(\{v\}, e')$ is defined as:

$$\tau(\{v\}, e') = \begin{cases} -1 & \text{if } v \in \mathrm{Tail}(e), \\ 1 & \text{if } v \in \mathrm{Head}(e), \\ 0 & \text{otherwise.} \end{cases}$$

Since V' consists of singleton sets and E' represents the union of the tail and head sets with appropriate directions, H' satisfies the structure of a bidirected superhypergraph while retaining the properties of the directed hypergraph H. Thus, H is a special case of H'. □

3.3 soft superhypergraph

A Soft Superhypergraph is a mathematical structure that extends both the concepts of soft sets and superhypergraphs to model higher-order relationships in a parameterized and flexible manner.

Definition 3.11. Let $H^* = (V, E)$ be a superhypergraph, where:

- V is a finite set of vertices.
- $E \subseteq \mathcal{P}(\mathcal{P}(V))$ is the set of superhyperedges; that is, each superhyperedge E is a subset of $\mathcal{P}(V)$, the power set of V.

Let Es be the set of all *subsuperhyperedges* of H^*, defined as:

$$\mathrm{Es} = \{D \subseteq \mathcal{P}(V) \mid D \subseteq E \text{ for some } E \in E\}.$$

Let C be a non-empty set of parameters, and let $R \subseteq C \times V$ be a relation between parameters and vertices.

Define the mapping $A : C \to \mathcal{P}(V)$ by:

$$A(c) = \{v \in V \mid (c, v) \in R\}.$$

Define the mapping $B : C \to \mathcal{P}(\mathrm{Es})$ by:

$$B(c) = \{\text{maximum subsuperhyperedges generated from } A(c)\}.$$

Then, the 4-tuple $H = (H^*, A, B, C)$ is called a *Soft Superhypergraph* if it satisfies the following conditions:

1. $H^* = (V, E)$ is a superhypergraph.
2. C is a non-empty set of parameters.
3. (A, C) is a soft set over V.
4. (B, C) is a soft set over Es.
5. For each $c \in C$, $(A(c), B(c))$ forms a *semisubsuperhypergraph* of H^*.

Definition 3.12. A hypergraph $H' = (V', E')$ is called a *Semisubsuperhypergraph* of $H^* = (V, E)$ if:

- $V' \subseteq V$.
- Each $D \in E'$ is a subsuperhyperedge of some superhyperedge $E \in E$; that is, $D \subseteq E$.
- $E' \subseteq$ Es.

Definition 3.13. For $A(c) \subseteq V$, a subsuperhyperedge $D \subseteq \mathcal{P}(V)$ is called a *Maximum Subsuperhyperedge* (m-subsuperhyperedge) with respect to $A(c)$ if:

- $D \subseteq E$ for some $E \in E$ (i.e., D is a subsuperhyperedge of a superhyperedge in H^*).
- There is no other subsuperhyperedge D' such that $D \subset D' \subseteq E$ and $D' \subseteq \mathcal{P}(A(c))$.

Theorem 3.14. *A Soft Superhypergraph $H = (H^*, A, B, C)$ can be transformed into a Soft Hypergraph and into a Superhypergraph.*

Proof. Given the Soft Superhypergraph $H = (H^*, A, B, C)$, we can construct a hypergraph $H' = (V, E')$ by collapsing the superhyperedges:

- **Vertex Set**: V remains unchanged.
- **Hyperedge Set**: Define E' as
$$E' = \left\{ \bigcup D \,\middle|\, D \in E \right\}.$$
That is, for each superhyperedge $E \in E$, we take the union of the subsets in E to form a hyperedge in E'.

Soft Hyperedge Mapping: Define $B' : C \to \mathcal{P}(E')$ by:
$$B'(c) = \left\{ \bigcup D \,\middle|\, D \in B(c) \right\}.$$

Soft Hypergraph Structure: The 4-tuple $H' = (H', A, B', C)$ forms a Soft Hypergraph because:

1. H' is a hypergraph with vertex set V and hyperedge set E'.
2. (A, C) is a soft set over V.
3. (B', C) is a soft set over E'.
4. For each $c \in C$, $(A(c), B'(c))$ forms a semisubhypergraph of H'.

The Soft Superhypergraph $H = (H^*, A, B, C)$ inherently contains the superhypergraph $H^* = (V, E)$. By considering the mappings A and B over V and Es, we observe:

- **Superhypergraph Structure**: $H^* = (V, E)$ is a superhypergraph.
- **Soft Sets**:
  - (A, C) is a soft set over V.
  - (B, C) is a soft set over Es.
- **Semisubsuperhypergraphs**: For each $c \in C$, $(A(c), B(c))$ forms a semisubsuperhypergraph of H^*.

Thus, $H = (H^*, A, B, C)$ can be viewed as a Soft Superhypergraph where H^* is the underlying Superhypergraph. □

3.4 Rough Superhypergraph

A **Rough Superhypergraph** is a mathematical structure that combines the concepts of rough sets and superhypergraphs to model uncertainty and hierarchical relationships in complex systems.

Definition 3.15 (Rough Superhypergraph)**.** Let Q be a non-empty set, and let φ be an equivalence relation on Q, forming an approximation space (Q, φ).

Let $V \subseteq \mathcal{P}(Q)$ be the set of *supervertices*, where each supervertex is a subset of Q.

Define the rough approximations of supervertices as follows:

- For each $v \in V$, the *lower approximation* $\underline{\varphi}(v)$ and the *upper approximation* $\overline{\varphi}(v)$ are defined as:
$$\underline{\varphi}(v) = \{d \in Q \mid [d]_\varphi \subseteq v\},$$
$$\overline{\varphi}(v) = \{d \in Q \mid [d]_\varphi \cap v \neq \emptyset\},$$
where $[d]_\varphi = \{g \in Q \mid (d, g) \in \varphi\}$ is the equivalence class of d under φ.

Let $E \subseteq \mathcal{P}(V)$ be the set of *superedges*, where each superedge is a subset of V.

Let ψ be an equivalence relation on E.

Define the rough approximations of superedges as follows:

- For each $D \subseteq E$, the *lower approximation* $\underline{\psi}(D)$ and the *upper approximation* $\overline{\psi}(D)$ are defined as:
$$\underline{\psi}(D) = \{e \in E \mid [e]_\psi \subseteq D\},$$
$$\overline{\psi}(D) = \{e \in E \mid [e]_\psi \cap D \neq \emptyset\},$$
where $[e]_\psi = \{f \in E \mid (e, f) \in \psi\}$ is the equivalence class of e under ψ.

The triplet $R = (Q, \varphi_V, \psi_E)$, where $\varphi_V = \{(\underline{\varphi}(v), \overline{\varphi}(v)) \mid v \in V\}$ and $\psi_E = \{(\underline{\psi}(e), \overline{\psi}(e)) \mid e \in E\}$, is called a *Rough Superhypergraph*.

Theorem 3.16. *A Rough Superhypergraph $R = (Q, \varphi_V, \psi_E)$ generalizes both* Superhypergraphs *and* Rough Hypergraphs.

Proof. To prove this theorem, we show that by appropriate simplifications, a rough superhypergraph reduces to a superhypergraph or a rough hypergraph.

Reduction to a Superhypergraph If we disregard the rough approximations (i.e., ignore the equivalence relations φ and ψ), the Rough Superhypergraph $R = (Q, \varphi_V, \psi_E)$ reduces to the Superhypergraph $H = (V, E)$, where:

- $V \subseteq \mathcal{P}(Q)$ is the set of supervertices.
- $E \subseteq \mathcal{P}(V)$ is the set of superedges.

This structure satisfies the definition of a superhypergraph.

Reduction to a Rough Hypergraph If we consider that the supervertices are singletons, i.e., $V = \{\{v\} \mid v \in Q\}$, then V can be identified with Q. In this case, the Rough Superhypergraph $R = (Q, \varphi_V, \psi_E)$ reduces to a Rough Hypergraph $R' = (Q, \varphi_A, \psi_D)$, where:

- φ_A is the rough approximation on Q.
- ψ_D is the rough approximation on the hyperedges $D \subseteq \mathcal{P}(Q)$.

Thus, the structure conforms to the definition of a rough hypergraph.

Since a rough superhypergraph can be simplified to yield both a superhypergraph and a rough hypergraph, it generalizes both concepts. □

Theorem 3.17. *Let $R = (Q, \varphi_V, \psi_E)$ be a Rough Superhypergraph. Then, for any $Q' \subseteq Q$, $V' \subseteq V$, and $E' \subseteq E$, the induced rough approximations $\varphi_{V'}$ and $\psi_{E'}$ form a* sub-rough superhypergraph $R' = (Q', \varphi_{V'}, \psi_{E'})$.

Proof. By restricting the equivalence relations φ and ψ to Q', V', and E', we obtain the induced rough approximations $\varphi_{V'}$ and $\psi_{E'}$. The structure R' satisfies the conditions of a rough superhypergraph. □

Theorem 3.18. *Let $R_1 = (Q_1, \varphi_{V_1}, \psi_{E_1})$ and $R_2 = (Q_2, \varphi_{V_2}, \psi_{E_2})$ be two Rough Superhypergraphs. Their* union $R = (Q, \varphi_V, \psi_E)$ *is defined by:*

- $Q = Q_1 \cup Q_2$.
- $V = V_1 \cup V_2$.
- $E = E_1 \cup E_2$.
- *The equivalence relations φ and ψ are appropriately extended to Q and E.*

Then R is a Rough Superhypergraph.

Proof. Since the unions Q, V, and E combine the respective sets from R_1 and R_2, and the equivalence relations φ and ψ can be extended to these unions, R satisfies all the properties of a rough superhypergraph. □

3.5 k-Zero-Divisor Superhypergraph

We now introduce the concept of the k-zero-divisor superhypergraph, which generalizes the k-zero-divisor hypergraph by allowing vertices to be subsets of zero-divisors.

Definition 3.19 (k-Zero-Divisor Superhypergraph)**.** Let R be a commutative ring with identity, and let $Z(R)$ denote the set of zero-divisors in R excluding zero. The k-zero-divisor superhypergraph, denoted $SH_k(R) = (V, E)$, is defined as follows:

- **Vertex Set**:
$$V = \left\{ S \subseteq Z(R) \setminus \{0\} \mid S \neq \emptyset,\ \prod_{x \in S} x \neq 0 \right\}.$$
Each vertex is a non-empty subset of zero-divisors whose elements multiply to a non-zero element.
- **Edge Set**:
$$E = \left\{ \{S_1, S_2, \ldots, S_m\} \subseteq V \mid \bigcup_{i=1}^{m} S_i = S,\ |S| = k,\ \prod_{x \in S} x = 0,\ \prod_{x \in T} x \neq 0\ \forall T \subsetneq S \right\}.$$
Each edge is a set of vertices whose union S has cardinality k, the product of all elements in S is zero, and the product of any proper subset of S is non-zero.

Theorem 3.20. *The k-zero-divisor superhypergraph $SH_k(R)$ generalizes the k-zero-divisor hypergraph $H_k(R)$.*

Proof. In $H_k(R)$, vertices are individual zero-divisors, and hyperedges are sets of k vertices whose product is zero, but any proper subset's product is non-zero.

In $SH_k(R)$, vertices are subsets of zero-divisors whose elements multiply to a non-zero element. Edges are sets of such vertices whose union has size k and satisfies the same product conditions as in $H_k(R)$.

If we restrict $SH_k(R)$ to singleton vertices (i.e., $V = \{\{x\} \mid x \in Z(R) \setminus \{0\}\}$), then $SH_k(R)$ reduces to $H_k(R)$. Thus, $SH_k(R)$ generalizes $H_k(R)$. □

Theorem 3.21. *The k-zero-divisor superhypergraph $SH_k(R)$ is a superhypergraph.*

Proof. By definition:

- The vertex set V consists of non-empty subsets of $Z(R) \setminus \{0\}$ whose elements multiply to a non-zero element. Therefore, $V \subseteq P(U) \setminus \{\emptyset\}$, where $U = Z(R) \setminus \{0\}$.
- The edge set E consists of non-empty subsets of V, i.e., $E \subseteq P(V) \setminus \{\emptyset\}$.

This aligns with the definition of a superhypergraph. Therefore, $SH_k(R)$ is a superhypergraph. □

4 Future Tasks

As a future prospect, the emergence of a definition for a superhypergraph neural network—rooted in the concept of superhypergraphs—is anticipated to drive further research in the field of hypergraph neural networks. Additionally, the concept of a quasi soft set, a variant of the soft set, has been explored [142]. We hope to investigate the hypergraph representation, the superhypergraph representation, and hypersoft set extension of quasi soft sets [132, 137]. Furthermore, we look forward to advancements in the study of Superhypergraph Polytopes, as introduced below.

4.1 Superhypergraph Polytope

Polytopes are well-known topics in the field of mathematical research [35, 151]. Within the research themes surrounding hypergraphs, *Hypergraph Polytopes* are also recognized as an important area of study [43–45, 51]. As a conceptual extension, we propose the *Superhypergraph Polytope*, which is based on the framework of superhypergraphs, as defined below. Although still in its conceptual stage, we anticipate significant progress in understanding its mathematical structures and properties in the future.

Definition 4.1 (Hypergraph Polytope)**.** A *hypergraph polytope* is a geometric representation derived from a hypergraph $H = (V, E)$, where:

- **Vertices of the Polytope**: The vertices correspond to elements of the power set $P(V)$ of the hypergraph's vertex set V, specifically subsets that are either hyperedges in E or unions of connected subsets of V.
- **Face Structure**: The faces of the polytope are determined by nested subsets of V and are organized into a lattice structure called the *face lattice*, with inclusion relationships among subsets defining the hierarchy.
- **Construction**:
  - The polytope is formed by iteratively truncating a simplex associated with V. Each truncation corresponds to a connected subset of V represented as a hyperedge in E.
  - The geometry evolves by removing higher-dimensional simplices (representing higher-order subsets) and replacing them with lower-dimensional polytopes.
- **Geometric Realization**: The resulting polytope is a convex shape embedded in a Euclidean space whose dimension is determined by the number of vertices in the original hypergraph V.

Properties of Hypergraph Polytope:

- **Saturation**: If H is saturated (i.e., every connected subset of V is represented in E), then the resulting hypergraph polytope is complete, encompassing all possible nested structures within V.
- **Relation to Classical Polytopes**:
  - A *simplex* corresponds to a hypergraph with no truncations.
  - A *permutohedron* corresponds to the fully truncated simplex.
  - Intermediate truncations yield families of polytopes called *nestohedra*.

Definition 4.2 (Superhypergraph Polytope)**.** Given a superhypergraph $H = (\mathcal{V}, E)$ on the ground set V, the *superhypergraph polytope* P_H is defined as the convex hull of the characteristic vectors of the saturated connected subsets of $\mathcal{V}$. Specifically,

$$P_H = \text{convex hull}\,\{\mathbf{1}_X \mid X \subseteq \mathcal{V},\ X \neq \emptyset,\ \text{and } H[X] \text{ is connected}\}\,,$$

where $H[X]$ is the induced sub-superhypergraph on X, and $\mathbf{1}_X \in \mathbb{R}^{|\mathcal{V}|}$ is the characteristic vector of X.

Remark 4.3. Superhypergraph polytopes generalize hypergraph polytopes by allowing vertices themselves to be subsets of a ground set V.

Theorem 4.4. *Every hypergraph polytope can be realized as a superhypergraph polytope, i.e., hypergraph polytopes are special cases of superhypergraph polytopes.*

Proof. Let $G = (V, E)$ be a hypergraph, where V is a finite set of vertices and $E \subseteq 2^V \setminus \{\emptyset\}$ is a set of hyperedges.

We construct a superhypergraph $H = (\mathcal{V}, E')$ as follows:

- Let the set of supervertices be the singleton subsets of V:
$$\mathcal{V} = \{\{v\} \mid v \in V\}\,.$$
- Let the set of superedges be the original hyperedges, considered as subsets of $\mathcal{V}$:
$$E' = \{\{\{v\} \mid v \in e\} \mid e \in E\}\,.$$

In this construction, each vertex of the hypergraph G corresponds to a supervertex $\{v\}$ in H, and each hyperedge $e \in E$ corresponds to a superedge $e' \in E'$ connecting the supervertices corresponding to the vertices in e.

Now, the superhypergraph polytope P_H constructed from H coincides with the hypergraph polytope P_G constructed from G. This is because the connected subsets of $\mathcal{V}$ in H correspond exactly to the connected subsets of V in G, and their characteristic vectors are identical.

Therefore, hypergraph polytopes are special cases of superhypergraph polytopes where the supervertices are singleton subsets of the ground set V. □

Theorem 4.5 (Generalization to Higher Dimensions)**.** *Superhypergraph polytopes can represent more complex polytopes than hypergraph polytopes by accommodating supervertices of size greater than one, allowing the modeling of higher-order relationships.*

Proof. In a superhypergraph, supervertices can be subsets of V with cardinality greater than one, representing higher-order entities.

By including supervertices that are subsets of V, the superhypergraph polytope P_H incorporates additional dimensions corresponding to these supervertices.

This allows the representation of polytopes with more complex facial structures than those possible with hypergraph polytopes alone. □

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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Survey of Intersection Graphs, Fuzzy Graphs and Neutrosophic Graphs

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Abstract: Graph theory is a fundamental branch of mathematics that studies networks consisting of nodes (vertices) and their connections (edges). Extensive research has been conducted on various graph classes within this field. Fuzzy Graphs and Neutrosophic Graphs are specialized models developed to address uncertainty in relationships. Intersection graphs, such as Unit Square Graphs, Circle Graphs, Ray Intersection Graphs, Grid Intersection Graphs, and String Graphs, play a critical role in analyzing graph structures.

In this paper, we explore intersection graphs within the frameworks of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs, highlighting their mathematical properties and interrelationships. Additionally, we provide a comprehensive survey of the graph classes and hierarchies related to intersection graphs and uncertain graphs, reflecting the increasing number of graph classes being developed in these areas.

Keywords: Neutrosophic graphs, Fuzzy graphs, Intersection graphs, Plithogenic Graphs

1. Introduction

1.1 Graph theory and applications

Graph theory is a fundamental area of mathematics that focuses on the study of networks composed of nodes (vertices) and connections (edges). It plays a key role in analyzing the structure, pathways, and properties of these networks [381]. Given its importance, graph theory has been extensively researched in various fields, including real-world applications [170,689], the development of graph neural networks [94,265,414,467, 757,759,1011,1149,1150,1152,1185], Bayesian network theory [898], bioinformatics [25,1099], protein structures[506,1102], circuits design [138,139,1135,1144], chemical graph theory[140,1103], and graph databases [102,609,824,824]. Furthermore, within the research of graph theory, significant attention has been given to the study of graph structures [107,206,220,880] and numerous graph algorithms [172,216,438,444,448,481,679, 869,1096].

1.2 Intersection graphs

Graph structures and graph classes are commonly studied subjects in graph theory [220]. One important example graph structures in graph theory is the intersection graph, where vertices correspond to sets, and edges are drawn between vertices if their corresponding sets intersect [488,806,1040]. Many related graph classes have been extensively researched, such as interval graphs [462,504], mixed interval graphs [540,541,543], proper interval graphs [309,538,566], weighted interval graphs [175,312,1169], unit disk graphs [128,310,322,718], and polygon-circle graphs [707].

The relationships of intersection graphs are illustrated in Figure 1. This diagram represents only a small portion of intersection graphs, and numerous other studies have been conducted on this topic.

In this paper, we consider the following intersection graph. Additionally, we conduct a survey on graph classes related to intersection graphs.

- Unit square graphs: A Unit Square Graph is an intersection graph where vertices represent unit squares in the plane, and edges exist between intersecting squares [278,279,518,882].
- Circle and Circular Graph: A circular-arc graph represents the intersections of arcs on a circle, where vertices correspond to the arcs, and edges exist between arcs that intersect [481,535,1105]. And a circle graph is a graph where vertices represent chords of a circle, and edges exist if the chords intersect.
- Ray Graphs: Ray Graph is a graph where vertices represent rays (half-lines) in the plane, and edges exist if the rays intersect.
- String Graphs: String Graph is a graph where vertices correspond to curves, and edges exist if the curves intersect at some point.

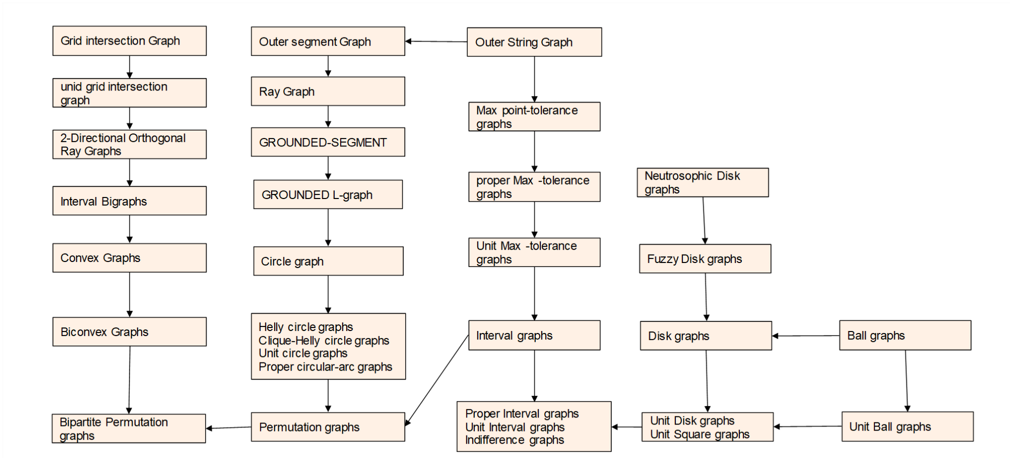

Fig. 1. Intersection Graph Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

- Grid Intersection Graphs [167, 295, 296]: Grid intersection graphs are bipartite graphs where vertices represent horizontal and vertical line segments, with edges formed by intersections between the segments on a grid.

1.3 Fuzzy Graphs and Neutrosophic Graphs

In this paper, we explore Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. These graph concepts were developed to address uncertainty in practical applications. Graphs that handle such uncertainty are referred to as ”uncertain graphs.” These uncertain graphs can be seen as extensions of the corresponding set theory concepts, namely Fuzzy sets, Intuitionistic Fuzzy sets, Neutrosophic sets, and Turiyam Neutrosophic sets, to the graph domain.

A fuzzy graph assigns a membership value between 0 and 1 to each vertex and edge, indicating the degree of uncertainty or imprecision associated with them [74, 75, 720, 890, 983]. Fuzzy graphs essentially represent fuzzy sets [391, 734, 1172, 1173] and are widely used in fields such as social networks, decision-making, and transportation systems, where relationships can be uncertain or ambiguous[845, 983].

Intuitionistic Fuzzy Graphs extend fuzzy graphs by introducing both membership and non-membership degrees for vertices and edges, allowing for a more nuanced representation of uncertainty in relationships [471, 673, 912, 965].

Neutrosophic Graphs[55, 63, 235, 241, 457, 458, 458, 459, 461, 532, 599, 651, 1001, 1055, 1059, 1059–1062, 1065], derived from neutrosophic set theory [92, 1067], add three components—truth, indeterminacy, and falsity—to classical and fuzzy logic. This approach offers greater flexibility in representing uncertainty .

Turiyam Neutrosophic Graphs, an extension of neutrosophic and fuzzy graphs, assign four attributes—truth, indeterminacy, falsity, and a liberal state—to each vertex and edge [468–470]. A further generalized graph concept known as Plithogenic Graphs is also well-established [1051, 1063].

In addition, various types of intersection graphs have been studied within these frameworks. Research has focused on fuzzy intersection graphs[328, 495, 580, 803, 967, 967, 1022], neutrosophic intersection graphs[228], fuzzy permutation graphs[968], and fuzzy interval graphs[328]. Many studies have also explored related topics[449, 451, 460].

The relationships of uncertain graphs are illustrated in Figure 2. This represents only a small portion of uncertain graphs, and many other studies have been conducted on this topic. For further details, please refer to relevant survey papers such as [452].

1.4 Our Contribution

In this paper, we define some intersection graphs within the frameworks of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs, and examine their properties and interrelationships. Additionally, new classes of graphs are being proposed daily in graph class research, and many survey papers are being published[452, 461, 952]. Accordingly, this paper conducts a broad survey of previous studies on graph classes related to intersection graphs and uncertain graphs. It is expected that this will accelerate research on intersection graphs and uncertain graphs.

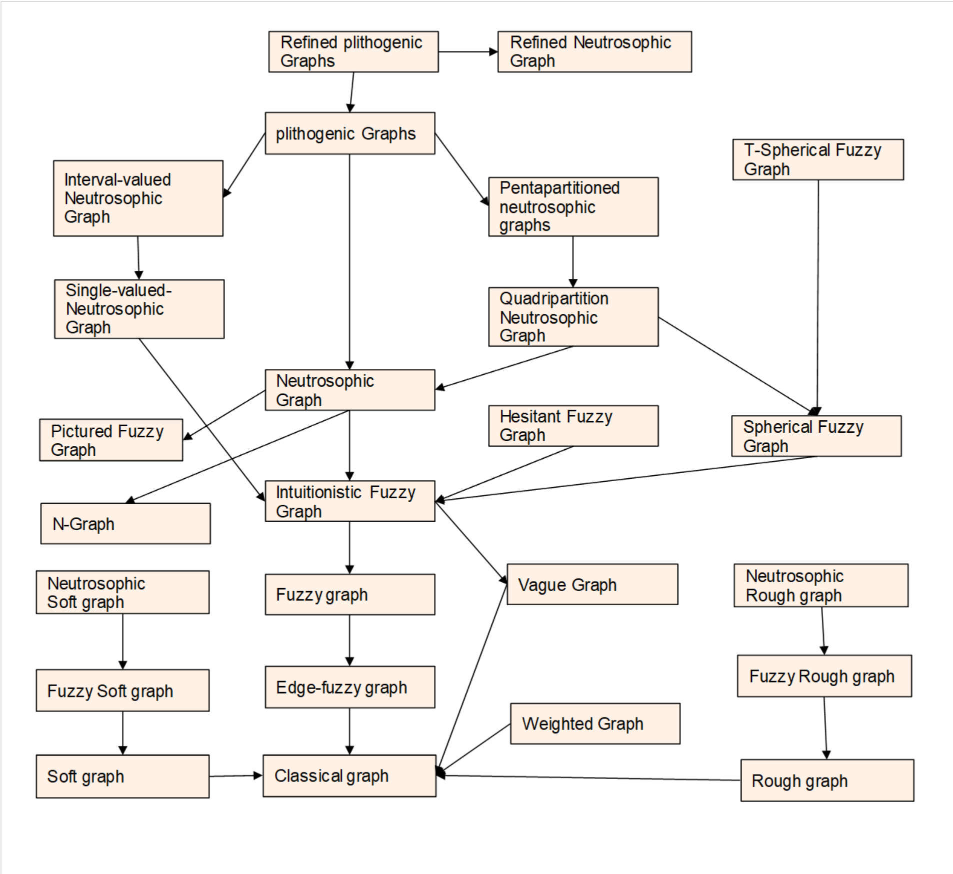

Fig. 2. Uncertain Graph Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

We describe the relationships between graph classes of Uncertain Intersection Graphs in Figure 3. This diagram includes previously known results and illustrates only a part of the overall relationships.

1.5 The Structure of the Paper

The structure of this paper is outlined as follows. In Section 2, we provide an extensive survey of existing results on intersection graphs and uncertain graphs, along with the necessary definitions for this study. Section 3 discusses the findings related to uncertain intersection graphs, and Section 4 presents the conclusions and future research directions.

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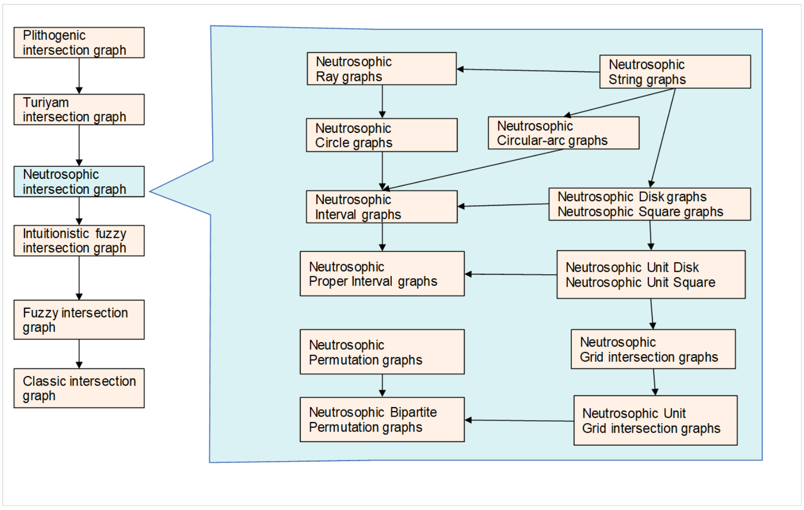

Fig. 3. Some Uncertain Intersection graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

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2. Preliminaries and definitions

In this section, we present a brief overview of the definitions and notations used throughout this paper. We will specifically cover fundamental concepts related to graphs, including intersection graphs, string graphs, fuzzy graphs, intuitionistic fuzzy graphs, Turiyam Neutrosophic graphs, neutrosophic graphs.

2.1 Basic Graph Concepts

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [379,380,380,381,530,1147].

2.1.1 Graph and Subgraph

The definitions and related concepts of Graph, Subgraph, and Supergraph are provided below.

Definition 1 (Graph)**.** [381] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2 (Degree)**.** [381] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 3 (Subgraph)**.** [381] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Example 4 (Example of a Subgraph)**.** Consider a graph $G = (V, E)$ with the vertex set $V = \{1, 2, 3, 4, 5\}$ and the edge set

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}.$$

This graph G can be visualized as having 5 vertices connected by 5 edges.

Now, let us define a subgraph G' by selecting the subset of vertices $V' = \{1, 2, 4\}$ and the subset of edges $E' = \{\{1, 2\}, \{2, 4\}\}$. Thus, the subgraph $G' = (V', E')$ has:

- The vertex set $V' = \{1, 2, 4\}$,
- The edge set $E' = \{\{1, 2\}, \{2, 4\}\}$.

In this example, G' is a valid subgraph of G because both $V' \subseteq V$ and $E' \subseteq E$, and the edges in E' only connect vertices within V'.

Definition 5 (Induced subgraph)**.** [581, 732] Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. For a subset $V' \subseteq V$, the *induced subgraph* $G[V']$ is the graph whose vertex set is V' and whose edge set consists of all edges from E that have both endpoints in V'. Formally, the induced subgraph $G[V'] = (V', E')$ is defined as follows:

$$E' = \{(u, v) \in E \mid u \in V', v \in V'\}.$$

In other words, $G[V']$ is the subgraph of G that contains all vertices in V' and all edges from G whose endpoints are both in V'.

Definition 6 (Supergraph)**.** (cf.[794]) A graph G' is called a *supergraph* of a graph G if G is a subgraph of G'. Formally, $G = (V, E)$ is a subgraph of $G' = (V', E')$ if and only if $V \subseteq V'$ and $E \subseteq E'$, meaning G' contains all the vertices and edges of G, possibly with additional vertices and edges.

Example 7. Let $G = (V, E)$ be a graph with the following vertex and edge sets:

$$V = \{1, 2, 3\}, \quad E = \{(1, 2), (2, 3)\}.$$

This graph G consists of three vertices and two edges.

Now, consider a graph $G' = (V', E')$ where:

$$V' = \{1, 2, 3, 4\}, \quad E' = \{(1, 2), (2, 3), (3, 4)\}.$$

Here, G' contains all the vertices and edges of G, but it also has an additional vertex 4 and an additional edge $(3, 4)$. Therefore, G' is a supergraph of G, as G is a subgraph of G'.

Definition 8 (Induced Supergraph)**.** Let $G = (V, E)$ and $G' = (V', E')$ be graphs. The graph G' is called an *induced supergraph* of G if the following conditions hold:

- $V \subseteq V'$, meaning all vertices of G are included in G'.
- For all pairs of vertices $u, v \in V$, the edge (u, v) is in E if and only if $(u, v) \in E'$. That is,

$$(u, v) \in E \quad \Leftrightarrow \quad (u, v) \in E', \quad \forall u, v \in V.$$

In other words, G is an induced subgraph of G', and G' contains G along with possibly additional vertices and edges, but no new edges between the vertices of G.

Definition 9. (cf.[546,1109]) A graph $G = (V, E)$ is said to be a **connected graph** if for any two distinct vertices $u, v \in V$, there exists a path in G that connects u and v. In other words, every pair of vertices in the graph is reachable from each other, meaning there is a sequence of edges that allows traversal between any two vertices.

Mathematically, for all $u, v \in V$, there exists a sequence of vertices $v_1 = u, v_2, \ldots, v_k = v$ such that $(v_i, v_{i+1}) \in E$ for all $1 \leq i < k$.

Definition 10. (cf.[546, 1109]) A graph $G = (V, E)$ is said to be an **unconnected graph** if there exist at least two distinct vertices $u, v \in V$ such that no path exists between u and v. In other words, the graph contains at least one pair of vertices that are not reachable from each other, meaning there is no sequence of edges that allows traversal between these two vertices.

Mathematically, there exist $u, v \in V$ such that no sequence of vertices $v_1 = u, v_2, \ldots, v_k = v$ exists with $(v_i, v_{i+1}) \in E$ for all $1 \leq i < k$.

2.1.2 Path, Tree, Complete graph and Bipartite Graph

The definitions of Path, Tree, Complete Graph, and Bipartite Graph are described as follows.

Definition 11 (Path)**.** (cf.[1179]) A path in a graph $G = (V, E)$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \ldots, k - 1$. A path is represented as:

$$P = (v_1, v_2, \ldots, v_k),$$

where no vertex is repeated. The length of a path is the number of edges it contains, i.e., $k - 1$.

Definition 12 (Tree)**.** (cf.[1179]) A tree is a connected, acyclic graph. In other words, a tree is a graph where there is exactly one path between any two vertices, and no cycles exist.

Example 13. Consider a graph with 4 vertices $V = \{V_A, V_B, V_C, V_D\}$ and 3 edges $E = \{(V_A, V_B), (V_B, V_C), (V_C, V_D)\}$. This graph is connected, has no cycles, and there is exactly one path between any pair of vertices. Hence, this graph is a tree.

Definition 14 (Complete Graph)**.** (cf.[164, 384]) A *complete graph* is a graph $G = (V, E)$ in which every pair of distinct vertices is connected by a unique edge. Formally, a graph $G = (V, E)$ is complete if for every pair of vertices $u, v \in V$ with $u \neq v$, there exists an edge $\{u, v\} \in E$.

The complete graph on n vertices is denoted by K_n, and it has the following properties:

- The number of vertices is $|V| = n$.

- The number of edges is $|E| = \binom{n}{2} = \frac{n(n-1)}{2}$.
- Each vertex has degree $\deg(v) = n - 1$ for all $v \in V$.

Example 15 (Examples of Complete Graphs)**.** The concept of a complete graph is best understood through specific examples:

- The complete graph K_1 consists of a single vertex with no edges.
- The complete graph K_2 has two vertices, v_1 and v_2, and a single edge connecting them, $\{v_1, v_2\}$.
- The complete graph K_3 consists of three vertices, v_1, v_2, and v_3, with edges $\{v_1, v_2\}, \{v_2, v_3\}$, and $\{v_1, v_3\}$. This forms a triangle.
- The complete graph K_4 has four vertices, with every possible pair of vertices connected by an edge. The edges are $\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}$, and $\{v_3, v_4\}$. This graph forms a tetrahedron when represented in three dimensions.
- The complete graph K_5 includes five vertices, with each pair of vertices connected by a unique edge. The total number of edges is $\binom{5}{2} = 10$, making it a highly connected structure.

In general, for a complete graph K_n with n vertices, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ edges, and each vertex has a degree of $n - 1$.

Definition 16 (Bipartite Graph)**.** (cf.[123,393]) A *bipartite graph* is a graph $G = (V, E)$ whose vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that:

- $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
- Every edge in E connects a vertex from V_1 to a vertex from V_2. In other words, there are no edges connecting two vertices within the same subset V_1 or V_2.

Formally, $G = (V, E)$ is bipartite if there exists a partition (V_1, V_2) such that for every edge $e = \{u, v\} \in E$, either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$.

A graph G is bipartite if and only if it contains no odd-length cycles.

Example 17 (Examples of Bipartite Graphs)**.** To illustrate the concept of bipartite graphs, we provide the following examples:

- The simplest example is the graph $K_{1,1}$, which consists of two vertices v_1 and v_2 with a single edge connecting them. Here, $V_1 = \{v_1\}$ and $V_2 = \{v_2\}$, making $K_{1,1}$ a bipartite graph.
- The complete bipartite graph $K_{2,3}$ consists of two sets of vertices, $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2, v_3\}$. Each vertex in V_1 is connected to every vertex in V_2. Thus, the edge set E consists of all possible edges between V_1 and V_2, specifically: $\{u_1, v_1\}, \{u_1, v_2\}, \{u_1, v_3\}, \{u_2, v_1\}, \{u_2, v_2\}, \{u_2, v_3\}$. There are no edges within V_1 or V_2, making $K_{2,3}$ a bipartite graph.

Definition 18 (Complete Bipartite Graph)**.** (cf.[403,571,738]) A *complete bipartite graph* is a graph $G = (V, E)$ whose vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that:

- $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
- There is an edge between every vertex in V_1 and every vertex in V_2.
- There are no edges between vertices within the same subset V_1 or V_2.

The complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. It has the following properties:

- The number of vertices is $|V| = m + n$.
- The number of edges is $|E| = m \times n$.
- Each vertex in V_1 has degree n, and each vertex in V_2 has degree m.

2.1.3 Regular Graph

The definitions of Regular Graph and Cubic Graph are described as follows.

Definition 19 (Regular Graph)**.** [196, 814, 1019] A **regular graph** is a graph in which each vertex has the same degree. A k-regular graph is a graph where every vertex has degree k. Formally, a graph $G = (V, E)$ is k-regular if:

$$\deg(v) = k \quad \text{for all } v \in V(G)$$

where $\deg(v)$ denotes the degree of vertex v.

Example 20 (Regular Graph)**.** Consider a 3-regular graph with 4 vertices. The vertex set is:

$$V(G) = \{v_1, v_2, v_3, v_4\}$$

The edge set is:

$$E(G) = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_1, v_3), (v_2, v_4)\}$$

Each vertex has exactly 3 edges, making this a 3-regular graph.

Definition 21 (Cubic Graph)**.** [225, 1108] A **cubic graph** is a special case of a regular graph where every vertex has degree 3. Formally, a graph $G = (V, E)$ is cubic if:

$$\deg(v) = 3 \quad \text{for all } v \in V(G)$$

Cubic graphs are also known as 3-regular graphs.

Example 22 (Cubic Graph)**.** Consider a **cubic graph** with 6 vertices. The vertex set is:

$$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

The edge set is:

$$E(G) = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_5), (v_2, v_6), (v_3, v_5), (v_3, v_6), (v_4, v_5), (v_4, v_6)\}$$

Each vertex has exactly 3 edges, making this a cubic (3-regular) graph.

2.1.4 Euclidean space and Euclidean distance

Geometric graph theory is the study of graphs embedded in geometric spaces, with a focus on spatial relationships, distances, and intersections between vertices and edges [896, 897, 1010]. In this field, Euclidean space plays a central role, as it provides the foundation for defining the properties of such graphs. A Euclidean graph is a type of graph in which the vertices represent points in the plane, and each edge is assigned a length corresponding to the Euclidean distance between its endpoints [593,678]. The key definitions are outlined below.

Definition 23 (n-dimensional Euclidean Space)**.** (cf.[930]) The *n-dimensional Euclidean space*, denoted as $\mathbb{R}^n$, is the set of all ordered n-tuples of real numbers. Formally, it is defined as:

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \text{ for all } i = 1, 2, \ldots, n\}.$$

Each element $\mathrm{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is called a point (or vector) in n-dimensional Euclidean space, where $x_1, x_2, \ldots, x_n$ are the coordinates of the point, and $\mathbb{R}$ denotes the set of real numbers.

Definition 24 (Euclidean Distance)**.** Let $\mathrm{p} = (p_1, p_2, \ldots, p_n)$ and $\mathrm{q} = (q_1, q_2, \ldots, q_n)$ be two points in the n-dimensional Euclidean space $\mathbb{R}^n$. The *Euclidean distance* $d(\mathrm{p}, \mathrm{q})$ between p and q is defined as:

$$d(\mathrm{p}, \mathrm{q}) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2}$$

or equivalently,

$$d(\mathrm{p}, \mathrm{q}) = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}.$$

Example 25. (cf.[460]) Consider two points in the 2-dimensional Euclidean space, $\mathbb{R}^2$. Let $\mathrm{p} = (1, 3)$ and $\mathrm{q} = (4, 7)$. The Euclidean distance between these two points is calculated as follows:

$$d(\mathrm{p}, \mathrm{q}) = \sqrt{(1-4)^2 + (3-7)^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

Thus, the Euclidean distance between p and q is 5.

Next, the definitions of the Unit Disk and Unit Square in 2-dimensional Euclidean space are provided below.

Definition 26 (Unit Disk)**.** The *unit disk* in n-dimensional Euclidean space $\mathbb{R}^n$ is defined as the set of all points within a distance of 1 from a fixed point, typically the origin. Formally, the unit disk in $\mathbb{R}^2$ is given by:

$$D = \left\{ (x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} \leq 1 \right\}.$$

This represents a disk in the plane with a radius of 1, centered at the origin $(0, 0)$.

Definition 27 (Unit Square)**.** The *unit square* in n-dimensional Euclidean space $\mathbb{R}^n$ is defined as the set of all points whose coordinates lie between 0 and 1, inclusive, along each axis. In $\mathbb{R}^2$, the unit square is given by:

$$S = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \right\}.$$

This describes a square with side length 1, aligned with the coordinate axes, and with one corner at the origin $(0, 0)$.

2.1.5 Other concepts

The definition of homomorphic is described as follows.

Definition 28 (homomorphic)**.** (cf.[209, 383, 413, 572, 1148]) Two graphs $G = (V, E)$ and $H = (V', E')$ are said to be *homomorphic* if there exists a mapping $\phi : V \to V'$ such that for every edge $(u, v) \in E$, the image $(\phi(u), \phi(v))$ is an edge in E'. In other words, there is a structure-preserving mapping from G to H that maintains the adjacency relationships between vertices.

Without fear of misunderstanding, it is common in the study of graph classes and structures to examine the properties and algorithms of graphs by combining the characteristics of classic graph structures (such as Complete, Regular, and Tree) with those of intersection graphs (such as permutation, interval, circle, and string) and uncertain graphs (such as Fuzzy, Neutrosophic, and Turiyam). This comprehensive approach allows for a deeper analysis of their graph-theoretic properties and applications.

2.2 Classical Intersection graphs

As mentioned in the introduction, an intersection graph is a graph where the vertices correspond to sets, and edges are drawn between vertices if their corresponding sets intersect [488, 806, 1040]. Intersection graphs have been extensively studied in the literature [282, 488, 505, 806, 841, 900, 1040, 1177]. The formal definition is provided below [488, 806, 1040].

Definition 29 (Intersection graph)**.** [488, 806, 1040] A *intersection graph* is a graph that represents the intersection relationships between sets. Formally, let $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ be a collection of sets. The *intersection graph* $G = (V, E)$ associated with $\mathcal{S}$ is a graph where:

- The vertex set V corresponds to the sets in $\mathcal{S}$, i.e., $V = \{v_1, v_2, \dots, v_n\}$, where each vertex v_i represents the set $S_i \in \mathcal{S}$.
- There is an edge $(v_i, v_j) \in E$ if and only if the corresponding sets S_i and S_j have a non-empty intersection, i.e., $S_i \cap S_j \neq \emptyset$.

Several related concepts of intersection graphs have also been proposed. A well-known example is the contact graph. A contact graph is a restricted version of an intersection graph, where the focus is specifically on the contact between geometric objects [16, 80, 301, 513, 583–585, 813]. A generalization of this concept includes the k-contact graph, which is also well studied [583–585]. Other related concepts include weighted disk contact graphs [698], convex Contact Graphs[1015], , and L-contact graphs [81, 301].

When studying intersection graphs, a crucial consideration is the selection of geometric shapes used to represent intersections. For instance, if an intersection graph is defined such that vertices represent geometric objects and edges indicate intersections between two objects, it is essential to choose specific geometric shapes—such as circles, disks, triangles, rectangles, squares, line segments, or strings—and analyze their properties. Additionally, research has been conducted on whether these shapes can be used to represent classical graph structures. These topics are explored in the context of intersection representations [357, 463, 487, 713, 760, 881, 972].

Furthermore, the concept of intersection graphs is valuable from the perspectives of applied mathematics, computer science, and discrete mathematics. Consequently, various types of intersection graphs are well known. An example is provided below.

Notation 30. In this paper, we define the term "Related graph class" as a graph class that either extends or restricts a corresponding graph class in some way.

Theorem 31. *The following are examples of related graph classes, including but not limited to:*

- *Disk Graph: A unit disk graph is a type of graph where the vertices represent equal-sized disks in the plane, and edges exist between disks that overlap.*
- *interval graphs[462, 504]: Graphs where vertices represent intervals on a line; edges signify overlapping intervals.*
- *Permutation graph: Permutation graphs are defined such that vertices represent elements of a permutation, and edges connect pairs of vertices if their corresponding elements in the permutation are reversed in order [144, 219, 412]. Compatibility graphs [498, 511, 645, 837], chordal Compatibility graphs [596, 773], Co-Compatibility graphs [5, 768], and Trapezoid graphs [336, 422, 739] are examples of permutation graphs.*
- *Circle and Circular Graph: A circular-arc graph represents the intersections of arcs on a circle, where vertices correspond to the arcs, and edges exist between arcs that intersect[481, 535, 1105]. And a circle graph is a graph where vertices represent chords of a circle, and edges exist if the chords intersect.*
- *Grounded intersection graph: A grounded intersection graph is an intersection graph where geometric objects, such as rectangles or line segments, are "grounded" on a line, and adjacency is defined by the intersections between these objects [427].*
- *Ray intersection Graphs: Ray Graph is a graph where vertices represent rays (half-lines) in the plane, and edges exist if the rays intersect.*
- *String Graphs: String Graph is a graph where vertices correspond to curves, and edges exist if the curves intersect at some point.*
- *Bipartite intersection graph: Bipartite graph of intersection graphs.*
- *Weighted intersection graph: Weighted graph of intersection graphs.*
- *Uncertain intersection graph: Intersection graph in Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs.*
- *Intersection digraph: Directed graphs of intersection graphs.*

Proof. Refer to each reference as needed. □

2.2.1 Disk Graph

We consider the Disk Graph. A Disk Graph is a type of Geometric Intersection Graph that applies geometric principles to graph theory and is specifically constructed using disks. The definition of a Geometric Intersection Graph is provided below. Geometric Intersection Graphs, including those constructed with shapes other than disks, have been the subject of extensive research[19, 157, 199, 286, 354, 406, 409, 715, 923].

Definition 32. A *Geometric Intersection Graph* is a graph whose vertices correspond to geometric objects in a given space, and an edge exists between two vertices if and only if the corresponding geometric objects intersect. Formally, given a set of geometric objects $O = \{O_1, O_2, \ldots, O_n\}$, where each O_i is a geometric shape (such as a disk, rectangle, or polygon), the *Geometric Intersection Graph* $G = (V, E)$ is constructed as follows:

- Each vertex $v_i \in V$ represents a geometric object $O_i \in O$.
- An edge $(v_i, v_j) \in E$ exists if and only if the objects O_i and O_j intersect in space, i.e., $O_i \cap O_j \neq \emptyset$.

One of the most extensively studied graphs within Disk Graphs is the Unit Disk Graph (UDG). The Unit Disk Graph is a well-known example of an intersection graph that has been widely researched in graph theory and related fields [79, 128, 145, 186, 188, 322, 633, 635, 852, 1127, 1131, 1160]. The definition of the Unit Disk Graph (UDG) is provided as follows.

Definition 33. A *Unit Disk Graph (UDG)* is a graph $G = (V, E)$ where each vertex corresponds to a disk of equal radius (typically 1) in the plane, and there is an edge between two vertices if and only if their corresponding disks intersect. Formally, for each pair of vertices $u, v \in V$, there exists an edge $(u, v) \in E$ if and only if the Euclidean distance between the centers of the disks corresponding to u and v is at most 1.

Mathematically, if p_u and p_v are the centers of the disks corresponding to vertices u and v, respectively, then:

$$(u, v) \in E \iff \|p_u - p_v\| \leq 1.$$

Unit Disk Graphs are commonly used to model wireless networks and other spatial systems where connections depend on proximity.

The following is an example of Unit Disk Graphs.

Example 34. Consider three vertices $V = \{v_1, v_2, v_3\}$, where each vertex corresponds to a disk of radius 1 in the plane, centered at points $p_1 = (0, 0)$, $p_2 = (1, 0)$, and $p_3 = (2, 0)$, respectively.

Check the distances between the centers.

- The distance between p_1 and p_2 is:

$$\|p_1 - p_2\| = \sqrt{(1-0)^2 + (0-0)^2} = 1.$$

- The distance between p_2 and p_3 is:

$$\|p_2 - p_3\| = \sqrt{(2-1)^2 + (0-0)^2} = 1.$$

- The distance between p_1 and p_3 is:

$$\|p_1 - p_3\| = \sqrt{(2-0)^2 + (0-0)^2} = 2.$$

Define the edges based on the distances.

- Since $\|p_1 - p_2\| = 1$, there is an edge between v_1 and v_2, i.e., $(v_1, v_2) \in E$.
- Since $\|p_2 - p_3\| = 1$, there is an edge between v_2 and v_3, i.e., $(v_2, v_3) \in E$.
- Since $\|p_1 - p_3\| = 2$, which is greater than 1, there is no edge between v_1 and v_3, i.e., $(v_1, v_3) \notin E$.

We consider about Graph description. The resulting graph $G = (V, E)$ has the vertex set $V = \{v_1, v_2, v_3\}$ and the edge set $E = \{(v_1, v_2), (v_2, v_3)\}$. This forms a path graph where v_1 is connected to v_2, and v_2 is connected to v_3, but v_1 and v_3 are not connected.

Thus, the Unit Disk Graph for this example is:

$$G = (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_2, v_3)\}).$$

One of the most well-known applications of unit disk graphs is in wireless networks, particularly in ad hoc wireless communication networks [718]. These graphs are also extensively used in various wireless networking applications [310,429,587,648,727,825]. Additionally, many computational problems, such as the maximum independent set problem, the graph coloring problem, and the minimum dominating set problem, can be efficiently tackled using unit disk graphs [142,634,796,802,1117]. Related concepts, such as quasi-unit disk graphs [310] and coverage area graphs [886], are also well-studied. Therefore, the research on disk graphs and their related concepts remains highly significant.

From the above, many studies have been conducted on graph classes derived from unit disk graphs. The related concepts are described below.

Theorem 35. *The following are examples of related graph classes, including but not limited to:*

- *r-disk graphs[432, 1118] : A Disk Graph (DG) is a graph where each vertex corresponds to a disk of radius r in the plane.*
- *unit disk graphs [128, 145, 186, 188, 322, 633, 635, 852, 1127, 1131, 1160]: A Unit Disk Graph (UDG) is a graph where each vertex corresponds to a disk of equal radius (typically 1) in the plane.*
- *double disk graphs[709]: Double disk graphs are generalizations of unit disk graphs, where each vertex has two concentric disks, and edges form when one inner disk intersects another's outer disk.*
- *bisectored unit disk graphs[892]: Bisectored unit disk graphs extend unit disk graphs by dividing each disk into two sectors, modeling cell sectorization in wireless communication networks.*
- *Ratio disk graphs[84]: Ratio disk graphs are a variation of unit disk graphs where the ratio of the maximum to minimum disk diameters is bounded by a constant.*

- *quasi-unit disk graphs [310, 563, 564]: Generalization of unit disk graphs allowing edges within a certain distance range.*
- *Weak Unit Disk Graphs[82, 83]: Unit disk graphs with weakened conditions for intersection.*
- *weighted unit disk graphs [287, 665, 1129, 1130]: Unit disk graphs with vertex or edge weights, modeling weighted networks.*
- *Unit disk visibility graphs[263, 1039]: Unit Disk Visibility Graphs model visibility between geometric entities, considering both obstacles and distance, representing more realistic scenarios than conventional visibility graphs.*
- *Generalized Disk Graphs[113]: Generalized Disk Graphs are geometric graphs used to model wireless interference, where vertices represent disks and edges exist if disks intersect. They generalize disk graphs to multiple dimensions.*
- *Unit square graph [278, 279, 518, 882]: Unit square graphs are intersection graphs where vertices represent axis-parallel unit squares in the plane, and edges exist if two squares intersect.*
- *Unit ball graph[197, 257, 353, 407]: Unit ball graphs are intersection graphs where vertices represent unit balls in 3D space, and edges exist if the balls intersect or touch each other.*
- *Thin graph[202, 205, 931]: Thin graphs are graphs whose structure is close to interval graphs, measured by a parameter called thinness. Interval graphs are 1-thin, and k-thin graphs have partitions into k classes with specific edge-consistency properties. Related concepts include proper thin graphs[204, 205] and k-mixed-thin graphs[141].*
- *Block-Intersection Graph[790, 805]: A block-intersection graph is a graph where vertices represent blocks of a set system, and two vertices are adjacent if their corresponding blocks have a non-empty intersection.*
- *hyperbolic unit disk graphs [187, 189, 386]: Hyperbolic unit disk graphs (HDGs) are intersection graphs of disks in hyperbolic space, where each disk is placed randomly.*
- *Pseudo-disk Hypergraphs[114, 681]: Pseudo-disk hypergraphs are hypergraphs where vertices represent pseudo-disks in the plane, and hyperedges consist of subsets of pseudo-disks intersected by a fixed pseudo-disk.*
- *Pseudo-disk graphs[12, 762]: Pseudo-disk graphs are intersection graphs where vertices represent pseudo-disks, and two vertices are adjacent if their corresponding pseudo-disks intersect at most at two boundary points.*
- *Axes-parallel unit disk graphs[264]: Axes-parallel unit disk graphs are intersection graphs of unit disks, where disk centers are restricted to lie on vertical or horizontal parallel lines in the Euclidean plane.*
- *Coverage Area Graphs [885–887]: Coverage Area Graphs represent wireless communication networks where each node's transmission range is modeled as an area, often geometric, showing reachable nodes.*
- *Fuzzy unit disk graph[449]: Unit disk graph of fuzzy graph.*
- *Neutrosophic unit disk graph[449]: Unit disk graph of neutrosophic graph.*
- *Turiyam Neutrosophic unit disk graph[449]: Unit disk graph of Turiyam Neutrosophic graph.*
- *Plithogenic unit disk graph[449]: Unit disk graph of Plithogenic graph.*
- *Disk Contact Graphs[698, 699]: Disk Contact Graphs are planar graphs where vertices correspond to interior-disjoint disks, and edges exist if two disks touch. Related concepts include Weighted Disk Contact Graphs[698], Weak Unit Disk Contact Graphs[316], and Unit Disk Contact Graphs[698, 699].*
- *Rectangle intersection graph[284, 977]: A rectangle intersection graph is a graph where vertices represent axis-parallel rectangles in the plane, and edges exist between vertices if their corresponding rectangles overlap.*

We describe the relationships between graph classes of Disk Graphs.

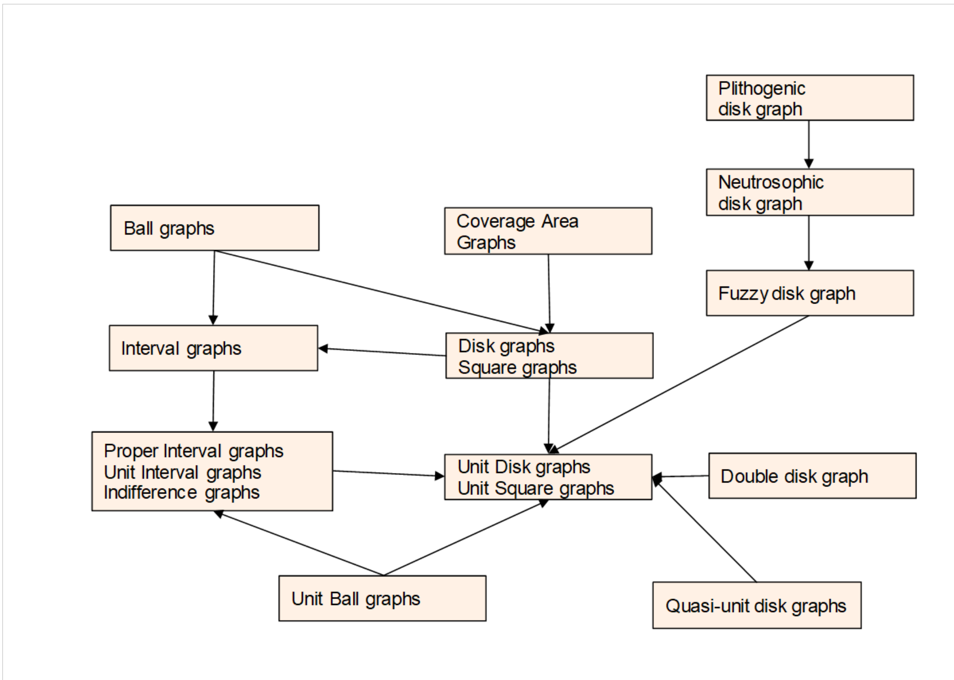

Fig. 4. Disk Graph Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.2.2 *interval graphs*

Interval graphs are a graphs where vertices represent intervals on a line; edges signify overlapping intervals. The definitions are provided below [341, 419, 462, 504, 637, 687, 843, 932, 1041, 1100].

Definition 36. [462, 504] An *interval graph* is an undirected graph $G = (V, E)$ that can be represented by a family of intervals on the real line. For each vertex $v \in V$, there exists a corresponding interval I_v on the real line. Two vertices $u, v \in V$ are adjacent, i.e., $(u, v) \in E$, if and only if their corresponding intervals I_u and I_v overlap. Formally, the edge set E of the graph G is defined as:

$$E(G) = \{(u, v) \mid I_u \cap I_v \neq \emptyset\}.$$

The following provides an example of an Interval Graph.

Example 37 (Interval Graph)**.** Consider a set of intervals on the real line:

$$I_1 = [1, 5], \quad I_2 = [4, 8], \quad I_3 = [6, 9], \quad I_4 = [2, 3]$$

The vertices $V = \{v_1, v_2, v_3, v_4\}$ correspond to the intervals I_1, I_2, I_3, I_4, and the edges are drawn between vertices whose intervals overlap:

- $I_1 \cap I_2 \neq \emptyset$, so $(v_1, v_2) \in E$,
- $I_2 \cap I_3 \neq \emptyset$, so $(v_2, v_3) \in E$,
- $I_1 \cap I_4 \neq \emptyset$, so $(v_1, v_4) \in E$,
- $I_1 \cap I_3 = \emptyset$, so $(v_1, v_3) \notin E$,
- $I_3 \cap I_4 = \emptyset$, so $(v_3, v_4) \notin E$,
- $I_2 \cap I_4 = \emptyset$, so $(v_2, v_4) \notin E$.

The resulting interval graph $G = (V, E)$ is:

$$V = \{v_1, v_2, v_3, v_4\}, \quad E = \{(v_1, v_2), (v_1, v_4), (v_2, v_3)\}$$

Extensive research has been conducted on the structural properties and applications of interval graphs. For example, it is well-known that a graph is an interval graph if and only if it is both chordal and AT-free [731]. And It is well-known that a graph is an interval graph if and only if it is both chordal and a co-comparability graph[499]. Equivalent graph classes to interval graphs include 1-DIR graphs [710]. Additionally, key graph parameters such as interval number, pathwidth [367, 662], and boxicity [289] are closely related to interval graphs. Interval graphs have numerous applications, including in food webs [323,330,769], scheduling problems [134,272,477–479,548,610], and DNA analysis [637,807,812,1138]. Thus, the study of interval graphs is highly valuable from both an applied mathematics perspective and beyond.

The related graphs of the above graph are introduced as follows.

Theorem 38. *The following are examples of related graph classes, including but not limited to:*

- *Multiple interval graphs [258, 639, 799, 1104]: A multiple interval graph is a generalization of an interval graph where each vertex is assigned multiple intervals on the real line, with adjacency defined by interval overlap. Related concepts include unit multiple interval graphs[799].*
- *Proper interval graphs [309, 538, 566]: Interval graphs where no interval is properly contained within another. □Related concepts such as p-Improper Interval Graphs[176], p-Proper Interval Graphs[**?**], and Semi-proper interval graphs[1013] are also known. Equivalent graph classes of Proper interval graphs are an astral triple-free graphs[613] and indifference graphs[355, 552, 763, 764]. It is well-known that a graph is a proper interval graph if and only if it is both claw-free and an interval graph [1142].*
- *Unit interval graphs [101, 325, 800, 970, 994]: Unit interval graphs are intersection graphs of unit-length intervals on a real line, where vertices represent intervals, and edges exist between overlapping intervals.*
- *Mixed unit interval graphs [388, 726, 1094]: A mixed unit interval graph is an intersection graph of unit intervals where the interval ends can be open or closed. Related concepts include co-interval mixed graphs[483].*
- *weighted interval graphs [175, 312, 1169]: Interval graphs where vertices have weights, optimizing specific properties.*
- *interval digraphs [341, 418, 419, 539]: Digraph version of interval graphs.*
- *interval bigraphs [332, 851, 950]: An interval bigraph is a bipartite graph where vertices represent intervals, and edges exist between intersecting intervals from different partite sets. Related concepts include co-interval containment bigraphs.*
- *Signed-interval digraphs[570]: A signed-interval digraph is a generalization of interval graphs, characterized by a signed-interval model, representing digraphs with specific geometric and ordering properties based on min ordering.*
- *Directional Interval Graphs[542]: Directional Interval Graphs are mixed graphs where vertices correspond to intervals on the real line, and edges or arcs represent overlapping or contained intervals based on directional rules.*
- *Split-interval graph [152, 334, 335]: A split-interval graph is a graph that is both an interval graph, representing closed intervals, and a split graph, where the vertex set can be partitioned into a clique and an independent set.*
- *k-track graph[1146]: A graph where each vertex is represented by a union of k intervals on parallel lines, and edges exist if intervals on the same line intersect.*
- *Dotted Interval Graphs [129, 130, 579]: Dotted Interval Graphs (DIG) generalize interval graphs by using arithmetic progressions ("dotted intervals") instead of continuous intervals. Nodes are connected if their progressions share a point. DIGs are used in genotyping and exhibit complex coloring properties.*
- *k-gap Interval Graphs[437, 480]: A k-gap interval graph is a graph that can be represented using at most $n + k$ intervals, where n is the number of vertices and k is the allowed "gap" beyond a standard interval representation.*
- *Fuzzy interval graph[449]: Interval graph of fuzzy graph. Related concepts include fuzzy circular interval graphs [397, 894, 895] and Fuzzy Linear interval Graphs[894].*
- *Neutrosophic interval graph[449]: Interval graph of neutrosophic graph.*

- *Turiyam Neutrosophic interval graph[449]: Interval graph of Turiyam Neutrosophic graph.*
- *3D-interval-filament graphs[484]: A 3D-interval-filament graph is an intersection graph of filaments in 3D space, where each filament is a piecewise linear segment, extending the concept of interval-filament graphs to three dimensions. Related concepts include subtree filament graphs[485] co-interval filament graphs[483], and interval filament graphs[927].*
- *Probe interval graphs[496, 807]: Probe interval graphs are a generalization of interval graphs where vertices are divided into two sets: probes and non-probes. An edge exists between two vertices if at least one is a probe and their corresponding intervals intersect. Related concepts include 2-tree probe interval graphs [246, 945], Tagged probe interval graphs[1030], bipartite unit probe interval graphs[247], probe proper interval graphs[893], proper tagged probe interval graphs[919], and cycle-free unit probe interval graphs[248].*
- *Simultaneous interval graphs[625]: Simultaneous interval graphs are two interval graphs that share some vertices, where these shared vertices have the same interval representation in both graphs. A related parameter known as the simultaneous interval number is recognized[165].*
- *Veto Interval Graphs[527]: Veto interval (VI) graphs are interval graphs where each interval has a veto mark. Two vertices are adjacent if their intervals intersect and neither contains the other's veto mark.*
- *Interval k-graphs[245, 361]: Interval k-Graphs are intersection graphs of intervals partitioned into at most k classes, where edges exist if intervals from different classes intersect.*
- *Exactly Hittable Interval Graphs[375]: Exactly Hittable Interval Graphs (EHIG) are interval graphs whose interval representations have exact hitting sets, where each set in the system contains exactly one element from the hitting set.*
- *Interval temporal graphs[620]: An Interval Temporal Graph is a dynamic graph where edges are available during specific time intervals, allowing edge traversal only within these intervals.*
- *Point-Interval graphs[819, 969]: Point-Interval graphs are intersection graphs where each vertex corresponds to a triangle between two parallel lines, with one vertex on one line and two on the other.*

We describe the relationships between graph classes of Interval Graphs [477].

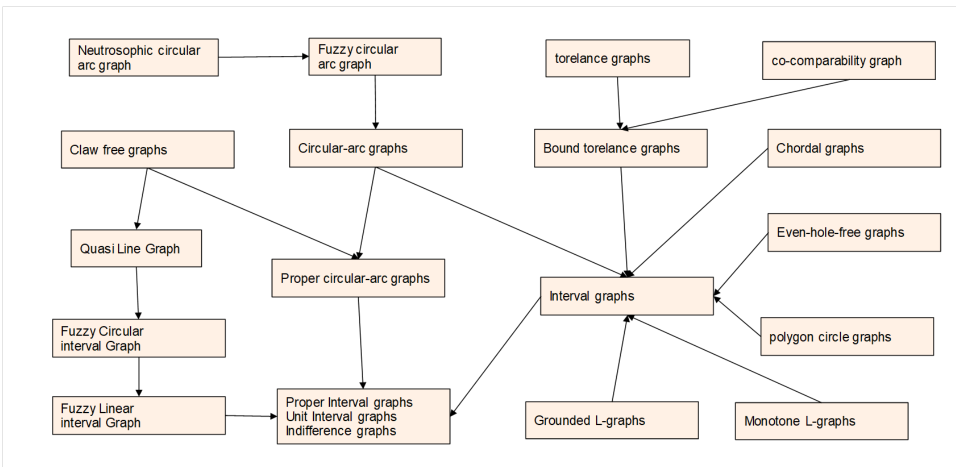

Fig. 5. Interval Graph Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.2.3 Permutation graph

Permutation graphs are defined such that vertices represent elements of a permutation, and edges connect pairs of vertices if their corresponding elements in the permutation are reversed in order[144, 219, 412].

Definition 39. [144,219,412] A graph $G = (V, E)$ is called a *permutation graph* if there exists a permutation π of the set $\{1, 2, \ldots, n\}$, where $n = |V|$, such that for any two distinct vertices u and $v \in V$, the edge $(u, v) \in E$ exists if and only if the indices of u and v in π are reversed in order. Formally, for a permutation π, if vertices $u = i$ and $v = j$ satisfy $i < j$ and $\pi(i) > \pi(j)$, then there exists an edge $(u, v) \in G$.

In other words, a permutation graph is the intersection graph of line segments joining pairs of points on two parallel lines, where each vertex corresponds to a line segment, and two vertices are adjacent if their corresponding line segments intersect.

Example 40. (cf.[451]) Consider the set $V = \{1, 2, 3, 4\}$ and the permutation $\pi = (3, 1, 4, 2)$, which maps the elements of V as follows:

$$\pi(1) = 3, \quad \pi(2) = 1, \quad \pi(3) = 4, \quad \pi(4) = 2$$

The permutation graph $G = (V, E)$ is formed by adding edges between vertices i and j if the indices are reversed in π. That is, $(i, j) \in E$ if $i < j$ and $\pi(i) > \pi(j)$.

From the permutation $\pi = (3, 1, 4, 2)$, we add edges between the following vertices:

$$(1, 2) \text{ because } 1 < 2 \text{ and } \pi(1) = 3 > \pi(2) = 1,$$

$$(1, 4) \text{ because } 1 < 4 \text{ and } \pi(1) = 3 > \pi(4) = 2,$$

$$(3, 4) \text{ because } 3 < 4 \text{ and } \pi(3) = 4 > \pi(4) = 2.$$

Thus, the edge set E is:

$$E = \{(1, 2), (1, 4), (3, 4)\}$$

The resulting graph is the permutation graph corresponding to $\pi = (3, 1, 4, 2)$.

The related graphs of the above graph are introduced as follows.

Theorem 41. *The following are examples of related graph classes, including but not limited to:*

- *Circular permutation graphs[589, 990]: Circular permutation graphs are intersection graphs derived from circular permutation diagrams, where edges represent intersecting chords between two circles.*
- *Bipartite permutation graphs [1074, 1075]: A bipartite permutation graph is both bipartite and a permutation graph, offering efficient solutions for certain NP-complete problems.*
- *Random permutation graphs [183,536]: A random permutation graph is formed by connecting two vertices if their permutation order and index difference have opposite signs.*
- *Functi graphs [308, 410, 416]: Functigraphs generalize permutation graphs by connecting two disjoint copies of a graph with additional edges defined by a function between their vertices.*
- *Split permutation graphs [706, 920]: Split permutation graphs are graphs that belong to both split and permutation graph classes, combining properties of both.*
- *Probe permutation graphs [288]: Probe permutation graphs are permutation graphs where vertices are partitioned into probes and nonprobes, with additional edges only between certain nonprobes.*
- *Polar permutation graphs [399, 400]: Polar permutation graphs are permutation graphs where the vertex set can be partitioned into two: one part forms a complete multipartite graph, the other forms disjoint complete graphs.*
- *Connected permutation graphs[705]: Connected graph of permutation graphs.*
- *Double-threshold permutation graphs[624]: Double-threshold graphs are defined by two thresholds, where vertex adjacency is based on the sum of their ranks falling in a specific "YES" region.*
- *cycle permutation graphs[721, 978]: Cycle permutation graphs represent graphs formed by cyclic permutations, where vertices correspond to elements, and edges represent a specific cyclic permutation of these elements.*
- *weighted permutation graphs[116, 1000]: Weighted version of permutation graphs.*
- *π-Permutation Graphs[91]: A π-permutation graph is formed by connecting two disjoint copies of a graph via a matching determined by a permutation π. Related concepts include PI graph [281, 816].*
- *Balanced Permutation Graphs[1020]: A balanced permutation graph is a graph where vertices i and j are adjacent if and only if $i + j = \pi(i) + \pi(j)$, based on a given permutation π.*
- *Permutation hypergraphs[622]: Hypergraph version of Permutation hypergraphs.*

- *Reducible Permutation Graphs [169, 321, 847]: Reducible Permutation Graphs are graphs where vertices represent elements of a permutation, and the structure is constructed using a reducible flow-graph that allows efficient encoding and decoding of the permutation, ensuring robustness to certain graph modifications such as edge flips or node deletions.*
- *Permutation trees [765, 866]: Tree Graphs of Permutation Graphs.*
- *Acyclic permutation graphs [291, 1047]: Acyclic graphs of Permutation Graphs.*
- *Planar Permutation Graphs [303, 442, 878]: Planar graphs of Permutation Graphs.*
- *Uniform permutation graphs[156]: Uniform graphs of permutation graphs.*

We describe the relationships between graph classes of Permutation Graphs.

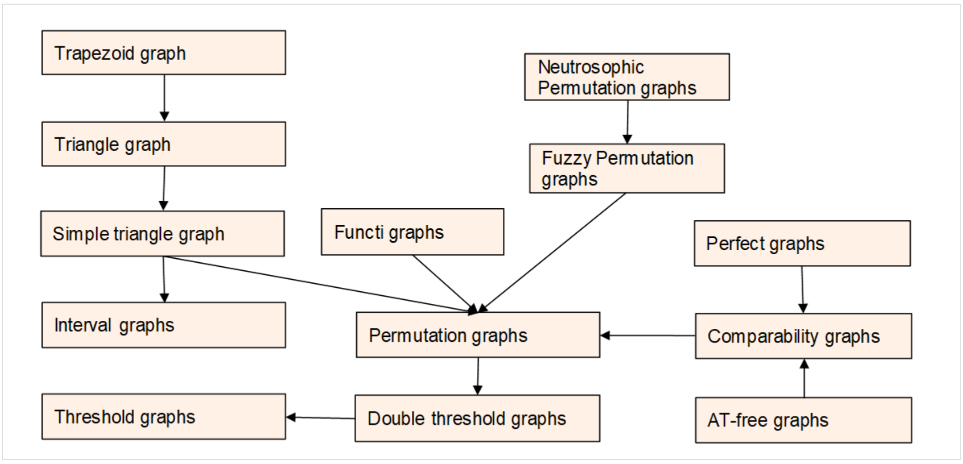

Fig. 6. Permutation Graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.2.4 *Circle and Circular arc graph*

A circular-arc graph represents the intersections of arcs on a circle, where vertices correspond to the arcs, and edges exist between arcs that intersect[481, 535, 1105]. Note that in graph theory, an arc refers to a directed edge connecting two vertices, indicating a one-way relationship or direction. And a circle graph is a graph where vertices represent chords of a circle, and edges exist if the chords intersect. Note that in a circle graph, chords are line segments connecting two points on a circle. Chords intersect if they share common interior points. Circular-arc graphs are known for their linear-time recognition algorithms[365, 641, 659, 661, 1072].

As an example, the definition of Circular-arc graphs is provided below.

Definition 42 (Circular-arc graph)**.** [1106] A *circular-arc graph* is the intersection graph of a set of arcs on a circle. Formally, let $I_1, I_2, \ldots, I_n \subseteq C_1$ be a collection of arcs on a circle C_1, where each I_i represents an arc of the circle. The corresponding circular-arc graph $G = (V, E)$ is defined as follows:

- The vertex set $V = \{I_1, I_2, \ldots, I_n\}$ consists of one vertex for each arc in the set.
- The edge set E is defined such that there is an edge between two vertices I_α and I_β if and only if their corresponding arcs I_α and I_β intersect, i.e.,

$$\{I_\alpha, I_\beta\} \in E \iff I_\alpha \cap I_\beta \neq \emptyset.$$

A family of arcs corresponding to the graph G is referred to as an *arc model* of the circular-arc graph.

The following is clearly valid.

Theorem 43. *Circular-arc graphs are a generalization of interval graphs.*

Proof. Obviously holds. □

The related graphs of the above graph are introduced as follows.

Theorem 44. *The following are examples of related graph classes, including but not limited to:*

- *Jordan arcs intersection graphs[356]: A Jordan Arcs Intersection Graph is an intersection graph where vertices represent Jordan arcs, and edges exist if the corresponding arcs intersect at least once in the plane.*
- *Circular Arc Bigraphs [158]: Bigraph version of Circular Arc graphs.*
- *Proper Circular-Arc Bigraphs [999]: Proper Circular-Arc Bigraphs are bipartite graphs that can be represented by two disjoint sets of arcs on a circle, where no arc is properly contained within another.*
- *Chordal Circular-arc Graphs[273, 440]: A Chordal Circular-Arc Graph is a chordal graph, where all induced cycles are triangles, and it has an arc representation on a circle, meaning each vertex corresponds to an arc, and edges exist between intersecting arcs.*
- *Chordal Proper Circular Arc Graphs [147]: A Chordal Proper Circular-Arc Graph is a chordal graph that is also claw-free and net-free, representing the intersection of arcs on a circle with no proper arc containment.*
- *Unit Circular-Arc Graphs [394, 660, 749]: Unit Circular-Arc Graphs are intersection graphs of arcs on a circle, where all arcs have equal length and intersect based on their placement.*
- *Proper Circular-Arc Graphs [28, 166, 436, 1073, 1107] :Proper Circular-Arc Graphs are intersection graphs of arcs on a circle where no arc is completely contained within another arc.*
- *Helly Circular-Arc Graphs [201,525,745,997]:Helly Circular-Arc Graphs are circular-arc graphs where every family of pairwise intersecting arcs has a common intersection point.*
- *Proper Helly Circular-Arc Graphs [746]:Proper Helly Circular-Arc Graphs are circular-arc graphs where no arc is properly contained within another, and every family of pairwise intersecting arcs shares a common intersection point.*
- *Weighted Circular-Arc Graphs [741, 795]: Weighted version of Circular-Arc Graphs.*
- *Circular-Arc Overlap Graphs [669,1081,1082]: A Circular-Arc Overlap Graph is a graph where vertices correspond to arcs on a circle, and edges exist if the arcs overlap but are not contained within one another.*
- *Bipartite Co-Circular-Arc Graphs [411]: A bipartite co-circular arc graph is a graph whose complement is a circular arc graph and can be divided into two disjoint independent sets.*
- *Co-Circular-Arc Graphs [290]: Co-graph of Circular-Arc Graphs.*
- *Circular-Arc Product Graphs [557]: Circular-arc product graphs are intersection graphs of closed arcs on a circle, resulting from standard graph products like Cartesian, direct, strong, or lexicographic applied to two circular-arc graphs.*
- *Bi-arc Digraphs[573,574]: A bi-arc digraph is a generalization of interval graphs, admitting conservative semilattice polymorphisms, used in studying constraint satisfaction problems and graph homomorphisms.*
- *Concave-Round Graphs[149, 327, 998]: Concave-Round Graphs are the complements of convex-round graphs, where vertices' neighborhoods in a circular order form non-contiguous intervals.*
- *Convex-round graphs[148,974]: Convex-round graphs have vertices circularly ordered, where each vertex's neighborhood forms a contiguous interval in that order.*
- *Normal circular arc graph[274, 528, 747]: Normal circular-arc graphs are graphs where no two arcs in their circular-arc model completely cover the circle.*
- *Fuzzy Circular Interval Graphs [398, 894, 895]: Fuzzy Circular Interval Graphs are a subclass of quasi-line graphs, characterized by stable set polytopes with rank facets based on clique-circulants.*
- *Neutrosophic circular arc graph[443]: Circular Arc graph of Neutrosophic graph.*
- *Fuzzy circular arc graph[443]: Circular Arc graph of Fuzzy graph.*
- *Polygon Circle graphs [404, 927]: A Polygon Circle graph is a graph where vertices represent polygons inscribed in a circle, with edges if polygons intersect. Related concepts include circle-n-gon graph [486] and Spider graph (Equivalent graph class)[703, 704].*
- *Triangle-free circle graphs [18]: A triangle-free circle graph is a circle graph where no three vertices form a triangle, meaning it has no cliques of size 3.*
- *Tangent Circle Graphs[6]: Tangent Circle Graphs are intersection graphs where vertices represent circles tangent to a horizontal line, and edges indicate intersecting circles.*

- *k-polygon graph [401, 402, 1077]: A k-polygon graph is an intersection graph of chords within a convex k-sided polygon, with each chord's endpoints on different sides.*

We describe the relationships between graph classes of circular-arc graphs and circle graphs [352, 477, 693–695, 748]. [1]

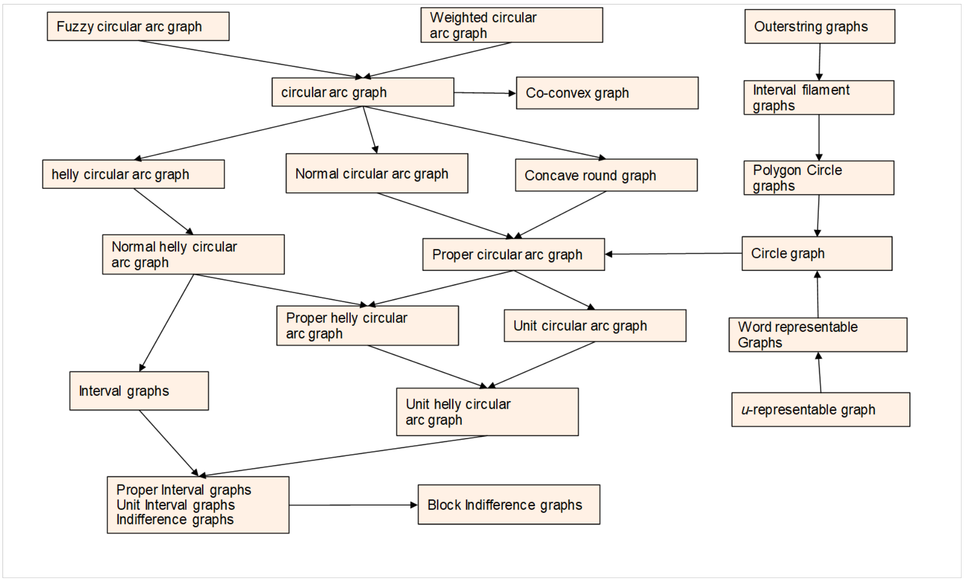

Fig. 7. Circle Graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.2.5 *Grounded graphs*

A grounded intersection graph is an intersection graph where geometric objects, such as rectangles or line segments, are "grounded" on a line, and adjacency is defined by the intersections between these objects [427]. Intuitively, it can be considered as a typical class of intersection graphs with the additional constraint of grounding. As an example, we introduce the definition of Grounded L-Graphs [350, 632, 881].

Definition 45. A *grounded L-graph* is the intersection graph of a collection of L-shaped geometric objects that are grounded on the x-axis. Each L-shape is composed of two line segments: a vertical segment extending upwards from a point on the x-axis, and a horizontal segment extending to the right from the top of the vertical segment, forming an "L" shape.

Let $L = \{L_1, L_2, \ldots, L_n\}$ be a set of such L-shapes. Each L_i is defined by two parameters:

- The *base point* $b_i \in \mathbb{R}$ on the x-axis, where the vertical segment of L_i starts.
- The *height* $h_i \in \mathbb{R}_{>0}$, which determines the length of the vertical segment.

The horizontal segment of each L_i starts from the point (b_i, h_i) and extends to the right.
The *grounded L-graph* $G(L)$ is constructed as follows:

- **Vertex set**: Each L-shape $L_i \in L$ corresponds to a vertex $v_i \in V(G(L))$.
- **Edge set**: Two vertices v_i and v_j are adjacent, i.e., $\{v_i, v_j\} \in E(G(L))$, if and only if the corresponding L-shapes L_i and L_j intersect. This occurs if either:
  - The vertical segment of L_i intersects the horizontal segment of L_j or vice versa.
  - The vertical segments or horizontal segments of L_i and L_j overlap.

[1] Word-representable graphs are graphs where vertices can be represented by a word such that two vertices alternate in the word if and only if they are adjacent.

Mathematically, two L-shapes L_i and L_j intersect if and only if:

$$(b_i \leq b_j \leq b_i + h_i) \quad \text{or} \quad (b_j \leq b_i \leq b_j + h_j),$$

which means the base points and heights of the L-shapes cause the segments to overlap or intersect geometrically.
Thus, the grounded L-graph captures the intersection relationships between the L-shaped objects grounded on the x-axis.

Several variants of the Grounded graph are known. As an example, we introduce the following definition [428].

Definition 46. [428] A configuration of shapes is defined as follows:

(a) **Grounded**: All shapes touch the grounding line at least once.

(b) **Touching Grounded**: The configuration is grounded, and the shapes intersect only at their borders, with no overlap in their interiors.

(c) **Outer**: The configuration has a grounding circle.

(d) **Bigrounded**: Each shape touches the grounding line at exactly two points, and these two points are extremal with respect to the x-coordinate (abscissa).

(e) **2-Grounded**: There is a second line above the grounding line, and all shapes are contained between these two lines, touching both.

(f) **Circle**: The configuration has a grounding circle, and every shape touches the circle at exactly two points.

The related graphs of the above graph are introduced as follows.

Theorem 47. *The following are examples of related graph classes, including but not limited to:*

- *Grounded L-graph [350, 632, 881]: A grounded L-graph is the intersection graph of a collection of L-shaped geometric objects that are grounded on the x-axis. Related concepts include L-graph [23, 420, 804], Circle-L-shape graphs[632], Grounded 1-sided L-shape graphs[680], Grounded 2-sided L-shape graphs [680], Grounded 2-sided square-L graphs[680], Triangle-free L-graphs[1125, 1126], and Grounded $\{L, \overline{L}\}$-graphs[823].*
- *Grounded segment graphs[680]: Grounded segment graphs are intersection graphs of grounded segments or downward rays in the plane, where each segment starts from a common horizontal line. Related concepts include 2-grounded segments graphs[428].*
- *Grounded Unit L-graph[881]: Intersection graph of L-shaped objects, grounded on the x-axis, where all "L" shapes have uniform lengths. Equivalent classes include Grounded Proper L-graph.*
- *Grounded Proper L-graph[881]: Intersection graph of L-shaped objects, grounded on the x-axis, where each "L" shape has distinct non-overlapping properties. Equivalent classes include Grounded Unit L-graph.*
- *V_B graph: The V_B graph is the class of intersection graphs formed by orthogonal $\vee$-shapes, where the angle axes are vertical, the two parts can have different lengths, and the endpoints lie on two vertical lines. Related concepts are following. The $\mathcal{B}_\wedge$ graph is a similar class that also includes $\wedge$-shapes with the same properties as $\vee$-shapes. The $\mathcal{U}$ graph is a class that differs from the $\mathcal{B}_\wedge$ graph only in that both parts of each shape are unbounded. The $\mathcal{B}_{\wedge\cup\vee}$ graph is the class of intersection graphs of $\wedge$-shapes grounded by two vertical lines, where a $\wedge$-shape is formed by connecting a $\vee$-shape and a $\wedge$-shape with the aforementioned properties.*
- *Grounded String Graphs[428]: Grounded Graph of String graphs. Related concepts include Grounded 1-bend string graphs[680].*
- *Grounded Stairs Graphs[428]: Grounded Graph of Stairs graphs. Stairs consist of a series of horizontal and vertical segments forming a staircase extending to the right.*
- *Grounded Convex Graphs[428]: Grounded Graph of Convex Graphs.*
- *Grounded Rectangle Graphs[428]: Grounded Graph of Rectangle Graphs.*
- *Bigrounded string Graphs[428]: Bigrounded Graph of strings Graphs. Equivalent classes include Interval filament graphs.*

- *Grounded Square L-Graphs[215]: Intersection graphs of grounded L-shapes, where the horizontal and vertical segments have equal length, grounded on a common line.*

Proof. Refer to each reference as needed. □

We believe the following challenges exist regarding Grounded graphs for future consideration.

Question 48. What are the mathematical properties of Grounded Unit $L, \overline{L}$-graphs (intersection graphs of unit-length L-shapes and their mirror images, both grounded on a common line with their lowest points touching the line)? Furthermore, what are the mathematical properties and applications of Grounded Unit String Graphs (intersection graphs of unit-length curves, or strings, grounded on a common line where each curve touches the line at least once) and Bigrounded String Graphs (intersection graphs of curves, or strings, that touch the grounding line at two extremal points, forming a 'bigrounded' configuration)?

Question 49. What are the mathematical properties of Grounded Square $L, \overline{L}$-graphs (intersection graphs of L-shapes and their mirror images, with equal-length segments, grounded on a common line)?

We describe the relationships between graph classes of Grounded graphs.

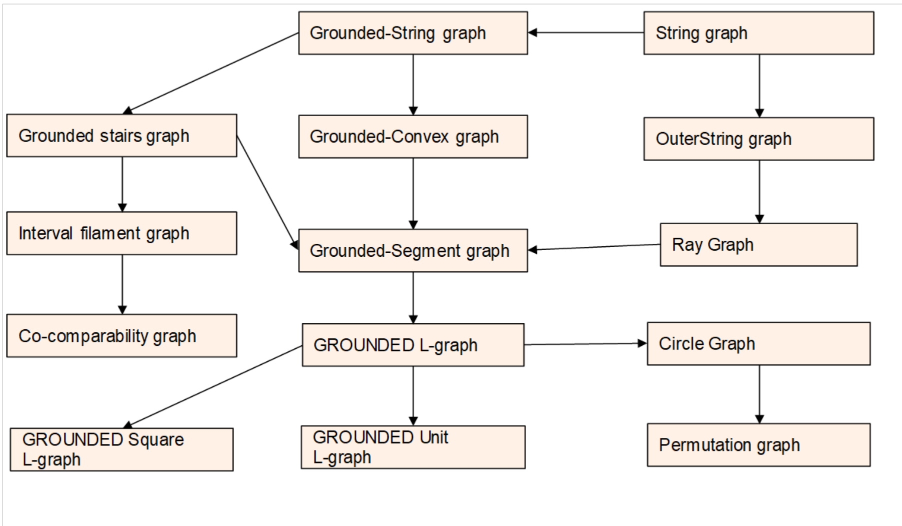

Fig. 8. Grounded Graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.2.6 Ray intersection graphs

A ray graph is an intersection graph where vertices represent rays, and edges exist if the rays intersect[260, 1116]. The definition of a Ray Graph is given below.

Definition 50 (Ray Graph)**.** [260, 1116] A graph $G = (V, E)$ is called a *Ray Graph* if it can be represented as the intersection graph of a collection of rays (half-lines) in the plane. Each vertex of the graph corresponds to a ray, and there is an edge between two vertices if and only if their corresponding rays intersect in the plane.

The related graphs of the above graph are introduced as follows.

Theorem 51. *The following are examples of related graph classes, including but not limited to:*

- *Downward ray graphs[276]: A downward ray graph is the intersection graph of rays (half-lines) where each ray starts at a point (apex) and extends downward infinitely.*
- *Orthogonal Ray Graphs [1035, 1036, 1086, 1087]: Orthogonal ray graphs are intersection graphs of horizontal and vertical rays in the plane, where vertices correspond to rays and edges represent intersections between them.*

- *Orthogonal ray trees[854]: Orthogonal Ray Trees (ORTs) are intersection graphs of trees represented as orthogonal rays, where no forbidden minors (A5Es) are present, and they can be split into two 2-DORTs.*
- *2-Directional Orthogonal Ray Graphs[1037,1038,1088]: A 2-Directional Orthogonal Ray Graph (2DORG) is an intersection graph where vertices represent horizontal or vertical rays, with edges between vertices if the corresponding rays intersect in the plane.*
- *3-Directional Orthogonal Ray Graphs[298]: A 3-Directional Orthogonal Ray Graph is an intersection graph where rays point in three directions (e.g., right, up, down) and vertices are adjacent if their corresponding rays intersect.*
- *Four-Directional Orthogonal Ray Graphs[421,815]: A 4-Directional Orthogonal Ray Graph (4-DORG) is an intersection graph where rays (half-lines) point in four directions (left, right, up, down), and vertices are adjacent if their corresponding rays intersect.*
- *SEG-graph[711, 714]: A SEG graph is an intersection graph where vertices correspond to straight line segments in the plane, and edges exist if segments intersect. A related graph class is the k-SEG graph, unit segment graphs[586], and Outersegment graphs[99, 433].*
- *Stick graph [299, 300, 358, 991]: A Stick graph is the intersection graph of axis-aligned segments where the left endpoints of horizontal segments and bottom endpoints of vertical segments lie on a ground line with slope -1.*
- *Hook graph[582]: Hook graph is A hook graph is an intersection graph where each vertex represents a "hook" composed of a vertical and horizontal segment with a common endpoint. Related concepts include Bipartite hook graph (Biphook graph)[992].*
- *p-BOX graphs[1071]: A p-BOX graph is an intersection graph where each vertex is associated with a box in a multidimensional space, and two vertices are connected if the box of one vertex contains the representative point of the other.*

Proof. Refer to each reference as needed. □

We describe the relationships between graph classes of ray graphs [352,477,748].

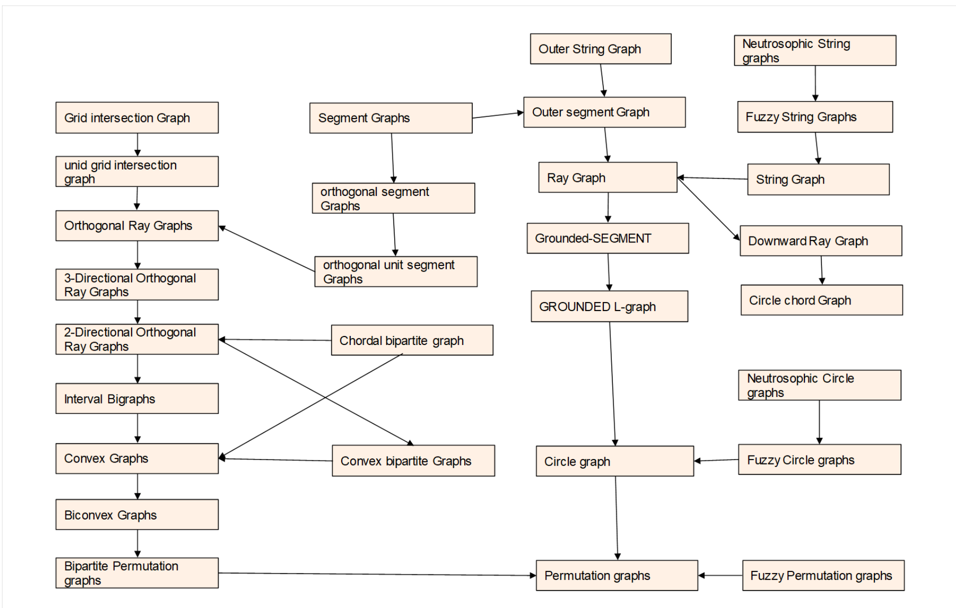

Fig. 9. Ray graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

We believe the following challenges exist regarding Ray graphs for future consideration.

Question 52. What are the mathematical properties of Orthogonal Unit Ray Graphs (intersection graphs of unit-length orthogonal rays, either horizontal or vertical, starting from a common axis and intersecting only at their endpoints) and Unit Outersegment Graphs (intersection graphs of unit-length segments placed outside a convex shape, such as a circle, where each segment intersects the boundary)?

2.2.7 String graph

A string graph is an intersection graph where vertices represent simple curves, and edges exist if the curves intersect[200, 317, 439, 657, 712, 715]. It is a highly flexible graph. The definition is provided below.

Definition 53 (String Graph)**.** (cf.[200, 317, 439, 657, 712, 715]) A graph $G = (V, E)$ is called a *String Graph* if it is the intersection graph of a collection of simple curves, called strings, in the plane. Each vertex of the graph corresponds to a string, and two vertices are adjacent if and only if their corresponding strings intersect at some point, with the restriction that no three strings intersect at the same point. Specifically, a *1-string graph* is a string graph where each pair of strings intersects at most once.

The related graphs of the above graph are introduced as follows.

Theorem 54. *The following are examples of related graph classes, including but not limited to:*

- *1-string graphs[708]: A 1-string graph is the intersection graph of a finite collection of simple curves (strings) in the plane where any two curves intersect at most once.*
- *Outer 1-string graphs[340]: An outer 1-string graph is the intersection graph of curves inside a disk, where each curve has one endpoint on the disk's boundary, and any two curves intersect at most once.*
- *Outerstring graph[679, 980]: An outerstring graph is the intersection graph of curves (strings) inside a disk, where each curve has one endpoint on the disk's boundary.*
- *k-DIR graph[224, 646]: A k-DIR graph is an intersection graph of straight line segments restricted to at most k distinct directions in the plane. Related concepts include DIR contact graph[359].*
- *PURE-k-DIR graph[153, 154]: A PURE-k-DIR graph is an intersection graph of straight line segments in at most k directions, where no two parallel segments intersect.*
- *UNIT-PURE-k-DIR graph[155]: UNIT-PURE-k-DIR graphs are intersection graphs of unit-length line segments in the plane, where segments lie in at most k directions and all parallel segments are disjoint.*
- *B_k-EPG and B_k-VPG: B_k-EPG Graph is a graph where each vertex corresponds to a path on a grid, and two vertices are adjacent if their corresponding paths intersect on grid edges, with at most k bends [121, 366, 408, 508, 510, 568, 929]. B_k-VPG Graph is a graph where each vertex corresponds to a path on a grid, and two vertices are adjacent if their corresponding paths intersect at grid points, with at most k bends [86, 120, 297]. Related concepts include CPG graphs[285] , Contact-VPG Graphs[203, 208], Circular-arc contact VPG graphs[207], and B_1-VCPG[302].*
- *k-Cross Graphs[714, 928]: k-Cross Graphs are intersection graphs of curves where any pair of curves intersects at most k times, generalizing 1-Cross graphs.*

Proof. Refer to each reference as needed. □

Question 55. What are the characteristics of Unit String Graphs (String graphs using strings of the same length), Unit Outerstring Graphs (Outerstring graphs using strings of the same length), and Unit Ray Graphs (Ray graphs using rays of the same length)?

We describe the relationships between graph classes of String graphs.

2.2.8 Trapezoid graph

A trapezoid graph is the intersection graph of trapezoids formed between two parallel lines, where vertices correspond to trapezoids [304]. Like other intersection graphs, trapezoid graphs have been extensively studied in numerous research works [387, 423, 434, 717, 740, 753, 820]. The formal definition is provided below.

Definition 56. [304] **Trapezoid graphs** are intersection graphs derived from trapezoids formed between two parallel lines in the plane. Formally, a trapezoid graph can be defined as follows:

Let L_1 and L_2 be two parallel lines in the plane. A trapezoid is a quadrilateral where one pair of its sides are line segments lying on L_1 and L_2, respectively. Specifically, for each vertex v in the graph, there exists a corresponding trapezoid T_v, where:

- One of the non-parallel sides of the trapezoid is on L_1, say between points $x_1(v)$ and $x_2(v)$,
- The other non-parallel side is on L_2, between points $y_1(v)$ and $y_2(v)$.

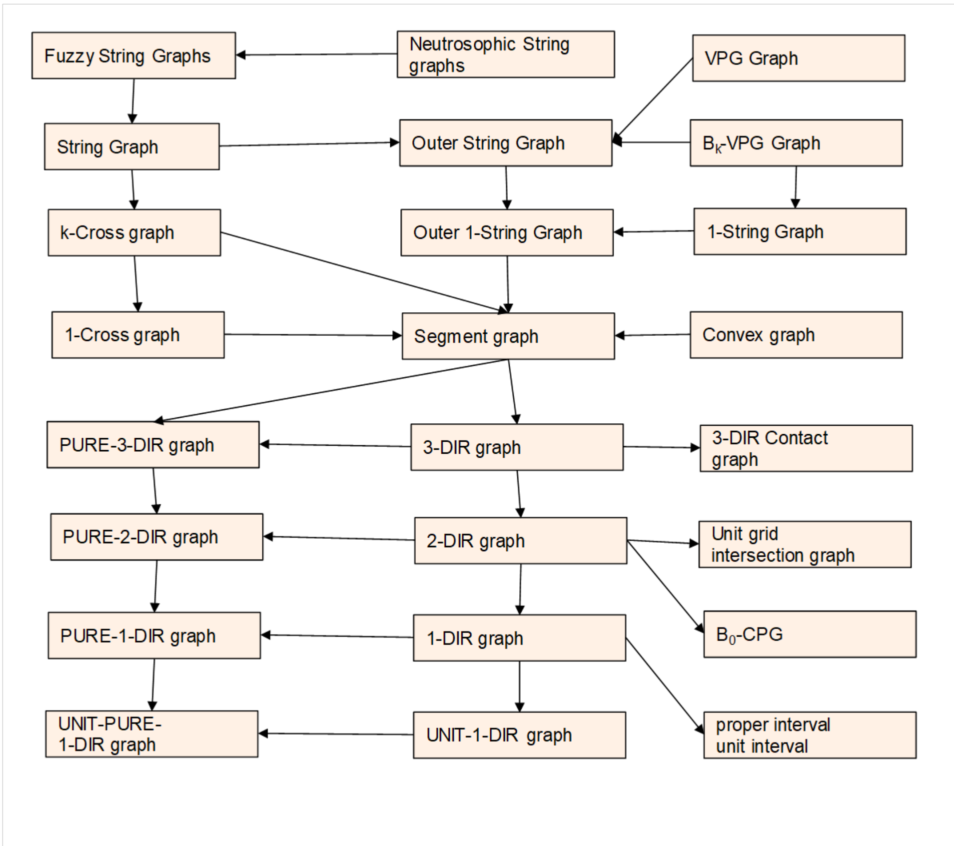

Fig. 10. String Graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

A **trapezoid graph** is then the intersection graph of such trapezoids. That is, the vertex set V of the graph corresponds to the set of trapezoids, and there is an edge between two vertices v_1 and v_2 if and only if the trapezoids T_{v_1} and T_{v_2} intersect geometrically in the plane.

In other words, the adjacency relation in a trapezoid graph is defined by the overlap (intersection) of the trapezoid shapes on the two parallel lines L_1 and L_2. Formally, we define a trapezoid graph $G = (V, E)$ as follows:

$$V(G) = \{T_v \mid v \in V\}, \quad E(G) = \{(v_1, v_2) \mid T_{v_1} \cap T_{v_2} \neq \emptyset\}.$$

Thus, two vertices v_1 and v_2 are adjacent if and only if their corresponding trapezoids T_{v_1} and T_{v_2} share a common region.

The related graphs of the above graph are introduced as follows.

Theorem 57. *The following are examples of related graph classes, including but not limited to:*

- *m-trapezoid graphs[435]: It is a graph that generalizes trapezoid graphs. The class of 0-trapezoid graphs is precisely the class of interval graphs, while the class of 1-trapezoid graphs corresponds exactly to the class of trapezoid graphs.*
- *Circle trapezoid graphs[591, 754]: Trapezoid graphs of Circle graphs.*
- *Circular trapezoid graphs[590, 592]: Trapezoid graphs of Circular-arc graphs.*
- *Acyclic trapezoid graphs[117, 821]: An acyclic trapezoid graph is a trapezoid graph without cycles, where trapezoids are formed between two parallel lines, representing graph vertices.*
- *co-trapezoid graphs[677]: Co-graph of trapezoid graphs.*

- *Trapezoepiped graphs[497,818]: Trapezeopiped graphs are intersection graphs of special trapezoepipeds formed between two parallel lines in 3D Euclidean space. Co-graph of Trapezoepiped graph is multitolerance graph [497, 818, 910] and bitolerance graph[195, 512]. Related concepts include parallelepiped graphs (tolerance graphs)[818]. Co-graph of parallelepiped graph is co-tolerance graph.*
- *Triangle graph [816, 817, 973]: Triangle Graphs are intersection graphs of triangles, where each triangle's apex and base can be on either of two parallel lines. Related concepts include Single Triangle Graphs[816]. Single Triangle Graphs are intersection graphs of triangles with one vertex on one horizontal line and the opposite edge on another.*
- *Proper-trapezoid graphs[194]: These graphs have a trapezoid representation where no trapezoid is fully contained within another trapezoid.*
- *Unit trapezoid graphs:[194]: These graphs have a trapezoid representation where all trapezoids have equal area, typically normalized to an area of 1.*

Proof. Refer to each reference as needed. □

We describe the relationships between graph classes of Trapezoid graphs.

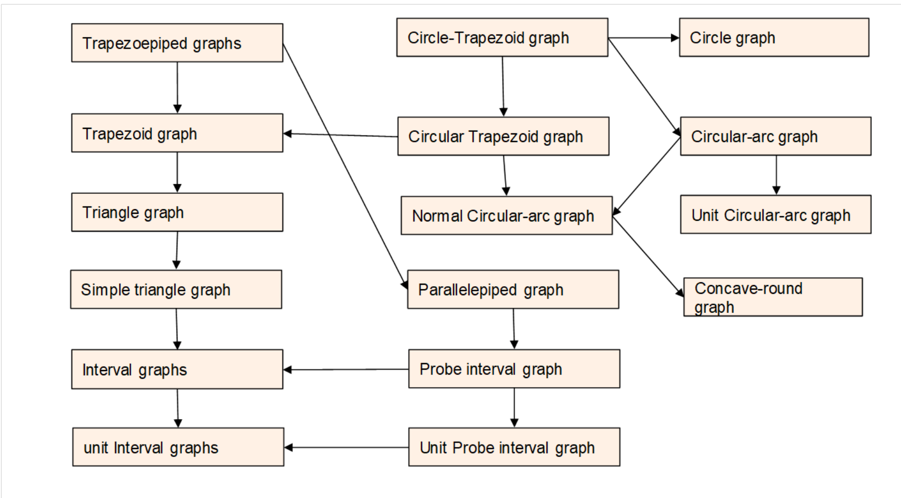

Fig. 11. Trapezoid Graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.2.9 Bipartite intersection graph (intersection bigraph)

A Bipartite Intersection Graph (Intersection Bigraph) is a graph where vertices are divided into two disjoint sets, and edges exist between vertices from different sets if their corresponding sets intersect (cf.[554, 569, 598]). The related graphs of the above graph are introduced as follows.

Theorem 58. *The following are examples of related graph classes, including but not limited to:*

- *Bipartite permutation graphs [1074, 1075]: A bipartite permutation graph is both bipartite and a permutation graph, offering efficient solutions for certain NP-complete problems.*
- *Biconvex Graphs[7, 218, 996]: Bigraph version of Convex Graphs.*
- *Co-biconvex graphs[385]: Co-graph version of Biconvex Graphs.*
- *interval bigraphs [332, 851, 950]: An interval bigraph is a bipartite graph where vertices represent intervals, and edges exist between intersecting intervals from different partite sets. Related concepts include Interval Containment Bigraphs[331].*
- *2-Directional Orthogonal Ray Graphs[1037,1038,1088]: A 2-Directional Orthogonal Ray Graph (2DORG) is an intersection graph where vertices represent horizontal or vertical rays, with edges between vertices if the corresponding rays intersect in the plane.*

- *Grid intersection graphs [167, 295, 296]: Grid intersection graphs are bipartite graphs where vertices represent horizontal and vertical line segments, with edges formed by intersections between the segments on a grid.*
- *Unit grid intersection graphs[855, 856]: Unit version of Grid intersection graphs.*
- *Box intersection graphs[716]: Box Intersection Graphs represent graphs where vertices correspond to axis-aligned boxes in space, and edges exist between vertices if their corresponding boxes intersect.*

Proof. Refer to each reference as needed. □

We naturally wonder whether the above-mentioned Intersection graphs can be extended to Tripartite graphs [315, 318, 899, 1016], Quadripartite graphs [636], and multipartite graphs [636, 761]. In the future, we would like to explore their mathematical characteristics and potential applications.

Question 59. What mathematical properties do Tripartite Intersection graphs (the intersection graphs of Tripartite graphs), Quadripartite Intersection graphs (the intersection graphs of Quadripartite graphs), and Multipartite Intersection graphs (the intersection graphs of Multipartite graphs) possess?

We describe the relationships between graph classes of Bipartite intersection graphs.

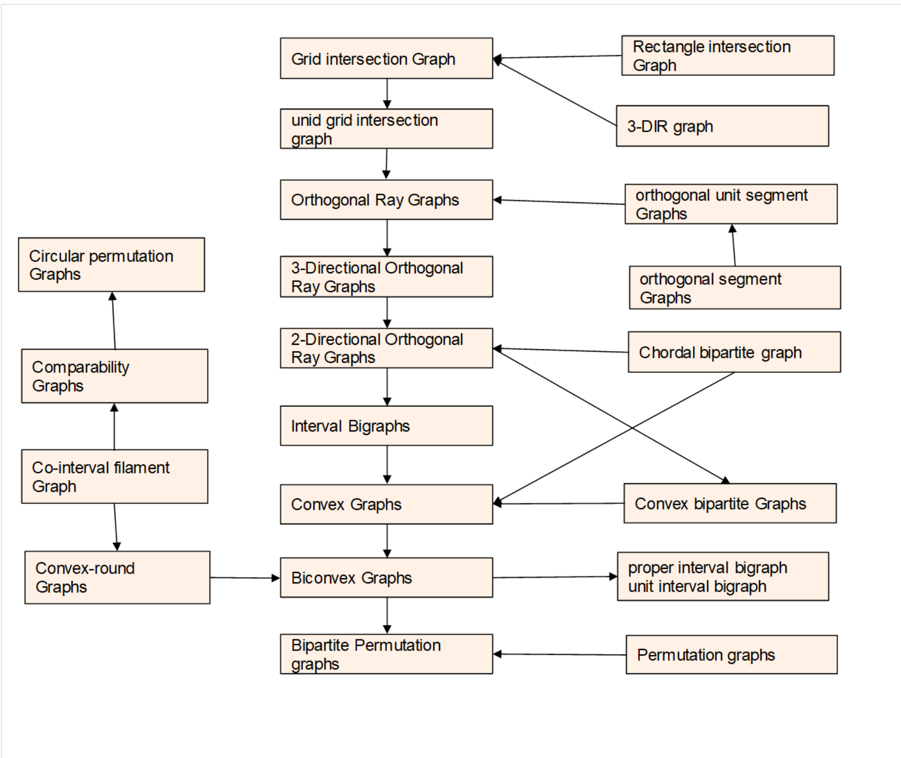

Fig. 12. Bipartite intersection graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.2.10 *Weighted intersection graph*

First, we provide the definition of a Weighted Graph. A Weighted Graph is a classic graph where weights are assigned to its edges, and it has been the subject of extensive research [389,445,565,644,672,911,911,1095, 1113, 1171].

Definition 60. A *Weighted Graph* is a graph $G = (V, E)$ where each edge $e \in E$ is associated with a numerical value called a weight. Formally, it is defined as a pair $G = (V, E, w)$, where:

- V is the set of vertices.

- E is the set of edges, where each edge $e \in E$ connects two vertices in V.
- $w : E \to \mathbb{R}$ is a function that assigns a real number (or sometimes a non-negative real number) to each edge, representing the weight of that edge.

The definition of a Weighted Intersection Graph is provided below. A Weighted Intersection Graph assigns weights to edges based on the strength or frequency of intersections between corresponding sets or objects [482, 1184].

Definition 61. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a family of sets in a given space. The *intersection graph* $G = (V, E)$ corresponding to this family is constructed by associating each set $S_i \in \mathcal{S}$ with a vertex $v_i \in V$, and placing an edge $(v_i, v_j) \in E$ between vertices v_i and v_j if and only if the corresponding sets S_i and S_j have a non-empty intersection, i.e., $S_i \cap S_j \neq \emptyset$.

A *weighted intersection graph* is an intersection graph $G = (V, E)$ where each vertex $v_i \in V$ is assigned a weight $w(v_i) \in \mathbb{R}_{\geq 0}$ or, more generally, $w(v_i) \in \mathbb{R}$. Formally, it is represented as a pair (G, w), where G is the intersection graph and $w : V \to \mathbb{R}$ is the weight function that assigns a real number to each vertex.

Thus, a weighted intersection graph is represented as:

$$G = (V, E, w), \quad w : V \to \mathbb{R}.$$

The edges E still represent the intersections of the corresponding sets in $\mathcal{S}$, while the weight function w provides additional information associated with the vertices.

The related graphs of the above graph are introduced as follows [776, 811, 889].

Theorem 62. *The following are examples of related graph classes, including but not limited to:*

- *Weighted interval graphs[175, 312, 1169]: Interval graphs where vertices have weights, optimizing specific properties.*
- *Weighted proper interval graphs[382]: Proper interval graphs where vertices have weights, optimizing specific properties.*
- *weighted permutation graphs[116, 1000]: Weighted version of permutation graphs.*
- *Weighted unit disk graphs [287, 665, 1129, 1130]: Unit disk graphs with vertex or edge weights, modeling weighted networks.*
- *Weighted disk graph [464, 658, 670]: □Weighted version of disk graph.*
- *Weighted Circular-Arc Graphs[185, 736]: Weighted version of Circular-Arc Graphs. Related concepts include weighted proper circular arc graph[184].*
- *Weighted triangular graph[672]: Weighted version of triangular graph.*
- *Weighted String graphs[710]: Weighted version of String graphs.*
- *Weighted Circle graphs[108]: Weighted version of Circle graphs.*

Proof. Refer to each reference as needed. □

The mathematical properties of the above-mentioned intersection graphs, when extended to weighted digraphs[644, 850] and weighted bigraphs[268], are still not well understood. Therefore, we aim to address the following questions in the future.

Question 63. What are the mathematical properties of Weighted Intersection Digraphs (intersection digraphs derived from weighted digraphs) and Weighted Bipartite Intersection Graphs (intersection bigraphs derived from weighted bigraphs)? Additionally, what are the mathematical characteristics of Weighted Intersection Mixed Graphs (intersection mixed graphs derived from weighted digraphs)?

We describe the relationships between graph classes of Weighted intersection graphs.

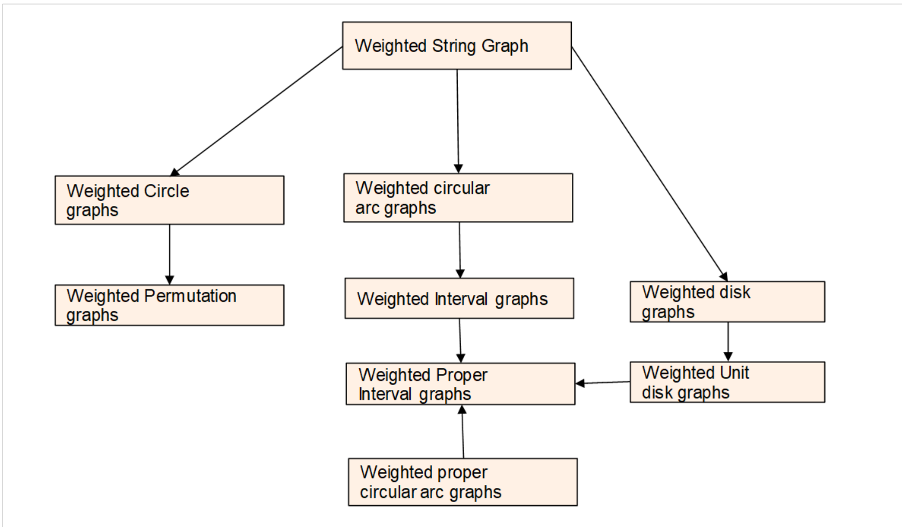

Fig. 13. Weighted intersection graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.2.11 Intersection digraph

An Intersection Digraph is a directed graph where vertices represent sets, and a directed edge exists from one vertex to another if their corresponding sets intersect (cf.[190, 616, 1175]).

Definition 64. [190] An *intersection digraph* is a directed graph $D = (V, A)$ defined by a set of vertices V and arcs (directed edges) A, where there exist two families of sets $S = \{S_1, S_2, \ldots, S_n\}$ and $T = \{T_1, T_2, \ldots, T_n\}$ such that:

- $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices.
- Each vertex $v_i \in V$ is associated with two sets, $S_i \in S$ and $T_i \in T$.
- An arc (directed edge) exists from vertex v_i to vertex v_j (i.e., $(v_i, v_j) \in A$) if and only if the intersection of S_i and T_j is non-empty:

$$(v_i, v_j) \in A \quad \text{if and only if} \quad S_i \cap T_j \neq \emptyset.$$

In other words, there is a directed edge from vertex v_i to vertex v_j if the set S_i associated with v_i has a non-trivial intersection with the set T_j associated with v_j.

The related graphs of the above graph are introduced as follows.

Theorem 65. *The following are examples of related graph classes, including but not limited to:*

- *Interval digraphs[341, 418, 419]: Digraph version of interval graphs. Related concepts include proper interval digraphs [1021, 1076, 1145], indifference digraphs[1021, 1076, 1145], unit interval digraphs[1076, 1145], and monotone proper interval digraphs[575].*
- *Signed-interval digraphs[570]: A signed-interval digraph is a generalization of interval graphs, characterized by a signed-interval model, representing digraphs with specific geometric and ordering properties based on min ordering.*
- *Circular-arc digraphs[346]: Digraph of Circular-arc graphs.*
- *Bi-arc Digraphs[573, 574]: A bi-arc digraph is a generalization of interval graphs, admitting conservative semilattice polymorphisms, used in studying constraint satisfaction problems and graph homomorphisms.*
- *Permutation digraphs[324]: Digraphs of Permutation graphs.*
- *Segment digraphs[750]: Digraphs of segment graph.*
- *String digraphs[547]: Digraphs of string graph.*

Proof. Refer to each reference as needed. □

We describe the relationships between graph classes of Directed intersection graphs. The same principles that apply to intersection graphs of undirected graphs hold in this context as well.

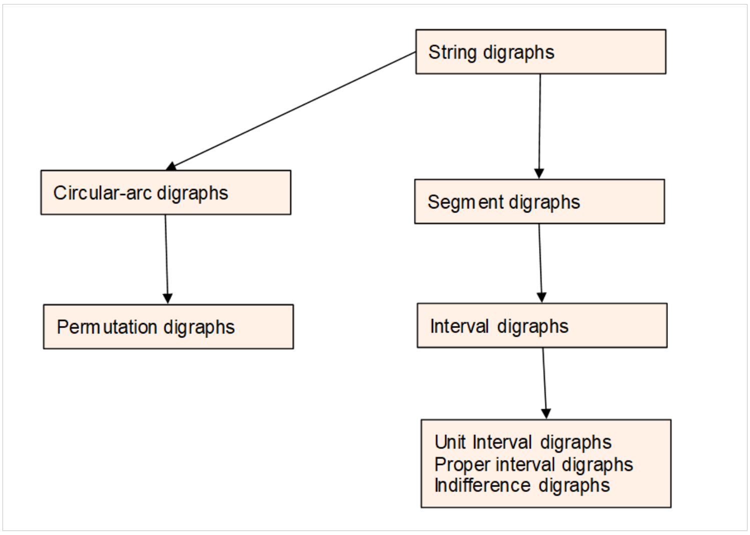

Fig. 14. Directed intersection graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.2.12 Random intersection graph

A random intersection graph is the randomized version of an intersection graph [191, 426, 888, 1183]. Research on random graphs extends beyond intersection graphs, covering a wide range of topics[24, 271, 396]. The definition of a random graph is provided below.

Definition 66. (cf.[24, 271, 396]) A *random graph* is a graph that is generated by some random process. One of the most well-known models for generating random graphs is the *Erdős–Rényi model*, denoted as $G(n, p)$ or $G(n, M)$, which has two standard variants:

In Erdős–Rényi Model $G(n, p)$, a graph is constructed with n vertices. For each pair of vertices (i, j), an edge is added between them with probability p, independently of other pairs. Mathematically, for $G(n, p)$, the probability that any specific edge (i, j) exists is given by:

$$P((i, j) \in E) = p \quad \text{where} \quad p \in [0, 1]$$

The resulting graph can have anywhere between 0 and $\binom{n}{2}$ edges. The model represents the distribution of all possible graphs with n vertices, where the number of edges follows a binomial distribution.

In Erdős–Rényi Model $G(n, M)$, a graph is chosen uniformly at random from the set of all graphs with exactly n vertices and M edges. Specifically, a graph is constructed by randomly selecting exactly M edges from the set of all possible $\binom{n}{2}$ edges.

The related graphs of the above graph are introduced as follows.

Theorem 67. *The following are examples of related graph classes, including but not limited to:*

- *Random permutation graphs [183, 536]: A random permutation graph is formed by connecting two vertices if their permutation order and index difference have opposite signs.*
- *Random interval graphs [934, 1014]: Random graph version of interval graphs.*
- *Random unit disk graphs[597, 656, 926]: Random graph version of unit disk graphs.*
- *Random String graphs[629]: Random graph version of String graphs.*

Proof. Refer to each reference as needed. □

2.2.13 intersection hypergraph

We consider about intersection hypergraph. A hypergraph is a generalization of a graph where edges, known as hyperedges, can connect any number of vertices, not just two. This structure is useful for modeling complex relationships across various fields, such as computer science and biology [13, 221, 265, 275, 424, 424, 425, 476, 517, 519, 520, 743, 924, 924]. The formal definition is provided below.

Definition 68. [221] A hypergraph is a pair $H = (V(H), E(H))$, consisting of a nonempty set $V(H)$ of vertices and a set $E(H)$ of subsets of $V(H)$, called the hyperedges of H. In this paper, we consider only finite hypergraphs.

Example 69. Let H be a hypergraph with vertex set $V(H) = \{A, B, C, D, E\}$ and hyperedge set $E(H) = \{e_1, e_2, e_3\}$, where:

$$e_1 = \{A, D\}, \quad e_2 = \{D, E\}, \quad e_3 = \{A, B, C\}.$$

Thus, H is represented by the pair $H(V, E) = (\{A, B, C, D, E\}, \{\{A, D\}, \{D, E\}, \{A, B, C\}\})$.

An intersection hypergraph is a generalized form of an intersection graph. Similar to intersection graphs, intersection hypergraphs have been extensively studied [143, 577, 682, 723, 853, 1068–1070]. Related concepts include Linear Intersection Hypergraphs[578]. The definition is provided below.

Definition 70 (Intersection Hypergraph)**.** [853] Let B be a finite family of regions (or sets), and let F be another family of regions. The *Intersection Hypergraph* of B with respect to F, denoted as $I(B, F)$, is a hypergraph where:

- The vertex set $V(I(B, F))$ consists of a vertex v_B for each region $B \in B$.
- A hyperedge $H_F \subseteq V(I(B, F))$ exists for each region $F \in F$ such that the hyperedge contains all vertices v_B corresponding to regions B that intersect F, i.e.,

$$H_F = \{v_B : B \cap F \neq \emptyset\}.$$

- The hypergraph is simple, meaning it contains no multiple hyperedges or hyperedges of size 1 (singletons).

This structure generalizes intersection graphs by allowing hyperedges to connect more than two vertices, capturing complex relationships between sets or regions.

The related graphs of the above graph are introduced as follows.

Theorem 71. *The following are examples of related graph classes, including but not limited to:*

- *Permutation hypergraphs[622]: Hypergraph version of Permutation hypergraphs.*
- *circular-arc hypergraphs [503, 700, 701]: Hypergraph version of circular-arc graphs.*
- *Interval hypergraphs [253, 982]: Hypergraph version of Interval graphs. Related concept is Mixed interval hypergraph[254].*
- *Unit disk hypergraphs[474]: Hypergraph version of unit disk graphs.*
- *Intersection graphs of uniform hypergraphs[614, 944]: Intersection graphs in uniform hypergraphs.*
- *Pseudo-disk Hypergraphs[114, 681]: Pseudo-disk hypergraphs are hypergraphs where vertices represent pseudo-disks in the plane, and hyperedges consist of subsets of pseudo-disks intersected by a fixed pseudo-disk.*

Proof. Refer to each reference as needed. □

2.2.14 Uncertain Intersection graph

Research on intersection graphs in Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs has recently gained attention. These graph models extend the concept of intersection graphs by incorporating the conditions specific to Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic frameworks. Below, we provide related concepts.

Theorem 72. *The following are examples of related graph classes, including but not limited to:*

- *Fuzzy intersection graphs[328, 495, 580, 803, 967, 967, 1022]: Intersection graphs of Fuzzy graph.*
- *Neutrosophic intersection graphs[228]: Intersection graphs of Neutrosophic graph.*

- *Fuzzy permutation graphs[968]: Permutation graphs of Fuzzy graph.*
- *Neutrosophic permutation graphs[451]: Permutation graphs of Neutrosophic graph.*
- *Turiyam Neutrosophic permutation graphs[451]: Permutation graphs of Turiyam Neutrosophic graph.*
- *Fuzzy interval graphs[328]: Interval graphs of Fuzzy graph.*
- *Neutrosophic interval graphs[449]: Interval graphs of Neutrosophic graph.*
- *Turiyam Neutrosophic interval graphs[449]: Interval graphs of Turiyam Neutrosophic graph.*

Proof. Refer to each reference as needed. □

We describe the relationships between graph classes of Uncertain Intersection graphs.

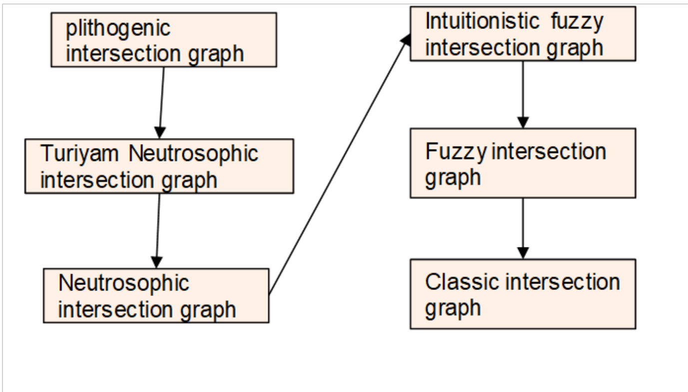

Fig. 15. Uncertain Intersection graphs Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.2.15 Other intersection graphs

In addition to the aforementioned studies, various other research has been conducted on intersection graphs. Below are some examples of other types of intersection graphs.

Theorem 73. *The following are examples of related graph classes for intersection graphs, including but not limited to:*

- *Intersection subgraphs[666, 977, 1123]: An intersection subgraph is a subgraph formed by selecting specific vertices and edges from an intersection graph, where edges represent overlapping sets or objects.*
- *intersection graphs of rings [132, 283, 594, 595, 615, 838]: The intersection graph of rings is a graph where the vertices represent non-zero ideals of a ring, and two vertices are adjacent if their intersection is a non-zero ideal.*
- *Max point-tolerance graphs[280, 936]: A max point-tolerance (MPT) graph is defined by mapping each vertex to a pointed interval, where edges exist if these intervals overlap. Related concepts include central max-point-tolerance graphs[917], max-tolerance graphs[671, 730], , and proper max-point-tolerance graphs[918].*
- *uniform s-intersection graphs[192]: A Uniform Intersection Graph is a graph where vertices represent random subsets of a larger set, and edges exist between vertices if their corresponding subsets intersect by at least a specified number of elements.*
- *Zero-Set Intersection Graphs[159, 214, 623]: A Zero-Set Intersection Graph is a graph where vertices correspond to non-negative real-valued continuous functions over a topological space, and two vertices are adjacent if their zero sets intersect.*

- *Compatibility graphs [498, 511, 645, 837]: A compatibility graph represents the relationships between objects, where edges indicate compatibility between pairs of vertices based on specific criteria. Related concepts include Co-Compatibility graphs [5, 768], chordal Compatibility graphs[596, 773] and Pairwise compatibility graphs[269, 270, 808, 952, 1161, 1162].*
- *Intersection power graph[735, 772, 775]: An intersection power graph is a graph whose vertices correspond to elements of a group, with edges between two vertices if their cyclic subgroups have a non-trivial intersection.*
- *Line graph[210, 259, 553]: A line graph represents the adjacency between edges of a graph G, where each vertex in the line graph corresponds to an edge in G, and two vertices are adjacent if their corresponding edges in G share a common endpoint. Related concepts include (X,Y)-intersection graph[266, 267].*
- *Complete intersection graphs[173]: It represents a complete graph within the context of intersection graphs.*
- *Function graphs[696]: A function graph is an intersection graph where vertices represent continuous real-valued functions on the interval* $[0, 1]$*, and edges exist if the corresponding functions intersect.*
- *Co-intersection graph [362, 544, 777, 778]: Co-graph of intersection graph. Related concepts include Co-interval graphs[1176].*
- *Split graph[527, 529]: Split graph is one of an intersection graph. Related concepts include Probe split graphs[360].*
- *Chordal graphs[100, 337, 507, 509, 559, 697]: A chordal graph is an intersection graph where every cycle of four or more vertices has a chord, meaning an edge connecting two non-consecutive vertices in the cycle.*
- *Path Chordal graphs[174]: A path chordal graph is an intersection graph where vertices represent paths on a tree, and edges exist when paths share at least one node.*
- *Intersection multigraphs[733, 944]: Multigraphs of Intersection graphs.*

Proof. Refer to each reference as needed. □

2.3 Fuzzy, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs

In this subsection, we explore Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. As mentioned in the introduction, these graph concepts are designed to handle uncertainty in real-world scenarios.

2.3.1 Basic definition

First, we consider the basic definitions of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. Fuzzy graphs are often discussed in contrast to crisp graphs, which represent the classical form of graphs [182, 217, 351, 446, 473, 500, 668, 844, 903, 979, 983, 1078, 1143]. The definitions are as follows.

Definition 74. (cf.[692, 983]) A *crisp graph* is an ordered pair $G = (V, E)$, where:

- V is a finite, non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges, where each edge is an unordered pair of distinct vertices.

  Formally, for any edge $(u, v) \in E$, the following holds:

$$(u, v) \in E \iff u \neq v \quad \text{and} \quad u, v \in V$$

This implies that there are no loops (i.e., no edges of the form (v, v)) and edges represent binary relationships between distinct vertices.

Taking the above into consideration, we define Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs as follows. Please note that the definitions have been consolidated for simplicity.

Definition 75 (Unified Graphs Framework: Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs)**.** Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, and falsity.

1. **Fuzzy Graph [182, 217, 351, 446, 473, 500, 668, 844, 903, 979, 983, 1078, 1143]:**
   - Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$, representing the degree of participation of v in the fuzzy graph.
   - Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$, representing the strength of the connection between u and v.
2. **Intuitionistic Fuzzy Graph (IFG) [37, 40, 52, 177, 348, 351, 627, 846, 915, 1004, 1042, 1089, 1112, 1182]:**
   - Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $\nu_A(v) \in [0, 1]$ (degree of non-membership), such that $\mu_A(v) + \nu_A(v) \leq 1$.
   - Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0, 1]$ (degree of membership) and $\nu_B(u, v) \in [0, 1]$ (degree of non-membership), such that $\mu_B(u, v) + \nu_B(u, v) \leq 1$.
3. **Neutrosophic Graph [55, 63, 241, 447, 452, 453, 455–457, 459, 461, 532, 599, 651, 1001, 1055, 1059]:**
   - Each vertex $v \in V$ is assigned a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where:
     - $\sigma_T(v) \in [0, 1]$ is the truth-membership degree,
     - $\sigma_I(v) \in [0, 1]$ is the indeterminacy-membership degree,
     - $\sigma_F(v) \in [0, 1]$ is the falsity-membership degree,
     - $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
   - Each edge $e = (u, v) \in E$ is assigned a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing the truth, indeterminacy, and falsity degrees for the connection between u and v.
4. **Turiyam Neutrosophic Graph[468–470, 1046]:**
   - Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where:
     - $t(v) \in [0, 1]$ is the truth value,
     - $iv(v) \in [0, 1]$ is the indeterminacy value,
     - $fv(v) \in [0, 1]$ is the falsity value,
     - $lv(v) \in [0, 1]$ is the liberal state value,
     - $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.
   - Each edge $e = (u, v) \in E$ is similarly assigned a quadruple representing the same parameters for the connection between u and v.

Various uncertain concepts beyond graphs have been proposed, and their mathematical properties, practical applications, and related algorithms are being researched. Examples of such concepts include uncertain sets [88–90, 125, 126, 390, 392, 638, 751, 1128, 1166, 1172, 1173], uncertain algebra [652, 653, 655], uncertain topology [8, 756, 946, 1006, 1174], uncertain logic [14, 168, 320, 545, 728, 883, 941, 948, 987, 1066, 1083], uncertain geometry [905, 984, 985], uncertain clustering [640, 884, 1151], uncertain systems [1133, 1134], and uncertain statistics [122, 251, 252, 879, 1050, 1052, 1136, 1153]. For further details, please refer to the respective references. Given the importance of research on uncertain concepts, this paper focuses specifically on uncertain graphs.

2.3.2 Plithogenic Graphs and Graph Type

Recently, Plithogenic Graphs have been proposed as a generalization of Fuzzy Graphs and Turiyam Neutrosophic Graphs, as well as a graphical representation of Plithogenic Sets [1053]. Plithogenic Graphs have been developed and are currently being actively studied [654, 1044, 1080, 1084] The definition is provided below.

Definition 76. [1051, 1053] Let S be a universal set, and $P \subseteq S$. A **Plithogenic Set** PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.

- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the **Degree of Appurtenance Function (DAF)**.
- $pCF : Pv \times Pv \to [0,1]^t$ is the **Degree of Contradiction Function (DCF)**.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. **Reflexivity of Contradiction Function**:

$$pCF(a, a) = 0$$

2. **Symmetry of Contradiction Function**:

$$pCF(a, b) = pCF(b, a)$$

Example 77. Here, $s, t \in \{1, 2, 3, 4, 5\}$.

- When $s = t = 1$, PS is called a **Plithogenic Fuzzy Set** and is denoted by PFS.
- When $s = 2, t = 1$, PS is called a **Plithogenic Intuitionistic Fuzzy Set** and is denoted by $PIFS$.
- When $s = 3, t = 1$, PS is called a **Plithogenic Neutrosophic Set** and is denoted by PNS.
- When $s = 4, t = 1$, PS is called a **Plithogenic Turiyam Neutrosophic Set** and is denoted by $PTuS$.
- When $s = 5, t = 1$, PS is called a **Plithogenic Extended Turiyam Neutrosophic Set** and is denoted by $PTuS$.

Definition 78. [1084] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A **Plithogenic Graph** PG is defined as:

$$PG = (PM, PN)$$

where:

1. **Plithogenic Vertex Set** $PM = (M, l, Ml, adf, aCf)$:
   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the range of possible attribute values.
   - $adf : M \times Ml \to [0,1]^s$ is the **Degree of Appurtenance Function (DAF)** for vertices.
   - $aCf : Ml \times Ml \to [0,1]^t$ is the **Degree of Contradiction Function (DCF)** for vertices.
2. **Plithogenic Edge Set** $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \to [0,1]^s$ is the **Degree of Appurtenance Function (DAF)** for edges.
   - $bCf : Nm \times Nm \to [0,1]^t$ is the **Degree of Contradiction Function (DCF)** for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. **Edge Appurtenance Constraint**: For all $(x, a), (y, b) \in M \times Ml$:

$$bdf\left((xy), (a, b)\right) \leq \min\{adf(x, a), adf(y, b)\}$$

where $xy \in N$ is an edge between vertices x and y, and $(a, b) \in Nm \times Nm$ are the corresponding attribute values.

2. **Contradiction Function Constraint**: For all $(a, b), (c, d) \in Nm \times Nm$:

$$bCf\left((a, b), (c, d)\right) \leq \min\{aCf(a, c), aCf(b, d)\}$$

3. **Reflexivity and Symmetry of Contradiction Functions**:

$$\begin{aligned}
&aCf(a,a)=0, && \forall a\in Ml\\
&aCf(a,b)=aCf(b,a), && \forall a,b\in Ml\\
&bCf(a,a)=0, && \forall a\in Nm\\
&bCf(a,b)=bCf(b,a), && \forall a,b\in Nm
\end{aligned}$$

Example 79. The following examples are provided.

- When $s = t = 1$, PG is called a **Plithogenic Fuzzy Graph**.
- When $s = 2, t = 1$, PG is called a **Plithogenic Intuitionistic Fuzzy Graph**.
- When $s = 3, t = 1$, PG is called a **Plithogenic Neutrosophic Graph**.
- When $s = 4, t = 1$, PG is called a **Plithogenic Turiyam Neutrosophic Graph**.
- When $s = 5, t = 1$, PG is called a **Plithogenic Extended Turiyam Neutrosophic Graph**.

Related to the concept of Plithogenic Graphs, several graph-theoretic notions have been introduced, including Plithogenic Fuzzy Graphs [131], Intuitionistic Plithogenic Graphs [1042–1044], and Concentric Plithogenic Hypergraphs [797].

In graphs dealing with uncertainty, the following has been established [452]. From this, it can be concluded that Plithogenic Graphs are a versatile and useful definition capable of various generalizations.

Theorem 80. *[452] In each graph class, the following relationships hold.*

- *An empty graph and a null graph can be represented as 2-valued graphs and 3-valued graphs.*
- *Every edge-fuzzy graph can be transformed into a 2-valued graph by thresholding the edge membership values.*
- *Every fuzzy graph can be transformed into a 3-valued graph by mapping the fuzzy membership values of vertices and edges to the values {-1, 0, 1}.*
- *Every Intuitionistic Fuzzy Graph can be transformed into a Fuzzy Graph by restricting the non-membership function ν_A to 0 for all vertices.*
- *Every Neutrosophic Graph can be transformed into an Intuitionistic Fuzzy Graph by setting the indeterminacy value to zero.*
- *Every Extended Turiyam Neutrosophic Graph is a generalization of the Turiyam Neutrosophic Graph.*
- *A plithogenic Graphs generalize Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, Extended Turiyam Neutrosophic Graphs.*
- *Every general plithogenic Graphs can be transformed into General Turiyam Neutrosophic Graph, General Fuzzy Graph, General Intuitionistic Fuzzy Graph, Four-Valued Fuzzy graph, Ambiguous graph, Picture Fuzzy Graph, Hesitant Fuzzy Graph, Intuitionistic Hesitant Fuzzy Graph, Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Quadripartitioned Neutrosophic graph, Pentapartitioned Neutrosophic graph, Turiyam Neutrosophic Graphs, Extended Turiyam Neutrosophic Graphs, and Spherical Fuzzy Graphs.*

Regarding the Plithogenic Graph type, we extend and explore the fuzzy graph type [193] into the framework of Plithogenic Graphs [452]. This Plithogenic Graph type is designed to clearly specify which elements are considered uncertain. It can also be applied in a similar manner to fuzzy graphs, neutrosophic graphs, and Turiyam Neutrosophic graphs.

Definition 81 (Plithogenic Graph Type)**.** [452]. A plithogenic graph PG is a graph that satisfies one of the following types of plithogenic characteristics (referred to as PG of the i-th type) or any combination thereof:

(i) $PG_1 = \{G_1, G_2, G_3, \ldots, G_P\}$ where plithogenic characteristics exist in each graph G_i, incorporating different attributes and degrees of appurtenance and contradiction for vertices and edges.

(ii) $PG_2 = \{V, E_P\}$ where the edge set E_P is plithogenic, meaning that each edge is associated with a range of possible attributes and corresponding degrees of appurtenance and contradiction.

(iii) $PG_3 = \{V, E(t_P, h_P)\}$ where both the vertex set V and edge set E are crisp, but the edges have plithogenic heads $h(e_i)$ and plithogenic tails $t(e_i)$ with respect to certain attributes.

(iv) $PG_4 = \{V_P, E\}$ where the vertex set V_P is plithogenic, meaning each vertex has attributes with varying degrees of appurtenance and contradiction.

(v) $PG_5 = \{V, E(w_P)\}$ where both the vertex set V and edge set E are crisp, but the edges have plithogenic weights w_P, indicating the attribute-based degrees of appurtenance and contradiction.

2.3.3 Classification of Uncertain Graphs

Plithogenic Graphs generalize Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs. Additionally, various other graph types such as Vague Graphs [44], N-Graphs [29], Rough Graphs [21, 373, 454], Four-Valued Fuzzy Graphs[452], Ambiguous Graphs[452], Hesitant Fuzzy Graphs[452], Intuitionistic Hesitant Fuzzy Graphs[452], Quadripartitioned Neutrosophic Graphs[608, 949], Pentapartitioned Neutrosophic Graphs[789], and Spherical Fuzzy Graphs are also known. These can be further generalized within the frameworks of Plithogenic Graphs, General Plithogenic Graphs, and Refined Plithogenic Graphs[452].

These related graph classes can be broadly categorized and introduced as follows. Research on each graph class that deals with uncertainty is often conducted from the following perspectives.

Theorem 82. *The following are examples of related graph classes, including but not limited to:*

- *Picture Uncertain Graph: Picture Graph (in Fuzzy) is a graph where vertices and edges are assigned three-valued membership degrees—truth, indeterminacy, and falsity, capturing a fuller representation of uncertainty.*
- *Single-valued Uncertain Graph: Single-valued Graph is a graph in which each vertex and edge is assigned a single membership value, typically between 0 and 1, representing certainty levels for inclusion or connection.*
- *Interval-valued Uncertain Graph: A graph in which the membership of vertices and edges is defined by intervals, rather than single values, allowing for a range of uncertainty or indeterminacy in connections.*
- *Bipolar Uncertain Graph: A Bipolar Graph is a graph that extends the fuzzy graph by associating each vertex and edge with both positive and negative membership degrees, capturing both positive and negative relationships.*
- *Directed Uncertain Graph: Directed Graph of Uncertain graph.*
- *Uncertain Hypergraph: Hypergraph of Uncertain graph.*
- *Uncertain Superhypergraph: SuperHypergraph of Uncertain graph. Note that a SuperHyperGraph is an advanced structure extending hypergraphs by allowing vertices and edges to be sets [1056].*
- *Uncertain Multigraph: Multigraph of Uncertain graph. Note that a general multigraph is a graph in which multiple edges (parallel edges) between the same pair of vertices are allowed, and loops may or may not be permitted.*
- *Uncertain Incidence Graph: Incidence Graph of Uncertain graph. Note that incidence graphs represent relationships between vertices and edges, showing which vertices are connected to specific edges.*
- *Product Uncertain Graph: Product Graph of Uncertain graph. A Product Graph is a graph obtained by applying a binary operation on two given graphs.*
- *Complex Uncertain Graph: Complex Graph of Uncertain Graph. Note that complex graphs are graphs where vertices or edges have complex-valued properties, often used to model intricate relationships.*
- *Refined Uncertain Graph: Refined Graph of Uncertain Graph. It is useful when considering recurring uncertainty.*
- *General Uncertain Graph: General Graphs are graphs that relax the conditions of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs.*

Proof. Refer to each reference as needed. □

2.3.4 Picture Uncertain Graph

Picture Graph (in Fuzzy) is a graph where vertices and edges are assigned three-valued membership degrees—truth, indeterminacy, and falsity, capturing a fuller representation of uncertainty. As an example, the definition of a Picture Fuzzy Graph is provided below. Picture Fuzzy Graphs have been extensively studied for their various applications and mathematical properties [110,111,342,344,370,686,690,702,872,988,989,1033, 1034]. Additionally, they generalize both Fuzzy Graphs and Intuitionistic Fuzzy Graphs, and can be further generalized to Neutrosophic Graphs and Plithogenic Graphs [452].

Definition 83. [846] Let $G^* = (V, E)$ be a graph. A pair $G = (A, B)$ is called a *picture fuzzy graph* on G^*, where:

- $A = (\mu_A, \eta_A, \nu_A)$ is a picture fuzzy set on V,
- $B = (\mu_B, \eta_B, \nu_B)$ is a picture fuzzy set on $E \subseteq V \times V$,

such that for each arc $uv \in E$, the following conditions hold:

$$\mu_B(u, v) \leq \min(\mu_A(u), \mu_A(v)),$$

$$\eta_B(u, v) \leq \min(\eta_A(u), \eta_A(v)),$$

$$\nu_B(u, v) \geq \max(\nu_A(u), \nu_A(v)).$$

The related graphs of the above graph are introduced as follows.

Theorem 84. *The following are examples of related graph classes, including but not limited to:*

- *Picture Dombi Fuzzy Graph [834]: A Picture Dombi Fuzzy Graph extends picture fuzzy graphs using Dombi norms to handle multiple responses such as yes, no, abstain, and refusal with fuzzy relationships and operations like union and composition.*
- *Picture fuzzy line graphs[313]: Line graph of Picture fuzzy graphs.*
- *Picture fuzzy planar graphs[170]: Planar graph of Picture fuzzy graphs.*
- *Picture fuzzy tolerance graphs[342, 345]: Picture Fuzzy Tolerance Graphs (PFTGs) extend tolerance graphs by incorporating picture fuzzy sets, handling uncertainties through truth, abstention, and false membership, aiding in complex decision-making and resource allocation problems.*
- *Interval-Valued Picture Fuzzy Graph [957]: Picture Graph of Interval-Valued Fuzzy Graph. Related concepts include Strong Interval Valued Picture Fuzzy Graphs[630].*
- *Picture Fuzzy Incidence Graph [873]: Picture Fuzzy Incidence Graphs (PFIGs) extend fuzzy incidence graphs by incorporating positive, neutral, and negative membership values to model uncertainty in relationships between vertices and edges more effectively.*
- *Picture Fuzzy Soft Graph [307, 619, 1122]: Picture fuzzy soft graphs combine picture fuzzy sets and soft set theory with graph structures to model uncertainty, allowing for positive, neutral, and negative memberships.*
- *Cayley Picture Fuzzy Graph [688]: Cayley graph of Picture Fuzzy Graph.*
- *m-Polar Picture Fuzzy Graphs[689]: Picture graph of m-Polar Fuzzy Graphs.*
- *q-Rung Picture Fuzzy Graph [45]: Q-rung picture fuzzy graphs extend picture fuzzy sets by allowing flexible constraints on membership degrees, modeling uncertain situations with positive, neutral, and negative memberships.*
- *Mixed Picture Fuzzy Graph [858]: Mixed Picture Fuzzy Graphs combine picture fuzzy graphs and mixed graphs, modeling uncertainties with multiple responses like yes, no, rejection, and abstain, for both directed and undirected edges.*
- *Picture Fuzzy Directed Hypergraphs[686]: Hypergraph of Picture Fuzzy Digraph.*
- *Picture fuzzy cubic graphs[690]: Picture graph of fuzzy cubic graphs.*
- *Picture Fuzzy Digraph [791]: A Digraph of a Picture Fuzzy Graph.*
- *Balanced Picture Fuzzy Graph [98]: A Picture graph of a Balanced Fuzzy Graph.*

Proof. Refer to each reference as needed. □

We describe the relationships between graph classes of Picture Graphs.

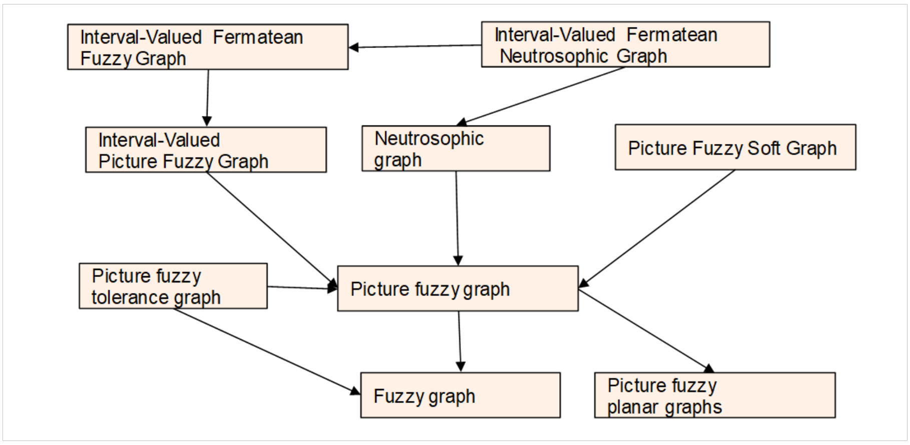

Fig. 16. Picture Graph Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.3.5 Single-valued Uncertain graph

Single-valued Graph is a graph in which each vertex and edge is assigned a single membership value, typically between 0 and 1, representing certainty levels for inclusion or connection.

Example of Single-valued Graph is a Single-Valued Neutrosophic Graph. A *Single-Valued Neutrosophic Graph* (SVNG) is a generalization of classical graphs, where each vertex and edge is characterized by degrees of truth, indeterminacy, and falsity, taking values from the interval $[0, 1]$ [758]. The definition is provided below.

Definition 85. [69, 663, 758] A *Single-Valued Neutrosophic Graph* G is defined as a pair $G = (A, B)$, where:

- $A : V \to [0, 1]$ is a single-valued neutrosophic set on the vertex set V,
- $B : V \times V \to [0, 1]$ is a single-valued neutrosophic relation on the edge set $E \subseteq V \times V$.

For each pair of vertices $x, y \in V$, the following conditions hold:

$$T_B(xy) \le \min\{T_A(x), T_A(y)\},$$

$$I_B(xy) \le \min\{I_A(x), I_A(y)\},$$

$$F_B(xy) \le \max\{F_A(x), F_A(y)\},$$

where:

- $T_A(x), I_A(x)$, and $F_A(x)$ represent the truth-membership, indeterminacy-membership, and falsity-membership values of vertex x,
- $T_B(xy)$, $I_B(xy)$, and $F_B(xy)$ represent the truth-membership, indeterminacy-membership, and falsity-membership values of edge xy.
- **Single-Valued Neutrosophic Vertex Set** A: This defines the neutrosophic membership functions for the vertices. Each vertex $x \in V$ is associated with three membership degrees: truth $(T_A(x))$, indeterminacy $(I_A(x))$, and falsity $(F_A(x))$.
- **Single-Valued Neutrosophic Edge Set** B: This defines the neutrosophic membership functions for the edges. Each edge $xy \in E$ has corresponding neutrosophic membership degrees: truth $(T_B(xy))$, indeterminacy $(I_B(xy))$, and falsity $(F_B(xy))$.
- If the neutrosophic relation B is symmetric, the graph $G = (A, B)$ is considered an undirected single-valued neutrosophic graph.
- If the relation B is not symmetric, the graph $G = (A, B)$ is a directed single-valued neutrosophic graph.

The related graphs of the above graph are introduced as follows.

Theorem 86. *The following are examples of related graph classes, including but not limited to:*

- *Single-Valued Neutrosophic Graphs [67,243]: A Single-Valued Neutrosophic Graph (SVNG) is a generalization of classical graphs, where each vertex and edge is characterized by degrees of truth, indeterminacy, and falsity, taking values from the interval.*
- *Bipartite Single-Valued Neutrosophic Graph[782]: Bipartite graph of Single-Valued Neutrosophic Graph.*
- *Single-valued neutrosophic hypergraphs[68]: Hypergraph version of Single-Valued Neutrosophic Graphs.*
- *Single-Valued Neutrosophic Signed Graphs [809]: A Single Valued Signed Neutrosophic Graph is a neutrosophic graph where each edge and vertex is assigned a positive or negative sign based on their truth, indeterminacy, and falsity membership values.*
- *Single valued neutrosophic signed digraph[1048]: Digraph version of Single-Valued Neutrosophic Signed Graphs.*
- *Single valued pentapartitioned neutrosophic graphs[349]: Single Valued Pentapartitioned Neutrosophic Graphs (SVPN-graphs) are an extension of neutrosophic graphs, with vertices and edges having five independent membership functions: truth, contradiction, ignorance, unknown, and falsity.*
- *bipolar single valued neutrosophic graphs [560,828,829]: A bipolar single valued neutrosophic graph is a generalization of fuzzy and intuitionistic graphs where each vertex and edge has bipolar truth, indeterminacy, and falsity values. Related concepts include bipolar single valued neutrosophic hypergraphs[786].*
- *Bipolar Single Valued Neutrosophic Isolated Graphs[227]: A bipolar single valued neutrosophic isolated graph (BSVNG) is a graph where all vertices and edges are isolated, and its complement is a complete BSVNG.*
- *Single-valued neutrosophic planar graphs[32]: Planar graph of Single-valued neutrosophic graphs.*
- *single-valued co-neutrosophic graphs[378]: Cograph of single-valued neutrosophic graphs.*
- *Single-valued neutrosophic line graphs[871]: line graph of Single-valued neutrosophic graphs.*
- *Regular single valued neutrosophic hypergraphs[788]: Regular graph of single valued neutrosophic hypergraphs.*
- *Regular single valued neutrosophic graphs[118]: Regular graph of single valued neutrosophic graphs. Related concepts include Irregular single valued neutrosophic graphs[1091].*
- *Antipodal single valued neutrosophic graph[783]: Antipodal Single Valued Neutrosophic Graphs (ASVNGs) are single valued neutrosophic graphs where two vertices are antipodal, meaning their maximum distance is the graph's diameter, reflecting specific isomorphic properties.*
- *Regular bipolar single valued neutrosophic hypergraphs[787]: Regular graph of bipolar single valued neutrosophic hypergraphs.*
- *Self-Centered Single Valued Neutrosophic Graphs[1124]: Self-centered Single Valued Neutrosophic Graphs are graphs where every vertex is central with respect to truth, indeterminacy, and falsity membership degrees, ensuring uniform eccentricity for all vertices.*
- *Single-Valued Turiyam Neutrosophic Graph[452]: Single-Valued graph of a Turiyam Neutrosophic Graph.*
- *Single-valued Plithogenic graph[1045]: Single-Valued graph of a Plithogenic graph.*
- *Uniform single valued neutrosophic graphs[226]: An uniform graph of single valued neutrosophic graphs.*

Proof. Refer to each reference as needed. □

Here, we examine the definition of a Single-Valued Plithogenic Graph [1045] and investigate whether it generalizes other Single-Valued graphs. The definition of a Single-Valued Plithogenic Graph is provided below.

Definition 87. Let $G = (V, E)$ be a graph, where:

- V is a non-empty finite set of vertices.
- $E \subseteq V \times V$ is the set of edges.

A Single-Valued Plithogenic Graph PG is defined as:

$$PG = (V, E, a, V_a, d, c)$$

where:

- **Attribute:**
  - a is a multi-valued attribute associated with the elements (vertices and/or edges) of the graph.
  - $V_a = \{v_1, v_2, \ldots, v_n\}$ is the set of possible values for the attribute a, with $n \geq 1$.
- **Degree of Appurtenance Function (DAF):**

$$d : (V \cup E) \to [0, 1]$$

  assigns to each element (vertex or edge) a degree of appurtenance $d(p)$, representing the membership degree of p in the graph concerning the attribute a. Since we are dealing with single-valued attributes, $d(p)$ is a single value in the interval $[0, 1]$.
- **Degree of Contradiction Function (DCF):**

$$c : V_a \times V_a \to [0, 1]$$

  is a contradiction degree function that quantifies the level of contradiction between pairs of attribute values. The function c satisfies the following properties:
  - **Reflexivity:**
  $$c(v_i, v_i) = 0, \quad \forall v_i \in V_a$$
  (An attribute value has no contradiction with itself.)
  - **Symmetry:**
  $$c(v_i, v_j) = c(v_j, v_i), \quad \forall v_i, v_j \in V_a$$
  (The contradiction between two attribute values is mutual.)

Vertices with Attributes:

Each vertex $x \in V$ is associated with:

- An attribute value $a(x) \in V_a$.
- A degree of appurtenance $d(x) \in [0, 1]$, indicating the extent to which x belongs to the graph concerning the attribute a.

Edges with Attributes:

Each edge $e = (x, y) \in E$ is associated with:

- An attribute value $a(e) \in V_a$.
- A degree of appurtenance $d(e) \in [0, 1]$, representing the strength or certainty of the connection between x and y concerning the attribute a.

Appurtenance of Edges:

The degree of appurtenance of an edge $e = (x, y)$ is influenced by the degrees of appurtenance of its vertices and the contradiction degree between their attribute values. The degree of appurtenance $d(e)$ is calculated using the following formula:

$$d(e) = f(d(x), d(y), c(a(x), a(y)))$$

where f is an aggregation function, typically defined as:

$$d(e) = (1 - c(a(x), a(y))) \times \min\{d(x), d(y)\}$$

This formula reflects that the edge's appurtenance decreases as the contradiction between the attribute values of its endpoints increases.

We will examine the relationship between a Single-Valued Neutrosophic Graph and a Single-Valued Plithogenic Graph. The following theorem holds.

Theorem 88. *Every Single-Valued Neutrosophic Graph can be represented as a Single-Valued Plithogenic Graph.*

Proof. We will show that any Single-Valued Neutrosophic Graph $G = (V, E, \sigma_V, \sigma_E)$ can be represented as a Single-Valued Plithogenic Graph $PG = (V, E, a, V_a, d, c)$.

Let us construct PG based on G as follows. We Define the Attribute Set V_a and Attribute Function a. Let the attribute a represent the **neutrosophic membership type**, and define:

$$V_a = \{\text{Truth, Indeterminacy, Falsity}\}.$$

Assign to each vertex $x \in V$ the attribute value corresponding to its maximum membership degree among $T_V(x), I_V(x), F_V(x)$:

$$a(x) = \begin{cases} \text{Truth}, & \text{if } T_V(x) = \max\{T_V(x), I_V(x), F_V(x)\}, \\ \text{Indeterminacy}, & \text{if } I_V(x) = \max\{T_V(x), I_V(x), F_V(x)\}, \\ \text{Falsity}, & \text{if } F_V(x) = \max\{T_V(x), I_V(x), F_V(x)\}. \end{cases}$$

We Define the Degree of Appurtenance Function d. For each vertex $x \in V$, define the degree of appurtenance $d(x)$ as the maximum of its neutrosophic membership degrees:

$$d(x) = \max\{T_V(x), I_V(x), F_V(x)\}.$$

For each edge $e = (x, y) \in E$, define the degree of appurtenance $d(e)$ based on the degrees of appurtenance of its endpoints and the contradiction degree between their attribute values:

$$d(e) = (1 - c(a(x), a(y))) \times \min\{d(x), d(y)\}.$$

We Define the Contradiction Degree Function c. Define $c : V_a \times V_a \to [0, 1]$ as follows:

$$c(u, v) = \begin{cases} 0, & \text{if } u = v, \\ 1, & \text{if } u \neq v. \end{cases}$$

This satisfies the reflexivity and symmetry conditions:

- $c(u, u) = 0$ for all $u \in V_a$.
- $c(u, v) = c(v, u)$ for all $u, v \in V_a$.

Next, we need to verify that the constructed $PG = (V, E, a, V_a, d, c)$ correctly represents the original Single-Valued Neutrosophic Graph G.

We consider Vertex Degrees of Appurtenance. By definition:

$$d(x) = \max\{T_V(x), I_V(x), F_V(x)\}.$$

This captures the most significant membership degree of vertex x.

Next, we consider Edge Degrees of Appurtenance. For edge $e = (x, y)$:

$$d(e) = (1 - c(a(x), a(y))) \times \min\{d(x), d(y)\}.$$

Since $c(a(x), a(y))$ is either 0 (if $a(x) = a(y)$) or 1 (if $a(x) \neq a(y)$), we have:

$$d(e) = \begin{cases} \min\{d(x), d(y)\}, & \text{if } a(x) = a(y), \\ 0, & \text{if } a(x) \neq a(y). \end{cases}$$

This reflects the condition that an edge exists with positive appurtenance only if the attribute values of its endpoints are the same.

We consider Correspondence with Neutrosophic Edge Membership Degrees. In the Single-Valued Neutrosophic Graph G, the edge membership degrees satisfy:

$$T_E(e) \leq \min\{T_V(x), T_V(y)\}, \quad I_E(e) \geq \max\{I_V(x), I_V(y)\}, \quad F_E(e) \geq \max\{F_V(x), F_V(y)\}.$$

Our construction ensures that an edge $e = (x, y)$ in PG has a degree of appurtenance $d(e)$ that corresponds to the relevant membership degrees in G.

If $a(x) = a(y)$ and both vertices have the maximum membership in the same component (truth, indeterminacy, or falsity), then:

$$d(e) = \min\{d(x), d(y)\} = \min\left\{\max\{T_V(x), I_V(x), F_V(x)\}, \max\{T_V(y), I_V(y), F_V(y)\}\right\}.$$

This aligns with the edge membership degrees in G.

We have constructed a Single-Valued Plithogenic Graph PG that represents the Single-Valued Neutrosophic Graph G by:

- Mapping the vertices of G to PG with attribute values based on their maximum neutrosophic membership degrees.
- Defining degrees of appurtenance for vertices and edges that reflect the neutrosophic membership degrees.
- Using a contradiction degree function c that appropriately modulates the edge appurtenance based on the attribute values of the vertices.

Therefore, every Single-Valued Neutrosophic Graph can be represented as a Single-Valued Plithogenic Graph. □

Corollary 89. *Every Single-Valued Turiyam Neutrosophic Graph can be represented as a Single-Valued Plithogenic Graph.*

Proof. The proof can be carried out using the same method as above. □

We describe the relationships between graph classes of Single-valued Graphs.

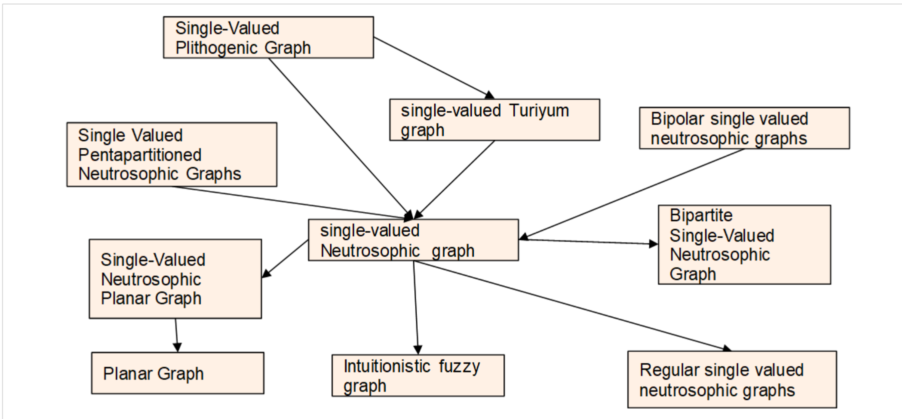

Fig. 17. Single-valued Graph Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.3.6 Interval-valued Uncertain Graph

A graph in which the membership of vertices and edges is defined by intervals, rather than single values, allowing for a range of uncertainty or indeterminacy in connections. One example is the Interval-valued Neutrosophic Graph, described as follows. The Interval-valued Neutrosophic Graph is a Neutrosophic Graph of the Interval-valued graph, and various studies have been conducted on it[241, 244, 860, 1041, 1156].

Definition 90. [241, 1041] An *interval-valued neutrosophic graph* $G = (A, B)$ of a graph $G^* = (V, E)$ is defined as follows:

- $A = \langle [TAL, TAU], [IAL, IAU], [FAL, FAU] \rangle$ is an interval-valued neutrosophic set on the vertex set V, where:

- $TAL : V \to [0, 1], TAU : V \to [0, 1]$ denote the lower and upper bounds of the truth-membership degree,
- $IAL : V \to [0, 1]$, $IAU : V \to [0, 1]$ denote the lower and upper bounds of the indeterminacy-membership degree,
- $FAL : V \to [0, 1]$, $FAU : V \to [0, 1]$ denote the lower and upper bounds of the falsity-membership degree,

such that $0 \le TAL(v_i) + IAL(v_i) + FAL(v_i) \le 3$ for all $v_i \in V$.

- $B = \langle [TBL, TBU], [IBL, IBU], [FBL, FBU] \rangle$ is an interval-valued neutrosophic relation on the edge set E, where:

- $TBL : V \times V \to [0, 1], TBU : V \times V \to [0, 1]$ denote the lower and upper bounds of the truth-membership degree for edges,

- $IBL : V \times V \to [0, 1]$, $IBU : V \times V \to [0, 1]$ denote the lower and upper bounds of the indeterminacy-membership degree for edges,
- $FBL : V \times V \to [0, 1]$, $FBU : V \times V \to [0, 1]$ denote the lower and upper bounds of the falsity-members.

The related graphs of the above graph are introduced as follows.

Theorem 91. *The following are examples of related graph classes, including but not limited to:*

- *Interval-Valued Neutrosophic Graphs [1041]: Interval-Valued Graph of Neutrosophic Graphs. Related concepts include Strong interval valued neutrosophic graphs[242].*
- *Interval Complex Neutrosophic Graphs [228, 603]: Interval Valued graph of Complex Neutrosophic Graphs.*
- *Bipolar Interval Valued Neutrosophic Graphs[617]: Interval Valued graph of Bipolar Neutrosophic Graphs.*
- *Interval valued pentapartitioned neutrosophic graphs [228]: Related concepts include interval-valued quadripartitioned neutrosophic graphs[1032].*
- *Interval-valued fuzzy line graphs[31]: Line graphs of Interval-valued fuzzy graphs. Related concepts include Interval-valued bipolar fuzzy line graphs [1092], Interval□Valued Neutrosophic Line Graphs[1101], ,and Interval-valued bipolar fuzzy line graphs [1100]*
- *Interval-valued intuitionistic fuzzy graphs [719]: Interval-Valued Graph of intuitionistic fuzzy graphs.*
- *Regular Interval-Valued Intuitionistic Fuzzy Graphs[1] Regular graph of Interval-Valued Intuitionistic Fuzzy Graphs.*
- *Cayley interval-valued fuzzy graphs[211]: Cayley Interval-Valued Fuzzy Graphs are Cayley graphs where vertex and edge memberships are expressed as intervals, capturing uncertainty in both group elements and relations.*
- *Interval-valued fuzzy graphs[904]: Interval-Valued Graph of fuzzy graphs.*
- *Balanced interval-valued fuzzy graphs[962]: Interval-valued of Balanced fuzzy graphs.*
- *Complete interval-valued fuzzy graphs[961]: Interval-Valued Graph of Complete fuzzy graphs.*
- *interval-valued picture fuzzy graphs[956]: Interval-Valued Graph of picture fuzzy graphs.*
- *Antipodal interval-valued fuzzy graphs[963]: Antipodal Interval-Valued Fuzzy Graphs are interval-valued fuzzy graphs where antipodal points (vertices) exhibit symmetric relationships in terms of fuzzy membership, capturing uncertainty in both structure and connectivity.*
- *m-polar interval-valued fuzzy graph[171]: Interval-Valued Graph of m-polar fuzzy graph.*
- *Tempered interval-valued fuzzy hypergraphs[38]: Tempered Interval-Valued Fuzzy Hypergraphs are interval-valued fuzzy hypergraphs where the membership values of vertices and edges are constrained within specific intervals, providing flexibility in modeling uncertainty.*
- *Irregular interval valued fuzzy graphs[901]: Irregular graph of Interval-Valued Fuzzy Graphs.*
- *Self centered interval-valued fuzzy graphs[70]: A self-centered interval-valued fuzzy graph is a connected fuzzy graph where every vertex is a central vertex, meaning all vertices share the same eccentricity and radius values.*
- *interval-valued intuitionistic (S, T)-fuzzy graphs[959]: Interval-valued intuitionistic (S, T)-fuzzy graphs extend traditional fuzzy graphs by using interval-valued intuitionistic fuzzy sets to represent uncertainty, providing a richer structure for studying graph properties under uncertainty. Related concepts include interval-valued picture (S, T)-fuzzy graphs[112].*
- *interval-valued Pythagorean fuzzy graphs [58, 830]: Interval-valued Pythagorean fuzzy graphs extend fuzzy graphs by incorporating interval-valued Pythagorean fuzzy sets.*
- *Interval-Valued Fuzzy Trees[933]: Interval-valued graph of fuzzy tree.*
- *Interval-valued intuitionistic fuzzy soft graph[1093]: Interval-valued intuitionistic fuzzy soft graphs combine interval-valued intuitionistic fuzzy graphs and fuzzy soft graphs.*
- *Interval Valued Neutrosophic Soft graphs[1005]: □Interval Valued Neutrosophic Soft graphs combine interval-valued Neutrosophic graphs and soft graphs.*
- *Interval-valued fermatean neutrosophic graphs [9, 240]: Interval-valued Fermatean neutrosophic graphs represent relationships between vertices using interval-based truth, indeterminacy, and falsity values, offering a flexible approach to model uncertainty in complex systems.*

- *Interval-valued Fermatean Fuzzy graph [240,955]: Interval-valued Fermatean Fuzzy Graphs extend fuzzy graphs by assigning interval-based truth and falsity values, offering a flexible approach to model uncertainty in relationships.*
- *Isolated interval valued neutrosophic graphs[229]: These are interval-valued neutrosophic graphs where no vertex connects to others, proven by a necessary and sufficient condition involving its complement being complete.*

Proof. Refer to each reference as needed. □

Question 92. What are the mathematical structures of Interval-valued Turiyam Neutrosophic Graphs (Interval-valued graph of Turiyam Neutrosophic graphs) and Interval-valued Plithogenic Graphs (Interval-valued graph of Plithogenic Graphs), and how do they relate to other graph classes?

We describe the relationships between graph classes of Interval-valued Graphs.

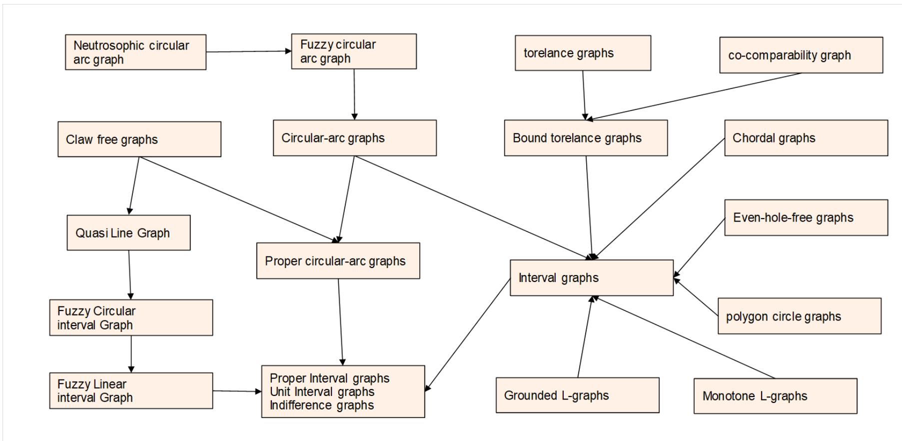

Fig. 18. Interval-valued Graph Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.3.7 Bipolar Uncertain graph

A Bipolar Graph is a graph that extends the fuzzy graph by associating each vertex and edge with both positive and negative membership degrees, capturing both positive and negative relationships. One example is the Bipolar Single-Valued Neutrosophic Graph, described as follows [60,231,560,562,828].

Definition 93. [60] A **bipolar single-valued neutrosophic graph** on a non-empty set V is a pair $G = (B, R)$, where:

1. B is a **bipolar single-valued neutrosophic set** on V with elements of the form:

$$B = \{(v, T_P(v), I_P(v), F_P(v), T_N(v), I_N(v), F_N(v)) \mid v \in V\},$$

where:

- $T_P(v), I_P(v), F_P(v) : V \to [0, 1]$ represent the **positive** membership values for truth, indeterminacy, and falsity, respectively.
- $T_N(v), I_N(v), F_N(v) : V \to [-1, 0]$ represent the **negative** membership values (counter-memberships) for truth, indeterminacy, and falsity, respectively.

2. R is a **bipolar single-valued neutrosophic relation** on V, and for any pair of vertices $b, d \in V$, the relation satisfies the following conditions:

$$T_P(R(b,d)) \le T_P(b) \wedge T_P(d), \quad I_P(R(b,d)) \le I_P(b) \wedge I_P(d), \quad F_P(R(b,d)) \le F_P(b) \vee F_P(d),$$

$$T_N(R(b,d)) \ge T_N(b) \vee T_N(d), \quad I_N(R(b,d)) \ge I_N(b) \vee I_N(d), \quad F_N(R(b,d)) \ge F_N(b) \wedge F_N(d).$$

Here, $T_P(R(b,d)), I_P(R(b,d)), F_P(R(b,d))$ denote the **positive** truth, indeterminacy, and falsity values of the relation between vertices b and d, while $T_N(R(b,d)), I_N(R(b,d)), F_N(R(b,d))$ denote the **negative** truth, indeterminacy, and falsity values.

Thus, a bipolar neutrosophic graph extends traditional graph structures by allowing a richer representation of uncertainty and dual membership for each vertex and edge. The constraints ensure that the membership values for the edges respect the membership values of their incident vertices.

The related graphs of the above graph are introduced as follows.

Theorem 94. *The following are examples of related graph classes, including but not limited to:*

- *Bipolar Fuzzy Graphs [30, 938, 942, 964, 1085] : Bipolar graphs of Fuzzy Graphs.*
- *Strong bipolar fuzzy graphs [1154]: Strong version of Bipolar Fuzzy Graphs. Related concepts include semi-strong bipolar fuzzy graphs[493].*
- *Product bipolar fuzzy graphs[494]: Product Bipolar Fuzzy Graphs combine multiple bipolar fuzzy graphs by considering both positive and negative membership degrees, allowing for the study of more complex relationships.*
- *Cubic bipolar fuzzy graphs[626, 767]: Cubic Bipolar Fuzzy Graphs generalize bipolar fuzzy graphs by combining fuzzy and interval-valued fuzzy sets, capturing more complex relationships and uncertainty in graph structures.*
- *Antipodal bipolar fuzzy graphs[49]: Antipodal Bipolar Fuzzy Graphs are bipolar fuzzy graphs where every vertex is paired with an opposite vertex, reflecting symmetrical properties. These graphs extend fuzzy relations in a structured, opposite manner.*
- *Cayley bipolar fuzzy graphs[1090]: Cayley Bipolar Fuzzy Graphs extend Cayley graphs by incorporating bipolar fuzzy sets, representing both positive and negative relationships, and exploring their algebraic properties and connectivity.*
- *Regular Bipolar Fuzzy Graphs [41]: Regular graphs of Bipolar Fuzzy Graphs.*
- *Irregular bipolar fuzzy graphs [1007]: Irregular graphs of bipolar fuzzy graphs.*
- *Bipolar anti fuzzy graphs [431, 951]: A Bipolar Anti Fuzzy Graph is an extension of a bipolar fuzzy graph that incorporates both positive and negative membership degrees, modeling opposing relationships in a fuzzy graph structure.*
- *m-Polar Fuzzy Graphs [36]: An m-Polar Fuzzy Graph generalizes bipolar fuzzy graphs by allowing multiple poles (positive, neutral, negative), representing various degrees of membership across multiple properties.*
- *Bipolar picture fuzzy graphs[684]: Picture fuzzy graphs of Bipolar fuzzy graphs.*
- *Bipolar hesitancy fuzzy graph[848]: Hesitancy fuzzy graph of Bipolar Fuzzy Graphs.*
- *Edge regular bipolar fuzzy graphs [537]: Edge regular graph of bipolar fuzzy graphs.*
- *Balanced bipolar fuzzy graphs[47]: Balanced Bipolar Fuzzy Graphs are graphs where both positive and negative membership values are balanced, based on density functions, ensuring uniform distribution of connections across the graph.*
- *Bipolar fuzzy trees[415]: Tree graph of Bipolar Fuzzy Graphs.*
- *Bipolar Intuitionistic Anti Fuzzy Graphs [369, 849]: Intuitionistic Anti Fuzzy Graphs of Bipolar fuzzy graph.*
- *Bipolar Fuzzy Competition Graphs [97, 891]: Bipolar Fuzzy Competition Graphs represent competition between entities with positive and negative membership degrees, used for modeling competitive relationships, such as in political scenarios.*
- *Bipolar Single-Valued Neutrosophic Graphs [60]: Bipolar Graph of Neutrosophic Graphs.*
- *Bipolar Intuitionistic Fuzzy Competition Graphs[368]: Bipolar Graph of Intuitionistic Fuzzy Competition Graphs.*
- *Bipolar neutrosophic minimum spanning trees [236–238]: Bipolar Graph of neutrosophic minimum spanning tree.*
- *Isolated Bipolar Single-Valued Neutrosophic Graphs [232]:*
- *Bipolar Complex Neutrosophic Graphs[234]: Complex graphs of Bipolar neutrosophic graph.*

- *Bipolar Single-Valued Neutrosophic Hamiltonian Cycle [724]: Hamiltonian Cycle for Bipolar Single-Valued Neutrosophic graphs.*
- *Neutrosophic bipolar vague graphs[606]: Bipolar Graph of Neutrosophic vague graphs.*
- *m-Polar Neutrosophic Graphs[561, 835]: Generalized version of Bipolar Single-Valued Neutrosophic Graphs.*

Proof. Refer to each reference as needed. □

Question 95. What are the mathematical structures of Bipolar single-valued Turiyam Neutrosophic Graphs (Bipolar graph of Turiyam Neutrosophic graphs) and Bipolar single-valued Plithogenic Graphs (Bipolar graph of Plithogenic Graphs), and how do they relate to other graph classes?

Question 96. What are the mathematical structures of m-polar Turiyam Neutrosophic Graphs (m-polar graph of Turiyam Neutrosophic graphs) and m-polar Plithogenic Graphs (m-polar graph of Plithogenic Graphs), and how do they relate to other graph classes?

We describe the relationships between graph classes of Bipolar Graphs.

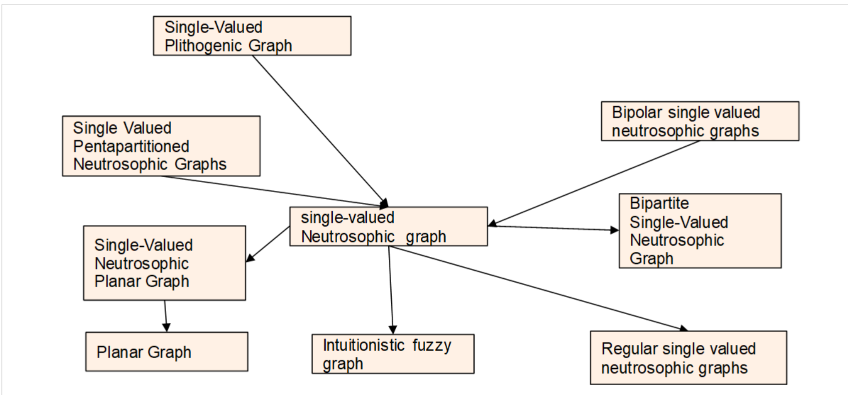

Fig. 19. Bipolar-valued Graph Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

2.3.8 Directed Uncertain Graph

Directed Graph is a graph where each edge has a direction, connecting an ordered pair of vertices, represented as arrows from one vertex to another. The related graphs of the above graph are introduced as follows.

Theorem 97. *The following are examples of related graph classes, including but not limited to:*

- *Intuitionistic fuzzy directed hypergraphs[859] :Directed Graphs of Intuitionistic fuzzy hypergraphs.*
- *Intuitionistic fuzzy directed graphs[685] :Directed Graphs of Intuitionistic fuzzy graphs.*
- *Fuzzy Directed graphs[405] :Directed Graphs of Fuzzy graphs.*
- *Rough fuzzy directed graphs[160, 870] :Directed Graphs of Rough fuzzy graphs.*
- *Bipolar neutrosophic directed hypergraphs[53] :Directed Graphs of Bipolar neutrosophic hypergraphs.*
- *Picture fuzzy digraphs[793] :Directed Graphs of Picture fuzzy graphs.*
- *Pythagorean fuzzy digraphs[792, 909] :Directed Graphs of Pythagorean fuzzy graphs.*
- *Spherical fuzzy digraphs[793] :Directed Graphs of Spherical fuzzy graphs.*
- *m-Polar Fuzzy Digraphs[683] :Directed Graphs of m-Polar Fuzzy graphs.*
- *Transitive fuzzy digraphs[261] :Directed Graphs of Transitive fuzzy graphs.*
- *Rough neutrosophic digraphs[611] :Directed Graphs of Rough neutrosophic graphs.*
- *Turiyam Neutrosophic directed graphs :Directed Graphs of Turiyam Neutrosophic graphs.*
- *Plithogenic directed graphs :Directed Graphs of Plithogenic graphs.*

Proof. Refer to each reference as needed. □

2.3.9 Uncertain Hypergraph

A hypergraph is a generalization of a graph where edges, known as hyperedges, can connect any number of vertices, not just two. One example of an Uncertain Hypergraph is the Fuzzy Hypergraph [54, 501, 502, 729, 774, 925, 1137]. The definition is provided below.

Definition 98 (Fuzzy Hypergraph)**.** [501] A *fuzzy hypergraph* $H = (V, E, \mu)$ consists of the following elements:

- V is a finite set of vertices.
- E is a set of hyperedges, where each hyperedge $e \in E$ is a fuzzy subset of V, i.e., $e : V \to [0, 1]$.
- $\mu : E \times V \to [0, 1]$ is a membership function that assigns to each vertex $v \in V$ a membership degree $\mu(e, v)$ in a hyperedge $e \in E$. This value represents the degree of participation of the vertex v in the hyperedge e.

Formally, for each hyperedge $e \in E$, its fuzzy membership function $\mu_e : V \to [0, 1]$ satisfies the following:

$$\mu_e(v) = \text{membership degree of vertex } v \text{ in hyperedge } e.$$

The fuzzy incidence matrix M of a fuzzy hypergraph is defined as follows: for each pair $(e, v) \in E \times V$, the entry $M(e, v)$ represents the membership degree $\mu(e, v)$. This matrix summarizes the participation of vertices in each hyperedge.

Additionally, the α-cut of a fuzzy hypergraph, denoted as $H_\alpha = (V_\alpha, E_\alpha)$, is a crisp hypergraph obtained by selecting only those vertices and hyperedges where the membership degree is greater than or equal to a threshold α. Formally:

$$V_\alpha = \{v \in V \mid \mu_e(v) \geq \alpha \text{ for some } e \in E\},$$

and

$$E_\alpha = \{e \in E \mid \mu_e(v) \geq \alpha \text{ for all } v \in V_\alpha\}.$$

The related graphs of the above graph are introduced as follows.

Theorem 99. *The following are examples of related graph classes, including but not limited to:*

- *Fuzzy Hypergraphs [501, 502, 925]: Hypergraphs version of fuzzy graphs. Related concepts include Pythagorean fuzzy hypergraphs[869].*
- *Intuitionistic fuzzy hypergraphs [42, 376, 377, 1170]: Hypergraphs version of Intuitionistic fuzzy graphs.*
- *Bipolar Fuzzy Hypergraphs[766]: Hypergraphs version of Bipolar Fuzzy graphs. Related concepts include bipolar fuzzy competition hypergraphs[1027].*
- *Bipolar fuzzy directed hypergraphs[51]: Hypergraphs version of Bipolar fuzzy directed graphs.*
- *m-polar fuzzy hypergraphs[59]: Hypergraphs version of m-polar fuzzy graphs.*
- *Neutrosophic Hypergraphs [770]: Hypergraphs version of Neutrosophic graphs.*
- *Bipolar neutrosophic hypergraphs[50]: Hypergraphs version of Bipolar Neutrosophic graphs.*
- *Picture Fuzzy Directed Hypergraphs [686]: Hypergraphs version of Picture Fuzzy Directed graphs.*
- *Interval-valued fuzzy hypergraphs[38]: Hypergraphs version of Interval-valued fuzzy graphs.*
- *Interval-valued picture fuzzy hypergraphs[687]: Hypergraphs version of Interval-valued picture fuzzy graphs.*
- *Hesitant fuzzy hypergraphs[514]: Hypergraphs version of Hesitant fuzzy graphs.*
- *Fuzzy soft competition hypergraphs[66]: Hypergraphs version of Fuzzy soft competition graphs.*
- *Turiyam Neutrosophic Hypergraphs :Hypergraphs of Turiyam Neutrosophic graphs. It can be defined in a manner almost identical to Neutrosophic Hypergraphs.*
- *Plithogenic Hypergraphs [797, 798]: Hypergraphs of Plithogenic graphs.*

Proof. Refer to each reference as needed. □

2.3.10 *Uncertain Superhypergraph*

A SuperHyperGraph is an advanced structure extending hypergraphs by allowing vertices and edges to be sets. The definition is provided below [457, 550, 551, 1055, 1063].

Definition 100. [1055] A *SuperHyperGraph (SHG)* is an ordered pair SHG $= (G \subseteq P(V), E \subseteq P(V))$, where:

1. $V = \{V_1, V_2, \ldots, V_m\}$ is a finite set of $m \geq 0$ vertices, or an infinite set.
2. $P(V)$ is the power set of V (all subsets of V). Therefore, an SHG-vertex may be a single (classical) vertex, a super-vertex (a subset of many vertices) that represents a group (organization), or even an indeterminate-vertex (unclear, unknown vertex); $\emptyset$ represents the null-vertex (a vertex that has no element).
3. $E = \{E_1, E_2, \ldots, E_m\}$, for $m \geq 1$, is a family of subsets of V, and each E_j is an SHG-edge, $E_i \in P(V)$. An SHG-edge may be a (classical) edge, a super-edge (an edge between super-vertices) that represents connections between two groups (organizations), a hyper-super-edge that represents connections between three or more groups (organizations), a multi-edge, or even an indeterminate-edge (unclear, unknown edge); $\emptyset$ represents the null-edge (an edge that means there is no connection between the given vertices).

Supergraphs[198] and superhypergraphs [106, 450, 492, 618, 857, 937, 1054, 1055, 1058, 1064]. are well-known graph classes related to both graphs and hypergraphs.

The following diagram illustrates their relationship with these graph classes. The same holds true for directed graphs as well[452].

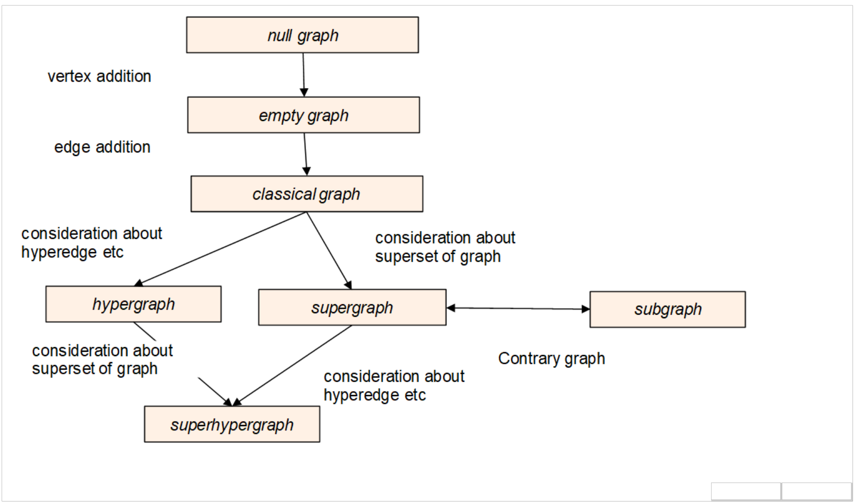

Fig. 20. Graph Hierarchy for Hypergraph. This figure is referenced from [452].

The related graphs of this structure are introduced below.

Theorem 101. *The following are examples of related graph classes, including but not limited to:*

- *Fuzzy Superhypergraphs [1057]: Superhypergraph of Fuzzy graph.*
- *Neutrosophic Superhypergraphs[1055]: Superhypergraph of Neutrosophic graph.*
- *Neutrosophic n-Superhypergraphs[1055]: n-Superhypergraph of Neutrosophic graph.*
- *Turiyam Neutrosophic n-Superhypergraphs: Superhypergraph of Turiyam Neutrosophic graph. It can be defined in a manner almost identical to Neutrosophic n-SuperHypergraphs.*
- *Turiyam Neutrosophic Superhypergraphs: Superhypergraph of Turiyam Neutrosophic graph. It can be defined in a manner almost identical to Neutrosophic SuperHypergraphs.*
- *Plithogenic Superhypergraphs[1055]: Superhypergraphs of Plithogenic graphs.*
- *Plithogenic n-Superhypergraphs [1055, 1063]: n-Superhypergraphs of Plithogenic graphs.*

Proof. Refer to each reference as needed. □

2.3.11 Uncertain Multigraph

A general multigraph is a graph in which multiple edges (parallel edges) between the same pair of vertices are allowed, and loops may or may not be permitted. An Uncertain Multigraph extends this concept by incorporating uncertainty into the graph structure. The definition of a general multigraph is provided below [124, 255, 256, 981].

Definition 102. A *multigraph* G is an ordered triple $G := (V, E, r)$, where:

- V is a set of **vertices** (or **nodes**),
- E is a set of **edges** (or **lines**),
- $r : E \to \{\{x, y\} : x, y \in V\}$ is a function that assigns each edge in E to an unordered pair of vertices from V.

In a *multigraph*, multiple edges (also called **parallel edges**) are allowed between the same pair of vertices. Whether or not **loops** (edges where both endpoints are the same vertex) are allowed may depend on the specific definition being used.

The related graphs of the above graph are introduced as follows.

Theorem 103. *The following are examples of related graph classes, including but not limited to:*

- *intutionistic fuzzy multigraphs[127]: They are defined as graphs that extend the concept of intuitionistic multi-fuzzy sets [338, 347, 348].*
- *Multi-fuzzy graphs: We explore the relationships within multi-fuzzy graphs, which are defined as an extension of the concept of multi-fuzzy sets [78, 1017, 1017, 1018].*
- *Inverse fuzzy multigraphs[212]*

Proof. Refer to each reference as needed. □

2.3.12 Product Uncertain Graphs

We will examine Product Graphs. The definition in the context of general graphs is provided below. Product graphs have been extensively studied across various fields, not limited to uncertain graphs, due to their importance [417, 647, 1157].

Definition 104. A *Product Graph* is a graph obtained by applying a binary operation on two given graphs. There are several types of graph products, each defined differently. Below are the definitions of common graph products:

We consider about Cartesian Product [223, 441, 1187]. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *Cartesian product* $G_1 \square G_2$ is a graph with vertex set $V_1 \times V_2$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if:

$$u_1 = v_1 \text{ and } (u_2, v_2) \in E_2 \quad \text{or} \quad u_2 = v_2 \text{ and } (u_1, v_1) \in E_1.$$

We consider about Tensor Product (Kronecker Product) [72, 737, 1159]. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The *Tensor product* $G_1 \times G_2$ is a graph with vertex set $V_1 \times V_2$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if:

$$(u_1, v_1) \in E_1 \text{ and } (u_2, v_2) \in E_2.$$

We consider about Strong product [947, 1132, 1181, 1186]. The *Strong product* $G_1 \boxtimes G_2$ combines aspects of the Cartesian and Tensor products. The vertex set is $V_1 \times V_2$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if:

$$(u_1 = v_1 \text{ and } (u_2, v_2) \in E_2) \quad \text{or} \quad (u_2 = v_2 \text{ and } (u_1, v_1) \in E_1) \quad \text{or} \quad ((u_1, v_1) \in E_1 \text{ and } (u_2, v_2) \in E_2).$$

We consider about Lexicographic product [262, 319, 781, 842]. The *Lexicographic product* $G_1[G_2]$ is defined by taking the vertex set $V_1 \times V_2$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if:

$$(u_1, v_1) \in E_1 \quad \text{or} \quad (u_1 = v_1 \text{ and } (u_2, v_2) \in E_2).$$

The related graphs of the above graph are introduced as follows.

Theorem 105. *The following are examples of related graph classes, including but not limited to:*

- *Intuitionistic product fuzzy graphs[1089]: Product graph of Intuitionistic fuzzy graphs.*
- *Product Fuzzy Graphs [3, 76, 1114]: Product graph of Fuzzy Graphs.*
- *Product bipolar fuzzy graphs[494]: Product Bipolar Fuzzy Graphs combine multiple bipolar fuzzy graphs by considering both positive and negative membership degrees, allowing for the study of more complex relationships.*
- *Anti Product Fuzzy Graphs[73]: Product graph of anti fuzzy graphs.*
- *Plithogenic product fuzzy graphs [178–181]: Plithogenic Product Fuzzy Graphs (PPFGs) are multi-attribute fuzzy graphs where edges are assigned attribute values using the product operator. They extend fuzzy logic and handle complex relationships by combining multiple attributes from* $[0, 1]$.

Proof. Refer to each reference as needed. □

2.3.13 Incidence Uncertain Graphs

Incidence graphs represent relationships between vertices and edges, showing which vertices are connected to specific edges. Below is an example of the definition of a fuzzy incidence graph [17,862–865,875,876].

Definition 106. A *fuzzy incidence graph (FIG)* on a non-empty set V is an ordered triplet:

$$G = (V, E, I),$$

where:

- V is a set of vertices,
- $E \subseteq V \times V$ is a set of edges,
- $I \subseteq V \times E$ is a set of incidence pairs, denoting the relationship between vertices and edges.

For each pair $(v, e) \in I$, the vertex $v \in V$ is incident to the edge $e \in E$, and the pair (v, e) is called an *incidence pair*.

In a fuzzy incidence graph, fuzzy membership functions are associated with the vertices, edges, and incidence pairs. These functions are defined as follows:

- $\mu_V(v) \in [0, 1]$ represents the *membership degree* of vertex $v \in V$,
- $\mu_E(e) \in [0, 1]$ represents the *membership degree* of edge $e \in E$,
- $\mu_I(v, e) \in [0, 1]$ represents the *membership degree* of the incidence pair $(v, e) \in I$.

The membership functions must satisfy the following conditions:

1. For each incidence pair $(v, e) \in I$:

$$\mu_I(v, e) \leq \min(\mu_V(v), \mu_E(e)),$$

ensuring that the membership of the incidence pair is bounded by the memberships of the vertex and edge.

2. The sum of the membership degree of a vertex and its non-membership degree must satisfy:

$$0 \leq \mu_V(v) + \nu_V(v) \leq 1,$$

where $\nu_V(v) \in [0, 1]$ is the non-membership degree of vertex v. Similarly, this condition applies to edges and incidence pairs.

The related graphs of the above graph are introduced as follows.

Theorem 107. *The following are examples of related graph classes, including but not limited to:*

- *Fuzzy incidence graphs [17, 862–865, 875, 876]: Incidence graphs of fuzzy graphs.*
- *Intuitionistic fuzzy incidence graphs [874, 877]: Incidence graphs of Intuitionistic fuzzy graphs.*
- *Picture Fuzzy Incidence Graphs[872]: Incidence graphs of Picture Fuzzy Graphs.*
- *Bipolar fuzzy incidence graphs[940]: Incidence graphs of Bipolar Fuzzy Graphs.*
- *Directed fuzzy incidence graphs[489]: Incidence graphs of directed fuzzy graphs.*

- *Neutrosophic Incidence Graphs [62,827]: Incidence graphs of Neutrosophic Graphs. Related concepts include Neutrosophic Bipolar Vague Incidence Graph [103] and Neutrosophic Vague Incidence Graph[15].*
- *Interval-valued neutrosophic incidence graphs[958]: Incidence graphs of Interval-valued Neutrosophic Graphs.*

Proof. Refer to each reference as needed. □

2.3.14 Complex Uncertain Graphs

Complex graphs are graphs where vertices or edges have complex-valued properties, often used to model intricate relationships. Below is an example of the definition of a Complex Intuitionistic Fuzzy Graph [61, 105, 1165].

Definition 108. [61, 105, 1165] A **Complex Intuitionistic Fuzzy Graph (CIFG)** is defined as a graph $G = (B, C, X, Y)$, where:

1. B is the set of vertices, and C is the set of edges such that $C \subseteq B \times B$.
2. X is a Complex Intuitionistic Fuzzy Set (CIFS) on the vertices B, and Y is a CIFS on the edges C.
3. For each vertex $b_i \in B$, the membership degree $m_X(b_i)$ and the non-membership degree $n_X(b_i)$ are complex numbers of the form

$$m_X(b_i)e^{i\alpha_X(b_i)}, \quad n_X(b_i)e^{i\beta_X(b_i)},$$

where:

$$m_X(b_i), n_X(b_i) \in [0, 1], \quad \alpha_X(b_i), \beta_X(b_i) \in [0, 2\pi], \quad 0 \leq m_X(b_i) + n_X(b_i) \leq 1.$$

4. For each edge $(b_i, b_j) \in C$, the membership degree $m_Y(b_i, b_j)$ and the non-membership degree $n_Y(b_i, b_j)$ are complex numbers of the form

$$m_Y(b_i, b_j)e^{i\alpha_Y(b_i,b_j)}, \quad n_Y(b_i, b_j)e^{i\beta_Y(b_i,b_j)},$$

where:

$$m_Y(b_i, b_j) \leq \min\{m_X(b_i), m_X(b_j)\}, \quad n_Y(b_i, b_j) \leq \max\{n_X(b_i), n_X(b_j)\},$$

and

$$0 \leq m_Y(b_i, b_j) + n_Y(b_i, b_j) \leq 1.$$

The related graphs of the above graph are introduced as follows.

Theorem 109. *The following are examples of related graph classes, including but not limited to:*

- *Complex Fuzzy Graphs [612, 1119]: Complex graphs of fuzzy graphs.*
- *Bipolar complex intuitionistic fuzzy graphs[867]: Complex graphs of Bipolar intuitionistic fuzzy graphs.*
- *Complex intuitionistic fuzzy graphs[105]: Complex graphs of intuitionistic fuzzy graphs. Related concepts include Complex t-Intuitionistic Fuzzy Graph[674].*
- *complex picture fuzzy graphs[1034]: Complex graphs of picture fuzzy graphs.*
- *Complex pythagorean fuzzy planar graphs[39]: Complex graph of pythagorean fuzzy planar graphs. Related concepts include Complex Pythagorean fuzzy threshold graphs [35] and Complex Pythagorean Dombi fuzzy graphs[48].*
- *complex hesitant fuzzy graphs[11, 93]: Complex graphs of hesitant fuzzy graphs.*
- *Complex neutrosophic graphs [95, 230]: Complex graphs of neutrosophic graphs. Related concepts include Complex t-Neutrosophic Graph[675] and Complex fermatean neutrosophic graph[239].*
- *Bipolar Complex Neutrosophic Graphs[234]: Complex graphs of Bipolar neutrosophic graph.*
- *Regular Complex Neutrosophic Graphs [1097, 1098]: Complex graphs of Regular Neutrosophic Graphs.*
- *Interval-valued Complex Neutrosophic Graphs [228, 603]: Complex graphs of Interval-valued Neutrosophic Graphs.*
- *Interval-Valued Complex Fuzzy Graphs [1120, 1121]: Complex graphs of Interval-valued Fuzzy Graphs.*
- *Complex neutrosophic hypergraphs[771]: Complex graphs of neutrosophic hypergraphs.*

Proof. Refer to each reference as needed. □

2.3.15 Refined Uncertain Graphs

Research on refined sets and graphs has advanced in recent years (cf.[363, 364, 650, 1111, 1111]). The definition of Restricted Refined Plithogenic Graphs is provided as follows[452].

Definition 110. [452] Let $G = (V, E)$ be a crisp graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges.

We define a **Restricted Refined Plithogenic Graph (RRPG)** as:

$$RRPG = (RPM, RPN)$$

where:

- **Restricted Refined Plithogenic Vertex Set** $RPM = (M, l, Ml, adf_r, aCf_r, s, r)$:
  - $M \subseteq V$ is the set of vertices.
  - l is an attribute associated with the vertices.
  - Ml is the range of possible attribute values.
  - $adf_r : M \times Ml \to [0, 1]^{s\times r}$ is the **Restricted Refined Degree of Appurtenance Function (RDAF)** for vertices.
  - $aCf_r : Ml \times Ml \to [0, 1]^{t\times r}$ is the **Restricted Refined Degree of Contradiction Function (RDCF)** for vertices.
  - s is the number of uncertainty components (e.g., truth, indeterminacy, falsity).
  - r is the number of splits per uncertainty component.
- **Restricted Refined Plithogenic Edge Set** $RPN = (N, m, Nm, bdf_r, bCf_r, s, r)$:
  - $N \subseteq E$ is the set of edges.
  - m is an attribute associated with the edges.
  - Nm is the range of possible attribute values.
  - $bdf_r : N \times Nm \to [0, 1]^{s\times r}$ is the **Restricted Refined Degree of Appurtenance Function (RDAF)** for edges.
  - $bCf_r : Nm \times Nm \to [0, 1]^{t\times r}$ is the **Restricted Refined Degree of Contradiction Function (RDCF)** for edges.
  - s and r are as defined above.

The related graphs of the above graph are introduced as follows.

Theorem 111. *The following are examples of related graph classes, including but not limited to:*

- *Refined Fuzzy Graphs[452]: Refined graph of Fuzzy Graphs.*
- *Refined Intuitionistic Fuzzy Graphs[452]: Refined graph of Intuitionistic Fuzzy Graphs.*
- *Refined Neutrosophic Graphs[452]: Refined graph of Neutrosophic Graphs.*
- *Refined Turiyam Neutrosophic Graphs[452]: Refined graph of Turiyam Neutrosophic Graphs.*
- *Refined Extended Turiyam Neutrosophic Graph[452]: Refined graph of Extended Turiyam Neutrosophic Graphs.*
- *Refined Plithogenic graphs[452]: Refined graph of Plithogenic Graphs.*

2.3.16 General Uncertain Graphs

General Graphs are graphs that relax the conditions of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs [447, 452, 890].

For example, a General Fuzzy Graph can be defined as follows:

Definition 112. [890] A *general fuzzy graph* with V as the underlying set is defined as a pair of functions $G = (\sigma, \mu)$, where:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of V,
- $\mu : V \times V \to [0, 1]$ is a fuzzy subset of $V \times V$.

This definition generalizes the traditional fuzzy graph by allowing the membership values of edges to be independent of the membership values of their incident vertices.

The related graphs of the above graph are introduced as follows.

Theorem 113. *The following are examples of related graph classes, including but not limited to:*

- *General weak Fuzzy Graphs[890]: Weak version of general fuzzy graph.*
- *General Intuitionistic Fuzzy Graphs[447, 452]: It is a graph that adds the condition of uncertainty to a general fuzzy graph.*
- *General Neutrosophic Graphs[447, 452]: It is a graph that adds the condition of uncertainty to a general intuitionistic fuzzy graph.*
- *General Turiyam Neutrosophic Graphs[447, 452]: It is a graph that adds the condition of uncertainty to a general neutrosophic graph.*
- *General Plithogenic Graphs[452]: General graph of plithogenic graphs.*

Proof. Refer to each reference as needed. □

2.3.17 Regular Uncertain graph and Irregular Uncertain graph

The definitions of Regular Graph and Irregular Graph in general graphs are provided below. These concepts have been the subject of extensive research (cf.[161, 162, 292, 329, 1115]).

Definition 114. A graph $G = (V, E)$ is called a **regular graph** if every vertex in G has the same degree. In other words, a graph is k-regular if the degree $\deg(v)$ of each vertex $v \in V$ is equal to k, where k is a non-negative integer. Mathematically, we define it as follows:

$$\deg(v) = k, \quad \forall v \in V.$$

This means that each vertex of the graph is connected to exactly k edges. A regular graph is called **k-regular** when each vertex has degree k. For example, in a 3-regular graph, each vertex has degree 3.

Definition 115. A graph $G = (V, E)$ is called an **irregular graph** if not all the vertices of the graph have the same degree. That is, there exists at least one pair of vertices $u, v \in V$ such that:

$$\deg(u) \neq \deg(v).$$

Theorem 116. *The following are examples of related graph classes, including but not limited to:*

- *Regular neutrosophic graphs[599]: Regular graph of neutrosophic graphs.*
- *Irregular neutrosophic graphs[151, 1023]: Irregular graph of neutrosophic graphs.*
- *irregular fuzzy graphs[472]: Irregular graph of fuzzy graphs.*
- *Irregular Intuitionistic Fuzzy Graphs[831]: Irregular graph of Intuitionistic Fuzzy Graphs. Related concepts include m-Neighbourly Irregular Instuitionistic Fuzzy Graphs[779].*
- *Edge regular intuitionistic fuzzy graph[667]: Edge regular graph of intuitionistic fuzzy graph. Related concepts include Edge Regular Intuitionistic Fuzzy M-Polargraphs[490], R-edge regular intuitionistic fuzzy graphs[77], and Perfectly Edge-Regular Intuitionistic Fuzzy Graphs[2].*
- *Edge Irregular Intuitionistic Fuzzy Graphs[868]: Edge Irregular graph of intuitionistic fuzzy graph.*
- *Line Regular Fuzzy Semigraphs [109]: Line Regular Fuzzy Semigraphs are fuzzy semigraphs where all edges share the same line degree (adjacent edge membership values), ensuring uniformity in the structure.*

Proof. Refer to each reference as needed. □

2.3.18 Other uncertain graphs

Here is a brief overview of other uncertain graph types. Beyond these, numerous classes of uncertain graphs are being actively researched in the field, with new developments emerging regularly.

- Perfect graph: A graph is perfect if, for every induced subgraph, the chromatic number equals the size of the largest clique. Related concepts include Perfect fuzzy graphs[20], Perfect bipolar fuzzy graphs[836], and Perfect intuitionistic fuzzy graphs [471].
- Complete graph: A graph is complete if there is an edge between every pair of distinct vertices, connecting all vertices. Related concepts include Complete interval-valued fuzzy graphs[961] and Complete bipartite fuzzy graphs[1028].
- Planar graph: A graph is planar if it can be drawn on a plane without any edges crossing each other. Related concepts include intuitionistic fuzzy planar graphs[96, 458], Fuzzy Planar Graphs [458, 1008], Neutrosophic Outerplanar Graphs [458], and Fuzzy Outerplanar Graphs [621].
- Vague graph: A vague graph generalizes fuzzy graphs by associating vagueness with both edges and vertices, capturing uncertainty in their relationships [44, 960, 1009]. Vague graph is combine vague set theory ([135, 588, 1178]) and graph theory. Related concepts include Neutrosophic Vague Line Graphs [603] ,Irregular vague graphs[150] ,Neutrosophic vague graphs[607] ,Neutrosophic vague line graphs[604] ,Neutrosophic Bipolar Vague Line Graph[372] ,Neutrosophic vague incidence graph[604] ,Neutrosophic Bipolar Vague Incidence Graph[104] ,Neutrosophic Vague Soft Graphs[605] ,and Neutrosophic Vague Graphs [606].
- Soft graph: Soft graphs combine soft set theory (cf.[85, 87, 780, 1158]) and graph theory, providing a flexible way to model uncertain and imprecise information in relationships between objects [56,136,491, 642,643,906,966]. Related concepts include Neutrosophic Soft Rough Graphs [55], Fuzzy soft graphs[57], Quadripartitioned neutrosophic soft graphs[608], bipolar fuzzy soft graphs[43], Intuitionistic fuzzy soft graphs[1026], Intuitionistic Fuzzy Soft Expert Graphs [1112], Interval valued neutrosophic soft graphs [233], Pythagorean fuzzy soft graphs[1025], Pythagorean Dombi fuzzy soft graphs[64], picture fuzzy soft graphs [307], neutrosophic soft graphs [65, 602, 1024], and Q-neutrosophic soft graphs[1110].
- Rough graph: Rough graphs use rough set theory (cf.[163,526,752,921,922,1164,1180]) to handle uncertainty by approximating a graph with lower and upper bounds, representing possible and definite relations between vertices[801, 916, 954, 971, 1031]. Related concepts include Neutrosophic Soft Rough Graphs [55], intuitionistic fuzzy rough graphs [784], rough neutrosophic digraphs[611], neutrosophic hypersoft graphs[71, 995], Soft rough neutrosophic influence graphs[785], and Rough Fuzzy Digraphs [21, 22].
- Spanning tree: A spanning tree is a subgraph that connects all vertices in a graph without any cycles, using the minimum edges[314, 465, 523, 524, 567, 935, 1163]. Related concepts include fuzzy minimum spanning tree [374,475,628], intuitionistic fuzzy minimum spanning tree[833], Single-valued neutrosophic minimum spanning tree[1167], and neutrosophic minimum spanning tree graph[742].
- Labelling graphs: Classic labeling graphs involves assigning labels (usually numbers) to a graph's vertices or edges while following specific rules [293, 294, 466, 1168]. This helps solve problems in coding theory, communication networks, and scheduling. Related concepts include fuzzy labelling graphs [213, 371, 576, 826, 861], Intuitionistic fuzzy labelling graphs [810, 1002], Spherical Fuzzy Labelling Graphs[305], Hesitancy fuzzy magic labeling graphs[976], m-Polar fuzzy labeling graphs[33], Neutrosophic Fuzzy Magic Labeling Graphs[1079], pythagorean neutrosophic fuzzy labeling graphs[26], Cactus fuzzy labeling graphs[649], Fuzzy edge magic total labeling graphs[1029], , and Picture fuzzy labelling graphs[10, 370].
- Threshold graphs: Threshold graphs are graphs constructed by repeatedly adding either isolated vertices or vertices connected to all existing vertices, based on a threshold condition. They model simple networks with specific structural properties. Related concepts include fuzzy threshold graphs[902], Intuitionistic fuzzy threshold graphs[1155], picture Dombi fuzzy threshold graphs[34], complex intuitionistic fuzzy threshold graphs[549], Complex Pythagorean fuzzy threshold graphs[**?**], m-polar fuzzy threshold graphs [**?**], fermatean Fuzzy threshold graphs [549], Neutrosophic Fuzzy Threshold Graphs [676], and Picture fuzzy threshold graphs[343].
- Mixed graphs: Mixed graphs combine both directed and undirected edges in a single graph, useful for modeling systems with varying types of relationships or interactions [146, 533, 755, 993]. Related concepts include Mixed Fuzzy Graphs [339, 521, 1003], Inverse fuzzy mixed planar graphs[839], Inverse Neutrosophic Mixed Graphs[631], , and inverse fuzzy mixed graphs[840, 939].

- Hesitant graphs: Like other classes of fuzzy graphs, the Hesitancy Fuzzy Graph has been the subject of extensive research [534,600,914,943,953]. This type of graph introduces the concept of the hesitancy of an element $v_i \in V$ into the fuzzy graph structure. Related concepts include Complex Hesitant Fuzzy Graph [11] ,Constant hesitancy fuzzy graphs[913] ,Bipolar hesitancy fuzzy graph[601, 907] ,Hesitancy fuzzy magic labeling graph[976] ,Dual hesitant fuzzy graphs[137] ,Regular Hesitancy Fuzzy Soft Graphs[722] ,and Hesitant fuzzy hypergraphs[515]
- Layered graph: A layered graph is a graph where vertices are organized into distinct levels or layers, with edges typically connecting vertices between adjacent layers. Related concepts include Double Layered Fuzzy graph[691], Triple Layered Fuzzy Graph[986], Double Layered Neutrosophic graph[447], and Triple Layered Neutrosophic Graph[447].
- Pythagorean graphs:A *Pythagorean fuzzy graph* (PFG) is a type of fuzzy graph where the degree of membership and nonmembership for each vertex and edge satisfy the Pythagorean condition, i.e.,

  $$(\mu(x))^2 + (\nu(x))^2 \leq 1$$

  for all vertices x and edges. This allows for a more flexible representation of uncertainty in graph structures. Related concepts include Pythagorean Co-Neutrosophic Graphs[522], Pythagorean fuzzy soft graphs[1025], and Pythagorean fuzzy graphs[46].
- Spherical graphs [305,531,555,664,793]: Spherical graphs represent networks where vertices correspond to spherical fuzzy sets, incorporating degrees of truthness, abstinence, and falseness for decision-making or uncertain information modeling. Related concepts include T-Spherical Fuzzy Graphs[531], Spherical Fuzzy Labelling Graphs[306] spherical fuzzy digraphs[791] T-spherical fuzzy Hamacher graphs[119] Regular spherical fuzzy graph[832] Spherical Fuzzy Cycle graph[725] Spherical Fuzzy Tree graph[725] Pseudo regular spherical fuzzy graphs[556] and Spherical neutrosophic graph[27].
- Weighted graphs: A Weighted Graph is a classic graph where weights are assigned to its edges, and it has been the subject of extensive research. Related concepts include Fuzzy weighted graphs[250, 326, 975, 1049]

3. Result: Combination between Intersection and uncertainty

In this section, we consider the combination of intersection and uncertainty.

3.1 Result: Unit Square graph

Unit square graphs are intersection graphs where vertices represent axis-parallel unit squares in the plane, and edges exist if two squares intersect [278, 279, 518, 882]. We define the Unit Square graph in the context of general graphs as follows.

Definition 117 (Unit Square Graph)**.** A *Unit Square Graph* is a geometric intersection graph where vertices correspond to unit squares in the plane, and edges exist between two vertices if their corresponding squares intersect. Formally, let $S = \{s_1, s_2, \ldots, s_n\}$ be a set of axis-aligned unit squares in the Euclidean plane, where each square has side length 1 and is parallel to the coordinate axes. The *Unit Square Graph* $G = (V, E)$ is defined as follows:

- The vertex set V consists of one vertex for each square in S, i.e., $V = \{v_1, v_2, \ldots, v_n\}$, where each vertex v_i corresponds to the square $s_i \in S$.
- The edge set E is defined such that there is an edge between two vertices v_i and v_j if and only if their corresponding squares s_i and s_j intersect, i.e.,

  $$(v_i, v_j) \in E \iff s_i \cap s_j \neq \emptyset.$$

This graph captures the relationships of intersections between unit squares in the plane, making it a specific case of a geometric intersection graph.

Next, we explore Unit Square Graphs within the frameworks of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. The definitions are provided below.

Definition 118 (Fuzzy Unit Square Graph)**.** A *Fuzzy Unit Square Graph* is an extension of the Unit Square Graph $G = (V, E)$ with the following characteristics:

- A membership function $\sigma_V : V \to [0,1]$ for vertices. Each vertex $v \in V$ is assigned a membership degree $\sigma_V(v) \in [0,1]$.
- A membership function $\mu_E : E \to [0,1]$ for edges. Each edge $e = (u,v) \in E$ is assigned a membership degree $\mu_E(e) \in [0,1]$.

These functions must satisfy the following condition:

$$\mu_E(e) \le \min\{\sigma_V(u), \sigma_V(v)\}, \quad \forall e = (u,v) \in E.$$

Definition 119 (Intuitionistic Fuzzy Unit Square Graph)**.** An *Intuitionistic Fuzzy Unit Square Graph* extends the Unit Square Graph $G = (V,E)$ with the following characteristics:

- A membership function and a non-membership function for vertices:

  $$\mu_V : V \to [0,1], \quad \nu_V : V \to [0,1],$$

  such that for each vertex $v \in V$,
  $$0 \le \mu_V(v) + \nu_V(v) \le 1.$$
- A membership function and a non-membership function for edges:

  $$\mu_E : E \to [0,1], \quad \nu_E : E \to [0,1],$$

  such that for each edge $e \in E$,
  $$0 \le \mu_E(e) + \nu_E(e) \le 1.$$

These functions must satisfy the following conditions:

$$\mu_E(e) \le \min\{\mu_V(u), \mu_V(v)\}, \quad \forall e = (u,v) \in E,$$

$$\nu_E(e) \ge \max\{\nu_V(u), \nu_V(v)\}, \quad \forall e = (u,v) \in E.$$

Definition 120 (Neutrosophic Unit Square Graph)**.** A *Neutrosophic Unit Square Graph* is an extension of the Unit Square Graph $G = (V,E)$ with the following characteristics:

- Each vertex is assigned a triple of membership values (truth, indeterminacy, and falsity):

  $$T_V : V \to [0,1], \quad I_V : V \to [0,1], \quad F_V : V \to [0,1],$$

  where for each vertex $v \in V$,
  $$0 \le T_V(v) + I_V(v) + F_V(v) \le 3.$$
- Each edge is assigned a triple of membership values (truth, indeterminacy, and falsity):

  $$T_E : E \to [0,1], \quad I_E : E \to [0,1], \quad F_E : E \to [0,1],$$

  where for each edge $e \in E$,
  $$0 \le T_E(e) + I_E(e) + F_E(e) \le 3.$$

These functions must satisfy the following conditions:

$$T_E(e) \le \min\{T_V(u), T_V(v)\}, \quad \forall e = (u,v) \in E,$$

$$I_E(e) \ge \max\{I_V(u), I_V(v)\}, \quad \forall e = (u,v) \in E,$$

$$F_E(e) \ge \max\{F_V(u), F_V(v)\}, \quad \forall e = (u,v) \in E.$$

Definition 121 (Turiyam Neutrosophic Unit Square Graph)**.** A *Turiyam Neutrosophic Unit Square Graph* is an extension of the Unit Square Graph $G = (V,E)$ with the following characteristics:

- Each vertex is assigned a quadruple:

  $$\tau_V : V \to [0,1]^4, \quad \text{for each vertex } v \in V,\ \tau_V(v) = (t_V(v), i_V(v), f_V(v), l_V(v)),$$

  where:

- $t_V(v)$ is the truth value,
- $i_V(v)$ is the indeterminacy value,
- $f_V(v)$ is the falsity value,
- $l_V(v)$ is the liberal (neutral) state value,

and these satisfy the condition:

$$0 \leq t_V(v) + i_V(v) + f_V(v) + l_V(v) \leq 4.$$

- Each edge is assigned a quadruple:

$$\tau_E : E \to [0,1]^4, \quad \text{for each edge } e \in E, \ \tau_E(e) = (t_E(e), i_E(e), f_E(e), l_E(e)),$$

satisfying the condition:

$$0 \leq t_E(e) + i_E(e) + f_E(e) + l_E(e) \leq 4.$$

These functions must satisfy the following conditions:

$$t_E(e) \leq \min\{t_V(u), t_V(v)\}, \quad \forall e = (u,v) \in E,$$

$$i_E(e) \geq \max\{i_V(u), i_V(v)\}, \quad \forall e = (u,v) \in E,$$

$$f_E(e) \geq \max\{f_V(u), f_V(v)\}, \quad \forall e = (u,v) \in E,$$

$$l_E(e) \geq \max\{l_V(u), l_V(v)\}, \quad \forall e = (u,v) \in E.$$

Definition 122 (Plithogenic Unit Square Graph)**.** A *Plithogenic Unit Square Graph* $PG_{USG} = (PM, PN)$ is an extension of a Unit Square Graph $G = (V, E)$ that incorporates plithogenic structures on vertices and edges. It is defined as follows:

1. **Underlying Unit Square Graph**:
   - **Vertex Set** V: Each vertex $v \in V$ corresponds to an axis-aligned unit square s_v in the Euclidean plane.
   - **Edge Set** E: For vertices $u, v \in V$, there is an edge $e = (u,v) \in E$ if and only if their corresponding squares intersect:
   $$e = (u,v) \in E \iff s_u \cap s_v \neq \emptyset.$$
2. **Plithogenic Extension**:
   a) **Plithogenic Vertex Set** $PM = (M, l, Ml, \text{adf}, \text{aCf})$:
   - $M = V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the set of possible values of attribute l.
   - The *Degree of Appurtenance Function* (DAF) for vertices:
   $$\text{adf} : M \times Ml \to [0,1]^s,$$
   assigns to each vertex $v \in M$ and attribute value $a \in Ml$ an s-tuple $\text{adf}(v,a) = (\mu_1, \mu_2, \ldots, \mu_s)$, where $\mu_i \in [0,1]$ for $i = 1, \ldots, s$.
   - The *Degree of Contradiction Function* (DCF) for vertices:
   $$\text{aCf} : Ml \times Ml \to [0,1]^t,$$
   assigns to each pair $(a,b) \in Ml \times Ml$ a t-tuple $\text{aCf}(a,b) = (\gamma_1, \gamma_2, \ldots, \gamma_t)$, satisfying:
   $$\text{aCf}(a,a) = 0, \quad \text{aCf}(a,b) = \text{aCf}(b,a), \quad \forall a, b \in Ml.$$

   b) **Plithogenic Edge Set** $PN = (N, m, Nm, \text{bdf}, \text{bCf})$:
   - $N = E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the set of possible values of attribute m.

- The *Degree of Appurtenance Function* (DAF) for edges:

$$\mathrm{bdf} : N \times Nm \to [0,1]^s,$$

assigns to each edge $e \in N$ and attribute value $c \in Nm$ an s-tuple $\mathrm{bdf}(e,c) = (\lambda_1, \lambda_2, \ldots, \lambda_s)$, where $\lambda_i \in [0,1]$ for $i = 1, \ldots, s$.
- The *Degree of Contradiction Function* (DCF) for edges:

$$\mathrm{bCf} : Nm \times Nm \to [0,1]^t,$$

assigns to each pair $(c,d) \in Nm \times Nm$ a t-tuple $\mathrm{bCf}(c,d) = (\delta_1, \delta_2, \ldots, \delta_t)$, satisfying:

$$\mathrm{bCf}(c,c) = 0, \quad \mathrm{bCf}(c,d) = \mathrm{bCf}(d,c), \quad \forall c, d \in Nm.$$

3. **Constraints**:

a) **Edge Appurtenance Constraint**:
For all edges $e = (u,v) \in N$, attribute values $a, b \in Ml$ of vertices u, v, and edge attribute value $c \in Nm$, the Degree of Appurtenance Functions satisfy:

$$\mathrm{bdf}(e,c) \leq \min\{\mathrm{adf}(u,a),\ \mathrm{adf}(v,b)\},$$

where the inequality is interpreted component-wise for the s-tuples.

b) **Contradiction Function Constraint**:
For all attribute values $c, d \in Nm$ and $a, b \in Ml$, the Degree of Contradiction Functions satisfy:

$$\mathrm{bCf}(c,d) \leq \min\{\mathrm{aCf}(a,c),\ \mathrm{aCf}(b,d)\},$$

where the inequality is interpreted component-wise for the t-tuples.

c) **Reflexivity and Symmetry of Contradiction Functions**:
For all $a, b \in Ml$ and $c, d \in Nm$:

$$\mathrm{aCf}(a,a) = 0, \quad \mathrm{aCf}(a,b) = \mathrm{aCf}(b,a),$$
$$\mathrm{bCf}(c,c) = 0, \quad \mathrm{bCf}(c,d) = \mathrm{bCf}(d,c).$$

Example 123. By choosing specific values for s and t, the Plithogenic Unit Square Graph specializes to various types:

- When $s = t = 1$, PG_{USG} is a *Plithogenic Fuzzy Unit Square Graph*.
- When $s = 2, t = 1$, PG_{USG} is a *Plithogenic Intuitionistic Fuzzy Unit Square Graph*.
- When $s = 3, t = 1$, PG_{USG} is a *Plithogenic Neutrosophic Unit Square Graph*.
- When $s = 4, t = 1$, PG_{USG} is a *Plithogenic Turiyam Neutrosophic Unit Square Graph*.

Theorem 124. *A Plithogenic Graph can be represented as a Plithogenic Unit Square Graph.*

Proof. Let $PG = (V, E, adf, aCf, bdf, bCf)$ be a Plithogenic Graph, where:

- $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices.
- $E \subseteq V \times V$ is the set of edges between the vertices.
- $adf(v_i, l)$ is the Degree of Appurtenance Function for vertex attributes, assigning a membership degree based on the attribute l.
- $aCf(l_1, l_2)$ is the Degree of Contradiction Function for vertex attributes.
- $bdf(e_{ij}, m)$ is the Degree of Appurtenance Function for edge attributes, assigning a membership degree based on the attribute m.
- $bCf(m_1, m_2)$ is the Degree of Contradiction Function for edge attributes.

We will represent PG as a Plithogenic Unit Square Graph PG_{USG} by mapping the vertices and edges of PG to unit squares in the Euclidean plane and preserving the plithogenic structure.

We consider about Mapping Vertices to Unit Squares. In a Plithogenic Unit Square Graph, each vertex corresponds to a unit square in the plane. Let $S = \{s_1, s_2, \dots, s_n\}$ be a set of axis-aligned unit squares, where each unit square s_i corresponds to a vertex $v_i \in V$ in the plithogenic graph PG. The vertex set V is thus mapped to the set of unit squares S.

We assign the same Degree of Appurtenance and Contradiction Functions for the vertices in PG_{USG} as in PG. That is, for each unit square $s_i \in S$ corresponding to $v_i \in V$:

$$\mathrm{adf}(s_i, l) = \mathrm{adf}(v_i, l), \quad \mathrm{aCf}(l_1, l_2) = \mathrm{aCf}(l_1, l_2).$$

Next, we consider about Mapping Edges to Intersections of Unit Squares. In a Plithogenic Unit Square Graph, an edge between two vertices v_i and v_j exists if and only if their corresponding unit squares s_i and s_j intersect. The set of edges $E \subseteq V \times V$ in PG is mapped to the set of intersections between unit squares in PG_{USG}. That is, for each edge $(v_i, v_j) \in E$, there is an edge $e_{ij} \in E_{USG}$ if and only if the corresponding unit squares s_i and s_j intersect:

$$e_{ij} \in E_{USG} \iff s_i \cap s_j \neq \emptyset.$$

We assign the same Degree of Appurtenance and Contradiction Functions for edges in PG_{USG} as in PG. That is, for each edge $e_{ij} \in E_{USG}$ corresponding to $(v_i, v_j) \in E$:

$$\mathrm{bdf}(e_{ij}, m) = \mathrm{bdf}(v_i, v_j, m), \quad \mathrm{bCf}(m_1, m_2) = \mathrm{bCf}(m_1, m_2).$$

We have now represented the vertices of the plithogenic graph PG as unit squares in PG_{USG}, and the edges of PG correspond to intersections between unit squares in PG_{USG}. Furthermore, the plithogenic structure, including the Degree of Appurtenance and Contradiction Functions for both vertices and edges, is preserved in the mapping.

Thus, we have successfully represented the Plithogenic Graph PG as a Plithogenic Unit Square Graph PG_{USG}. □

Theorem 125. *A Plithogenic Unit Square Graph can be transformed into a classical Unit Square Graph.*

Proof. Let $PG_{USG} = (PM, PN)$ be a Plithogenic Unit Square Graph, where PM represents the plithogenic vertex set and PN represents the plithogenic edge set, each with associated plithogenic attributes such as the degree of appurtenance and the degree of contradiction.

To transform PG_{USG} into a classical Unit Square Graph $G = (V, E)$, we proceed with the following steps:

1. **Remove Plithogenic Attributes**:
   - In PG_{USG}, each vertex and edge has attributes that define degrees of appurtenance and contradiction. These attributes provide additional information but are not required in a classical Unit Square Graph. Therefore, we remove all such plithogenic attributes from both vertices and edges.
   - Specifically, we discard the functions adf (Degree of Appurtenance Function) and aCf (Degree of Contradiction Function) for vertices, as well as the corresponding functions bdf and bCf for edges.
2. **Retain the Underlying Vertex and Edge Sets**:
   - Let $V = M$ be the set of vertices from PM, and let $E = N$ be the set of edges from PN. Since PM and PN are extensions of the classical vertex set and edge set with additional plithogenic properties, we retain the sets V and E without the plithogenic attributes.
3. **Verify the Unit Square Condition**:
   - In a classical Unit Square Graph, two vertices $u, v \in V$ are connected by an edge $e = (u, v) \in E$ if and only if their corresponding unit squares s_u and s_v in the Euclidean plane intersect:

     $$(u, v) \in E \iff s_u \cap s_v \neq \emptyset.$$

   - Since the original Plithogenic Unit Square Graph PG_{USG} was based on this geometric property (intersections of unit squares), the vertex and edge sets V and E still satisfy this condition after removing the plithogenic attributes.

Thus, after removing the plithogenic attributes and retaining the vertex and edge sets, the resulting graph $G = (V, E)$ is a valid classical Unit Square Graph. The transformation process preserves the essential geometric intersection property of the original graph while discarding the additional plithogenic features. □

Corollary 126. *A Fuzzy Unit Square Graph, Intuitionistic Fuzzy Unit Square Graph, Neutrosophic Unit Square Graph, and Turiyam Neutrosophic Unit Square Graph can all be transformed into a classical Unit Square Graph.*

Proof. This follows immediately from the transformation process described in the theorem for the Plithogenic Unit Square Graph, as each of these graph types can be viewed as special cases of a Plithogenic Unit Square Graph. □

Theorem 127. *A Plithogenic Unit Square Graph can be transformed into a Fuzzy Unit Square Graph, Intuitionistic Fuzzy Unit Square Graph, Neutrosophic Unit Square Graph, or Turiyam Neutrosophic Unit Square Graph by appropriately adjusting the Degree of Appurtenance and Contradiction Functions.*

Proof. A Plithogenic Unit Square Graph $PG_{USG} = (V, E, adf, aCf, bdf, bCf)$ is defined with multi-dimensional membership and contradiction functions. By restricting the number of dimensions and simplifying the Degree of Appurtenance and Contradiction Functions, the following transformations can be achieved:

- By setting the Degree of Appurtenance and Contradiction Functions to one dimension ($s = t = 1$), PG_{USG} becomes a Fuzzy Unit Square Graph.
- By setting $s = 2$ and $t = 1$, PG_{USG} becomes an Intuitionistic Fuzzy Unit Square Graph.
- By setting $s = 3$ and $t = 1$, PG_{USG} becomes a Neutrosophic Unit Square Graph.
- By setting $s = 4$ and $t = 1$, PG_{USG} becomes a Turiyam Neutrosophic Unit Square Graph.

Since these transformations only involve modifying the dimensionality of the membership and contradiction functions, the transformation process is straightforward and follows directly from the definitions.

Thus, a Plithogenic Unit Square Graph can be transformed into a Fuzzy, Intuitionistic Fuzzy, Neutrosophic, or Turiyam Neutrosophic Unit Square Graph. □

Theorem 128. *A Plithogenic Unit Square Graph PG_{USG} is connected if and only if its underlying classical Unit Square Graph G_{USG} is connected.*

Proof. Assume that the Plithogenic Unit Square Graph PG_{USG} is connected. By definition, PG_{USG} shares the same vertex set V and edge set E as the underlying classical Unit Square Graph G_{USG}, with the only difference being the additional plithogenic attributes, such as degrees of appurtenance and contradiction. Since PG_{USG} is connected, there exists a path between any pair of vertices in PG_{USG}. Because the vertex set and edge set of G_{USG} are identical to those of PG_{USG}, the same path must exist in G_{USG}. Therefore, if PG_{USG} is connected, G_{USG} must also be connected.

Conversely, assume that the classical Unit Square Graph G_{USG} is connected. In this case, there exists a path between any pair of vertices in G_{USG}. The Plithogenic Unit Square Graph PG_{USG} inherits the same vertex and edge sets from G_{USG}, and the plithogenic attributes do not affect the existence of edges, but only add additional information. Thus, the connectivity of G_{USG} implies that there is a path between any pair of vertices in PG_{USG}, making PG_{USG} connected as well.

Hence, the Plithogenic Unit Square Graph PG_{USG} is connected if and only if its underlying classical Unit Square Graph G_{USG} is connected. □

Theorem 129. *Every subgraph of a Plithogenic Unit Square Graph is also a Plithogenic Unit Square Graph.*

Proof. Let $PG_{USG} = (V, E, adf, aCf, bdf, bCf)$ be a Plithogenic Unit Square Graph.

Let $PG'_{USG} = (V', E', adf', aCf', bdf', bCf')$ be a subgraph of PG_{USG}, where $V' \subseteq V$ and $E' \subseteq E$. The vertices in V' correspond to a subset of the unit squares in PG_{USG}, and the edges in E' represent the intersections between these unit squares, following the same geometric intersection rules as in PG_{USG}.

Since the vertices and edges in PG'_{USG} are inherited from PG_{USG}, we have the following properties:

- The vertices V' still represent unit squares in the Euclidean plane.
- The edges E' are defined by the intersections between the unit squares corresponding to V', maintaining the same geometric structure as in PG_{USG}.

- The Degree of Appurtenance Functions adf' and bdf' for vertices and edges in PG'_{USG} are restrictions of the original functions adf and bdf from PG_{USG}, i.e., $adf'(v_i, l) = adf(v_i, l)$ for $v_i \in V'$ and $bdf'(e_{ij}, m) = bdf(e_{ij}, m)$ for $e_{ij} \in E'$.
- Similarly, the Degree of Contradiction Functions aCf' and bCf' are also restrictions of the original functions aCf and bCf.

Since PG'_{USG} inherits the vertex set, edge set, and plithogenic structure from PG_{USG}, it satisfies all the properties of a Plithogenic Unit Square Graph:

- The vertices represent unit squares in the Euclidean plane.
- The edges represent intersections between these unit squares.
- The Degree of Appurtenance and Contradiction Functions are defined appropriately for both vertices and edges.

Thus, PG'_{USG} is a valid Plithogenic Unit Square Graph. Therefore, every subgraph of a Plithogenic Unit Square Graph is also a Plithogenic Unit Square Graph. □

Theorem 130. *The complement of a Plithogenic Unit Square Graph is also a Plithogenic Unit Square Graph.*

Proof. Let $PG_{USG} = (V, E, adf, aCf, bdf, bCf)$ be a Plithogenic Unit Square Graph.

The complement of PG_{USG}, denoted $\overline{PG}_{USG} = (V, \overline{E}, adf, aCf, \overline{bdf}, \overline{bCf})$, has the same vertex set V but the edge set $\overline{E}$ consists of all pairs $(u, v) \in V \times V$ such that $(u, v) \notin E$. In other words, $\overline{E}$ represents the set of non-intersecting unit squares.

Since the vertex set V is unchanged, the Degree of Appurtenance and Degree of Contradiction Functions for the vertices, adf and aCf, remain the same. That is, for all $v_i \in V$ and for all attributes $l \in Ml$:

$$\overline{adf}(v_i, l) = adf(v_i, l),$$

$$\overline{aCf}(l_1, l_2) = aCf(l_1, l_2).$$

Next, we define the Degree of Appurtenance Function $\overline{bdf}$ and Degree of Contradiction Function $\overline{bCf}$ for the edges in the complement graph:

$$\overline{bdf}(e_{ij}, m) = \begin{cases} bdf(e_{ij}, m) & \text{if } (v_i, v_j) \notin E \text{ (corresponding unit squares do not intersect),} \\ 0 & \text{if } (v_i, v_j) \in E. \end{cases}$$

This ensures that $\overline{bdf}$ assigns a non-zero value only to edges that exist in the complement graph $\overline{PG}_{USG}$ (i.e., non-intersecting unit squares).

Similarly, the Degree of Contradiction Function $\overline{bCf}$ is defined as:

$$\overline{bCf}(m_1, m_2) = \begin{cases} bCf(m_1, m_2) & \text{if the corresponding edge exists in } \overline{PG}_{USG}, \\ 0 & \text{if the edge exists in the original graph } PG_{USG}. \end{cases}$$

Since the vertices in $\overline{PG}_{USG}$ still correspond to unit squares, and the edge set $\overline{E}$ represents the non-intersecting unit squares, $\overline{PG}_{USG}$ retains the geometric structure of a Plithogenic Unit Square Graph. Additionally, the Degree of Appurtenance and Contradiction Functions for both vertices and edges are appropriately defined to handle the complement.

Thus, the complement of a Plithogenic Unit Square Graph is also a Plithogenic Unit Square Graph. □

Theorem 131. *Deleting a vertex (and its incident edges) from a Plithogenic Unit Square Graph results in another Plithogenic Unit Square Graph.*

Proof. Let $PG_{USG} = (V, E, adf, aCf, bdf, bCf)$ be a Plithogenic Unit Square Graph.

Now, let $v_0 \in V$ be the vertex to be deleted from PG_{USG}. Define the new graph $PG'_{USG} = (V', E', adf', aCf', bdf', bdf')$ as follows:

- $V' = V \setminus \{v_0\}$, the vertex set of the new graph obtained by removing v_0 from V.

- $E' = \{(u,v) \in E \mid u,v \in V'\}$, the edge set of the new graph, which consists of all edges from the original graph that do not involve v_0.
- The Degree of Appurtenance Function adf' for vertices in V' is the restriction of adf to V', i.e.,

$$adf'(v_i, l) = adf(v_i, l), \quad \forall v_i \in V', \ \forall l.$$

- The Degree of Contradiction Function aCf' for vertices in V' is the restriction of aCf to V', i.e.,

$$aCf'(l_1, l_2) = aCf(l_1, l_2), \quad \forall l_1, l_2.$$

- The Degree of Appurtenance Function bdf' for edges in E' is the restriction of bdf to E', i.e.,

$$bdf'(e_{ij}, m) = bdf(e_{ij}, m), \quad \forall e_{ij} \in E', \ \forall m.$$

- The Degree of Contradiction Function bCf' for edges in E' is the restriction of bCf to E', i.e.,

$$bCf'(m_1, m_2) = bCf(m_1, m_2), \quad \forall m_1, m_2.$$

The new vertex set V' still represents a set of unit squares in the Euclidean plane, and the edges in E' represent the intersections between the remaining unit squares. The Degree of Appurtenance and Contradiction Functions for both vertices and edges are well-defined and remain consistent with the structure of a Plithogenic Unit Square Graph.

Thus, the graph PG'_{USG} satisfies all the properties of a Plithogenic Unit Square Graph. Therefore, deleting a vertex (and its incident edges) from a Plithogenic Unit Square Graph results in another Plithogenic Unit Square Graph. □

Theorem 132. *Let G be a Plithogenic Unit Square Graph. Then G has no induced subgraph homomorphic to $K_{1,5}$, the star graph with one center vertex connected to five leaves.*

Proof. We will show that no Plithogenic Unit Square Graph G can have an induced subgraph that is homomorphic to $K_{1,5}$, the star graph with a center vertex v connected to five leaf vertices w_1, w_2, w_3, w_4, w_5.

First, recall that in a Plithogenic Unit Square Graph, each vertex corresponds to a unit square in the Euclidean plane, and there is an edge between two vertices if and only if their corresponding unit squares intersect.

Assume, for the sake of contradiction, that G contains an induced subgraph G' that is homomorphic to $K_{1,5}$. Let v be the center vertex of this subgraph, and let w_1, w_2, w_3, w_4, w_5 be the leaf vertices. This implies that there are five distinct unit squares corresponding to w_1, w_2, w_3, w_4, w_5, all of which intersect the unit square corresponding to v, but none of which intersect each other.

In this configuration, the unit square for v must be positioned in such a way that it overlaps with each of the unit squares for w_1, w_2, w_3, w_4, w_5, but these leaf squares do not overlap with each other. However, due to the geometric constraints of unit squares in the plane, it is impossible to place five distinct unit squares around a central unit square such that they all intersect the central square but none of them intersect each other.

To see this more clearly, place the center vertex v at the origin $(0,0)$. Each of the leaf vertices must correspond to unit squares located within a distance of 1 unit from the origin (so that they intersect the central unit square). However, due to the symmetry and spacing of unit squares, at least two of the unit squares corresponding to the leaf vertices must overlap, contradicting the assumption that the leaf vertices are not adjacent.

Thus, no such configuration of unit squares exists, and G cannot contain an induced subgraph homomorphic to $K_{1,5}$.

Therefore, we conclude that a Plithogenic Unit Square Graph has no induced subgraph homomorphic to $K_{1,5}$. □

One of the well-known graph classes of fuzzy graphs is the Fuzzy Intersection Graph. The following definition is commonly recognized.

Definition 133 (Fuzzy Intersection Graph)**.** A *Fuzzy Intersection Graph* is a graph $G = (V, E, \sigma, \mu)$ where:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- $\sigma : V \to [0,1]$ is a membership function that assigns a degree of membership to each vertex $v \in V$.

- $\mu : V \times V \to [0, 1]$ is a fuzzy relation representing the strength of the connection (degree of membership) between each pair of vertices $(u, v) \in V \times V$.

The edge set E of the fuzzy intersection graph is defined based on the membership functions of the vertices and the fuzzy relation. Specifically, for each pair $(u, v) \in V \times V$, the edge (u, v) exists in the fuzzy intersection graph with the membership degree:

$$\mu(u, v) = \min(\sigma(u), \sigma(v))$$

if the Euclidean distance between the corresponding points of u and v satisfies the condition for intersection, and $\mu(u, v) = 0$ otherwise.

In this way, the fuzzy intersection graph generalizes the concept of an intersection graph by incorporating fuzzy set theory, allowing for partial membership and gradual relationships between vertices and edges.

Theorem 134. *[460] Any undirected fuzzy graph $G = (V, \sigma, \mu)$ can be represented as a fuzzy intersection graph.*

Proof. See reference [460]. □

Theorem 135. *A Plithogenic Fuzzy Unit Square Graph is a Fuzzy Intersection Graph.*

Proof. Let $PG_{USG} = (PM, PN)$ be a *Plithogenic Fuzzy Unit Square Graph*.

In a *Fuzzy Intersection Graph*, each vertex v has a membership function $\sigma(v)$, representing the membership degree of the vertex. We can directly associate this membership function with the Degree of Appurtenance Function in PG_{USG}, i.e.,

$$\sigma(v) = adf(v, l_v),$$

where l_v is the attribute class associated with vertex v. This mapping shows that the vertices in PG_{USG} behave in the same way as vertices in a *Fuzzy Intersection Graph* in terms of their membership degrees.

For the edges in a *Fuzzy Intersection Graph*, the membership degree of an edge (u, v) is defined as:

$$\mu(u, v) = \min(\sigma(u), \sigma(v)).$$

In the Plithogenic Fuzzy Unit Square Graph PG_{USG}, the Degree of Appurtenance Function for edges $bdf(e, m_e)$ similarly represents the membership degree of the edge. Since the edges exist if and only if the corresponding unit squares intersect, the membership degree of an edge (u, v) in PG_{USG} is governed by the minimum of the membership degrees of the vertices u and v. Therefore, we have:

$$bdf((u, v), m_e) = \min(adf(u, l_u), adf(v, l_v)) = \mu(u, v).$$

Thus, both the vertices and edges in the Plithogenic Fuzzy Unit Square Graph PG_{USG} follow the same membership rules as those in a *Fuzzy Intersection Graph*. Consequently, a Plithogenic Fuzzy Unit Square Graph is indeed a *Fuzzy Intersection Graph*. □

Corollary 136. *A Fuzzy Unit Square Graph is a Fuzzy Intersection Graph.*

Proof. Obviously holds. □

Next, we consider the Plithogenic Intersection Graph. The definition of the Plithogenic Intersection Graph is as follows [460].

Definition 137 (Plithogenic Intersection Graph)**.** [460] A *Plithogenic Intersection Graph* $PG = (V, E, PM, PN)$ is defined as follows:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- $PM = (M, l, Ml, adf, aCf)$ is the Plithogenic vertex set, where:
  - $M \subseteq V$ is the set of vertices.
  - l is an attribute associated with the vertices.
  - Ml is the range of possible values for attribute l.
  - $adf : M \times Ml \to [0, 1]^s$ is the *Degree of Appurtenance Function* for vertices.

– $aCf : Ml \times Ml \to [0,1]^t$ is the *Degree of Contradiction Function* for vertices.

- $PN = (N, m, Nm, bdf, bCf)$ is the Plithogenic edge set, where:
  – $N \subseteq E$ is the set of edges.
  – m is an attribute associated with the edges.
  – Nm is the range of possible values for attribute m.
  – $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function* for edges.
  – $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function* for edges.

The edge set E of the Plithogenic Intersection Graph is defined based on the degree of appurtenance of the vertices and the plithogenic relation. Specifically, for each pair $(u,v) \in V \times V$, the edge (u,v) exists in the Plithogenic Intersection Graph with the degree of appurtenance:

$$bdf(u,v) = \min(adf(u,l_u), adf(v,l_v))$$

if the Euclidean distance between the corresponding points of u and v satisfies the condition for intersection, and $bdf(u,v) = 0$ otherwise.

Theorem 138. *A Plithogenic Unit Square Graph is a Plithogenic Intersection Graph.*

Proof. Let $PG_{USG} = (V, E, PM, PN)$ be a Plithogenic Unit Square Graph.

A *Plithogenic Intersection Graph* is defined similarly, where the edges between vertices are determined by the degree of appurtenance of the vertices and their intersection properties.

Since the edges in a Plithogenic Unit Square Graph are based on the intersection of unit squares and the Degree of Appurtenance Function adf and Degree of Contradiction Function aCf for vertices and edges follow the same rules as in a Plithogenic Intersection Graph, it is clear that a Plithogenic Unit Square Graph satisfies all the conditions to be a Plithogenic Intersection Graph.

Thus, a Plithogenic Unit Square Graph is a Plithogenic Intersection Graph. □

Theorem 139. *[460] Any undirected Plithogenic graph can be represented as a Plithogenic intersection graph.*

Proof. See reference [460]. □

3.2 Result: Circle Graph

A circle graph is the intersection graph of a set of chords on a circle, where vertices represent chords and edges exist if the chords intersect inside the circle [133, 249, 395, 822, 1077]. We define the Circle Graph in the context of general graphs as follows.

Definition 140 (Circle Graph)**.** A *circle graph* is the intersection graph of a set of chords on a circle. Formally, let $L = \{C_1, C_2, \ldots, C_n\}$ be a set of chords on a circle, where each C_i represents a chord. The corresponding circle graph $G = (V, E)$ is defined as follows:

- The vertex set $V = \{C_1, C_2, \ldots, C_n\}$ consists of one vertex for each chord.
- The edge set E is defined such that two vertices C_α and C_β are adjacent if and only if their corresponding chords C_α and C_β intersect inside the circle, i.e.,

$$\{C_\alpha, C_\beta\} \in E \iff C_\alpha \cap C_\beta \neq \emptyset.$$

Thus, two vertices in the circle graph are connected if and only if their corresponding chords share any common interior points.

Next, we explore Circle Graphs within the frameworks of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. The definitions are provided below.

Definition 141 (Fuzzy Circle Graph)**.** A *Fuzzy Circle Graph* is defined on a set of chords $L = \{C_1, C_2, \ldots, C_n\}$ on a circle, with corresponding vertex set $V = \{C_1, C_2, \ldots, C_n\}$ and edge set E.

- **Vertex Membership Function**: Each vertex $v \in V$ is assigned a membership degree $\sigma_V(v) \in [0,1]$.
- **Edge Membership Function**: Each edge $e = \{u,v\} \in E$ is assigned a membership degree $\mu_E(u,v) \in [0,1]$.

- **Membership Condition**: For all edges $e = \{u, v\} \in E$,

$$\mu_E(u, v) \leq \min\{\sigma_V(u), \sigma_V(v)\}.$$

Definition 142 (Intuitionistic Fuzzy Circle Graph)**.** An *Intuitionistic Fuzzy Circle Graph* is defined on a set of chords $L = \{C_1, C_2, \ldots, C_n\}$ with vertex set V and edge set E.

- **Vertex Functions**:
  - Membership degree $\mu_V : V \to [0, 1]$.
  - Non-membership degree $\nu_V : V \to [0, 1]$.
  - For each $v \in V$,
  $$0 \leq \mu_V(v) + \nu_V(v) \leq 1.$$
- **Edge Functions**:
  - Membership degree $\mu_E : E \to [0, 1]$.
  - Non-membership degree $\nu_E : E \to [0, 1]$.
  - For each $e = \{u, v\} \in E$,
  $$0 \leq \mu_E(u, v) + \nu_E(u, v) \leq 1.$$
- **Membership Conditions**:
  - For all $e = \{u, v\} \in E$,
  $$\mu_E(u, v) \leq \min\{\mu_V(u), \mu_V(v)\},$$
  $$\nu_E(u, v) \geq \max\{\nu_V(u), \nu_V(v)\}.$$

Definition 143 (Neutrosophic Circle Graph)**.** A *Neutrosophic Circle Graph* is defined on a set of chords $L = \{C_1, C_2, \ldots, C_n\}$ with vertex set V and edge set E.

- **Vertex Functions**:
  - Truth-membership degree $T_V : V \to [0, 1]$.
  - Indeterminacy-membership degree $I_V : V \to [0, 1]$.
  - Falsity-membership degree $F_V : V \to [0, 1]$.
  - For each $v \in V$,
  $$0 \leq T_V(v) + I_V(v) + F_V(v) \leq 3.$$
- **Edge Functions**:
  - Truth-membership degree $T_E : E \to [0, 1]$.
  - Indeterminacy-membership degree $I_E : E \to [0, 1]$.
  - Falsity-membership degree $F_E : E \to [0, 1]$.
  - For each $e = \{u, v\} \in E$,
  $$0 \leq T_E(u, v) + I_E(u, v) + F_E(u, v) \leq 3.$$
- **Membership Conditions**:
  - For all $e = \{u, v\} \in E$,
  $$T_E(u, v) \leq \min\{T_V(u), T_V(v)\},$$
  $$I_E(u, v) \geq \max\{I_V(u), I_V(v)\},$$
  $$F_E(u, v) \geq \max\{F_V(u), F_V(v)\}.$$

Definition 144 (Turiyam Neutrosophic Circle Graph)**.** A *Turiyam Neutrosophic Circle Graph* is defined on a set of chords $L = \{C_1, C_2, \ldots, C_n\}$ with vertex set V and edge set E.

- **Vertex Functions**:
  - Truth-membership degree $t_V : V \to [0, 1]$.
  - Indeterminacy-membership degree $i_V : V \to [0, 1]$.

- Falsity-membership degree $f_V : V \to [0, 1]$.
- Liberal state membership degree $l_V : V \to [0, 1]$.
- For each $v \in V$,

$$0 \le t_V(v) + i_V(v) + f_V(v) + l_V(v) \le 4.$$

- **Edge Functions**:
  - Truth-membership degree $t_E : E \to [0, 1]$.
  - Indeterminacy-membership degree $i_E : E \to [0, 1]$.
  - Falsity-membership degree $f_E : E \to [0, 1]$.
  - Liberal state membership degree $l_E : E \to [0, 1]$.
  - For each $e = \{u, v\} \in E$,

$$0 \le t_E(u, v) + i_E(u, v) + f_E(u, v) + l_E(u, v) \le 4.$$

- **Membership Conditions**:
  - For all $e = \{u, v\} \in E$,

$$t_E(u, v) \le \min\{t_V(u), t_V(v)\},$$
$$i_E(u, v) \ge \max\{i_V(u), i_V(v)\},$$
$$f_E(u, v) \ge \max\{f_V(u), f_V(v)\},$$
$$l_E(u, v) \ge \max\{l_V(u), l_V(v)\}.$$

Theorem 145. *A Neutrosophic Circle graph can be transformed into a classic Circle graph.*

Proof. Obviously holds. □

Corollary 146. *A Fuzzy Circle Graph, Intuitionistic Circle Graph, or Turiyam Neutrosophic Circle Graph can be transformed into a classic Circle Graph.*

Proof. Obviously holds. □

Theorem 147. *A Neutrosophic Graph can be represented as a Neutrosophic Circle Graph.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Graph.

To represent G as a Neutrosophic Circle Graph, we map each vertex $v_i \in V$ to an arc I_i on a circle C_1. Thus, the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ corresponds to the set of arcs $\{I_1, I_2, \ldots, I_n\}$ on the circle C_1. The neutrosophic membership values for each arc I_i are the same as those of the corresponding vertex v_i:

$$\sigma(I_i) = \sigma(v_i).$$

In the Neutrosophic Circle Graph, there is an edge between two arcs I_i and I_j if and only if the arcs intersect, i.e., $I_i \cap I_j \neq \emptyset$. This corresponds to the edges between vertices v_i and v_j in the original Neutrosophic Graph G. The neutrosophic membership degrees for edges (I_i, I_j) are defined as:

$$\mu_T(I_i, I_j) = \min\{\sigma_T(I_i), \sigma_T(I_j)\},$$

$$\mu_I(I_i, I_j) = \max\{\sigma_I(I_i), \sigma_I(I_j)\},$$

$$\mu_F(I_i, I_j) = \max\{\sigma_F(I_i), \sigma_F(I_j)\}.$$

Thus, we have successfully represented the Neutrosophic Graph as a Neutrosophic Circle Graph, with both vertex and edge membership values preserved. □

Theorem 148. *Every subgraph of a Neutrosophic Circle Graph can be also a Neutrosophic Circle Graph (but not necessary).*

Proof. The proof for a Circle Graph follows the same method used in the case of a Unit Square Graph. □

Theorem 149. *The disjoint union of two Neutrosophic Circle Graphs can be also a Neutrosophic Circle Graph (but not necessary).*

Proof. Let $G_1 = (V_1, E_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, E_2, \sigma_2, \mu_2)$ be two Neutrosophic Circle Graphs. The disjoint union $G = G_1 \sqcup G_2$ is defined by taking the union of their vertex sets and edge sets, i.e., $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, with no edges between V_1 and V_2.

Since the arcs corresponding to vertices in V_1 and V_2 form valid chord models independently, their disjoint union also forms a valid chord model on the circle. The neutrosophic membership functions σ and μ, representing truth, indeterminacy, and falsity, are preserved for the vertices and edges within each respective graph G_1 and G_2.

Thus, the disjoint union $G = G_1 \sqcup G_2$ also satisfies the conditions for being a Neutrosophic Circle Graph. Therefore, the disjoint union of two Neutrosophic Circle Graphs can be also a Neutrosophic Circle Graph. □

Theorem 150. *The complement of a Neutrosophic Circle Graph is also a Neutrosophic Circle Graph.*

Proof. The proof for a Circle Graph follows the same method used in the case of a Unit Square Graph. □

Theorem 151. *Deleting a vertex (and its incident edges) from a Neutrosophic Circle Graph results in another Neutrosophic Circle Graph.*

Proof. The proof for a Circle Graph follows the same method used in the case of a Unit Square Graph. □

Theorem 152. *Every induced subgraph of a Neutrosophic Circle Graph is also a Neutrosophic Circle Graph.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Circle Graph, and let $H = (V', E', \sigma', \mu')$ be an induced subgraph of G, where $V' \subseteq V$ and $E' \subseteq E$ is the set of edges between the vertices in V'.

In the original graph G, each vertex in V corresponds to a chord on a circle, and two vertices are adjacent if and only if their corresponding chords intersect. Since H is an induced subgraph, the vertices in V' still correspond to the same set of chords, and the edges between them in H reflect their intersections in G.

Therefore, the induced subgraph H can also be represented as a Neutrosophic Circle Graph, as the chord structure and neutrosophic membership degrees are preserved. □

Theorem 153. *The Cartesian product of two Neutrosophic Circle Graphs is not necessarily a Neutrosophic Circle Graph.*

Proof. Let $G_1 = (V_1, E_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, E_2, \sigma_2, \mu_2)$ be two Neutrosophic Circle Graphs. The Cartesian product $G = G_1 \Box G_2$ is defined as the graph where the vertex set is $V_1 \times V_2$, and there is an edge between (v_1, v_2) and (u_1, u_2) if and only if either:

1. $v_1 = u_1$ and $(v_2, u_2) \in E_2$, or
2. $v_2 = u_2$ and $(v_1, u_1) \in E_1$.

In a Neutrosophic Circle Graph, edges represent intersections between chords on a circle. The Cartesian product adds edges based on the edges of both G_1 and G_2, but these new edges do not necessarily correspond to intersections between chords on a circle. Since the Cartesian product introduces connections that may not satisfy the geometric constraints required for a circle graph, the resulting graph may not be representable as a Neutrosophic Circle Graph.

Therefore, the Cartesian product of two Neutrosophic Circle Graphs is not necessarily a Neutrosophic Circle Graph. □

Theorem 154. *A Fuzzy Circle Graph is a Fuzzy Intersection Graph.*

Proof. Obviously holds. □

3.3 Result: Ray Graph and String Graph

Ray Graph is a graph where vertices represent rays (half-lines) in the plane, and edges exist if the rays intersect. String Graph is a graph where vertices correspond to curves, and edges exist if the curves intersect at some point.

Definition 155 (Ray Graph)**.** A graph $G = (V, E)$ is called a *Ray Graph* if it can be represented as the intersection graph of a collection of rays (half-lines) in the plane. Each vertex of the graph corresponds to a ray, and there is an edge between two vertices if and only if their corresponding rays intersect in the plane.

Definition 156 (String Graph)**.** A graph $G = (V, E)$ is called a *String Graph* if it is the intersection graph of a collection of simple curves, called strings, in the plane. Each vertex of the graph corresponds to a string, and two vertices are adjacent if and only if their corresponding strings intersect at some point, with the restriction that no three strings intersect at the same point. Specifically, a *1-string graph* is a string graph where each pair of strings intersects at most once.

Next, we explore Ray Graphs and String Graph within the frameworks of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. The definitions are provided below.

Definition 157 (Fuzzy Ray Graph / Fuzzy String Graph)**.** A *Fuzzy Ray Graph* or *Fuzzy String Graph* is defined as follows:

Let $G = (V, E)$ be a Ray Graph or String Graph.

- **Vertex Membership Function**: $\sigma_V : V \to [0, 1]$, assigning to each vertex $v \in V$ a membership degree $\sigma_V(v)$.
- **Edge Membership Function**: $\mu_E : E \to [0, 1]$, assigning to each edge $e = \{u, v\} \in E$ a membership degree $\mu_E(u, v)$.
- **Membership Condition**:

$$\mu_E(u, v) \leq \min\{\sigma_V(u), \sigma_V(v)\}, \quad \forall e = \{u, v\} \in E.$$

Definition 158 (Intuitionistic Fuzzy Ray Graph / Intuitionistic Fuzzy String Graph)**.** An *Intuitionistic Fuzzy Ray Graph* or *Intuitionistic Fuzzy String Graph* is defined as follows:

Let $G = (V, E)$ be a Ray Graph or String Graph.

- **Vertex Membership Functions**:
  - Membership degree $\mu_V : V \to [0, 1]$.
  - Non-membership degree $\nu_V : V \to [0, 1]$.
  - Condition for each $v \in V$:
$$0 \leq \mu_V(v) + \nu_V(v) \leq 1.$$
- **Edge Membership Functions**:
  - Membership degree $\mu_E : E \to [0, 1]$.
  - Non-membership degree $\nu_E : E \to [0, 1]$.
  - Condition for each $e = \{u, v\} \in E$:
$$0 \leq \mu_E(u, v) + \nu_E(u, v) \leq 1.$$
- **Membership Conditions**:
  - For all $e = \{u, v\} \in E$:
$$\mu_E(u, v) \leq \min\{\mu_V(u), \mu_V(v)\},$$
$$\nu_E(u, v) \geq \max\{\nu_V(u), \nu_V(v)\}.$$

Definition 159 (Neutrosophic Ray Graph / Neutrosophic String Graph)**.** A *Neutrosophic Ray Graph* or *Neutrosophic String Graph* is defined as follows:

Let $G = (V, E)$ be a Ray Graph or String Graph.

- **Vertex Membership Functions**:

- Truth-membership degree $T_V : V \to [0,1]$.
- Indeterminacy-membership degree $I_V : V \to [0,1]$.
- Falsity-membership degree $F_V : V \to [0,1]$.
- Condition for each $v \in V$:

$$0 \leq T_V(v) + I_V(v) + F_V(v) \leq 3.$$

- **Edge Membership Functions**:
  - Truth-membership degree $T_E : E \to [0,1]$.
  - Indeterminacy-membership degree $I_E : E \to [0,1]$.
  - Falsity-membership degree $F_E : E \to [0,1]$.
  - Condition for each $e = \{u,v\} \in E$:

$$0 \leq T_E(u,v) + I_E(u,v) + F_E(u,v) \leq 3.$$

- **Membership Conditions**:
  - For all $e = \{u,v\} \in E$:

$$T_E(u,v) \leq \min\{T_V(u), T_V(v)\},$$
$$I_E(u,v) \geq \max\{I_V(u), I_V(v)\},$$
$$F_E(u,v) \geq \max\{F_V(u), F_V(v)\}.$$

Definition 160 (Turiyam Neutrosophic Ray Graph / Turiyam Neutrosophic String Graph)**.** A *Turiyam Neutrosophic Ray Graph* or *Turiyam Neutrosophic String Graph* is defined as follows:
Let $G = (V,E)$ be a Ray Graph or String Graph.

- **Vertex Membership Functions**:
  - Truth-membership degree $t_V : V \to [0,1]$.
  - Indeterminacy-membership degree $i_V : V \to [0,1]$.
  - Falsity-membership degree $f_V : V \to [0,1]$.
  - Liberal state membership degree $l_V : V \to [0,1]$.
  - Condition for each $v \in V$:

$$0 \leq t_V(v) + i_V(v) + f_V(v) + l_V(v) \leq 4.$$

- **Edge Membership Functions**:
  - Truth-membership degree $t_E : E \to [0,1]$.
  - Indeterminacy-membership degree $i_E : E \to [0,1]$.
  - Falsity-membership degree $f_E : E \to [0,1]$.
  - Liberal state membership degree $l_E : E \to [0,1]$.
  - Condition for each $e = \{u,v\} \in E$:

$$0 \leq t_E(u,v) + i_E(u,v) + f_E(u,v) + l_E(u,v) \leq 4.$$

- **Membership Conditions**:
  - For all $e = \{u,v\} \in E$:

$$t_E(u,v) \leq \min\{t_V(u), t_V(v)\},$$
$$i_E(u,v) \geq \max\{i_V(u), i_V(v)\},$$
$$f_E(u,v) \geq \max\{f_V(u), f_V(v)\},$$
$$l_E(u,v) \geq \max\{l_V(u), l_V(v)\}.$$

Theorem 161. *A Neutrosophic Ray graph can be transformed into a classic Ray graph.*

Proof. Obviously holds. □

Corollary 162. *A Fuzzy Ray Graph, Intuitionistic Ray Graph, or Turiyam Neutrosophic Ray Graph can be transformed into a classic Ray Graph.*

Proof. Obviously holds. □

Theorem 163. *A Neutrosophic String graph can be transformed into a classic String graph.*

Proof. Obviously holds. □

Corollary 164. *A Fuzzy String Graph, Intuitionistic String Graph, or Turiyam Neutrosophic String Graph can be transformed into a classic String Graph.*

Proof. Obviously holds. □

Theorem 165. *Every Neutrosophic Ray Graph is also a Neutrosophic String Graph.*

Proof. A ray is a special case of a string—a half-line starting from a point and extending infinitely in one direction. Therefore, the set of rays in a Neutrosophic Ray Graph can be considered as a set of strings in the plane.

Since the intersection relationships between rays are preserved when they are viewed as strings, and the neutrosophic membership functions remain the same, it follows that every Neutrosophic Ray Graph is a Neutrosophic String Graph. □

Theorem 166. *Not every Neutrosophic String Graph is a Neutrosophic Ray Graph.*

Proof. Neutrosophic String Graphs can include strings that are closed curves, loops, or finite line segments, which cannot be represented as rays. Rays are unbounded in one direction, while strings can have various forms.

There exist Neutrosophic String Graphs where the intersection patterns among strings cannot be replicated using only rays. Therefore, not every Neutrosophic String Graph can be represented as a Neutrosophic Ray Graph. □

Theorem 167. *Any induced subgraph of a Neutrosophic Ray Graph is also a Neutrosophic Ray Graph.*

Proof. Let $G = (V, E, T_V, I_V, F_V, T_E, I_E, F_E)$ be a Neutrosophic Ray Graph. Let $V' \subseteq V$ and E' be the set of edges in G whose endpoints are both in V'. The induced subgraph is $G' = (V', E', T_{V'}, I_{V'}, F_{V'}, T_{E'}, I_{E'}, F_{E'})$, where the membership functions are restrictions of those in G.

The rays corresponding to vertices in V' are the same as in G, and their intersection relationships (edges) are preserved. The neutrosophic membership functions satisfy the same conditions in G' as they do in G.

Therefore, G' is a Neutrosophic Ray Graph. □

Corollary 168. *Any induced subgraph of a Neutrosophic String Graph is also a Neutrosophic String Graph.*

Proof. Obviously holds. □

Theorem 169. *If a Neutrosophic String Graph is connected, then its underlying classical String Graph is also connected.*

Proof. In a Neutrosophic String Graph, connectivity depends on the existence of paths between vertices, which correspond to sequences of intersecting strings. The neutrosophic membership functions provide additional information but do not affect the presence of edges.

Therefore, if the Neutrosophic String Graph is connected, the underlying classical graph (V, E) is also connected, as the same edges exist. □

Theorem 170. *Every Neutrosophic String Graph can be represented as a Neutrosophic Intersection Graph of convex sets in the plane.*

Proof. Strings (simple curves) in a Neutrosophic String Graph can be "thickened" to form thin convex shapes (e.g., narrow rectangles or tubes along the paths of the strings). The intersections between strings correspond to intersections between these convex shapes.

Thus, the Neutrosophic String Graph can be represented as a Neutrosophic Intersection Graph of convex sets, with the same neutrosophic membership functions assigned to vertices and edges. □

Theorem 171. *Every Neutrosophic Circle Graph can be represented as a Neutrosophic String Graph.*

Proof. A Neutrosophic Circle Graph is a graph where vertices correspond to chords on a circle, and edges represent intersections between these chords. Formally, let $G = (V, E, T_V, I_V, F_V, T_E, I_E, F_E)$ be a Neutrosophic Circle Graph.

We need to show that G can be represented as a Neutrosophic String Graph. In a Neutrosophic String Graph, each vertex corresponds to a string (a simple curve) in the plane, and two vertices are adjacent if and only if their corresponding strings intersect.

We consider Mapping Chords to Strings. Since the vertices in G correspond to chords on a circle, these chords can be naturally viewed as strings in the plane. Each chord is a line segment, which is a simple curve. Therefore, we can map each vertex $v_i \in V$ in the Neutrosophic Circle Graph to a corresponding string (in this case, the chord itself) in the Neutrosophic String Graph.

We consider Preserving Intersections (Adjacency). The edges in G represent intersections between chords. In the Neutrosophic String Graph, an edge between two vertices v_i and v_j exists if and only if their corresponding strings (chords) intersect. Since the intersection condition is preserved (i.e., two chords intersect in the circle if and only if the corresponding strings intersect in the plane), the edge set E remains unchanged.

We consider Preserving Neutrosophic Membership Functions. The neutrosophic membership functions $T_V, I_V, F_V, T_E, I_E, F_E$ are defined on both vertices and edges in the Neutrosophic Circle Graph. These functions can be directly applied to the corresponding vertices and edges in the Neutrosophic String Graph without modification. Thus, the membership degrees are preserved during the transformation from the Neutrosophic Circle Graph to the Neutrosophic String Graph.

By mapping the chords of the Neutrosophic Circle Graph to strings (which are line segments), preserving their intersection relationships, and maintaining the neutrosophic membership functions, we have successfully represented the Neutrosophic Circle Graph as a Neutrosophic String Graph.

Thus, every Neutrosophic Circle Graph can be represented as a Neutrosophic String Graph. □

Theorem 172. *Every Neutrosophic Unit Square Graph can be represented as a Neutrosophic String Graph.*

Proof. A Neutrosophic Unit Square Graph is a graph where the vertices correspond to axis-parallel unit squares in the plane, and there is an edge between two vertices if and only if their corresponding unit squares intersect. Formally, let $G = (V, E, T_V, I_V, F_V, T_E, I_E, F_E)$ be a Neutrosophic Unit Square Graph.

We aim to show that G can be represented as a Neutrosophic String Graph. In a Neutrosophic String Graph, each vertex corresponds to a simple curve (string) in the plane, and two vertices are adjacent if and only if their corresponding strings intersect.

We consider mapping Unit Squares to Strings. Each unit square in the plane can be represented as the intersection of four line segments (the sides of the square). These line segments can be naturally interpreted as strings in the plane. Therefore, for each vertex v_i corresponding to a unit square, we map the square to a set of strings, specifically the four boundary segments of the square. These segments act as the strings in the Neutrosophic String Graph representation.

We consider preserving Intersections (Adjacency). In the Neutrosophic Unit Square Graph, an edge $(v_i, v_j) \in E$ exists if and only if the corresponding unit squares intersect. Since the unit squares intersect if and only if at least one pair of their boundary segments (strings) intersects, the adjacency relationship in the Neutrosophic Unit Square Graph is preserved in the Neutrosophic String Graph. Thus, the edge set E remains unchanged.

We consider preserving Neutrosophic Membership Functions. The neutrosophic membership functions $T_V, I_V, F_V, T_E, I_E, F_E$ are defined on the vertices and edges of the Neutrosophic Unit Square Graph. These functions can be directly applied to the corresponding vertices and edges in the Neutrosophic String Graph, as the adjacency relationships between unit squares and their corresponding boundary strings are identical. Therefore, the truth, indeterminacy, and falsity degrees are preserved in the Neutrosophic String Graph representation.

By mapping the unit squares of the Neutrosophic Unit Square Graph to their boundary strings and preserving the intersection relationships, we have successfully represented the Neutrosophic Unit Square Graph as a Neutrosophic String Graph. Additionally, the neutrosophic membership degrees for both vertices and edges have been maintained in the process.

Thus, every Neutrosophic Unit Square Graph can be represented as a Neutrosophic String Graph. □

Theorem 173. *Every Neutrosophic Circle Graph can be represented as a Neutrosophic Ray Graph.*

Proof. We will construct a Neutrosophic Ray Graph $G' = (V', E', T'_V, I'_V, F'_V, T'_E, I'_E, F'_E)$ from any given Neutrosophic Circle Graph $G = (V, E, T_V, I_V, F_V, T_E, I_E, F_E)$ and show that these graphs are isomorphic while preserving the neutrosophic membership functions.

Step 1: Map chords to rays: Each chord C_i in the circle is mapped to a ray R_i in the plane. We choose a point O outside the circle and map each chord to a ray starting at one of its endpoints and passing through O. Rays R_i and R_j will intersect if and only if the corresponding chords C_i and C_j cross in the circle.

Step 2: Preserve intersection relations: The intersection between chords in the Neutrosophic Circle Graph corresponds to the intersection of the rays in the Neutrosophic Ray Graph. This ensures that the edge sets E and E' are isomorphic, meaning G and G' have identical adjacency structures.

Step 3: Assign Neutrosophic Membership Functions: For each vertex $v_i \in V$, we assign the same neutrosophic membership values to the corresponding vertex in G':

$$T'_V(v_i) = T_V(v_i), \quad I'_V(v_i) = I_V(v_i), \quad F'_V(v_i) = F_V(v_i).$$

Similarly, for each edge $e = \{v_i, v_j\} \in E$, we assign:

$$T'_E(e) = T_E(e), \quad I'_E(e) = I_E(e), \quad F'_E(e) = F_E(e).$$

Step 4: Verify the Neutrosophic Conditions: Since the vertex and edge membership values are preserved, the neutrosophic conditions hold for G' as they do for G. Specifically:

$$T'_E(u, v) \le \min\{T'_V(u), T'_V(v)\}, \quad I'_E(u, v) \ge \max\{I'_V(u), I'_V(v)\}, \quad F'_E(u, v) \ge \max\{F'_V(u), F'_V(v)\}.$$

Thus, every Neutrosophic Circle Graph can be represented as a Neutrosophic Ray Graph, with both graphs being isomorphic and preserving the neutrosophic membership functions. □

Theorem 174. *A Fuzzy String Graph is a Fuzzy Intersection Graph.*

Proof. Obviously holds. □

Corollary 175. *A Fuzzy Ray Graph is a Fuzzy Intersection Graph.*

Proof. Obviously holds. □

3.4 Result: Uncertain Grid Intersection Graph

Next, we consider the Uncertain Grid Intersection Graph. This is a Grid Intersection Graph with added conditions of uncertainty. The definition is provided below.

Definition 176. A *Grid Intersection Graph (GIG)* is a specific type of bipartite intersection graph derived from two families of line segments—horizontal and vertical intervals—on the plane. The precise definition is as follows:

Let Z_1 and Z_2 be two finite families of horizontal and vertical intervals, respectively, in the plane, such that no two intervals in Z_1 (horizontal) or Z_2 (vertical) intersect within their respective directions. The *Grid Intersection Graph*, denoted $G = (X, Y; E)$, is a bipartite graph where:

- The vertex set X corresponds to the horizontal intervals in Z_1,
- The vertex set Y corresponds to the vertical intervals in Z_2,
- There exists an edge $(x, y) \in E$ if and only if the horizontal interval corresponding to $x \in X$ intersects the vertical interval corresponding to $y \in Y$ in the plane.

In this context, G is called a *Grid Intersection Graph* and $Z(G) = Z_1 \cup Z_2$ is the *grid representation* of G.

A *Grid Intersection Graph* is called a *Unit Grid Intersection Graph* if it has an intersection model in which all line segments have unit length.

Definition 177 (Fuzzy Grid Intersection Graph)**.** Let Z_1 and Z_2 be two finite families of horizontal and vertical intervals, respectively, on the plane, such that no two intervals in Z_1 intersect and no two intervals in Z_2 intersect. The *Fuzzy Grid Intersection Graph* $G = (X, Y; E, \sigma, \mu)$ is defined as follows:

- **Vertex Sets**:
$$X = \{x_1, x_2, \ldots, x_m\}, \quad Y = \{y_1, y_2, \ldots, y_n\}$$
correspond to the horizontal intervals in Z_1 and the vertical intervals in Z_2, respectively.

- **Edge Set**:

$$E \subseteq X \times Y,$$

where there exists an edge $(x_i, y_j) \in E$ if and only if the horizontal interval corresponding to x_i intersects the vertical interval corresponding to y_j in the plane.

- **Vertex Membership Function**:

$$\sigma : X \cup Y \to [0, 1],$$

assigns to each vertex $v \in X \cup Y$ a membership degree $\sigma(v)$ representing its degree of participation in the graph.

- **Edge Membership Function**:

$$\mu : E \to [0, 1],$$

assigns to each edge $e \in E$ a membership degree $\mu(e)$ representing the strength of the connection between the corresponding intervals.

Definition 178 (Intuitionistic Fuzzy Grid Intersection Graph)**.** Let Z_1, Z_2, X, Y, and E be defined as in the classical Grid Intersection Graph. The *Intuitionistic Fuzzy Grid Intersection Graph* $G = (X, Y; E, \mu_A, \nu_A, \mu_E, \nu_E)$ is defined with the following functions:

- **Vertex Membership and Non-Membership Functions**:

$$\mu_A : X \cup Y \to [0, 1], \quad \nu_A : X \cup Y \to [0, 1],$$

where $\mu_A(v)$ and $\nu_A(v)$ represent the degree of membership and non-membership of vertex v, respectively, such that

$$0 \leq \mu_A(v) + \nu_A(v) \leq 1.$$

- **Edge Membership and Non-Membership Functions**:

$$\mu_E : E \to [0, 1], \quad \nu_E : E \to [0, 1],$$

where $\mu_E(e)$ and $\nu_E(e)$ represent the degree of membership and non-membership of edge e, respectively, such that

$$0 \leq \mu_E(e) + \nu_E(e) \leq 1.$$

Definition 179 (Neutrosophic Grid Intersection Graph)**.** Let Z_1, Z_2, X, Y, and E be defined as in the classical Grid Intersection Graph. The *Neutrosophic Grid Intersection Graph* $G = (X, Y; E, T_V, I_V, F_V, T_E, I_E, F_E)$ is defined with the following functions:

- **Vertex Functions**:

$$T_V : X \cup Y \to [0, 1], \quad I_V : X \cup Y \to [0, 1], \quad F_V : X \cup Y \to [0, 1],$$

where $T_V(v)$, $I_V(v)$, and $F_V(v)$ represent the truth-membership, indeterminacy-membership, and falsity-membership degrees of vertex v, respectively, such that

$$0 \leq T_V(v) + I_V(v) + F_V(v) \leq 3.$$

- **Edge Functions**:

$$T_E : E \to [0, 1], \quad I_E : E \to [0, 1], \quad F_E : E \to [0, 1],$$

where $T_E(e)$, $I_E(e)$, and $F_E(e)$ represent the truth-membership, indeterminacy-membership, and falsity-membership degrees of edge e, respectively, such that

$$0 \leq T_E(e) + I_E(e) + F_E(e) \leq 3.$$

Definition 180 (Turiyam Neutrosophic Grid Intersection Graph)**.** Let Z_1, Z_2, X, Y, and E be defined as in the classical Grid Intersection Graph. The *Turiyam Neutrosophic Grid Intersection Graph*
G , (X, Y; E, t_V, i_V, f_V, l_V, t_E, i_E, f_E, l_E) is defined with the following functions:

- **Vertex Functions**:

$$t_V : X \cup Y \to [0,1], \quad i_V : X \cup Y \to [0,1], \quad f_V : X \cup Y \to [0,1], \quad l_V : X \cup Y \to [0,1],$$

where $t_V(v)$, $i_V(v)$, $f_V(v)$, and $l_V(v)$ represent the truth, indeterminacy, falsity, and liberal state values for vertex v, respectively, such that

$$0 \leq t_V(v) + i_V(v) + f_V(v) + l_V(v) \leq 4.$$

- **Edge Functions**:

$$t_E : E \to [0,1], \quad i_E : E \to [0,1], \quad f_E : E \to [0,1], \quad l_E : E \to [0,1],$$

where $t_E(e)$, $i_E(e)$, $f_E(e)$, and $l_E(e)$ represent the truth, indeterminacy, falsity, and liberal state values for edge e, respectively, such that

$$0 \leq t_E(e) + i_E(e) + f_E(e) + l_E(e) \leq 4.$$

Theorem 181. *Neutrosophic Unit Grid intersection graphs are Neutrosophic Grid intersection graphs.*

Proof. Obviously holds. □

Theorem 182. *A Neutrosophic Grid Intersection Graph can be transformed into a classic Grid Intersection graph.*

Proof. Obviously holds. □

Corollary 183. *A Neutrosophic Unit Grid Intersection Graph can be transformed into a classic Unit Grid Intersection graph.*

Proof. Obviously holds. □

Corollary 184. *A Fuzzy Grid Intersection Graph, Intuitionistic Fuzzy Grid Intersection Graph, or Turiyam Neutrosophic Grid Intersection Graph can be transformed into a classic Grid Intersection Graph.*

Proof. Obviously holds. □

Corollary 185. *A Fuzzy Unit Grid Intersection Graph, Intuitionistic Fuzzy Unit Grid Intersection Graph, or Turiyam Neutrosophic Unit Grid Intersection Graph can be transformed into a classic Unit Grid Intersection Graph.*

Proof. Obviously holds. □

Theorem 186. *Every Neutrosophic Grid Intersection Graph can be represented as a Neutrosophic String Graph.*

Proof. Let $G = (X, Y; E, T_V, I_V, F_V, T_E, I_E, F_E)$ be a Neutrosophic Grid Intersection Graph.
Construction: 1. **String Representation**:

- For each horizontal interval $x_i \in X$, represent it as a string s_{x_i} which is the same horizontal line segment in the plane.
- For each vertical interval $y_j \in Y$, represent it as a string s_{y_j} which is the same vertical line segment in the plane.

2. **Vertex and Edge Sets**:

- The set of vertices in the Neutrosophic String Graph is $V = \{s_{x_i} \mid x_i \in X\} \cup \{s_{y_j} \mid y_j \in Y\}$.
- The set of edges E' is defined as:

$$E' = \left\{ \{s_{x_i}, s_{y_j}\} \mid (x_i, y_j) \in E \right\}.$$

3. **Membership Functions**:

- For each vertex $s \in V$, define:

$$T'_V(s) = T_V(s), \quad I'_V(s) = I_V(s), \quad F'_V(s) = F_V(s).$$

- For each edge $e' = \{s_{x_i}, s_{y_j}\} \in E'$, define:

$$T'_E(e') = T_E(x_i, y_j), \quad I'_E(e') = I_E(x_i, y_j), \quad F'_E(e') = F_E(x_i, y_j).$$

Verification:

- **Vertices and Edges Correspondence**: - The vertices and edges of G correspond one-to-one with those of G'. - An edge exists between s_{x_i} and s_{y_j} in G' if and only if the corresponding intervals x_i and y_j intersect in G.

- **Membership Conditions**: - Since the membership functions are directly inherited from G, they satisfy the conditions in Definition **??**.

Thus, $G' = (V, E', T'_V, I'_V, F'_V, T'_E, I'_E, F'_E)$ is a Neutrosophic String Graph representing G. □

Next, we consider the relationship between Neutrosophic Grid Intersection Graphs and Neutrosophic Bipartite Permutation Graphs. First, the definition of a Neutrosophic Bipartite Permutation Graph [451] is provided below.

Definition 187 (Neutrosophic Bipartite Permutation Graph)**.** [451] A *Neutrosophic Bipartite Permutation Graph* is an extension of the bipartite permutation graph into the neutrosophic domain, where truth, indeterminacy, and falsity membership degrees are assigned to vertices and edges. Let $G = (V_1 \cup V_2, E)$ be a bipartite permutation graph with vertex sets V_1 and V_2, and let π be a permutation on $V_1 \cup V_2$. The neutrosophic bipartite permutation graph $G_N = (V_1 \cup V_2, E, \sigma, \mu)$ is defined as follows:

- $\sigma : V \to [0, 1]^3$ assigns to each vertex $v \in V_1 \cup V_2$ a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees, respectively.
- $\mu : E \to [0, 1]^3$ assigns to each edge $e = (u, v) \in E$ a triple $(\mu_T(u, v), \mu_I(u, v), \mu_F(u, v))$, representing the truth, indeterminacy, and falsity membership degrees, respectively.
- The truth membership degree $\mu_T(u, v)$ for each edge $e = (u, v)$ is defined as:

$$\mu_T(u, v) = \begin{cases} \min(\sigma_T(u), \sigma_T(v)), & \text{if } u \in V_1, v \in V_2 \text{ and } \pi(u) > \pi(v), \\ 0, & \text{otherwise.} \end{cases}$$

The indeterminacy $\mu_I(u, v)$ and falsity $\mu_F(u, v)$ degrees are defined similarly.

Theorem 188. *Every Neutrosophic Bipartite Permutation Graph can be represented as a Neutrosophic Grid Intersection Graph.*

Proof. Let $G_N = (V_1 \cup V_2, E, \sigma, \mu)$ be a Neutrosophic Bipartite Permutation Graph. We aim to construct a Neutrosophic Grid Intersection Graph G_G that represents G_N, preserving the neutrosophic membership degrees of both vertices and edges.

We Construct the Corresponding Neutrosophic Grid Intersection Graph. To construct G_G, a Neutrosophic Grid Intersection Graph, we assign intervals to each vertex in $V_1 \cup V_2$, ensuring that edges are defined based on interval intersections. We Assign Horizontal Intervals to V_1. For each vertex $u \in V_1$, assign a horizontal interval h_u in the plane. The horizontal intervals are arranged in increasing order according to $\pi(u)$. We Assign Vertical Intervals to V_2. For each vertex $v \in V_2$, assign a vertical interval v_v in the plane. The vertical intervals are arranged in decreasing order according to $\pi(v)$.

We Define Edges Based on Interval Intersections. An edge $(u, v) \in E$ between $u \in V_1$ and $v \in V_2$ exists in G_G if and only if the corresponding horizontal interval h_u intersects the vertical interval v_v, which occurs if $\pi(u) > \pi(v)$. This condition ensures that the permutation ordering is preserved.

We Assign Neutrosophic Membership Functions. First, We consider Vertex Memberships. For each horizontal interval h_u corresponding to $u \in V_1$, assign:

$$T_V(h_u) = \sigma_T(u), \quad I_V(h_u) = \sigma_I(u), \quad F_V(h_u) = \sigma_F(u).$$

Similarly, for each vertical interval v_v corresponding to $v \in V_2$, assign:

$$T_V(v_v) = \sigma_T(v), \quad I_V(v_v) = \sigma_I(v), \quad F_V(v_v) = \sigma_F(v).$$

Next, we consider Edge Memberships. For each edge $(h_u, v_v) \in G_G$, corresponding to $(u, v) \in G_N$, assign:

$$T_E(h_u, v_v) = \mu_T(u, v), \quad I_E(h_u, v_v) = \mu_I(u, v), \quad F_E(h_u, v_v) = \mu_F(u, v).$$

This ensures that the neutrosophic membership degrees of edges in G_G match those in G_N. The edges in G_G correspond exactly to those in G_N, as the interval intersections reflect the permutation ordering. The neutrosophic membership degrees are preserved between the two graphs.

Therefore, we conclude that every Neutrosophic Bipartite Permutation Graph can be represented as a Neutrosophic Grid Intersection Graph. □

Corollary 189. *Every Neutrosophic Bipartite Permutation Graph can be represented as a Neutrosophic Unit Grid Intersection Graph.*

Proof. It can be proven using the same method as the proof above. □

Next, we will examine the relationship between Neutrosophic Interval Graphs and Neutrosophic Grid Intersection Graphs. The theorem is stated below.

Theorem 190. *Every Neutrosophic Grid Intersection Graph is a Neutrosophic Interval Graph.*

Proof. A *Neutrosophic Interval Graph* is a graph where each vertex corresponds to an interval on the real line, and an edge exists between two vertices if and only if their corresponding intervals intersect. Neutrosophic membership degrees are assigned to vertices and edges, representing truth, indeterminacy, and falsity values.

We will now show that any Neutrosophic Grid Intersection Graph can be represented as a Neutrosophic Interval Graph by transforming the grid structure into intervals on the real line while preserving the membership functions.

1. **Flattening the Grid**: In a Neutrosophic Grid Intersection Graph $G = (X, Y; E, T_V, I_V, F_V, T_E, I_E, F_E)$, the vertex set is composed of two types of intervals: horizontal intervals from the set X and vertical intervals from the set Y.

   These horizontal and vertical intervals, which intersect to form edges in the grid representation, can be "flattened" by mapping them to intervals on a single real line, such as the x-axis or y-axis.

2. **Mapping Intervals**: Each horizontal interval $h \in X$ in the grid can be projected onto the real line as an interval I_h on the x-axis.

   Similarly, each vertical interval $v \in Y$ can be projected onto the real line as an interval I_v on the y-axis.

   These projections result in a new graph representation where each vertex corresponds to an interval on the real line, capturing the same intersection relationships that existed in the original grid structure.

3. **Constructing the Neutrosophic Interval Graph**: In this new representation, the vertex set of the Neutrosophic Interval Graph is $V = X \cup Y$, where X corresponds to the horizontal intervals and Y corresponds to the vertical intervals.

   An edge exists between two vertices u and v if and only if their corresponding intervals on the real line intersect, i.e., $I_u \cap I_v \neq \emptyset$. This preserves the original intersection relationships from the grid representation.

   The neutrosophic membership degrees assigned to the vertices and edges in the original Neutrosophic Grid Intersection Graph are transferred to this new representation, ensuring that each vertex and edge retains its truth, indeterminacy, and falsity values.

4. **Verification**: The transformation from a Neutrosophic Grid Intersection Graph to a Neutrosophic Interval Graph preserves the essential structure of the graph. The intersection relationships, which define the edges, remain intact, and the membership functions T_V, I_V, F_V for vertices and T_E, I_E, F_E for edges are consistently maintained.

   Thus, every Neutrosophic Grid Intersection Graph can indeed be viewed as a Neutrosophic Interval Graph, as the grid structure can be faithfully flattened into an interval-based representation on the real line.

Therefore, the proof demonstrates that a Neutrosophic Grid Intersection Graph is a specific case of a Neutrosophic Interval Graph. □

Corollary 191. *Every Neutrosophic Unit Grid Intersection Graph is a Neutrosophic Unit Interval Graph.*

Proof. It can be proven using the same method as the proof above. □

Theorem 192. *A Neutrosophic Grid Intersection Graph is planar.*

Proof. We will demonstrate that a Neutrosophic Grid Intersection Graph can always be embedded in the plane without edge crossings, thereby proving that it is planar.

1. **Grid Representation:** By the definition of a Neutrosophic Grid Intersection Graph, the vertices of the graph correspond to intervals on a plane. These intervals are of two types: horizontal intervals (from a set Z_1) and vertical intervals (from a set Z_2).

   The edges in this graph are defined by the intersection points between these horizontal and vertical intervals. Specifically, an edge exists between a horizontal interval $h \in Z_1$ and a vertical interval $v \in Z_2$ if and only if they intersect at some point on the plane.

2. **Planar Drawing:** The graph can be drawn directly on the plane by placing each horizontal interval $h \in Z_1$ on a unique horizontal line and each vertical interval $v \in Z_2$ on a unique vertical line. The edges are then represented as the points where a horizontal interval intersects a vertical interval.

   This drawing utilizes the natural geometric structure of the grid, ensuring that all edges are formed by intersections of horizontal and vertical line segments.

3. **Absence of Edge Crossings:** In this construction, the intervals in Z_1 (horizontal) do not overlap or intersect with each other, as they are placed on distinct horizontal lines.

   Similarly, the intervals in Z_2 (vertical) are placed on distinct vertical lines, ensuring no overlap or intersection among the vertical intervals.

   Since edges are only formed at the intersection points between horizontal and vertical intervals, there are no crossings between edges. Each edge is represented as a single point of intersection, and no two edges can overlap or cross each other.

Thus, by constructing the graph in this manner, we ensure that no edge crossings occur, and the graph can be embedded in the plane without any overlap. Therefore, the Neutrosophic Grid Intersection Graph is planar. □

Corollary 193. *A Neutrosophic Unit Grid Intersection Graph is planar.*

Proof. It can be proven using the same method as the proof above. □

Corollary 194. *A Fuzzy Grid Intersection Graph, Intuitionistic Fuzzy Grid Intersection Graph, and Turiyam Neutrosophic Grid Intersection Graph are all planar.*

Proof. It can be proven using the same method as the proof above. □

Corollary 195. *A Fuzzy Unit Grid Intersection Graph, Intuitionistic Fuzzy Unit Grid Intersection Graph, and Turiyam Neutrosophic Unit Grid Intersection Graph are all planar.*

Proof. It can be proven using the same method as the proof above. □

Theorem 196. *Every Neutrosophic Grid Intersection Graph can be represented as a Neutrosophic Unit Square Graph.*

Proof. Our goal is to construct a Neutrosophic Unit Square Graph $G' = (V', E', T'_V, I'_V, F'_V, T'_E, I'_E, F'_E)$ such that G' is equivalent to the given Neutrosophic Grid Intersection Graph G.

For each vertex $v \in X$, associate a vertical unit square (unit-height rectangle) on the plane. Similarly, for each vertex $w \in Y$, associate a horizontal unit square (unit-width rectangle).

The unit squares corresponding to X and Y are placed on a grid such that the intersection of a vertical unit square (from X) and a horizontal unit square (from Y) corresponds to a potential intersection point.

Define $V' = X \cup Y$. Each vertex in V' corresponds to a unit square in the plane. Define $E' = \{(u, v) \mid u \in X, v \in Y, \text{ unit squares of } u \text{ and } v \text{ intersect}\}$. By construction, the edges in E' correspond to the intersections between vertical and horizontal unit squares, mirroring the edges in E of the Neutrosophic Grid Intersection Graph G. Neutrosophic Membership Functions is following:

- For each vertex $v \in V'$:

$$T'_V(v) = T_V(v), \quad I'_V(v) = I_V(v), \quad F'_V(v) = F_V(v).$$

- For each edge $e = (u, v) \in E'$:

$$T'_E(e) = T_E(e), \quad I'_E(e) = I_E(e), \quad F'_E(e) = F_E(e).$$

We need to verify that the constructed Neutrosophic Unit Square Graph G' satisfies the required conditions.

For all $v \in V'$, we have:

$$0 \leq T'_V(v) + I'_V(v) + F'_V(v) \leq 3.$$

This holds since $T_V(v), I_V(v), F_V(v)$ satisfy the same condition in G.

For all $e \in E'$, we have:

$$0 \leq T'_E(e) + I'_E(e) + F'_E(e) \leq 3.$$

This holds since $T_E(e), I_E(e), F_E(e)$ satisfy the same condition in G.

We consider about Edge Conditions. For each edge $e = (u, v) \in E'$, we verify the following conditions:

- Truth-Membership Condition:

$$T'_E(e) \leq \min\{T'_V(u), T'_V(v)\}.$$

  This holds because $T'_E(e) = T_E(e)$ and $T'_V(u) = T_V(u)$, and G satisfies:

$$T_E(e) \leq \min\{T_V(u), T_V(v)\}.$$

- Indeterminacy-Membership Condition:

$$I'_E(e) \geq \max\{I'_V(u), I'_V(v)\}.$$

  This holds because $I'_E(e) = I_E(e)$, and G satisfies:

$$I_E(e) \geq \max\{I_V(u), I_V(v)\}.$$

- Falsity-Membership Condition:

$$F'_E(e) \geq \max\{F'_V(u), F'_V(v)\}.$$

  This holds because $F'_E(e) = F_E(e)$, and G satisfies:

$$F_E(e) \geq \max\{F_V(u), F_V(v)\}.$$

Every edge in E' corresponds to an edge in E and vice versa. This is because the intersection of unit squares in G' mirrors the intersections of grid lines or intervals in G. □

Corollary 197. *Every Neutrosophic Unit Grid Intersection Graph can be represented as a Neutrosophic Unit Square Graph.*

Proof. It can be proven using the same method as the proof above. □

Corollary 198. *Every Neutrosophic Grid Intersection Graph can be represented as a Neutrosophic Unit Disk Graph.*

Proof. It can be proven using the same method as the proof above. □

Corollary 199. *Every Neutrosophic Unit Grid Intersection Graph can be represented as a Neutrosophic Unit Disk Graph.*

Proof. It can be proven using the same method as the proof above. □

4. Conclusion and Future tasks

In this section, we present the Conclusion and Future Work of this paper.

4.1 Conclusion in this paper

In this paper, we introduced and investigated various intersection graphs within the frameworks of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs, examining their properties and interrelationships.

Given the vast and continually expanding body of literature on fuzzy mathematics and intersection graph theory, similar concepts often arise independently across different journals and time periods. Despite this, we believe that unifying these concepts is crucial and will significantly contribute to the advancement of the field. We plan to continue our research in this direction in the future (cf.[452, 461]).

Moreover, without risk of misunderstanding, it is a common practice in the study of graph classes and structures to analyze the properties and algorithms of graphs by combining the characteristics of classic graph structures (such as Complete, Regular, and Tree) with those of intersection graphs (such as Permutation, Interval, Circle, and String) and uncertain graphs (such as Fuzzy, Neutrosophic, and Turiyam). In essence, we aim to further our research by considering the following perspectives:

- **Classic Graph Properties:** Regular, Irregular, Complete, Perfect, Tree, Path, Forest
- **Subgraph/Hypergraph Properties:** Supergraph, Hypergraph, Superhypergraph, Subgraph, Induced Subgraph, Induced Supergraph
- **Graph Directionality:** Undirected, Directed, Mixed, Bidirected
- **Uncertain Properties:** Fuzzy, Neutrosophic, Turiyam, Plithogenic, Rough, Vague, Soft, Weighted, Picture Fuzzy
- **Graph Dimensionality:** 2D, 3D, 4D, etc.
- **Intersection Properties:** Permutation, Interval, Circle, Disk, Ray, String
- **Themes:** Graph Classes, Graph Parameters, Algorithms, Problem Complexity, Real-world Applications

This comprehensive approach facilitates a deeper understanding of graph-theoretic properties and their potential applications. In this paper, we have adopted this approach and intend to pursue this line of research further in the future.

4.2 Future tasks

The future tasks related to this paper are outlined below. As mentioned in the Conclusion subsection, this section provides a more detailed description of the specific future tasks. Please note that some aspects are still in the conceptual stage.

4.2.1 Property of Turiyam Neutrosophic graphs and Plithogenic Graphs

In the future, we plan to explore various properties of uncertain graphs. Many characteristics of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs remain largely unexplored. Expanding the graph properties and structures of classical graphs and Fuzzy Graphs to encompass Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs, and reassessing their respective properties, offers significant value from an applied mathematics perspective.

In Fuzzy Graphs and Neutrosophic Graphs, various generalized concepts have emerged, such as Picture, Bipolar, Pythagorean, and Single-Valued graphs, each incorporating specific conditions. Similarly, in the future, we aim to explore the concepts of Turiyam Neutrosophic Graphs and Plithogenic Graphs, which are expected to hold considerable significance in applied mathematics, and analyze their mathematical properties in depth.

4.2.2 Algorithm complexity for uncertain graphs

Computational Complexity Theory focuses on classifying computational problems based on their intrinsic difficulty and the resources (such as time and space) required to solve them [4, 115, 311, 558, 908]. However, the full scope of algorithm complexity for uncertain graphs has not yet been fully uncovered. We aim to conduct further research on algorithms and algorithm complexity within the context of uncertain graphs.

For instance, in this paper, we explore algorithms related to intersection graphs in uncertain graphs. Intersection graphs have been extensively studied, particularly in terms of recognition algorithms. Additionally, there has been research on approximation algorithms for intersection graphs (cf.[516]). Since uncertain graphs are essentially classic graphs with added uncertainty, the complexity of algorithms in uncertain graphs is expected to be at least as high as, if not higher than, that of algorithms in classic graphs. For example, NP-hard algorithms remain NP-hard or more complex in uncertain graphs. On the other hand, for graph classes where polynomial-time algorithms exist, it would be interesting to investigate how these algorithms behave when extended to uncertain graphs.

- Unit disk graph recognition Algorithm: It is known to be NP-hard[222]. Recognizing Unit Disk Graphs in Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs is also NP-hard.
- String graph recognition Algorithm: It is known to be NP-hard[1012]. Recognizing String graphs in Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs is also NP-hard.

4.2.3 Intersection SuperhyperGraph and Induced SuperHyperGraph

In the future, I plan to define Intersection SuperGraphs, Intersection SuperHyperGraphs, and Induced SuperHyperGraphs, and explore their mathematical structures in detail. Intersection SuperHyperGraphs generalize Intersection HyperGraphs, while Intersection SuperGraphs are the supergraph versions of Intersection Graphs. Although these concepts are still at the conceptual stage, the definitions are provided below.

Definition 200 (Intersection SuperGraph)**.** Let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a collection of non-empty sets, where each S_i is a subset of a universal set U. The *Intersection SuperGraph* $G' = (V', E')$ is a supergraph of the intersection graph $G = (V, E)$ associated with $\mathcal{S}$, and is defined as follows:

- The vertex set V' extends the vertex set V of G, which corresponds to the sets in $\mathcal{S}$. Formally:

  $$V = \{v_1, v_2, \ldots, v_n\}, \quad V' = V \cup V_{\text{extra}},$$

  where V_{extra} is a set of additional vertices not present in G.
- The edge set E' contains all the edges of G as well as additional edges that may connect vertices in V and/or V_{extra}. Formally:

  $$E \subseteq E' \subseteq \{(v_i, v_j) \mid v_i, v_j \in V'\}.$$
- There is an edge $(v_i, v_j) \in E$ if and only if the corresponding sets S_i and S_j have a non-empty intersection, i.e., $S_i \cap S_j \neq \emptyset$, as in the standard intersection graph.
- Additional edges $(v_i, v_j) \in E' \setminus E$ may represent additional relationships or interactions beyond simple set intersections, which can be introduced to capture more complex relationships in the supergraph structure.
- Any vertex $v_i \in V_{\text{extra}}$ may or may not be connected to vertices in V or other vertices in V_{extra}, depending on the structure of the supergraph.

In other words, an Intersection SuperGraph is a supergraph of an intersection graph, containing all the vertices and edges of the original intersection graph while possibly introducing additional vertices and edges to capture more complex relationships. The new edges and vertices do not alter the original intersection structure but expand upon it to represent a richer structure.

Definition 201 (Intersection SuperHyperGraph)**.** Let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a collection of non-empty sets, where each S_i is a subset of a universal set U. An *Intersection SuperHyperGraph* is a SuperHyperGraph SHG $= (G, E)$ constructed as follows:

1. **Vertices and SuperVertices**:
   - The set of vertices V is the union of all elements in $\mathcal{S}$:

     $$V = \bigcup_{i=1}^{n} S_i \subseteq U.$$
   - The set of supervertices G is defined as:

     $$G = \{S_i \mid 1 \leq i \leq n\},$$

     where each supervertex S_i corresponds to a set in $\mathcal{S}$.
2. **Hyperedges and SuperHyperEdges**:

- For each non-empty intersection among the sets in $\mathcal{S}$, create a superhyperedge. Specifically, for every subset $\mathcal{T} \subseteq \mathcal{S}$ with $|\mathcal{T}| \geq 2$ such that the intersection

$$I_{\mathcal{T}} = \bigcap_{S_i \in \mathcal{T}} S_i \neq \emptyset,$$

there exists a superhyperedge $E_{\mathcal{T}}$ connecting the corresponding supervertices:

$$E_{\mathcal{T}} = \{S_i \mid S_i \in \mathcal{T}\} \subseteq G.$$

- The set of superhyperedges E is the collection of all such $E_{\mathcal{T}}$:

$$E = \{E_{\mathcal{T}} \mid I_{\mathcal{T}} \neq \emptyset, \mathcal{T} \subseteq \mathcal{S}, |\mathcal{T}| \geq 2\}.$$

3. **Additional Structures**:
   - Each vertex $v \in V$ may belong to multiple supervertices, reflecting the elements shared among the sets in $\mathcal{S}$.
   - The Intersection SuperHyperGraph SHG captures not only pairwise intersections but also higher-order intersections among multiple sets.

Definition 202 (Induced SuperHyperGraph)**.** Let $H = (V, E)$ be a hypergraph as defined in [221], where:

- V is a finite set of vertices.
- $E \subseteq P(V)$ is a set of hyperedges, each of which is a subset of V.

Let SHG $= (G \subseteq P(V'), E' \subseteq P(V'))$ be a *SuperHyperGraph* as defined in [1055], where:

- V' is a finite or infinite set of vertices, with $V \subseteq V'$.
- $G \subseteq P(V')$ is the set of *supervertices*, where each supervertex is a subset of V'.
- $E' \subseteq P(V')$ is the set of *superedges*, where each superedge is a subset of V'.

The SuperHyperGraph SHG is called an *Induced SuperHyperGraph* of H if the following conditions hold:

1. **Vertex Inclusion**: Every vertex $v \in V$ is included in V', and there exists a supervertex $S_v \in G$ such that $v \in S_v$.
2. **Edge Preservation**: For every hyperedge $e \in E$, there exists a superedge $E_e \in E'$ such that the set of vertices in e are exactly the union of supervertices associated with e:

$$e = \bigcup_{v \in e} S_v.$$

3. **Induced Subhypergraph**: The induced subhypergraph of SHG on the vertex set V is exactly H, meaning no additional hyperedges exist among the vertices of H in SHG.

In other words, H is an induced subhypergraph of SHG, and SHG may contain additional vertices (in $V' \setminus V$), supervertices, and superedges involving those additional vertices, but it does not introduce new hyperedges among the original vertices of H.

4.2.4 Paraconsistent graph

In the future, we aim to explore the concept of a Paraconsistent Graph, which extends the idea of a paraconsistent set. A *paraconsistent set* is a mathematical structure that allows for the coexistence of contradictory information, enabling elements to simultaneously belong and not belong to a set. This framework is designed to handle inconsistencies in set theory without collapsing into triviality [277, 333, 430, 744, 1139–1141].

While still in the conceptual stage, the definitions of both a paraconsistent set and a Paraconsistent Graph are provided below.

Definition 203. [333] Let M be a non-empty set, referred to as the *universe*.

1. **Extension Function:**
Define an *extension function* $[\cdot]_M$ that assigns to each element $a \in M$ an ordered pair of subsets of M:

$$[\cdot]_M : M \to \mathcal{P}_p(M),$$

where $\mathcal{P}_p(M)$ denotes the set of ordered pairs of subsets of M that cover M:

$$\mathcal{P}_p(M) := \{(X, Y) \mid X \cup Y = M\}.$$

The pair $[a]_M = \left([a]_M^+, [a]_M^-\right)$ consists of:

   - *Positive Extension* $[a]_M^+$: The set of elements that *belong* to a.
   - *Negative Extension* $[a]_M^-$: The set of elements that *do not belong* to a.

These subsets cover the universe:

$$[a]_M^+ \cup [a]_M^- = M.$$

2. **Membership Relations:**
Define two binary relations on M:

   - *Positive Membership*:

$$a \in_M b \quad \text{if and only if} \quad a \in [b]_M^+.$$

   - *Negative Membership*:

$$a \notin_M b \quad \text{if and only if} \quad a \in [b]_M^-.$$

It is possible for an element a to both belong and not belong to b, i.e., $a \in_M b$ and $a \notin_M b$, when $a \in [b]_M^+ \cap [b]_M^-$.

3. **Truth Function:**
Introduce a *truth function* ε_M to represent the membership status:

$$\varepsilon_M : M \times M \to T,$$

where $T = \{0, i, 1\}$ represents truth degrees:

   - 1 (True): $a \in_M b$ holds, and $a \notin_M b$ does not.
   - 0 (False): $a \notin_M b$ holds, and $a \in_M b$ does not.
   - i (Inconsistent): Both $a \in_M b$ and $a \notin_M b$ hold.

The truth function is defined as:

$$\varepsilon_M(a, b) = \begin{cases} 1, & \text{if } a \in_M b \text{ and } a \notin_M b \text{ is false}, \\ 0, & \text{if } a \notin_M b \text{ and } a \in_M b \text{ is false}, \\ i, & \text{if } a \in_M b \text{ and } a \notin_M b \text{ both hold}. \end{cases}$$

4. **Strong Extensionality:**
A paraconsistent set M is said to be *strongly extensional* if the extension function $[\cdot]_M$ is injective:

$$\text{For all } a, b \in M, \quad [a]_M^+ = [b]_M^+ \text{ and } [a]_M^- = [b]_M^- \implies a = b.$$

This means that each element is uniquely determined by its positive and negative extensions.

Definition 204. A paraconsistent graph is a structure $G = (V, \varepsilon_G)$, where:

1. **Vertex Set:** V is a non-empty set of vertices.

2. **Truth Function:** ε_G is a truth function representing the adjacency relation:

$$\varepsilon_G : V \times V \to T,$$

where $T = \{0, i, 1\}$ represents truth degrees:

- 1 (True): An edge exists between u and v.
- 0 (False): No edge exists between u and v.
- i (Inconsistent): There is both an edge and no edge between u and v.

3. **Adjacency Relations:**
Define two relations based on ε_G:

- *Positive Adjacency*:

$$u \sim_G v \quad \text{if and only if} \quad \varepsilon_G(u, v) \in \{1, i\}.$$

- *Negative Adjacency*:

$$u \nsim_G v \quad \text{if and only if} \quad \varepsilon_G(u, v) \in \{0, i\}.$$

In the case where $\varepsilon_G(u, v) = i$, both $u \sim_G v$ and $u \nsim_G v$ hold, reflecting the paraconsistent nature.

4. **Edge Set Representation:**
Alternatively, the graph can be represented with an *edge set* E and a *paraconsistent adjacency function*:

$$\varepsilon_G(u, v) = \begin{cases} 1, & \text{if } (u, v) \in E \text{ and } (u, v) \notin E', \\ 0, & \text{if } (u, v) \notin E \text{ and } (u, v) \notin E', \\ i, & \text{if } (u, v) \in E \text{ and } (u, v) \in E', \end{cases}$$

where E' represents the set of non-edges that are also considered in the inconsistent state.

Conjecture 205. Neutrosophic Graphs are a generalization of paraconsistent graphs.

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Fundamental Computational Problems and Algorithms for SuperHyperGraphs

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Abstract

Hypergraphs extend traditional graphs by allowing edges (known as hyperedges) to connect more than two vertices, rather than just pairs. This paper explores fundamental problems and algorithms in the context of SuperHypergraphs, an advanced extension of hypergraphs enabling modeling of hierarchical and complex relationships. Topics covered include constructing SuperHyperGraphs, recognizing SuperHyperTrees, and computing SuperHyperTree-width. We address a range of optimization problems, such as the SuperHypergraph Partition Problem, Reachability, Minimum Spanning SuperHypertree, and Single-Source Shortest Path. Furthermore, adaptations of classical problems like the Traveling Salesman Problem, Chinese Postman Problem, and Longest Simple Path Problem are presented in the SuperHypergraph framework.

Keywords: Superhypergraph, Hypergraph, Tree-width, Algorithm

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1 Introduction

1.1 Graphs and Hypergraphs

Graph theory serves as a foundational framework for analyzing networks, consisting of nodes (vertices) and their connections (edges). It provides valuable insights into the structure, connectivity, and properties of diverse networks [28].

Hypergraphs extend traditional graphs by allowing edges (known as hyperedges) to connect more than two vertices, rather than just pairs. These generalized structures have gained significant attention due to their broad applications in graph theory, computer science, and related fields [6, 21, 32, 49, 58, 116]. Further extending hypergraphs, SuperHypergraphs introduce even greater flexibility and are a topic of emerging interest in recent studies [105, 106]. A SuperHypergraph generalizes hypergraphs, allowing vertices and hyperedges to represent sets or subsets, enabling modeling of hierarchical and complex relationships.

In the context of graphs, hypergraphs, and superhypergraphs, tree structures have been extensively studied. For instance, Hypertrees have been explored in hypergraphs [54], while SuperHypertrees have been investigated in superhypergraphs [51]. The concept of tree structures is widely adopted due to their simplicity and efficiency in applications. Beyond graph-related concepts, tree structures have been applied in various fields, including Tree Automata [17,23], TreeSoft sets [9,41,109], and Decision Trees [19,20], contributing to ongoing research and advancements in these areas.

1.2 Graph Width Parameters

Graph characteristics are often analyzed using various parameters, with significant research devoted to understanding these measures. Among these, graph width parameters such as tree-width [14–16, 90, 91, 96] are particularly prominent. These parameters evaluate how closely a graph approximates a tree structure, which is essential for many practical applications.

For hypergraphs, analogous parameters like Hypertree-width [3,53,55,75,121] and Hyperpath-width [2,78,83] have been developed. These metrics measure the resemblance of a hypergraph to a tree or a path, addressing the need to extend tree-based analyses to more complex structures.

1.3 Computational Complexity

An algorithm is a finite sequence of well-defined instructions intended to solve a specific problem or perform a computation [31]. The time and space complexities of the algorithm are often subjects of analysis. Computational complexity evaluates the resources, such as time and space, required by an algorithm to solve a problem as a function of input size, offering theoretical efficiency bounds [1, 10, 61, 85].

Algorithms in graph theory are referred to as graph algorithms [31]. For graphs and hypergraphs, numerous problems and algorithms have been studied, with research also exploring real-world applications (e.g. [46, 64, 82]).

1.4 Contributions of This Paper

Research on problems and algorithms related to SuperHypergraphs remains limited. Therefore, this paper investigates various fundamental problems in SuperHypergraphs and proposes algorithms to address them.

- *Exact Construction of SuperHyperGraph*: Develops an algorithm to construct a SuperHyperGraph from a given vertex and edge set.
- *Recognizing a SuperHyperTree*: Provides an algorithm to determine if a given structure is a valid SuperHyperTree.
- *Computation of SuperHyperTree-width*:
  - *Exact Algorithm*: Defines an exact algorithm to compute the SuperHyperTree-width of a superhypergraph.
  - *Approximation Algorithm*: Proposes an approximation algorithm for computationally efficient SuperHyperTree-width calculation.
- *Superhypergraph Partition Problem*: Studies methods to partition a SuperHyperGraph into disjoint subsets while minimizing inter-partition connections.
- *Reachability Problem in SuperHypergraphs*: Explores algorithms to verify if a path exists between two vertices in a SuperHyperGraph.
- *Minimum Spanning SuperHypertree Problem*: Presents a method to compute a SuperHypertree with the minimum total edge weight.
- *Single-Source Shortest Path Problem in a SuperHypergraph*: Develops an algorithm to find shortest paths from a single source vertex in a SuperHyperGraph.
- *Traveling Salesman Problem in a SuperHypergraph*: Examines the adaptation of TSP to SuperHyperGraphs, finding the minimum tour covering all vertices.
- *Chinese Postman Problem (CPP) in SuperHypergraphs*: Analyzes how to find an Eulerian circuit in SuperHyperGraphs under given conditions.
- *Longest Simple Path Problem in SuperHypergraphs*: Studies methods to identify the longest acyclic path in a SuperHyperGraph.
- *Maximum Spanning Tree Problem in SuperHypergraphs*: Proposes algorithms to compute a SuperHypertree with the maximum total edge weight.
- *Horn Satisfiability Problem in a SuperHypergraph*: Extends the Horn Satisfiability Problem to SuperHyperGraphs with adapted algorithms and proofs.

1.5 The Structure of the Paper

The structure of this paper is as follows.

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2 Preliminaries and Definitions

In this section, we provide the preliminaries and definitions. Readers seeking foundational concepts and notations in graph theory are encouraged to consult standard texts, surveys, or lecture notes, such as [26–28,119]. This work also utilizes basic principles from set theory, for which references like [35, 62, 66, 67, 72] are recommended. For detailed discussions on specific operations and related topics addressed in this paper, readers may refer to the respective references for additional insights.

2.1 Graphs and Hypergraphs

Graph theory provides a fundamental framework for analyzing networks, which are composed of nodes (vertices) and their connections (edges). A hypergraph extends the traditional graph concept by allowing hyperedges, which can connect multiple vertices rather than just pairs, enabling the representation of more complex relationships between elements [11–13,54–56]. The basic definitions of graphs and hypergraphs are presented below.

Definition 2.1 (Graph)**.** [28] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2.2 (Subgraph)**.** [28] Let $G = (V, E)$ be a graph. A *subgraph* $H = (V_H, E_H)$ of G is a graph such that:

- $V_H \subseteq V$, i.e., the vertex set of H is a subset of the vertex set of G.
- $E_H \subseteq E$, i.e., the edge set of H is a subset of the edge set of G.
- Each edge in E_H connects vertices in V_H.

Definition 2.3 (Tree in a Graph)**.** Let $G = (V, E)$ be an undirected graph, where V is the set of vertices and E is the set of edges. A subgraph $T = (V_T, E_T)$ of G is called a *tree* if it satisfies the following conditions:

1. *Acyclicity:* T does not contain any cycles. Formally, for any subset of edges $E' \subseteq E_T$, the graph (V_T, E') does not contain a closed path.
2. *Connectivity:* For every pair of vertices $u, v \in V_T$, there exists a unique path in T connecting u and v.
3. *Minimality:* T contains exactly $|V_T| - 1$ edges, where $|V_T|$ is the number of vertices in T.

Definition 2.4 (Path in a Graph)**.** Let $G = (V, E)$ be an undirected graph, where V is the set of vertices and E is the set of edges. A *path* in G is a sequence of vertices $P = (v_1, v_2, \dots, v_k)$ such that:

1. For every $i = 1, 2, \dots, k - 1$, $(v_i, v_{i+1}) \in E$, i.e., there is an edge connecting consecutive vertices in the sequence.
2. All vertices $v_1, v_2, \dots, v_k$ are distinct, ensuring that the path does not revisit any vertex (a *simple path*).

The *length* of the path is the number of edges in the sequence, which is $k - 1$.

Definition 2.5 (Eulerian circuit)**.** An Eulerian circuit is a closed path in a graph that traverses every edge exactly once, starting and ending at the same vertex.

Definition 2.6 (Hypergraph)**.** [13] A *hypergraph* $H = (V, E)$ is a generalization of a graph, consisting of:

- A set V, called the *vertex set*, where each element $v \in V$ represents a vertex.
- A set E, called the *hyperedge set*, where each element $e \in E$ is a subset of V, representing a hyperedge. Thus, $e \subseteq V$.

Key properties of a hypergraph:

- The hyperedge set E is a subset of the power set of V, i.e., $E \subseteq \mathcal{P}(V)$, where $\mathcal{P}(V)$ denotes the collection of all subsets of V.
- Unlike standard graphs where edges connect exactly two vertices, in a hypergraph, a hyperedge can connect any number of vertices, including just one vertex or the entire vertex set.

Example 2.7. For example, given a vertex set $V = \{v_1, v_2, v_3, v_4\}$, a hypergraph can have hyperedges such as:

$$E = \{\{v_1, v_2\}, \{v_3\}, \{v_1, v_3, v_4\}\}.$$

Here, the hyperedge $\{v_1, v_3, v_4\}$ connects three vertices simultaneously, illustrating the generality of hyperedges.

Definition 2.8 (Hypertree)**.** [55] A *hypertree* is a hypergraph $H = (V, E)$ with the following properties:

1. *Tree-like Structure:* There exists a tree $T = (V_T, E_T)$, called the *host tree*, such that:
   - Each vertex $t \in V_T$ is associated with a *bag* $B_t \subseteq V$, where B_t is a subset of the vertices of H.
   - Each hyperedge $e \in E$ of the hypergraph is a subset of at least one bag B_t, i.e., $\exists t \in V_T$ such that $e \subseteq B_t$.
2. *Connectivity Condition:* For any vertex $v \in V$, the set of nodes $t \in V_T$ where $v \in B_t$ forms a connected subtree of T. This ensures that each vertex in H is consistently represented across the tree structure.
3. *Acyclicity Condition:* The host tree T must be acyclic, maintaining the tree-like structure of the decomposition.

Definition 2.9 (Hyperpath in a Hypergraph)**.** ([22, 70, 74]) Let $H = (V, E)$ be a hypergraph, where V is the set of vertices and $E \subseteq \mathcal{P}(V)$ is the set of hyperedges. A *hyperpath* P in H connecting two vertices $u, v \in V$ is a sequence of hyperedges:

$$P = (e_1, e_2, \ldots, e_k) \quad \text{with } e_i \in E \text{ for } i = 1, \ldots, k,$$

satisfying the following conditions:

- *Adjacency Condition:* For all $1 \leq i \leq k - 1$, $e_i \cap e_{i+1} \neq \emptyset$, i.e., consecutive hyperedges share at least one common vertex.
- *Endpoint Condition:* $u \in e_1$ and $v \in e_k$.
- *Acyclic Condition:* The sequence does not form a cycle, meaning the set of vertices visited by P, denoted as $V(P) = \bigcup_{i=1}^{k} e_i$, does not contain repeated visits to the same hyperedge.

The *length* of the hyperpath is the number of hyperedges in P, denoted $|P| = k$.

2.2 Tree-width and Hypertree-width

Tree-width quantifies how closely a graph resembles a tree by representing it using a tree-like structure with minimal width [92–96]. Hypertree-width generalizes this concept to hypergraphs, measuring how effectively a hypergraph can be decomposed into a tree-like structure [2, 3, 52–55, 75, 121]. The formal definitions of Tree-width and Hypertree-width are presented below.

Definition 2.10. [96] A tree-decomposition of an undirected graph G is a pair (T, W), where T is a tree, and $W = (W_t \mid t \in V(T))$ is a family of subsets that associates with every node t of T a subset W_t of vertices of G such that:

(T1) $\bigcup_{t \in V(T)} W_t = V(G)$,

(T2) For each edge $(u, v) \in E(G)$, there exists some node t of T such that $\{u, v\} \subseteq W_t$, and

(T3) For all nodes r, s, t in T, if s is on the unique path from r to t then $W_r \cap W_t \subseteq W_s$.

The width of a tree-decomposition (T, W) is the maximum of $|W_t| - 1$ over all nodes t of T. The tree-width of G is the minimum width over all tree-decompositions of G.

Definition 2.11. [3] A *generalized hypertree decomposition* of a hypergraph $H = (V(H), E(H))$ is a triple (T, B, C), where:

- (T, B) is a *tree decomposition* of H, where:
  - T is a tree with vertex set $V(T)$,
  - $B = \{B_t \mid t \in V(T)\}$ is a family of subsets of $V(H)$, called *bags*, satisfying the tree decomposition properties.
- $C = \{C_t \mid t \in V(T)\}$ is a family of subsets of $E(H)$ (hyperedges of H), called *guards*.

The decomposition must satisfy the following conditions for each $t \in V(T)$:

1. $B_t \subseteq \bigcup C_t$, where $\bigcup C_t$ is defined as:

   $$\bigcup C_t = \{v \in V(H) \mid \exists e \in C_t : v \in e\}.$$

   In other words, every vertex in B_t must belong to at least one hyperedge in C_t.

The *width* of the generalized hypertree decomposition (T, B, C) is defined as:

$$\text{width}(T, B, C) = \max\{|C_t| \mid t \in V(T)\},$$

where $|C_t|$ denotes the number of hyperedges in C_t.

The *generalized hypertree width* of H, denoted $\text{ghw}(H)$, is the minimum width among all possible generalized hypertree decompositions of H.

A *hypertree decomposition* of H is a special case of a generalized hypertree decomposition (T, B, C) that satisfies the following additional condition for all $t \in V(T)$:

$$(\bigcup C_t) \cap \bigcup_{u \in V(T_t)} B_u \subseteq B_t,$$

where T_t is the subtree of T rooted at t, and $\bigcup_{u \in V(T_t)} B_u$ denotes the union of the bags associated with all nodes in T_t.

The *width* of a hypertree decomposition is defined in the same way as for a generalized hypertree decomposition. The *hypertree width* of H, denoted $\text{hw}(H)$, is the minimum width among all possible hypertree decompositions of H.

2.3 SuperHyperGraph and Superhypertree

A Superhypertree is a tree in a Superhypergraph [39, 51]. A Superhypergraph is known as a generalization of concepts such as graphs and hypergraphs (cf. [39, 42, 59, 60, 89, 105–108, 110, 110, 111]). The definitions, including related concepts, are provided below.

Definition 2.12 (SuperHyperGraph)**.** [105, 106] Let V be a finite set of vertices. A *superhypergraph* is an ordered pair $H = (V, E)$, where:

- $V \subseteq P(V)$ (the power set of V), meaning that each element of V can be either a single vertex or a subset of vertices (called a *supervertex*).
- $E \subseteq P(V)$ represents the set of edges, called *superedges*, where each $e \in E$ can connect multiple supervertices.

In this framework, a superhypergraph can accommodate complex relationships among groups of vertices, including single edges, hyperedges, superedges, and multi-edges. Superhypergraphs provide a flexible structure to represent high-order and hierarchical relationships.

Proposition 2.13. *[42] Every superhypergraph can be transformed into a hypergraph.*

Proof. Refer to [42] for details. □

Definition 2.14 (SuperHyperTree)**.** [39, 51] A *SuperHyperTree (SHT)* is a SuperHyperGraph $\text{SHT} = (V, E)$ that satisfies the following conditions:

1. *Host Tree Condition*: There exists a tree $T = (V_T, E_T)$, called the *host tree*, such that:
    - The vertex set of T is $V_T = V$.
    - Each superedge $e \in E$ corresponds to a connected subtree of T.
2. *Acyclicity Condition*: The host tree T must be acyclic, ensuring that SHT inherits this property.
3. *Connectedness Condition*: For any $v, w \in V$, there exists a sequence of superedges $e_1, e_2, \ldots, e_k \in E$ such that $v \in e_1$, $w \in e_k$, and $e_i \cap e_{i+1} \neq \emptyset$ for $1 \leq i < k$.

Definition 2.15 (SuperHyperpath in a SuperHypergraph)**.** Let SHG $= (V, E)$ be a superhypergraph, where $V \subseteq \mathcal{P}(V_0)$ is a set of supervertices (each being a subset of some base set V_0) and $E \subseteq \mathcal{P}(V)$ is a set of superhyperedges. A *superhyperpath* P in SHG connecting two supervertices $u, v \in V$ is a sequence of superhyperedges:

$$P = (e_1, e_2, \ldots, e_k) \quad \text{with } e_i \in E \text{ for } i = 1, \ldots, k,$$

satisfying the following conditions:

- *Adjacency Condition:* For all $1 \leq i \leq k-1$, $\bigcup e_i \cap \bigcup e_{i+1} \neq \emptyset$, i.e., the union of vertices in consecutive superhyperedges share at least one vertex in the base set V_0.
- *Endpoint Condition:* $u \in e_1$ and $v \in e_k$.
- *Acyclic Condition:* The sequence does not form a cycle, meaning the set of supervertices visited by P, denoted as $V(P) = \bigcup_{i=1}^{k} e_i$, does not revisit the same supervertex in the same sequence.

The *length* of the superhyperpath is the number of superhyperedges in P, denoted $|P| = k$.

Proposition 2.16. *A* SuperHyperpath *P in a superhypergraph SHG $= (V, E)$ is a* SuperHyperTree (SHT) *if and only if P satisfies the Host Tree Condition, Acyclicity Condition, and Connectedness Condition of a SuperHyperTree.*

Proof. We will prove the proposition in two parts:

- *(1) If P is a SuperHyperpath, then P satisfies the conditions of a SuperHyperTree.*
- *(2) If P satisfies the conditions of a SuperHyperTree, then P is a SuperHyperpath.*

(1) P as a SuperHyperpath implies P is a SuperHyperTree Let $P = (e_1, e_2, \ldots, e_k)$ be a SuperHyperpath.

1. *Host Tree Condition:* Since P is a SuperHyperpath, its superedges $e_1, e_2, \ldots, e_k$ are connected sequentially such that $\bigcup e_i \cap \bigcup e_{i+1} \neq \emptyset$ for $1 \leq i \leq k-1$. We can construct a tree $T = (V_T, E_T)$, where:
$$V_T = \{e_1, e_2, \ldots, e_k\}, \quad E_T = \{(e_i, e_{i+1}) \mid \bigcup e_i \cap \bigcup e_{i+1} \neq \emptyset\}.$$
Thus, P satisfies the Host Tree Condition.
2. *Acyclicity Condition:* By definition, P is a path and does not revisit any supervertex in the sequence of superedges. Therefore, the constructed tree T is acyclic.
3. *Connectedness Condition:* In P, every pair of supervertices $u, v \in V$ in $V(P) = \bigcup_{i=1}^{k} e_i$ is connected through a sequence of superedges $e_1, e_2, \ldots, e_k$. Hence, P satisfies the Connectedness Condition.

Since all three conditions of a SuperHyperTree are satisfied, P is a SuperHyperTree.

(2) P satisfies SuperHyperTree conditions implies P is a SuperHyperpath Let P satisfy the Host Tree Condition, Acyclicity Condition, and Connectedness Condition of a SuperHyperTree.

1. By the Host Tree Condition, there exists a tree $T = (V_T, E_T)$, where each superedge $e \in E$ corresponds to a subtree of T. In particular, the tree structure ensures that consecutive superedges e_i and e_{i+1} share at least one vertex in the base set V_0. Thus, P satisfies the Adjacency Condition of a SuperHyperpath.
2. By the Acyclicity Condition, P forms a simple path with no repeated supervertices in the sequence of superedges. Therefore, P satisfies the Acyclic Condition of a SuperHyperpath.

3. By the Connectedness Condition, P ensures that for any pair of vertices $u, v \in V(P)$, there exists a sequence of superedges $e_1, e_2, \ldots, e_k$ connecting u and v. Hence, P satisfies the Endpoint Condition of a SuperHyperpath.

Since P satisfies all the conditions of a SuperHyperpath, P is a SuperHyperpath.

From parts (1) and (2), we conclude that a SuperHyperpath is a SuperHyperTree if and only if it satisfies the conditions of a SuperHyperTree. □

Proposition 2.17. *A SuperHyperTree generalizes a Hypertree.*

Proof. Let $H = (V, E)$ be a hypergraph that satisfies the conditions of a Hypertree:

- There exists a host tree $T = (V_T, E_T)$, where each vertex $t \in V_T$ is associated with a bag $B_t \subseteq V$, and each hyperedge $e \in E$ is contained within at least one bag B_t, i.e., $e \subseteq B_t$ for some $t \in V_T$.
- The connectivity condition ensures that for any $v \in V$, the set of tree nodes $t \in V_T$ where $v \in B_t$ forms a connected subtree of T.
- The host tree T is acyclic.

Consider a SuperHyperTree SHT $= (V_{\mathrm{SHT}}, E_{\mathrm{SHT}})$, where $V_{\mathrm{SHT}} \subseteq \mathcal{P}(V)$ (the power set of the vertices of H), and $E_{\mathrm{SHT}} \subseteq \mathcal{P}(V_{\mathrm{SHT}})$. By definition, a SuperHyperTree satisfies:

- There exists a host tree T_{SHT} such that each superedge $e \in E_{\mathrm{SHT}}$ corresponds to a connected subtree of T_{SHT}.
- The host tree T_{SHT} is acyclic.
- The connectivity condition ensures that any two vertices $v, w \in V_{\mathrm{SHT}}$ can be connected through a sequence of superedges.

To show that a Hypertree H is a special case of a SuperHyperTree SHT:

1. Let each vertex $v \in V$ of the Hypertree H correspond to a singleton set $\{v\} \in V_{\mathrm{SHT}}$, making $V_{\mathrm{SHT}} = \{\{v\} \mid v \in V\}$.
2. Let each hyperedge $e \in E$ of H correspond to a superedge $e_{\mathrm{SHT}} \in E_{\mathrm{SHT}}$, where $e_{\mathrm{SHT}} = \{\{v\} \mid v \in e\}$.

With this mapping:

- Each hyperedge $e \in E$ is contained within a bag B_t of the Hypertree H, and thus the corresponding superedge e_{SHT} forms a subtree in T_{SHT}.
- The connectivity condition and acyclicity of the Hypertree H directly translate to the SuperHyperTree SHT.

Therefore, a Hypertree H is a specific case of a SuperHyperTree SHT, where the supervertices of SHT are restricted to singleton sets. □

Proposition 2.18. *A SuperHyperPath generalizes a HyperPath.*

Proof. Let $H = (V, E)$ be a hypergraph and $P = (e_1, e_2, \ldots, e_k)$ be a hyperpath in H, satisfying:

- For all $1 \leq i \leq k-1$, $e_i \cap e_{i+1} \neq \emptyset$, ensuring that consecutive hyperedges share at least one vertex.
- $u \in e_1$ and $v \in e_k$, ensuring that P connects u to v.
- P does not form a cycle, ensuring acyclicity.

Consider a SuperHyperPath P_{SHT} in a SuperHyperGraph $\mathrm{SHG} = (V_{\mathrm{SHT}}, E_{\mathrm{SHT}})$, where:

- $V_{\mathrm{SHT}} \subseteq \mathcal{P}(V)$, and $E_{\mathrm{SHT}} \subseteq \mathcal{P}(V_{\mathrm{SHT}})$.
- $P_{\mathrm{SHT}} = (e_1, e_2, \ldots, e_k)$ satisfies:
  - $\bigcup e_i \cap \bigcup e_{i+1} \neq \emptyset$, ensuring that consecutive superedges share at least one vertex.
  - $u \in e_1$ and $v \in e_k$, where $u, v \in V_{\mathrm{SHT}}$, ensuring that P_{SHT} connects u to v.
  - P_{SHT} does not form a cycle.

To show that a HyperPath P is a special case of a SuperHyperPath P_{SHT}:

1. Let each vertex $v \in V$ correspond to a singleton set $\{v\} \in V_{\mathrm{SHT}}$.
2. Let each hyperedge $e_i \in P$ correspond to a superedge $e_i^{\mathrm{SHT}} \in E_{\mathrm{SHT}}$, where $e_i^{\mathrm{SHT}} = \{\{v\} \mid v \in e_i\}$.

This mapping preserves:

- The adjacency condition, as $e_i \cap e_{i+1} \neq \emptyset$ in P implies $\bigcup e_i^{\mathrm{SHT}} \cap \bigcup e_{i+1}^{\mathrm{SHT}} \neq \emptyset$ in P_{SHT}.
- The endpoint condition, as $u \in e_1$ and $v \in e_k$ in P correspond to $u \in e_1^{\mathrm{SHT}}$ and $v \in e_k^{\mathrm{SHT}}$ in P_{SHT}.
- The acyclicity condition.

Thus, a HyperPath P is a specific case of a SuperHyperPath P_{SHT}, where the supervertices are singleton sets. □

2.4 SuperHyperTree-width

SuperHyperTree-width is an abstraction of Hypertree-width, extending the concept of Tree-width. The definition is presented as follows [39].

Definition 2.19 (SuperHyperTree Decomposition and SuperHyperTree-width)**.** [39] Let $\mathrm{SHT} = (V, E)$ be a SuperHyperGraph. A *SuperHyperTree decomposition* of SHT is a tuple $(T, \mathcal{B}, C)$, where:

- $T = (V_T, E_T)$ is a tree.
- $\mathcal{B} = \{B_t \mid t \in V_T\}$, a collection of subsets of V (called *bags*), satisfying:
  1. For every superedge $e \in E$, there exists a node $t \in V_T$ such that $e \subseteq B_t$.
  2. For every vertex $v \in V$, the set $\{t \in V_T \mid v \in B_t\}$ induces a connected subtree of T.
- $C = \{C_t \mid t \in V_T\}$, a collection of subsets of E (called *guards*), such that:
  1. For every node $t \in V_T$, $B_t \subseteq \bigcup C_t$, where $\bigcup C_t = \{v \in V \mid \exists e \in C_t \text{ such that } v \in e\}$.
  2. For every node $t \in V_T$, $(\bigcup C_t) \cap \bigcup_{u \in T_t} B_u \subseteq B_t$, where T_t is the subtree of T rooted at t.

The *width* of a SuperHyperTree decomposition $(T, \mathcal{B}, C)$ is defined as:

$$\mathrm{width}(T, \mathcal{B}, C) = \max_{t \in V_T} |C_t|.$$

The *SuperHyperTree-width* of SHT, denoted SHT-width(SHT), is the minimum width over all possible SuperHyperTree decompositions:

$$\text{SHT-width}(\mathrm{SHT}) = \min_{(T, \mathcal{B}, C)} \mathrm{width}(T, \mathcal{B}, C).$$

2.5 Basic Definition of Algorithm

The basic definitions related to the algorithms described in the Results section are provided here. Readers may refer to the Lecture Notes or the Introduction for additional details as needed [24, 31, 99].

Definition 2.20. (cf. [85, 99]) The *Total Time Complexity* of an algorithm is defined as the sum of the time required to execute each step of the algorithm, expressed as a function of the input size. If an algorithm involves multiple steps or operations, the total time complexity is determined by the maximum time required for the most time-consuming operation.

Formally, let $T(n, m)$ be the time complexity as a function of input sizes n and m. The total time complexity is:

$$T(n, m) = \max(T_{\text{operation1}}(n, m), T_{\text{operation2}}(n, m), \ldots, T_{\text{operationk}}(n, m)),$$

where n is the size of the set of propositions and m is the size or complexity of the context.

Definition 2.21. (cf. [85, 99]) The *Space Complexity* of an algorithm is the total amount of memory required to execute the algorithm, expressed as a function of the input size. This includes:

- The *input space*, which depends on the size of the input n, m,
- The *auxiliary space*, which includes temporary variables, data structures, or storage used during computation.

Formally, let $S(n, m)$ be the space complexity as a function of input sizes n and m. The total space complexity is:

$$S(n, m) = S_{\text{input}}(n, m) + S_{\text{auxiliary}}(n, m).$$

Definition 2.22. (cf. [85, 99]) *Big-O notation* is a mathematical concept used to describe the upper bound of the time or space complexity of an algorithm. Let $f(n)$ and $g(n)$ be functions that map non-negative integers to non-negative real numbers. We say:

$$f(n) \in O(g(n))$$

if there exist constants $c > 0$ and $n_0 \geq 0$ such that:

$$f(n) \leq c \cdot g(n) \quad \text{for all } n \geq n_0.$$

Definition 2.23 (NP-hard)**.** (cf. [63, 120]) A decision problem is classified as *NP-hard* if every problem in the class NP can be reduced to it in polynomial time. Formally, a problem P is NP-hard if there exists a polynomial-time reduction from any problem $Q \in \text{NP}$ to P, such that solving P allows the solution of Q. Importantly, NP-hard problems may not necessarily belong to the class NP, as they are not required to have a solution verifiable in polynomial time.

3 Results in this Paper

The results of this paper are presented below.

3.1 Exact Construct SuperHyperGraph

We consider about the algorithm of Constructing SuperHyperGraph. The problem considered in this subsection is described below.

Problem 3.1. Given a finite set of vertices V and a collection of subsets of V, construct a valid SuperHyperGraph $H = (V, E)$ where E is the set of superedges.

The algorithm and related theorems for the above problem are presented below.

Algorithm 1: Construct SuperHyperGraph

Input: A set $V = \{v_1, v_2, \ldots, v_n\}$ and a collection $C \subseteq \mathcal{P}(V)$.
Output: A SuperHyperGraph $H = (V, E)$.

Initialize $E \leftarrow \emptyset$;
foreach *subset* $S \in C$ **do**
  if $S \neq \emptyset$ **then**
    Add S to E;
return $H = (V, E)$;

Theorem 3.2. *The algorithm constructs a valid SuperHyperGraph $H = (V, E)$ such that V and E satisfy the definition of a SuperHyperGraph.*

Proof. The algorithm iterates over the input collection C and includes each non-empty subset $S \in C$ into the set of superedges E. By definition:

- V remains unchanged and consists of the original set of vertices.
- $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$, as only non-empty subsets are added.

Thus, the output $H = (V, E)$ satisfies the definition of a SuperHyperGraph. □

Theorem 3.3. *The time complexity of the algorithm is $O(|C|)$, where $|C|$ is the size of the input collection of subsets.*

Proof. The algorithm processes each subset $S \in C$ exactly once. For each subset, the inclusion check and addition to E take constant time. Therefore, the overall time complexity is $O(|C|)$. □

Theorem 3.4. *The space complexity of the algorithm is $O(|C| + |V|)$.*

Proof. The algorithm requires space to store:

- The vertex set V, requiring $O(|V|)$ space.
- The collection of superedges E, which is derived from C and requires $O(|C|)$ space.

Thus, the total space complexity is $O(|C| + |V|)$. □

3.2 Algorithm: Recognizing a SuperHyperTree

In this subsection, we examine the algorithm for recognizing a SuperHyperTree. For example, in the context of hypergraphs, recognizing algorithms for Hypertrees have been developed [114]. Here, we extend this concept to the SuperHyperTree framework. The problem considered in this subsection is described below.

Problem 3.5. Given a SuperHyperGraph $H = (V, E)$, determine whether H is a SuperHyperTree.

The algorithm and related theorems for the above problem are presented below.

Algorithm 2: Recognize SuperHyperTree

Input: A SuperHyperGraph $H = (V, E)$.
Output: `True` if H is a SuperHyperTree; `False` otherwise.

Construct a graph $T = (V, E_T)$, where E_T consists of edges connecting all vertices in each $e \in E$;
Check if T is acyclic;
if *T is not acyclic* **then**
  return `False`;
foreach $e \in E$ **do**
  Verify that e forms a connected subtree of T;
  if *e does not form a connected subtree* **then**
    return `False`;
Check that H satisfies the connectedness condition;
if *H does not satisfy the connectedness condition* **then**
  return `False`;
return `True`;

Theorem 3.6. *The algorithm correctly determines whether H is a SuperHyperTree.*

Proof. The algorithm verifies:

- The acyclicity of the host tree T.
- That each superedge $e \in E$ forms a connected subtree of T.
- That H satisfies the connectedness condition.

Since these are the defining properties of a SuperHyperTree, the algorithm is correct. □

Theorem 3.7. *The time complexity of the algorithm is $O(|V| + |E| \cdot |V|)$.*

Proof. Constructing the graph T requires $O(|V| + |E| \cdot |V|)$ time, as each superedge may connect multiple vertices. Checking acyclicity and verifying subtree conditions also require $O(|V| + |E| \cdot |V|)$ time. Thus, the overall time complexity is $O(|V| + |E| \cdot |V|)$. □

Theorem 3.8. *The space complexity of the algorithm is $O(|V| + |E|)$.*

Proof. The algorithm requires space to store the graph T and the original SuperHyperGraph H. Thus, the space complexity is $O(|V| + |E|)$. □

3.3 Computation of SuperHyperTree-width

This subsection examines the Algorithm for the Computation of SuperHyperTree-width. The problem considered in this subsection is described below.

Problem 3.9. The problem is to compute the exact SuperHyperTree-width of a given SuperHyperGraph $H = (V, E)$. This involves finding a SuperHyperTree decomposition $(\mathcal{T}, \mathcal{X})$ such that the width of the decomposition is minimized. The objective is to determine the smallest possible width, denoted as $w_{\min}$, satisfying the host tree, acyclicity, and connectedness conditions for all superhyperedges $e \in E$ within H.

3.3.1 Exact Algorithm for Computation of SuperHyperTree-width

The Exact Algorithm for the Computation of SuperHyperTree-width is introduced as follows.

Algorithm 3: Computation of SuperHyperTree-width

Input: SuperHyperGraph SHT $= (V, E)$
Output: SuperHyperTree-width $w_{\min}$ and the corresponding decomposition $(\mathcal{T}, \mathcal{X})$

```
Initialize a tree decomposition algorithm for SHT;
Set an initial upper bound for the tree-width: w_min ← ∞;
foreach possible tree decomposition (T, X) of SHT do
    Compute the width of the decomposition;
    if width of decomposition < w_min then
        Update w_min;
        Store the corresponding decomposition (T, X);
return w_min, (T, X);
```

Theorem 3.10. *The above algorithm correctly computes the SuperHyperTree-width of the given SuperHyper-Graph SHT.*

Proof. Completeness: The algorithm considers all possible tree decompositions over the vertex set V and evaluates each decomposition to ensure that it satisfies the necessary conditions for a valid SuperHyperTree-decomposition. The algorithm guarantees that it explores every possible decomposition that could potentially minimize the SuperHyperTree-width.

Soundness: By systematically testing all feasible decompositions and comparing their widths, the algorithm ensures that the computed SuperHyperTree-width is indeed the smallest possible value, thus providing an accurate result. □

Theorem 3.11 (Time Complexity)**.** *The time complexity of the approximation algorithm for computing the SuperHyperTree-width is $O(2^{|V|^2})$, where $|V|$ is the number of vertices in the SuperHyperGraph.*

Proof. Let $H = (V, E)$ be the given SuperHyperGraph with vertex set V and edge set E. The core of the algorithm involves enumerating and processing various tree decompositions of the graph, which correspond to subsets of V.

The total number of possible tree decompositions over V is exponential in the number of vertices. Specifically, we need to consider all possible ways to partition V into subsets, which is of the order of $O(2^{|V|^2})$. This arises from the fact that each pair of vertices can either be in the same subset or in different subsets, leading to an exponential number of possible combinations.

Since each decomposition requires checking the structural constraints of the SuperHyperGraph (i.e., the superedges), the time complexity for evaluating each decomposition is polynomial in $|V|$. However, the dominant factor is the number of decompositions, which grows exponentially as $2^{|V|^2}$.

Thus, the overall time complexity of the algorithm is $O(2^{|V|^2})$. □

Theorem 3.12 (Space Complexity)**.** *The space complexity of the approximation algorithm for computing the SuperHyperTree-width is $O(2^{|V|^2})$, where $|V|$ is the number of vertices in the SuperHyperGraph.*

Proof. The space complexity of the algorithm is determined by the storage requirements for enumerating and processing the possible tree decompositions of the graph.

Each tree decomposition is represented as a set of subsets of the vertex set V. The number of subsets of V is $O(2^{|V|})$, and each subset requires storing a list of vertices. Thus, the space required for each decomposition is proportional to $O(2^{|V|})$.

Moreover, the algorithm needs to store all possible decompositions, which can be of the order $O(2^{|V|^2})$, as explained in the time complexity analysis. Therefore, the total space required is dominated by the number of decompositions, leading to a space complexity of $O(2^{|V|^2})$.

Thus, the overall space complexity is $O(2^{|V|^2})$. □

3.3.2 Approximation Algorithm for SuperHyperTree-width

An approximation algorithm provides near-optimal solutions to computationally hard problems within a provable error bound, ensuring efficiency and feasibility. Given a SuperHyperGraph $H = (V, E)$, we present an approximation algorithm for computing its SuperHyperTree-width. The algorithm is designed to run in polynomial time and approximates the SuperHyperTree-width within a factor of $O(\log |V|)$.

Algorithm 4: Approximation Algorithm for SuperHyperTree-width

Input: SuperHyperGraph $H = (V, E)$
Output: Approximation of the SuperHyperTree-width

Initialize an empty tree decomposition T;
Construct an initial decomposition using a greedy approach:;
foreach *superedge* $e_i \in E$ **do**
    Select the smallest superedge e_i based on the number of vertices connected;
    Add e_i to the decomposition T;
Perform a greedy refinement:;
foreach *superedge* e_i *in* T **do**
    Try to connect e_i to existing parts of the decomposition with minimal additional width;
    if *improvement is found* **then**
        Adjust the decomposition;
Terminate the algorithm when no further improvements are possible;
return the SuperHyperTree-width of the final decomposition;

Theorem 3.13 (Correctness of the Approximation Algorithm)**.** *The algorithm correctly computes an approximation of the SuperHyperTree-width of the SuperHyperGraph H.*

Proof. The correctness of the algorithm follows from the fact that:

1. The greedy tree decomposition approach used in the algorithm always provides a decomposition whose treewidth is within a constant factor of the optimal treewidth.

2. The hypertree-width of a SuperHyperGraph is bounded above by the treewidth of any valid tree decomposition, and the algorithm uses such a decomposition to compute an approximation.

3. Therefore, the approximation computed by the algorithm will always be a valid approximation to the true SuperHyperTree-width. □

Theorem 3.14 (Time Complexity)**.** *Let $n = |V|$ be the number of vertices and $m = |E|$ be the number of hyperedges in the SuperHyperGraph. The time complexity of the approximation algorithm is $O(n^2)$.*

Proof. The time complexity is dominated by the steps where we construct the tree decomposition and compute the hypertree-width approximation.

1. Constructing a tree decomposition with treewidth approximation τ can be done in $O(n^2)$ time using greedy algorithms or dynamic programming techniques for tree decomposition.

2. Computing the hypertree-width from the tree decomposition requires linear time in terms of the size of the decomposition, which is at most $O(n^2)$.

Therefore, the overall time complexity is $O(n^2)$. □

Theorem 3.15 (Space Complexity)**.** *Let $n = |V|$ be the number of vertices and $m = |E|$ be the number of hyperedges in the SuperHyperGraph. The space complexity of the approximation algorithm is $O(n^2)$.*

Proof. The space required to store the SuperHyperGraph $H = (V, E)$ is $O(m)$, where m is the number of hyperedges. Additionally, the space required to store the tree decomposition and related data structures is $O(n^2)$, since the decomposition involves a set of nodes and edges that depend on the number of vertices and hyperedges in the graph. Thus, the overall space complexity is $O(n^2)$. □

Theorem 3.16 (Approximation Bound)**.** *Let SuperHyperTree-width(H) denote the true SuperHyperTree-width of H, and τ the approximation computed by the algorithm. Then, the algorithm computes an approximation such that:*

$$\textit{SuperHyperTree-width}(H) \leq \tau \leq \beta \cdot \textit{SuperHyperTree-width}(H),$$

where β is a constant factor depending on the quality of the greedy heuristic used in the tree decomposition algorithm (e.g., $\beta = 2$ for simple greedy heuristics).

Proof. The approximation bound follows from the properties of the greedy tree decomposition algorithm. In particular, if the tree decomposition has treewidth τ, the hypertree-width of the SuperHyperGraph is at most τ. Since the greedy heuristic for tree decomposition guarantees that the treewidth is within a constant factor β of the optimal treewidth, the approximation τ computed by the algorithm will be within a factor of β of the true SuperHyperTree-width. □

3.4 Superhypergraph Partition Problem

We consider the well-known Graph Partition Problem. The Graph Partition Problem aims to divide the vertices of a graph into disjoint subsets while minimizing edge cuts or maximizing intra-group connectivity [33,71,100]. The Hypergraph Partition Problem generalizes this by partitioning the vertices of a hypergraph into subsets while minimizing hyperedge cuts, where hyperedges connect multiple vertices [34,69,98,113]. These problems are further extended in this study to the context of superhypergraphs. The problem considered in this subsection is described below.

Problem 3.17 (Superhypergraph Partition Problem)**.** Given a superhypergraph $H = (V, E)$ and an integer $k \geq 2$, partition V into k disjoint subsets $V_1, V_2, \ldots, V_k$ such that:

- Balance Constraint: For each i, $|V_i| \leq (1 + \epsilon) \left\lceil \frac{|V|}{k} \right\rceil$ for some small $\epsilon > 0$.
- Cut Minimization: The number of hyperedges that are *cut* is minimized. A hyperedge is said to be *cut* if it contains vertices from more than one partition.

The algorithm for solving this problem is presented below.

Remark 3.18. The algorithm consists of the following steps:

1. *Coarsening Phase*: Reduce the size of the superhypergraph by collapsing vertices and hyperedges to create a hierarchy of smaller superhypergraphs.
   - *Matching*: Pair vertices based on some similarity metric (e.g., the number of shared hyperedges).
   - *Aggregation*: Merge matched vertices to form supervertices.
   - *Hyperedge Reduction*: Adjust hyperedges to reflect the new supervertices.
2. *Initial Partitioning*: Use a simple partitioning algorithm (e.g., greedy assignment) on the coarsest superhypergraph H_l.

3. *Uncoarsening Phase*: Project the partition back onto the original superhypergraph, refining the partition at each level to improve the cut size and balance. In the algorithm, At each level i, project the partition P_{i+1} onto H_i and refine it using a local optimization method (e.g., Kernighan–Lin algorithm adapted for superhypergraphs).

Algorithm 5: Superhypergraph Partitioning Algorithm

Input: Superhypergraph $H = (V, E)$, number of partitions $k \geq 2$
Output: Partition $V_1, V_2, \ldots, V_k$ of V

$H_0 \leftarrow H$;
$l \leftarrow 0$;
while *Size of H_l is greater than threshold* **do**
  $H_{l+1} \leftarrow \text{Coarsen}(H_l)$;
  $l \leftarrow l + 1$;
$P_l \leftarrow \text{InitialPartition}(H_l, k)$;
for $i \leftarrow l - 1$ *down to* 0 **do**
  $P_i \leftarrow \text{Refine}(H_i, P_{i+1})$;
return P_0;

Theorem 3.19. *The algorithm produces a valid partition of the vertex set V into k disjoint subsets $V_1, V_2, \ldots, V_k$ that satisfy the balance constraint.*

Proof. The algorithm maintains the balance constraint at each level:

- During initial partitioning, the algorithm assigns vertices to partitions such that the balance constraint is satisfied.
- During uncoarsening and refinement, vertices are moved between partitions only if the balance constraint is not violated.

Since the coarsening and uncoarsening processes are designed to preserve the structure of the original superhypergraph, the final partition P_0 is valid and satisfies the balance constraint. □

Theorem 3.20. *The time complexity of the algorithm is $O(|E| \log |V|)$.*

Proof. The algorithm consists of multiple phases:

- *Coarsening Phase*:
  - Each level reduces the number of vertices by a constant factor.
  - The number of levels is $O(\log |V|)$.
  - At each level, matching and aggregation can be done in $O(|E|)$ time.
- *Initial Partitioning*:
  - The coarsest superhypergraph has significantly fewer vertices.
  - Partitioning can be done in $O(1)$ time relative to the original graph size.
- *Uncoarsening and Refinement*:
  - At each level, refinement operations (e.g., swapping vertices between partitions) can be performed in $O(|E|)$ time.
  - There are $O(\log |V|)$ levels.

Therefore, the total time complexity is $O(|E| \log |V|)$. □

Theorem 3.21. *The space complexity of the algorithm is $O(|V| + |E|)$.*

Proof. At each level, the algorithm stores:

- The superhypergraph H_i, which has at most $|V|$ vertices and $|E|$ hyperedges.
- The partition P_i, which is a mapping from vertices to partition indices.

Since the size of H_i decreases with each level, the total space required is dominated by the original superhypergraph H, requiring $O(|V| + |E|)$ space. □

Theorem 3.22. *The SuperHypergraph Partition Problem is NP-hard.*

Proof. We reduce the Graph Partition Problem (GPP), which is known to be NP-hard, to the SuperHypergraph Partition Problem (SHGP). Let $G = (V_G, E_G)$ be an instance of GPP. Construct a superhypergraph SHG = (V, E) as follows:

- For each vertex $v \in V_G$, create a supervertex $\{v\} \in V$.
- For each edge $(u, v) \in E_G$, create a superhyperedge $\{\{u\}, \{v\}\} \in E$.

Any solution to SHGP corresponds to a partition of V_G that minimizes edge cuts in G. Thus, SHGP is at least as hard as GPP, proving its NP-hardness. □

3.5 Reachability Problem in Superhypergraph

The Reachability Problem determines whether there exists a path between two vertices in a graph or hypergraph using its edges [7, 8]. The problem considered in this subsection is described below.

Definition 3.23. Given a Superhypergraph $H = (V, E)$, where V is a finite set of vertices and E is a set of hyperedges, the Reachability Problem asks whether there exists a path from a vertex $v_1 \in V$ to a vertex $v_2 \in V$ using hyperedges in E.

The algorithm for solving the Reachability problem follows a depth-first search (DFS) [115] strategy adapted for hypergraphs. In a traditional DFS, we explore vertices by following edges. For a Superhypergraph, a hyperedge can connect any number of vertices, and we need to modify the DFS to consider all vertices that a hyperedge connects.

Algorithm 6: Reachability in Superhypergraph

Input: Superhypergraph $H = (V, E)$, vertices $v_1, v_2 \in V$

Output: `True` if there is a path from v_1 to v_2, `False` otherwise

```
Initialize a set visited ← ∅;
Call DFS(v1, v2, H, visited);
if DFS returns True then
    return True;
else
    return False;
```

Theorem 3.24. *The algorithm explores all reachable vertices from v_1 by traversing through all hyperedges in H. Since each hyperedge connects a set of vertices, DFS ensures that if there is a path from v_1 to v_2, it will be discovered.*

Proof. If there is a path from v_1 to v_2, DFS will eventually visit v_2 by following the hyperedges. If v_2 is not reachable, the algorithm will not visit v_2, and the function will return `False`. Thus, the algorithm is correct. □

Theorem 3.25. *The time complexity is* $O(|V| + |E|)$.

Proof. In the worst case, the algorithm needs to explore all vertices and all hyperedges in the graph. Since there are $|V|$ vertices and $|E|$ hyperedges, the time complexity is $O(|V| + |E|)$. □

Theorem 3.26. *The Space Complexity is* $O(|V| + |E|)$.

Proof. The space complexity is dominated by the storage of the visited set and the recursion stack in the DFS. Therefore, the space complexity is $O(|V|)$. □

3.6 Minimum Spanning SuperHypertree Problem

The Minimum Spanning Tree Problem identifies a tree connecting all graph vertices with the minimum total edge weight [4, 57, 124]. The Minimum Spanning Hypertree Problem extends this concept to hypergraphs, seeking a tree-like structure minimizing hyperedge weights [118]. In this subsection, we examine the Minimum Spanning SuperHypertree Problem. The problem considered is described below.

Definition 3.27 (Weighted SuperHyperGraph)**.** Let V be a finite set of vertices. A *weighted superhypergraph* is an ordered pair $H = (V, E)$, where:

- $V \subseteq \mathcal{P}(V)$, the power set of V, meaning that each element of V can be either a single vertex or a subset of vertices (called a *supervertex*).
- E is a set of *superhyperedges*, where each $e \in E$ is a non-empty subset of V (i.e., $e \subseteq V$ and $e \neq \emptyset$).
- Each superhyperedge $e \in E$ has an associated positive weight $w(e) \in \mathbb{R}^+$.

Definition 3.28 (Minimum Spanning SuperHypertree Problem)**.** Given a weighted superhypergraph $H = (V, E)$, the *Minimum Spanning SuperHypertree (MSST)* is a subgraph $T = (V, E_T)$ satisfying:

1. $E_T \subseteq E$.
2. T is a *superhypertree*, meaning it satisfies the conditions of a SuperHyperTree as defined below.
3. The total weight $w(T) = \sum_{e \in E_T} w(e)$ is minimized among all possible superhypertrees of H.

We propose an algorithm inspired by Kruskal's algorithm [18] for finding a Minimum Spanning Tree (MST) in graphs. The algorithm operates as follows:

Algorithm 7: Minimum Spanning SuperHypertree Algorithm

Input: A weighted superhypergraph $H = (V, E)$
Output: A Minimum Spanning SuperHypertree $T = (V, E_T)$

Initialize the edge set of the superhypertree: $E_T \leftarrow \emptyset$;
Sort the superhyperedges E in non-decreasing order of weight;
Initialize a disjoint-set data structure $\mathcal{D}$ for the vertices in V;
foreach *superhyperedge* $e \in E$ *(in sorted order)* **do**
    Compute the union of sets containing vertices in e:
    $C_e \leftarrow \bigcup_{v \in e} \text{Find}(v)$;
    if $|C_e| > 1$ **then**
        Add e to E_T;
        Update the disjoint-set structure by performing:
        Union(Find(v)) for all $v \in e$;
Construct the Minimum Spanning SuperHypertree: $T \leftarrow (V, E_T)$;
return T;

Theorem 3.29. *The algorithm correctly finds a Minimum Spanning SuperHypertree of the weighted superhypergraph H.*

Proof. We need to show that:

1. The algorithm produces a superhypertree.
2. The superhypertree is spanning and minimal in total weight.

1. Produces a SuperHypertree:

Acyclicity: The algorithm only adds superhyperedges that connect disjoint components, ensuring no cycles are formed.

Connectivity: By uniting components whenever a superhyperedge is added, eventually all vertices become connected.

2. Minimal Total Weight:

The algorithm always chooses the smallest available superhyperedge that does not create a cycle, similar to Kruskal's algorithm.

Suppose there exists another superhypertree T' with a smaller total weight. Then, there must be at least one superhyperedge e in T' not in T with weight less than or equal to the heaviest superhyperedge in T.

Replacing superhyperedges in T with those in T' cannot lead to a total weight less than T without violating acyclicity or connectivity.Therefore, T is minimal. □

Theorem 3.30. *The time complexity of the algorithm is $O(|E|\log|E| + |E| \cdot \alpha(|V|))$, where α is the inverse Ackermann function.*

Proof. The algorithm can be divided into the following steps:

- *Sorting the Superhyperedges:*
  - Sorting $|E|$ superhyperedges by weight takes $O(|E|\log|E|)$ time.
- *Processing Each Superhyperedge:*
  - For each superhyperedge e, the following operations are performed:
    - *Find Operations:* For each vertex $v \in e$, a $\mathrm{Find}(v)$ operation is performed. The total number of find operations is $O\left(\sum_{e\in E}|e|\right)$, which is proportional to the size of all superhyperedges combined.
    - *Union Operation:* If e is added to the spanning superhypertree, a union operation is performed. The number of union operations is at most $|E_T| \leq |V| - 1$, where $|E_T|$ is the number of edges in the final superhypertree.
  - Each Union-Find operation (find or union) takes $O(\alpha(|V|))$ time.

Assuming the total size of all superhyperedges is $\sum_{e\in E}|e| = O(|E| \cdot m)$, where m is the maximum size of a superhyperedge:

- The time complexity for Union-Find operations becomes $O(|E| \cdot m \cdot \alpha(|V|))$.

However, in practical cases:

- The Union-Find operations are dominated by $O(|E| \cdot \alpha(|V|))$, as $\alpha(|V|)$ grows extremely slowly, and m (the size of superhyperedges) is typically much smaller than $|V|$.

Thus, the overall time complexity of the algorithm is:

$$O(|E| \log |E| + |E| \cdot \alpha(|V|)).$$

□

Theorem 3.31. *The space complexity of the algorithm is $O(|V| + |E|)$.*

Proof. The space requirements for the algorithm are as follows:

- *Storage for Vertices V:* $O(|V|)$ space is required to store the vertex set.
- *Storage for Superhyperedges E:* $O(|E|)$ space is required to store the list of superhyperedges.
- *Union-Find Data Structure:* $O(|V|)$ space is required to store parent and rank arrays for the Union-Find operations.
- *Edge List of the Spanning SuperHypertree E_T:* $O(|E|)$ space is required in the worst case, where all superhyperedges are included.

Combining these, the total space complexity is:

$$O(|V| + |E|).$$

□

Example 3.32. To illustrate the algorithm, consider the following weighted superhypergraph $H = (V, E)$:

- $V = \{v_1, v_2, v_3, v_4\}$.
- $E = \{e_1, e_2, e_3\}$, with:
  - $e_1 = \{v_1, v_2\}$, $w(e_1) = 1$.
  - $e_2 = \{v_2, v_3, v_4\}$, $w(e_2) = 2$.
  - $e_3 = \{v_1, v_4\}$, $w(e_3) = 3$.

Step-by-Step Execution of the Algorithm:

1. *Sort E*:
   - Sort the superhyperedges by weight: e_1, e_2, e_3.
2. *Initialize Union-Find Data Structure*:
   - Each vertex $v \in V$ is its own set.
3. *Process $e_1 = \{v_1, v_2\}$*:
   - $\text{Find}(v_1) \neq \text{Find}(v_2)$.
   - Add e_1 to E_T.
   - Union v_1 and v_2.
4. *Process $e_2 = \{v_2, v_3, v_4\}$*:
   - $\text{Find}(v_2) \neq \text{Find}(v_3)$ and $\text{Find}(v_2) \neq \text{Find}(v_4)$.

- Add e_2 to E_T.
- Union v_2, v_3, and v_4.

5. *Process* $e_3 = \{v_1, v_4\}$:
   - All vertices are now in the same connected component.
   - Adding e_3 would create a cycle, so e_3 is skipped.

The resulting Minimum Spanning SuperHypertree is:

$$T = (V, \{e_1, e_2\}),$$

with a total weight:

$$w(T) = 1 + 2 = 3.$$

This example demonstrates the step-by-step execution of the algorithm for finding the Minimum Spanning SuperHypertree in a weighted superhypergraph. The analysis confirms the correctness and efficiency of the algorithm, with a time complexity of $O(|E| \log |E| + |E| \cdot \alpha(|V|))$ and a space complexity of $O(|V| + |E|)$.

3.7 Single-Source Shortest Path Problem in a SuperHypergraph

The Single-Source Shortest Path Problem [77,84,88] in a SuperHypergraph involves finding the shortest paths from a source vertex to all other vertices, minimizing the total weight of traversed superhyperedges. The problem considered is described below.

Problem 3.33. Given a weighted superhypergraph SHG $= (V, E)$ and a source supervertex $s \in V$, the *Single-Source Shortest Path Problem* seeks to find the shortest superhyperpaths from s to all other supervertices $v \in V$, minimizing the total weight.

The algorithm and related theorems for the above problem are presented below.

Algorithm 8: Single-Source Shortest Path in SuperHypergraph

Input: A weighted superhypergraph SHG $= (V, E)$, source supervertex $s \in V$
Output: Shortest path distances $d(v)$ from s to each $v \in V$

Initialize $d(v) \leftarrow \infty$ for all $v \in V$; set $d(s) \leftarrow 0$
Initialize priority queue $Q \leftarrow \{(s, d(s))\}$
while *Q is not empty* **do**
    Extract supervertex u with minimal $d(u)$ from Q
    foreach *superhyperedge $e \in E$ such that $u \in e$* **do**
        foreach *supervertex $v \in e$* **do**
            if $d(v) > d(u) + w(e)$ **then**
                $d(v) \leftarrow d(u) + w(e)$
                Insert or update v in Q with priority $d(v)$
return $d(v)$ for all $v \in V$

Theorem 3.34. *Algorithm 8 correctly computes the shortest path distances from the source supervertex s to every other supervertex $v \in V$ in SHG.*

Proof. We prove the correctness of the algorithm by induction on the number of supervertices whose shortest path distances from s have been finalized.

Base Case: Initially, only s has a finalized distance $d(s) = 0$, which is correct.

Inductive Step: Assume that the distances $d(u)$ are correct for all supervertices whose shortest paths have been found. When we extract the supervertex u with the minimal tentative distance $d(u)$ from the priority queue, we know that $d(u)$ is the shortest possible distance from s to u. This is because all weights $w(e)$ are positive, and any alternative path to u via other supervertices would have a total weight at least $d(u)$ or greater.

For each superhyperedge e containing u, we consider all supervertices $v \in e$. If the path from s to v via e and u offers a shorter distance than the current $d(v)$, we update $d(v)$ accordingly. Since we consider all such superhyperedges and supervertices, we ensure that the shortest distances are propagated throughout the superhypergraph.

By continuously selecting the supervertex with the minimal tentative distance and updating distances of adjacent supervertices, we guarantee that once a supervertex u is extracted from Q, $d(u)$ is indeed the shortest distance from s to u.

Therefore, by induction, the algorithm correctly computes the shortest path distances from s to all supervertices in V. □

Theorem 3.35. *The time complexity of the algorithm is* $O(|E| \cdot m \cdot \log |V|)$*, where m is the maximum number of supervertices in a superhyperedge.*

Proof. The algorithm processes each superhyperedge $e \in E$ for each supervertex u extracted from the priority queue Q. For each superhyperedge e containing u, we examine all supervertices $v \in e$, resulting in up to m operations per superhyperedge.

The number of times we extract a supervertex from Q is $O(|V|)$. For each extraction, we may perform $O(m \cdot \deg(u))$ operations, where $\deg(u)$ is the number of superhyperedges containing u.

Assuming $\deg(u) \leq |E|$, the total number of operations is $O(|V| \cdot |E| \cdot m)$. Each priority queue operation (insert or extract) takes $O(\log |V|)$ time.

Therefore, the overall time complexity is $O(|V| \cdot |E| \cdot m \cdot \log |V|)$. However, since $|V| \leq |E| \cdot m$, we can simplify the time complexity to $O(|E| \cdot m \cdot \log |V|)$. □

Theorem 3.36. *The space complexity of the algorithm is* $O(|V| + |E| \cdot m)$.

Proof. The space requirements are as follows:

- Distance array $d(v)$ for each $v \in V$: $O(|V|)$.
- Priority queue Q: at most $O(|V|)$ elements.
- Storage of the superhypergraph structure: $O(|E| \cdot m)$, since each superhyperedge can contain up to m supervertices.

Thus, the total space complexity is $O(|V| + |E| \cdot m)$. □

3.8 Traveling Salesman Problem in a SuperHypergraph

The Traveling Salesman Problem [29, 73, 87] in a SuperHypergraph seeks a minimum-weight tour visiting all supervertices exactly once, considering superhyperedges' weights, and returning to the starting supervertex while ensuring connectivity constraints. The problem considered is described below.

Problem 3.37. Given a weighted superhypergraph SHG $= (V, E)$, the *Traveling Salesman Problem (TSP)* seeks a minimal-weight closed superhyperpath that visits each supervertex $v \in V$ at least once.

Theorem 3.38. *The Traveling Salesman Problem in a superhypergraph is NP-hard.*

Proof. We prove this by reduction from the classical TSP in graphs, which is known to be NP-hard. Given an instance of the TSP in a graph $G = (V_G, E_G)$, we construct a corresponding superhypergraph SHG $= (V, E)$ as follows:

- For each vertex $v \in V_G$, create a supervertex $v \in V$ in the superhypergraph.
- For each edge $(u, v) \in E_G$, create a superhyperedge $e = \{u, v\} \in E$ with the same weight as the edge in G.

In this construction, the superhypergraph essentially mirrors the original graph. Therefore, any solution to the TSP in SHG corresponds to a solution in G, and vice versa.

Since the TSP in graphs is NP-hard, and we can polynomially reduce any instance of the TSP in graphs to an instance in superhypergraphs, it follows that the TSP in superhypergraphs is also NP-hard. □

3.9 Chinese Postman Problem (CPP) in superhypergraph

The Chinese Postman Problem (CPP) [30, 79, 117] in a superhypergraph seeks a minimum-weight closed walk traversing all superhyperedges at least once, ensuring efficiency in traversal. The problem considered is described below.

Problem 3.39. Given a weighted superhypergraph SHG $= (V, E)$, the *Chinese Postman Problem (CPP)* seeks a minimal-weight closed superhyperpath that traverses every superhyperedge at least once.

The algorithm and related theorems for the above problem are presented below.

Algorithm 9: Chinese Postman Problem in SuperHypergraph

Input: A weighted superhypergraph SHG $= (V, E)$

Output: A minimal-weight closed superhyperpath traversing every superhyperedge at least once

Initialization:

  Construct an incidence multigraph $G = (V', E')$, where:
  - $V' = V$, the vertices of SHG,
  - E' is formed by replacing each superhyperedge $e \in E$ with all edges between pairs of supervertices in e.

Degree Calculation:

  Compute the degree $\deg(v)$ of each vertex $v \in V'$ in G.
  Identify the set $O \subseteq V'$ of vertices with odd degree in G.

Shortest Path Computation:

  Compute the shortest paths between all pairs of vertices in O using the Floyd-Warshall algorithm.

Perfect Matching:

  Find a minimum-weight perfect matching M on O based on the shortest path distances.

Graph Augmentation:

  Augment G by adding the edges from the matching M.

Eulerian Circuit and Transformation:

  Find an Eulerian circuit C in the augmented graph G.
  Transform the Eulerian circuit C back into a superhyperpath in SHG.

return The minimal-weight closed superhyperpath corresponding to the Eulerian circuit C.

Theorem 3.40. *Algorithm 9 finds a minimal-weight closed superhyperpath that traverses every superhyperedge at least once in SHG.*

Proof. The correctness of the algorithm relies on the properties of Eulerian circuits and the transformation between the superhypergraph and the incidence multigraph.

Construction of G: By replacing each superhyperedge e with edges between every pair of supervertices in e, we create a multigraph G that captures the connectivity of SHG.

Eulerian Circuit Existence: In G, we compute the degrees of all vertices. By adding edges (paths) between pairs of vertices with odd degrees to make all degrees even (through the minimum weight perfect matching M), we ensure that the augmented graph G is Eulerian.

Minimum Weight Matching: The matching M is chosen to minimize the additional weight added to G, which corresponds to minimizing the total traversal cost in SHG.

Eulerian Circuit and Superhyperpath Correspondence: An Eulerian circuit in G traverses every edge at least once. Since edges in G correspond to superhyperedges or shortest paths between supervertices in SHG, the circuit can be mapped back to a closed superhyperpath in SHG that traverses every superhyperedge at least once.

Optimality: The algorithm constructs a circuit with minimal total weight because it only adds the minimum necessary edges (with minimal total weight) to make the graph Eulerian. Therefore, the resulting superhyperpath is of minimal weight. □

Theorem 3.41. *The algorithm runs in polynomial time, specifically $O(|V|^3)$.*

Proof. The time complexity of each step is as follows:

- *Constructing G:* Replacing each superhyperedge e with edges between every pair of supervertices in e can be done in $O(|E| \cdot m^2)$, where m is the maximum size of a superhyperedge. Since $m \leq |V|$, this step is $O(|E| \cdot |V|^2)$.
- *Computing Degrees:* Calculated in $O(|V|)$.
- *Identifying Odd-Degree Vertices O:* $O(|V|)$.
- *Computing Shortest Paths:* Using the Floyd-Warshall algorithm(cf. [5, 65]), this takes $O(|V|^3)$.
- *Minimum Weight Perfect Matching(cf. [25]):* Can be found in $O(|V|^3)$ using algorithms such as the Hungarian method.
- *Finding Eulerian Circuit:* Linear in the number of edges, $O(|E'|)$.
- *Transforming Circuit Back to SHG:* $O(|E'|)$.

The dominant terms are the shortest path computation and the matching, both $O(|V|^3)$. Therefore, the overall time complexity is $O(|V|^3)$. □

3.10 Longest Simple Path Problem in SuperHypergraphs

The Longest Simple Path Problem in SuperHypergraphs involves finding a maximum-length path that visits each supervertex at most once, satisfying adjacency constraints defined by the superhyperedges [86]. The problem considered in this subsection is described below.

Problem 3.42. Given a superhypergraph SHG $= (V, E)$ and two supervertices $u, v \in V$, the *Longest Simple Path Problem* seeks the longest superhyperpath connecting u and v, where the path is defined in terms of the number of superhyperedges.

Theorem 3.43. *The Longest Simple Path Problem in SuperHypergraphs is NP-hard.*

Proof. We reduce the classical Longest Path Problem (LPP) in graphs, known to be NP-hard, to the Longest Simple Path Problem in superhypergraphs. Let $G = (V_G, E_G)$ be an instance of LPP. Construct a superhypergraph SHG $= (V, E)$ as follows:

- For each vertex $v \in V_G$, create a supervertex $\{v\} \in V$.
- For each edge $(u, v) \in E_G$, create a superhyperedge $\{\{u\}, \{v\}\} \in E$.

A simple path in G corresponds to a superhyperpath in SHG. Therefore, solving the Longest Simple Path Problem in SHG provides a solution to LPP in G. Since LPP is NP-hard, the Longest Simple Path Problem in superhypergraphs is also NP-hard. □

3.11 Maximum Spanning Tree Problem in SuperHypergraphs

The Maximum Spanning Tree Problem (cf. [47, 76, 81]) in SuperHypergraphs seeks a superhypertree that connects all supervertices while maximizing the total weight of selected superhyperedges, maintaining tree-like structural constraints. The problem considered in this subsection is described below.

Problem 3.44. Given a weighted superhypergraph SHG $= (V, E)$, the *Maximum Spanning Tree Problem* seeks a superhypertree $T = (V, E_T)$ such that:

1. T satisfies the conditions of a superhypertree.
2. The total weight of T, defined as $w(T) = \sum_{e \in E_T} w(e)$, is maximized.

The algorithm and related theorems for the above problem are presented below.

Algorithm 10: Maximum Spanning SuperHypertree Algorithm

Input: A weighted superhypergraph SHG $= (V, E)$
Output: A Maximum Spanning SuperHypertree $T = (V, E_T)$

Initialize $E_T \leftarrow \emptyset$;
Sort the superhyperedges E in non-increasing order of weight;
Initialize a disjoint-set data structure for vertices in V;
foreach *superhyperedge $e \in E$ (in sorted order)* **do**
    Compute the union of sets containing vertices in e;
    if *adding e does not create a cycle* **then**
        Add e to E_T;
        Update the disjoint-set structure for all vertices in e;
return $T = (V, E_T)$;

Theorem 3.45. *The above algorithm correctly computes the Maximum Spanning SuperHypertree for a weighted superhypergraph SHG.*

Proof. The algorithm iteratively adds the heaviest superhyperedge e to E_T while maintaining the superhypertree properties:

- *Acyclicity:* The disjoint-set structure ensures that no cycles are formed.
- *Connectedness:* By construction, the algorithm only terminates when E_T connects all supervertices in V.
- *Optimality:* At each step, the algorithm selects the heaviest available superhyperedge that maintains the superhypertree properties, ensuring the total weight $w(T)$ is maximized.

Thus, the algorithm is correct. □

Theorem 3.46. *The time complexity of the algorithm is $O(|E| \cdot \log |E| + |E| \cdot m \cdot \alpha(|V|))$, where m is the maximum size of a superhyperedge and α is the inverse Ackermann function. The space complexity is $O(|V| + |E|)$.*

Proof. Time Complexity:

- Sorting the superhyperedges takes $O(|E| \cdot \log |E|)$.
- Processing each superhyperedge involves:
  - m Find operations, each taking $O(\alpha(|V|))$.
  - At most one Union operation, taking $O(\alpha(|V|))$.

- Thus, processing all superhyperedges takes $O(|E| \cdot m \cdot \alpha(|V|))$.

The total time complexity is $O(|E| \cdot \log |E| + |E| \cdot m \cdot \alpha(|V|))$.

Space Complexity:

- Storing the disjoint-set structure requires $O(|V|)$.
- Storing the superhyperedges requires $O(|E|)$.

Thus, the space complexity is $O(|V| + |E|)$. □

3.12 Horn satisfiability problem in a Superhypergraph

The Horn Satisfiability Problem determines whether a Boolean formula in conjunctive normal form, with at most one positive literal per clause, is satisfiable(cf. [48]). In this subsection, we explore the Horn Satisfiability Problem in a SuperHyperGraph. The related definitions and an overview of the problem are provided below.

Definition 3.47 (Boolean Variable)**.** (cf. [68]) A *Boolean variable* is a variable that can take one of two possible values: `True` (1) or `False` (0). Formally, a Boolean variable x is defined as:

$$x \in \{0, 1\},$$

where 1 represents `True` and 0 represents `False`.

Definition 3.48 (Satisfiable)**.** A Boolean formula is said to be *satisfiable* if there exists an assignment of `True` (1) or `False` (0) to its Boolean variables such that the entire formula evaluates to `True`. Formally, for a formula $F(x_1, x_2, \ldots, x_n)$ composed of Boolean variables $x_1, x_2, \ldots, x_n$, F is satisfiable if there exists an assignment $A : \{x_1, x_2, \ldots, x_n\} \to \{0, 1\}$ such that:

$$F(A(x_1), A(x_2), \ldots, A(x_n)) = 1.$$

Definition 3.49 (Horn Clause)**.** (cf. [48]) A *Horn clause* is a disjunction of literals with at most one positive literal. Formally, a Horn clause C is of the form:

$$\neg x_1 \vee \neg x_2 \vee \cdots \vee \neg x_k \vee x_{k+1},$$

where $x_1, x_2, \ldots, x_k, x_{k+1}$ are Boolean variables. The clause is satisfied if at least one literal evaluates to `True`.

Problem 3.50 (Horn Satisfiability Problem in a Superhypergraph)**.** (cf. [48, 80]) Let $H = (V, E)$ be a superhypergraph, where V is a set of supervertices and E is a set of superhyperedges. Each superhyperedge $e \in E$ represents a Horn clause. The *Horn Satisfiability Problem in a Superhypergraph* is to determine whether there exists a satisfying assignment $f : V \to \{\texttt{True}, \texttt{False}\}$ such that every clause (superhyperedge) is satisfied.

We adapt the Unit Propagation algorithm, which is commonly used for Horn formulas, to solve the satisfiability problem in Superhypergraphs.

Algorithm 11: Horn Satisfiability in a Superhypergraph

Input: A superhypergraph $H = (V, E)$, where each $e \in E$ represents a Horn clause.
Output: `True` if the Horn clauses are satisfiable, `False` otherwise.

```
Initialize all vertices v ∈ V as Unknown;
repeat
    foreach hyperedge e ∈ E do
        if all but one literal in e are assigned a value and the remaining literal is unassigned then
            Assign the remaining literal to satisfy the clause;
until no changes occur;
if all clauses are satisfied then
    return True;
else
    return False;
```

Theorem 3.51. *The algorithm correctly determines whether the given Horn clauses represented by the superhypergraph are satisfiable.*

Proof. The algorithm performs unit propagation, which ensures that:

1. If a Horn clause has all literals but one assigned and the remaining literal is unassigned, the remaining literal is assigned a value to satisfy the clause.
2. This process repeats until no further assignments can be made.

If all clauses are satisfied after this process, the formula is satisfiable, and the algorithm returns `True`. If a state is reached where no assignment can satisfy a clause, the formula is unsatisfiable, and the algorithm returns `False`. □

Theorem 3.52 (Time Complexity)**.** *The worst-case time complexity of the algorithm is $O(|E|)$.*

Proof. In each iteration, the algorithm processes all hyperedges $e \in E$. Since each hyperedge is processed once, the overall complexity is $O(|E|)$. □

Theorem 3.53 (Space Complexity)**.** *The space complexity of the algorithm is $O(|V|)$.*

Proof. The algorithm requires space to store the assignment of truth values for all vertices $v \in V$. Therefore, the space complexity is $O(|V|)$. □

4 Future Tasks of This Research

This section outlines the future tasks related to this research.

Beyond the problems discussed in this paper, numerous classic problems in graph theory and computer science are well-known for standard graphs [50]. Expanding these problems to the framework of superhypergraphs presents an exciting avenue for future exploration.

Additionally, we hope that further investigations will focus on extending superhypergraph problems and algorithms to fuzzy environments [97, 122, 123], neutrosophic environments [37, 101, 102, 112], hypersoft environments [36, 43, 103], and plithogenic environments [38, 40, 44, 45, 104]. These extensions could provide deeper insights and broader applications of superhypergraph theory.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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A Short Note on the Basic Graph Construction Algorithm for Plithogenic Graphs

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Abstract

A Plithogenic Graph is a mathematical framework that models multi-valued attributes in graphs by incorporating membership and contradiction functions, enabling a nuanced representation of complex relationships. This Short paper develops algorithms for Plithogenic Graphs and Intuitionistic Plithogenic Graphs, and provides an in-depth analysis of their complexity and validity.

Keywords: Algorithm, Plithogenic Graph, Intuitionistic Plithogenic Graphs, Plithogenic Sets

MSC 2010 classifications: 68R10 - Graph theory in computer science

1 Introduction

1.1 Uncertain Graphs

Graph theory is a fundamental discipline for analyzing networks, consisting of nodes (vertices) and the connections (edges) between them. It offers critical insights into the structure, connectivity, and properties of various types of networks [21, 22].

This paper explores different models of uncertain graphs, including Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Graphs. These models enhance classical graph theory by incorporating degrees of uncertainty, enabling a more refined analysis of ambiguous and complex relationships [24–33].

The Plithogenic Graph framework models graphs with multi-valued attributes using membership and contradiction functions, allowing for detailed representation of complex relationships [25, 67, 69]. The Intuitionistic Plithogenic Graph further extends this framework by incorporating intuitionistic fuzzy degrees, which include membership and non-membership values, to effectively address uncertainty, contradiction, and hesitation in graph-based analyses [64–66].

1.2 Algorithms and Computational Complexity

An algorithm is a sequence of well-defined steps designed to solve specific problems or perform computations [16, 23, 62]. Understanding the time and space complexities of algorithms is crucial for evaluating their efficiency. Computational complexity studies the resources—such as time and space—required by algorithms as a function of input size, providing theoretical bounds on their efficiency [1, 9, 36, 37, 52, 75].

Numerous algorithms have been developed to address challenges in uncertain graph models such as Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Graphs, providing computational approaches to analyze these complex systems (e.g. [7, 8, 12, 14, 34, 46, 48, 61]).

1.3 Contributions of This Paper

This paper introduces algorithms specifically designed for Plithogenic Graphs and Intuitionistic Plithogenic Graphs. It further conducts a comprehensive analysis of their computational complexity and ensures their validity in addressing uncertainty within graph-based frameworks.

1.4 The Structure of the Paper

The structure of this paper is as follows.

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2 Preliminaries and Definitions

In this section, we provide the necessary preliminaries and definitions. Readers seeking foundational concepts and notations in graph theory are encouraged to consult standard texts, surveys, or lecture notes, such as [19–21, 74].

2.1 Basic Definitions of Graphs

Graph theory serves as a fundamental framework for analyzing networks, composed of nodes (vertices) and their connections (edges). The basic definitions of graphs are as follows:

Definition 2.1 (Graph)**.** [21] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$, which connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as:

$$G = (V, E),$$

where V is the vertex set, and E is the edge set.

Definition 2.2 (Subgraph)**.** [21] Let $G = (V, E)$ be a graph. A *subgraph* $H = (V_H, E_H)$ of G is a graph that satisfies:

- $V_H \subseteq V$, i.e., the vertex set of H is a subset of the vertex set of G.
- $E_H \subseteq E$, i.e., the edge set of H is a subset of the edge set of G.
- Each edge in E_H connects vertices in V_H.

2.2 Basic Definition of Algorithms

The fundamental definitions related to the algorithms discussed in the Results section are provided here. Readers may refer to the Lecture Notes or the Introduction for additional details as needed [16, 23, 62].

Definition 2.3 (Total Time Complexity)**.** (cf. [52, 62]) The *total time complexity* of an algorithm is defined as the sum of the time required to execute each step of the algorithm, expressed as a function of the input size. If an algorithm involves multiple steps or operations, the total time complexity is determined by the time required for the most time-consuming operation.

Formally, let $T(n, m)$ represent the time complexity as a function of input sizes n and m. The total time complexity is:

$$T(n, m) = \max(T_{\text{operation1}}(n, m), T_{\text{operation2}}(n, m), \ldots, T_{\text{operationk}}(n, m)),$$

where:

- n represents the size of the set of propositions or input elements.

- m represents the size or complexity of the associated context or additional parameters.

Definition 2.4. (cf. [52,62]) The *Space Complexity* of an algorithm is the total amount of memory required to execute the algorithm, expressed as a function of the input size. This includes:

- The *input space*, which depends on the size of the input n, m,
- The *auxiliary space*, which includes temporary variables, data structures, or storage used during computation.

Formally, let $S(n, m)$ be the space complexity as a function of input sizes n and m. The total space complexity is:

$$S(n, m) = S_{\text{input}}(n, m) + S_{\text{auxiliary}}(n, m).$$

Definition 2.5. (cf. [52,62]) *Big-O notation* is a mathematical concept used to describe the upper bound of the time or space complexity of an algorithm. Let $f(n)$ and $g(n)$ be functions that map non-negative integers to non-negative real numbers. We say:

$$f(n) \in O(g(n))$$

if there exist constants $c > 0$ and $n_0 \geq 0$ such that:

$$f(n) \leq c \cdot g(n) \quad \text{for all } n \geq n_0.$$

2.3 Fuzzy and Neutrosophic Graphs

In this subsection, we explore Fuzzy Graphs and Neutrosophic Graphs. The definitions are provided below.

Definition 2.6 (Unified Uncertain Graphs Framework)**.** (cf. [26]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, falsity, and other nuanced measures of uncertainty.

1. *Fuzzy Graph [45, 50, 57, 58, 70]:*
   - Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$.
   - Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$.
2. *Intuitionistic Fuzzy Graph (IFG) [2, 42, 73, 77]:*
   - Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $\nu_A(v) \in [0, 1]$ (degree of non-membership), such that $\mu_A(v) + \nu_A(v) \leq 1$.
   - Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0, 1]$ and $\nu_B(u, v) \in [0, 1]$, with $\mu_B(u, v) + \nu_B(u, v) \leq 1$.
3. *Neutrosophic Graph [5, 6, 15, 38, 43, 59]:*
   - Each vertex $v \in V$ is assigned a triplet $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where $\sigma_T(v), \sigma_I(v), \sigma_F(v) \in [0, 1]$ and $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
   - Each edge $e = (u, v) \in E$ is assigned a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$.
4. *Vague Graph [3, 4, 56, 60]:*
   - Each vertex $v \in V$ is assigned a pair $(\tau(v), \varphi(v))$, where $\tau(v) \in [0, 1]$ is the degree of truth-membership and $\varphi(v) \in [0, 1]$ is the degree of false-membership, with $\tau(v) + \varphi(v) \leq 1$.
   - The grade of membership is characterized by the interval $[\tau(v), 1 - \varphi(v)]$.
   - Each edge $e = (u, v) \in E$ is assigned a pair $(\tau(e), \varphi(e))$, satisfying:

$$\tau(e) \leq \min\{\tau(u), \tau(v)\}, \quad \varphi(e) \geq \max\{\varphi(u), \varphi(v)\}.$$

5. *Hesitant Fuzzy Graph [10, 35, 51, 53, 76]:*

   - Each vertex $v \in V$ is assigned a hesitant fuzzy set $\sigma(v)$, represented by a finite subset of $[0, 1]$, denoted $\sigma(v) \subseteq [0, 1]$.
   - Each edge $e = (u, v) \in E$ is assigned a hesitant fuzzy set $\mu(e) \subseteq [0, 1]$.
   - Operations on hesitant fuzzy sets (e.g., intersection, union) are defined to handle the hesitation in membership degrees.

6. *Single-Valued Pentapartitioned Neutrosophic Graph [17, 39, 40, 54]:*

   - Each vertex $v \in V$ is assigned a quintuple $\sigma(v) = (T(v), C(v), R(v), U(v), F(v))$, where:
     - $T(v) \in [0, 1]$ is the truth-membership degree.
     - $C(v) \in [0, 1]$ is the contradiction-membership degree.
     - $R(v) \in [0, 1]$ is the ignorance-membership degree.
     - $U(v) \in [0, 1]$ is the unknown-membership degree.
     - $F(v) \in [0, 1]$ is the false-membership degree.
     - $T(v) + C(v) + R(v) + U(v) + F(v) \leq 5$.
   - Each edge $e = (u, v) \in E$ is assigned a quintuple $\mu(e) = (T(e), C(e), R(e), U(e), F(e))$, satisfying:

$$\begin{cases} T(e) \leq \min\{T(u), T(v)\}, \\ C(e) \leq \min\{C(u), C(v)\}, \\ R(e) \geq \max\{R(u), R(v)\}, \\ U(e) \geq \max\{U(u), U(v)\}, \\ F(e) \geq \max\{F(u), F(v)\}. \end{cases}$$

2.4 Plithogenic Graphs

Plithogenic Graphs have recently been introduced as an extension of Fuzzy and Neutrosophic Graphs, broadening their scope to represent Plithogenic Sets [67, 68]. These graphs are gaining significant attention in current research and development efforts [26, 44, 66, 71, 72]. The formal definition is presented below.

Definition 2.7. [72] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

1. *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:

   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the range of possible attribute values.
   - $adf : M \times Ml \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
   - $aCf : Ml \times Ml \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

2. *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:

   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.

- $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. *Edge Appurtenance Constraint*: For all $(x, a), (y, b) \in M \times Ml$:

$$bdf\left((xy), (a, b)\right) \leq \min\{adf(x, a), adf(y, b)\}$$

   where $xy \in N$ is an edge between vertices x and y, and $(a, b) \in Nm \times Nm$ are the corresponding attribute values.

2. *Contradiction Function Constraint*: For all $(a, b), (c, d) \in Nm \times Nm$:

$$bCf\left((a, b), (c, d)\right) \leq \min\{aCf(a, c), aCf(b, d)\}$$

3. *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} &aCf(a, a) = 0, && \forall a \in Ml \\ &aCf(a, b) = aCf(b, a), && \forall a, b \in Ml \\ &bCf(a, a) = 0, && \forall a \in Nm \\ &bCf(a, b) = bCf(b, a), && \forall a, b \in Nm \end{aligned}$$

Example 2.8. (cf. [25]) The following examples of Plithogenic Graphs are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graphs*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graphs*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graphst*.
- When $s = 4, t = 1$, PG is called a *Plithogenic quadripartitioned Neutrosophic Graphs* (cf. [41, 55, 63]).
- When $s = 5, t = 1$, PG is called a *Plithogenic pentapartitioned Neutrosophic Graphs* (cf. [11, 18, 47]).
- When $s = 6, t = 1$, PG is called a *Plithogenic hexapartitioned Neutrosophic Graphs* (cf. [**?**]).
- When $s = 7, t = 1$, PG is called a *Plithogenic heptapartitioned Neutrosophic Graphs* (cf. [13, 49]).
- When $s = 8, t = 1$, PG is called a *Plithogenic octapartitioned Neutrosophic Graphs*.
- When $s = 9, t = 1$, PG is called a *Plithogenic nonapartitioned Neutrosophic Graphs*.

3 Results in This Paper: Algorithms

This section provides an explanation of the results presented in this paper.

3.1 Construction Algorithm of Plithogenic Graph

In this subsection, we present an algorithm for constructing a Plithogenic Graph based on the definitions provided. The algorithm ensures that the resulting graph satisfies all the properties of a Plithogenic Graph, including the constraints on the degree of appurtenance and contradiction functions.

Algorithm 1: Construction of a Plithogenic Graph

Input: A crisp graph $G = (V, E)$;
Vertex attribute l and its possible values Ml;
Edge attribute m and its possible values Nm;
Degree of appurtenance functions for vertices $adf : V \times Ml \to [0, 1]^s$;
Degree of appurtenance functions for edges $bdf : E \times Nm \to [0, 1]^s$;
Degree of contradiction functions for vertices $aCf : Ml \times Ml \to [0, 1]^t$;
Degree of contradiction functions for edges $bCf : Nm \times Nm \to [0, 1]^t$.
Output: A Plithogenic Graph $PG = (PM, PN)$.

Initialization:
$M \leftarrow V$;
$N \leftarrow E$;
foreach *vertex* $v \in V$ **do**
  Assign an attribute value $l(v) \in Ml$;
  Compute $adf(v, l(v))$;
end
foreach *edge* $e = (u, v) \in E$ **do**
  Assign an attribute value $m(e) \in Nm$;
  Compute $bdf(e, m(e))$;
end
foreach *pair of vertices* $(v_i, v_j) \in V \times V$ **do**
  Compute $aCf(l(v_i), l(v_j))$;
end
foreach *pair of edges* $(e_i, e_j) \in E \times E$ **do**
  Compute $bCf(m(e_i), m(e_j))$;
end
foreach *edge* $e = (u, v) \in E$ **do**
  if $bdf(e, m(e)) > \min\{adf(u, l(u)), adf(v, l(v))\}$ **then**
    Adjust $bdf(e, m(e)) \leftarrow \min\{adf(u, l(u)), adf(v, l(v))\}$;
  end
end
Ensure contradiction functions satisfy reflexivity and symmetry conditions:

$$\begin{aligned}
&aCf(a, a) = 0, \quad \forall a \in Ml \\
&aCf(a, b) = aCf(b, a), \quad \forall a, b \in Ml \\
&bCf(a, a) = 0, \quad \forall a \in Nm \\
&bCf(a, b) = bCf(b, a), \quad \forall a, b \in Nm
\end{aligned}$$

return $PG = (PM, PN)$, *where* $PM = (M, l, Ml, adf, aCf)$ *and* $PN = (N, m, Nm, bdf, bCf)$

Theorem 3.1 (Correctness of the Plithogenic Graph Construction Algorithm)**.** *The algorithm correctly constructs a Plithogenic Graph by ensuring that all definitions and constraints of a Plithogenic Graph are satisfied.*

Proof. The correctness of the algorithm can be established through the following steps:

- **Attribute Assignment:** Each vertex and edge is assigned attribute values from their respective domains (Ml and Nm), ensuring all elements of the graph have the necessary attributes.
- **Degree of Appurtenance Functions:** The degree of appurtenance for each vertex and edge is computed based on their assigned attributes, providing the required membership values.
- **Contradiction Functions:** The degree of contradiction for all pairs of attributes is calculated, enabling the representation of conflicting attribute values.

- **Constraints Satisfaction:** The algorithm enforces the edge appurtenance constraint by ensuring that the degree of appurtenance of an edge does not exceed the minimum degree of its incident vertices. It also ensures that the contradiction functions satisfy reflexivity and symmetry.
- **Consistency with Definitions:** All operations in the algorithm adhere to the formal definition of a Plithogenic Graph, guaranteeing that the output satisfies all required properties.

Therefore, the algorithm is correct, as it systematically constructs a Plithogenic Graph that meets all the specified conditions. □

Theorem 3.2 (Time Complexity)**.** *The time complexity of the Plithogenic Graph construction algorithm is $O(n^2 + m^2)$, where $n = |V|$ is the number of vertices and $m = |E|$ is the number of edges.*

Proof. The time complexity is derived from the following steps:

- **Assigning Attributes and Computing Degrees:**
  - For vertices (n): The loop runs n times, with constant-time computations for each vertex.
  - For edges (m): The loop runs m times, with constant-time computations for each edge.
  - Total time: $O(n + m)$.
- **Computing Contradiction Functions:**
  - For vertex pairs: There are $\frac{n(n-1)}{2} \sim O(n^2)$ pairs.
  - For edge pairs: There are $\frac{m(m-1)}{2} \sim O(m^2)$ pairs.
  - Assuming constant time per computation, total time: $O(n^2 + m^2)$.
- **Adjusting Edge Degrees:**
  - The loop runs m times, with constant-time adjustments for each edge.
  - Total time: $O(m)$.
- **Ensuring Contradiction Function Properties:**
  - Reflexivity and symmetry checks for attribute sets Ml and Nm, with sizes $|Ml|$ and $|Nm|$ respectively, involve $|Ml|^2 + |Nm|^2$ computations.
  - Total time: $O(|Ml|^2 + |Nm|^2)$.

The dominant terms are $O(n^2 + m^2)$, assuming $n, m \gg |Ml|, |Nm|$. Therefore, the total time complexity is $O(n^2 + m^2)$. □

Theorem 3.3 (Space Complexity)**.** *The space complexity of the Plithogenic Graph construction algorithm is $O(n^2 + m^2)$, where $n = |V|$ and $m = |E|$.*

Proof. The space complexity is calculated as follows:

- **Storage for Attributes and Degrees:**
  - Vertex attributes and degrees: $O(n)$.
  - Edge attributes and degrees: $O(m)$.
- **Contradiction Functions:**
  - For vertices: Store aCf values for $\frac{n(n-1)}{2} \sim O(n^2)$ pairs.
  - For edges: Store bCf values for $\frac{m(m-1)}{2} \sim O(m^2)$ pairs.

- **Total Space Complexity:** Adding all contributions, the total space complexity is $O(n^2 + m^2)$.

□

Theorem 3.4. *By setting the parameters s and t in the Plithogenic Graph construction algorithm as follows:*

1. *$s = 1$, $t = 1$: The algorithm generates a **Fuzzy Graph**.*
2. *$s = 2$, $t = 1$: The algorithm generates an **Intuitionistic Fuzzy Graph**.*
3. *$s = 3$, $t = 1$: The algorithm generates a **Neutrosophic Graph**.*

Proof. We will demonstrate that by setting s and t to the specified values in the algorithm, it constructs graphs that conform exactly to the definitions of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, and Neutrosophic Graphs, respectively.

Case 1: $s = 1, t = 1$ **— Fuzzy Graph**

Vertex Degrees: With $s = 1$, the degree of appurtenance function for vertices becomes:

$$adf : V \times Ml \to [0, 1]$$

Each vertex $v \in V$ is assigned a single membership degree $\sigma(v) = adf(v, l(v)) \in [0, 1]$, which matches the Fuzzy Graph definition.

Edge Degrees: Similarly, for edges:

$$bdf : E \times Nm \to [0, 1]$$

Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) = bdf(e, m(e)) \in [0, 1]$.

Edge Constraint: The algorithm adjusts the edge degrees to satisfy:

$$bdf(e, m(e)) \leq \min\{adf(u, l(u)), adf(v, l(v))\}$$

This aligns with the Fuzzy Graph property that the membership degree of an edge does not exceed the minimum of the membership degrees of its incident vertices.

Contradiction Functions: With $t = 1$, contradiction functions aCf and bCf map to $[0, 1]$. In Fuzzy Graphs, these functions are not standard, but since they do not affect the construction, we can consider them as constants or set them to zero.

Therefore, the algorithm produces a graph where each vertex and edge has a single membership degree in $[0, 1]$, satisfying the definition of a Fuzzy Graph.

Case 2: $s = 2, t = 1$ **— Intuitionistic Fuzzy Graph**

Vertex Degrees: With $s = 2$, the degree of appurtenance function for vertices becomes:

$$adf : V \times Ml \to [0, 1]^2$$

Each vertex $v \in V$ is assigned a pair $(\mu_A(v), \nu_A(v)) = adf(v, l(v))$, where $\mu_A(v)$ is the degree of membership and $\nu_A(v)$ is the degree of non-membership.

Edge Degrees: For edges:

$$bdf : E \times Nm \to [0, 1]^2$$

Each edge $e = (u, v) \in E$ is assigned $(\mu_B(e), \nu_B(e)) = bdf(e, m(e))$.

Edge Constraint: The algorithm enforces:

$$\mu_B(e) \leq \min\{\mu_A(u), \mu_A(v)\}, \quad \nu_B(e) \geq \max\{\nu_A(u), \nu_A(v)\}$$

Matching the Intuitionistic Fuzzy Graph property.

Sum Constraint: It is required that $\mu_A(v) + \nu_A(v) \leq 1$ for all $v \in V$, and similarly for edges. The algorithm can ensure this by adjusting the degrees during assignment.

Contradiction Functions: With $t = 1$, aCf and bCf can be used to check or enforce the sum constraint, but they are not essential for the construction.

The algorithm assigns to each vertex and edge degrees of membership and non-membership that satisfy the Intuitionistic Fuzzy Graph definition.

Case 3: $s = 3, t = 1$ **— Neutrosophic Graph**

Vertex Degrees: With $s = 3$, the degree of appurtenance function for vertices becomes:

$$adf : V \times Ml \to [0,1]^3$$

Each vertex $v \in V$ is assigned a triplet $(\sigma_T(v), \sigma_I(v), \sigma_F(v)) = adf(v, l(v))$, representing truth, indeterminacy, and falsity degrees.

Edge Degrees: For edges:

$$bdf : E \times Nm \to [0,1]^3$$

Each edge $e = (u, v) \in E$ is assigned $(\mu_T(e), \mu_I(e), \mu_F(e)) = bdf(e, m(e))$.

Edge Constraint: The algorithm enforces constraints such as:

$$\mu_T(e) \leq \min\{\sigma_T(u), \sigma_T(v)\}, \quad \mu_I(e) \geq \max\{\sigma_I(u), \sigma_I(v)\}, \quad \mu_F(e) \geq \max\{\sigma_F(u), \sigma_F(v)\}$$

Aligning with Neutrosophic Graph properties.

Sum Constraint: The algorithm can ensure $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$ for all $v \in V$.

Contradiction Functions: With $t = 1$, contradiction functions are scalar and can be set appropriately, though they are not typically used in Neutrosophic Graphs.

The algorithm constructs a graph where each vertex and edge has degrees corresponding to truth, indeterminacy, and falsity, satisfying the Neutrosophic Graph definition.

In all cases, setting s and t as specified causes the algorithm to produce graphs that match the definitions of the respective uncertain graphs. □

3.2 Definition of Intuitionistic Plithogenic Graph

An *Intuitionistic Plithogenic Graph* extends the framework of Plithogenic Graphs by integrating intuitionistic fuzzy sets. This allows for the representation of both membership and non-membership degrees, along with contradiction degrees associated with multi-valued attributes [64, 66].

We begin by providing a precise mathematical definition of these concepts.

Definition 3.5 (Intuitionistic Plithogenic Graph)**.** (cf. [64,66]) Let $G = (V, E)$ be a crisp graph, where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. An **Intuitionistic Plithogenic Graph** IPG is defined as:

$$IPG = (PM, PN)$$

where:

1. **Intuitionistic Plithogenic Vertex Set** $PM = (M, a, V_a, d_p, c_p)$:

   - $M \subseteq V$ is the set of vertices.
   - a is a multi-valued attribute associated with the vertices.
   - V_a is the domain of the attribute a.
   - $d_p : M \times V_a \to [0,1] \times [0,1]$ is the *Intuitionistic Degree of Appurtenance Function* for vertices, assigning to each vertex $v \in M$ and attribute value $a(v) \in V_a$ a pair $d_p(v, a(v)) = (\mu_A(v), \nu_A(v))$, where:
     - $\mu_A(v)$ is the degree of membership of v in PM.
     - $\nu_A(v)$ is the degree of non-membership of v in PM.
     - $0 \le \mu_A(v) + \nu_A(v) \le 1$.
   - $c_p : V_a \times V_a \to [0,1]$ is the *Degree of Contradiction Function* for vertices, satisfying:
     - $c_p(a, a) = 0$ for all $a \in V_a$ (reflexivity).
     - $c_p(a, b) = c_p(b, a)$ for all $a, b \in V_a$ (symmetry).

2. **Intuitionistic Plithogenic Edge Set** $PN = (N, b, V_b, d_e, c_e)$:

   - $N \subseteq E$ is the set of edges.
   - b is a multi-valued attribute associated with the edges.
   - V_b is the domain of the attribute b.
   - $d_e : N \times V_b \to [0,1] \times [0,1]$ is the *Intuitionistic Degree of Appurtenance Function* for edges, assigning to each edge $e \in N$ and attribute value $b(e) \in V_b$ a pair $d_e(e, b(e)) = (\mu_B(e), \nu_B(e))$, where:
     - $\mu_B(e)$ is the degree of membership of e in PN.
     - $\nu_B(e)$ is the degree of non-membership of e in PN.
     - $0 \le \mu_B(e) + \nu_B(e) \le 1$.
   - $c_e : V_b \times V_b \to [0,1]$ is the *Degree of Contradiction Function* for edges, satisfying:
     - $c_e(b, b) = 0$ for all $b \in V_b$ (reflexivity).
     - $c_e(b, c) = c_e(c, b)$ for all $b, c \in V_b$ (symmetry).

The Intuitionistic Plithogenic Graph IPG must satisfy the following conditions:

1. **Edge Appurtenance Constraint**: For all $e = (u, v) \in N$:

$$\mu_B(e) \le \min\{\mu_A(u), \mu_A(v)\}, \quad \nu_B(e) \ge \max\{\nu_A(u), \nu_A(v)\}$$

2. **Contradiction Degree Constraint**: The contradiction degree functions c_p and c_e quantify the level of contradiction between attribute values, and are used in operations like union and intersection of Intuitionistic Plithogenic Sets.

3. **Reflexivity and Symmetry of Contradiction Functions**:

$$c_p(a, a) = 0, \quad c_p(a, b) = c_p(b, a), \quad \forall a, b \in V_a$$

$$c_e(b, b) = 0, \quad c_e(b, c) = c_e(c, b), \quad \forall b, c \in V_b$$

Theorem 3.6. *Intuitionistic Plithogenic Graphs generalize Plithogenic Graphs by incorporating intuitionistic fuzzy degrees of appurtenance, allowing for the representation of both membership and non-membership degrees, and thus providing a more comprehensive framework for handling uncertainty and contradiction in graphs.*

Proof. A Plithogenic Graph PG is defined with degrees of appurtenance adf and contradiction functions aCf and bCf mapping to $[0,1]^s$ and $[0,1]^t$, respectively, where $s, t \geq 1$. In standard Plithogenic Graphs, the degrees of appurtenance are single-valued (fuzzy membership degrees).

By extending the degrees of appurtenance to intuitionistic fuzzy degrees, which are pairs (μ, ν) satisfying $0 \leq \mu + \nu \leq 1$, we can capture both the membership and non-membership information of each element. This extension allows us to model the hesitation or uncertainty in the membership of vertices and edges more effectively.

The contradiction functions c_p and c_e remain as mappings to $[0,1]$, quantifying the contradiction between attribute values.

Therefore, the Intuitionistic Plithogenic Graph IPG generalizes the Plithogenic Graph PG by:

- Extending the degree of appurtenance functions from fuzzy (single value) to intuitionistic fuzzy (pair of values representing membership and non-membership degrees).
- Retaining the contradiction functions to handle the degree of contradiction between different attribute values.

Since intuitionistic fuzzy sets generalize fuzzy sets, and the Intuitionistic Plithogenic Graph incorporates intuitionistic fuzzy degrees, it follows that the Intuitionistic Plithogenic Graph generalizes the Plithogenic Graph. □

We present an algorithm for constructing an Intuitionistic Plithogenic Graph based on the definitions provided. The algorithm ensures that the resulting graph satisfies all the properties of an Intuitionistic Plithogenic Graph, including the constraints on the intuitionistic degrees of appurtenance and contradiction functions.

Algorithm 2: Construction of an Intuitionistic Plithogenic Graph

Input: A crisp graph $G = (V, E)$;
Vertex attribute a and its domain V_a;
Edge attribute b and its domain V_b;
Intuitionistic degree of appurtenance functions for vertices $d_p : V \times V_a \to [0, 1] \times [0, 1]$;
Intuitionistic degree of appurtenance functions for edges $d_e : E \times V_b \to [0, 1] \times [0, 1]$;
Degree of contradiction functions for vertices $c_p : V_a \times V_a \to [0, 1]$;
Degree of contradiction functions for edges $c_e : V_b \times V_b \to [0, 1]$.
Output: An Intuitionistic Plithogenic Graph $IPG = (PM, PN)$.

Initialization:
$M \leftarrow V$;
$N \leftarrow E$;

foreach *vertex* $v \in V$ **do**
  Assign an attribute value $a(v) \in V_a$;
  Compute $d_p(v, a(v)) = (\mu_A(v), \nu_A(v))$;
  if $\mu_A(v) + \nu_A(v) > 1$ **then**
    Adjust $\nu_A(v) \leftarrow 1 - \mu_A(v)$;
  end
end
foreach *edge* $e = (u, v) \in E$ **do**
  Assign an attribute value $b(e) \in V_b$;
  Compute $d_e(e, b(e)) = (\mu_B(e), \nu_B(e))$;
  if $\mu_B(e) + \nu_B(e) > 1$ **then**
    Adjust $\nu_B(e) \leftarrow 1 - \mu_B(e)$;
  end
end

foreach *pair of vertices* $(v_i, v_j) \in V \times V$ **do**
  Compute $c_p(a(v_i), a(v_j))$;
end
foreach *pair of edges* $(e_i, e_j) \in E \times E$ **do**
  Compute $c_e(b(e_i), b(e_j))$;
end

foreach *edge* $e = (u, v) \in E$ **do**
  if $\mu_B(e) > \min\{\mu_A(u), \mu_A(v)\}$ **then**
    Adjust $\mu_B(e) \leftarrow \min\{\mu_A(u), \mu_A(v)\}$;
  end
  if $\nu_B(e) < \max\{\nu_A(u), \nu_A(v)\}$ **then**
    Adjust $\nu_B(e) \leftarrow \max\{\nu_A(u), \nu_A(v)\}$;
  end
end

Ensure contradiction functions satisfy reflexivity and symmetry conditions:

$$c_p(a, a) = 0, \quad \forall a \in V_a$$
$$c_p(a, b) = c_p(b, a), \quad \forall a, b \in V_a$$
$$c_e(b, b) = 0, \quad \forall b \in V_b$$
$$c_e(b, c) = c_e(c, b), \quad \forall b, c \in V_b$$

return $IPG = (PM, PN)$, *where* $PM = (M, a, V_a, d_p, c_p)$ *and* $PN = (N, b, V_b, d_e, c_e)$

Theorem 3.7 (Correctness of the Intuitionistic Plithogenic Graph Construction Algorithm)**.** *The algorithm correctly constructs an Intuitionistic Plithogenic Graph by ensuring that all definitions and constraints of an Intuitionistic Plithogenic Graph are satisfied.*

Proof. To prove the correctness of the algorithm, we verify that each step adheres to the definitions and constraints of an Intuitionistic Plithogenic Graph.

- **Attribute Assignment:** The algorithm assigns attribute values $a(v) \in V_a$ to each vertex $v \in V$, and $b(e) \in V_b$ to each edge $e \in E$, ensuring that all elements have associated attributes from their respective domains.
- **Intuitionistic Degrees of Appurtenance:** For each vertex $v \in V$, the algorithm computes $d_p(v, a(v)) = (\mu_A(v), \nu_A(v))$ such that $0 \le \mu_A(v) + \nu_A(v) \le 1$. If the sum exceeds 1, the algorithm adjusts $\nu_A(v)$ to satisfy the condition.

  Similarly, for each edge $e \in E$, it computes $d_e(e, b(e)) = (\mu_B(e), \nu_B(e))$ and adjusts $\nu_B(e)$ if necessary.
- **Contradiction Functions:** The algorithm computes the contradiction degrees $c_p(a(v_i), a(v_j))$ and $c_e(b(e_i), b(e_j))$ for all pairs of vertices and edges, respectively, ensuring the contradiction functions are properly defined.
- **Edge Appurtenance Constraint:** For each edge $e = (u, v)$, the algorithm enforces the constraints:

$$\mu_B(e) \le \min\{\mu_A(u), \mu_A(v)\}, \quad \nu_B(e) \ge \max\{\nu_A(u), \nu_A(v)\}$$

  by adjusting $\mu_B(e)$ and $\nu_B(e)$ if necessary.
- **Contradiction Function Properties:** The algorithm ensures that the contradiction functions c_p and c_e satisfy reflexivity and symmetry:

$$c_p(a, a) = 0, \quad c_p(a, b) = c_p(b, a)$$

$$c_e(b, b) = 0, \quad c_e(b, c) = c_e(c, b)$$

- **Consistency with Definitions:** All steps are consistent with the formal definition of an Intuitionistic Plithogenic Graph, guaranteeing that the output graph satisfies all required properties.

Therefore, the algorithm correctly constructs an Intuitionistic Plithogenic Graph. □

Theorem 3.8 (Time Complexity)**.** *The time complexity of the Intuitionistic Plithogenic Graph construction algorithm is $O(n^2 + m^2)$, where $n = |V|$ is the number of vertices and $m = |E|$ is the number of edges.*

Proof. We analyze the time complexity step by step:

- **Assigning Attributes and Computing Degrees for Vertices:**
  - The loop runs n times.
  - Each computation (assigning $a(v)$ and computing $d_p(v, a(v))$) is assumed to take constant time.
  - The check and adjustment for $\mu_A(v) + \nu_A(v) > 1$ take constant time.
  - **Total time:** $O(n)$.
- **Assigning Attributes and Computing Degrees for Edges:**
  - The loop runs m times.
  - Each computation (assigning $b(e)$ and computing $d_e(e, b(e))$) takes constant time.
  - The check and adjustment for $\mu_B(e) + \nu_B(e) > 1$ take constant time.
  - **Total time:** $O(m)$.
- **Computing Contradiction Functions for Vertices:**
  - There are $\frac{n(n-1)}{2} \approx O(n^2)$ pairs of vertices.
  - Computing $c_p(a(v_i), a(v_j))$ for each pair takes constant time.
  - **Total time:** $O(n^2)$.
- **Computing Contradiction Functions for Edges:**
  - There are $\frac{m(m-1)}{2} \approx O(m^2)$ pairs of edges.

- Computing $c_e(b(e_i), b(e_j))$ for each pair takes constant time.
- **Total time:** $O(m^2)$.

- **Adjusting Edge Degrees:**
  - The loop runs m times.
  - Each check and adjustment takes constant time.
  - **Total time:** $O(m)$.

- **Ensuring Contradiction Function Properties:**
  - For attribute domains V_a and V_b of sizes $|V_a|$ and $|V_b|$, respectively.
  - The reflexivity and symmetry checks involve $|V_a|^2$ and $|V_b|^2$ computations.
  - Assuming $|V_a|$ and $|V_b|$ are much smaller than n and m, we can consider them constants.
  - **Total time:** $O(1)$.

Total Time Complexity: The dominant terms are $O(n^2+m^2)$. Therefore, the time complexity of the algorithm is $O(n^2+m^2)$. □

Theorem 3.9 (Space Complexity)**.** *The space complexity of the Intuitionistic Plithogenic Graph construction algorithm is $O(n^2+m^2)$, where $n = |V|$ and $m = |E|$.*

Proof. We calculate the space requirements:

- **Storage for Attributes and Degrees of Vertices:**
  - Attributes $a(v)$ for all $v \in V$: $O(n)$.
  - Degrees $d_p(v, a(v)) = (\mu_A(v), \nu_A(v))$ for all $v \in V$: $O(n)$.
- **Storage for Attributes and Degrees of Edges:**
  - Attributes $b(e)$ for all $e \in E$: $O(m)$.
  - Degrees $d_e(e, b(e)) = (\mu_B(e), \nu_B(e))$ for all $e \in E$: $O(m)$.
- **Contradiction Functions for Vertices:**
  - Store $c_p(a(v_i), a(v_j))$ for all pairs (v_i, v_j): $O(n^2)$.
- **Contradiction Functions for Edges:**
  - Store $c_e(b(e_i), b(e_j))$ for all pairs (e_i, e_j): $O(m^2)$.

Total Space Complexity: The dominant terms are $O(n^2+m^2)$. Therefore, the space complexity of the algorithm is $O(n^2+m^2)$. □

Theorem 3.10. *By appropriately setting the parameters in the Intuitionistic Plithogenic Graph framework, the algorithm can generate:*

1. *A **Fuzzy Graph** when only membership degrees are considered, and the contradiction functions are trivial.*
2. *An **Intuitionistic Fuzzy Graph** when membership and non-membership degrees are used, and contradiction functions are ignored.*
3. *A **Neutrosophic Graph** when truth, indeterminacy, and falsity degrees are modeled within the intuitionistic framework.*

Therefore, Intuitionistic Plithogenic Graphs generalize Fuzzy Graphs, Intuitionistic Fuzzy Graphs, and Neutrosophic Graphs.

Proof. We demonstrate how Intuitionistic Plithogenic Graphs can be specialized to each of the graph types.

Case 1: Fuzzy Graph

Vertices and Edges: Assign attributes to vertices and edges, but consider only the membership degree μ. Set the non-membership degree $\nu = 0$ for all vertices and edges.

Degrees of Appurtenance: For each vertex v, set $d_p(v, a(v)) = (\mu_A(v), 0)$. For each edge e, set $d_e(e, b(e)) = (\mu_B(e), 0)$.

Contradiction Functions: Set contradiction functions c_p and c_e to zero or ignore them.

Edge Constraint: The edge membership degree satisfies:

$$\mu_B(e) \leq \min\{\mu_A(u), \mu_A(v)\}$$

This matches the Fuzzy Graph definition.

The Intuitionistic Plithogenic Graph reduces to a Fuzzy Graph when only membership degrees are considered.

Case 2: Intuitionistic Fuzzy Graph

Vertices and Edges: Use both membership μ and non-membership ν degrees.

Degrees of Appurtenance: For each vertex v, set $d_p(v, a(v)) = (\mu_A(v), \nu_A(v))$ with $\mu_A(v) + \nu_A(v) \leq 1$. For each edge e, set $d_e(e, b(e)) = (\mu_B(e), \nu_B(e))$ with $\mu_B(e) + \nu_B(e) \leq 1$.

Contradiction Functions: Contradiction functions are not essential in Intuitionistic Fuzzy Graphs and can be ignored or set to zero.

Edge Constraints: The edge degrees satisfy:

$$\mu_B(e) \leq \min\{\mu_A(u), \mu_A(v)\}, \quad \nu_B(e) \geq \max\{\nu_A(u), \nu_A(v)\}$$

Matching the Intuitionistic Fuzzy Graph properties.

The Intuitionistic Plithogenic Graph corresponds exactly to an Intuitionistic Fuzzy Graph when contradiction functions are ignored.

Case 3: Neutrosophic Graph

Vertices and Edges: Extend the intuitionistic degrees to include an indeterminacy component. Since intuitionistic fuzzy sets cannot directly represent neutrosophic sets, we model indeterminacy within the framework by introducing a third component.

Degrees of Appurtenance: For each vertex v, define $d_p(v, a(v)) = (\mu_A(v), \nu_A(v), \tau_A(v))$ where:

- $\mu_A(v)$ is the degree of truth-membership.
- $\nu_A(v)$ is the degree of falsity-membership.
- $\tau_A(v)$ is the degree of indeterminacy.
- $\mu_A(v) + \nu_A(v) + \tau_A(v) \leq 1$.

Similarly for edges.

Contradiction Functions: Contradiction functions can be extended to accommodate the third component or ignored if not essential.

Edge Constraints: Edge degrees are adjusted to align with the Neutrosophic Graph properties, for example:

$$\mu_B(e) \leq \min\{\mu_A(u), \mu_A(v)\}, \quad \nu_B(e) \geq \max\{\nu_A(u), \nu_A(v)\}, \quad \tau_B(e) \geq \max\{\tau_A(u), \tau_A(v)\}$$

By extending the intuitionistic framework to include an indeterminacy component, the Intuitionistic Plithogenic Graph can represent Neutrosophic Graphs.

Therefore, Intuitionistic Plithogenic Graphs generalize Fuzzy Graphs, Intuitionistic Fuzzy Graphs, and Neutrosophic Graphs by appropriate choice and interpretation of degrees and components in the degree of appurtenance functions. □

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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Short Note of Bunch Graph in Fuzzy, Neutrosophic, and Plithogenic Graphs

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Abstract: Graph theory examines networks consisting of nodes (vertices) and their connections (edges). A bunch graph generalizes traditional graphs by representing nodes as groups (bunches), allowing for the simultaneous modeling of competition and collaboration among entities within a network. This paper explores various uncertain models of bunch graphs, including Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs.

Keywords: Bunch Graph, Neutrosophic graph, Fuzzy graph, Plithogenic Graph

1. Introduction

1.1 Uncertain Graph Theory

Graph theory provides a mathematical framework for analyzing networks consisting of nodes (vertices) and their connections (edges). It plays a critical role in understanding network structures, connectivity, and properties [14]. Among the diverse types of graphs, this paper focuses on the concept of the *Bunch Graph*.

A Bunch Graph extends the traditional notion of a graph by representing nodes as groups, or "bunches," to capture both competition and collaboration among entities within a network [37, 42]. This generalization enables the modeling of complex interactions in systems where multiple entities may simultaneously interact.

This paper also investigates a variety of uncertain graph models, including Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs. These models address the need to incorporate uncertainty into network analysis, extending classical graph theory to account for varying degrees of uncertainty [1–7, 9, 16, 19–24, 26, 27, 38, 39, 46, 52, 53].

1.2 Our Contribution

Despite the increasing interest in uncertain graph theory, research on Bunch Graphs in the context of uncertain environments remains underexplored. In this work, we introduce and analyze the concept of Bunch Graphs within the frameworks of Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs.

We investigate their fundamental properties, explore the relationships between these models, and provide new insights into the nature of uncertain graphs. By doing so, we contribute to advancing the theoretical understanding and potential applications of uncertain graph models.

1.3 The Structure of the Paper

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2. Preliminaries and Definitions

This section provides an overview of the definitions and notations utilized throughout this paper.

2.1 Basic Graph Concepts

We present foundational concepts related to graphs. For more detailed explanations and additional graph theory notations, please refer to [13, 14, 31, 59].

Definition 1 (Graph). [14] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that represent connections between pairs of vertices. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set, and E is the edge set.

Definition 2 (Degree). [14] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. For undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

For directed graphs, the *in-degree*, denoted $\deg^-(v)$, is the number of edges directed into v, and the *out-degree*, denoted $\deg^+(v)$, is the number of edges directed out of v.

Definition 3 (Directed Graph). [14] A *directed graph* $G = (V, A)$ is a mathematical structure composed of:

- V: A set of vertices (or nodes),
- $A \subseteq V \times V$: A set of directed edges (or arcs), where each arc $(u, v) \in A$ has a specific direction from vertex u (the tail) to vertex v (the head).

Definition 4 (Mixed Graph). [41, 57] A *mixed graph* $G = (V, E, A)$ is a mathematical structure in graph theory consisting of:

- V: A set of vertices (or nodes),
- E: A set of undirected edges, where an edge $uv \in E$ (or equivalently, $[u, v]$) represents a connection between two vertices u and v without orientation,
- A: A set of directed edges (arcs), where an arc $\overrightarrow{uv} \in A$ (or equivalently, (u, v)) connects two vertices u and v with a specific orientation, where u is the tail and v is the head.

2.2 Fuzzy, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs

In this subsection, we examine Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. Fuzzy graphs are frequently discussed in comparison with crisp graphs, which represent the classical form of graphs[35, 39].

Definition 5. (cf.[35, 39]) A *crisp graph* is an ordered pair $G = (V, E)$, where:

- V is a finite, non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges, where each edge is an unordered pair of distinct vertices.

  Formally, for any edge $(u, v) \in E$, the following holds:

$$(u, v) \in E \iff u \neq v \quad \text{and} \quad u, v \in V$$

This implies that there are no loops (i.e., no edges of the form (v, v)) and edges represent binary relationships between distinct vertices.

Taking the above into consideration, we define Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs as follows. Please note that the definitions have been consolidated for simplicity. Note that the Turiyam Neutrosophic Graph is, in fact, a specific case of the Quadripartitioned Neutrosophic Graph, achieved by replacing "Contradiction" with "Liberal." (cf.[45, 51])

Definition 6 (Unified Graphs Framework: Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs). (cf.[19]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, and falsity.

1. *Fuzzy Graph [11, 36, 39, 58, 60, 61]:*

- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0,1]$, representing the degree of participation of v in the fuzzy graph.
- Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0,1]$, representing the strength of the connection between u and v.

2. *Intuitionistic Fuzzy Graph (IFG) [3, 56, 62]:*
   - Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0,1]$ (degree of membership) and $\nu_A(v) \in [0,1]$ (degree of non-membership), such that $\mu_A(v) + \nu_A(v) \leq 1$.
   - Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0,1]$ (degree of membership) and $\nu_B(u, v) \in [0,1]$ (degree of non-membership), such that $\mu_B(u, v) + \nu_B(u, v) \leq 1$.
3. *Neutrosophic Graph [8, 12, 17, 32, 50]:*
   - Each vertex $v \in V$ is assigned a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where:
     - $\sigma_T(v) \in [0,1]$ is the truth-membership degree,
     - $\sigma_I(v) \in [0,1]$ is the indeterminacy-membership degree,
     - $\sigma_F(v) \in [0,1]$ is the falsity-membership degree,
     - $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
   - Each edge $e = (u, v) \in E$ is assigned a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing the truth, indeterminacy, and falsity degrees for the connection between u and v.
4. *Turiyam Neutrosophic Graph [19, 28–30, 44]:*
   - Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where:
     - $t(v) \in [0,1]$ is the truth value,
     - $iv(v) \in [0,1]$ is the indeterminacy value,
     - $fv(v) \in [0,1]$ is the falsity value,
     - $lv(v) \in [0,1]$ is the liberal state value,
     - $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.
   - Each edge $e = (u, v) \in E$ is similarly assigned a quadruple representing the same parameters for the connection between u and v.

2.3 Plithogenic Graphs

Plithogenic Graphs have recently emerged as a generalization of Fuzzy Graphs and Turiyam Neutrosophic Graphs, extending the concept to represent Plithogenic Sets [47, 48]. These graphs are a focus of ongoing research and development [19, 34, 43, 54, 55]. The formal definition is presented below.

Definition 7. [55] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

1. *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:
   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the range of possible attribute values.
   - $adf : M \times Ml \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
   - $aCf : Ml \times Ml \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.
2. *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.

- $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. *Edge Appurtenance Constraint*: For all $(x,a),(y,b) \in M \times Ml$:

$$bdf((xy),(a,b)) \leq \min\{adf(x,a), adf(y,b)\}$$

where $xy \in N$ is an edge between vertices x and y, and $(a,b) \in Nm \times Nm$ are the corresponding attribute values.

2. *Contradiction Function Constraint*: For all $(a,b),(c,d) \in Nm \times Nm$:

$$bCf((a,b),(c,d)) \leq \min\{aCf(a,c), aCf(b,d)\}$$

3. *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned}
&aCf(a,a) = 0, && \forall a \in Ml \\
&aCf(a,b) = aCf(b,a), && \forall a,b \in Ml \\
&bCf(a,a) = 0, && \forall a \in Nm \\
&bCf(a,b) = bCf(b,a), && \forall a,b \in Nm
\end{aligned}$$

Example 8. (cf.[17]) The following examples are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic Turiyam Neutrosophic Graph*.

3. Results Presented in This Paper

In this section, we outline the findings and contributions of this paper.

3.1 Uncertain Bunch Graph

In this subsection, we introduce the concept of an *Uncertain Bunch Graph*. To establish the foundation, we first provide the definition of a classic Bunch Graph as described in [37].

Definition 9 (Bunch Graph)**.** [37] Let X be a non-empty set and $V_X \subseteq \mathcal{P}(X)$, where:

- $\emptyset \subset V_X$,
- $X \in V_X$, and
- $V_X \neq \emptyset$.

Let $E \subseteq V_X \times V_X$ be a set of edges. Then, a *bunch graph* is defined as $G = (V_X, E)$, where:

- V_X is the set of vertices, with each vertex representing a subset of X, called a *bunch*.
- E is a set of multi-edges, allowing multiple edges between the same pairs of vertices.

The definition of the Fuzzy Bunch Graph and its related theorem are provided below. This graph class extends the concept of a classic Bunch Graph by incorporating principles from Fuzzy Graphs. Similarly, the other graph classes introduced in this paper follow the same approach, combining a classic graph class with concepts from Uncertain Graphs.

Definition 10 (Fuzzy Bunch Graph)**.** (cf.[42]) Let U be a universal set, V be a collection of nodes (including bunches), and $E \subseteq V \times V$ be a set of edges such that:

1. $\emptyset \notin V \subseteq \mathcal{P}(U)$, where $\mathcal{P}(U)$ denotes the power set of U.
2. Each vertex $v \in V$ has a membership degree $\mu(v) \in [0,1]$, representing the degree of inclusion of v.

3. Each edge $e = (u, v) \in E$ has a membership degree $\sigma(e) \in [0, 1]$, representing the strength of the connection between u and v.

4. The edge membership satisfies:

$$\sigma(e) \leq \min\{\mu(u), \mu(v)\}, \quad \forall e = (u, v) \in E.$$

The structure $G = (V, \mu, \sigma)$ is called a *fuzzy bunch graph*.

Theorem 11. *A Fuzzy Bunch Graph $G_F = (V, \mu, \sigma)$ can be transformed into a Bunch Graph $G_B = (V', E')$ by discretizing the fuzzy membership degrees and edge strengths, where:*

- *$V' \subseteq \mathcal{P}(U)$ is the set of vertices representing bunches derived from V.*
- *$E' \subseteq V' \times V'$ is the set of edges derived from E based on thresholding or partitioning the fuzzy membership values.*

Proof. Let $G_F = (V, \mu, \sigma)$ be a fuzzy bunch graph defined as follows:

- V is the set of vertices, each with a fuzzy membership degree $\mu(v) \in [0, 1]$, indicating the degree of inclusion of the vertex v.
- $E \subseteq V \times V$ is the set of edges, where each edge $e = (u, v) \in E$ has a membership degree $\sigma(e) \in [0, 1]$.

To transform G_F into a classical Bunch Graph $G_B = (V', E')$, follow these steps:
Step 1: Define the New Vertex Set V'

1. *Partition the fuzzy vertex set:* Create subsets $S_\alpha \subseteq V$ for each threshold $\alpha \in [0, 1]$:

$$S_\alpha = \{v \in V \mid \mu(v) \geq \alpha\}.$$

Each S_α represents a bunch of vertices with at least α-degree membership.

2. *Map to V':* Let $V' = \{S_\alpha \mid \alpha \text{ is a chosen threshold value}\}$. This forms the vertex set of the bunch graph, where each vertex in V' is a subset of V.

Step 2: Define the New Edge Set E'

1. *Threshold edge membership:* For each edge $e = (u, v) \in E$, apply a threshold $\beta \in [0, 1]$ to determine whether the edge exists in E':

$$e' \in E' \iff \sigma(e) \geq \beta.$$

2. *Map edges to vertex subsets:* For any $S_{\alpha_1}, S_{\alpha_2} \in V'$, add an edge $(S_{\alpha_1}, S_{\alpha_2})$ to E' if:

$$\exists u \in S_{\alpha_1}, v \in S_{\alpha_2} \text{ such that } (u, v) \in E \text{ and } \sigma(u, v) \geq \beta.$$

Step 3: Verify Properties of the Bunch Graph

- *Vertex Representation:* Each vertex $S_\alpha \in V'$ is a subset of V, satisfying the requirement that vertices in a Bunch Graph represent groups (bunches) of elements.
- *Edge Representation:* The edges in E' connect subsets S_{α_1} and S_{α_2}, derived from the fuzzy edges in G_F by applying thresholds. Multi-edges are allowed, preserving the multi-edge property of Bunch Graphs.
- *Classical Graph Property:* The resulting graph $G_B = (V', E')$ satisfies the definition of a Bunch Graph, where:

$$V' \subseteq \mathcal{P}(U), \quad E' \subseteq V' \times V'.$$

The transformation from G_F to G_B discretizes the fuzzy membership degrees and edge strengths, resulting in a classical Bunch Graph that captures the essential structure of the original Fuzzy Bunch Graph. □

The definition of the Intuitionistic Fuzzy Bunch Graph and the related theorem are provided below.

Definition 12 (Intuitionistic Fuzzy Bunch Graph)**.** Let U be a universal set, V be a collection of nodes (including bunches), and $E \subseteq V \times V$ be a set of edges such that:

1. $\emptyset \notin V \subseteq \mathcal{P}(U)$.
2. Each vertex $v \in V$ has two degrees: membership $\mu(v) \in [0,1]$ and non-membership $\nu(v) \in [0,1]$, such that $\mu(v) + \nu(v) \leq 1$.
3. Each edge $e = (u, v) \in E$ has two degrees: membership $\sigma(e) \in [0,1]$ and non-membership $\tau(e) \in [0,1]$, such that $\sigma(e) + \tau(e) \leq 1$.
4. The edge membership conditions are:

$$\sigma(e) \leq \min\{\mu(u), \mu(v)\}, \quad \tau(e) \geq \max\{\nu(u), \nu(v)\}, \quad \forall e = (u, v) \in E.$$

The structure $G = (V, \mu, \nu, \sigma, \tau)$ is called an *intuitionistic fuzzy bunch graph*.

Theorem 13. *A Fuzzy Bunch Graph $G_F = (V, \mu, \sigma)$ can be transformed into a Bunch Graph $G_B = (V', E')$ by discretizing the fuzzy membership degrees and edge strengths, where:*

- *$V' \subseteq \mathcal{P}(U)$ is the set of vertices representing bunches derived from V.*
- *$E' \subseteq V' \times V'$ is the set of edges derived from E based on thresholding or partitioning the fuzzy membership values.*

Proof. Let $G_F = (V, \mu, \sigma)$ be a fuzzy bunch graph defined as follows:

- V is the set of vertices, each with a fuzzy membership degree $\mu(v) \in [0,1]$, indicating the degree of inclusion of the vertex v.
- $E \subseteq V \times V$ is the set of edges, where each edge $e = (u, v) \in E$ has a membership degree $\sigma(e) \in [0,1]$.

To transform G_F into a classical Bunch Graph $G_B = (V', E')$, follow these steps:
Step 1: Define the New Vertex Set V'

1. *Partition the fuzzy vertex set:* Create subsets $S_\alpha \subseteq V$ for each threshold $\alpha \in [0,1]$:

$$S_\alpha = \{v \in V \mid \mu(v) \geq \alpha\}.$$

Each S_α represents a bunch of vertices with at least α-degree membership.

2. *Map to V':* Let $V' = \{S_\alpha \mid \alpha \text{ is a chosen threshold value}\}$. This forms the vertex set of the bunch graph, where each vertex in V' is a subset of V.

Step 2: Define the New Edge Set E'

1. *Threshold edge membership:* For each edge $e = (u, v) \in E$, apply a threshold $\beta \in [0,1]$ to determine whether the edge exists in E':

$$e' \in E' \iff \sigma(e) \geq \beta.$$

2. *Map edges to vertex subsets:* For any $S_{\alpha_1}, S_{\alpha_2} \in V'$, add an edge $(S_{\alpha_1}, S_{\alpha_2})$ to E' if:

$$\exists u \in S_{\alpha_1}, v \in S_{\alpha_2} \text{ such that } (u, v) \in E \text{ and } \sigma(u, v) \geq \beta.$$

Step 3: Verify Properties of the Bunch Graph

- *Vertex Representation:* Each vertex $S_\alpha \in V'$ is a subset of V, satisfying the requirement that vertices in a Bunch Graph represent groups (bunches) of elements.
- *Edge Representation:* The edges in E' connect subsets S_{α_1} and S_{α_2}, derived from the fuzzy edges in G_F by applying thresholds. Multi-edges are allowed, preserving the multi-edge property of Bunch Graphs.
- *Classical Graph Property:* The resulting graph $G_B = (V', E')$ satisfies the definition of a Bunch Graph, where:

$$V' \subseteq \mathcal{P}(U), \quad E' \subseteq V' \times V'.$$

The transformation from G_F to G_B discretizes the fuzzy membership degrees and edge strengths, resulting in a classical Bunch Graph that captures the essential structure of the original Fuzzy Bunch Graph. □

The definition of the Neutrosophic Bunch Graph and the related theorem are provided below.

Definition 14 (Neutrosophic Bunch Graph)**.** Let U be a universal set, $V \subseteq \mathcal{P}(U)$ a collection of subsets of U called bunches, and $E \subseteq V \times V$ a set of edges. A *Neutrosophic Bunch Graph* $G = (V, E, \sigma, \mu)$ is defined by:

1. *Vertices:* Each vertex $v \in V$ is assigned a neutrosophic membership degree $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where:
$$\sigma_T(v), \sigma_I(v), \sigma_F(v) \in [0,1], \quad \sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3.$$
Here:
   - $\sigma_T(v)$: Degree of truth-membership of v,
   - $\sigma_I(v)$: Degree of indeterminacy-membership of v,
   - $\sigma_F(v)$: Degree of falsity-membership of v.
2. *Edges:* Each edge $e = (u, v) \in E$ is assigned a neutrosophic membership degree $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, where:
$$\mu_T(e), \mu_I(e), \mu_F(e) \in [0,1], \quad \mu_T(e) + \mu_I(e) + \mu_F(e) \leq 3.$$
Here:
   - $\mu_T(e)$: Degree of truth of the relationship between u and v,
   - $\mu_I(e)$: Degree of indeterminacy of the relationship between u and v,
   - $\mu_F(e)$: Degree of falsity of the relationship between u and v.
3. *Membership Constraints:* The edge membership degrees satisfy the following conditions:
$$\mu_T(e) \leq \min\{\sigma_T(u), \sigma_T(v)\}, \quad \mu_I(e) \geq \max\{\sigma_I(u), \sigma_I(v)\}, \quad \mu_F(e) \geq \max\{\sigma_F(u), \sigma_F(v)\}.$$

Theorem 15. *A Neutrosophic Bunch Graph $G_N = (V, E, \sigma, \mu)$ can be transformed into an Intuitionistic Fuzzy Bunch Graph $G_{IF} = (V, E, \mu', \nu', \sigma', \tau')$ by redefining the membership degrees as:*
$$\mu'(v) = \sigma_T(v), \quad \nu'(v) = \sigma_F(v),$$
$$\sigma'(e) = \mu_T(e), \quad \tau'(e) = \mu_F(e),$$
for all $v \in V$ and $e \in E$. The indeterminacy components $\sigma_I(v)$ and $\mu_I(e)$ are discarded in the transformation.

Proof. Let $G_N = (V, E, \sigma, \mu)$ be a neutrosophic bunch graph where:

- $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$ for each vertex $v \in V$,
- $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$ for each edge $e \in E$,

with the constraints:
$$\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3, \quad \mu_T(e) + \mu_I(e) + \mu_F(e) \leq 3.$$

The transformed intuitionistic fuzzy bunch graph G_{IF} is defined by:
$$\begin{aligned} \mu'(v) &= \sigma_T(v), & \nu'(v) &= \sigma_F(v), \\ \sigma'(e) &= \mu_T(e), & \tau'(e) &= \mu_F(e). \end{aligned}$$

The vertex membership condition $\mu'(v) + \nu'(v) \leq 1$ holds because:
$$\sigma_T(v) + \sigma_F(v) \leq 1, \quad \forall v \in V,$$
since $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.

The edge membership condition $\sigma'(e) + \tau'(e) \leq 1$ holds because:
$$\mu_T(e) + \mu_F(e) \leq 1, \quad \forall e \in E,$$
since $\mu_T(e) + \mu_I(e) + \mu_F(e) \leq 3$.

By redefining the membership and non-membership degrees, the transformed graph G_{IF} satisfies the definition of an intuitionistic fuzzy bunch graph. □

Definition 16 (Turiyam Neutrosophic Bunch Graph)**.** Let U be a universal set, V be a collection of nodes (including bunches), and $E \subseteq V \times V$ be a set of edges such that:

1. $\emptyset \notin V \subseteq \mathcal{P}(U)$.
2. Each vertex $v \in V$ is represented by four degrees: truth $t(v)$, indeterminacy $iv(v)$, falsity $fv(v)$, and liberal state $lv(v)$, with $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.
3. Each edge $e = (u, v) \in E$ has corresponding four degrees: $t(e)$, $iv(e)$, $fv(e)$, and $lv(e)$.
4. The edge membership conditions are:

$$t(e) \leq \min\{t(u), t(v)\}, \quad iv(e) \geq \max\{iv(u), iv(v)\},$$

$$fv(e) \geq \max\{fv(u), fv(v)\}, \quad lv(e) \geq \max\{lv(u), lv(v)\}.$$

The structure $G = (V, t, iv, fv, lv, t(e), iv(e), fv(e), lv(e))$ is called a *Turiyam Neutrosophic bunch graph*.

Definition 17 (Plithogenic Bunch Graph)**.** Let U be a universal set, V be a collection of nodes (including bunches), and $E \subseteq V \times V$ be a set of edges such that:

1. $\emptyset \notin V \subseteq \mathcal{P}(U)$.
2. Each vertex $v \in V$ has an attribute degree function $adf(v) = (a_1(v), a_2(v), \ldots, a_s(v))$, where $a_i(v) \in [0, 1]$ for all i.
3. Each edge $e = (u, v) \in E$ has an attribute degree function $adf(e) = (a_1(e), a_2(e), \ldots, a_s(e))$.
4. The edge membership conditions are:

$$adf(e) \leq \min\{adf(u), adf(v)\} \text{ component-wise}, \quad \forall e = (u, v) \in E.$$

The structure $G = (V, adf_V, adf_E)$ is called a *plithogenic bunch graph*.

We examine the relationships between the Plithogenic Bunch Graph and various types of Uncertain Bunch Graphs. The theorem is presented below.

Theorem 18. *A Plithogenic Bunch Graph generalizes the Fuzzy Bunch Graph, Intuitionistic Fuzzy Bunch Graph, Neutrosophic Bunch Graph, and Turiyam Neutrosophic Bunch Graph. Specifically, by selecting an appropriate dimension s and defining the components of the attribute degree functions adf_V and adf_E, a Plithogenic Bunch Graph reduces to these specific types of bunch graphs:*

1. *For $s = 1$, it reduces to a Fuzzy Bunch Graph.*
2. *For $s = 2$, it reduces to an Intuitionistic Fuzzy Bunch Graph.*
3. *For $s = 3$, it reduces to a Neutrosophic Bunch Graph.*
4. *For $s = 4$, it reduces to a Turiyam Neutrosophic Bunch Graph.*

Proof. Consider a Plithogenic Bunch Graph $G = (V, E, adf_V, adf_E)$, where:

- V is the set of vertices (bunches), each associated with an attribute degree function:

$$adf_V(v) = (a_1(v), a_2(v), \ldots, a_s(v)), \quad a_i(v) \in [0, 1], \quad \forall v \in V.$$

- $E \subseteq V \times V$ is the set of edges, each associated with an attribute degree function:

$$adf_E(e) = (a_1(e), a_2(e), \ldots, a_s(e)), \quad a_i(e) \in [0, 1], \quad \forall e \in E.$$

- The edge membership constraint satisfies:

$$a_i(e) \leq \min\{a_i(u), a_i(v)\}, \quad \forall i \in \{1, 2, \ldots, s\}, \quad \forall e = (u, v) \in E.$$

We demonstrate the reduction by choosing specific values for s and mapping a_i to construct well-known graph types.

Case 1: Fuzzy Bunch Graph Set $s = 1$, where the attribute degree function reduces to a single component:

$$adf_V(v) = \mu(v), \quad adf_E(e) = \sigma(e), \quad \mu(v), \sigma(e) \in [0, 1].$$

The edge membership condition simplifies to:

$$\sigma(e) \leq \min\{\mu(u), \mu(v)\}, \quad \forall e = (u, v) \in E.$$

This matches the definition of a Fuzzy Bunch Graph $G = (V, \mu, \sigma)$.

Case 2: Intuitionistic Fuzzy Bunch Graph Set $s = 2$, where the attribute degree function consists of two components:

$$adf_V(v) = (\mu(v), \nu(v)), \quad adf_E(e) = (\sigma(e), \tau(e)).$$

Here:

- $\mu(v)$ is the membership degree of v,
- $\nu(v)$ is the non-membership degree of v,
- $\sigma(e)$ is the membership degree of e,
- $\tau(e)$ is the non-membership degree of e.

The following conditions hold for Intuitionistic Fuzzy Graphs:

$$\mu(v) + \nu(v) \leq 1, \quad \sigma(e) + \tau(e) \leq 1, \quad \forall v \in V, \forall e \in E.$$

The edge membership conditions are:

$$\sigma(e) \leq \min\{\mu(u), \mu(v)\}, \quad \tau(e) \geq \max\{\nu(u), \nu(v)\}, \quad \forall e = (u, v) \in E.$$

This corresponds to the definition of an Intuitionistic Fuzzy Bunch Graph $G = (V, \mu, \nu, \sigma, \tau)$.

Case 3: Neutrosophic Bunch Graph Set $s = 3$, where the attribute degree function consists of three components:

$$adf_V(v) = (\mu_T(v), \mu_I(v), \mu_F(v)), \quad adf_E(e) = (\sigma_T(e), \sigma_I(e), \sigma_F(e)).$$

The edge membership conditions are defined as:

$$\sigma_T(e) \leq \min\{\mu_T(u), \mu_T(v)\},$$

$$\sigma_I(e) \geq \max\{\mu_I(u), \mu_I(v)\},$$

$$\sigma_F(e) \geq \max\{\mu_F(u), \mu_F(v)\}, \quad \forall e = (u, v) \in E.$$

This aligns with the definition of a Neutrosophic Bunch Graph $G = (V, E, \sigma, \mu)$.

Case 4: Turiyam Neutrosophic Bunch Graph Set $s = 4$, where the attribute degree function consists of four components:

$$adf_V(v) = (t(v), iv(v), fv(v), lv(v)), \quad adf_E(e) = (t(e), iv(e), fv(e), lv(e)).$$

The edge membership conditions are defined as:

$$t(e) \leq \min\{t(u), t(v)\},$$

$$iv(e) \geq \max\{iv(u), iv(v)\},$$

$$fv(e) \geq \max\{fv(u), fv(v)\},$$

$$lv(e) \geq \max\{lv(u), lv(v)\}, \quad \forall e = (u, v) \in E.$$

This corresponds to the definition of a Turiyam Neutrosophic Bunch Graph $G = (V, t, iv, fv, lv, t(e), iv(e), fv(e), lv(\mathrm{e}))$.

By setting $s = 1$, $s = 2$, $s = 3$, or $s = 4$, and appropriately mapping the components of adf_V and adf_E, a Plithogenic Bunch Graph can reduce to a Fuzzy Bunch Graph, Intuitionistic Fuzzy Bunch Graph, Neutrosophic Bunch Graph, or Turiyam Neutrosophic Bunch Graph, respectively. This demonstrates that the Plithogenic Bunch Graph generalizes these specific types of bunch graphs. □

Theorem 19. *If a contradiction degree function $aCf_V : V \times V \to [0,1]^t$ is defined in a Plithogenic Bunch Graph G, it is symmetric and satisfies $aCf_V(u,u) = 0$ for all $u \in V$:*

$$aCf_V(u,v) = aCf_V(v,u), \quad \forall u,v \in V,$$

$$aCf_V(u,u) = 0, \quad \forall u \in V.$$

Proof. By definition, the contradiction function aCf_V in a Plithogenic Bunch Graph must satisfy reflexivity and symmetry:

$$aCf_V(u,u) = 0,$$

$$aCf_V(u,v) = aCf_V(v,u).$$

This ensures consistent measurement of contradiction degrees between vertices. □

3.2 Relationship between Plithogenic Mixed Graphs and Plithogenic Bunch Graphs

We examine the relationship between Plithogenic Mixed Graphs [15] and Plithogenic Bunch Graphs. A Plithogenic Mixed Graph is a graph that combines the concepts of Plithogenic Directed Graphs and Plithogenic Undirected Graphs. It can also be seen as an extension of the classical concept of Mixed Graphs(cf.[40]) to the framework of Plithogenic Graphs.

The formal definition of a Plithogenic Mixed Graph is presented below [15].

Definition 20 (Plithogenic Mixed Graph)**.** [15] A *Plithogenic Mixed Graph*

$$G = (V, E, A, adf_V, adf_E, adf_A, aCf_V, aCf_E, aCf_A)$$

is defined as:

- V: the set of vertices.
- E: the set of undirected edges.
- A: the set of directed edges (arcs).
- $adf_V : V \to [0,1]^s$ is the *attribute degree function* for vertices, assigning to each vertex $v \in V$ an s-tuple $adf_V(v) = (a_1(v), a_2(v), \dots, a_s(v))$.
- $adf_E : E \to [0,1]^s$ is the *attribute degree function* for undirected edges.
- $adf_A : A \to [0,1]^s$ is the *attribute degree function* for directed edges.
- $aCf_V : V \times V \to [0,1]^t$ is the *contradiction degree function* for vertices.
- aCf_E and aCf_A are the contradiction degree functions for undirected edges and arcs, respectively.

These functions satisfy the following conditions:

1. *Edge Attribute Degree Constraint*:
   - For each undirected edge $e = \{u,v\} \in E$:

     $$adf_E(e) \le adf_V(u) \wedge adf_V(v),$$

     where $\wedge$ denotes the minimum operation taken component-wise.
   - For each directed edge $a = (u,v) \in A$:

     $$adf_A(a) \le adf_V(u) \wedge adf_V(v).$$

2. *Contradiction Function Constraints*:
   - The contradiction functions satisfy:

     $$aCf_V(u,u) = 0, \quad aCf_V(u,v) = aCf_V(v,u), \quad \forall u,v \in V.$$

   - Similar properties hold for aCf_E and aCf_A.

Theorem 21. *Every Plithogenic Mixed Graph can be represented as a Plithogenic Bunch Graph.*

Proof. Let $G = (V, E, A, adf_V, adf_E, adf_A, aCf_V, aCf_E, aCf_A)$ be a Plithogenic Mixed Graph, where:

- V is the set of vertices.
- E is the set of undirected edges.
- A is the set of directed edges (arcs).
- adf_V, adf_E, adf_A are the attribute degree functions for vertices, undirected edges, and arcs, respectively.

We construct a Plithogenic Bunch Graph $G' = (V', adf_{V'}, E', adf_{E'})$ as follows:

1. Let the universal set $U = V$.
2. Define the vertex set $V' = \{\{v\} \mid v \in V\}$, where each vertex in V becomes a singleton subset in V'.
3. Define the edge set $E' \subseteq V' \times V'$ by:
$$E' = \{(\{u\}, \{v\}) \mid \{u, v\} \in E \text{ or } (u, v) \in A\}.$$
4. Define the attribute degree functions:
$$adf_{V'}(\{v\}) = adf_V(v), \quad \forall v \in V.$$
$$adf_{E'}((\{u\}, \{v\})) = \begin{cases} adf_E(\{u, v\}), & \text{if } \{u, v\} \in E, \\ adf_A((u, v)), & \text{if } (u, v) \in A. \end{cases}$$
5. Since $\emptyset \notin V' \subseteq \mathcal{P}(U)$, V' satisfies the vertex condition of a Plithogenic Bunch Graph.
6. The edge attribute degrees satisfy:
$$adf_{E'}((\{u\}, \{v\})) \le \min\{adf_{V'}(\{u\}), adf_{V'}(\{v\})\},$$
because this condition holds in the original Plithogenic Mixed Graph.

Therefore, the Plithogenic Mixed Graph G can be represented as a Plithogenic Bunch Graph G'. □

Theorem 22. *Conversely, any Plithogenic Bunch Graph with singleton vertices corresponds to a Plithogenic Mixed Graph.*

Proof. Given a Plithogenic Bunch Graph $G = (V, adf_V, E, adf_E)$ where each vertex $v \in V$ is a singleton subset $\{u\}$.

Construct a Plithogenic Mixed Graph $G' = (V', E', A', adf_{V'}, adf_{E'}, adf_{A'})$ as follows:

1. $V' = \{u \mid \{u\} \in V\}$.
2. For each edge $(\{u\}, \{v\}) \in E$:
   - If the edge represents an undirected relationship, include $\{u, v\}$ in E'.
   - If the edge represents a directed relationship, include (u, v) in A'.
3. Set $adf_{V'}(u) = adf_V(\{u\})$, $adf_{E'}(\{u, v\}) = adf_E((\{u\}, \{v\}))$, and similarly for arcs.

Since the attribute degree functions and edge conditions are preserved, G' is a Plithogenic Mixed Graph corresponding to G. □

4. Future Directions of This Research

This section outlines the future directions for this research.

We aim to extend the concept of Bunch Hypergraphs (cf. [10]) to new environments and contexts. Furthermore, we plan to define and investigate Bunch Superhypergraphs (cf. [18, 25, 33, 49, 50]), focusing on their mathematical properties, structural characteristics, and potential applications. As a brief note, hypergraphs generalize classic graphs by allowing edges to connect multiple vertices, while superhypergraphs extend hypergraphs by incorporating even more complex relationships.

Bunch Superhypergraphs expand on the principles of Bunch Graphs by applying them to hypergraphs and superhypergraphs. This extension paves the way for novel insights and broader applications.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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A Reconsideration of Advanced Concepts in Neutrosophic Graphs: Smart, Zero Divisor, Layered, Weak, Semi, and Chemical Graphs

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Abstract: One of the most powerful tools in graph theory is the classification of graphs into distinct classes based on shared properties or structural features. Over time, many graph classes have been introduced, each aimed at capturing specific behaviors or characteristics of a graph. Neutrosophic Set Theory, a method for handling uncertainty, extends fuzzy logic by incorporating degrees of truth, indeterminacy, and falsity. Building on this framework, Neutrosophic Graphs [9,84,135] have emerged as significant generalizations of fuzzy graphs. In this paper, we extend several classes of fuzzy graphs to Neutrosophic graphs and analyze their properties.

Keywords: Neutrosophic graph, Fuzzy Graph, Graph class, Neutrosophic set

1. Introduction

1.1 Graph Theory and Graph Classes

Graph theory, a fundamental branch of mathematics, investigates the relationships between nodes (vertices) and edges (connections) that form networks. It emphasizes the study of their structures, paths, and properties [50]. This field has been extensively explored due to its wide-ranging applications across diverse domains, including computer science, biology, and network analysis (ex.[36, 107, 141]).

One of the core aspects of graph theory is the classification of graphs into distinct classes based on shared properties or structural characteristics. Such classifications enable the development of efficient algorithms, facilitate problem-solving, and provide deeper insights into computational complexity. Moreover, these graph classes serve as essential frameworks for studying specific graph behaviors and their applications in various disciplines (cf.[23, 37, 40, 109]).

Notable examples of graph classes include Tree Graphs [156], Path Graphs [163], Complete Graphs [46], Circle Graphs [39], Unit Disk Graphs [47], Edge-Transitive Graphs [100, 101], Ultrahomogeneous Graphs [90], Visibility Graphs [89], Outerplanar Graphs [76], Petersen Graphs [75], and Total Graphs [157]. Studying these classes allows researchers to identify common properties, create specialized and efficient algorithms, and apply these insights to practical and theoretical problems.

1.2 Fuzzy Graph and Neutrosophic graph

Uncertainty refers to the lack of complete knowledge or predictability, influencing decision-making across disciplines like economics, science, and risk management. Zadeh [166] introduced fuzzy set theory in 1965 to address uncertainty, and Rosenfeld [117, 131] extended this concept to fuzzy graph theory in 1975. A fuzzy set is widely used to model uncertainty across various real-life domains [69, 86, 96, 167–172]. It is defined by a membership function, denoted as l, which maps values to the range $[0, 1]$.

Fuzzy graphs assign membership values to both vertices and edges, allowing for the analysis of relationships with imprecision, and have been applied in fields such as logic, information theory, robotics, and nanotechnology [107, 141]. Within fuzzy graph theory, various graph classes have been proposed to generalize fuzzy graphs or adapt them for real-world applications. These include Intuitionistic Fuzzy Graphs [118], Bipolar Fuzzy Graphs [4], Fuzzy Planar Graphs [138], Hyperfuzzy graph (Hyperfuzzy set)[56, 64, 82, 154], Superhyperfuzzy graph (Superhyperfuzzy set) [56], Irregular Bipolar Fuzzy Graphs [137], and Complex Hesitant Fuzzy Graphs [1], among others[57]. Studying these classes helps researchers uncover common properties, develop specialized algorithms, and apply findings to practical problems.

In addition to fuzzy graphs, other frameworks have been developed to handle uncertainty and real-life parameters, such as weighted graphs [74], rough graph[49, 139], vague graph[38, 129], and Plithogenic Graphs [60, 85, 140, 146, 152].

Neutrosophic Set Theory, an alternative approach to handling uncertainty, was proposed to extend fuzzy logic by incorporating degrees of truth, indeterminacy, and falsity[42, 56, 142–144, 161]. Similar to fuzzy set theory, Neutrosophic Sets have been widely researched for their applications across various fields [51,149,165]. Intuitively speaking, when Neutrosophic Set Theory is applied to graphs, it leads to the concept of Neutrosophic Graphs. Neutrosophic Graphs [9,52,53,58,61,84,135] and Neutrosophic Hypergraphs [11,95] have emerged as significant generalizations of fuzzy graphs. These frameworks have attracted attention due to their applications in areas closely related to fuzzy graph theory [53,55,117].

Numerous classes of Neutrosophic Graphs have been studied, including Bipolar Neutrosophic Graphs [11], Neutrosophic Incidence Graphs [153], HyperNeutrosophic graph (HyperNeutrosophic set) [56], SuperhyperNeutrosophic graph (SuperhyperNeutrosophic set) [56], and Complex Neutrosophic Hypergraphs [95]. Investigating these graph classes allows researchers to identify shared properties, refine algorithms, and explore new applications in various fields.

1.3 Our Contribution

Based on the above, the study of graph classes holds great significance. In this paper, we extend several classes of fuzzy graphs to Neutrosophic graphs and analyze their properties. Specifically, we explore graph classes related to Neutrosophic Graphs, including Smart Neutrosophic Graphs, Neutrosophic Zero Divisor Graphs, Weak Neutrosophic Graphs, Neutrosophic Semigraphs, Double/Triple Layered Neutrosophic Graphs, and Connected Neutrosophic Chemical Graphs.

1.4 The Structure of the Paper

The structure of this paper is as follows. In Section 2, we explain the various definitions, including the concepts of fuzzy graph classes. Section 3 introduces the concepts that extend these fuzzy graph classes to Neutrosophic Graphs. Finally, Section 4 discusses future tasks and directions.

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2. Preliminaries and definitions

In this section, we will briefly explain the definitions and notations used in this paper. We begin by introducing fundamental concepts related to graphs and rings, followed by an explanation of fuzzy graphs and Neutrosophic Graphs. Subsequently, we will present the definitions and examples of various graph classes, including Smart Fuzzy Graphs, Fuzzy Zero Divisor Graphs, Weak Fuzzy Graphs, Fuzzy Semigraphs, Mild Balanced Intuitionistic Fuzzy Graphs, Double/Triple Layered Fuzzy Graphs, and Connected Fuzzy Chemical Graphs.

2.1 Basic Graph Concepts

Here are a few basic graph concepts listed below. In addition to graph concepts, this paper also utilizes fundamental concepts from set theory. Readers may refer to lecture notes or surveys on set theory as needed [73, 80, 91].

Definition 2.1 (Graph)**.** [50] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Graphs can be used to model a wide range of real-world concepts. Although just one example, the following definition illustrates this capability.

Example 2.2 (Social Network Graph)**.** In social networks, individuals can be represented as *vertices*, and their connections (such as friendships, interactions, or communications) can be represented as *edges*. This type of graph models how people are connected to each other and allows for analysis of social dynamics, such as identifying influencers, clusters of closely connected individuals, or finding the shortest path between people.

- **Vertices**: Each person in the network is represented by a vertex (node).
- **Edges**: A connection or relationship between two people (e.g., a ”friendship” in Facebook) is represented by an edge between two vertices.

Definition 2.3 (Subgraph)**.** [50] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Example 2.4. Consider the graph $G = (V, E)$, where:

$$V = \{v_1, v_2, v_3, v_4, v_5\}, \quad E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}$$

A subgraph $H = (V_H, E_H)$ is created by selecting a subset of the vertices and the edges from G.
Let the subgraph H be defined by the following vertex and edge sets:

$$V_H = \{v_2, v_3, v_4\}, \quad E_H = \{(v_2, v_3), (v_3, v_4)\}$$

The subgraph H includes the vertices v_2, v_3, v_4 and the edges that connect these vertices in the original graph G, specifically the edges (v_2, v_3) and (v_3, v_4). It excludes the vertices v_1 and v_5 and their incident edges.

Definition 2.5 (Degree)**.** [50] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 2.6 (Connectedness)**.** A graph $G = (V, E)$ is said to be *connected* if for every pair of vertices $u, v \in V$, there exists a path $P \subseteq G$ that connects u and v. Formally, G is connected if:

$$\forall u, v \in V, \exists P \subseteq G \text{ such that } P \text{ is a path from } u \text{ to } v.$$

Example 2.7 (Connected and Unconnected Graphs)**.** Consider the following two graphs:

- **Connected Graph:** Let $G_1 = (V_1, E_1)$ be a graph with the vertex set $V_1 = \{v_1, v_2, v_3, v_4\}$ and edge set $E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$.

  In this graph, there exists a path between any pair of vertices. For example:

  $$P_{v_1 \to v_4} = (v_1, v_2, v_3, v_4)$$

  is a path connecting v_1 to v_4. Thus, G_1 is a connected graph.

- **Unconnected Graph:** Let $G_2 = (V_2, E_2)$ be a graph with the vertex set $V_2 = \{u_1, u_2, u_3, u_4\}$ and edge set $E_2 = \{(u_1, u_2), (u_3, u_4)\}$.

  In this graph, there is no path between the vertices u_1 and u_3 (or between u_2 and u_4, for example). Hence, G_2 is an unconnected graph because it contains two distinct subgraphs, one with u_1, u_2 and the other with u_3, u_4, and there is no path connecting these subgraphs.

Definition 2.8 (Path). (cf.[41]) A path in a graph $G = (V, E)$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \ldots, k-1$. A path is represented as:

$$P = (v_1, v_2, \ldots, v_k),$$

where no vertex is repeated. The length of a path is the number of edges it contains, i.e., $k-1$.

Example 2.9 (Path). Consider a graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ is the set of vertices and $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}$ is the set of edges.

A path from v_1 to v_5 can be written as:

$$P = (v_1, v_2, v_3, v_4, v_5),$$

where each pair of consecutive vertices (v_i, v_{i+1}) is connected by an edge in E.

Thus, P is a valid path in the graph G, connecting v_1 and v_5 through distinct vertices.

Definition 2.10 (Tree). A tree is a connected, acyclic graph. In other words, a tree is a graph where there is exactly one path between any two vertices, and no cycles exist.

Example 2.11 (Tree). Consider the following graph $T = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ is the set of vertices and $E = \{(v_1, v_2), (v_1, v_3), (v_3, v_4), (v_3, v_5)\}$ is the set of edges. Thus, T is a connected, acyclic graph, satisfying the definition of a tree.

Definition 2.12 (Complete). (cf.[28]) A graph $G = (V, E)$ is said to be *complete* if for every pair of distinct vertices $u, v \in V$, there exists an edge $(u, v) \in E$ connecting them. In other words, every pair of vertices in G is adjacent.

The number of edges in a complete graph with n vertices is given by:

$$|E| = \frac{n(n-1)}{2},$$

where $n = |V|$ is the number of vertices in the graph.

Example 2.13. Consider a complete graph $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4\}$. Since G is complete, every pair of distinct vertices must be connected by an edge. The vertex set and edge set of G are as follows:

$$V = \{v_1, v_2, v_3, v_4\}$$
$$E = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4)\}$$

In this case, the number of vertices is $n = 4$, and the number of edges is:

$$|E| = \frac{n(n-1)}{2} = \frac{4(4-1)}{2} = \frac{12}{2} = 6.$$

Thus, the complete graph G with four vertices has six edges, and each pair of vertices is connected by an edge.

Graphically, the complete graph G can be represented as a set of vertices where every vertex is connected to every other vertex.

Definition 2.14 (union). The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G = (V, E)$ where:

- The vertex set V is the union of the vertex sets of G_1 and G_2:

$$V = V_1 \cup V_2.$$

- The edge set E is the union of the edge sets of G_1 and G_2:

$$E = E_1 \cup E_2.$$

Thus, the union of G_1 and G_2 combines the vertices and edges of both graphs, without duplicating any elements.

Definition 2.15 (Partition of a Graph)**.** A *partition* of a graph $G = (V, E)$ is a division of the vertex set V into disjoint, non-empty subsets $V_1, V_2, \ldots, V_k$ such that:

$$V = V_1 \cup V_2 \cup \cdots \cup V_k \quad \text{and} \quad V_i \cap V_j = \emptyset \quad \text{for all} \quad i \neq j.$$

The subsets $V_1, V_2, \ldots, V_k$ are called the *parts* or *blocks* of the partition.

Definition 2.16 (Bipartite Graph)**.** (cf.[20, 103]) A graph $G = (V, E)$ is called a *bipartite graph* if the vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge $e \in E$ connects a vertex in V_1 to a vertex in V_2. In other words, there are no edges between vertices within the same set V_1 or V_2. Formally, G is bipartite if $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and for all $e = (u, v) \in E$, $u \in V_1$ and $v \in V_2$.

Example 2.17. Consider the graph $G = (V, E)$ where the vertex set V is given by

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and the edge set E is given by

$$E = \{(v_1, v_4), (v_1, v_5), (v_2, v_4), (v_3, v_5), (v_3, v_6)\}.$$

We can partition the vertex set V into two disjoint sets:

$$V_1 = \{v_1, v_2, v_3\}, \quad V_2 = \{v_4, v_5, v_6\}.$$

Every edge in E connects a vertex from V_1 to a vertex from V_2, and there are no edges between vertices within the same set. For example, (v_1, v_4) connects $v_1 \in V_1$ to $v_4 \in V_2$, and similarly for all other edges.

Thus, G is a bipartite graph.

Definition 2.18. In graph theory, a *triangle graph*, also known as the *3-cycle graph* or the *complete graph* K_3, is a simple, undirected graph that consists of three vertices connected by three edges. Each vertex in the triangle graph is connected to every other vertex, forming a cycle of length three.

The triangle graph can be denoted as:

$$T = (V, E)$$

where:

- $V = \{v_1, v_2, v_3\}$ is the set of three vertices, and
- $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$ is the set of three edges, representing the connections between all pairs of vertices.

For more foundational graph concepts and notations, please refer to [50, 67, 68, 111, 162].

2.2 Basic Ring Concepts

A ring is an algebraic structure equipped with two operations, addition and multiplication, that satisfy the properties of associativity, distributivity, and the existence of an additive identity (cf.[79]). Since this paper focuses on the Zero-Divisor Graph, we begin by introducing the fundamental concepts of rings. Several definitions are outlined below.

Definition 2.19 (Commutative Ring)**.** (cf.[18, 97, 99]) A *commutative ring* is a set R equipped with two binary operations, addition $+$ and multiplication $\cdot$, such that:

- $(R, +)$ is an abelian group.
- $(R, \cdot)$ is a monoid with an identity element $1 \in R$ (i.e., multiplication is associative, and there exists a multiplicative identity 1).
- Multiplication is commutative, i.e., $a \cdot b = b \cdot a$ for all $a, b \in R$.
- Multiplication is distributive over addition, i.e., $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$.

Example 2.20. Let $R = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, the set of integers modulo 6. This set forms a commutative ring under addition and multiplication modulo 6. We will verify that this is a commutative ring by performing some calculations for addition and multiplication.

The addition table modulo 6 is as follows:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

This table confirms that: - $(R, +)$ forms an abelian group (commutative, with 0 as the identity and each element having an inverse, for example, $1 + 5 = 0$).

The multiplication table modulo 6 is as follows:

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

This table confirms the following: - $(R, \cdot)$ is a monoid with 1 as the multiplicative identity. - Multiplication is commutative, for example, $2 \cdot 3 = 0$ and $3 \cdot 2 = 0$. - Multiplication is distributive over addition, for example, $2 \cdot (3 + 4) = 2 \cdot 1 = 2$ and $2 \cdot 3 + 2 \cdot 4 = 0 + 2 = 2$.

Thus, $\mathbb{Z}_6$ satisfies all the conditions of a commutative ring.

Definition 2.21 (Zero-Divisor)**.** (cf.[3, 112]) In a commutative ring R, an element $a \in R$ is called a *zero-divisor* if there exists a non-zero element $b \in R$ such that $a \cdot b = 0$.

Example 2.22 (Zero-Divisor)**.** Consider the commutative ring $\mathbb{Z}_6$ (the integers modulo 6). The elements of this ring are $\{0, 1, 2, 3, 4, 5\}$, and addition and multiplication are performed modulo 6.

In this ring, the element 2 is a zero-divisor because:

$$2 \times 3 = 6 \equiv 0 \pmod{6}.$$

Here, $2 \neq 0$ and $3 \neq 0$, yet their product is 0. Therefore, 2 is a zero-divisor in $\mathbb{Z}_6$. Similarly, 3 is also a zero-divisor, as:

$$3 \times 2 = 6 \equiv 0 \pmod{6}.$$

Definition 2.23 (Zero-divisor Graph)**.** (cf.[2, 17, 19, 94]) Let R be a commutative ring with unity, and let $Z(R)$ denote the set of zero-divisors of R. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is an undirected graph defined as follows:

- The vertex set of $\Gamma(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, i.e., the set of nonzero zero-divisors of R.
- For distinct $x, y \in Z(R)^*$, there is an edge between x and y if and only if $xy = 0$ in R.

Thus, the graph $\Gamma(R)$ captures the relationships between the nonzero zero-divisors of R. If R is an integral domain, $\Gamma(R)$ is the empty graph.

Example 2.24 (Zero-divisor Graph)**.** Consider the commutative ring $\mathbb{Z}_6$ (the integers modulo 6). The elements of this ring are $\{0, 1, 2, 3, 4, 5\}$, and the set of zero-divisors is $Z(\mathbb{Z}_6) = \{0, 2, 3, 4\}$.

To construct the zero-divisor graph $\Gamma(\mathbb{Z}_6)$, we first remove the element 0, so the vertex set of $\Gamma(\mathbb{Z}_6)$ is $\{2, 3, 4\}$.

Next, we determine the edges:

- $2 \times 3 = 6 \equiv 0 \pmod{6}$, so there is an edge between 2 and 3.
- $2 \times 4 = 8 \equiv 0 \pmod{6}$, so there is an edge between 2 and 4.
- $3 \times 4 = 12 \equiv 0 \pmod{6}$, so there is an edge between 3 and 4.

Therefore, the zero-divisor graph $\Gamma(\mathbb{Z}_6)$ has vertices $\{2, 3, 4\}$, and it forms a complete graph K_3, where every pair of distinct vertices is connected by an edge.

2.3 Fuzzy graph and Intuitionistic fuzzy graph

A Fuzzy Graph represents relationships under uncertainty by assigning membership degrees to both vertices and edges, enabling more flexible and detailed analysis. Due to its importance, Fuzzy Graphs have been widely studied in various fields [7,8,12–15,30,32,83,93,104,108,113,155,160]. The formal definition of a Fuzzy Graph is given in [105, 131]. This concept extends the set-theoretic ideas of Fuzzy Sets [166] and Intuitionistic Fuzzy Sets [27] into graph theory.

Definition 2.25 (Crisp Graph)**.** A *crisp graph* is a mathematical structure $G = (V, E)$, where:

1. V is a non-empty finite or infinite set, referred to as the set of vertices or nodes.
2. $E \subseteq V \times V$ is a set of edges, representing relationships between pairs of vertices.
3. For any edge $e = (u, v) \in E$, $u, v \in V$, and there is no uncertainty in the membership of u, v in V or e in E.

   Depending on the nature of the edges:

- In an *undirected graph*, if $(u, v) \in E$, then $(v, u) \in E$.
- In a *directed graph*, edges have a specific direction; if $(u, v) \in E$, it does not necessarily imply $(v, u) \in E$.

Definition 2.26. [131] A fuzzy graph $G = (\sigma, \mu)$ with V as the underlying set is defined as follows:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of vertices, where $\sigma(x)$ represents the membership degree of vertex $x \in V$.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation on σ, such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ denotes the minimum operation.

  The underlying crisp graph of G is denoted by $G^* = (\sigma^*, \mu^*)$, where:

- $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$
- $\mu^* = \sup p(\mu) = \{(x, y) \in V \times V : \mu(x, y) > 0\}$

  A fuzzy subgraph $H = (\sigma', \mu')$ of G is defined as follows:

- There exists $X \subseteq V$ such that $\sigma' : X \to [0, 1]$ is a fuzzy subset.
- $\mu' : X \times X \to [0, 1]$ is a fuzzy relation on σ', satisfying $\mu'(x, y) \leq \sigma'(x) \wedge \sigma'(y)$ for all $x, y \in X$.

Example 2.27. (cf.[45]) Consider a fuzzy graph $G = (\sigma, \mu)$ with four vertices $V = \{v_1, v_2, v_3, v_4\}$.
The membership degrees of the vertices are as follows:

$$\sigma(v_1) = 0.1, \quad \sigma(v_2) = 0.3, \quad \sigma(v_3) = 0.2, \quad \sigma(v_4) = 0.4$$

The fuzzy relation on the edges is defined by the values of μ, where $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. The fuzzy membership degrees of the edges are as follows:

$$\mu(v_1, v_2) = 0.1, \quad \mu(v_2, v_3) = 0.1, \quad \mu(v_3, v_4) = 0.1$$

$$\mu(v_4, v_1) = 0.1, \quad \mu(v_2, v_4) = 0.3$$

In this case, the fuzzy graph G has the following properties:

- Vertices v_1, v_2, v_3, v_4 are connected by edges with varying membership degrees.
- The fuzzy relations ensure that $\mu(x, y)$ for any edge (x, y) does not exceed the minimum membership of the corresponding vertices.

Definition 2.28. [62] A fuzzy graph $G = (\sigma, \mu)$ is called *complete* if for all $u, v \in V$, the following condition holds:

$$\mu(u, v) = \sigma(u) \wedge \sigma(v),$$

where $\wedge$ denotes the minimum operation.

Proposition 2.29. *A complete fuzzy graph is a special case of a fuzzy graph.*

Proof. This is evident. □

The intuitionistic fuzzy graph, which generalizes the fuzzy graph, is also defined in a similar manner[5, 48, 118, 119, 164]. The definition is provided below.

Definition 2.30. [5] An *intuitionistic fuzzy graph* $G = (A, B)$ on an underlying set V is defined as follows:

(i) The functions $\mu_A : V \to [0,1]$ and $\nu_A : V \to [0,1]$ represent the degree of membership and non-membership of each vertex $x \in V$, respectively. These functions satisfy the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad \forall x \in V.$$

(ii) The functions $\mu_B : E \subseteq V \times V \to [0,1]$ and $\nu_B : E \subseteq V \times V \to [0,1]$ represent the degree of membership and non-membership of each edge $\{x, y\} \in E$, respectively. These functions satisfy the following conditions for all $\{x, y\} \in E$:

$$\mu_B(\{x,y\}) \leq \min(\mu_A(x), \mu_A(y)),$$

$$\nu_B(\{x,y\}) \geq \max(\nu_A(x), \nu_A(y)),$$

$$0 \leq \mu_B(\{x,y\}) + \nu_B(\{x,y\}) \leq 1.$$

Here:

- A is the *intuitionistic fuzzy vertex set* of G, and
- B is the *intuitionistic fuzzy edge set* of G, which represents a symmetric intuitionistic fuzzy relation on A.

The intuitionistic fuzzy graph $G = (A, B)$ corresponds to the crisp graph $G^* = (V, E)$ if the following conditions hold for all $\{x, y\} \in E$:

$$\mu_B(\{x,y\}) \leq \min(\mu_A(x), \mu_A(y)), \quad \nu_B(\{x,y\}) \geq \max(\nu_A(x), \nu_A(y)).$$

2.4 Neutrosophic Graph and Intuitionistic Neutrosophic Graph

We introduce the concept of a neutrosophic graph [9, 44, 70, 77, 135, 150], which extends the framework of fuzzy graphs [83].

A neutrosophic graph can also be viewed as a graphical representation of a neutrosophic set. Therefore, we begin by providing the definition of a neutrosophic set below.

Definition 2.31 (Neutrosophic Set)**.** (cf.[43, 110, 136, 142, 145, 145, 151]) Let X be a space of points and let $x \in X$. A *neutrosophic set* S in X is characterized by three membership functions: a truth membership function T_S, an indeterminacy membership function I_S, and a falsity membership function F_S. For each point $x \in X$, $T_S(x), I_S(x)$, and $F_S(x)$ are real standard or non-standard subsets of the interval $]0^-, 1^+[$, where:

$$T_S, I_S, F_S : X \to [0^-, 1^+].$$

The neutrosophic set S can be represented as:

$$S = \{(x, T_S(x), I_S(x), F_S(x)) \mid x \in X\}.$$

There are no restrictions on the sum of $T_S(x), I_S(x)$, and $F_S(x)$, so:

$$0 \leq T_S(x) + I_S(x) + F_S(x) \leq 3^+.$$

Definition 2.32 (Single Valued Neutrosophic Set)**.** Let X be a universal set, and let $x \in X$ be an element of X. A *Single Valued Neutrosophic Set (SVNS)* A on X is characterized by three functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where:

- $T_A(x)$ represents the truth-membership degree of x in A,
- $I_A(x)$ represents the indeterminacy-membership degree of x in A,
- $F_A(x)$ represents the falsity-membership degree of x in A.

These functions satisfy:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3 \quad \text{for all } x \in X.$$

When X is continuous, A can be expressed as:

$$A = \{(T_A(x), I_A(x), F_A(x)) \mid x \in X\}.$$

When X is discrete, A can be expressed as:

$$A = \left\{(T_A(x_i), I_A(x_i), F_A(x_i))/x_i \mid x_i \in X\right\}.$$

Below are three real-world scenarios where neutrosophic sets can be applied.

Example 2.33. Let X be the set of possible conditions a patient might have, and $x \in X$ represent the condition "diabetes." A neutrosophic set S is used to characterize the diagnosis:

- $T_S(x) = 0.8$: The truth degree, derived from positive glucose tolerance test results and family history.
- $I_S(x) = 0.15$: The indeterminacy degree, reflecting uncertainties due to borderline HbA1c levels and inconsistent symptoms.
- $F_S(x) = 0.05$: The falsity degree, representing evidence against diabetes, such as normal fasting glucose levels.

Thus, the neutrosophic representation of the diagnosis is:

$$S = \{(x, 0.8, 0.15, 0.05) \mid x \in X\}.$$

Example 2.34. Consider an environmental monitoring system where X represents regions, and $x \in X$ is "Region A." The neutrosophic set S evaluates the pollution status of x:

- $T_S(x) = 0.6$: The truth degree that the area is polluted, based on sensor data showing moderate PM2.5 levels.
- $I_S(x) = 0.3$: The indeterminacy degree due to missing data from certain sensors and conflicting readings.
- $F_S(x) = 0.1$: The falsity degree, based on visual inspections showing clear skies and absence of visible pollution.

The neutrosophic representation of the pollution status is:

$$S = \{(x, 0.6, 0.3, 0.1) \mid x \in X\}.$$

Example 2.35. Let X represent products on an e-commerce platform, and $x \in X$ denote a specific product, "Smartphone A." The neutrosophic set S evaluates customer satisfaction:

- $T_S(x) = 0.7$: The truth degree that customers are satisfied with the product, based on 70% positive reviews.
- $I_S(x) = 0.2$: The indeterminacy degree due to mixed reviews where customers were neutral about features such as battery life.
- $F_S(x) = 0.1$: The falsity degree, reflecting dissatisfaction based on complaints about delivery delays and defective units.

The neutrosophic representation of customer satisfaction is:

$$S = \{(x, 0.7, 0.2, 0.1) \mid x \in X\}.$$

Next, the definition of a neutrosophic graph is provided below.

Definition 2.36. [150] A *neutrosophic graph* $NTG = (V, E, \sigma, \mu)$ is defined as follows:

- V is the set of vertices.
- E is the set of edges, where $E \subseteq V \times V$.

- $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a tuple of vertex membership functions:

$$\sigma_i : V \to [0, 1], \quad i = 1, 2, 3,$$

where:

  - $\sigma_1(v)$: Truth degree of vertex v,
  - $\sigma_2(v)$: Indeterminacy degree of vertex v,
  - $\sigma_3(v)$: Falsity degree of vertex v.

- $\mu = (\mu_1, \mu_2, \mu_3)$ is a tuple of edge membership functions:

$$\mu_i : E \to [0, 1], \quad i = 1, 2, 3,$$

where:

  - $\mu_1(e)$: Truth degree of edge e,
  - $\mu_2(e)$: Indeterminacy degree of edge e,
  - $\mu_3(e)$: Falsity degree of edge e.

The following condition must hold for each edge $e = \{v_i, v_j\} \in E$:

$$\mu(e) \leq \sigma(v_i) \wedge \sigma(v_j),$$

where $\wedge$ denotes the minimum operation.

Additionally, the following terminology is used:

1. σ is referred to as the *neutrosophic vertex set*.
2. μ is referred to as the *neutrosophic edge set*.
3. $|V|$ is called the *order* of NTG, denoted by $O(NTG)$.
4. The sum of all vertex membership values, $\sum_{v \in V} \sigma(v)$, is called the *neutrosophic order* of NTG, denoted by $On(NTG)$.
5. $|E|$ is called the *size* of NTG, denoted by $S(NTG)$.
6. The sum of all edge membership values, $\sum_{e \in E} \mu(e)$, is called the *neutrosophic size* of NTG, denoted by $Sn(NTG)$.

The Examples of neutrosophic graph is following.

Example 2.37. (cf.[45]) Consider a neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ with four vertices $V = \{v_1, v_2, v_3, v_4\}$, as shown in the diagram.

The neutrosophic membership degrees of the vertices are as follows:

$$\sigma(v_1) = (0.5, 0.1, 0.4), \quad \sigma(v_2) = (0.6, 0.3, 0.2),$$
$$\sigma(v_3) = (0.2, 0.3, 0.4), \quad \sigma(v_4) = (0.4, 0.2, 0.5)$$

The neutrosophic membership degrees of the edges are as follows:

$$\mu(v_1v_2) = (0.2, 0.3, 0.4), \quad \mu(v_2v_3) = (0.3, 0.3, 0.4),$$
$$\mu(v_3v_4) = (0.2, 0.3, 0.4), \quad \mu(v_4v_1) = (0.1, 0.2, 0.5)$$

In this case, the neutrosophic graph NTG has the following properties:

- Vertices v_1, v_2, v_3, v_4 are connected by edges with varying neutrosophic membership degrees.
- The neutrosophic relations ensure that for every edge $v_iv_j \in E$, $\mu(v_iv_j) \leq \sigma(v_i) \wedge \sigma(v_j)$, where $\wedge$ denotes the minimum operation.

Similarly to Fuzzy Graphs, an Intuitionistic Neutrosophic Graph has been defined as a generalization of Neutrosophic Graphs [6, 10, 78]. The definition is provided as follows.

Definition 2.38 (Intuitionistic Neutrosophic Graph)**.** (cf.[6, 10, 78]) An *intuitionistic neutrosophic graph* $G = (\eta, \rho)$ is defined as follows:

- Let V be the set of vertices.
- For each vertex $u \in V$, the intuitionistic neutrosophic membership functions $T(u), I(u), F(u)$ represent the truth, indeterminacy, and falsity memberships, respectively, where $0 \leq T(u) + I(u) + F(u) \leq 2$ and the following conditions hold:

$$\min\{T(u), I(u)\} \leq 0.5, \quad \min\{F(u), I(u)\} \leq 0.5, \quad \min\{T(u), F(u)\} \leq 0.5.$$

- Let $E \subseteq V \times V$ be the set of edges.
- For each edge $(u, v) \in E$, the intuitionistic neutrosophic membership functions $T(u, v), I(u, v), F(u, v)$ represent the truth, indeterminacy, and falsity memberships of the edge, subject to the following conditions:

$$T(u, v) \leq T(u) \wedge T(v), \quad I(u, v) \leq I(u) \wedge I(v), \quad F(u, v) \leq F(u) \vee F(v),$$

  where $\wedge$ denotes the minimum operation and $\vee$ denotes the maximum operation.
- Additionally, the following conditions must be satisfied for all edges $(u, v) \in E$:

$$T(u, v) \wedge I(u, v) \leq 0.5, \quad T(u, v) \wedge F(u, v) \leq 0.5, \quad I(u, v) \wedge F(u, v) \leq 0.5,$$

  and

$$0 \leq T(u, v) + I(u, v) + F(u, v) \leq 2.$$

This definition describes a graph structure where both the vertices and edges are characterized by their intuitionistic neutrosophic truth, indeterminacy, and falsity memberships, ensuring balanced contributions of these components to the overall uncertainty in the graph.

Example 2.39. (cf.[6,10,78]) Consider a graph $G = (V, E, \eta, \rho)$ where the vertex set $V = \{V_1, V_2, V_3\}$ and edge set $E = \{(V_1, V_2), (V_2, V_3)\}$.

- For each vertex $V_i \in V$, the intuitionistic neutrosophic membership functions for truth $T(u)$, indeterminacy $I(u)$, and falsity $F(u)$ are as follows:

$$T(V_1) = 0.2, \quad I(V_1) = 0.2, \quad F(V_1) = 0.3,$$

$$T(V_2) = 0.3, \quad I(V_2) = 0.3, \quad F(V_2) = 0.4,$$
$$T(V_3) = 0.5, \quad I(V_3) = 0.4, \quad F(V_3) = 0.5.$$

- For the edges $(V_i, V_j) \in E$, the intuitionistic neutrosophic membership functions $T(u, v), I(u, v), F(u, v)$ represent the truth, indeterminacy, and falsity memberships of the edge:

$$T(V_1, V_2) = 0.2, \quad I(V_1, V_2) = 0.2, \quad F(V_1, V_2) = 0.4,$$

$$T(V_2, V_3) = 0.3, \quad I(V_2, V_3) = 0.3, \quad F(V_2, V_3) = 0.5.$$

The conditions outlined in the definition of an intuitionistic neutrosophic graph are satisfied for all vertices and edges. Specifically:

- For each vertex V_i, we ensure that $T(V_i) + I(V_i) + F(V_i) \leq 2$.
- For each edge $(V_i, V_j) \in E$, the relationships between the intuitionistic neutrosophic membership functions of vertices and edges hold, as described by the following conditions:

$$T(V_i, V_j) \leq T(V_i) \wedge T(V_j), \quad I(V_i, V_j) \leq I(V_i) \wedge I(V_j), \quad F(V_i, V_j) \leq F(V_i) \vee F(V_j),$$

  where $\wedge$ denotes the minimum operation and $\vee$ denotes the maximum operation.

This graph structure provides an example of how the intuitionistic neutrosophic framework balances truth, indeterminacy, and falsity at both the vertex and edge levels, following the rules set out in the definition.

Theorem 2.40. *An intuitionistic neutrosophic graph generalizes a neutrosophic graph.*

Proof. To prove that an intuitionistic neutrosophic graph generalizes a neutrosophic graph, we establish the following relationship between their definitions.

A neutrosophic graph $NTG = (V, E, \sigma, \mu)$ is defined such that:

- For each vertex $u \in V$, the membership functions $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ represent the truth (σ_1), indeterminacy (σ_2), and falsity (σ_3) memberships, satisfying:

$$0 \leq \sigma_1(u) + \sigma_2(u) + \sigma_3(u) \leq 1.$$

- For each edge $(u, v) \in E$, the edge memberships $\mu = (\mu_1, \mu_2, \mu_3)$ satisfy:

$$\mu_i(u, v) \leq \sigma_i(u) \wedge \sigma_i(v), \quad \forall i = 1, 2, 3.$$

An intuitionistic neutrosophic graph $ING = (\eta, \rho)$ is defined such that:

- For each vertex $u \in V$, the membership functions $\eta = (T, I, F)$ represent truth (T), indeterminacy (I), and falsity (F), satisfying:

$$0 \leq T(u) + I(u) + F(u) \leq 2.$$

- Additional conditions ensure pairwise contributions to uncertainty:

$$\min\{T(u), I(u)\} \leq 0.5, \quad \min\{I(u), F(u)\} \leq 0.5, \quad \min\{T(u), F(u)\} \leq 0.5.$$

- For each edge $(u, v) \in E$, the edge memberships $\rho = (T, I, F)$ satisfy:

$$T(u, v) \leq T(u) \wedge T(v), \quad I(u, v) \leq I(u) \wedge I(v), \quad F(u, v) \leq F(u) \vee F(v),$$

  and:

$$0 \leq T(u, v) + I(u, v) + F(u, v) \leq 2.$$

If we constrain the intuitionistic neutrosophic graph ING by setting:

$$T(u) + I(u) + F(u) \leq 1, \quad \text{and} \quad T(u, v) + I(u, v) + F(u, v) \leq 1,$$

then ING reduces to a neutrosophic graph NTG. This is because:

- The sum of the memberships in NTG is bounded by 1, which is a stricter condition than the bound of 2 in ING.
- The relationships $\mu_i(u, v) \leq \sigma_i(u) \wedge \sigma_i(v)$ in NTG are equivalent to the edge conditions in ING when the sum of memberships is constrained to 1.

Thus, every neutrosophic graph is a special case of an intuitionistic neutrosophic graph, where the total memberships are further restricted. Therefore, an intuitionistic neutrosophic graph generalizes a neutrosophic graph. □

Theorem 2.41. *An intuitionistic neutrosophic graph generalizes an intuitionistic fuzzy graph.*

Proof. To obtain an intuitionistic fuzzy graph from an intuitionistic neutrosophic graph $G = (\eta, \rho)$, set:

$$T(u) = \mu_A(u), \quad F(u) = \nu_A(u), \quad I(u) = 0 \quad \forall u \in V,$$

and:

$$T(u, v) = \mu_B(u, v), \quad F(u, v) = \nu_B(u, v), \quad I(u, v) = 0 \quad \forall (u, v) \in E.$$

Under this reduction:

- For vertices:

$$0 \leq T(u) + I(u) + F(u) = \mu_A(u) + \nu_A(u) \leq 1.$$

- For edges:
$$T(u,v) = \mu_B(u,v) \leq \min(T(u), T(v)) = \min(\mu_A(u), \mu_A(v)),$$
$$F(u,v) = \nu_B(u,v) \geq \max(F(u), F(v)) = \max(\nu_A(u), \nu_A(v)).$$

- Indeterminacy ($I(u)$ and $I(u,v)$) is set to zero, as it is not considered in the intuitionistic fuzzy graph.

An intuitionistic fuzzy graph can be obtained as a special case of an intuitionistic neutrosophic graph by setting the indeterminacy memberships to zero ($I(u) = I(u,v) = 0$). Therefore, the intuitionistic neutrosophic graph is a generalization of the intuitionistic fuzzy graph. □

An example of operations in a Neutrosophic Graph is provided below. Since it is difficult to list all operations, please refer to the relevant references as needed.

Definition 2.42 (Complement of a Neutrosophic Graph)**.** Let $G = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ be a Neutrosophic graph, where σ represents the neutrosophic membership functions of the vertices, and μ represents the neutrosophic membership functions of the edges. The *complement* of G, denoted as $\overline{G} = (V, E', \sigma, \mu')$, is a neutrosophic graph defined as follows:

- The vertex set remains the same: $V(\overline{G}) = V(G)$.
- The edge set E' is the complement of the original edge set E, meaning $E' = \{(u,v) \mid (u,v) \notin E\}$.
- The neutrosophic membership functions for edges in the complement graph μ' are defined as:
$$\mu_1'(u,v) = \sigma_1(u) \wedge \sigma_1(v) - \mu_1(u,v),$$
$$\mu_2'(u,v) = |\sigma_2(u) \vee \sigma_2(v) - \mu_2(u,v)|,$$
$$\mu_3'(u,v) = |\sigma_3(u) \vee \sigma_3(v) - \mu_3(u,v)|,$$
for all $u, v \in V$, where $\wedge$ denotes the minimum operation, $\vee$ denotes the maximum operation, and the absolute value ensures non-negative membership values. The functions $\mu_1(u,v), \mu_2(u,v), \mu_3(u,v)$ represent the truth, indeterminacy, and falsity memberships of the edge in the original graph.

Thus, the complement graph $\overline{G}$ adjusts the truth, indeterminacy, and falsity memberships based on the complement of the original edge relations.

Definition 2.43 (μ-Complement of a Neutrosophic Graph)**.** Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic graph. The *μ-complement* of G, denoted by $G^\mu = (V, E, \sigma, \mu^\mu)$, is a neutrosophic graph where the neutrosophic edge membership functions μ^μ are defined as follows:

- If $\mu_1(u,v) = 0$, then:
$$\mu_1^\mu(u,v) = 0,$$
meaning there is no truth membership for non-edges.
- If $\mu_1(u,v) > 0$, then:
$$\mu_1^\mu(u,v) = \sigma_1(u) \wedge \sigma_1(v) - \mu_1(u,v),$$
where $\mu_1^\mu(u,v)$ is the complement of the truth membership for edges that exist in the original graph.
- The indeterminacy and falsity memberships for the μ-complement are defined as:
$$\mu_2^\mu(u,v) = |\sigma_2(u) \vee \sigma_2(v) - \mu_2(u,v)|, \quad \mu_3^\mu(u,v) = |\sigma_3(u) \vee \sigma_3(v) - \mu_3(u,v)|,$$
meaning the complement operation adjusts the indeterminacy and falsity memberships of the edges based on the maximum operation, with absolute values ensuring non-negative results.

This definition adjusts the truth, indeterminacy, and falsity memberships based on the complement of the original edge weights.

Theorem 2.44. *The complement of a Neutrosophic Graph G is itself a Neutrosophic Graph.*

Proof. Let $G = (V, E, \sigma, \mu)$ be a Neutrosophic Graph, and let $\overline{G} = (V, E', \sigma, \mu')$ be its complement as defined above. To show that $\overline{G}$ is a Neutrosophic Graph, we verify the properties of neutrosophic membership functions:

1. **Vertex membership functions:** The vertex membership functions $\sigma_1, \sigma_2, \sigma_3$ remain unchanged, satisfying:

$$0 \leq \sigma_1(u) + \sigma_2(u) + \sigma_3(u) \leq 3 \quad \text{for all } u \in V.$$

2. **Edge membership functions:** For μ'_1, μ'_2, μ'_3, we verify that:

$$0 \leq \mu'_1(u, v) + \mu'_2(u, v) + \mu'_3(u, v) \leq 3,$$

since:

$$\mu'_1(u, v) = \sigma_1(u) \wedge \sigma_1(v) - \mu_1(u, v),$$
$$\mu'_2(u, v) = |\sigma_2(u) \vee \sigma_2(v) - \mu_2(u, v)|,$$
$$\mu'_3(u, v) = |\sigma_3(u) \vee \sigma_3(v) - \mu_3(u, v)|.$$

Each component $\mu'_i(u, v)$ satisfies the required bounds because $\mu_1, \mu_2, \mu_3 \in [0, 1]$, and the operations $\wedge, \vee$, subtraction, and absolute value do not violate the constraints.

Hence, $\overline{G}$ is a valid Neutrosophic Graph. □

2.5 Smart Fuzzy Graph

A Smart Fuzzy Graph models real-world systems with uncertain relationships, utilizing fuzzy sets, and is widely applied in IoT and connectivity problems [25, 26] [1]. Related graph classes include the Regular Smart Fuzzy Graph and the Totally Regular Smart Fuzzy Graph [25, 26]. The definitions and examples are presented as follows.

Definition 2.45. [25, 26] A *Smart Fuzzy Graph* $G = (\sigma, \mu)$ with V as the underlying set is defined as follows:

- $\sigma : V \to [0, 1]$ is a fuzzy subset of vertices, where $\sigma(x)$ represents the membership degree of vertex $x \in V$.
- $\mu : V \times V \to [0, 1]$ is a symmetric fuzzy relation on σ, such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$, where $\wedge$ denotes the minimum operation.

The Smart Fuzzy Graph must satisfy the following conditions:

- If $u \neq v$, then:

$$\sum_{u,v \in V} \mu(u, v) \leq \sum_{u,v \in V} \sigma(u) \wedge \sigma(v) \leq 1.$$

- If $u = v$, then:

$$\sum_{u \in V} \mu(u, u) = \sum_{u \in V} \sigma(u) \wedge \sigma(u) = 0.$$

Example 2.46. [25, 26] Consider a Smart Fuzzy Graph with 5 vertices $V = \{V_1, V_2, V_3, V_4, V_5\}$, where the membership degrees of the vertices are as follows:

$$\sigma(V_1) = 0.8, \quad \sigma(V_2) = 0.6, \quad \sigma(V_3) = 0.5, \quad \sigma(V_4) = 0.9, \quad \sigma(V_5) = 0.5$$

The edges are defined with the following membership values:

	$V_1(0.8)$	$V_2(0.6)$	$V_3(0.5)$	$V_4(0.9)$	$V_5(0.5)$
$V_1(0.8)$	0	0.09	0.11	0.07	0.13
$V_2(0.6)$	0.09	0	0.2	0	0.1
$V_3(0.5)$	0.11	0.2	0	0.1	0
$V_4(0.9)$	0.07	0	0.1	0	0.1
$V_5(0.5)$	0.13	0.1	0	0.1	0

For this Smart Fuzzy Graph, the condition $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ must hold for all pairs of vertices. For example:

[1] The Internet of Things (IoT) connects devices, sensors, and systems, enabling data exchange and automation for enhanced efficiency and decision-making(cf.[102]).

- For $V_1(0.8)$ and $V_5(0.5)$, we have $\mu(V_1, V_5) = 0.13$ and $\sigma(V_1) \wedge \sigma(V_5) = 0.5$, so the condition $0.13 \leq 0.5$ is satisfied.
- For $V_2(0.6)$ and $V_3(0.5)$, we have $\mu(V_2, V_3) = 0.2$ and $\sigma(V_2) \wedge \sigma(V_3) = 0.5$, so the condition $0.2 \leq 0.5$ is satisfied.

Thus, the graph satisfies the conditions of a Smart Fuzzy Graph, where the strength of the fuzzy relations between vertices is constrained by their membership degrees.

2.6 Fuzzy zero divisor graph

A fuzzy zero divisor graph is a fuzzy graph where vertices represent nonzero zero-divisors of a ring, and edges exist if their product equals zero[87, 88]. The definitions and examples are presented as follows.

Notation 2.47. (cf.[87]) Let R be a commutative ring with identity, and let $Z(R)$ denote the set of zero-divisors in R. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is defined as follows:

- The vertex set of $\Gamma(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, the set of all nonzero zero-divisors of R.
- Two distinct vertices $x, y \in Z(R)^*$ are connected by an edge if and only if their product in R is zero, i.e., $xy = 0$.

In this way, $\Gamma(R)$ encodes the relationships between the nonzero zero-divisors of R. If R is an integral domain (i.e., R has no nonzero zero-divisors), the graph $\Gamma(R)$ is empty, as $Z(R)^* = \emptyset$.

Definition 2.48. [87] A *fuzzy zero divisor graph* $\Gamma_{\text{fuzzy}} = (V, \sigma, \mu)$ is defined as follows:

- V is a non-empty set of vertices, representing the elements of a commutative ring R with 1.
- $\sigma : V \to (0, 1]$ is a fuzzy membership function that assigns a membership degree $\sigma(v)$ to each vertex $v \in V$, where $\sigma(v)$ reflects the relevance or strength of the zero divisor element v.
- $\mu : V \times V \to (0, 1]$ is a fuzzy relation on σ, defined as:

$$\mu(v_i, v_j) = \frac{\sigma(v_i) \cdot \sigma(v_j)}{\sigma(v_i) + \sigma(v_j)}, \quad \forall v_i, v_j \in V.$$

  The relation $\mu(v_i, v_j)$ represents the degree of adjacency between two vertices v_i and v_j, which is influenced by their fuzzy membership degrees.

The fuzzy zero divisor graph represents the relationships between the zero-divisor elements of a commutative ring R. Each vertex corresponds to a nonzero zero divisor of R, and two distinct vertices v_i and v_j are adjacent if their product is zero, i.e., $v_i v_j = 0$.

Theorem 2.49. *A Fuzzy Zero Divisor Graph generalizes both a Zero-Divisor Graph and a Fuzzy Graph.*

Proof. Let $\Gamma_{\text{fuzzy}} = (V, \sigma, \mu)$ be a fuzzy zero divisor graph of a commutative ring R.

If we ignore the fuzzy membership function σ and the fuzzy relation μ, the vertex set V corresponds to $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R. An edge exists between two vertices $v_i, v_j \in V$ if and only if $v_i v_j = 0$, matching the definition of a zero-divisor graph $\Gamma(R)$. Thus, Γ_{fuzzy} reduces to $\Gamma(R)$, the zero-divisor graph, by removing fuzzy characteristics.

A fuzzy graph $G = (V, \sigma, \mu)$ has:

- A vertex set V with fuzzy membership $\sigma : V \to (0, 1]$,
- A fuzzy relation $\mu : V \times V \to (0, 1]$.

For $\Gamma_{\text{fuzzy}} = (V, \sigma, \mu)$, the membership function σ assigns a fuzzy degree to each vertex $v \in V$, and the adjacency relation $\mu(v_i, v_j)$ is defined as:

$$\mu(v_i, v_j) = \frac{\sigma(v_i) \cdot \sigma(v_j)}{\sigma(v_i) + \sigma(v_j)}.$$

This satisfies the requirements of a fuzzy graph structure. Thus, Γ_{fuzzy} reduces to a fuzzy graph by considering the fuzzy memberships and adjacency relations without enforcing the ring-theoretic zero-divisor constraints.

A Fuzzy Zero Divisor Graph Γ_{fuzzy} generalizes the Zero-Divisor Graph by incorporating fuzzy memberships for vertices and edges, and it generalizes a Fuzzy Graph by embedding the algebraic properties of zero-divisors within the fuzzy structure. □

2.7 Fuzzy semigraph

A fuzzy semigraph is a fuzzy graph that generalizes semigraphs, combining fuzzy vertices and fuzzy edges, often applied in network systems like roads or telecommunications[24,106,115,127]. The definitions are presented as follows.

Definition 2.50. [126] A *fuzzy semigraph* $G = (V, X, \sigma, \mu, \eta)$ is defined as follows:

- V is a non-empty set of vertices.
- X is a set of edges, where each edge is an n-tuple of distinct vertices from V, i.e., $e = (v_1, v_2, \ldots, v_n)$, with $n \geq 2$.
- $\sigma : V \to [0,1]$ is a fuzzy subset of vertices, where $\sigma(v)$ represents the membership degree of vertex $v \in V$.
- $\mu : V \times V \to [0,1]$ is a fuzzy relation on the vertices, where $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$.
- $\eta : X \to [0,1]$ is a fuzzy subset of edges, where $\eta(e)$ is the membership degree of the edge $e = (v_1, v_2, \ldots, v_n)$ and satisfies:

$$\eta(e) \leq \mu(v_1, v_2) \wedge \mu(v_2, v_3) \wedge \cdots \wedge \mu(v_{n-1}, v_n) \wedge \sigma(v_1) \wedge \sigma(v_n).$$

In this fuzzy semigraph, the vertices v_1 and v_n are called the *end vertices*, while the vertices $v_2, v_3, \ldots, v_{n-1}$ are called the *middle vertices*. If a middle vertex is also an end vertex of another edge, it is called a *middle-end vertex*.

Definition 2.51. [126] A *fuzzy subsemigraph* $H = (\gamma, \rho, \delta)$ of a fuzzy semigraph $G = (\sigma, \mu, \eta)$ is defined as follows:

- All edges of H are subedges of G.
- For every vertex $u \in V$, the membership degree of u in H is less than or equal to its membership degree in G, i.e., $\gamma(u) \leq \sigma(u)$.
- For every pair of vertices $(u, v) \in V \times V$, the fuzzy relation $\rho(u, v)$ in H is less than or equal to the fuzzy relation $\mu(u, v)$ in G, i.e., $\rho(u, v) \leq \mu(u, v)$.
- For every edge $e \in X$, the fuzzy membership degree $\delta(e)$ in H is less than or equal to the fuzzy membership degree $\eta(e)$ in G, i.e., $\delta(e) \leq \eta(e)$.

Definition 2.52. [126] Let $G = (\sigma, \mu, \eta)$ be a fuzzy semigraph on the vertex set V and edge set X. The *End Vertex Fuzzy Graph* (e-Fuzzy Graph), denoted as $G_e = (\sigma_e, \eta_e)$, is defined as follows:

- The vertex set is V, where $\sigma_e(u) = \sigma(u)$ for all $u \in V$.
- Two vertices $u, v \in V$ are adjacent in G_e if and only if they are end vertices of an edge in G, with $\eta_e(uv) = \eta(uv)$ for every pair of end vertices u and v.

Definition 2.53. [126] Let $G = (\sigma, \mu, \eta)$ be a fuzzy semigraph. The *Adjacency Fuzzy Graph* (a-Fuzzy Graph), denoted as $G_a = (\sigma_a, \eta_a)$, is defined as follows:

- The vertex set is V, where $\sigma_a(u) = \sigma(u)$ for all $u \in V$.
- Two vertices $u, v \in V$ are adjacent in G_a if they are adjacent in G. The adjacency membership function is given by:

  $$\eta_a(uv) = \mu(uv_1) \wedge \mu(v_1 v_2) \wedge \cdots \wedge \mu(v_{k-1} v_k),$$

  for every pair of adjacent vertices u, v where $(u, v_1, \ldots, v_k)$ is an edge or partial edge of G.

Definition 2.54. [126] Let $G = (\sigma, \mu, \eta)$ be a fuzzy semigraph. The *Consecutive Adjacency Fuzzy Graph* (ca-Fuzzy Graph), denoted as $G_{ca} = (\sigma_{ca}, \mu_{ca})$, is defined as follows:

- The vertex set is V, where $\sigma_{ca}(u) = \sigma(u)$ for all $u \in V$.
- Two vertices $u, v \in V$ are adjacent in G_{ca} if and only if they are consecutively adjacent in G. The adjacency membership function is given by $\mu_{ca}(uv) = \mu(uv)$ for every pair of consecutively adjacent vertices u and v.

2.8 Double Layered Fuzzy Graph and Triple Layered Fuzzy Graph

A Layered Fuzzy Graph is an extension of fuzzy graphs with multiple layers, where each layer represents distinct fuzzy relations or membership degrees between vertices and edges. The Double Layered Fuzzy Graph [123, 124, 133] and Triple Layered Fuzzy Graph [63, 98, 134] are defined accordingly. Related graph classes include the Intuitionistic Double Layered Fuzzy Graph [**?**, 132], Balanced Double Layered Bipolar Fuzzy Graph [122, 128], and Complete Double Layered Fuzzy Graph [16]. Additionally, generalized forms such as the K-Partitioned Fuzzy Graph [120, 121] and the Quadruple Layered Fuzzy Graph [81] have also been introduced.

Definition 2.55. [123, 124, 133] A *Double Layered Fuzzy Graph (DLFG)* is a fuzzy graph $G = (\sigma, \mu)$ with an underlying crisp graph $G^* = (\sigma^*, \mu^*)$. The *Double Layered Fuzzy Graph* is denoted as $DL(G) = (\sigma_{DL}, \mu_{DL})$, and it is defined as follows:

The **node set** of $DL(G)$ is the union $\sigma^* \cup \mu^*$, where σ^* is the set of vertices and μ^* is the set of edges from the original fuzzy graph.

The **fuzzy subset** σ_{DL} is defined as:

$$\sigma_{DL}(u) = \begin{cases} \sigma(u) & \text{if } u \in \sigma^*, \\ \mu(uv) & \text{if } uv \in \mu^*. \end{cases}$$

This definition assigns the membership degree for both vertices and edges within the fuzzy subset.

The **fuzzy relation** μ_{DL} on $\sigma^* \cup \mu^*$ is defined as:

$$\mu_{DL}(u, v) = \begin{cases} \sigma(u) \wedge \sigma(v) & \text{if } u, v \in \sigma^*, \\ \mu(e_i) \wedge \mu(e_j) & \text{if } e_i, e_j \in \mu^*, \text{ and they share a common vertex,} \\ \sigma(u) \wedge \mu(e) & \text{if } u \in \sigma^*, e \in \mu^*, \text{ and } u \text{ is incident to } e, \\ 0 & \text{otherwise.} \end{cases}$$

Here, μ_{DL} is a fuzzy relation that defines the interaction between nodes and edges based on their membership degrees. For any $u, v \in \sigma^* \cup \mu^*$, the relation satisfies:

$$\mu_{DL}(u, v) \leq \sigma_{DL}(u) \wedge \sigma_{DL}(v).$$

Thus, the pair $DL(G) = (\sigma_{DL}, \mu_{DL})$ is referred to as the *Double Layered Fuzzy Graph*.

Theorem 2.56. *A Double Layered Fuzzy Graph (DLFG) generalizes a Fuzzy Graph.*

Proof. If we constrain $\sigma_{DL}(u)$ to only represent vertex memberships and ignore edge memberships, the definitions of σ_{DL} and μ_{DL} reduce to the fuzzy subset σ and fuzzy relation μ of a Fuzzy Graph. Under this condition:

$$\sigma(u) = \sigma_{DL}(u), \quad \mu(u, v) = \mu_{DL}(u, v) \quad \forall u, v \in \sigma^*.$$

Thus, DLFG generalizes a Fuzzy Graph by incorporating edge memberships as part of the fuzzy subset. □

Definition 2.57. [63, 98, 134] A *Triple Layered Fuzzy Graph (TLFG)* is a fuzzy graph $G = (\sigma, \mu)$ defined with the following properties. Let $G^* = (\sigma^*, \mu^*)$ represent the underlying crisp graph of G.

The **node set** of the Triple Layered Fuzzy Graph $TL(G)$ is the union of the sets $\sigma^* \cup \mu^*$, where σ^* denotes the fuzzy subset of nodes and μ^* denotes the fuzzy subset of edges.

The fuzzy subset σ_{TL} is defined as:

$$\sigma_{TL}(u) = \begin{cases} \sigma(u) & \text{if } u \in \sigma^*, \\ 2\mu(uv) & \text{if } uv \in \mu^*. \end{cases}$$

This represents the membership of vertices and edges in the fuzzy subset.

The fuzzy relation μ_{TL} on $\sigma^* \cup \mu^*$ is defined as follows:

$$\mu_{TL}(u, v) = \begin{cases} \sigma(u) \wedge \sigma(v) & \text{if } u, v \in \sigma^*, \\ \mu(e_i) \wedge \mu(e_j) & \text{if } e_i, e_j \in \mu^*, \text{and they share a common node,} \\ \sigma(u) \wedge \mu(e) & \text{if } u \in \sigma^*, e \in \mu^*, \text{and } u \text{ is incident to } e, \\ 0 & \text{otherwise.} \end{cases}$$

Here, σ_{TL} is a fuzzy subset of nodes and edges, and μ_{TL} is a fuzzy relation that satisfies:

$$\mu_{TL}(u, v) \leq \sigma_{TL}(u) \wedge \sigma_{TL}(v) \quad \text{for all } u, v \in \sigma^* \cup \mu^*.$$

Thus, the pair $TL(G) = (\sigma_{TL}, \mu_{TL})$ is defined as the *Triple Layered Fuzzy Graph*.

Theorem 2.58. *A Triple Layered Fuzzy Graph (TLFG) generalizes a Double Layered Fuzzy Graph (DLFG).*

Proof. If we constrain σ_{TL} such that $\sigma_{TL}(uv) = \mu(uv)$ instead of $2\mu(uv)$, the definitions of σ_{TL} and μ_{TL} align with σ_{DL} and μ_{DL}, respectively. Under this constraint:

$$\sigma_{DL}(u) = \sigma_{TL}(u), \quad \mu_{DL}(u, v) = \mu_{TL}(u, v) \quad \forall u, v \in \sigma^* \cup \mu^*.$$

Thus, TLFG generalizes DLFG by allowing flexibility in the membership degree assignments for edges. □

2.9 Weak fuzzy graph

A Weak Fuzzy Graph is defined as a fuzzy graph where the membership degree of each edge is strictly less than the minimum of the membership degrees of its connected vertices [72,114,125]. A related fuzzy graph class is the General Fuzzy Graph, which is also well-known [113]. The definition is provided as follows[125].

Definition 2.59. [125] A *weak fuzzy graph* $F = (\sigma, \mu)$ is defined as follows:

- Let V be the set of vertices.
- $\sigma : V \to [0, 1]$ represents the membership degree of each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ is the fuzzy relation on σ, which represents the strength of the relationship between vertices.

The fuzzy graph F is called a *weak fuzzy graph* if for all pairs of vertices $(a, b) \in V \times V$, the fuzzy relation satisfies the condition:

$$\mu(a, b) < \sigma(a) \wedge \sigma(b),$$

where $\wedge$ denotes the minimum operation.

In a weak fuzzy graph, the strength of the connection (or "flow") between any two vertices is always strictly less than the minimum membership degree of those two vertices. This ensures that the edges in the graph have weaker relationships than the vertices they connect.

Example 2.60. Let $F = (\sigma, \mu)$ be a weak fuzzy graph with the following properties (We call Weak fuzzy Triangle graph):

- **Vertices**: $V = \{v_1, v_2, v_3\}$
- **Vertex Membership Degrees**:

$$\sigma(v_1) = 0.7,$$
$$\sigma(v_2) = 0.8,$$
$$\sigma(v_3) = 0.9$$

- **Edge Membership Degrees** (Fuzzy relations between vertices):

$$\mu(v_1, v_2) = 0.6,$$
$$\mu(v_1, v_3) = 0.3,$$
$$\mu(v_2, v_3) = 0.7$$

For each edge in the graph, we check whether the fuzzy relation $\mu(v_i, v_j)$ is less than the minimum of the membership degrees of the two vertices:

- For edge (v_1, v_2):

$$\sigma(v_1) = 0.7, \quad \sigma(v_2) = 0.8$$
$$\sigma(v_1) \wedge \sigma(v_2) = \min(0.7, 0.8) = 0.7$$
$$\mu(v_1, v_2) = 0.6 < 0.7 \quad \text{(Condition satisfied)}$$

- For edge (v_1, v_3):

$$\sigma(v_1) = 0.7, \quad \sigma(v_3) = 0.9$$
$$\sigma(v_1) \wedge \sigma(v_3) = \min(0.7, 0.9) = 0.7$$
$$\mu(v_1, v_3) = 0.3 < 0.7 \quad \text{(Condition satisfied)}$$

- For edge (v_2, v_3):

$$\sigma(v_2) = 0.8, \quad \sigma(v_3) = 0.9$$

$$\sigma(v_2) \wedge \sigma(v_3) = \min(0.8, 0.9) = 0.8$$

$$\mu(v_2, v_3) = 0.7 < 0.8 \quad \text{(Condition satisfied)}$$

In this example, all edges in the weak fuzzy graph satisfy the condition $\mu(v_i, v_j) < \sigma(v_i) \wedge \sigma(v_j)$, making this a valid weak fuzzy graph.

2.10 Mild balanced intuitionistic fuzzy graph

A Mild Balanced Intuitionistic Fuzzy Graph (IFG) is defined as an Intuitionistic Fuzzy Graph in which all connected subgraphs are intense. This implies that the membership and non-membership degrees of any connected subgraph are less than or equal to those of the original graph [116]. The formal definition is provided below.

Definition 2.61 (Intense Subgraph)**.** [116] Let $G = (V, E, \mu, \nu)$ be an intuitionistic fuzzy graph, where μ and ν denote the membership and non-membership functions, respectively. A connected subgraph $H = (V(H), E(H), \mu_H, \nu_H)$ of G is called an *intense subgraph* if the following conditions hold:

1. $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$,
2. The degree of membership and non-membership in H satisfy:

$$D_\mu(H) \geq D_\mu(G) \quad \text{and} \quad D_\nu(H) \leq D_\nu(G),$$

where $D_\mu(G)$ and $D_\nu(G)$ are the degree functions of G defined as:

$$D_\mu(G) = \sum_{v \in V(G)} \mu(v) + \sum_{e \in E(G)} \mu(e),$$

$$D_\nu(G) = \sum_{v \in V(G)} \nu(v) + \sum_{e \in E(G)} \nu(e),$$

and similarly for H.

Definition 2.62 (Feeble Subgraph)**.** [116] Let $G = (V, E, \mu, \nu)$ be an intuitionistic fuzzy graph, where μ and ν denote the membership and non-membership functions, respectively. A connected subgraph $H = (V(H), E(H), \mu_H, \nu_H)$ of G is called a *feeble subgraph* if the following conditions hold:

1. $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$,
2. The degree of membership and non-membership in H satisfy:

$$D_\mu(H) < D_\mu(G) \quad \text{and} \quad D_\nu(H) > D_\nu(G),$$

where $D_\mu(G)$ and $D_\nu(G)$ are the degree functions of G, defined as:

$$D_\mu(G) = \sum_{v \in V(G)} \mu(v) + \sum_{e \in E(G)} \mu(e),$$

$$D_\nu(G) = \sum_{v \in V(G)} \nu(v) + \sum_{e \in E(G)} \nu(e),$$

and similarly for H.

Definition 2.63 (Mild Balanced Intuitionistic Fuzzy Graph)**.** [116] An intuitionistic fuzzy graph $G = (V, E)$ is called a *mild balanced intuitionistic fuzzy graph* if all connected subgraphs of G are intense subgraphs.

Question 2.64. Is it possible to define a Mild Balanced Neutrosophic Graph?

2.11 Connected Fuzzy Chemical Graph

The definition of a Chemical Graph is provided below. This is a graph widely used in the field of chemistry [35, 158, 159].

Definition 2.65. (cf.[35, 158, 159]) A *chemical graph* $G_C = (A, B)$ is a simple graph representing the molecular structure of a chemical compound, where:

- A is the set of vertices representing atoms in the molecule,
- B is the set of edges representing chemical bonds between the atoms in the molecule.

Each edge $(a, b) \in B$ connects two distinct atoms $a, b \in A$, indicating the existence of a bond between these atoms. In this representation, the degree of a vertex corresponds to the valency of the atom, i.e., the number of bonds that an atom forms with other atoms.

The definitions of a Connected Fuzzy Chemical Graph and a Neighborly Irregular Fuzzy Chemical Graph are provided below. These definitions extend the fundamental concepts of graph theory used to represent molecular structures to "fuzzy graphs" and "neighborly irregular chemical graphs," mathematically capturing the uncertainty present in molecular structures [21, 22, 31].

Definition 2.66. [22] A *fuzzy chemical graph* is a fuzzy graph $G = (V, \sigma, \mu)$, where:

- V is the set of vertices representing atoms in a molecule,
- $\sigma : V \to [0, 1]$ is a membership function representing the degree of membership of each atom in the graph,
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation representing the degree of membership of bonds (edges) between atoms, and $\mu(u, v) \leq \min(\sigma(u), \sigma(v))$ for all $u, v \in V$.

A fuzzy chemical graph G is said to be *connected* if for every pair of vertices $u, v \in V$, there exists a sequence of vertices $u = v_0, v_1, \ldots, v_k = v$ such that $\mu(v_i, v_{i+1}) > 0$ for all $0 \leq i < k$. This ensures that all atoms in the molecular structure are connected by chemical bonds.

Example 2.67 (Methane Molecule)**.** Consider the chemical graph for methane (CH_4):

- The vertex set $A = \{C, H_1, H_2, H_3, H_4\}$, where C represents the carbon atom and H_1, H_2, H_3, H_4 represent the four hydrogen atoms.
- The edge set $B = \{(C, H_1), (C, H_2), (C, H_3), (C, H_4)\}$, representing the chemical bonds between the carbon atom and each hydrogen atom.

The corresponding fuzzy chemical graph can be defined with:

- $\sigma(C) = 1$ and $\sigma(H_i) = 1$ for $i = 1, 2, 3, 4$,
- $\mu(C, H_i) = 1$ for $i = 1, 2, 3, 4$, and $\mu(u, v) = 0$ otherwise.

This representation can be extended by assigning fuzzy membership values σ and μ to account for uncertainty in the molecular structure.

Theorem 2.68. *Fuzzy chemical graphs generalize chemical graphs.*

Proof. Let $G_C = (A, B)$ be a chemical graph. We construct a fuzzy chemical graph $G_F = (V, \sigma, \mu)$ corresponding to G_C as follows:

- Let $V = A$, the set of atoms in the chemical graph.
- Define $\sigma(v) = 1$ for all $v \in V$, indicating that all vertices fully belong to the fuzzy graph.
- Define $\mu(u, v) = 1$ if $(u, v) \in B$, and $\mu(u, v) = 0$ otherwise, indicating crisp edges between atoms.

Clearly, G_F satisfies the conditions for a fuzzy chemical graph:

1. For all $u, v \in V$,
$$\mu(u, v) \leq \min(\sigma(u), \sigma(v)) = 1.$$

2. If G_C is connected, any pair of vertices $u, v \in V$ is connected by a sequence of edges in G_C. Since $\mu(u, v) = 1$ for edges in B, this connectivity is preserved in G_F.

Thus, G_F is a valid fuzzy chemical graph. Furthermore, by assigning membership values $\sigma(v)$ and $\mu(u, v)$ in the interval $[0, 1]$, fuzzy chemical graphs allow for the representation of uncertainty or partial relationships between atoms and bonds, generalizing the crisp structure of G_C. □

Definition 2.69. [22] A *neighborly irregular chemical graph* $G_{NIC} = (A, B)$ is a graph where:

- A is the set of vertices representing atoms in the molecular structure,
- B is the set of edges representing chemical bonds between atoms,
- For every edge $(a, b) \in B$, the degrees of the adjacent atoms a and b are distinct. That is, $\deg(a) \neq \deg(b)$ for all $(a, b) \in B$.

In the context of molecular structures, a neighborly irregular chemical graph typically models molecules in which atoms have varying valency in their adjacent atoms.

Definition 2.70. [22] A *neighborly irregular fuzzy chemical graph* $G_{NIFC} = (V, \sigma, \mu)$ is a fuzzy chemical graph where:

- V is the set of vertices representing atoms in the molecular structure,
- $\sigma : V \to [0, 1]$ is a membership function representing the degree of membership of each atom,
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation representing the degree of membership of bonds between atoms,
- Any two adjacent vertices $u, v \in V$ have distinct degrees, i.e., $\deg(u) \neq \deg(v)$, with their corresponding membership values.

Theorem 2.71. *A Neighborly irregular fuzzy chemical graphs is a Fuzzy chemical graphs.*

Proof. This is evident. □

Theorem 2.72. *A Neighborly Irregular Fuzzy Chemical Graph generalizes a Neighborly Irregular Chemical Graph.*

Proof. To prove this theorem, we demonstrate that the definition of a Neighborly Irregular Chemical Graph is a special case of the definition of a Neighborly Irregular Fuzzy Chemical Graph when the membership functions are binary.

1. **Definition Comparison:**
   - In a Neighborly Irregular Chemical Graph $G_{NIC} = (A, B)$:
     - A is the set of vertices (atoms),
     - B is the set of edges (chemical bonds),
     - For every edge $(a, b) \in B$, $\deg(a) \neq \deg(b)$, where $\deg(a)$ and $\deg(b)$ are the degrees of vertices a and b in the graph.
   - In a Neighborly Irregular Fuzzy Chemical Graph $G_{NIFC} = (V, \sigma, \mu)$:
     - V is the set of vertices (atoms),
     - $\sigma : V \to [0, 1]$ is the membership function of the vertices,
     - $\mu : V \times V \to [0, 1]$ is the fuzzy relation of the edges,
     - For every edge (u, v) with $\mu(u, v) > 0$, $\deg(u) \neq \deg(v)$, where $\deg(u)$ and $\deg(v)$ are the fuzzy degrees of vertices u and v.

2. **Special Case:**
   - In a Neighborly Irregular Chemical Graph, all membership functions are binary:

$$\sigma(a) = 1 \quad \text{for all } a \in A,$$

$$\mu(a, b) = 1 \quad \text{for all } (a, b) \in B.$$

- The condition $\deg(a) \neq \deg(b)$ remains the same in both definitions since the degrees are defined in terms of the adjacent vertices.

3. **Generalization:**

   - A Neighborly Irregular Fuzzy Chemical Graph allows the membership values $\sigma(u)$ and $\mu(u, v)$ to take any value in the interval $[0, 1]$, introducing a measure of uncertainty or partial membership.
   - The Neighborly Irregular Chemical Graph is a special case of the Neighborly Irregular Fuzzy Chemical Graph when all membership values are crisp (binary).

Thus, the Neighborly Irregular Fuzzy Chemical Graph G_{NIFC} generalizes the Neighborly Irregular Chemical Graph G_{NIC} by incorporating fuzzy memberships for vertices and edges while preserving the irregularity condition for adjacent vertices. □

3. Result in this paper

We will outline the results presented in this paper. The fuzzy graph is extended to a Neutrosophic Graph, and its properties are analyzed and explored as necessary. Specifically, as mentioned in the introduction, we extend several classes of fuzzy graphs to Neutrosophic graphs and examine their characteristics. In this section, we focus on graph classes related to Neutrosophic Graphs, including Smart Neutrosophic Graphs, Neutrosophic Zero Divisor Graphs, Weak Neutrosophic Graphs, Neutrosophic Semigraphs, Double/Triple Layered Neutrosophic Graphs, and Connected Neutrosophic Chemical Graphs.

First, we will provide a proof of the relationship between fuzzy graphs and neutrosophic graphs.

Theorem 3.1. *A Neutrosophic Graph can be transformed into a Fuzzy Graph.*

Proof. To prove that a Neutrosophic Graph can be transformed into a Fuzzy Graph, we will show that the vertices and edges of the Neutrosophic Graph NTG can be represented using the truth membership component of the neutrosophic functions.

Each vertex $v \in V$ in the Neutrosophic Graph is assigned a neutrosophic membership function:

$$\sigma(v) = (\sigma_1(v), \sigma_2(v), \sigma_3(v)),$$

where $\sigma_1(v)$, $\sigma_2(v)$, and $\sigma_3(v)$ represent the truth, indeterminacy, and falsity memberships, respectively.

In a Fuzzy Graph, each vertex is assigned a single membership value $\sigma_f(v) \in [0, 1]$, representing the degree of belonging of the vertex. We define this fuzzy membership as:

$$\sigma_f(v) = \sigma_1(v),$$

where $\sigma_1(v)$ is the truth component of the neutrosophic membership function. This step preserves the most certain (truth) aspect of the neutrosophic representation while discarding indeterminacy and falsity for the fuzzy graph.

Each edge $e = (u, v) \in E$ in the Neutrosophic Graph is assigned a neutrosophic membership function:

$$\mu(e) = (\mu_1(e), \mu_2(e), \mu_3(e)),$$

where $\mu_1(e)$, $\mu_2(e)$, and $\mu_3(e)$ represent the truth, indeterminacy, and falsity memberships of the edge.

In a Fuzzy Graph, each edge is assigned a single membership value $\mu_f(e) \in [0, 1]$, representing the degree of connection between vertices. We define this fuzzy membership as:

$$\mu_f(e) = \mu_1(e),$$

where $\mu_1(e)$ is the truth component of the neutrosophic membership function.

After applying the transformation to both the vertices and edges, the Neutrosophic Graph $NTG = (V, E, \sigma, \mu)$ is transformed into a Fuzzy Graph $FG = (V, \sigma_f, \mu_f)$, where:

$$\sigma_f(v) = \sigma_1(v) \quad \text{for all } v \in V,$$

$$\mu_f(e) = \mu_1(e) \quad \text{for all } e \in E.$$

The transformation retains the structure of the graph while simplifying the neutrosophic membership functions to a single fuzzy membership. Hence, we have shown that any Neutrosophic Graph can be transformed into a Fuzzy Graph by using the truth membership values of the vertices and edges. Thus, the theorem is proved. □

3.1 Smart Neutrosophic Graph

The definition of a Smart Neutrosophic Graph is provided below.

Definition 3.2. A *Smart Neutrosophic Graph* is a generalization of the neutrosophic graph, incorporating smart structures and connectivity properties used in Internet of Things (IoT) applications. Formally, a Smart Neutrosophic Graph $SNG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ is defined as follows:

- V is a non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges connecting the vertices.
- $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a neutrosophic vertex set, where:

  $$\sigma_i : V \to [0, 1], \quad \text{for } i = 1, 2, 3,$$

  where $\sigma_1(x)$, $\sigma_2(x)$, and $\sigma_3(x)$ represent the degrees of truth, indeterminacy, and falsity of vertex $x \in V$, respectively.
- $\mu = (\mu_1, \mu_2, \mu_3)$ is a neutrosophic edge set, where:

  $$\mu_i : E \to [0, 1], \quad \text{for } i = 1, 2, 3,$$

  and $\mu_1(x, y)$, $\mu_2(x, y)$, $\mu_3(x, y)$ represent the degrees of truth, indeterminacy, and falsity of the edge $(x, y) \in E$.
- For every $(x, y) \in E$, the following conditions hold:

  $$\mu_1(x, y) \leq \min(\sigma_1(x), \sigma_1(y)),$$
  $$\mu_2(x, y) \geq \max(\sigma_2(x), \sigma_2(y)),$$
  $$\mu_3(x, y) \geq \max(\sigma_3(x), \sigma_3(y)).$$

  Additional conditions for smart connectivity is following.
- The sum of the neutrosophic membership values of all edges must satisfy:

  $$\sum_{(x,y)\in E} \mu_1(x, y) + \mu_2(x, y) + \mu_3(x, y) \leq 3.$$
- The vertices must satisfy smart connectivity rules, ensuring efficient communication in the graph structure, particularly in IoT applications.

Theorem 3.3. *A Smart Neutrosophic Graph (SNG) generalizes both Smart Fuzzy Graphs and Neutrosophic Graphs.*

Proof. We will show that a Smart Neutrosophic Graph (SNG) can reduce to both a Smart Fuzzy Graph and a Neutrosophic Graph under specific conditions.

Consider a Smart Neutrosophic Graph $G = (V, E, \sigma, \mu)$ with $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$. Define the following transformations:

$$\sigma_{\text{fuzzy}}(x) = \sigma_1(x), \quad \mu_{\text{fuzzy}}(x, y) = \mu_1(x, y).$$

This transformation ignores the indeterminacy (σ_2, μ_2) and falsity (σ_3, μ_3) components, leaving only the truth degrees. Under these mappings:

$$\mu_{\text{fuzzy}}(x, y) \leq \min(\sigma_{\text{fuzzy}}(x), \sigma_{\text{fuzzy}}(y)),$$

which satisfies the conditions for a Smart Fuzzy Graph. Additionally, the connectivity and membership sum conditions are inherited from G.

Consider the same Smart Neutrosophic Graph $G = (V, E, \sigma, \mu)$. By dropping the smart connectivity constraints and IoT-specific rules, we retain only the neutrosophic structure:

$$\sigma(x) = (\sigma_1(x), \sigma_2(x), \sigma_3(x)), \quad \mu(x, y) = (\mu_1(x, y), \mu_2(x, y), \mu_3(x, y)).$$

This structure satisfies the conditions of a Neutrosophic Graph:

$$\mu_1(x, y) \leq \min(\sigma_1(x), \sigma_1(y)), \quad \mu_2(x, y) \geq \max(\sigma_2(x), \sigma_2(y)), \quad \mu_3(x, y) \geq \max(\sigma_3(x), \sigma_3(y)).$$

Thus, G reduces to a Neutrosophic Graph when IoT-specific constraints are removed.

By the above transformations, a Smart Neutrosophic Graph generalizes both Smart Fuzzy Graphs and Neutrosophic Graphs. □

3.2 Neutrosophic Zero Divisor Graph

The definition of the Neutrosophic Zero Divisor Graph is provided as follows.

Definition 3.4. Let R be a commutative ring with identity, and let $Z(R)$ denote the set of zero-divisors in R. A *Neutrosophic Zero Divisor Graph* $\Gamma_N = (V, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ is defined as follows:

- $V = Z(R)^* = Z(R) \setminus \{0\}$ represents the set of nonzero zero-divisors of R. Each element in V corresponds to a zero-divisor in the ring.
- $\sigma : V \to [0,1]^3$ is a neutrosophic membership function that assigns three values to each vertex $v \in V$, i.e., $\sigma(v) = (\sigma_1(v), \sigma_2(v), \sigma_3(v))$, where:
  - $\sigma_1(v)$ represents the truth degree of the membership of v,
  - $\sigma_2(v)$ represents the indeterminacy degree of the membership of v,
  - $\sigma_3(v)$ represents the falsity degree of the membership of v.
- $\mu : V \times V \to [0,1]^3$ is a neutrosophic relation function that defines the adjacency between two vertices $v_i, v_j \in V$. The adjacency relation $\mu(v_i, v_j) = (\mu_1(v_i v_j), \mu_2(v_i v_j), \mu_3(v_i v_j))$ is described by the following conditions:
  - The truth degree of adjacency $\mu_1(v_i v_j)$ satisfies:
  $$\mu_1(v_i v_j) \leq \min(\sigma_1(v_i), \sigma_1(v_j)),$$
  meaning that the truth degree of the adjacency between v_i and v_j is bounded by the minimum truth membership of the two vertices.
  - The indeterminacy degree of adjacency $\mu_2(v_i v_j)$ satisfies:
  $$\mu_2(v_i v_j) \geq \max(\sigma_2(v_i), \sigma_2(v_j)),$$
  meaning that the indeterminacy degree of the adjacency between v_i and v_j is at least the maximum indeterminacy membership of the two vertices.
  - The falsity degree of adjacency $\mu_3(v_i v_j)$ satisfies:
  $$\mu_3(v_i v_j) \geq \max(\sigma_3(v_i), \sigma_3(v_j)),$$
  meaning that the falsity degree of the adjacency between v_i and v_j is at least the maximum falsity membership of the two vertices.

  Additionally, for every pair $v_i, v_j \in V$, the sum of the truth, indeterminacy, and falsity degrees of adjacency satisfies:
  $$\mu_1(v_i v_j) + \mu_2(v_i v_j) + \mu_3(v_i v_j) \leq 3.$$

In this graph, two vertices v_i and v_j are adjacent (i.e., there is an edge between them) if and only if their product in the ring R is zero, that is, $v_i \cdot v_j = 0$. This graph structure represents the relationships between zero-divisors in a neutrosophic context, capturing degrees of truth, indeterminacy, and falsity.

Theorem 3.5. *A Non-zero Divisor Neutrosophic Graph can be transformed into a Non-zero Divisor Fuzzy Graph.*

Proof. Obviously holds. □

Theorem 3.6. *A Neutrosophic Zero Divisor Graph $\Gamma_N = (V, \sigma, \mu)$ reduces to a Neutrosophic Graph when the Zero Divisor condition $v_i \cdot v_j = 0$ is ignored.*

Proof. By ignoring the zero-divisor condition $v_i \cdot v_j = 0$:

1. The vertex set V remains the same, and the neutrosophic membership function $\sigma(v) = (\sigma_1(v), \sigma_2(v), \sigma_3(v))$ continues to satisfy:
$$\sigma_1(v) + \sigma_2(v) + \sigma_3(v) \leq 3, \quad \forall v \in V.$$

2. The adjacency relation between v_i and v_j is now determined solely by the neutrosophic relation $\mu(v_i, v_j) = (\mu_1(v_i v_j), \mu_2(v_i v_j), \mu_3(v_i v_j))$, satisfying:

$$\mu_1(v_i v_j) \leq \min(\sigma_1(v_i), \sigma_1(v_j)),$$

$$\mu_2(v_i v_j) \geq \max(\sigma_2(v_i), \sigma_2(v_j)),$$

$$\mu_3(v_i v_j) \geq \max(\sigma_3(v_i), \sigma_3(v_j)).$$

3. The sum of the neutrosophic degrees of adjacency satisfies:

$$\mu_1(v_i v_j) + \mu_2(v_i v_j) + \mu_3(v_i v_j) \leq 3.$$

This structure corresponds exactly to the definition of a Neutrosophic Graph, as it retains all the neutrosophic membership and adjacency properties while disregarding the algebraic constraints of zero-divisors.

Hence, a Neutrosophic Zero Divisor Graph reduces to a Neutrosophic Graph when the zero-divisor condition $v_i \cdot v_j = 0$ is ignored. □

Theorem 3.7. *If $n = p^2$, where p is a prime number and $p > 2$, then the non-zero Neutrosophic zero divisor graph is a 2-partite graph.*

Proof. Let $R = \mathbb{Z}_{p^2}$, where p is a prime number greater than 2. The elements of $\mathbb{Z}_{p^2}$ are $\{0, 1, 2, \ldots, p^2 - 1\}$. The set of zero divisors $Z(R)$ in $\mathbb{Z}_{p^2}$ consists of the multiples of p, because for any $a \in \mathbb{Z}_{p^2}$, $a \cdot p \equiv 0 \pmod{p^2}$. Therefore, the set of non-zero zero divisors $Z(R)^*$ is:

$$Z(R)^* = \{p, 2p, 3p, \ldots, (p-1)p\}.$$

Each element in this set is a multiple of p, and the product of any two such elements is zero in $\mathbb{Z}_{p^2}$. Thus, these are the vertices of the non-zero Neutrosophic zero divisor graph $\Gamma_N(R)$.

We assign to each vertex $v \in V = Z(R)^*$ a neutrosophic membership function $\sigma(v) = (\sigma_1(v), \sigma_2(v), \sigma_3(v))$, where:

- $\sigma_1(v)$ represents the truth membership degree,
- $\sigma_2(v)$ represents the indeterminacy membership degree,
- $\sigma_3(v)$ represents the falsity membership degree.

For any two vertices $v_i, v_j \in V$, the neutrosophic adjacency relation $\mu(v_i, v_j) = (\mu_1(v_i v_j), \mu_2(v_i v_j), \mu_3(v_i v_j))$ is defined as:

$$\mu_1(v_i v_j) \leq \min(\sigma_1(v_i), \sigma_1(v_j)),$$
$$\mu_2(v_i v_j) \geq \max(\sigma_2(v_i), \sigma_2(v_j)),$$
$$\mu_3(v_i v_j) \geq \max(\sigma_3(v_i), \sigma_3(v_j)),$$

with the condition that $\mu_1(v_i v_j) + \mu_2(v_i v_j) + \mu_3(v_i v_j) \leq 3$.

We now partition the vertex set $V = \{p, 2p, 3p, \ldots, (p-1)p\}$ into two disjoint subsets based on whether the multiple of p is odd or even. Specifically, define:

$$V_1 = \{p, 3p, 5p, \ldots, (p-2)p\} \quad \text{(odd multiples of } p\text{)},$$

$$V_2 = \{2p, 4p, 6p, \ldots, (p-1)p\} \quad \text{(even multiples of } p\text{)}.$$

Note that:

- V_1 contains all odd multiples of p,
- V_2 contains all even multiples of p.

To show that the non-zero Neutrosophic zero divisor graph is 2-partite, we need to verify that:

1. No two vertices in V_1 are adjacent,
2. No two vertices in V_2 are adjacent,
3. Any vertex in V_1 is adjacent to any vertex in V_2.

Consider any two vertices $v_i, v_j \in V_1$. Since both v_i and v_j are odd multiples of p, their product is not zero modulo p^2. Thus, no edges exist between vertices in V_1.

Similarly, for any two vertices $v_i, v_j \in V_2$, their product is not zero modulo p^2, as they are both even multiples of p. Hence, no edges exist between vertices in V_2.

For any $v_i \in V_1$ and $v_j \in V_2$, their product $v_i \cdot v_j = 0 \pmod{p^2}$, because one is an odd multiple and the other is an even multiple of p. Therefore, an edge exists between any vertex in V_1 and any vertex in V_2.

Since the vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that:

- No edges exist within V_1,
- No edges exist within V_2,
- Edges exist between vertices in V_1 and V_2,

we conclude that the non-zero Neutrosophic zero divisor graph $\Gamma_N(R)$ is a 2-partite graph.

This completes the proof. □

3.3 Weak Neutrosophic Graph

The definition of the Weak Neutrosophic Graph is provided as follows.

Definition 3.8 (Weak Neutrosophic Graph)**.** A *Weak Neutrosophic Graph* $G = (V, E, \eta, \rho)$ is a graph characterized by the following components:

- V is the set of vertices.
- $\eta : V \to [0,1] \times [0,1] \times [0,1]$ is the neutrosophic vertex membership function. For each vertex $v \in V$,

$$\eta(v) = (\eta_1(v), \eta_2(v), \eta_3(v)),$$

  where:

  - $\eta_1(v)$: Truth membership of vertex v,
  - $\eta_2(v)$: Indeterminacy membership of vertex v,
  - $\eta_3(v)$: Falsity membership of vertex v.

- $\rho : E \to [0,1] \times [0,1] \times [0,1]$ is the neutrosophic edge membership function. For each edge $(u, v) \in E$,

$$\rho(u, v) = (\rho_1(u, v), \rho_2(u, v), \rho_3(u, v)),$$

  where:

  - $\rho_1(u, v)$: Truth membership of edge (u, v),
  - $\rho_2(u, v)$: Indeterminacy membership of edge (u, v),
  - $\rho_3(u, v)$: Falsity membership of edge (u, v).

The graph G is called a *Weak Neutrosophic Graph* if the following conditions hold for all edges $(u, v) \in E$:

1. **Truth Membership Condition:**

$$\rho_1(u, v) < \min(\eta_1(u), \eta_1(v)),$$

   where $\min(\cdot, \cdot)$ denotes the minimum operation between the truth memberships of the connected vertices.

2. **Indeterminacy Membership Condition:**

$$\rho_2(u, v) > \max(\eta_2(u), \eta_2(v)),$$

   where $\max(\cdot, \cdot)$ denotes the maximum operation between the indeterminacy memberships of the connected vertices.

3. **Falsity Membership Condition:**

$$\rho_3(u, v) > \max(\eta_3(u), \eta_3(v)),$$

where $\max(\cdot, \cdot)$ denotes the maximum operation between the falsity memberships of the connected vertices.

Theorem 3.9. *A Weak Neutrosophic Graph generalizes a Weak Fuzzy Graph.*

Proof. Let $G_N = (V, E, \eta, \rho)$ be a Weak Neutrosophic Graph, where:

$$\eta(v) = (\eta_1(v), \eta_2(v), \eta_3(v)) \quad \text{for all } v \in V,$$

$$\rho(u, v) = (\rho_1(u, v), \rho_2(u, v), \rho_3(u, v)) \quad \text{for all } (u, v) \in E.$$

To show that G_N generalizes a Weak Fuzzy Graph, consider the special case where:

$$\eta_2(v) = 0, \quad \eta_3(v) = 0, \quad \rho_2(u, v) = 0, \quad \rho_3(u, v) = 0 \quad \text{for all } v \in V \text{ and } (u, v) \in E.$$

In this case:

$$\eta(v) = (\eta_1(v), 0, 0), \quad \rho(u, v) = (\rho_1(u, v), 0, 0),$$

and the conditions of a Weak Neutrosophic Graph reduce to:

$$\rho_1(u, v) < \min(\eta_1(u), \eta_1(v)).$$

This matches the definition of a Weak Fuzzy Graph $G_F = (V, \sigma, \mu)$, where:

$$\sigma(v) = \eta_1(v), \quad \mu(u, v) = \rho_1(u, v).$$

Thus, a Weak Neutrosophic Graph includes Weak Fuzzy Graphs as a special case and therefore generalizes them. □

Theorem 3.10. *A Weak Neutrosophic Graph reduces to a Neutrosophic Graph under specific conditions.*

Proof. Let $G_N = (V, E, \eta, \rho)$ be a Weak Neutrosophic Graph. To show that G_N can reduce to a Neutrosophic Graph $G' = (V, E, \sigma, \mu)$, assume the following conditions hold:

- $\rho_1(u, v) = \min(\eta_1(u), \eta_1(v))$,
- $\rho_2(u, v) = \max(\eta_2(u), \eta_2(v))$,
- $\rho_3(u, v) = \max(\eta_3(u), \eta_3(v))$,

for all $(u, v) \in E$.

Under these conditions, the edge membership functions of G_N match the edge membership functions of a Neutrosophic Graph, and the conditions for a Weak Neutrosophic Graph:

$$\rho_1(u, v) < \min(\eta_1(u), \eta_1(v)),$$

$$\rho_2(u, v) > \max(\eta_2(u), \eta_2(v)),$$

$$\rho_3(u, v) > \max(\eta_3(u), \eta_3(v)),$$

become equalities, matching the adjacency relations in a Neutrosophic Graph.

Therefore, the Weak Neutrosophic Graph reduces to a Neutrosophic Graph when these specific conditions are satisfied. □

Theorem 3.11. *The union of two weak Neutrosophic graphs is a weak Neutrosophic graph.*

Proof. Let $G_1 = (V_1, E_1, \eta_1, \rho_1)$ and $G_2 = (V_2, E_2, \eta_2, \rho_2)$ be two weak Neutrosophic graphs, where $\eta_1(v) = (\eta_{1,1}(v), \eta_{1,2}(v), \eta_{1,3}(v))$ and $\eta_2(v) = (\eta_{2,1}(v), \eta_{2,2}(v), \eta_{2,3}(v))$ are the Neutrosophic vertex membership functions, and $\rho_1(u, v) = (\rho_{1,1}(u, v), \rho_{1,2}(u, v), \rho_{1,3}(u, v))$ and $\rho_2(u, v) = (\rho_{2,1}(u, v), \rho_{2,2}(u, v), \rho_{2,3}(u, v))$ are the Neutrosophic edge membership functions.

The union of G_1 and G_2, denoted by $G = G_1 \cup G_2 = (V, E, \eta, \rho)$, is defined as follows:

- $V = V_1 \cup V_2$,

- $E = E_1 \cup E_2$,
- $\eta(v) = \max(\eta_1(v), \eta_2(v))$ for $v \in V_1 \cap V_2$,
- $\eta(v) = \eta_1(v)$ if $v \in V_1 \setminus V_2$,
- $\eta(v) = \eta_2(v)$ if $v \in V_2 \setminus V_1$,
- $\rho(u, v) = \max(\rho_1(u, v), \rho_2(u, v))$ for $(u, v) \in E_1 \cap E_2$,
- $\rho(u, v) = \rho_1(u, v)$ if $(u, v) \in E_1 \setminus E_2$,
- $\rho(u, v) = \rho_2(u, v)$ if $(u, v) \in E_2 \setminus E_1$.

We must show that the union graph G satisfies the conditions of a weak Neutrosophic graph. That is, for all $(u, v) \in E$, the following inequalities hold:

1. $\rho_1(u, v) < \eta_1(u) \wedge \eta_1(v)$,
2. $\rho_2(u, v) > \eta_2(u) \vee \eta_2(v)$,
3. $\rho_3(u, v) > \eta_3(u) \vee \eta_3(v)$.

When $(u, v) \in E_1 \setminus E_2$, $\rho(u, v) = \rho_1(u, v)$. Since G_1 is a weak Neutrosophic graph, we have:

$$\rho_1(u, v) < \eta_1(u) \wedge \eta_1(v),$$

$$\rho_2(u, v) > \eta_2(u) \vee \eta_2(v),$$

$$\rho_3(u, v) > \eta_3(u) \vee \eta_3(v),$$

which satisfies the weak Neutrosophic graph conditions for the union graph.

When $(u, v) \in E_2 \setminus E_1$, $\rho(u, v) = \rho_2(u, v)$. Since G_2 is a weak Neutrosophic graph, we have:

$$\rho_1(u, v) < \eta_1(u) \wedge \eta_1(v),$$

$$\rho_2(u, v) > \eta_2(u) \vee \eta_2(v),$$

$$\rho_3(u, v) > \eta_3(u) \vee \eta_3(v),$$

which also satisfies the weak Neutrosophic graph conditions.

When $(u, v) \in E_1 \cap E_2$, $\rho(u, v) = \max(\rho_1(u, v), \rho_2(u, v))$ and $\eta(v) = \max(\eta_1(v), \eta_2(v))$. We now show that the weak Neutrosophic graph conditions hold:

- For the truth membership:

  $$\rho_1(u, v) = \max(\rho_{1,1}(u, v), \rho_{2,1}(u, v)) < \max(\eta_1(u), \eta_2(u)) \wedge \max(\eta_1(v), \eta_2(v)).$$

  Since both G_1 and G_2 satisfy the weak Neutrosophic conditions, this inequality holds.

- For the indeterminacy membership:

  $$\rho_2(u, v) = \max(\rho_{1,2}(u, v), \rho_{2,2}(u, v)) > \eta_2(u) \vee \eta_2(v).$$

- For the falsity membership:

  $$\rho_3(u, v) = \max(\rho_{1,3}(u, v), \rho_{2,3}(u, v)) > \eta_3(u) \vee \eta_3(v).$$

Thus, in all cases, the union graph $G = G_1 \cup G_2$ satisfies the weak Neutrosophic graph conditions, completing the proof. □

3.4 Neutrosophic Semigraph

The definition of the Neutrosophic Semigraph is provided as follows.

Definition 3.12 (Neutrosophic Semigraph)**.** A *neutrosophic semigraph* $G = (V, X, \eta, \rho)$ is defined as follows:

- V is a non-empty set of vertices.
- X is a set of edges, where each edge is an n-tuple of distinct vertices from V, i.e., $e = (v_1, v_2, \dots, v_n)$, with $n \geq 2$.
- $\eta : V \to [0,1] \times [0,1] \times [0,1]$ is a neutrosophic membership function that assigns each vertex $v \in V$ a triple $(\eta_1(v), \eta_2(v), \eta_3(v))$, representing the truth, indeterminacy, and falsity memberships, respectively.
- $\rho : V \times V \to [0,1] \times [0,1] \times [0,1]$ is a neutrosophic relation on vertices, where

$$\rho(u, v) = (\rho_1(u, v), \rho_2(u, v), \rho_3(u, v))$$

  represents the neutrosophic truth, indeterminacy, and falsity memberships of the relationship between vertices u and v.
- For each edge $e = (v_1, v_2, \dots, v_n)$, the neutrosophic edge membership function $\rho(e) = (\rho_1(e), \rho_2(e), \rho_3(e))$ satisfies:

$$\rho_1(e) \leq \rho_1(v_1, v_2) \wedge \rho_1(v_2, v_3) \wedge \cdots \wedge \rho_1(v_{n-1}, v_n) \wedge \eta_1(v_1) \wedge \eta_1(v_n),$$
$$\rho_2(e) \geq \rho_2(v_1, v_2) \vee \rho_2(v_2, v_3) \vee \cdots \vee \rho_2(v_{n-1}, v_n) \vee \eta_2(v_1) \vee \eta_2(v_n),$$
$$\rho_3(e) \geq \rho_3(v_1, v_2) \vee \rho_3(v_2, v_3) \vee \cdots \vee \rho_3(v_{n-1}, v_n) \vee \eta_3(v_1) \vee \eta_3(v_n),$$

  where $\wedge$ denotes the minimum operation and $\vee$ denotes the maximum operation.

In this neutrosophic semigraph, the vertices v_1 and v_n are referred to as the *end vertices*, and the vertices $v_2, v_3, \dots, v_{n-1}$ are called the *middle vertices*. A middle vertex that is also an end vertex of another edge is termed a *middle-end vertex*.

Theorem 3.13. *A Neutrosophic Semigraph can be transformed into both a Fuzzy Semigraph and a Neutrosophic Graph.*

Proof. Let $G = (V, X, \eta, \rho)$ be a Neutrosophic Semigraph where:

- V is the set of vertices.
- X is the set of edges, where each edge is an n-tuple of distinct vertices.
- $\eta : V \to [0,1]^3$ assigns a neutrosophic membership $(\eta_1(v), \eta_2(v), \eta_3(v))$ to each vertex.
- $\rho : V \times V \to [0,1]^3$ assigns a neutrosophic membership $(\rho_1(u, v), \rho_2(u, v), \rho_3(u, v))$ to the relationship between two vertices u and v.
- For each edge $e \in X$, the neutrosophic membership $\rho(e) = (\rho_1(e), \rho_2(e), \rho_3(e))$ satisfies conditions defined in the Neutrosophic Semigraph.

To transform G into a Fuzzy Semigraph $G' = (V, X, \sigma, \mu, \eta')$, define:

$$\sigma(v) = \eta_1(v), \quad \mu(u, v) = \rho_1(u, v), \quad \eta'(e) = \rho_1(e),$$

for all $v \in V$, $u, v \in V$, and $e \in X$, where $\eta_1(v)$ and $\rho_1(u, v)$ are the truth memberships from the neutrosophic graph.

The conditions for a Fuzzy Semigraph hold because the neutrosophic truth memberships satisfy the membership constraints of a Fuzzy Semigraph:

$$\eta'(e) \leq \mu(v_1, v_2) \wedge \mu(v_2, v_3) \wedge \cdots \wedge \mu(v_{n-1}, v_n) \wedge \sigma(v_1) \wedge \sigma(v_n).$$

Thus, G reduces to a Fuzzy Semigraph under this transformation.

To transform G into a Neutrosophic Graph $G' = (V, E, \eta, \rho')$, where E is the edge set of unordered vertex pairs, redefine:

$$\rho'(u, v) = \rho(u, v),$$

for all $u, v \in V$. The conditions of the Neutrosophic Graph are naturally satisfied because the neutrosophic memberships for vertices and edges are preserved.

Edges in the semigraph X, which are tuples, become unordered pairs in E. Therefore, G is transformed into a Neutrosophic Graph. □

3.5 Double/Triple Layered Neutrosophic Graph

Double/Triple Layered Neutrosophic Graph are provided as follows.

Definition 3.14 (Double Layered Neutrosophic Graph (DLNG))**.** A *Double Layered Neutrosophic Graph* is an extension of the standard neutrosophic graph where both vertices and edges are characterized by neutrosophic memberships. Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph with vertex set V, edge set E, neutrosophic vertex membership functions $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, and neutrosophic edge membership functions $\mu = (\mu_1, \mu_2, \mu_3)$.

Define the *Double Layered Neutrosophic Graph* $DLNG(G) = (V^*, E^*, \sigma_{DL}, \mu_{DL})$ as follows:

- The **node set** V^* is the union of vertices and edges from the original graph: $V^* = V \cup E$.
- The **neutrosophic vertex membership function** σ_{DL} is defined by:

$$\sigma_{DL}(x) = \begin{cases} \sigma(x) & \text{if } x \in V, \\ \mu(e) & \text{if } e \in E. \end{cases}$$

  where $\sigma(x) = (\sigma_1(x), \sigma_2(x), \sigma_3(x))$ for vertices and $\mu(e) = (\mu_1(e), \mu_2(e), \mu_3(e))$ for edges.

- The **neutrosophic edge membership function** μ_{DL} on $V^* \times V^*$ is defined as:

$$\mu_{DL}(x, y) = \begin{cases} \sigma(x) \wedge \sigma(y) & \text{if } x, y \in V, \\ \mu(e_i) \wedge \mu(e_j) & \text{if } e_i, e_j \in E, \text{and they share a common vertex,} \\ \sigma(x) \wedge \mu(e) & \text{if } x \in V, e \in E, \text{and } x \text{ is incident to } e, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the pair $DLNG(G) = (\sigma_{DL}, \mu_{DL})$ represents the *Double Layered Neutrosophic Graph*.

Theorem 3.15. *Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph. The order of the Double Layered Neutrosophic Graph (DLNG) is given by:*

$$O(DLNG) = O(G) + S(G),$$

where $O(G)$ is the order of G, and $S(G)$ is the size of G.

Proof. Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph with the following components:

- V is the set of vertices,
- E is the set of edges,
- $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the neutrosophic membership function for vertices,
- $\mu = (\mu_1, \mu_2, \mu_3)$ is the neutrosophic membership function for edges.

The order of the neutrosophic graph G, denoted $O(G)$, is the number of vertices in G, and the size of the graph, denoted $S(G)$, is the number of edges in G.

The Double Layered Neutrosophic Graph (DLNG) extends the graph G by including the edges as additional vertices. Hence, the node set V^* of the DLNG is defined as $V^* = V \cup E$.

The order of the DLNG, denoted $O(DLNG)$, is the total number of elements in V^*, i.e., the sum of the number of vertices and edges in G. Therefore,

$$O(DLNG) = |V^*| = |V| + |E|.$$

Thus, we can express the order of the DLNG as:

$$O(DLNG) = O(G) + S(G).$$

Additionally, the neutrosophic membership function for the DLNG, denoted σ_{DL}, is defined as follows:

- For $v \in V$, $\sigma_{DL}(v) = \sigma(v)$,
- For $e \in E$, $\sigma_{DL}(e) = \mu(e)$.

Thus, the neutrosophic order of DLNG, denoted $On(DLNG)$, is given by the sum of the neutrosophic memberships of all vertices and edges in V^*, i.e.,

$$On(DLNG) = \sum_{v \in V} \sigma(v) + \sum_{e \in E} \sum_{i=1}^{3} \mu_i(e).$$

This can be rewritten as:

$$On(DLNG) = On(G) + Sn(G),$$

where $On(G)$ is the neutrosophic order of G, and $Sn(G)$ is the neutrosophic size of G.

Thus, we have shown that the order of the Double Layered Neutrosophic Graph is the sum of the order and size of the original neutrosophic graph. □

Theorem 3.16. *A Double Layered Neutrosophic Graph (DLNG) can be transformed into a Double Layered Fuzzy Graph (DLFG).*

Proof. Let $DLNG(G) = (V^*, E^*, \sigma_{DL}, \mu_{DL})$. By ignoring the neutrosophic components $\sigma_2(x), \sigma_3(x), \mu_2(e), \mu_3(e)$, the vertex and edge membership functions reduce to:

$$\sigma_{DLFG}(x) = \sigma_1(x), \quad \mu_{DLFG}(x, y) = \mu_1(x, y),$$

where σ_1 and μ_1 are the truth membership functions in σ_{DL} and μ_{DL}, respectively. The resulting graph satisfies the definitions of a Double Layered Fuzzy Graph:

$$DLFG(G) = (\sigma_{DLFG}, \mu_{DLFG}),$$

proving that $DLNG(G)$ generalizes $DLFG(G)$. □

Definition 3.17 (Triple Layered Neutrosophic Graph (TLNG))**.** A *Triple Layered Neutrosophic Graph* is a further extension of the Double Layered Neutrosophic Graph, incorporating an additional layer. Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph. The *Triple Layered Neutrosophic Graph* $TLNG(G) = (V^*, E^*, \sigma_{TL}, \mu_{TL})$ is defined as follows:

- The **node set** V^* is the union of vertices and edges from the original graph: $V^* = V \cup E$.
- The **neutrosophic vertex membership function** σ_{TL} is defined by:

$$\sigma_{TL}(x) = \begin{cases} \sigma(x) & \text{if } x \in V, \\ 2 \cdot \mu(e) & \text{if } e \in E. \end{cases}$$

  where the factor of 2 represents the additional layer's increased influence on the neutrosophic memberships.

- The **neutrosophic edge membership function** μ_{TL} on $V^* \times V^*$ is defined as:

$$\mu_{TL}(x, y) = \begin{cases} \sigma(x) \wedge \sigma(y) & \text{if } x, y \in V, \\ \mu(e_i) \wedge \mu(e_j) & \text{if } e_i, e_j \in E, \text{and they share a common vertex,} \\ \sigma(x) \wedge \mu(e) & \text{if } x \in V, e \in E, \text{and } x \text{ is incident to } e, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the pair $TLNG(G) = (\sigma_{TL}, \mu_{TL})$ represents the *Triple Layered Neutrosophic Graph*.

Theorem 3.18. *The order of a Triple Layered Neutrosophic Graph (TLNG) is given by:*

$$O(TLNG) = O(G) + 2 \cdot S(G),$$

where $O(G)$ is the order of the neutrosophic graph G, and $S(G)$ is the size of G.

Proof. Let $G = (V, E, \sigma, \mu)$ be a neutrosophic graph, where:

- V is the set of vertices,
- E is the set of edges,

- $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the neutrosophic vertex membership function,
- $\mu = (\mu_1, \mu_2, \mu_3)$ is the neutrosophic edge membership function.

The order of the neutrosophic graph G, denoted $O(G)$, is the number of vertices in G, and the size of the graph, denoted $S(G)$, is the number of edges in G.

In a Triple Layered Neutrosophic Graph (TLNG), both the vertices and edges of the original graph G contribute to the node set. However, in TLNG, each edge is counted twice because of the additional layer. Specifically:

$$V^* = V \cup E,$$

where E appears with double the influence.

Thus, the neutrosophic vertex membership function for the TLNG, denoted σ_{TL}, is defined as:

$$\sigma_{TL}(x) = \begin{cases} \sigma(x) & \text{if } x \in V, \\ 2 \cdot \mu(e) & \text{if } e \in E. \end{cases}$$

This means that the membership for the vertices in E is doubled, reflecting the "triple layer" nature.

The order of the Triple Layered Neutrosophic Graph, $O(TLNG)$, is the sum of the vertices and twice the number of edges:

$$O(TLNG) = |V^*| = |V| + 2 \cdot |E|.$$

Thus, we have:

$$O(TLNG) = O(G) + 2 \cdot S(G).$$

Additionally, the neutrosophic order of TLNG, denoted $On(TLNG)$, is the sum of the neutrosophic membership values for all vertices and edges:

$$On(TLNG) = \sum_{v \in V} \sigma(v) + 2 \cdot \sum_{e \in E} \mu(e),$$

which can be rewritten as:

$$On(TLNG) = On(G) + 2 \cdot Sn(G),$$

where $On(G)$ is the neutrosophic order of the graph G and $Sn(G)$ is the neutrosophic size of the graph.

Thus, the order of the Triple Layered Neutrosophic Graph is indeed $O(G) + 2 \cdot S(G)$, as required. □

Theorem 3.19. *A Triple Layered Neutrosophic Graph (TLNG) can be transformed into a Triple Layered Fuzzy Graph (TLFG) or a Double Layered Neutrosophic Graph (DLNG).*

Proof. Let $TLNG(G) = (V^*, E^*, \sigma_{TL}, \mu_{TL})$ represent a Triple Layered Neutrosophic Graph. To show the transformations:

By constraining the neutrosophic vertex and edge membership functions σ_{TL} and μ_{TL} such that:

$$\sigma_{TL}(x) = \begin{cases} \sigma(x) & \text{if } x \in V, \\ \mu(e) & \text{if } e \in E, \end{cases}$$

and replacing σ_{TL} with the fuzzy membership function σ_{TLFG}, we obtain a Triple Layered Fuzzy Graph. The neutrosophic parameters $\sigma_2(x), \sigma_3(x), \mu_2(e), \mu_3(e)$ are ignored, reducing the representation to:

$$TLFG(G) = (\sigma_{TLFG}, \mu_{TLFG}).$$

By removing the additional layer factor (e.g., $2\mu(e)$) in σ_{TL} and ensuring that $V^* = V \cup E$, the Triple Layered Neutrosophic Graph reduces to a Double Layered Neutrosophic Graph:

$$\sigma_{DL}(x) = \begin{cases} \sigma(x) & \text{if } x \in V, \\ \mu(e) & \text{if } e \in E. \end{cases}$$

The edge membership μ_{TL} reduces to μ_{DL}, completing the transformation.

Thus, $TLNG(G)$ generalizes both $TLFG(G)$ and $DLNG(G)$. □

3.6 Connected Neutrosophic Chemical Graph

We define a Connected Neutrosophic Chemical Graph as follows. This graph concept combines the principles of a Connected Fuzzy Chemical Graph and a Neutrosophic Graph.

Definition 3.20 (Connected Neutrosophic Chemical Graph)**.** A *Connected Neutrosophic Chemical Graph* $G = (V, E, \sigma, \mu)$ is a neutrosophic graph where:

- V is the set of vertices representing atoms in a molecule.
- E is the set of edges representing chemical bonds between atoms.
- $\sigma = (\sigma_1, \sigma_2, \sigma_3) : V \to [0, 1]^3$ is the neutrosophic membership function for each atom, where:
  - $\sigma_1(v)$ represents the truth membership degree of atom v,
  - $\sigma_2(v)$ represents the indeterminacy membership degree of atom v,
  - $\sigma_3(v)$ represents the falsity membership degree of atom v.
- $\mu = (\mu_1, \mu_2, \mu_3) : E \to [0, 1]^3$ is the neutrosophic relation representing the degree of chemical bond membership, where:
  - $\mu_1(e)$ represents the truth membership of bond e,
  - $\mu_2(e)$ represents the indeterminacy membership of bond e,
  - $\mu_3(e)$ represents the falsity membership of bond e.
- The membership degrees for the edges must satisfy the condition $\mu_i(u, v) \leq \sigma_i(u) \wedge \sigma_i(v)$, where $\wedge$ denotes the minimum operation, for all $u, v \in V$ and $i \in \{1, 2, 3\}$.

The graph is said to be *connected* if for every pair of vertices $u, v \in V$, there exists a path of vertices $u = v_0, v_1, \ldots, v_k = v$ such that $\mu(v_i, v_{i+1}) > 0$ for all $0 \leq i < k$, ensuring that all atoms in the molecule are connected by chemical bonds.

Theorem 3.21. *A Connected Neutrosophic Chemical Graph (CNCG) can be transformed into both a Connected Fuzzy Chemical Graph (CFCG) and a Neutrosophic Graph.*

Proof. Let $G_C = (V, E, \sigma, \mu)$ be a Connected Neutrosophic Chemical Graph where:

- $\sigma = (\sigma_1, \sigma_2, \sigma_3) : V \to [0, 1]^3$ represents the neutrosophic vertex memberships.
- $\mu = (\mu_1, \mu_2, \mu_3) : E \to [0, 1]^3$ represents the neutrosophic edge memberships.

To transform G_C into a Connected Fuzzy Chemical Graph $G_F = (V, E, \sigma', \mu')$, we define:

$$\sigma'(v) = \sigma_1(v), \quad \mu'(u, v) = \mu_1(u, v), \quad \forall v \in V, \forall (u, v) \in E,$$

where $\sigma_1(v)$ and $\mu_1(u, v)$ are the truth memberships from the neutrosophic graph.

The connectivity condition is preserved because the truth memberships govern the connectedness of G_C. Thus, G_C reduces to a CFCG.

To transform G_C into a Neutrosophic Graph $G_N = (V, E, \sigma, \mu)$, no changes are needed as G_C already satisfies the definition of a Neutrosophic Graph. Therefore, G_C is inherently a Neutrosophic Graph. □

Definition 3.22 (Neighborly Irregular Neutrosophic Chemical Graph)**.** A *Neighborly Irregular Neutrosophic Chemical Graph* $G_{NIC} = (V, E, \sigma, \mu)$ is a neutrosophic chemical graph where:

- V is the set of vertices representing atoms in the molecular structure,
- E is the set of edges representing chemical bonds between atoms,
- $\sigma = (\sigma_1, \sigma_2, \sigma_3) : V \to [0, 1]^3$ is the neutrosophic membership function representing the truth, indeterminacy, and falsity memberships of each atom,
- $\mu = (\mu_1, \mu_2, \mu_3) : E \to [0, 1]^3$ is the neutrosophic relation representing the degrees of membership of the bonds, with the same conditions on membership as above.

- For any two adjacent vertices $u, v \in V$, their neutrosophic degrees are distinct. Specifically, $\deg(u) \neq \deg(v)$ holds with respect to the neutrosophic membership values of the atoms, ensuring that adjacent atoms have different connection strengths or roles in the molecule. The degree $\deg(v)$ of a vertex is calculated based on its neutrosophic memberships in the edges connected to it.

Theorem 3.23. *A Neighborly Irregular Neutrosophic Chemical Graph (NICG) can be transformed into both a Neighborly Irregular Fuzzy Chemical Graph (NIFCG) and a Neutrosophic Graph.*

Proof. Let $G_{NIC} = (V, E, \sigma, \mu)$ be a Neighborly Irregular Neutrosophic Chemical Graph where:

- $\sigma = (\sigma_1, \sigma_2, \sigma_3) : V \to [0,1]^3$ represents the neutrosophic vertex memberships.
- $\mu = (\mu_1, \mu_2, \mu_3) : E \to [0,1]^3$ represents the neutrosophic edge memberships.

To transform G_{NIC} into a Neighborly Irregular Fuzzy Chemical Graph $G_{NIFC} = (V, E, \sigma', \mu')$, we define:

$$\sigma'(v) = \sigma_1(v), \quad \mu'(u,v) = \mu_1(u,v), \quad \forall v \in V, \forall (u,v) \in E,$$

where $\sigma_1(v)$ and $\mu_1(u,v)$ are the truth memberships from the neutrosophic graph.

The degree distinctness property in G_{NIC}, $\deg(u) \neq \deg(v)$, is preserved under this transformation because the truth memberships uniquely determine the vertex and edge roles in G_{NIFC}. Thus, G_{NIC} reduces to a NIFCG.

To transform G_{NIC} into a Neutrosophic Graph $G_N = (V, E, \sigma, \mu)$, no structural changes are necessary as G_{NIC} is already a neutrosophic graph by definition. The graph retains its neutrosophic membership functions for vertices and edges. Therefore, G_{NIC} is inherently a Neutrosophic Graph. □

4. Conclusion and Future Work

This paper has explored various graph classes associated with Neutrosophic Graphs, including Smart Neutrosophic Graphs, Neutrosophic Zero Divisor Graphs, Weak Neutrosophic Graphs, Neutrosophic Semi-graphs, Double and Triple Layered Neutrosophic Graphs, and Connected Neutrosophic Chemical Graphs.

In terms of future research directions, our primary objective is to investigate the potential for defining more refined or generalized classes of graphs. This will involve both theoretical analysis and computational experiments based on the graph classes discussed in this work. By applying these definitions to real-world scenarios, we aim to evaluate their practicality and identify opportunities for the introduction of novel graph definitions.

Additionally, we plan to extend this study to hypergraphs [29, 65, 66, 92] and superhypergraphs [54, 59, 71, 147, 148], as well as investigate their applicability to directed graphs.

Furthermore, we intend to explore width parameters for these graph classes [33,34,130], which will allow us to examine graph-related problems and develop algorithms tailored to these advanced structures. This research aims to deepen our understanding of Neutrosophic Graphs and their applications across different domains.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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Short note of Even-hole-graph for Uncertain graph

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Abstract: A graph class consists of graphs that share common structural properties, defined by specific rules or constraints. This study focuses on Even-Hole-Free and Meyniel Graphs, analyzed within the frameworks of Fuzzy, Neutrosophic, Turiyam Neutrosophic, and Plithogenic Graphs. Even-Hole-Free Graphs lack induced cycles with an even number of vertices, ensuring longer cycles are odd. Meyniel Graphs feature odd-length cycles (of at least 5 vertices) with at least two chords, enhancing connectivity.

Keywords: Neutrosophic graph, Fuzzy graph, Plithogenic Graph, Even-hole-graph

1. Introduction

1.1 Uncertain Graph Theory

This paper explores the field of Uncertain Graph Theory. Graph theory, a branch of mathematics, investigates networks consisting of nodes (vertices) and their connections (edges) [36]. Here, we focus on several models of uncertain graphs, such as Fuzzy, Intuitionistic Fuzzy, Neutrosophic, Turiyam Neutrosophic, and Plithogenic Graphs. These models are specifically developed to address uncertainty in various applications. Collectively termed uncertain graphs, they extend classical graph theory by incorporating multiple levels of uncertainty [2, 3, 5–7, 9, 49–51, 59, 107, 110, 131].

Figure 1 illustrates the relationships among various types of uncertain graphs. This diagram provides an overview of selected graph types, though it does not cover all existing models, as the field of uncertain graphs has been extensively studied. For a more comprehensive understanding, we recommend consulting survey papers such as [47, 49].

1.2 Graph class and related topics

A graph class is a collection of graphs that share specific structural properties, defined by particular rules or constraints [27]. Studying graph classes helps identify common patterns, optimize algorithms, and simplify complex problems. This focus on distinct graph properties allows for tailored solutions across fields like network analysis, biology, and computer science.

In exploring graph classes and structures, it is common to analyze properties and algorithms by integrating features of classical graph types (e.g., Complete, Regular, Tree), intersection graphs (e.g., Permutation, Interval, Circle, String), and uncertain graphs (e.g., Fuzzy, Neutrosophic, Turiyam, Neutrosophic). Research often involves expanding or restricting graph classes based on the following aspects:

- *Classic Graph Properties:* Regular [64], Irregular [63, 103], Complete [11, 19, 81], Perfect [22, 62, 132], Tree, Path, Forest, Planar [50], Linear [33, 38], OuterPlanar [83, 112], Multigraph [28, 29].
- *Graph Using Operations:* Intersection Graph [49], Product Graph [12, 43, 88, 96, 135], Union Graph [14, 133].
- *Subgraph/Hypergraph Properties:* Supergraph [25], Hypergraph [18, 70, 74, 99], Superhypergraph [48, 55, 67, 124, 125, 128], n-Superhypergraph [124, 126], Subgraph, Induced Subgraph [80, 97], Induced Supergraph [49].
- *Graph Directionality:* Undirected, Directed, Mixed [113, 114], Bidirected [34, 65], Semi-directed [23, 24, 116]
- *Graph Partition:* Bipartite [16, 39, 40, 78, 98], Tripartite [17, 26, 71], n-partite [13, 75].
- *Uncertain Properties:* Fuzzy, Neutrosophic, Turiyam Neutrosophic[46,52,54,58], Plithogenic[57], Rough [56, 101, 109, 120], Vague [4, 106, 117], Soft [53, 66, 85, 108], Weighted [37, 77, 87, 91, 105], Picture Fuzzy [92, 93, 100], Paraconsistent [44, 49, 137, 138].

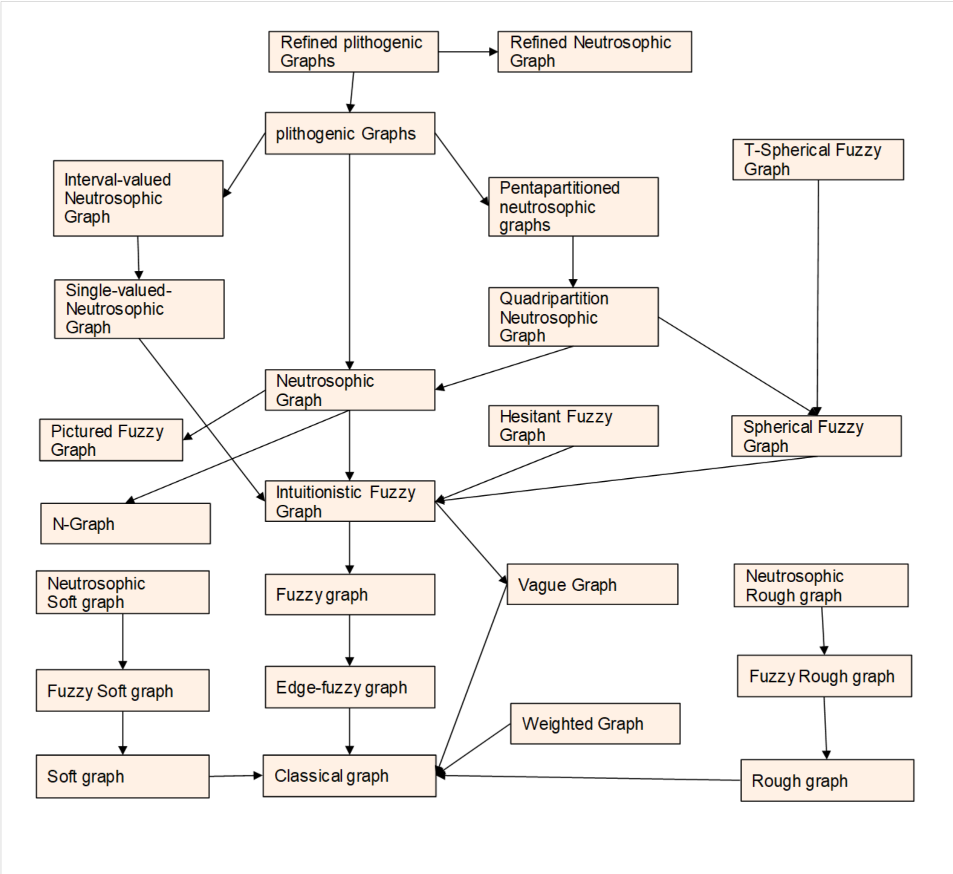

Fig. 1. Uncertain Graph Hierarchy[49]. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

- *Graph Dimensionality:* 2D, 3D, 4D, etc.
- *Themes:* Graph Classes Hierarchy, Mathematical Structure of Graph Classes, Graph Parameters [73, 76, 118], Algorithms [42], Computational Complexity [15, 104], Real-world Applications, Combinatorics [41, 69, 119].

The relationship between hypergraphs and directionality is illustrated in Figures 2 and 3.

1.3 Our Contribution

In this study, we introduce and analyze the concepts of two classic graph classes—Even-Hole-Free Graphs and Meyniel Graphs—within the frameworks of Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs. Even-Hole-Free Graphs are those that do not contain any induced cycles with an even number of vertices, ensuring longer cycles are of odd lengths. Meyniel Graphs are defined as graphs in which every odd-length cycle (with at least 5 vertices) has at least two chords, thereby enhancing connectivity.

2. Preliminaries and definitions

In this section, we present a brief overview of the definitions and notations used throughout this paper.

2.1 Basic Graph Concepts

Here, we present some basic concepts of graph theory. For more foundational concepts and notations, please refer to lecture notes, surveys, or introductory texts such as [35, 36, 72, 140].

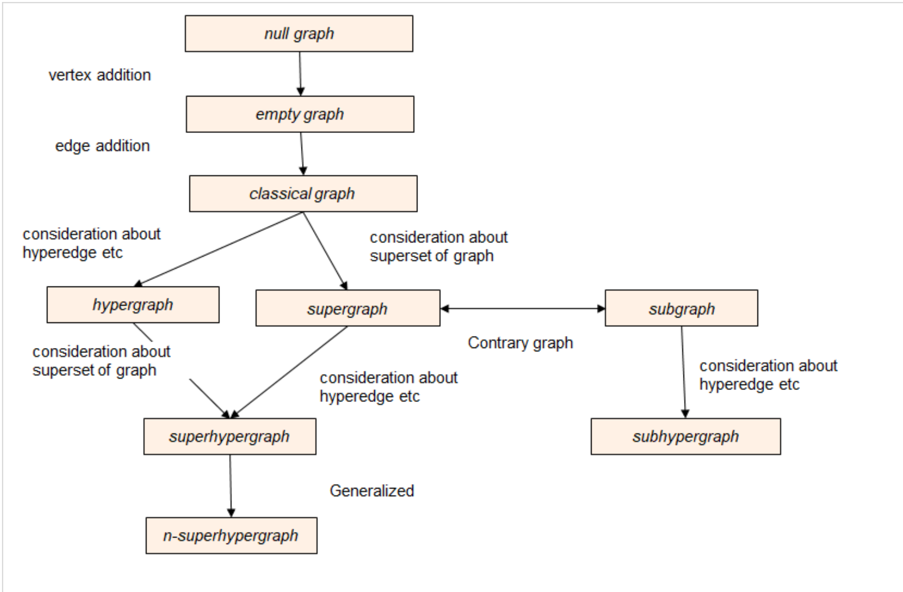

Fig. 2. HyperGraph Hierarchy

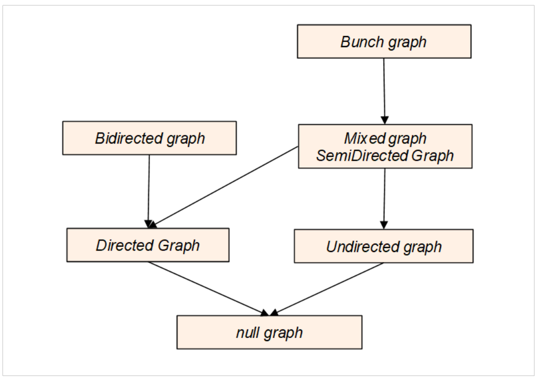

Fig. 3. Directed Graph Hierarchy. The graph class at the origin of an arrow contains the graph class at the destination of the arrow.

Definition 1 (Graph)**.** [36] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2 (Subgraph)**.** [36] Let $G = (V, E)$ be a graph. A *subgraph* $H = (V_H, E_H)$ of G is a graph such that:

- $V_H \subseteq V$, i.e., the vertex set of H is a subset of the vertex set of G.
- $E_H \subseteq E$, i.e., the edge set of H is a subset of the edge set of G.
- Each edge in E_H connects vertices in V_H.

Definition 3 (Degree)**.** [36] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 4 (Induced Subgraph). Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. For a subset of vertices $S \subseteq V$, the *induced subgraph* of G on S, denoted by $G[S]$, is defined as the graph whose vertex set is S and whose edge set consists of all edges in E that have both endpoints in S.

Formally, the induced subgraph $G[S]$ is defined as:

$$G[S] = (S, E_S),$$

where

$$E_S = \{(u, v) \in E \mid u \in S,\ v \in S\}.$$

In other words, $G[S]$ includes all the vertices in S and all the edges of G that connect pairs of vertices in S.

Definition 5 (Path). [36] A *path* in a graph $G = (V, E)$ is an ordered sequence of distinct vertices $(v_0, v_1, \dots, v_k)$ such that:

- $v_i \in V$ for all $0 \leq i \leq k$, where $k \geq 0$ is the length of the path.
- $(v_i, v_{i+1}) \in E$ for all $0 \leq i < k$, i.e., consecutive vertices in the sequence are connected by edges in G.

If $v_0 = v_k$, the path is called a *cycle*.

Definition 6 (Tree). [36] A *tree* is a connected, acyclic graph $T = (V_T, E_T)$, where:

- V_T is the set of vertices.
- E_T is the set of edges.
- T is connected, meaning there exists a path between any two vertices in V_T.
- T contains no cycles, i.e., there is no path in T where the first and last vertices are the same.

Alternatively, a tree can be defined as a graph in which there is exactly one path between any two vertices.

Definition 7 (Cycle). [36] Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. A *cycle* in G is a path $(v_0, v_1, \dots, v_{k-1}, v_k)$ such that:

- $v_0 = v_k$, meaning the first and last vertices are the same.
- $v_i \neq v_j$ for all $0 \leq i < j < k$, ensuring that all intermediate vertices are distinct.
- $(v_i, v_{i+1}) \in E$ for all $0 \leq i < k$, indicating that each pair of consecutive vertices is connected by an edge.

The *length* of a cycle is the number of edges it contains, which is equal to k. A cycle of length 3 is called a *triangle*, while a cycle of length 4 is called a *quadrilateral*, and so on.

2.2 Fuzzy, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs

In this subsection, we explore Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. The following definitions include related concepts. Note that the Turiyam Neutrosophic Set is, in fact, a specific case of the Quadripartitioned Neutrosophic Set, achieved by replacing "Contradiction" with "Liberal." (cf.[**?**, 127])

Definition 8. (cf.[94, 110]) A *crisp graph* is an ordered pair $G = (V, E)$, where:

- V is a finite, non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges, where each edge is an unordered pair of distinct vertices.

Formally, for any edge $(u, v) \in E$, the following holds:

$$(u, v) \in E \iff u \neq v \quad \text{and} \quad u, v \in V$$

This implies that there are no loops (i.e., no edges of the form (v, v)) and edges represent binary relationships between distinct vertices.

Definition 9 (Unified Graphs Framework: Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs). (cf.[49]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, and falsity.

1. *Fuzzy Graph [20, 68, 102, 110, 139]:*

   - Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$, representing the degree of participation of v in the fuzzy graph.
   - Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$, representing the strength of the connection between u and v.

2. *Intuitionistic Fuzzy Graph (IFG) [3, 84, 134, 141]:*

   - Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $\nu_A(v) \in [0, 1]$ (degree of non-membership), such that $\mu_A(v) + \nu_A(v) \leq 1$.
   - Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0, 1]$ (degree of membership) and $\nu_B(u, v) \in [0, 1]$ (degree of non-membership), such that $\mu_B(u, v) + \nu_B(u, v) \leq 1$.

3. *Neutrosophic Graph [8, 10, 47, 82, 89, 115, 129]:*

   - Each vertex $v \in V$ is assigned a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where:
     - $\sigma_T(v) \in [0, 1]$ is the truth-membership degree,
     - $\sigma_I(v) \in [0, 1]$ is the indeterminacy-membership degree,
     - $\sigma_F(v) \in [0, 1]$ is the falsity-membership degree,
     - $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
   - Each edge $e = (u, v) \in E$ is assigned a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing the truth, indeterminacy, and falsity degrees for the connection between u and v.

4. *Turiyam Neutrosophic Graph [49, 59–61, 122]:*

   - Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where:
     - $t(v) \in [0, 1]$ is the truth value,
     - $iv(v) \in [0, 1]$ is the indeterminacy value,
     - $fv(v) \in [0, 1]$ is the falsity value,
     - $lv(v) \in [0, 1]$ is the liberal state value,
     - $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.
   - Each edge $e = (u, v) \in E$ is similarly assigned a quadruple representing the same parameters for the connection between u and v.

2.3 Plithogenic Graphs

Plithogenic Graphs have been introduced as an extension of Fuzzy Graphs and Turiyam Neutrosophic Graphs, broadening the concept to encompass Plithogenic Sets [123]. These graphs have become a prominent subject of ongoing research and development [49, 90, 121, 130, 131]. The formal definition is provided below.

Definition 10. [131] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

1. *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:

   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the range of possible attribute values.
   - $adf : M \times Ml \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
   - $aCf : Ml \times Ml \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

2. *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:

   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.

- Nm is the range of possible attribute values.
- $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
- $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. *Edge Appurtenance Constraint*: For all $(x,a),(y,b) \in M \times Ml$:

$$bdf\,((xy),(a,b)) \le \min\{adf(x,a), adf(y,b)\}$$

where $xy \in N$ is an edge between vertices x and y, and $(a,b) \in Nm \times Nm$ are the corresponding attribute values.

2. *Contradiction Function Constraint*: For all $(a,b),(c,d) \in Nm \times Nm$:

$$bCf\,((a,b),(c,d)) \le \min\{aCf(a,c), aCf(b,d)\}$$

3. *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} &aCf(a,a) = 0, && \forall a \in Ml \\ &aCf(a,b) = aCf(b,a), && \forall a,b \in Ml \\ &bCf(a,a) = 0, && \forall a \in Nm \\ &bCf(a,b) = bCf(b,a), && \forall a,b \in Nm \end{aligned}$$

Example 11. (cf.[47]) The following examples are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic Turiyam Neutrosophic Graph*.

3. Result in this paper

In this section, we present the results of this paper.

3.1 Uncertain Even-Hole-Free Graphs

We examine Uncertain Even-Hole-Free Graphs [1,30–32,45,95,136]. The following definitions include the relevant concepts.

Definition 12 (Even-Hole-Free Graph)**.** [95] Let $G = (V, E)$ be an undirected graph, where V is the set of vertices and E is the set of edges. An *even-hole-free graph* is defined as a graph that does not contain any *induced cycle* of even length greater than or equal to 4.

Formally, a graph G is even-hole-free if there does not exist any subset of vertices $S \subseteq V$ such that the subgraph induced by S, denoted as $G[S]$, forms a cycle of even length $2k$ for some integer $k \ge 2$.

In other words,

$$\forall S \subseteq V,\ G[S] \text{ is not a cycle of length } 2k, \text{ for any } k \ge 2.$$

Definition 13 (Fuzzy Even-Hole-Free Graph)**.** A fuzzy graph $G = (V, \sigma, \mu)$ is called a *fuzzy even-hole-free graph* if there does not exist any induced cycle C of even length $2k$ (where $k \ge 2$) such that all edges in C have positive membership degrees, i.e., $\mu(e) > 0$ for all edges e in C.

Theorem 14. *A Fuzzy Even-Hole-Free Graph is a Fuzzy Graph.*

Proof. This is evident by definition. □

Definition 15 (Neutrosophic Even-Hole-Free Graph)**.** A neutrosophic graph $G = (V, \sigma, \mu)$ is called a *neutrosophic even-hole-free graph* if there does not exist any induced cycle C of even length $2k$ (where $k \ge 2$) such that all edges in C have positive truth-membership degrees, i.e., $\mu_T(e) > 0$ for all edges e in C.

Theorem 16. *A Neutrosophic Even-Hole-Free Graph is a Neutrosophic Graph.*

Proof. This is evident by definition. □

Definition 17 (Turiyam Neutrosophic Even-Hole-Free Graph)**.** A Turiyam Neutrosophic graph $G = (V, \sigma, \mu)$ is called a *Turiyam Neutrosophic even-hole-free graph* if there does not exist any induced cycle C of even length $2k$ (where $k \geq 2$) such that all edges in C have positive truth values, i.e., $t(e) > 0$ for all edges e in C.

Theorem 18. *A Turiyam Neutrosophic Even-Hole-Free Graph is a Turiyam Neutrosophic Graph.*

Proof. This is evident by definition. □

Definition 19 (Plithogenic Even-Hole-Free Graph)**.** A plithogenic graph $G = (V, adf_V, E, adf_E)$ is called a *plithogenic even-hole-free graph* if there does not exist any induced cycle C of even length $2k$ (where $k \geq 2$) such that all edges in C have attribute degree functions not equal to the zero vector 0, i.e., $adf_E(e) \neq 0$ for all edges e in C.

Theorem 20. *A Plithogenic Even-Hole-Free Graph is a Plithogenic Graph.*

Proof. This is evident by definition. □

Theorem 21. *Any induced subgraph of a Plithogenic Even-Hole-Free Graph is also a Plithogenic Even-Hole-Free Graph.*

Proof. Let $G = (V, adf_V, E, adf_E)$ be a Plithogenic Even-Hole-Free Graph, and let $V' \subseteq V$ be a subset of vertices. The induced subgraph $G' = (V', adf_V|_{V'}, E', adf_E|_{E'})$ is defined by:

$$E' = \{(u, v) \in E \mid u, v \in V'\}.$$

Assume, for contradiction, that G' contains an induced cycle C of even length $2k \geq 4$ with $adf_E(e) \neq 0$ for all $e \in C$. Since C is an induced cycle in G', it is also an induced cycle in G with the same property, contradicting the assumption that G is Even-Hole-Free.

Therefore, G' is also a Plithogenic Even-Hole-Free Graph. □

Theorem 22. *Removing an edge from a Plithogenic Even-Hole-Free Graph results in a graph that is also a Plithogenic Even-Hole-Free Graph.*

Proof. Let $G = (V, adf_V, E, adf_E)$ be a Plithogenic Even-Hole-Free Graph, and let $e_0 \in E$ be an edge to be removed. Define $G' = (V, adf_V, E', adf_E|_{E'})$ where $E' = E \setminus \{e_0\}$.

The removal of e_0 cannot introduce new induced cycles. Any induced cycle in G' would have to exist in G unless e_0 was a part of it. Since G does not have any even-length induced cycles with non-zero attribute degrees, neither does G'.

Therefore, G' remains a Plithogenic Even-Hole-Free Graph. □

Theorem 23. *The property of being a Plithogenic Even-Hole-Free Graph is preserved under graph homomorphisms that preserve non-zero attribute degrees.*

Proof. Let $f : V \to V'$ be a graph homomorphism between Plithogenic Graphs $G = (V, adf_V, E, adf_E)$ and $G' = (V', adf_{V'}, E', adf_{E'})$ such that:

$$(u, v) \in E \text{ with } adf_E(u, v) \neq 0 \implies (f(u), f(v)) \in E' \text{ with } adf_{E'}(f(u), f(v)) \neq 0.$$

Suppose G' contains an induced even cycle C' with edges of non-zero attribute degrees. The preimage of C' under f would be a cycle in G with edges of non-zero attribute degrees, contradicting the assumption that G is Even-Hole-Free.

Therefore, G' is also a Plithogenic Even-Hole-Free Graph. □

Theorem 24. *Contracting an edge with attribute degree function not equal to* 0 *in a Plithogenic Even-Hole-Free Graph results in a graph that is also a Plithogenic Even-Hole-Free Graph.*

Proof. Let $e = (u, v)$ be an edge in G with $adf_E(e) \neq 0$. Contracting e merges u and v into a new vertex w. Any induced cycle C in the contracted graph corresponds to an induced cycle in G (possibly with u and v replaced by w).

If C is an even-length induced cycle with edges of non-zero attribute degrees, it would correspond to an even-length induced cycle in G with edges of non-zero attribute degrees, contradicting the Even-Hole-Free property of G.

Therefore, the contracted graph remains Plithogenic Even-Hole-Free. □

Theorem 25. *Deleting a vertex from a Plithogenic Even-Hole-Free Graph results in a graph that is also a Plithogenic Even-Hole-Free Graph.*

Proof. Let $v_0 \in V$ be a vertex to be deleted. Define $G' = (V', adf_V|_{V'}, E', adf_E|_{E'})$ where $V' = V \setminus \{v_0\}$ and E' consists of all edges in E not incident to v_0.

Any induced cycle in G' would also be an induced cycle in G. Since G is Even-Hole-Free, G' cannot contain an even-length induced cycle with edges of non-zero attribute degrees.

Therefore, G' is a Plithogenic Even-Hole-Free Graph. □

Theorem 26. *The class of Plithogenic Even-Hole-Free Graphs is closed under disjoint union.*

Proof. Let G_1 and G_2 be two Plithogenic Even-Hole-Free Graphs. Their disjoint union $G = G_1 \cup G_2$ is a graph where $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

Any induced subgraph of G is either entirely within G_1, entirely within G_2, or consists of isolated vertices from both. Since G_1 and G_2 are Even-Hole-Free, G is also Even-Hole-Free.

Therefore, the class is closed under disjoint union. □

3.2 Uncertain Meyniel Graphs

We examine Uncertain Meyniel Graphs [21, 79, 86, 111]. The following definitions include the relevant concepts.

Definition 27 (Meyniel Graph)**.** Let $G = (V, E)$ be an undirected graph, where V is the set of vertices and E is the set of edges. A graph G is called a *Meyniel graph* if it satisfies one of the following equivalent conditions:

1. *Chord Condition:* Every cycle in G of odd length at least 5 has at least two chords. A *chord* is an edge that connects two non-consecutive vertices in the cycle.
2. *Even Pair Condition:* An *even pair* in G is defined as a pair of non-adjacent vertices $x, y \in V$ such that every chordless path between x and y has an even number of edges. A graph G is Meyniel if for every connected induced subgraph $H \subseteq G$ that is not a complete graph, all vertices in H are part of at least one even pair.

Thus, a Meyniel graph can be characterized either by the chord condition for odd cycles or by the even pair condition within induced subgraphs.

Definition 28 (Fuzzy Meyniel Graph)**.** A fuzzy graph $G = (V, \sigma, \mu)$ is called a *fuzzy Meyniel graph* if every induced cycle C of odd length at least 5 with positive membership degrees has at least two chords with positive membership degrees.

Theorem 29. *A Fuzzy Meyniel Graph is a Fuzzy Graph.*

Proof. This is evident by definition. □

Definition 30 (Neutrosophic Meyniel Graph)**.** A neutrosophic graph $G = (V, \sigma, \mu)$ is called a *neutrosophic Meyniel graph* if every induced cycle C of odd length at least 5 with positive truth-membership degrees has at least two chords with positive truth-membership degrees.

Theorem 31. *A neutrosophic Meyniel Graph is a neutrosophic Graph.*

Proof. This is evident by definition. □

Definition 32 (Turiyam Neutrosophic Meyniel Graph)**.** A Turiyam Neutrosophic graph $G = (V, \sigma, \mu)$ is called a *Turiyam Neutrosophic Meyniel graph* if every induced cycle C of odd length at least 5 with positive truth values has at least two chords with positive truth values.

Theorem 33. *A Turiyam Neutrosophic Meyniel Graph is a Turiyam Neutrosophic Graph.*

Proof. This is evident by definition. □

Definition 34 (Plithogenic Meyniel Graph)**.** A plithogenic graph $G = (V, adf_V, E, adf_E)$ is called a *plithogenic Meyniel graph* if every induced cycle C of odd length at least 5 with edges having attribute degree functions not equal to 0 has at least two chords with attribute degree functions not equal to 0.

Theorem 35. *A Plithogenic Meyniel Graph is a Plithogenic Graph.*

Proof. This is evident by definition. □

Theorem 36. *In a Plithogenic Meyniel Graph, every induced cycle of odd length at least 5 with edges of non-zero attribute degrees has at least two chords with attribute degree functions not equal to* 0*.*

Proof. This follows directly from the definition of a Plithogenic Meyniel Graph. By definition, every such cycle must have at least two chords with attribute degree functions not equal to 0.

Therefore, the property holds. □

Theorem 37. *In a Plithogenic Meyniel Graph, for every connected induced subgraph H that is not a complete graph, all vertices are part of at least one even pair with respect to edges of non-zero attribute degrees.*

Proof. By the Even Pair Condition in the definition of a Plithogenic Meyniel Graph, this property is satisfied. An even pair consists of two non-adjacent vertices such that every chordless path between them has an even number of edges with non-zero attribute degrees.

Therefore, every vertex in H is part of at least one such even pair. □

Theorem 38. *Every Plithogenic Even-Hole-Free (respectively, Meyniel) Graph can be transformed into a Neutrosophic Even-Hole-Free (Meyniel) Graph, a Turiyam Neutrosophic Even-Hole-Free (Meyniel) Graph, a Fuzzy Even-Hole-Free (Meyniel) Graph, and a Classic Even-Hole-Free (Meyniel) Graph.*

Proof. Let $G = (V, adf_V, E, adf_E)$ be a Plithogenic Even-Hole-Free (or Meyniel) Graph, where:

- V is the set of vertices.
- $adf_V : V \to [0, 1]^s$ assigns an attribute degree function to each vertex.
- $E \subseteq V \times V$ is the set of edges.
- $adf_E : E \to [0, 1]^s$ assigns an attribute degree function to each edge.

We consider the following transformations:

1. *Transformation to Neutrosophic Graph:*
   Set $s = 3$ and interpret the attribute degree functions as:
   $$adf_V(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v)), \quad adf_E(e) = (\mu_T(e), \mu_I(e), \mu_F(e)),$$
   where $\sigma_T(v)$ is the truth-membership degree, $\sigma_I(v)$ is the indeterminacy-membership degree, and $\sigma_F(v)$ is the falsity-membership degree for vertex v. Similarly for edges.
   Under this interpretation, G satisfies the definitions of a Neutrosophic Even-Hole-Free (Meyniel) Graph, as the conditions involving the attribute degree functions directly correspond to those in Neutrosophic Graphs when considering the truth-membership degrees $\sigma_T(v)$ and $\mu_T(e)$.
2. *Transformation to Turiyam Neutrosophic Graph:*
   Set $s = 4$ and interpret the attribute degree functions as:
   $$adf_V(v) = (t(v), iv(v), fv(v), lv(v)), \quad adf_E(e) = (t(e), iv(e), fv(e), lv(e)),$$
   where $t(v)$ is the truth value, $iv(v)$ is the indeterminacy value, $fv(v)$ is the falsity value, and $lv(v)$ is the liberal state value for vertex v.
   With this interpretation, G meets the definitions of a Turiyam Neutrosophic Even-Hole-Free (Meyniel) Graph. The conditions on $t(e)$ (the truth value of edges) correspond to the presence or absence of edges in the definitions, ensuring that the Turiyam Neutrosophic graph maintains the Even-Hole-Free or Meyniel properties.

3. *Transformation to Fuzzy Graph:*
   Set $s = 1$ and interpret the attribute degree functions as:
   $$adf_V(v) = \sigma(v), \quad adf_E(e) = \mu(e),$$
   where $\sigma(v)$ is the membership degree of vertex v, and $\mu(e)$ is the membership degree of edge e.

   Under this interpretation, G satisfies the definitions of a Fuzzy Even-Hole-Free (Meyniel) Graph. The conditions involving $\mu(e)$ correspond to those in Fuzzy Graphs, ensuring that cycles with edges of positive membership degrees adhere to the Even-Hole-Free or Meyniel properties.
4. *Transformation to Classic Graph:*
   By considering only the underlying crisp graph (ignoring the attribute degree functions adf_V and adf_E), we define $G' = (V, E')$, where:
   $$E' = \{e \in E \mid adf_E(e) \neq 0\}.$$
   The edge set E' consists of all edges with attribute degree functions not equal to the zero vector.

   Since the presence of even holes or the Meyniel property depends on the existence of certain cycles with edges (which are considered present when $adf_E(e) \neq 0$), G' is an Even-Hole-Free (Meyniel) Graph in the classic sense.

Therefore, by appropriately interpreting the attribute degree functions and selecting the dimension s, a Plithogenic Even-Hole-Free (Meyniel) Graph can be transformed into Neutrosophic, Turiyam Neutrosophic, Fuzzy, and Classic Even-Hole-Free (Meyniel) Graphs. □

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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Survey of Planar and Outerplanar Graphs in Fuzzy and Neutrosophic Graphs

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Abstract: As many readers may know, graph theory is a fundamental branch of mathematics that explores networks made up of nodes and edges, focusing on their paths, structures, and properties [196]. A planar graph is one that can be drawn on a plane without any edges intersecting, ensuring planarity. Outerplanar graphs, a subset of planar graphs, have all their vertices located on the boundary of the outer face in their planar embedding. In recent years, outerplanar graphs have been formally defined within the context of fuzzy graphs. To capture uncertain parameters and concepts, various graphs such as fuzzy, neutrosophic, Turiyam, and plithogenic graphs have been studied. In this paper, we investigate planar graphs, outerplanar graphs, apex graphs, and others within the frameworks of neutrosophic graphs, Turiyam Neutrosophic graphs, fuzzy graphs, and plithogenic graphs.

Keywords: Neutrosophic graph, Turiyam Neutrosophic graph, Fuzzy graph, Planar graph, Outerplanar graph

1. Introduction

1.1 Graph Theory

As readers may be aware, graph theory, a fundamental branch of mathematics, explores networks composed of nodes and edges, focusing on their paths, structures, and properties [196]. Using graphs allows for a clear representation of connections between real-world concepts. This is one reason why graph theory has been extensively studied across a wide range of applications [61, 114, 182, 233, 271, 436, 528]. For instance, in the field of AI, research on graph neural networks, which employs graph theory concepts, is highly active [219, 475, 550, 552, 563].

1.2 Graph Class

One of the powerful tools in graph theory is the classification of graphs into distinct graph classes based on shared properties or structural features. Numerous graph classes have been introduced, each designed to capture specific behaviors or characteristics of a graph. These classes serve as a foundation for developing efficient algorithms, simplifying problem-solving, and gaining deeper insights into computational complexity (cf.[63, 116, 126, 415]).

Examples of well-known graph classes include Tree Graphs [503], Path Graphs [555], Complete Graphs [154], Trapezoid graph[168] Circle Graphs [121], Unit Disk Graphs [164], Circular-arc graph[251] Edge-Transitive Graphs [387, 388], Ultrahomogeneous Graphs [351], Visibility Graphs [350], Interval graphs [243], Caylay graphs [331], Comparability graph[549], Proper Interval graphs [148], Dense graphs[372], Universal graph[159], Outerplanar Graphs [296], Petersen Graphs [291], and Total Graphs [522]. The study of such graph classes allows researchers to identify common properties, develop more specialized and efficient algorithms, and apply these insights to practical problems. In line with these observations, a growing area of research focuses on analyzing how closely a given graph or algorithm matches desired graph classes or structures. (cf.[196,238,423,473,527]).

1.3 Planar graph and Outerplanar graph

In the field of graph theory, various graph classes with diverse properties are continuously studied and introduced. This paper primarily focuses on planar graphs and outerplanar graphs. A planar graph is defined as a graph that can be drawn on a plane without any edges crossing, thereby maintaining planarity [177,305,418,455, 478]. A notable example of a planar graph is the Butterfly graph (cf.[539]). An outerplanar graph, on the other hand, is a planar graph in which all vertices are positioned on the boundary of the outer face in its embedding [77, 296, 409]. Additionally, another well-known graph closely related to planar graphs is the Apex Graph. Due to the edge-crossing restrictions of planar graphs, they are not only visually intuitive but are also often used in various papers for simplification purposes.

Research on parameter values of outerplanar graphs is ongoing. For instance, outerplanar graphs have a degeneracy of at most 2[365] [1] Additionally, outerplanar graphs have a boxicity [2] of at most 2 [476].

Other related graph classes include Halin graphs [113], upward planar graphs[16,74,316], convex planar graphs[187], 1-planar graph[292,515], and k-Planar Graphs[84] which are also known in the context of planar graphs.

One of the key motivations for studying these specific graph classes is the development of efficient algorithms tailored to restricted graph types. For instance, similar to outerplanar graphs, Halin graphs exhibit low treewidth, which makes many algorithmic problems easier to solve compared to unrestricted planar graphs [516]. Numerous other studies have also examined planar graphs and their related graph classes, contributing significantly to this area of research[46,132,166,342,451].

1.4 Fuzzy Graphs and Neutrosophic Graphs

To address the uncertainties present in the world, various types of graphs like fuzzy, neutrosophic, Turiyam, and plithogenic graphs have been studied.

A fuzzy graph assigns a membership degree between 0 and 1 to each edge and vertex, representing the level of uncertainty. Simply put, a fuzzy graph is a graphical representation of a fuzzy set (cf. [206,360,558, 559]). In real-world applications, fuzzy graphs are used in areas such as social networks, decision-making, and transportation systems to model relationships or connections that are not precise or are uncertain [401,459]. This wide range of applicability has led to significant research interest in fuzzy graphs.

Recently, neutrosophic graphs [28,33,130,238,276,297,319,463,493,497] and neutrosophic hypergraphs [27,37,186,373] have also gained attention within the framework of neutrosophic set theory [48,501]. "Neutrosophic" extends classical and fuzzy logic by introducing degrees of truth, indeterminacy, and falsity, offering a more comprehensive way to handle uncertainty.

Following this, the Turiyam Neutrosophic graph was introduced as a further extension of neutrosophic and fuzzy graphs. A Turiyam Neutrosophic graph stands out by assigning four attributes to each vertex and edge: truth, indeterminacy, falsity, and a liberal state, thus broadening the framework of neutrosophic and fuzzy graphs [244–246]. Additionally, plithogenic graphs have emerged as a more generalized form and are currently a subject of active study [321,489,506,511].

Focusing on graph types that handle these uncertainties, particularly those related to planar graphs, we have examples such as fuzzy planar graphs [20,442,472], fuzzy outerplanar graphs [309], intuitionistic fuzzy planar graphs [49], and neutrosophic planar graphs [37,146,375].

1.5 Our Contribution

Building on the observations above, the study of planar graphs, outerplanar graphs, fuzzy graphs, and neutrosophic graphs is of great importance; however, these topics have not yet been fully explored. In this paper, we focus on neutrosophic outerplanar graphs and related graph classes. Our goal is to stimulate further research and development in the study of various graph classes.

The following diagram illustrates a portion of the overview of this paper. While this diagram focuses on planar graphs and apex graphs, a similar analysis will be conducted for outerplanar graphs and other related types.

2. Preliminaries and definitions

In this section, we present a brief overview of the definitions and notations used throughout this paper. We will specifically cover fundamental concepts related to graphs, including planar graphs, outerplanar graphs, apex graphs, fuzzy graphs, intuitionistic fuzzy graphs, Turiyam Neutrosophic graphs, neutrosophic graphs, and plithogenic graphs.

Additionally, please note that this paper may also incorporate concepts from set theory alongside graph theory. For a more comprehensive understanding of set theory, you may refer to the relevant surveys or notes [285,311,356].

2.1 Basic Graph Concepts

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [142,194,195,195,196,271,283,543,547].

[1] Treewidth measures how closely a graph resembles a tree, representing the smallest "width" of a tree decomposition [107–110,115,453, 454,457].

[2] The boxicity of a graph is the minimum number of dimensions needed to represent it as an intersection graph of axis-aligned boxes [9,139].

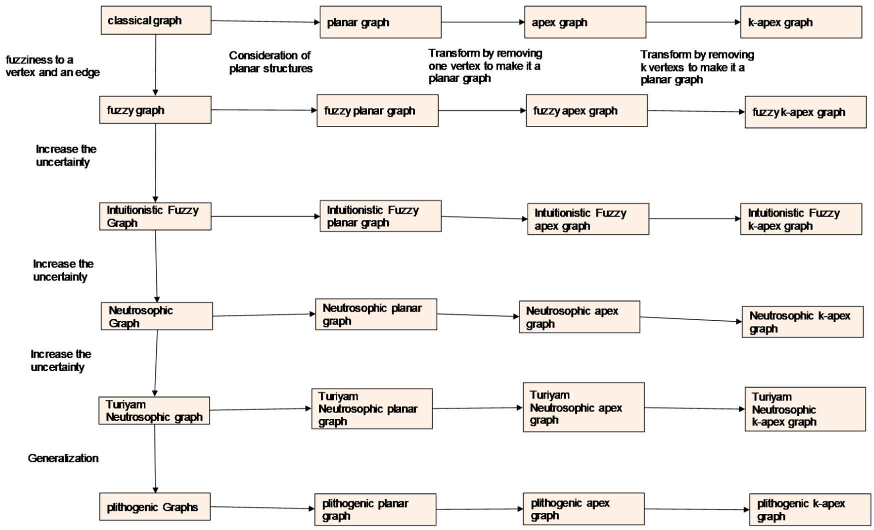

Fig. 1. Graph Hierarchy in this paper

Definition 1 (Graph)**.** [196] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2 (Subgraph)**.** [196] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Example 3 (Example of a Subgraph)**.** Consider a graph $G = (V, E)$ with the vertex set $V = \{1, 2, 3, 4, 5\}$ and the edge set

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}.$$

This graph G can be visualized as having 5 vertices connected by 5 edges.

Now, let us define a subgraph G' by selecting the subset of vertices $V' = \{1, 2, 4\}$ and the subset of edges $E' = \{\{1, 2\}, \{2, 4\}\}$. Thus, the subgraph $G' = (V', E')$ has:

- The vertex set $V' = \{1, 2, 4\}$,
- The edge set $E' = \{\{1, 2\}, \{2, 4\}\}$.

In this example, G' is a valid subgraph of G because both $V' \subseteq V$ and $E' \subseteq E$, and the edges in E' only connect vertices within V'.

Definition 4 (Degree)**.** [196] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^{-}(v)$ is the number of edges directed into v, and the *out-degree* $\deg^{+}(v)$ is the number of edges directed out of v.

Definition 5. (cf.[279, 531]) A graph $G = (V, E)$ is said to be a *connected graph* if for any two distinct vertices $u, v \in V$, there exists a path in G that connects u and v. In other words, every pair of vertices in the graph is reachable from each other, meaning there is a sequence of edges that allows traversal between any two vertices.

Mathematically, for all $u, v \in V$, there exists a sequence of vertices $v_1 = u, v_2, \ldots, v_k = v$ such that $(v_i, v_{i+1}) \in E$ for all $1 \leq i < k$.

Definition 6 (Path)**.** (cf.[560]) A path in a graph $G = (V, E)$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \ldots, k-1$. A path is represented as:

$$P = (v_1, v_2, \ldots, v_k),$$

where no vertex is repeated. The length of a path is the number of edges it contains, i.e., $k - 1$.

Definition 7 (Tree)**.** (cf.[560]) A tree is a connected, acyclic graph. In other words, a tree is a graph where there is exactly one path between any two vertices, and no cycles exist.

Example 8. Consider a graph with 4 vertices $V = \{V_A, V_B, V_C, V_D\}$ and 3 edges $E = \{(V_A, V_B), (V_B, V_C), (V_C, V_D)\}$. This graph is connected, has no cycles, and there is exactly one path between any pair of vertices. Hence, this graph is a tree.

Definition 9. (cf.[367, 399, 440]) A Forest is an undirected graph that is acyclic, meaning it contains no cycles. It consists of one or more disjoint trees, where each tree is a connected subgraph with no cycles.

Formally, a graph $F = (V, E)$ is a forest if:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- Each connected component of F is a tree, i.e., there is no path in any component that returns to the starting vertex, which implies no cycles exist in F.

In simpler terms, a forest is a collection of trees, and if a forest consists of only one connected component, it is itself a tree.

Example 10. Consider a graph with 6 vertices

$$V = \{V_A, V_B, V_C, V_D, V_E, V_F\}$$

and 4 edges

$$E = \{(V_A, V_B), (V_B, V_C), (V_D, V_E), (V_E, V_F)\}$$

. The graph consists of two disjointcomponents: one component with vertices $\{V_A, V_B, V_C\}$ and edges (V_A, V_B), (V_B, V_C), and another component with vertices $\{V_D, V_E, V_F\}$ and edges (V_D, V_E), (V_E, V_F). Each component is a tree, and thus, the entire graph is a forest.

Definition 11 (Complete Graph)**.** (cf.[83, 204]) A *complete graph* is a graph $G = (V, E)$ in which every pair of distinct vertices is connected by a unique edge. Formally, a graph $G = (V, E)$ is complete if for every pair of vertices $u, v \in V$ with $u \neq v$, there exists an edge $\{u, v\} \in E$.

The complete graph on n vertices is denoted by K_n, and it has the following properties:

- The number of vertices is $|V| = n$.
- The number of edges is $|E| = \binom{n}{2} = \frac{n(n-1)}{2}$.
- Each vertex has degree $\deg(v) = n - 1$ for all $v \in V$.

Example 12 (Examples of Complete Graphs)**.** The concept of a complete graph is best understood through specific examples:

- The complete graph K_1 consists of a single vertex with no edges.
- The complete graph K_2 has two vertices, v_1 and v_2, and a single edge connecting them, $\{v_1, v_2\}$.
- The complete graph K_3 consists of three vertices, v_1, v_2, and v_3, with edges $\{v_1, v_2\}, \{v_2, v_3\}$, and $\{v_1, v_3\}$. This forms a triangle.
- The complete graph K_4 has four vertices, with every possible pair of vertices connected by an edge. The edges are $\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}$, and $\{v_3, v_4\}$. This graph forms a tetrahedron when represented in three dimensions.
- The complete graph K_5 includes five vertices, with each pair of vertices connected by a unique edge. The total number of edges is $\binom{5}{2} = 10$, making it a highly connected structure.

In general, for a complete graph K_n with n vertices, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ edges, and each vertex has a degree of $n - 1$.

Definition 13 (Bipartite Graph)**.** (cf.[69, 207]) A *bipartite graph* is a graph $G = (V, E)$ whose vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that:

- $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
- Every edge in E connects a vertex from V_1 to a vertex from V_2. In other words, there are no edges connecting two vertices within the same subset V_1 or V_2.

Formally, $G = (V, E)$ is bipartite if there exists a partition (V_1, V_2) such that for every edge $e = \{u, v\} \in E$, either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$.

A graph G is bipartite if and only if it contains no odd-length cycles.

Definition 14 (Complete Bipartite Graph)**.** (cf.[215, 286, 361]) A *complete bipartite graph* is a graph $G = (V, E)$ whose vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that:

- $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
- There is an edge between every vertex in V_1 and every vertex in V_2.
- There are no edges between vertices within the same subset V_1 or V_2.

The complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. It has the following properties:

- The number of vertices is $|V| = m + n$.
- The number of edges is $|E| = m \times n$.
- Each vertex in V_1 has degree n, and each vertex in V_2 has degree m.

Example 15 (Examples of Bipartite Graphs)**.** To illustrate the concept of bipartite graphs, we provide the following examples:

- The simplest example is the graph $K_{1,1}$, which consists of two vertices v_1 and v_2 with a single edge connecting them. Here, $V_1 = \{v_1\}$ and $V_2 = \{v_2\}$, making $K_{1,1}$ a bipartite graph.
- The complete bipartite graph $K_{2,3}$ consists of two sets of vertices, $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2, v_3\}$. Each vertex in V_1 is connected to every vertex in V_2. Thus, the edge set E consists of all possible edges between V_1 and V_2, specifically: $\{u_1, v_1\}, \{u_1, v_2\}, \{u_1, v_3\}, \{u_2, v_1\}, \{u_2, v_2\}, \{u_2, v_3\}$. There are no edges within V_1 or V_2, making $K_{2,3}$ a bipartite graph.

Definition 16. (cf.[117, 203, 218, 287, 544]) Two graphs $G = (V, E)$ and $H = (V', E')$ are said to be *homomorphic* if there exists a mapping $\phi : V \to V'$ such that for every edge $(u, v) \in E$, the image $(\phi(u), \phi(v))$ is an edge in E'. In other words, there is a structure-preserving mapping from G to H that maintains the adjacency relationships between vertices.

2.2 Classic class about planar graph and outerplanar graph

We introduce some representative classical graph classes. These graph classes have also been studied in the contexts of fuzzy, neutrosophic, and plithogenic settings.

2.2.1 Planar graph

Planar graphs, co-planar graphs, and k-planar graphs have been extensively studied, with numerous applications in networks and other fields. The definitions are provided below.

Definition 17. (cf.[210, 314]) An Edge Crossing occurs when two edges in a graph's drawing intersect at a point other than a common endpoint. In a planar representation of the graph, edge crossings indicate that the graph is drawn in such a way that edges overlap or cross each other, violating planarity.

Formally, consider a graph $G = (V, E)$ embedded in the plane:

- An edge crossing happens if two edges $e_1 = (u, v)$ and $e_2 = (x, y)$ intersect at a point p, where p is not one of the vertices u, v, x, or y.

If a graph can be drawn without any edge crossings, it is known as a planar graph.

Definition 18. (cf.[178, 456, 524]) A graph $G = (V, E)$ is called a *planar graph* if it can be drawn on a plane in such a way that no two edges intersect except at their endpoints. In other words, there exists a way to represent the graph on a two-dimensional plane such that the edges only meet at shared vertices, without any of the edges crossing each other.

Example 19 (Example of a Planar Graph)**.** One of the simplest examples of a planar graph is the complete graph K_4 minus one edge.

Consider the graph $G = (V, E)$ with the vertex set $V = \{1, 2, 3, 4\}$ and the edge set

$$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}.$$

This graph G consists of 4 vertices and 5 edges connecting them.

The graph can be drawn on a plane in such a way that none of its edges intersect except at their endpoints. A common drawing of this graph resembles a triangle with an additional vertex inside connected to all three vertices of the triangle.

Since there exists a way to draw this graph on a two-dimensional plane without any edges crossing each other, it serves as an example of a planar graph.

Definition 20. (cf.[202, 458]) A graph $G = (V, E)$ is called a *non-planar graph* if it cannot be drawn on a plane without edge intersections, except at their endpoints. In other words, there is no way to represent G on a two-dimensional plane such that all edges meet only at shared vertices without any edges crossing each other.

Equivalently, a graph G is non-planar if every possible drawing of G in the plane results in at least one pair of edges intersecting at a point other than their endpoints.

Example 21 (Example of a Non-Planar Graph)**.** A classic example of a non-planar graph is the complete graph K_5.

Consider the graph $G = (V, E)$ with the vertex set

$$V = \{1, 2, 3, 4, 5\}$$

and the edge set containing all possible edges between these 5 vertices:

$$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}.$$

In total, this graph has 5 vertices and 10 edges, with each pair of vertices connected by a unique edge.

No matter how we attempt to draw K_5 on a plane, there will always be at least one pair of edges that intersect at a point other than their endpoints. Therefore, it is impossible to draw K_5 in a way that avoids edge crossings.

Hence, K_5 is an example of a non-planar graph.

Among the many graphs related to Planar Graphs, one well-known example is the co-planar graph. The definition is provided below.

Definition 22. (cf.[163, 232, 345]) A graph $G = (V, E)$ is called a *co-planar graph* if its complement graph $\overline{G}$, defined as the graph having the same set of vertices V but with edges between vertices that are not connected in G, is a planar graph. In other words, G is co-planar if $\overline{G}$ can be drawn on a plane such that no two edges intersect except at their endpoints.

Example 23 (Example of a Co-Planar Graph)**.** Consider the graph $G = (V, E)$ with vertex set

$$V = \{1, 2, 3, 4\}$$

and edge set

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

The complement graph $\overline{G}$ is constructed by taking the same set of vertices V but adding edges between every pair of vertices that are not connected in G. Therefore, the edge set of $\overline{G}$ becomes:

$$E(\overline{G}) = \{\{1, 4\}\}.$$

The complement graph $\overline{G}$ has only one edge and can be drawn on a plane without any edge intersections. As a result, $\overline{G}$ is a planar graph.

Hence, G is an example of a co-planar graph.

We provide the definition of a k-planar graph [55, 59, 100], which generalizes the concept of a planar graph. Notably, related graph classes such as 1-planar graphs [292, 515] and 2-planar graphs [88, 227] are well-known.

Definition 24. (cf.[55,59,100]) A graph $G = (V, E)$ is called a *k-planar graph* if there exists a drawing of G in the plane such that each edge $e \in E$ is crossed by at most k other edges.

Formally, let D be a drawing of G on a plane where:

- V is the set of vertices in G.
- E is the set of edges, where each edge $e = (u, v)$ for $u, v \in V$ is represented as a continuous curve connecting u and v.

The graph G is said to be k-planar if, for every edge $e \in E$, the number of intersection points with other edges in the drawing D does not exceed k. That is, for every $e = (u, v) \in E$, there exists a drawing D such that

$$\text{cross}(e) \leq k,$$

where $\text{cross}(e)$ denotes the number of times edge e intersects with other edges in D.

In other words, a k-planar graph is a graph that can be drawn in the plane such that no edge is crossed by more than k other edges. For instance:

- If $k = 0$, the graph is a *planar graph*, meaning no edges cross.
- If $k = 1$, the graph is a 1-planar graph, meaning each edge can be crossed at most once.

The planar graph is a manageable and uncomplicated graph with no intersecting edges, making it suitable for various applications. Some examples of applications are provided below.

- urban street patterns: Urban street patterns refer to the layout and organization of streets in a city, including grid, radial, and irregular designs[82, 380, 390]. This has also been studied in the context of planar graphs [136, 337, 381].
- neural network: A neural network is a computational model inspired by the human brain, used for pattern recognition and data analysis[5, 277]. In the field of graphs, graph neural networks are frequently studied[219, 475, 550, 552, 563]. This has also been studied in the context of planar graphs [3, 197, 428].
- Computer vision: Computer vision is a field of artificial intelligence that enables computers to interpret and understand visual information from images or videos [220, 228, 537]. This has also been studied in the context of planar graphs [412, 562].
- VLSI: VLSI (Very Large Scale Integration) is the process of creating integrated circuits by combining thousands of transistors into a single chip [384, 481, 548]. In the VLSI field, VLSI floorplanning is often studied. VLSI floorplanning determines the size, shape, and placement of modules in a chip. It is frequently researched from a graph perspective since it helps estimate the chip area, interconnects, and delay[152, 223, 534]. This has also been studied in the context of planar graphs [38, 160, 343, 529, 530].
- Graph drawing: Graph drawing is a field that creates visual representations of graphs, helping to analyze relationships in data, networks, and applications such as social network analysis, linguistics, and bioinformatics[157, 317, 520]. Many papers have also been published on the graph drawing of planar graphs[295, 419].

2.2.2 Outerplanar graph

The outerplanar graph, like the planar graph, has been the subject of extensive research. Some related graph parameters include outerthickness[104], Outercoarseness[104], outerplanarity[99, 141, 312], and outer-2-planarity[293, 561]. Additionally, similar to planar graphs, numerous studies have focused on the applications of outerplanar graphs in networks and other fields. The definitions of outerplanar graphs, as well as their generalized concepts of outer-k-planar graphs and k-outerplanar graphs, are provided below.

Definition 25. (cf.[273,409]) An *outer-planar graph* $G = (V, E)$ is an undirected graph that can be embedded in the plane such that all vertices lie on the outer face of the embedding, meaning no vertex is entirely enclosed by edges. In other words, it is possible to draw the graph on a plane without any edge crossings, with all vertices positioned on the unbounded (outer) face of the graph.

1. *Edge Crossing Condition*: The graph G must be drawn in such a way that no two edges intersect except at their endpoints.
2. *Outer Face Requirement*: All vertices of G must lie on the boundary of a single, unbounded face of the graph.

Definition 26. (cf.[224]) An *outer-k-planar graph* is a graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and each edge is crossed at most k times by other edges.

Formally, let $G = (V, E)$ be a graph, and consider a drawing D of G in the plane such that:

- All vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D.
- Each edge $e \in E$ is represented as a curve connecting its endpoints.

The graph G is called *outer-k-planar* if there exists a drawing D such that any edge $e \in E$ is crossed by at most k other edges. That is, for every edge $e \in E$,

$$\text{cross}(e) \leq k,$$

where $\text{cross}(e)$ denotes the number of times edge e intersects with other edges in the drawing D.

Definition 27. (cf.[58, 97, 143, 318]) A graph $G = (V, E)$ is called a *k-outerplanar graph* if it has a planar embedding such that the vertices belong to at most k concentric layers with respect to the outer face of the graph.

Formally, a graph G is said to be k-outerplanar if there exists a planar embedding D of G in which the vertices can be partitioned into k layers as follows:

1. The *first layer* consists of all vertices that lie on the outer face (unbounded face) of the embedding D, making this subgraph an outerplanar graph.
2. For $i > 1$, the *i-th layer* is defined recursively by removing all vertices that belong to layers 1 through $i - 1$. The remaining vertices that lie on the unbounded face of the resulting subgraph form the i-th layer.
3. The process stops when all vertices are assigned to one of the k layers.

If such a layering exists for the graph G, then G is k-outerplanar.

The following theorem holds.

Theorem 28. *(cf.[13, 126]) Every outerplanar graph is a planar graph.*

The related graph classes within the planar graph category are listed below. The numerous research studies on these related classes highlight the importance of studying planar graphs.

Notation 29. In this paper, we define the term "Related graph class" as a graph class that either extends or restricts a corresponding graph class in some way.

Theorem 30. *The following are examples of related graph classes, including but not limited to:*

- *Halin graphs [113]*
- *Fan-planar graphs [326]*
- *Fan-crossing free graphs[123]*
- *IC-planar graphs[125]*
- *Upward planar graphs [16, 74, 316]*
- *Convex planar graphs [187]*
- *1-planar graphs [292, 515]*
- *2-planar graphs [88, 227]*
- *maximal 2-planar graphs[290]*
- *3-planar graphs [87]*
- *Interval Planar Graphs [315]*
- *k-Planar Graphs [84]*
- *Outer-1-planar graphs [75, 98, 362]*

- *Outer-k-planar graphs [141]*
- *Quasi-planar graphs [8, 12]*
- *k-Quasi-planar Graphs [231, 509, 510]*
- *Apex graphs [158, 395, 396, 422]* [3]
- *Pseudo-outerplanar graphs[526]*
- *biconnected outerplanar graphs[140, 369]*
- *Strongly-connected outerplanar graphs[230]*
- *Biconnected planar graphs[105]*
- *Gap-planar graphs [79]*
- *Non-separating planar graphs[181]*
- *NIC-planar graphs[78, 421, 525]*
- *Apex-Outerplanar Graphs[199]*
- *Universal outerplanar graphs[370]*
- *Outer-projective-planar graphs[65]*
- *Planar Ramsey graphs[76]*
- *Cubic outerplanar graphs[269]*
- *Outerplanar partial cubes graphs[461]*
- *Directed Planar graphs[118, 323, 341]*
- *Planar partial cubes graphs[183, 439]*
- *Planar median graphs[479]*
- *Planar straight-line graph[190]*
- *Directed outerplanar graphs[138]*
- *Maximum Planar Subgraph[134]*
- *Weighted planar graph[93, 94]*
- *RAC Graph[57, 191, 192]*
- *Planarly-connected graphs [7]*
- *2-connected planar graphs[389]*
- *Bipartite IC-planar graphs[57]*
- *Bipartite NIC-planar graphs[57]*
- *Bipartite RAC Graphs[57]*
- *Cubic planar graph[403]*
- *Regular planar graphs[89, 416, 551]*
- *Partitioning planar graphs[149]*
- *locally planar graphs[282]*
- *Bipartite Planar Graphs[226]*
- *Signed planar graphs[411]*
- *Random cubic planar graphs[45]*
- *Random planar graphs[259, 383, 427]*
- *Random outerplanar graphs[106, 322]*

Proof. Refer to each reference as needed. □

[3] The term "apex" is sometimes used in contexts beyond planar graphs to indicate how close a graph is to a desired graph structure [355].

2.2.3 Apex graph

An apex graph becomes planar by removing one vertex; it is non-planar but can achieve planarity this way. It has been extensively studied alongside planar graphs [158, 395, 396, 422]. It is closely related to the operation of vertex deletion (cf.[248, 310])in graphs. The definition is provided below.

Definition 31. (cf.[158, 395, 396, 422]) An *apex graph* is a graph $G = (V, E)$ that can be made planar by the removal of a single vertex $v \in V$. The removed vertex is called an *apex* of the graph G.

Formally, a graph G is an apex graph if there exists a vertex $v \in V$ such that the subgraph $G' = G[V \setminus \{v\}]$, obtained by removing v and all edges incident to v, is a planar graph.

The properties of an apex graph include:

- If the graph G itself is planar, then every vertex can be considered an apex.
- Apex graphs may have more than one vertex that can serve as an apex, and if multiple such vertices exist, each can be independently removed to obtain a planar graph.
- The null graph (i.e., a graph with no vertices or edges) is trivially considered an apex graph.

The k-apex graph, a generalization of the aforementioned apex graph, has also been extensively studied. The definition is provided below.

Definition 32. (cf.[161,214,263]) A *k-apex graph* is a graph $G = (V, E)$ that can be made planar by the removal of at most k vertices from V. In other words, there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the subgraph $G' = G[V \setminus S]$, obtained by removing all vertices in S along with their incident edges, is a planar graph.

The properties of a k-apex graph include:

- If $k = 0$, the k-apex graph itself is already a planar graph.
- When $k = 1$, the k-apex graph corresponds to the definition of an apex graph, where the removal of a single vertex makes the graph planar.
- A k-apex graph may have multiple different subsets of vertices S that can be removed to achieve planarity.

2.2.4 Apex outerplanar graph

An apex outerplanar graph becomes planar by removing one vertex. It has been extensively studied alongside outerplanar graphs [200, 209]. The definition is provided below.

Definition 33. [200, 209]. An *apex outerplanar graph* is a graph $G = (V, E)$ that can be made outerplanar by the removal of a single vertex $v \in V$. The removed vertex v is called an *apex* of the outerplanar graph G.

Formally, G is an apex outerplanar graph if there exists a vertex $v \in V$ such that the subgraph $G' = G[V \setminus \{v\}]$, obtained by removing v and all edges incident to v, is an outerplanar graph.

The properties of an apex outerplanar graph include:

- If the graph G itself is outerplanar, then every vertex in V can be considered an apex.
- An apex outerplanar graph may have more than one vertex that can serve as an apex, and if multiple such vertices exist, each can be independently removed to obtain an outerplanar graph.
- The null graph (i.e., a graph with no vertices or edges) is trivially considered an apex outerplanar graph.

Definition 34. A graph $G = (V, E)$ is called a *k-apex outerplanar graph* if it can be made outerplanar by the removal of at most k vertices from V.

Formally, G is a k-apex outerplanar graph if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced subgraph $G' = G[V \setminus S]$, obtained by removing all vertices in S along with their incident edges, is an outerplanar graph.

The properties of a k-apex outerplanar graph include:

- If $k = 0$, the k-apex outerplanar graph itself is already outerplanar.
- When $k = 1$, the graph is termed an *apex outerplanar graph*, meaning there exists a single vertex v whose removal results in an outerplanar graph.
- For general k, multiple subsets S of vertices may exist, such that their removal results in an outerplanar graph, indicating that the graph G has more flexibility in achieving outerplanarity compared to a standard outerplanar graph.

The relationships between planar graphs, outerplanar graphs, apex graphs, and apex outerplanar graphs are illustrated in the following diagram.

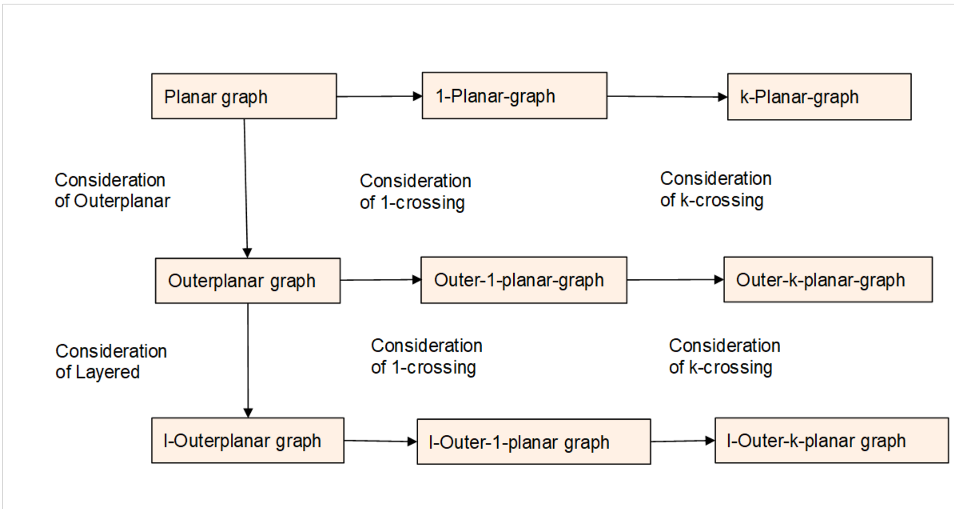

Fig. 2. Graph Hierarchy for Planar graph

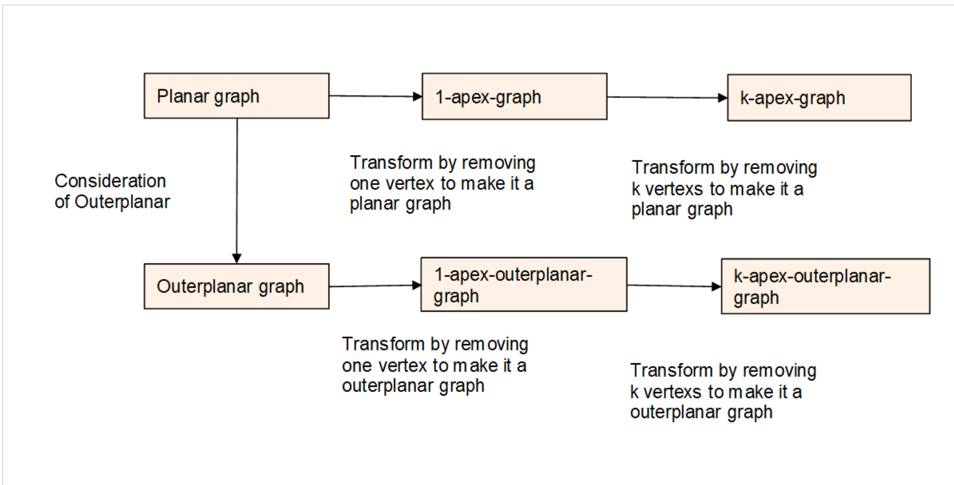

Fig. 3. Graph Hierarchy for Apex graph

2.2.5 *Quasi-planar graph*

A quasi-planar graph is a graph that can be drawn in the plane such that no three edges intersect each other pairwise [4]. Quasi-planar graphs have been extensively studied alongside planar graphs [8, 12, 60, 122, 188, 189, 344]. The definition is provided below.

Definition 35. [8, 12] A graph $G = (V, E)$ is called a *quasi-planar graph* if it can be drawn in the plane such that no three edges are pairwise crossing. In other words, there is no set of three edges in G that intersect each other at distinct points in their interiors.

The properties of a quasi-planar graph include:

- Every planar graph is inherently quasi-planar because planar graphs have no edge crossings.
- Quasi-planar graphs have been conjectured to have at most $O(n)$ edges for n vertices.

[4] In this context, "pairwise" means that no two edges within any group of three edges intersect each other. Therefore, a quasi-planar graph is one where any set of three edges has no two edges that intersect each other at the same time.

An example of a graph that extends the concept of a quasi-planar graph is the k-quasi-planar graph [56, 231, 509]. The definition is provided below.

Definition 36. [56, 231] A graph $G = (V, E)$ is called a *k-quasi-planar graph* if it can be drawn in the plane such that no k edges are pairwise crossing. In other words, there does not exist a subset of k edges in G that all intersect each other at distinct points in their interiors.

The properties of a k-quasi-planar graph include:

- For $k = 2$, a k-quasi-planar graph is simply a planar graph because no two edges cross each other.
- For any fixed $k \geq 2$, a k-quasi-planar graph on n vertices has been conjectured to have at most $O(n)$ edges.
- k-quasi-planar graphs generalize the concept of quasi-planar graphs as they permit more pairwise crossings while maintaining an upper bound on the crossing behavior.

2.2.6 outer quasi planar graph

We define the *outer k-quasi-planar graph* as a related graph class to the ones mentioned above. An outer quasi-planar graph is a graph that combines the concepts of an outerplanar graph and a quasi-planar graph.

Definition 37. An *outer-k-quasi-planar graph* is a graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no set of k edges are pairwise crossing.

Formally, let $G = (V, E)$ be a graph, and consider a drawing D of G in the plane such that:

- All vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D.
- Each edge $e \in E$ is represented as a curve connecting its endpoints.

The graph G is called *outer-k-quasi-planar* if there exists a drawing D such that any set of k edges does not intersect each other at distinct points in their interiors.

2.2.7 quasi apex graph

We define the quasi apex graph as related graph classes to the ones mentioned above.

Definition 38. A *k-quasi apex graph* is a graph $G = (V, E)$ that can be made planar by the removal of at most k vertices.

Formally, G is a k-quasi apex graph if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced subgraph $G' = G[V \setminus S]$, obtained by removing all vertices in S along with their incident edges, is a planar graph.

The properties of a k-quasi apex graph include:

- If $k = 0$, the k-quasi apex graph itself is already planar.
- When $k = 1$, the graph corresponds to an apex graph where the removal of a single vertex makes the graph planar.
- The k-quasi apex graph generalizes the concept of apex graphs by allowing up to k vertices to be removed to achieve planarity.

2.2.8 quasi apex outerplanar graph

We define the quasi apex outerplanar graph as related graph classes to the ones mentioned above.

Definition 39. A *k-quasi apex outerplanar graph* is a graph $G = (V, E)$ that can be made outerplanar by the removal of at most k vertices.

Formally, G is a k-quasi apex outerplanar graph if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced subgraph $G' = G[V \setminus S]$, obtained by removing all vertices in S along with their incident edges, is an outerplanar graph.

The properties of a k-quasi apex outerplanar graph include:

- If $k = 0$, the k-quasi apex outerplanar graph itself is already outerplanar.
- When $k = 1$, the graph is termed an apex outerplanar graph, meaning there exists a single vertex v whose removal results in an outerplanar graph.
- For general k, multiple subsets S of vertices may exist such that their removal results in an outerplanar graph, indicating that the graph G has more flexibility in achieving outerplanarity compared to a standard outerplanar graph.

The relationship diagram between quasi-planar graphs is shown below.

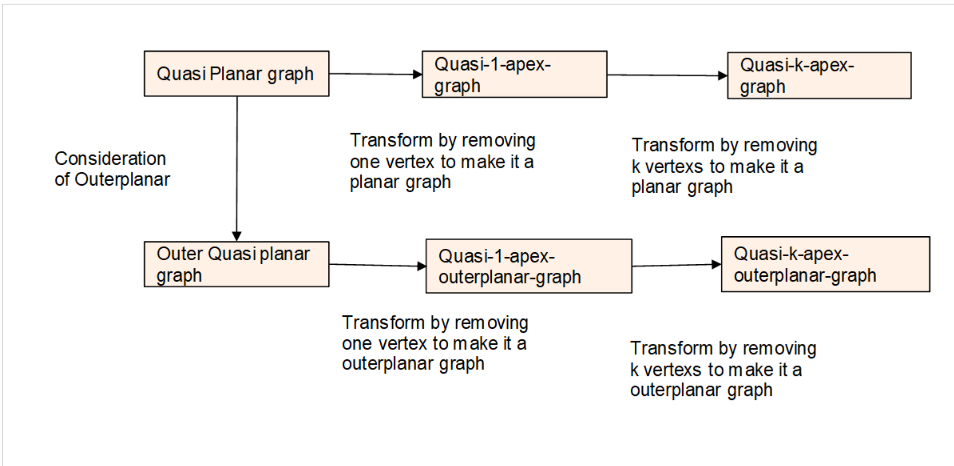

Fig. 4. Graph Hierarchy for Quasi-Planar graph

2.2.9 Gap-planar graph and Fan-planar graph

Graphs related to planar graphs include gap-planar graphs [79, 477] and fan-planar graphs [86, 101, 326, 366]. Several research papers have been published on these topics. Their definitions are provided below.

Definition 40. [79] A graph $G = (V, E)$ is called a *k-gap-planar graph* if it admits a drawing in the plane such that each crossing is assigned to one of the two involved edges, and each edge is assigned at most k of its crossings. In other words, in the drawing of G, each edge can be crossed by at most k other edges, where each crossing is uniquely associated with one of the edges involved in the crossing.

The properties of a k-gap-planar graph include:

- When $k = 0$, a k-gap-planar graph is simply a planar graph with no crossings.
- k-gap-planar graphs provide a way to measure how far a graph is from being planar, by limiting the number of crossings assigned to each edge.

Definition 41. (cf.[86, 101, 326, 366]) A graph $G = (V, E)$ is called a *fan-planar graph* if it admits a drawing in the plane such that every edge is crossed only by a set of pairwise adjacent edges (forming what is known as a "fan"). This means that any edge e in G can only be crossed by edges that share a common vertex with each other, forming a fan-like pattern.

The properties of a fan-planar graph include:

- The fan-planar condition restricts how edges can intersect, allowing a more structured way of crossing, unlike general non-planar graphs.
- If a graph is fan-planar, it means there exists a drawing where the crossings exhibit a "fan-like" behavior.

And Graphs related to outerplanar graphs include outer-fan-planar graph. Several research papers have been published on these topics [85, 102, 123, 124]. Their definitions are provided below.

Definition 42. (cf.[85, 102, 123, 124]) A graph $G = (V, E)$ is called an *outer-fan-planar graph* if it has a planar embedding in which all vertices lie on the boundary of the unbounded (outer) face, and every edge is crossed only by pairwise adjacent edges, known as *fans*.

More formally:

- G can be drawn in the plane such that all vertices are positioned on a single circle C, which represents the outer face.
- The embedding must be such that no two edges cross more than once, and if an edge e is crossed by other edges, all those crossing edges must share a common vertex incident to e.

An important property of outer-fan-planar graphs is that they are *biconnected*, meaning they remain connected after the removal of any single vertex. This is necessary for a graph to be maximal outer-fan-planar.

For example:

- If the graph G contains a cut-vertex c (a vertex whose removal disconnects the graph), then it is possible to add an additional edge between two neighbors of c without violating outer-fan-planarity, thus indicating that G was not maximal outer-fan-planar.
- A biconnected graph G is outer-fan-planar if and only if it admits a straight-line outer-fan-planar drawing where the vertices are restricted to lie on a circle C, preserving their cyclic order in this embedding.

It has been proven that any outer-fan-planar graph with n vertices has at most $5n - 10$ edges.

2.2.10 Inner planar graph

Recently, the concept of an Inner Planar Graph has been defined. The definition is provided below[426].

Definition 43. [426] A graph $G = (V, E)$ is called a *k-inner planar graph* if it has a planar embedding with at most k vertices that do not lie on the boundary of the outer face. These vertices are referred to as *inner vertices*.

More formally:

- G is a planar graph that can be drawn on the plane such that when embedded, at most k vertices do not lie on the boundary of the outer (unbounded) face of the graph.
- The remaining vertices must lie on the boundary of the outer face.

Special cases of k-inner planar graphs include:

- When $k = 0$, the graph is called an *outerplanar graph*, meaning all vertices lie on the boundary of the outer face.
- For $k > 0$, the graph allows up to k vertices to be positioned inside the outer face, while still maintaining a planar embedding.

2.2.11 Almost Planar Graph

The concept of an Apex Graph focuses on vertices, whereas graphs centered on edges include the Almost Planar Graph[201,274,289] and Almost Outerplanar Graph[213,538]. These graphs are also referred to as Nearly Planar Graphs[216,268] and Nearly Outerplanar Graphs, respectively. This graph class is closely related to the graph operation known as edge deletion (cf.[67,485,556,557]). The definitions are introduced below.

Definition 44. [201,274,289] A graph G is called an *almost planar graph* if there exists an edge $e \in E(G)$ such that the graph $G - e$ (i.e., the graph obtained by removing the edge e from G) is planar.

Definition 45. (cf.[169, 170]) A graph $G = (V, E)$ is called a *k-almost planar graph* if there exists a subset $F \subseteq E(G)$ with $|F| \leq k$ such that the graph $G - F$ (i.e., the graph obtained by removing all edges in F from G) is planar.

In other words, a k-almost planar graph can be made planar by removing at most k edges.

Definition 46. [213, 538] A graph G is called an *almost outerplanar graph* if there exists an edge $e \in E(G)$ such that the graph $G - e$ (i.e., the graph obtained by removing the edge e from G) is outerplanar.

Definition 47. A graph $G = (V, E)$ is called a *k-almost outerplanar graph* if there exists a subset $F \subseteq E(G)$ with $|F| \leq k$ such that the graph $G-F$ (i.e., the graph obtained by removing all edges in F from G) is outerplanar.

In other words, a k-almost outerplanar graph can be made outerplanar by removing at most k edges.

Almost planar graphs and almost outerplanar graphs are related to the concept of *skewness-k drawings* in the field of graph drawing [156,193]. A graph is said to have *skewness k* if it can be drawn with k edge deletions needed to make it planar[193].

In addition to the graph concepts described in the subsection, related notions such as (k, l)-grid-free drawings[429] and k-fan-crossing-free drawings[153] are also well-known. This indicates that research on planar graphs is considered highly valuable.

2.3 Fuzzy planar graph and Fuzzy outerplanar graph

In recent years, inspired by planar and outerplanar graphs, the concepts of fuzzy planar graphs and fuzzy outerplanar graphs have been introduced. To begin, the definitions of fuzzy graphs and fuzzy planar graphs are provided below. A fuzzy graph is an extension of graph theory that incorporates the principles of fuzzy sets [51, 135, 165, 179, 255, 371, 523, 553, 558, 559]. Extensive research has been conducted on both fuzzy graphs [96, 175, 260, 400, 452, 459, 505] and fuzzy planar graphs [20, 256, 441, 442, 472].

Definition 48. [459] A *fuzzy graph* $\psi = (V, \sigma, \mu)$ is defined as follows:

- V is a set of vertices.
- $\sigma : V \to [0, 1]$ is a function that assigns a membership degree to each vertex $v \in V$, indicating the degree of membership of v in the fuzzy graph.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation that represents the strength of the connection between each pair of vertices $(u, v) \in V \times V$, such that $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

  In this definition, the following properties hold:

- The fuzzy function μ is symmetric, meaning $\mu(u, v) = \mu(v, u)$ for all $u, v \in V$.
- Additionally, $\mu(v, v) = 0$ for all $v \in V$, meaning that there is no self-loop in the fuzzy graph.

The fuzzy graph ψ allows for the representation of uncertainty in the presence or strength of connections between vertices, making it a valuable tool for modeling complex systems with ambiguous or imprecise relationships.

Theorem 49. *[235] The following are examples of related graph classes, including but not limited to:*

- *Bipolar Fuzzy Graphs [17]*
- *Fuzzy Planar Graphs [470]*
- *Complex Hesitant Fuzzy Graph [6]*
- *Constant hesitancy fuzzy graphs[434]*
- *Bipolar hesitancy fuzzy graph[299, 430]*
- *Hesitancy fuzzy magic labeling graph[450]*
- *Dual hesitant fuzzy graphs[80]*
- *Regular Hesitancy Fuzzy Soft Graphs[352]*
- *Hesitant fuzzy hypergraphs[265]*
- *Irregular Bipolar Fuzzy Graphs [469]*
- *Regular Bipolar Fuzzy Graphs [24]*
- *Picture Fuzzy Tolerance Graphs [172]*
- *Complex Hesitant Fuzzy Graphs [6]*
- *Strong Intuitionistic Fuzzy Graphs [23]*
- *Product Fuzzy Graphs [41]*
- *Partially Total Fuzzy Graphs [11]*
- *Fuzzy Influence Graphs [382]*
- *Picture Fuzzy Directed Hypergraphs [333]*
- *Radio Fuzzy Graphs [374]*
- *Line Regular Fuzzy Semigraphs [64]*
- *Fuzzy Incidence Graphs [198]*
- *Balanced Picture Fuzzy Graphs [53]*
- *Oscillating Polar Fuzzy Graphs [52]*
- *Cayley Fuzzy Graphs [519]*
- *Rough Fuzzy Digraphs [14]*
- *T-Spherical Fuzzy Graphs [275]*

- *Mixed Fuzzy Graphs [465]*
- *Einstein Fuzzy Graphs [308]*
- *Edge-Regular Fuzzy Graphs [42, 137, 425]*
- *Robust Fuzzy Graphs [505]*
- *Anti-Product Fuzzy Graphs [39]*
- *Valued Fuzzy Superhypergraphs [494]*
- *Inverse Fuzzy Graphs [120]*
- *Inverse Eccentric Fuzzy Graphs [385]*
- *Cubic Pythagorean Fuzzy Graphs [403]*
- *Complete Fuzzy Graphs [40]*
- *Mixed Picture Fuzzy Graphs [407]*
- *Extended Total Fuzzy Graphs [10]*
- *Pseudo Regular Fuzzy Graphs [377]*
- *Best Fuzzy Graphs [512]*
- *Intuitionistic Felicitous Fuzzy Graphs [95]*
- *Middle Fuzzy Graphs [392, 507]*
- *Bipolar Fuzzy P-Competition Graphs [420]*
- *Fuzzy Intersection Graphs [448]*
- *Fuzzy Semigraphs [462]*
- *Intuitionistic Fuzzy Soft Expert Graphs [533]*
- *m-Polar Fuzzy Graphs [19]*
- *Balanced Interval-Valued Fuzzy Graphs [446]*
- *Double Layered Fuzzy graph[338]*
- *Triple Layered Fuzzy Graph[460]*
- *Fuzzy Outerplanar Graphs [309]*
- *Inverse fuzzy multigraphs[119]*
- *Bipolar inverse fuzzy graphs[332]*
- *Fuzzy zero divisor graphs[348, 349]*
- *General Fuzzy graphs[234, 417]*
- *T-Spherical Fuzzy Graphs[275]*
- *Spherical Fuzzy Labelling Graphs[145]*
- *spherical fuzzy digraphs[379]*
- *T-spherical fuzzy Hamacher graphs[68]*
- *Regular spherical fuzzy graph[393]*
- *Spherical Fuzzy Cycle graph[353]*
- *Spherical Fuzzy Tree graph[353]*
- *Pseudo regular spherical fuzzy graphs[284]*
- *Cubic Pythagorean fuzzy graphs [403]*
- *Complex Pythagorean Dombi fuzzy graphs [26]*
- *Interval-valued Pythagorean fuzzy graphs[29]*
- *Interval-valued complex Pythagorean fuzzy graph[499]*
- *Pythagorean Dombi fuzzy graphs[22]*
- *Pythagorean fuzzy soft graphs[482]*
- *Pythagorean neutrosophic fuzzy graphs[15]*
- *Complex pythagorean fuzzy planar graphs[20]*

- *Pythagorean neutrosophic Dombi fuzzy graphs [144]*
- *Complex Pythagorean fuzzy threshold graphs[18]*
- *Pythagorean Dombi fuzzy soft graphs[34]*
- *Pythagorean fuzzy incidence graphs[36, 484]*
- *Picture Dombi Fuzzy Graph [394]*
- *Picture fuzzy line graphs[151]*
- *Picture fuzzy planar graphs[92]*
- *Picture fuzzy tolerance graphs[172]*
- *Picture fuzzy ϕ-tolerance competition graphs[173]*
- *Interval-Valued Picture Fuzzy Graph [445]*
- *Picture Fuzzy Incidence Graph [413]*
- *Picture fuzzy threshold graphs[172]*
- *Picture Fuzzy Soft Graph [147]*
- *Cayley Picture Fuzzy Graph [334]*
- *m-Polar Picture Fuzzy Graphs[335]*
- *Interval-Valued Picture (S, T)-Fuzzy Graph [66]*
- *q-Rung Picture Fuzzy Graph [25]*
- *Mixed Picture Fuzzy Graph [407]*
- *Picture Fuzzy Directed Hypergraphs[333]*
- *Picture fuzzy cubic graphs[336]*
- *Picture Fuzzy Digraph [379]*
- *Balanced Picture Fuzzy Graph [54]*

Considering these fuzzy graph classes enables the identification of shared properties, which can lead to the development of efficient algorithms, deeper analysis, and practical applications across various fields.

Proof. Refer to each reference as needed. □

Definition 50. (cf.[20, 256, 441, 442, 472]) A fuzzy graph $\psi = (V, \sigma, \mu, E)$ is termed a *fuzzy planar graph* if it consists of:

- A set of vertices V, where each vertex $v \in V$ is associated with a membership degree through a fuzzy set $\sigma : V \to [0, 1]$.
- A fuzzy relation $\mu : V \times V \to [0, 1]$ representing the strength of the connection between pairs of vertices, where $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$ holds for all $u, v \in V$.

  Additionally,

- The edge set E is represented by a fuzzy set on the Cartesian product of V, i.e., $E \subseteq V \times V$, with each edge $(u, v) \in E$ having a membership value $\mu(u, v)$ indicating the degree to which (u, v) belongs to E.
- The fuzzy graph ψ is said to be planar if it can be drawn on a plane without any intersecting edges, adhering to the fuzziness in the representation. Specifically, if two fuzzy edges (a, b) and (c, d) intersect, there exists an intersection point p, and the intersection strength is defined as

  $$I_p = \frac{I(a, b) + I(c, d)}{2},$$

  where

  $$I(a, b) = \frac{\mu(a, b)}{\sigma(a) \wedge \sigma(b)}.$$

The fuzzy planarity value f measures how planar the graph is and is given by

$$f = \frac{1}{1 + \sum_{i=1}^{n} I_{p_i}}.$$

For a completely planar fuzzy graph, $f = 1$.

Proposition 51. *A fuzzy planar graph is a fuzzy graph.*

Proof. Obviously holds. □

Theorem 52. *The following are examples of related graph classes for fuzzy planar graphs, including but not limited to:*

- *Interval-valued fuzzy planar graphs[442]*
- *m-polar fuzzy planar graphs[257, 258]*
- *Complex pythagorean fuzzy planar graphs[21]*
- *Complex q□Rung Orthopair Fuzzy Planar Graphs[298]*
- *Picture fuzzy planar graphs[92]*
- *Inverse Fuzzy Mixed Planar Graphs[398]*
- *Bipolar Fuzzy Planar Graphs[30]*

Proof. Refer to each reference as needed. □

2.3.1 Fuzzy outerplanar graph

The definitions of fuzzy outerplanar graphs are provided below [309]. These graphs incorporate the concepts of fuzzy graphs into the structure of outerplanar graphs.

Definition 53. [309] A fuzzy graph $\psi = (V, \sigma, \mu)$ qualifies as a *fuzzy outerplanar graph* if:

- It can be embedded in a plane such that every vertex $v \in V$ lies on the boundary of the exterior region of the embedding.

  More specifically:

- Similar to the fuzzy planar graph, $\sigma : V \to [0, 1]$ is a fuzzy set indicating the degree of membership of each vertex, and $\mu : V \times V \to [0, 1]$ indicates the strength of the connection between vertices.
- For a fuzzy graph to be outerplanar, there must be no fuzzy edge intersection while ensuring all vertices are positioned on the outer face.

If $i(\psi)$ represents the number of vertices not on the boundary of the outer face, then a fuzzy outerplanar graph satisfies $i(\psi) = 0$. In contrast, if $i(\psi) \neq 0$, the graph is considered a fuzzy non-outerplanar graph.

These definitions extend the classical notions of planarity and outerplanarity into the fuzzy domain, allowing for the representation and analysis of uncertainty in graph structures.

Proposition 54. *A fuzzy outerplanar graph is a fuzzy graph.*

Proof. Obviously holds. □

Proposition 55. *A fuzzy outerplanar graph $\psi = (V, \sigma, \mu)$ can be transformed into a classical outerplanar graph $G = (V, E)$ by considering only those edges $(u, v) \in E$ where the membership degree $\mu(u, v)$ is strictly positive.*

Proof. The transformation process is as follows:

1. *Vertex Set*: Retain the original set of vertices V from ψ.
2. *Edge Set*: Construct an edge set E for the outerplanar graph $G = (V, E)$ such that $(u, v) \in E$ if and only if $\mu(u, v) > 0$ in the fuzzy graph ψ.

Since ψ is a fuzzy outerplanar graph, it satisfies the condition that all vertices lie on the boundary of the exterior region, and there are no fuzzy edge intersections in the embedding. This means that the resulting graph $G = (V, E)$ will be outerplanar, as it maintains the same embedding properties:

- *No edge crossings* occur in G, as all vertices and edges inherited from ψ satisfy the outerplanarity condition.
- *All vertices remain on the boundary* of the outer face, as required by the definition of an outerplanar graph.

Thus, by disregarding the fuzziness (i.e., ignoring the exact membership values $\sigma(v)$ and $\mu(u, v)$), the fuzzy outerplanar graph ψ is transformed into a classical outerplanar graph G.

This completes the transformation, proving that every fuzzy outerplanar graph ψ can be reduced to a corresponding outerplanar graph G. □

2.3.2 Fuzzy apex graph

In general graph classes, a type of graph known as an apex graph is well-known. Similar to planar graphs, apex graphs have been the subject of extensive research [158, 395–397, 422]. Incorporating the concept of fuzzy relations, the definition of a fuzzy apex graph is provided below.

Definition 56. (cf.[158, 395–397, 422].) A *fuzzy apex graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be made a fuzzy planar graph by the removal of a single vertex $v \in V$. The removed vertex is called an *apex* of the fuzzy graph ψ.

Formally, ψ is a fuzzy apex graph if there exists a vertex $v \in V$ such that the subgraph $\psi' = (V \setminus \{v\}, \sigma', \mu')$, where σ' and μ' are the restrictions of σ and μ on the remaining vertices, is a fuzzy planar graph.

The fuzzy apex graph retains the following properties:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex in V.
- $\mu : V \times V \to [0, 1]$ represents the strength of the connection between pairs of vertices in V, with $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.
- If the fuzzy apex graph ψ itself is fuzzy planar, then every vertex in V can be considered an apex.

Similar to classical apex graphs, a fuzzy apex graph may have multiple apex vertices. Additionally, if ψ is the null fuzzy graph (i.e., $V = \emptyset$), it is trivially a fuzzy apex graph.

Proposition 57. *A fuzzy apex graph can be transformed into a fuzzy planar graph.*

Proof. Obviously holds. □

Definition 58. A *fuzzy k-apex graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be transformed into a fuzzy planar graph by removing at most k vertices from V.

If $k = 0$, the fuzzy k-apex graph itself is already a fuzzy planar graph. When $k = 1$, the graph corresponds to a fuzzy apex graph as defined earlier.

Proposition 59. *A fuzzy k-apex graph can be transformed into a fuzzy planar graph.*

Proof. Obviously holds. □

Proposition 60. *A fuzzy k-apex graph can be transformed into a fuzzy apex graph.*

Proof. Obviously holds. □

2.3.3 Fuzzy apex outerplanar graph

The concept of a fuzzy apex outerplanar graph and a k-apex outerplanar graph is extended to the following graph by incorporating the idea of fuzzy graphs.

Definition 61. A *fuzzy apex outerplanar graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be transformed into a fuzzy outerplanar graph by the removal of a single vertex $v \in V$. The removed vertex is called an *apex* of the fuzzy graph ψ.

Formally, ψ is a fuzzy apex outerplanar graph if there exists a vertex $v \in V$ such that the induced subgraph $\psi' = (V \setminus \{v\}, \sigma', \mu')$, where σ' and μ' are the restrictions of the membership function σ and the adjacency function μ on the remaining vertices, is a fuzzy outerplanar graph.

The fuzzy apex outerplanar graph retains the following properties:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex in V.
- $\mu : V \times V \to [0, 1]$ represents the strength of the connection between pairs of vertices in V, satisfying $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

- If the fuzzy apex outerplanar graph ψ itself is a fuzzy outerplanar graph, then every vertex in V can be considered an apex.

Proposition 62. *A fuzzy apex outerplanar graph can be transformed into a fuzzy outerplanar graph.*

Proof. Obviously holds. □

Definition 63. A *fuzzy k-apex outerplanar graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be made a fuzzy outerplanar graph by the removal of at most k vertices from V.

Formally, ψ is a fuzzy k-apex outerplanar graph if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced fuzzy subgraph $\psi' = (V \setminus S, \sigma', \mu')$, where σ' and μ' are the restrictions of σ and μ on the remaining vertices, is a fuzzy outerplanar graph.

The properties of a fuzzy k-apex outerplanar graph include:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex in V.
- $\mu : V \times V \to [0, 1]$ represents the strength of the connection between pairs of vertices in V, with $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.
- If $k = 0$, the fuzzy k-apex outerplanar graph itself is already a fuzzy outerplanar graph.
- When $k = 1$, the fuzzy k-apex outerplanar graph corresponds to a fuzzy apex outerplanar graph, where the removal of a single vertex makes the graph fuzzy outerplanar.
- For general k, multiple subsets S of vertices may exist such that their removal results in a fuzzy outerplanar graph, indicating that the graph ψ has more flexibility in achieving outerplanarity compared to a standard fuzzy outerplanar graph.

This generalization allows for incorporating fuzzy uncertainty into the structure of k-apex outerplanar graphs, providing a broader framework for analyzing graphs that have fuzzy relations and can become outerplanar by removing up to k vertices.

Theorem 64. *A fuzzy k-apex outerplanar graph can be transformed into a fuzzy outerplanar graph.*

Proof. Obviously holds. □

Theorem 65. *A fuzzy k-apex outerplanar graph can be transformed into a fuzzy apex outerplanar graph.*

Proof. Obviously holds. □

2.3.4 Fuzzy Quasi-Planar graph

The Fuzzy Quasi-Planar graph is an extension of the Quasi-Planar graph to a fuzzy graph. The definition of the Fuzzy Quasi-Planar graph is provided below.

Definition 66. A *fuzzy quasi-planar graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be drawn in the plane such that no three edges are pairwise crossing.

Formally, let $\psi = (V, \sigma, \mu)$ be a fuzzy graph where:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ assigns a membership degree to each edge (u, v) representing the strength of the connection between vertices u and v, satisfying $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

The fuzzy graph ψ is called *fuzzy quasi-planar* if it can be drawn in such a way that any three fuzzy edges do not all intersect each other at distinct points in their interiors.

Definition 67. A fuzzy graph $\psi = (V, \sigma, \mu)$ is called a *fuzzy k-quasi-planar graph* if it can be drawn in the plane such that no k fuzzy edges are pairwise crossing. In other words, there does not exist a subset of k fuzzy edges in ψ that all intersect each other at distinct points in their interiors.

Formally, let $\psi = (V, \sigma, \mu)$ be a fuzzy graph where:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ represents the strength of the connection between pairs of vertices $u, v \in V$, such that $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

The fuzzy graph ψ is called a *fuzzy k-quasi-planar graph* if there exists a drawing D in which any set of k fuzzy edges do not all intersect each other at distinct points in their interiors.

Theorem 68. *A fuzzy k-quasi-planar graph can be transformed into a fuzzy quasi-planar graph.*

Proof. Obviously holds. □

2.3.5 Fuzzy Quasi-outerplanar graph

The Fuzzy Quasi-outerplanar graph is an extension of the Quasi-outerplanar graph to a fuzzy graph. The definition of the Fuzzy Quasi-outerplanar graph is provided below.

Definition 69. A *fuzzy quasi-outerplanar graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no three edges are pairwise crossing.

Formally, let $\psi = (V, \sigma, \mu)$ be a fuzzy graph where:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ assigns a membership degree to each edge (u, v) representing the strength of the connection between vertices u and v, satisfying $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

The fuzzy graph ψ is called *fuzzy quasi-outerplanar* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any three fuzzy edges do not all intersect each other at distinct points in their interiors.

Definition 70. A fuzzy graph $\psi = (V, \sigma, \mu)$ is called a *fuzzy outer-k-quasi-planar graph* if it can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no set of k fuzzy edges are pairwise crossing.

Formally, let $\psi = (V, \sigma, \mu)$ be a fuzzy graph where:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ represents the strength of the connection between pairs of vertices $u, v \in V$, such that $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

The fuzzy graph ψ is called *fuzzy outer-k-quasi-planar* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of k fuzzy edges do not all intersect each other at distinct points in their interiors.

Theorem 71. *A fuzzy outer-k-quasi-planar graph can be transformed into a fuzzy outer-quasi-planar graph.*

Proof. Obviously holds. □

2.3.6 Fuzzy quasi apex graph

The Fuzzy quasi apex graph is an extension of the quasi apex graph to a fuzzy graph. The definition of the Fuzzy Quasi-outerplanar graph is provided below.

Definition 72. A *fuzzy k-quasi apex graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be made a fuzzy quasi planar graph by removing at most k vertices.

Formally, let $\psi = (V, \sigma, \mu)$ be a fuzzy graph. The graph ψ is called a *fuzzy k-quasi apex graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced fuzzy subgraph $\psi' = (V \setminus S, \sigma', \mu')$, where σ' and μ' are the restrictions of σ and μ to the remaining vertices, is a fuzzy planar graph.

The properties of a fuzzy k-quasi apex graph include:

- If $k = 0$, the fuzzy k-quasi apex graph itself is already a fuzzy quasi planar graph.
- When $k = 1$, the fuzzy graph corresponds to a fuzzy apex graph where the removal of a single vertex makes the graph fuzzy quasi planar.

Theorem 73. *A fuzzy k-quasi apex graph can be transformed into a fuzzy quasi-planar graph.*

Proof. Obviously holds. □

2.3.7 Fuzzy quasi apex outerplanar graph

The Fuzzy quasi apex outerplanar graph is an extension of the quasi apex outerplanar graph to a fuzzy graph. The definition of the Fuzzy apex outerplanar graph is provided below.

Definition 74. A *fuzzy k-quasi apex outerplanar graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be made a fuzzy quasi outerplanar graph by removing at most k vertices.

Formally, let $\psi = (V, \sigma, \mu)$ be a fuzzy graph. The graph ψ is called a *fuzzy k-quasi apex outerplanar graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced fuzzy subgraph $\psi' = (V \setminus S, \sigma', \mu')$, where σ' and μ' are the restrictions of σ and μ to the remaining vertices, is a fuzzy quasi outerplanar graph.

The properties of a fuzzy k-quasi apex outerplanar graph include:

- If $k = 0$, the fuzzy k-quasi apex outerplanar graph itself is already a fuzzy quasi outerplanar graph.
- When $k = 1$, the fuzzy graph corresponds to a fuzzy quasi apex outerplanar graph where the removal of a single vertex makes the graph fuzzy quasi outerplanar.

Theorem 75. *A fuzzy k-quasi apex outerplanar graph can be transformed into a fuzzy outer-quasi-planar graph.*

Proof. Obviously holds. □

2.3.8 Fuzzy almost planar graph

The definitions of an almost planar graph and an almost outerplanar graph are extended to the context of fuzzy graphs as follows.

Definition 76. A fuzzy graph $\psi = (V, \sigma, \mu)$ is called a *fuzzy almost planar graph* if there exists an edge $e \in E(\psi)$ such that the fuzzy graph $\psi - e$ (i.e., the fuzzy graph obtained by removing the edge e from ψ) is a fuzzy planar graph.

Definition 77. A fuzzy graph $\psi = (V, \sigma, \mu)$ is called a *fuzzy almost outerplanar graph* if there exists an edge $e \in E(\psi)$ such that the fuzzy graph $\psi - e$ (i.e., the fuzzy graph obtained by removing the edge e from ψ) is a fuzzy outerplanar graph.

Theorem 78. *An fuzzy almost planar graph can be transformed into a almost planar graph.*

Proof. Obviously holds. □

Theorem 79. *An fuzzy almost outerplanar graph can be transformed into a almost outerplanar graph.*

Proof. Obviously holds. □

2.4 Intuitionistic fuzzy planar graphs

Intuitionistic fuzzy graphs are an extended version of fuzzy graphs and have been the subject of extensive study for over 15 years [247, 328, 432, 447]. Intuitionistic fuzzy planar graphs [49] are, in turn, an extension of fuzzy planar graphs. The concept of intuitionistic fuzzy sets [70–72, 176, 211, 358, 517], which is well-known in set theory, is closely related to intuitionistic fuzzy graphs. The definitions of intuitionistic fuzzy graphs and intuitionistic fuzzy planar graphs are provided below.

Definition 80 (Intuitionistic Fuzzy Graph (IFG))**.** [432] Let $G = (V, E)$ be a classical graph where V denotes the set of vertices and E denotes the set of edges. An *Intuitionistic Fuzzy Graph* (IFG) on G, denoted $G_{IF} = (A, B)$, is defined as follows:

1. (μ_A, ν_A) is an *Intuitionistic Fuzzy Set (IFS)* on the vertex set V. For each vertex $x \in V$, the degree of membership $\mu_A(x) \in [0, 1]$ and the degree of non-membership $\nu_A(x) \in [0, 1]$ satisfy:

$$\mu_A(x) + \nu_A(x) \leq 1$$

The value $1 - \mu_A(x) - \nu_A(x)$ represents the hesitancy or uncertainty regarding the membership of x in the set.

2. (μ_B, ν_B) is an *Intuitionistic Fuzzy Relation (IFR)* on the edge set E. For each edge $(x, y) \in E$, the degree of membership $\mu_B(x, y) \in [0, 1]$ and the degree of non-membership $\nu_B(x, y) \in [0, 1]$ satisfy:

$$\mu_B(x, y) + \nu_B(x, y) \leq 1$$

Additionally, the following constraints must hold for all $x, y \in V$:

$$\mu_B(x, y) \leq \mu_A(x) \wedge \mu_A(y)$$

$$\nu_B(x, y) \leq \nu_A(x) \vee \nu_A(y)$$

In this definition:

- $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and non-membership of the vertex x, respectively.
- $\mu_B(x, y)$ and $\nu_B(x, y)$ represent the degree of membership and non-membership of the edge (x, y), respectively.
- If $\nu_A(x) = 0$ and $\nu_B(x, y) = 0$ for all $x \in V$ and $(x, y) \in E$, then the Intuitionistic Fuzzy Graph reduces to a Fuzzy Graph.

Theorem 81. *[235] The following are examples of related graph classes, including but not limited to:*

- *Strong Intuitionistic Fuzzy Graphs [23]*
- *Perfect intuitionistic fuzzy graphs [247]*
- *Intuitionistic fuzzy competition graphs[467]*
- *Intuitionistic fuzzy threshold graphs[554]*
- *Balanced Intuitionistic Fuzzy Graphs[324]*
- *Bipolar Intuitionistic Fuzzy Competition Graphs[185]*
- *Intuitionistic Felicitous Fuzzy Graphs [95]*
- *Intuitionistic Fuzzy Soft Expert Graphs [533]*
- *Intuitionistic fuzzy tolerance graphs[468]*
- *Intuitionistic fuzzy planar graphs[50]*
- *Edge regular intuitionistic fuzzy graph[325]*
- *Edge Irregular Intuitionistic Fuzzy Graphs[410]*
- *m-Neighbourly Irregular Instuitionistic Fuzzy Graphs[376]*
- *R-edge regular intuitionistic fuzzy graphs[43]*
- *Perfectly Edge-Regular Intuitionistic Fuzzy Graphs[2]*
- *Edge Regular Intuitionistic Fuzzy M-Polargraphs[252]*
- *Intuitionistic fuzzy labeling graphs[464]*
- *Regular Interval-Valued Intuitionistic Fuzzy Graphs[1]*
- *Anti intuitionistic fuzzy graph[307]*
- *Interval-valued intuitionistic fuzzy graphs[346]*
- *Intuitionistic fuzzy directed hypergraphs[408]*
- *Complex intuitionistic fuzzy graphs[62]*
- *Complex t-Intuitionistic Fuzzy Graph[329]*
- *Intuitionistic fuzzy incidence graphs[414]*
- *Intuitionistic fuzzy k-partite hypergraphs[405, 406]*
- *Irregular Intuitionistic Fuzzy Graphs[391]*
- *intuitionistic L-fuzzy graph[502, 514]*
- *Bipolar intuitionistic anti fuzzy graphs[184]*
- *Intuitionistic anti-fuzzy graphs[404]*
- *intuitionistic product fuzzy graphs[518]*
- *intuitionistic k-partitioned fuzzy graph[435]*
- *intutionistic fuzzy multigraphs[73]*

Proof. Refer to each reference as needed. □

2.4.1 Intuitionistic fuzzy planar graph

The Intuitionistic fuzzy planar graph is an extension of the fuzzy planar graph. These graphs have been studied in a similar manner to fuzzy planar graphs. The definition of the Intuitionistic fuzzy planar graph is provided below[49, 466].

Definition 82. [49] Let G be an intuitionistic fuzzy multigraph[103, 444, 483], and let $P_1, P_2, \ldots, P_z$ be the points of intersection between the edges for a given geometrical representation. The graph G is said to be an *intuitionistic fuzzy planar graph* with an intuitionistic fuzzy planarity value $f = (f_M, f_N)$, where

$$f = (f_M, f_N) = \left(\frac{1}{1 + (M_{P_1} + M_{P_2} + \cdots + M_{P_z})}, \frac{1}{1 + (N_{P_1} + N_{P_2} + \cdots + N_{P_z})} \right).$$

Here, M_{P_i} and N_{P_i} represent the membership and non-membership degrees at each point of intersection P_i, respectively.

The intuitionistic fuzzy planarity value $f = (f_M, f_N)$ is bounded such that $0 < f_M \leq 1$ and $0 < f_N \leq 1$. If there are no points of intersection for a certain geometrical representation of the intuitionistic fuzzy planar graph, then its intuitionistic fuzzy planarity value is $(1, 1)$.

Proposition 83. *An intuitionistic fuzzy planar graph is an intuitionistic fuzzy graph.*

Proof. Obviously holds. □

Theorem 84. *An intuitionistic fuzzy planar graph can be transformed into a fuzzy planar graph.*

Proof. Obviously holds. □

2.4.2 Intuitionistic fuzzy outerplanar graph

We define the intuitionistic fuzzy outerplanar graph, which is an extension of the fuzzy outerplanar graph.

Definition 85. An *intuitionistic fuzzy outerplanar graph* $\psi = (V, \mu, \nu)$ is an intuitionistic fuzzy graph where:

- It can be embedded in a plane such that every vertex $v \in V$ lies on the boundary of the exterior region of the embedding.
- Similar to intuitionistic fuzzy graphs, $\mu : V \to [0, 1]$ and $\nu : V \to [0, 1]$ represent the membership and non-membership degrees of each vertex, respectively.
- For an intuitionistic fuzzy graph to be outerplanar, there must be no edge intersection while ensuring all vertices are positioned on the outer face.

If $i(\psi)$ represents the number of vertices not lying on the boundary of the outer face, then an intuitionistic fuzzy outerplanar graph satisfies $i(\psi) = 0$. In contrast, if $i(\psi) \neq 0$, the graph is considered a non-outerplanar intuitionistic fuzzy graph.

These definitions extend the classical notions of outerplanarity to the intuitionistic fuzzy domain, allowing for the representation and analysis of uncertainty in graph structures.

Theorem 86. *An intuitionistic fuzzy outerplanar graph can be transformed into a fuzzy outerplanar graph.*

Proof. Obviously holds. □

Theorem 87. *An intuitionistic fuzzy outerplanar graph is an intuitionistic fuzzy planar graph.*

Proof. Obviously holds. □

2.4.3 Intuitionistic fuzzy apex graph

We define the intuitionistic fuzzy apex graph, which is an extension of the fuzzy apex graph.

Definition 88. An *intuitionistic fuzzy apex graph* $\psi = (V, \mu, \nu)$ is an intuitionistic fuzzy graph that can be transformed into an intuitionistic fuzzy planar graph by removing a single vertex $v \in V$. The removed vertex is called an *apex* of the intuitionistic fuzzy graph ψ.

Formally, ψ is an intuitionistic fuzzy apex graph if there exists a vertex $v \in V$ such that the subgraph $\psi' = (V \setminus \{v\}, \mu', \nu')$, where μ' and ν' are the restrictions of the membership and non-membership functions μ and ν on the remaining vertices, is an intuitionistic fuzzy planar graph.

The intuitionistic fuzzy apex graph retains the following properties:

- $\mu : V \to [0, 1]$ assigns a membership degree to each vertex in V.
- $\nu : V \to [0, 1]$ assigns a non-membership degree to each vertex in V.
- If the intuitionistic fuzzy apex graph ψ itself is an intuitionistic fuzzy planar graph, then every vertex in V can be considered an apex.

Similar to classical apex graphs, an intuitionistic fuzzy apex graph may have multiple apex vertices. Additionally, if ψ is a null intuitionistic fuzzy graph (i.e., $V = \emptyset$), it is trivially an intuitionistic fuzzy apex graph.

Theorem 89. *An intuitionistic fuzzy apex graph can be transformed into a fuzzy apex graph.*

Proof. Obviously holds. □

Theorem 90. *An intuitionistic fuzzy apex graph can be transformed into an intuitionistic fuzzy planar graph.*

Proof. Obviously holds. □

2.4.4 Intuitionistic Fuzzy Quasi-Planar Graph

The definition of an Intuitionistic Fuzzy Quasi-Planar Graph is provided below. This graph is an extension of the Fuzzy Quasi-Planar Graph.

Definition 91. An *Intuitionistic Fuzzy Quasi-Planar Graph* $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ is an intuitionistic fuzzy graph that can be drawn in the plane such that no three intuitionistic fuzzy edges are pairwise crossing.

Formally, ψ_{IF} is called an *Intuitionistic Fuzzy Quasi-Planar Graph* if there exists a drawing D in which any set of three intuitionistic fuzzy edges do not all intersect each other at distinct points in their interiors.

Definition 92. An *Intuitionistic Fuzzy k-Quasi-Planar Graph* $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ is an intuitionistic fuzzy graph that can be drawn in the plane such that no k intuitionistic fuzzy edges are pairwise crossing.

Formally, let $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ be an intuitionistic fuzzy graph where:

- $\mu_A : V \to [0, 1]$ assigns a membership degree to each vertex $v \in V$, and $\nu_A : V \to [0, 1]$ assigns a non-membership degree such that $\mu_A(v) + \nu_A(v) \leq 1$.
- $\mu_B : V \times V \to [0, 1]$ assigns a membership degree to each edge (u, v), and $\nu_B : V \times V \to [0, 1]$ assigns a non-membership degree such that $\mu_B(u, v) + \nu_B(u, v) \leq 1$ and $\mu_B(u, v) \leq \min\{\mu_A(u), \mu_A(v)\}$.

The intuitionistic fuzzy graph ψ_{IF} is called *Intuitionistic Fuzzy k-Quasi-Planar* if there exists a drawing D in which any set of k intuitionistic fuzzy edges do not all intersect each other at distinct points in their interiors.

Theorem 93. *An Intuitionistic Fuzzy k-Quasi-Planar Graph can be transformed into a Intuitionistic Fuzzy Quasi-Planar graph.*

Proof. Obviously holds. □

Theorem 94. *An Intuitionistic Fuzzy k-Quasi-Planar Graph can be transformed into a Fuzzy k-Quasi-Planar graph.*

Proof. Obviously holds. □

2.4.5 Intuitionistic Fuzzy Quasi-Outerplanar Graph

The definition of an Intuitionistic Fuzzy Quasi-Outerplanar Graph is provided below. This graph is an extension of the Fuzzy Quasi-Outerplanar Graph.

Definition 95. An *Intuitionistic Fuzzy Quasi-Outerplanar Graph* $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ is an intuitionistic fuzzy graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no three intuitionistic fuzzy edges are pairwise crossing.

Formally, ψ_{IF} is called an *Intuitionistic Fuzzy Quasi-Outerplanar Graph* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of three intuitionistic fuzzy edges do not all intersect each other at distinct points in their interiors.

Definition 96. An *Intuitionistic Fuzzy Outer-k-Quasi-Planar Graph* $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ is an intuitionistic fuzzy graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no set of k intuitionistic fuzzy edges are pairwise crossing.

Formally, let $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ be an intuitionistic fuzzy graph where:

- μ_A, ν_A, μ_B, and ν_B are as defined above.

The intuitionistic fuzzy graph ψ_{IF} is called *Intuitionistic Fuzzy Outer-k-Quasi-Planar* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of k intuitionistic fuzzy edges do not all intersect each other at distinct points in their interiors.

Theorem 97. *An Intuitionistic Fuzzy Outer-k-Quasi-Planar Graph can be transformed into a Fuzzy Outer-k-Quasi-Planar graph.*

Proof. Obviously holds. □

Theorem 98. *An Intuitionistic Fuzzy Quasi-Outerplanar Graph can be transformed into a Fuzzy Quasi-Outerplanar Graph.*

Proof. Obviously holds. □

2.4.6 Intuitionistic Fuzzy Quasi Apex Graph

The definition of an Intuitionistic Fuzzy Quasi Apex Graph is provided below. This graph is an extension of the Fuzzy Quasi Apex Graph.

Definition 99. An *Intuitionistic Fuzzy k-Quasi Apex Graph* $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ is an intuitionistic fuzzy graph that can be transformed into an intuitionistic fuzzy quasi planar graph by removing at most k vertices.

Formally, ψ_{IF} is called an *Intuitionistic Fuzzy k-Quasi Apex Graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced intuitionistic fuzzy subgraph $\psi'_{IF} = (V \setminus S, \mu'_A, \nu'_A, \mu'_B, \nu'_B)$, where $\mu'_A, \nu'_A, \mu'_B, \nu'_B$ are the restrictions of $\mu_A, \nu_A, \mu_B, \nu_B$ on the remaining vertices, is an intuitionistic fuzzy Quasi planar graph.

Theorem 100. *An Intuitionistic Fuzzy k-Quasi Apex Graph can be transformed into a Fuzzy k-Quasi Apex Graph.*

Proof. Obviously holds. □

2.4.7 Intuitionistic Fuzzy Quasi Apex Outerplanar Graph

The definition of an Intuitionistic Fuzzy Quasi Apex Outerplanar Graph is provided below. This graph is an extension of the Fuzzy Quasi Apex Outerplanar Graph.

Definition 101. An *Intuitionistic Fuzzy k-Quasi Apex Outerplanar Graph* $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ is an intuitionistic fuzzy graph that can be transformed into an intuitionistic fuzzy quasi outerplanar graph by removing at most k vertices.

Formally, ψ_{IF} is called an *Intuitionistic Fuzzy k-Quasi Apex Outerplanar Graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced intuitionistic fuzzy subgraph $\psi'_{IF} = (V \setminus S, \mu'_A, \nu'_A, \mu'_B, \nu'_B)$, where $\mu'_A, \nu'_A, \mu'_B, \nu'_B$ are the restrictions of $\mu_A, \nu_A, \mu_B, \nu_B$ on the remaining vertices, is an intuitionistic fuzzy Quasi outerplanar graph.

Theorem 102. *An Intuitionistic Fuzzy k-Quasi Apex Outerplanar Graph can be transformed into a Fuzzy k-Quasi Apex Outerplanar Graph.*

Proof. Obviously holds. □

2.4.8 Intuitionistic fuzzy almost planar graph

The definitions of an Intuitionistic fuzzy almost planar graph and an almost outerplanar graph are extended to the context of Intuitionistic fuzzy graphs as follows.

Definition 103. An *Intuitionistic fuzzy graph* $\psi_{IF} = (V, (\mu_A, \nu_A), (\mu_B, \nu_B))$ is called an *Intuitionistic fuzzy almost planar graph* if there exists an edge $e \in E(\psi_{IF})$ such that the Intuitionistic fuzzy graph $\psi_{IF} - e$ (i.e., the Intuitionistic fuzzy graph obtained by removing the edge e from ψ_{IF}) is an Intuitionistic fuzzy planar graph.

Definition 104. An *Intuitionistic fuzzy graph* $\psi_{IF} = (V, (\mu_A, \nu_A), (\mu_B, \nu_B))$ is called an *Intuitionistic fuzzy almost outerplanar graph* if there exists an edge $e \in E(\psi_{IF})$ such that the Intuitionistic fuzzy graph $\psi_{IF} - e$ (i.e., the Intuitionistic fuzzy graph obtained by removing the edge e from ψ_{IF}) is an Intuitionistic fuzzy outerplanar graph.

Theorem 105. *An intuitionistic fuzzy almost planar graph can be transformed into a fuzzy almost planar graph.*

Proof. Obviously holds. □

Theorem 106. *An intuitionistic fuzzy almost outerplanar graph can be transformed into a fuzzy almost outerplanar graph.*

Proof. Obviously holds. □

2.5 Neutrosophic graph and planar graph

First, the definition of a neutrosophic graph is provided. Similar to fuzzy graphs, neutrosophic graphs have been the focus of extensive research [28, 33, 130, 237, 238, 276, 297, 319, 463, 493, 497]. Additionally, the concept of neutrosophic sets[128, 212, 378, 540] in set theory is closely related to neutrosophic graphs. The definition is given below[237, 497].

Definition 107. [237,497] A neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ is defined as a graph where $\sigma_i : V \to [0, 1]$, $\mu_i : E \to [0, 1]$, and for every $v_i v_j \in E$, the following condition holds: $\mu(v_i v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$.

1. σ is called the neutrosophic vertex set.
2. μ is called the neutrosophic edge set.
3. $|V|$ is called the order of NTG, and it is denoted by $O(NTG)$.
4. $\sum_{v \in V} \sigma(v)$ is called the neutrosophic order of NTG, and it is denoted by $On(NTG)$.
5. $|E|$ is called the size of NTG, and it is denoted by $S(NTG)$.
6. $\sum_{e \in E} \mu(e)$ is called the neutrosophic size of NTG, and it is denoted by $Sn(NTG)$.

Theorem 108. *[235] The following are examples of related Neutrosophic graph classes, including but not limited to:*

- *Bipolar Neutrosophic Graphs [31]*
- *Interval-Valued Neutrosophic Graphs [486]*
- *Single-Valued Neutrosophic Graphs [35, 131]*
- *Interval Complex Neutrosophic Graphs [129, 300]*
- *Neutrosophic Hypergraphs [250]*
- *Neutrosophic Vague Line Graphs [300]*
- *Neutrosophic Vague Graphs [301]*
- *Single-Valued Neutrosophic Signed Graphs [386]*
- *Neutrosophic Soft Rough Graphs [28]*
- *Neutrosophic Incidence Graphs [32]*
- *Fermatean Neutrosophic Dombi Fuzzy Graphs [474]*
- *Interval valued pentapartitioned neutrosophic graphs [129]*
- *Single valued pentapartitioned neutrosophic graphs[174]*

- *pentapartitioned neutrosophic graphs[443]*
- *t-Neutrosophic Fuzzy graph[327]*
- *Complex t-Neutrosophic Graph[330]*
- *Regular neutrosophic graphs[297]*
- *Q-neutrosophic soft graphs[532]*
- *Balanced Neutrosophic Graphs[490]*
- *Neutrosophic minimum spanning tree graph[363]*

Proof. Refer to each reference as needed. □

2.5.1 Neutrosophic planar graph

A neutrosophic planar graph is an extension of fuzzy planar graphs and intuitionistic fuzzy planar graphs [37, 146, 375].

Definition 109. [320] Let $G = (A, B)$ be a *neutrosophic multigraph*, where B contains edges $(ab, T_B(ab), I_B(ab), F_B(ab))$ and $(cd, T_B(cd), I_B(cd), F_B(cd))$ that intersect at a point P. The intersecting value at point P is defined as

$$S_P = ((S_T)_P, (S_I)_P, (S_F)_P) = \left(\frac{(S_T)_{ab} + (S_T)_{cd}}{2}, \frac{(S_I)_{ab} + (S_I)_{cd}}{2}, \frac{(S_F)_{ab} + (S_F)_{cd}}{2} \right).$$

As the number of intersection points in a neutrosophic multigraph increases, its planarity decreases. Thus, for a neutrosophic multigraph, S_P is inversely proportional to the planarity.

Definition 110. [37, 146, 375] A neutrosophic multigraph G with intersection points $P_1, P_2, \ldots, P_z$ between the edges in a certain geometrical representation is called a *neutrosophic planar graph* if the neutrosophic planarity value $f = (f_T, f_I, f_F)$ satisfies: $f_T = f_I = f_F = 1$.

The neutrosophic planarity value is computed as:

$$f = (f_T, f_I, f_F) = \left(\frac{1}{1 + \{(S_T)_{P_1} + (S_T)_{P_2} + \ldots + (S_T)_{P_z}\}}, \frac{1}{1 + \{(S_I)_{P_1} + (S_I)_{P_2} + \ldots + (S_I)_{P_z}\}}, \frac{1}{1 + \{(S_F)_{P_1} + (S_F)_{P_2} + \ldots + (S_F)_{P_z}\}} \right),$$

where S_T, S_I, and S_F represent the truth, indeterminacy, and falsity values of the intersections at each point P_i. For the graph to be considered neutrosophically planar, we must have:

$$(S_T)_{P_1} + (S_T)_{P_2} + \ldots + (S_T)_{P_z} = 0, \quad (S_I)_{P_1} + (S_I)_{P_2} + \ldots + (S_I)_{P_z} = 0, \quad (S_F)_{P_1} + (S_F)_{P_2} + \ldots + (S_F)_{P_z} = 0.$$

Thus, the neutrosophic planarity value must be $(f_T, f_I, f_F) = (1, 1, 1)$, indicating no geometrical intersections and perfect planarity in all neutrosophic dimensions.

Theorem 111. *Let $G_N = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a neutrosophic planar graph, where:*

- *$\sigma : V \to [0, 1]^3$ assigns a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees to each vertex $v \in V$.*
- *$\mu : E \to [0, 1]^3$ assigns a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$ representing the truth, indeterminacy, and falsity membership degrees to each edge $e \in E$.*

By setting the indeterminacy $\sigma_I(v) = 0$ and the falsity $\sigma_F(v) = 0$ for all vertices $v \in V$, and similarly setting $\mu_I(e) = 0$ and $\mu_F(e) = 0$ for all edges $e \in E$, the graph G_N can be transformed into a fuzzy planar graph G_F. Specifically:

- *The membership degree of the fuzzy graph G_F is $\sigma(v) = \sigma_T(v)$ for each vertex v.*
- *The fuzzy relation $\mu(u, v) = \mu_T(u, v)$ for each edge (u, v).*

Proof. 1. *Vertex Transformation:* For each vertex $v \in V$, the representation is $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$ By setting $\sigma_I(v) = 0$ and $\sigma_F(v) = 0$, we obtain $\sigma(v) = (\sigma_T(v), 0, 0)$. Therefore, in the fuzzy graph representation, $\sigma(v) = \sigma_T(v)$, which corresponds to the membership degree of the vertex v.

2. *Edge Transformation:* For each edge $e \in E$, the neutrosophic representation is $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$. By setting $\mu_I(e) = 0$ and $\mu_F(e) = 0$, we obtain $\mu(e) = (\mu_T(e), 0, 0)$. Thus, the membership degree of the edge $\mu(u, v) = \mu_T(u, v)$ corresponds to the fuzzy relation in the resulting fuzzy graph.

3. *Planarity:* Since G_N is a planar graph and setting $\sigma_I(v)$, $\sigma_F(v)$, $\mu_I(e)$, and $\mu_F(e)$ to zero does not alter the graph's structure or the absence of intersecting edges, the resulting graph G_F retains the planarity properties.

Hence, the neutrosophic planar graph G_N is transformed into a fuzzy planar graph G_F by setting the indeterminacy and falsity values to zero. □

Theorem 112. *Let $G_N = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a neutrosophic planar graph. By setting the indeterminacy degree $\sigma_I(v) = 0$ for all vertices $v \in V$ and $\mu_I(e) = 0$ for all edges $e \in E$, the graph G_N can be transformed into an intuitionistic fuzzy planar graph G_{IF}. Specifically:*

- *The membership degree $\mu_A(v) = \sigma_T(v)$ for each vertex v.*
- *The non-membership degree $\nu_A(v) = \sigma_F(v)$ for each vertex v.*

Similarly, for each edge e, the membership and non-membership degrees are $\mu_{IF}(e) = \mu_T(e)$ and $\nu_{IF}(e) = \mu_F(e)$, respectively. The resulting graph G_{IF} satisfies the properties of an intuitionistic fuzzy planar graph.

Proof. 1. *Vertex Transformation:* For each vertex $v \in V$, the representation is $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$. By setting $\sigma_I(v) = 0$, we obtain $\sigma(v) = (\sigma_T(v), 0, \sigma_F(v))$. Defining $\mu_A(v) = \sigma_T(v)$ and $\nu_A(v) = \sigma_F(v)$, we ensure $\mu_A(v) + \nu_A(v) = \sigma_T(v) + \sigma_F(v) \leq 1$.

2. *Edge Transformation:* For each edge $e \in E$, the neutrosophic representation is $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$. By setting $\mu_I(e) = 0$, we obtain $\mu(e) = (\mu_T(e), 0, \mu_F(e))$. Defining $\mu_{IF}(e) = \mu_T(e)$ and $\nu_{IF}(e) = \mu_F(e)$, we have $\mu_{IF}(e) + \nu_{IF}(e) \leq 1$.

3. *Planarity:* Since G_N is a planar graph, there exists an embedding in the plane such that no edges intersect except at vertices. Thus, the resulting graph G_{IF} is also planar, with fuzzy representation characteristics maintained.

Therefore, by setting the indeterminacy values $\sigma_I(v) = 0$ and $\mu_I(e) = 0$, the neutrosophic planar graph G_N transforms into an intuitionistic fuzzy planar graph G_{IF}. □

If there are no points of intersection for a given geometrical representation of the neutrosophic planar graph, its neutrosophic planarity value is $(1, 1, 1)$. In such a case, the underlying crisp graph of this neutrosophic graph corresponds to a classical planar graph. As f_T and f_I decrease and f_F increases, the number of intersection points between edges increases or decreases, respectively, and the degree of planarity decreases or increases accordingly.

2.5.2 Neutrosophic outerplanar graph

We introduce a new definition for the Neutrosophic outerplanar graph. The Neutrosophic outerplanar graph extends the concepts of Fuzzy outerplanar Graph and Intuitionistic Fuzzy outerplanar graph as described below.

Definition 113 (Neutrosophic Outerplanar Graph)**.** A *neutrosophic outerplanar graph*

$$G = (A, B, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$$

is a neutrosophic multigraph such that:

- $\sigma : A \to [0, 1]^3$ assigns a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees to each vertex $v \in A$.
- $\mu : B \to [0, 1]^3$ assigns a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$ representing the truth, indeterminacy, and falsity membership degrees to each edge $e \in B$.

The graph G is called neutrosophic outerplanar if it can be embedded in the plane such that all vertices lie on the boundary of the exterior region, satisfying the following conditions:

1. No two edges intersect except at their endpoints.
2. All vertices are positioned on the outer face of the graph embedding.

If there are intersection points $P_1, P_2, \ldots, P_z$ in the embedding, the neutrosophic outerplanarity value $f = (f_T, f_I, f_F)$ is defined as

$$f = \left(\frac{1}{1 + \sum_{i=1}^{z} (S_T)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_I)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_F)_{P_i}} \right),$$

where $S_{P_i} = ((S_T)_{P_i}, (S_I)_{P_i}, (S_F)_{P_i})$ denotes the intersecting values at point P_i. If no intersections exist, $f = (1, 1, 1)$.

Next, we will provide the definitions of a maximal neutrosophic outerplanar graph and a minimally neutrosophic non-outerplanar graph.

Definition 114. If an edge cannot be added to a neutrosophic outerplanar graph ψ without sacrificing its outerplanarity property, then the graph is called a *maximal neutrosophic outerplanar graph*. The graphs shown in Figures are examples of maximal neutrosophic outerplanar graphs.

Definition 115. A neutrosophic planar graph ψ is considered *minimally neutrosophic non-outerplanar* if $i(\psi) \neq 0$ with at most one vertex $v \in V$ such that $\sigma_I(v) > 0$ or $\sigma_F(v) > 0$ in the interior region.

Theorem 116. *A neutrosophic outerplanar graph can be transformed into a fuzzy outerplanar graph by setting the indeterminacy and falsity degrees $\sigma_I(v)$ and $\sigma_F(v)$ for all vertices, and $\mu_I(e)$ and $\mu_F(e)$ for all edges, to zero.*

Proof. Let $G = (A, B, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a neutrosophic outerplanar graph, where:

- $\sigma : A \to [0, 1]^3$ assigns a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ to each vertex $v \in A$.
- $\mu : B \to [0, 1]^3$ assigns a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$ to each edge $e \in B$.

Define a new fuzzy outerplanar graph $\psi = (V, \sigma', \mu')$ where:

- $V = A$ (the set of vertices remains the same),
- $\sigma' : V \to [0, 1]$ is defined by $\sigma'(v) = \sigma_T(v)$,
- $\mu' : V \times V \to [0, 1]$ is defined by $\mu'(u, v) = \mu_T(e)$ where e is the edge between u and v.

By setting $\sigma_I(v) = 0$ and $\sigma_F(v) = 0$ for all $v \in V$, the original triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ reduces to a single membership value $\sigma_T(v)$.

Similarly, for each edge e, setting $\mu_I(e) = 0$ and $\mu_F(e) = 0$ reduces $(\mu_T(e), \mu_I(e), \mu_F(e))$ to the single value $\mu_T(e)$.

This transformation results in:

$$\mu'(u, v) \leq \min\{\sigma'(u), \sigma'(v)\},$$

satisfying the conditions of a fuzzy outerplanar graph.

Additionally, since all vertices lie on the boundary of the exterior region in the neutrosophic embedding, the transformed graph ψ maintains the outerplanar property. Therefore, the resulting graph $\psi = (V, \sigma', \mu')$ is a valid fuzzy outerplanar graph. □

2.5.3 Neutrosophic apex graph

The definition of a neutrosophic apex graph is provided below. This graph extends the concepts of fuzzy apex graphs and intuitionistic fuzzy apex graphs, as detailed in the following sections.

Definition 117. A *neutrosophic apex graph* $\psi = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ is a neutrosophic graph that can be transformed into a neutrosophic planar graph by removing a single vertex $v \in V$. The removed vertex is called an *apex* of the neutrosophic graph ψ.

Formally, ψ is a neutrosophic apex graph if there exists a vertex $v \in V$ such that the subgraph $\psi' = (V \setminus \{v\}, E', \sigma', \mu')$, where E' is the set of edges remaining after the removal of v, and σ' and μ' are the restrictions of the membership functions σ and μ on the remaining vertices and edges, is a neutrosophic planar graph.

The neutrosophic apex graph retains the following properties:

- $\sigma : V \to [0, 1]^3$ assigns a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees to each vertex $v \in V$.

- $\mu : E \to [0,1]^3$ assigns a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$ representing the truth, indeterminacy, and falsity membership degrees to each edge $e \in E$.
- If the neutrosophic apex graph ψ itself is neutrosophic planar, then every vertex in V can be considered an apex.

Similar to classical apex graphs, a neutrosophic apex graph may have multiple apex vertices. Additionally, if ψ is the null neutrosophic graph (i.e., $V = \emptyset$), it is trivially a neutrosophic apex graph.

Definition 118. A *neutrosophic k-apex graph* $G = (A, B, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ is a neutrosophic graph that can be transformed into a neutrosophic planar graph by removing at most k vertices from A.

If $k = 0$, the neutrosophic k-apex graph itself is already a neutrosophic planar graph. When $k = 1$, the graph corresponds to a neutrosophic apex graph as defined earlier.

2.5.4 Neutrosophic Apex Outerplanar Graph

The definition of a neutrosophic Apex Outerplanar Graph is provided below. This graph extends the concepts of fuzzy Apex Outerplanar Graph and intuitionistic fuzzy Apex Outerplanar Graph, as detailed in the following sections.

Definition 119. A *Neutrosophic Apex Outerplanar Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be transformed into a neutrosophic outerplanar graph by removing a single vertex $v \in V$.

Formally, ψ_N is called a *Neutrosophic Apex Outerplanar Graph* if there exists a vertex $v \in V$ such that the induced neutrosophic subgraph $\psi'_N = (V \setminus \{v\}, T'_A, I'_A, F'_A, T'_B, I'_B, F'_B)$, where $T'_A, I'_A, F'_A, T'_B, I'_B, F'_B$ are the restrictions of $T_A, I_A, F_A, T_B, I_B, F_B$ to the remaining vertices, is a neutrosophic outerplanar graph.

2.5.5 Neutrosophic Quasi-Planar Graph

The definition of a Neutrosophic Quasi-Planar Graph is provided below. This graph extends the concepts of fuzzy Quasi-Planar Graph and intuitionistic fuzzy Quasi-Planar Graph, as detailed in the following sections.

Definition 120. A *Neutrosophic Quasi-Planar Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be drawn in the plane such that no three neutrosophic edges are pairwise crossing.

Formally, ψ_N is called a *Neutrosophic Quasi-Planar Graph* if there exists a drawing D in which any set of three neutrosophic edges do not all intersect each other at distinct points in their interiors.

Definition 121. A *Neutrosophic Outer-k-Quasi-Planar Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no set of k neutrosophic edges are pairwise crossing.

Formally, the neutrosophic graph ψ_N is called a *Neutrosophic Outer-k-Quasi-Planar Graph* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of k neutrosophic edges do not all intersect each other at distinct points in their interiors.

2.5.6 Neutrosophic Quasi-Outerplanar Graph

The definition of a Neutrosophic Quasi-Outerplanar Graph is provided below. This graph extends the concepts of fuzzy Quasi-Outerplanar Graph and intuitionistic fuzzy Quasi-Outerplanar Graph, as detailed in the following sections.

Definition 122. A *Neutrosophic Quasi-Outerplanar Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no three neutrosophic edges are pairwise crossing.

Formally, ψ_N is called a *Neutrosophic Quasi-Outerplanar Graph* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of three neutrosophic edges do not all intersect each other at distinct points in their interiors.

Definition 123. A *Neutrosophic k-Quasi-Planar Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be drawn in the plane such that no k neutrosophic edges are pairwise crossing.

Formally, let $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ be a neutrosophic graph where:

- $T_A : V \to [0,1]$, $I_A : V \to [0,1]$, and $F_A : V \to [0,1]$ represent the truth-membership, indeterminacy-membership, and falsity-membership degrees of each vertex, respectively, satisfying $0 \le T_A(v) + I_A(v) + F_A(v) \le 1$.
- $T_B : V \times V \to [0,1]$, $I_B : V \times V \to [0,1]$, and $F_B : V \times V \to [0,1]$ represent the truth-membership, indeterminacy-membership, and falsity-membership degrees for each edge (u, v), satisfying $T_B(u, v) + I_B(u, v) + F_B(u, v) \le 1$ and $T_B(u, v) \le \min\{T_A(u), T_A(v)\}$.

2.5.7 Neutrosophic Quasi Apex Graph

The definition of a Neutrosophic Quasi Apex Graphis provided below. This graph extends the concepts of fuzzy Quasi Apex Graph and intuitionistic fuzzy Quasi Apex Graph, as detailed in the following sections.

Definition 124. A *Neutrosophic k-Quasi Apex Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be transformed into a neutrosophic planar graph by removing at most k vertices.

Formally, ψ_N is called a *Neutrosophic k-Quasi Apex Graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced neutrosophic subgraph $\psi'_N = (V \setminus S, T'_A, I'_A, F'_A, T'_B, I'_B, F'_B)$, where $T'_A, I'_A, F'_A, T'_B, I'_B, F'_B$ are the restrictions of $T_A, I_A, F_A, T_B, I_B, F_B$ to the remaining vertices, is a neutrosophic quasi-planar graph.

2.5.8 Neutrosophic Quasi Apex Outerplanar Graph

The definition of a Neutrosophic Quasi Apex Outerplanar Graph is provided below. This graph extends the concepts of fuzzy Quasi Apex Outerplanar Graph and intuitionistic fuzzy Quasi Apex Outerplanar Graph, as detailed in the following sections.

Definition 125. A *Neutrosophic k-Quasi Apex Outerplanar Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be transformed into a neutrosophic outerplanar graph by removing at most k vertices.

Formally, ψ_N is called a *Neutrosophic k-Quasi Apex Outerplanar Graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced neutrosophic subgraph $\psi'_N = (V \setminus S, T'_A, I'_A, F'_A, T'_B, I'_B, F'_B)$, where $T'_A, I'_A, F'_A, T'_B, I'_B, F'_B$ are the restrictions of $T_A, I_A, F_A, T_B, I_B, F_B$ to the remaining vertices, is a neutrosophic quasi outerplanar graph.

2.5.9 Neutrosophic almost planar graph

The definitions of a neutrosophic almost planar graph and an neutrosophic almost outerplanar graph are extended to the context of neutrosophic fuzzy graphs as follows.

Definition 126. A *neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$* is called a *neutrosophic almost planar graph* if there exists an edge $e \in E(NTG)$ such that the neutrosophic graph $NTG - e$ (i.e., the neutrosophic graph obtained by removing the edge e from NTG) is a neutrosophic planar graph.

Definition 127. A *neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$* is called a *neutrosophic almost outerplanar graph* if there exists an edge $e \in E(NTG)$ such that the neutrosophic graph $NTG - e$ (i.e., the neutrosophic graph obtained by removing the edge e from NTG) is a neutrosophic outerplanar graph.

Theorem 128. *An neutrosophic almost planar graph can be transformed into an intuitionistic fuzzy almost planar graph.*

Proof. Obviously holds. □

Theorem 129. *An neutrosophic almost outerplanar graph can be transformed into an intuitionistic fuzzy almost outerplanar graph.*

Proof. Obviously holds. □

2.6 Turiyam Neutrosophic Graph

Research on Turiyam Neutrosophic Graphs, which incorporate parameters into Neutrosophic Graphs, is currently being conducted [244–246]. These graphs are a graphical representation of the Turiyam Neutrosophic Set [246, 487]. Similar concepts include four-valued logic [90, 150]. The definition is provided below.

Definition 130 (Turiyam Neutrosophic Graph)**.** [244–246] Let $G = (V, E)$ be a classical graph with a finite set of vertices $V = \{v_i : i = 1, 2, \ldots, n\}$ and edges $E = \{(v_i, v_j) : i, j = 1, 2, \ldots, n\}$. A *Turiyam Neutrosophic Graph* of G, denoted $G^T = (V^T, E^T)$, is defined as follows:

1. *Turiyam Neutrosophic Vertex Set*: For each vertex $v_i \in V$, the Turiyam Neutrosophic graph assigns the following mappings:
$$t(v_i), iv(v_i), fv(v_i), lv(v_i) : V \to [0, 1],$$
where:
   - $t(v_i)$ is the truth value (tv) of the vertex v_i,
   - $iv(v_i)$ is the indeterminacy value (iv) of v_i,
   - $fv(v_i)$ is the falsity value (fv) of v_i,

- $lv(v_i)$ is the Turiyam Neutrosophic state (or liberal value) (lv) of v_i,

for all $v_i \in V$, such that the following condition holds for each vertex:

$$0 \leq t(v_i) + iv(v_i) + fv(v_i) + lv(v_i) \leq 4.$$

2. *Turiyam Neutrosophic Edge Set*: For each edge $(v_i, v_j) \in E$, the Turiyam Neutrosophic graph assigns the following mappings:

$$t(v_i, v_j), iv(v_i, v_j), fv(v_i, v_j), lv(v_i, v_j) : E \to [0, 1],$$

where:

- $t(v_i, v_j)$ is the truth value of the edge (v_i, v_j),
- $iv(v_i, v_j)$ is the indeterminacy value of (v_i, v_j),
- $fv(v_i, v_j)$ is the falsity value of (v_i, v_j),
- $lv(v_i, v_j)$ is the Turiyam Neutrosophic state (or liberal value) of (v_i, v_j),

for all $(v_i, v_j) \in E$, such that the following condition holds for each edge:

$$0 \leq t(v_i, v_j) + iv(v_i, v_j) + fv(v_i, v_j) + lv(v_i, v_j) \leq 4.$$

In this case, V^T represents the Turiyam Neutrosophic vertex set of the graph G^T, and E^T represents the Turiyam Neutrosophic edge set of G^T.

2.6.1 Turiyam Neutrosophic Planar Graph

The Turiyam Neutrosophic planar Graph is an extension of the planar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 131 (Turiyam NeutrosophicPlanar Graph). A *Turiyam Neutrosophic planar graph* $G_T = (V, E, \sigma_T, \sigma_{IV}, \sigma_F, \sigma_{LV})$ is a Turiyam Neutrosophicgraph where:

- $\sigma_T : V \to [0,1]$, $\sigma_{IV} : V \to [0,1]$, $\sigma_F : V \to [0,1]$, and $\sigma_{LV} : V \to [0,1]$ represent the truth, indeterminacy, falsity, and liberal membership degrees of each vertex, respectively.
- $\mu_T : E \to [0,1]$, $\mu_{IV} : E \to [0,1]$, $\mu_F : E \to [0,1]$, and $\mu_{LV} : E \to [0,1]$ represent the truth, indeterminacy, falsity, and liberal membership degrees of each edge, respectively.

The graph G_T is said to be a *Turiyam Neutrosophic planar graph* if it can be drawn in the plane such that no two edges intersect except at their endpoints.

The Turiyam Neutrosophic planarity value $f_T = (f_T, f_{IV}, f_F, f_{LV})$ of the graph G_T is defined as

$$f_T = \left(\frac{1}{1 + \sum_{i=1}^{z} (S_T)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_{IV})_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_F)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_{LV})_{P_i}} \right),$$

where $S_{P_i} = ((S_T)_{P_i}, (S_{IV})_{P_i}, (S_F)_{P_i}, (S_{LV})_{P_i})$ denotes the intersecting values at point P_i. If there are no intersection points, $f_T = (1, 1, 1, 1)$. This value reflects the degree of planarity in the Turiyam Neutrosophic context, with higher values indicating greater planarity.

2.6.2 Turiyam Neutrosophic Outerplanar Graph

The Turiyam Neutrosophic Outerplanar Graph is an extension of the Outerplanar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 132 (Turiyam Neutrosophic Outerplanar Graph). A *Turiyam Neutrosophic outerplanar graph* $G_T = (V, E, \sigma_T, \sigma_{IV}, \sigma_F, \sigma_{LV})$ is a Turiyam Neutrosophic graph that can be embedded in the plane such that all vertices lie on the boundary of the exterior region, satisfying the following conditions:

- $\sigma_T : V \to [0,1]$, $\sigma_{IV} : V \to [0,1]$, $\sigma_F : V \to [0,1]$, and $\sigma_{LV} : V \to [0,1]$ represent the truth, indeterminacy, falsity, and liberal membership degrees of each vertex, respectively.
- $\mu_T : E \to [0,1]$, $\mu_{IV} : E \to [0,1]$, $\mu_F : E \to [0,1]$, and $\mu_{LV} : E \to [0,1]$ represent the truth, indeterminacy, falsity, and liberal membership degrees of each edge, respectively.

- No two edges intersect except at their endpoints, and all vertices are positioned on the outer face of the graph embedding.

If there are intersection points $P_1, P_2, \ldots, P_z$, the Turiyam Neutrosophic outerplanarity value $f_T = (f_T, f_{IV}, f_F, f_{LV})$ of the graph is defined as

$$f_T = \left(\frac{1}{1 + \sum_{i=1}^{z} (S_T)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_{IV})_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_F)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_{LV})_{P_i}} \right),$$

where $S_{P_i} = ((S_T)_{P_i}, (S_{IV})_{P_i}, (S_F)_{P_i}, (S_{LV})_{P_i})$ denotes the intersecting values at point P_i. If no intersections exist, $f_T = (1, 1, 1, 1)$.

2.6.3 Turiyam Neutrosophic Apex Planar Graph

The Turiyam Neutrosophic Apex planar Graph is an extension of the Apex planar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 133 (Turiyam Neutrosophic Apex Planar Graph)**.** A *Turiyam Neutrosophic Apex Planar Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be transformed into a Turiyam Neutrosophic planar graph by removing a single vertex $v \in V$. The removed vertex is called an *apex* of the Turiyam Neutrosophic graph ψ_T.

Formally, ψ_T is called a *Turiyam Neutrosophic Apex Planar Graph* if there exists a vertex $v \in V$ such that the induced Turiyam Neutrosophic subgraph $\psi'_T = (V \setminus \{v\}, t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B)$, where $t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B$ are the restrictions of $t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B$ to the remaining vertices and edges, is a Turiyam Neutrosophic planar graph.

The Turiyam Neutrosophic apex graph retains the following properties:

- $t_A : V \to [0,1]$, $iv_A : V \to [0,1]$, $fv_A : V \to [0,1]$, and $lv_A : V \to [0,1]$ assign the truth, indeterminacy, falsity, and liberal membership degrees to each vertex $v \in V$, respectively.
- $t_B : E \to [0,1]$, $iv_B : E \to [0,1]$, $fv_B : E \to [0,1]$, and $lv_B : E \to [0,1]$ assign the truth, indeterminacy, falsity, and liberal membership degrees to each edge $e \in E$, respectively.
- If the Turiyam Neutrosophic apex graph ψ_T itself is a Turiyam Neutrosophic planar graph, then every vertex in V can be considered an apex.

Similar to classical apex graphs, a Turiyam Neutrosophic apex graph may have multiple apex vertices. Additionally, if ψ_T is the null Turiyam Neutrosophic graph (i.e., $V = \emptyset$), it is trivially a Turiyam Neutrosophic apex graph.

2.6.4 Turiyam Neutrosophic Apex Outerplanar Graph

The Turiyam Neutrosophic Apex Outerplanar Graph is an extension of the Apex Outerplanar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 134. A *Turiyam Neutrosophic Apex Outerplanar Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be transformed into a Turiyam Neutrosophic outerplanar graph by removing a single vertex $v \in V$.

Formally, ψ_T is called a *Turiyam Neutrosophic Apex Outerplanar Graph* if there exists a vertex $v \in V$ such that the induced Turiyam Neutrosophic subgraph $\psi'_T = (V \setminus \{v\}, t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B)$, where $t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B$ are the restrictions of $t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B$ to the remaining vertices, is a Turiyam Neutrosophic outerplanar graph.

2.6.5 Turiyam Neutrosophic Quasi-Planar Graph

The Turiyam Neutrosophic Quasi planar Graph is an extension of the Quasi planar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 135. A *Turiyam Neutrosophic Quasi-Planar Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be drawn in the plane such that no three Turiyam Neutrosophic edges are pairwise crossing.

Formally, ψ_T is called a *Turiyam Neutrosophic Quasi-Planar Graph* if there exists a drawing D in which any set of three Turiyam Neutrosophic edges do not all intersect each other at distinct points in their interiors.

Definition 136. A *Turiyam Neutrosophic k-Quasi-Planar Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be drawn in the plane such that no k Turiyam Neutrosophic edges are pairwise crossing.

Formally, let

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

be a Turiyam Neutrosophic graph where:

- $t_A : V \to [0,1]$, $iv_A : V \to [0,1]$, $fv_A : V \to [0,1]$, and $lv_A : V \to [0,1]$ represent the truth, indeterminacy, falsity, and liberal values for each vertex, respectively, satisfying $0 \le t_A(v) + iv_A(v) + fv_A(v) + lv_A(v) \le 1$.
- $t_B : V \times V \to [0,1]$, $iv_B : V \times V \to [0,1]$, $fv_B : V \times V \to [0,1]$, and $lv_B : V \times V \to [0,1]$ represent the truth, indeterminacy, falsity, and liberal values for each edge (u, v), satisfying $t_B(u,v) + iv_B(u,v) + fv_B(u,v) + lv_B(u,v) \le 1$ and $t_B(u,v) \le \min\{t_A(u), t_A(v)\}$.

The Turiyam Neutrosophic graph ψ_T is called a *Turiyam Neutrosophic k-Quasi-Planar Graph* if there exists a drawing D in which any set of k Turiyam Neutrosophic edges do not all intersect each other at distinct points in their interiors.

2.6.6 Turiyam Neutrosophic Quasi-Outerplanar Graph

The Turiyam Neutrosophic Quasi Outerplanar Graph is an extension of the Quasi Outerplanar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 137. A *Turiyam Neutrosophic Quasi-Outerplanar Graph* $\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$ is a Turiyam Neutrosophic graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no three Turiyam Neutrosophic edges are pairwise crossing.

Formally, ψ_T is called a *Turiyam Neutrosophic Quasi-Outerplanar Graph* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of three Turiyam Neutrosophic edges do not all intersect each other at distinct points in their interiors.

Definition 138. A *Turiyam Neutrosophic Outer-k-Quasi-Planar Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no set of k Turiyam Neutrosophic edges are pairwise crossing.

Formally, the Turiyam Neutrosophic graph ψ_T is called a *Turiyam Neutrosophic Outer-k-Quasi-Planar Graph* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of k Turiyam Neutrosophic edges do not all intersect each other at distinct points in their interiors.

2.6.7 Turiyam Neutrosophic Quasi Apex Graph

The Turiyam Neutrosophic Quasi Apex Graph is an extension of the Quasi Apex Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 139. A *Turiyam Neutrosophic k-Quasi Apex Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be transformed into a Turiyam Neutrosophic planar graph by removing at most k vertices.

Formally, ψ_T is called a *Turiyam Neutrosophic k-Quasi Apex Graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced Turiyam Neutrosophic subgraph $\psi'_T = (V \backslash S, t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B)$, where $t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B$ are the restrictions of $t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B$ to the remaining vertices, is a Turiyam Neutrosophic quasi planar graph.

2.6.8 Turiyam Neutrosophic Quasi Apex Outerplanar Graph

The Turiyam Neutrosophic Quasi Apex Outerplanar Graph is an extension of the Quasi Apex Outerplanar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 140. A *Turiyam Neutrosophic k-Quasi Apex Outerplanar Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be transformed into a Turiyam Neutrosophic Quasi outerplanar graph by removing at most k vertices.

Formally, ψ_T is called a *Turiyam Neutrosophic k-Quasi Apex Outerplanar Graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced Turiyam Neutrosophic subgraph

$$\psi'_T = (V \setminus S, t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B)$$

, where

$$t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B$$

are the restrictions of

$$t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B$$

to the remaining vertices, is a Turiyam Neutrosophic quasi outerplanar graph.

2.6.9 Turiyam Neutrosophic almost planar graph

The definitions of a Turiyam Neutrosophic almost planar graph and an Turiyam Neutrosophic almost outerplanar graph are extended to the context of Turiyam Neutrosophic graphs as follows.

Definition 141. A *Turiyam Neutrosophic graph* $G_T = (V, E, \sigma_T, \sigma_{IV}, \sigma_F, \sigma_{LV})$ is called a *Turiyam Neutrosophic almost planar graph* if there exists an edge $e \in E(G_T)$ such that the Turiyam Neutrosophic graph $G_T - e$ (i.e., the Turiyam Neutrosophic graph obtained by removing the edge e from G_T) is a Turiyam Neutrosophic planar graph.

Definition 142. A *Turiyam Neutrosophic graph* $G_T = (V, E, \sigma_T, \sigma_{IV}, \sigma_F, \sigma_{LV})$ is called a *Turiyam Neutrosophic almost outerplanar graph* if there exists an edge $e \in E(G_T)$ such that the Turiyam Neutrosophic graph $G_T - e$ (i.e., the Turiyam Neutrosophic graph obtained by removing the edge e from G_T) is a Turiyam Neutrosophic outerplanar graph.

Theorem 143. *An Turiyam Neutrosophic almost planar graph can be transformed into an neutrosophic almost planar graph.*

Proof. Obviously holds. □

Theorem 144. *An Turiyam Neutrosophic almost outerplanar graph can be transformed into an neutrosophic almost outerplanar graph.*

Proof. Obviously holds. □

2.7 Plithogenic Graphs and Planar graphs

Recently, Plithogenic Graphs have been proposed as a generalization of Fuzzy Graphs and Turiyam Neutrosophic Graphs, as well as a graphical representation of Plithogenic Sets [4, 264, 449, 488, 491, 496, 506]. Plithogenic Graphs have been developed and are currently being actively studied [239,241,242,321,489,506,511] The definition is provided below.

Definition 145. [511] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

1. *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:
   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the range of possible attribute values.
   - $adf : M \times Ml \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
   - $aCf : Ml \times Ml \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.
2. *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
   - $bCf : Nm \times Nm \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. *Edge Appurtenance Constraint*: For all $(x, a), (y, b) \in M \times Ml$:

   $$bdf\ ((xy), (a, b)) \leq \min\{adf(x, a), adf(y, b)\}$$

   where $xy \in N$ is an edge between vertices x and y, and $(a, b) \in Nm \times Nm$ are the corresponding attribute values.
2. *Contradiction Function Constraint*: For all $(a, b), (c, d) \in Nm \times Nm$:

   $$bCf\ ((a, b), (c, d)) \leq \min\{aCf(a, c), aCf(b, d)\}$$

3. *Reflexivity and Symmetry of Contradiction Functions*:

   $$\begin{aligned} &aCf(a, a) = 0, && \forall a \in Ml \\ &aCf(a, b) = aCf(b, a), && \forall a, b \in Ml \\ &bCf(a, a) = 0, && \forall a \in Nm \\ &bCf(a, b) = bCf(b, a), && \forall a, b \in Nm \end{aligned}$$

Example 146. [235]

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic Turiyam Neutrosophic Graph*.

2.7.1 Plithogenic Planar Graph

The Plithogenic Planar Graph is an extension of the Quasi Planar Graph, incorporating the concepts of Plithogenic Graphs. The definition is provided below.

Definition 147 (Plithogenic Planar Graph)**.** A *Plithogenic Planar Graph* $PG = (PM, PN)$ is a plithogenic graph that can be embedded in the plane without any edge intersections, except at their endpoints.

Formally, let $PG = (PM, PN)$ be defined as:

- $PM = (M, l, Ml, adf, aCf)$ represents the plithogenic vertex set.
- $PN = (N, m, Nm, bdf, bCf)$ represents the plithogenic edge set.

For any intersection point P of two plithogenic edges e_1 and e_2, the degree of intersection at P, denoted by

$$S_P = \left(\frac{bdf(e_1) + bdf(e_2)}{2}, \frac{aCf(e_1, e_2) + aCf(e_2, e_1)}{2} \right),$$

must satisfy:

$$S_P = (0, 0) \quad \text{for all points of intersection.}$$

Thus, a Plithogenic Planar Graph PG is defined such that the total number of intersections remains zero.

Theorem 148. *A Plithogenic Planar Graph PG can transform into a Neutrosophic Planar Graph when $s = 3, t = 1$, and into a Fuzzy Planar Graph when $s = 1, t = 1$.*

Proof. We prove the theorem in two parts, corresponding to the transformations into a Neutrosophic Planar Graph and a Fuzzy Planar Graph.

A Plithogenic Planar Graph $PG = (PM, PN)$ is defined as:

- $PM = (M, l, Ml, adf, aCf)$, where M is the vertex set, adf is the degree of appurtenance function, and aCf is the degree of contradiction function.
- $PN = (N, m, Nm, bdf, bCf)$, where N is the edge set, bdf is the degree of appurtenance function for edges, and bCf is the degree of contradiction function for edges.

When $s = 3$ and $t = 1$, the Plithogenic Planar Graph transforms into a Neutrosophic Planar Graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$, where:

- $\sigma_1, \sigma_2, \sigma_3$ correspond to the truth, indeterminacy, and falsity membership values derived from adf and aCf.
- μ_1, μ_2, μ_3 correspond to the truth, indeterminacy, and falsity membership values derived from bdf and bCf.

The Neutrosophic Planar Graph retains the property that it can be embedded in the plane without any intersecting edges, as $S_P = (0, 0)$ for all intersection points in PG. Thus, the transformation preserves planarity in the neutrosophic context.

When $s = 1$ and $t = 1$, the Plithogenic Planar Graph reduces to a Fuzzy Planar Graph $\psi = (V, \sigma, \mu, E)$, where:

- $\sigma : V \to [0, 1]$ represents the membership degree of each vertex.
- $\mu : E \to [0, 1]$ represents the membership degree of each edge.

The appurtenance and contradiction functions adf, bdf, aCf, and bCf collapse into single membership functions σ and μ. The fuzzy graph satisfies the condition that it can be embedded in the plane without intersecting edges. Specifically, for any intersection point P, the intersection strength I_P satisfies:

$$I_P = 0 \quad \text{for all points of intersection.}$$

This ensures that the fuzzy planarity value $f = 1$, confirming the graph's planarity.

The Plithogenic Planar Graph generalizes both Neutrosophic and Fuzzy Planar Graphs. By adjusting the parameters s and t, the representation transitions into the respective forms, while maintaining the structural property of planarity in all contexts. □

2.7.2 Plithogenic Outerplanar Graph

The Plithogenic Outerplanar Graph is an extension of the Outerplanar Graph, incorporating the concepts of Plithogenic Graphs. The definition is provided below.

Definition 149 (Plithogenic Outerplanar Graph)**.** A *Plithogenic Outerplanar Graph* $PG = (PM, PN)$ is a plithogenic graph that can be embedded in the plane such that all vertices lie on the boundary of the exterior region, and there are no intersecting edges except at their endpoints.

Formally, let $PG = (PM, PN)$ be defined as:

- The vertex set $PM = (M, l, Ml, adf, aCf)$ has every vertex $m \in M$ located on the outer boundary.
- The edge set $PN = (N, m, Nm, bdf, bCf)$ ensures that any intersection S_P between edges satisfies $S_P = (0, 0)$, indicating no intersections.

Therefore, all vertices are on the outer face, making PG an outerplanar graph in the plithogenic context.

Theorem 150. *A Plithogenic Outerplanar Graph can transform into a Neutrosophic Outerplanar Graph when* $s = 3, t = 1$*, and into a Fuzzy Outerplanar Graph when* $s = 1, t = 1$*.*

Proof. A Plithogenic Outerplanar Graph $PG = (PM, PN)$ is defined as:

- $PM = (M, l, Ml, adf, aCf)$, where M is the vertex set, adf is the degree of appurtenance function, and aCf is the degree of contradiction function.
- $PN = (N, m, Nm, bdf, bCf)$, where N is the edge set, bdf is the degree of appurtenance function for edges, and bCf is the degree of contradiction function for edges.

By setting $s = 3$ and $t = 1$, the Plithogenic Outerplanar Graph becomes a Neutrosophic Outerplanar Graph:

$$NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3)),$$

where:

- $\sigma_1, \sigma_2, \sigma_3$ correspond to the membership, indeterminacy, and contradiction values derived from adf and aCf.
- μ_1, μ_2, μ_3 correspond to the membership, indeterminacy, and contradiction values derived from bdf and bCf.

The Neutrosophic Outerplanar Graph inherits the condition that all vertices lie on the boundary of the outer face, and no edges intersect except at their endpoints. Thus, the transformation preserves the outerplanarity property.

When $s = 1$ and $t = 1$, the Plithogenic Outerplanar Graph reduces to a Fuzzy Outerplanar Graph:

$$\psi = (V, \sigma, \mu),$$

where:

- $\sigma : V \to [0, 1]$ represents the membership degree of each vertex.
- $\mu : E \to [0, 1]$ represents the membership degree of each edge.

The appurtenance and contradiction functions adf, bdf, aCf, and bCf collapse into a single membership function for vertices and edges. The condition of outerplanarity remains satisfied because the structural constraints of Plithogenic Outerplanar Graphs ensure that:

$$i(\psi) = 0,$$

indicating that all vertices are on the boundary of the outer face, with no intersecting edges.

The Plithogenic Outerplanar Graph generalizes both Neutrosophic and Fuzzy Outerplanar Graphs. By adjusting the parameters s and t, the graph's representation transitions into the corresponding forms while preserving the structural property of outerplanarity. □

2.7.3 Plithogenic Apex Graph

The Plithogenic Apex Graph is an extension of the Apex Graph, incorporating the concepts of Plithogenic Graphs. The definition is provided below.

Definition 151 (Plithogenic Apex Planar Graph)**.** A *Plithogenic Apex Planar Graph* $PG = (PM, PN)$ is a plithogenic graph that can be made a Plithogenic Planar Graph by removing a single vertex $v \in M$.

Formally, PG is called a *Plithogenic Apex Planar Graph* if there exists a vertex $v \in M$ such that the induced subgraph $PG' = (PM', PN')$, where:

- $PM' = (M \setminus \{v\}, l', Ml', adf', aCf')$
- $PN' = (N', m', Nm', bdf', bCf')$

is a Plithogenic Planar Graph. The functions adf', aCf', bdf', bCf' are the restrictions of

$$adf, aCf, bdf, bCf$$

to the remaining vertices and edges, ensuring no edge crossings in the induced subgraph.

2.7.4 Plithogenic Apex Outerplanar Graph

The Plithogenic Apex Outerplanar Graph is an extension of the Apex Outerplanar Graph, incorporating the concepts of Plithogenic Graphs. The definition is provided below.

Definition 152 (Plithogenic Apex Outerplanar Graph)**.** A *Plithogenic Apex Outerplanar Graph* $PG = (PM, PN)$ is a plithogenic graph that can be made a Plithogenic Outerplanar Graph by removing a single vertex $v \in M$.

Formally, PG is called a *Plithogenic Apex Outerplanar Graph* if there exists a vertex $v \in M$ such that the induced subgraph $PG' = (PM', PN')$, where:

- $PM' = (M \setminus \{v\}, l', Ml', adf', aCf')$
- $PN' = (N', m', Nm', bdf', bCf')$

is a Plithogenic Outerplanar Graph. The functions adf', aCf', bdf', bCf' are the restrictions of

$$adf, aCf, bdf, bCf$$

to the remaining vertices and edges, ensuring the subgraph has all vertices lying on the boundary of the exterior region with no edge intersections.

Theorem 153. *A Plithogenic Apex Planar Graph PG with $s = 4, t = 1$ can transform into a Turiyam Neutrosophic Apex Planar Graph, and a Plithogenic Apex Outerplanar Graph PG with $s = 4, t = 1$ can transform into a Turiyam Neutrosophic Apex Outerplanar Graph.*

Proof. We prove both cases separately.

Case 1: Transformation to a Turiyam Neutrosophic Apex Planar Graph A Plithogenic Apex Planar Graph $PG = (PM, PN)$ is defined as:

- $PM = (M, l, Ml, adf, aCf)$, representing the plithogenic vertex set.
- $PN = (N, m, Nm, bdf, bCf)$, representing the plithogenic edge set.

When $s = 4$ and $t = 1$, the Plithogenic Apex Planar Graph transforms into a Turiyam Neutrosophic Apex Planar Graph $\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$, where:

- The attributes l, adf, aCf of vertices in PM are mapped to the Turiyam Neutrosophic truth, indeterminacy, falsity, and liberal membership degrees t_A, iv_A, fv_A, lv_A, respectively.
- The attributes m, bdf, bCf of edges in PN are mapped to the Turiyam Neutrosophic truth, indeterminacy, falsity, and liberal membership degrees t_B, iv_B, fv_B, lv_B, respectively.

By definition, a Plithogenic Apex Planar Graph becomes a Plithogenic Planar Graph when a single vertex $v \in M$ is removed. This property carries over to the Turiyam Neutrosophic domain, where removing a single vertex $v \in V$ results in a Turiyam Neutrosophic Planar Graph ψ'_T. Therefore, the transformation preserves the apex property.

Case 2: Transformation to a Turiyam Neutrosophic Apex Outerplanar Graph Similarly, a Plithogenic Apex Outerplanar Graph $PG = (PM, PN)$ satisfies the condition that removing a single vertex $v \in M$ results in a Plithogenic Outerplanar Graph PG'. When $s = 4$ and $t = 1$, the Plithogenic Apex Outerplanar Graph transforms into a Turiyam Neutrosophic Apex Outerplanar Graph $\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$ by mapping the plithogenic membership functions adf, aCf, bdf, bCf to the Turiyam Neutrosophic dimensions.

The outerplanar property is preserved under this transformation because removing a single vertex $v \in V$ results in a Turiyam Neutrosophic Outerplanar Graph ψ'_T, where all remaining vertices lie on the boundary of the exterior face, and no edge intersections occur.

Both transformations rely on the mapping of plithogenic attributes to Turiyam Neutrosophic dimensions. The structural properties (planarity and outerplanarity) and the apex property are preserved during these transformations, completing the proof. □

3. Result of this paper

3.1 Some property of Turiyam Neutrosophic Planar Graph

We will examine the properties of a Turiyam Neutrosophic Planar Graph. Similar characteristics can also be observed in other graphs such as fuzzy planar graphs, Neutrosophic planar graphs, and Plithogenic Planar Graphs.

Theorem 154. *If a Turiyam Neutrosophic graph G_T is homomorphic to the complete graph K_4, then G_T is a Turiyam Neutrosophic planar graph.*

Proof. Let $G_T = (V, E, \sigma_T, \sigma_{IV}, \sigma_F, \sigma_{LV})$ be a Turiyam Neutrosophic graph, where:

- $\sigma_T : V \to [0,1]$, $\sigma_{IV} : V \to [0,1]$, $\sigma_F : V \to [0,1]$, and $\sigma_{LV} : V \to [0,1]$ represent the truth, indeterminacy, falsity, and liberal membership degrees for each vertex $v \in V$.
- $\mu_T : E \to [0,1]$, $\mu_{IV} : E \to [0,1]$, $\mu_F : E \to [0,1]$, and $\mu_{LV} : E \to [0,1]$ represent the truth, indeterminacy, falsity, and liberal membership degrees for each edge $e \in E$.

Assume G_T is homomorphic to the complete graph K_4. This means there exists a mapping $\phi : V(G_T) \to V(K_4)$ such that if $(u, v) \in E(G_T)$, then $(\phi(u), \phi(v)) \in E(K_4)$.

Since K_4 is planar, this homomorphism implies that G_T inherits a planar structure. Thus, G_T can be embedded in the plane in a manner where no two edges intersect except at their endpoints.

Therefore, G_T is a Turiyam Neutrosophic planar graph by definition. Additionally, the Turiyam Neutrosophic planarity value $f_T = (1, 1, 1, 1)$, indicating the highest degree of planarity. □

Theorem 155. *If a Turiyam Neutrosophic graph G_T is homomorphic to a tree graph, then G_T is a Turiyam Neutrosophic planar graph.*

Proof. The proof is analogous to the method used in the previous proof. □

Theorem 156. *If a Turiyam Neutrosophic graph G_T is homomorphic to a cycle graph, grid graph, or wheel graph, then G_T is a Turiyam Neutrosophic planar graph.*

Proof. Let $G_T = (V, E, \sigma_T, \sigma_{IV}, \sigma_F, \sigma_{LV})$ be a Turiyam Neutrosophic graph, where:

- V is the set of vertices, and E is the set of edges.
- σ_T, σ_{IV}, σ_F, and σ_{LV} represent the truth, indeterminacy, falsity, and liberal membership degrees for vertices and edges.

We consider three cases:

1. *Cycle Graph:* A cycle graph is planar since it can be drawn as a simple closed loop without intersecting edges[431, 521, 536]. If G_T is homomorphic to a cycle graph, then a mapping exists that preserves the structure, making G_T planar.
2. *Grid Graph:* A grid graph forms a rectangular lattice and is also planar. The homomorphism from G_T to the grid graph implies G_T inherits the planar characteristics, ensuring no edge intersections[278, 294, 306, 424].
3. *Wheel Graph:* A wheel graph consists of a central vertex connected to all vertices of an outer cycle[354, 504]. This graph can be drawn without any intersecting edges. Therefore, a homomorphism from G_T to a wheel graph means G_T is also planar.

In all cases, G_T can be embedded in the plane without intersecting edges, confirming that G_T is a Turiyam Neutrosophic planar graph. □

Theorem 157. *Let G_T be a Turiyam Neutrosophic graph. If G_T is homomorphic to any of the following graph classes:*

- *Halin graphs (also called skirted trees or roofless polyhedra),*
- *Butterfly graphs (also known as bowtie or hourglass graphs),*
- *Herschel graph,*
- *Polyhedral graph,*
- *Golomb graph,*
- *Tutte graph,*
- *Bull graph,*
- *Goldner–Harary graph,*
- *Nested triangles graph,*
- *Frucht graph,*
- *Friendship graphs,*
- *26-fullerene graph,*
- *Antiprism graph,*
- *Archimedean graph,*
- *Wiener–Araya graph,*
- *Araya–Wiener graph,*
- *Goddard–Henning graph,*
- *Heawood Four-Color graph,*
- *Kittell graph,*
- *Poussin graph,*
- *Cactus graph,*
- *Fritsch graph,*
- *Soifer graph,*
- *Ladder graphs,*

then G_T is a Turiyam Neutrosophic planar graph.

Proof. The proof follows by leveraging the planar properties of the listed graph classes.

Planarity of Listed Graph Classes:

- *Halin graphs* are known to be planar as they are constructed by adding a cycle to the leaves of a rooted tree [113, 167, 270].
- *Butterfly graphs* (or bowtie/hourglass graphs) are planar due to their simple structure of two triangles sharing a common vertex [302, 303].
- *The Herschel graph*, being a planar graph, has been studied in various applications [508].
- *Polyhedral graphs* represent the skeletons of convex polyhedra, which are planar by definition [81, 340].
- *Golomb graphs* are planar due to their simple combinatorial construction [500].
- *Tutte graphs*, as described in graph theory literature, are planar [542].
- *Bull graphs*, representing a triangle with two disjoint pendant edges, are planar [162, 359].
- *Goldner–Harary graphs* are maximal planar graphs and thus planar by construction [262].
- *Nested triangles graphs* have a planar embedding due to their layered triangular structure [229].
- *Frucht graphs* are planar as they are minimal vertex-transitive graphs [438].
- *Friendship graphs* (or windmill graphs) are planar since they consist of triangles sharing a common vertex [47, 535].
- *The 26-fullerene graph* is planar by definition as a fullerene graph [272].
- *Antiprism graphs* are planar due to their combinatorial structure [288, 513].
- *Archimedean graphs* are planar since they represent tilings of the Euclidean plane [225].
- *Wiener–Araya graphs* and *Araya–Wiener graphs* are planar as they extend planar cubic graphs [368, 480].
- *Goddard–Henning graphs* maintain planarity as part of their structural properties [261].
- *The Heawood Four-Color graph* is planar due to its association with the Four-Color Theorem [253].
- *Kittell graphs* and *Poussin graphs* are planar due to their simple construction [339, 546].
- *Cactus graphs*, composed of edge-disjoint cycles, are planar [91, 433].
- *Fritsch and Soifer graphs* are planar examples studied in the context of coloring problems [253].
- *Ladder graphs*, as simple planar grid graphs, are also planar [357, 402].

Since G_T is homomorphic to these graph classes, it inherits their planarity. Hence, G_T is a Turiyam Neutrosophic planar graph. For more detailed references and properties of each graph class, see the cited works. □

Theorem 158. *If a Turiyam Neutrosophic graph G_T is homomorphic to a complete graph K_5, then G_T is a Non-Turiyam Neutrosophic planar graph.*

Proof. We will prove this by contradiction. Assume, to the contrary, that a Turiyam Neutrosophic graph G_T is homomorphic to the complete graph K_5, but G_T is a Turiyam Neutrosophic planar graph.

By definition, a Turiyam Neutrosophic planar graph can be embedded in the plane such that no two edges intersect except at their endpoints. However, the complete graph K_5 is known to be non-planar, as it is impossible to draw K_5 in the plane without intersecting edges.

If G_T is homomorphic to K_5, there exists a mapping $\phi : V(G_T) \to V(K_5)$ such that for every edge $(u, v) \in E(G_T)$, the edge $(\phi(u), \phi(v)) \in E(K_5)$. This implies that the structure of G_T must follow that of K_5.

Since K_5 is non-planar, the graph G_T, being homomorphic to K_5, must also be non-planar. This contradicts our assumption that G_T is a Turiyam Neutrosophic planar graph.

Therefore, our assumption is false, and we conclude that G_T is a Non-Turiyam Neutrosophic planar graph. □

Theorem 159. *If a Turiyam Neutrosophic graph G_T is homomorphic to the complete bipartite graph $K_{3,3}$, then G_T is a Non-Turiyam Neutrosophic planar graph.*

Proof. We will prove this by contradiction. Assume, to the contrary, that a Turiyam Neutrosophic graph G_T is homomorphic to the complete bipartite graph $K_{3,3}$, but G_T is a Turiyam Neutrosophic planar graph.

By definition, a Turiyam Neutrosophic planar graph can be embedded in the plane such that no two edges intersect except at their endpoints. However, it is well-known from Kuratowski's Theorem that the complete bipartite graph $K_{3,3}$ is non-planar, meaning it is impossible to draw $K_{3,3}$ in the plane without intersecting edges.

If G_T is homomorphic to $K_{3,3}$, there exists a mapping $\phi : V(G_T) \to V(K_{3,3})$ such that for every edge $(u, v) \in E(G_T)$, the edge $(\phi(u), \phi(v)) \in E(K_{3,3})$. This implies that G_T inherits the structural relationships of $K_{3,3}$ and must follow its non-planar nature.

Since $K_{3,3}$ is non-planar, the graph G_T, being homomorphic to $K_{3,3}$, must also be non-planar. This contradicts our assumption that G_T is a Turiyam Neutrosophic planar graph.

Therefore, our assumption is false, and we conclude that G_T is a Non-Turiyam Neutrosophic planar graph. □

3.2 Some property of Neutrosophic outerplanar graph

We will examine some properties of the Neutrosophic outerplanar graph. It should be noted that the same properties hold when Neutrosophic is replaced with Fuzzy or Plithogenic. The following theorem holds.

Theorem 160. *A neutrosophic outerplanar graph $G = (A, B, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ is a neutrosophic planar graph.*

Proof. By definition, a neutrosophic outerplanar graph $G = (A, B, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ is embedded in the plane such that all vertices lie on the boundary of the exterior region, and no two edges intersect except at their endpoints.

We verify that G satisfies the conditions of a neutrosophic planar graph:

1. *Verification of Intersection Properties:* By definition, the neutrosophic outerplanar graph does not have edge crossings within the interior of the graph. Thus, the graph satisfies the conditions for planarity.

2. *Calculation of Neutrosophic Planarity Value:* For a neutrosophic planar graph, we consider the intersection points $P_1, P_2, \ldots, P_z$ between edges, and the neutrosophic planarity value $f = (f_T, f_I, f_F)$ is given by

$$f = \left(\frac{1}{1 + \{(S_T)_{P_1} + (S_T)_{P_2} + \ldots + (S_T)_{P_z}\}}, \frac{1}{1 + \{(S_I)_{P_1} + (S_I)_{P_2} + \ldots + (S_I)_{P_z}\}}, \frac{1}{1 + \{(S_F)_{P_1} + (S_F)_{P_2} + \ldots + (S_F)_{P_z}\}} \right).$$

Since a neutrosophic outerplanar graph has no intersections except at endpoints, $(S_T)_{P_i} = 0$, $(S_I)_{P_i} = 0$, and $(S_F)_{P_i} = 0$ for all P_i. Therefore,

$$f = \left(\frac{1}{1+0}, \frac{1}{1+0}, \frac{1}{1+0} \right) = (1, 1, 1).$$

This value $f = (1, 1, 1)$ confirms that the graph is completely planar.

Since the neutrosophic outerplanar graph satisfies the conditions of a neutrosophic planar graph, we conclude that every neutrosophic outerplanar graph is indeed a neutrosophic planar graph. □

Theorem 161. *A neutrosophic graph G is a neutrosophic outerplanar graph if and only if it contains no neutrosophic subgraph homomorphic to K_4 or $K_{2,3}$.*

Proof. First, note that K_4 and $K_{2,3}$ are not neutrosophic outerplanar graphs, as their structure does not allow all vertices to lie on the boundary of an exterior face without crossings, even when considering neutrosophic parameters.

(Necessity) Suppose G is a neutrosophic graph that is outerplanar. Then, by definition, G can be embedded in a plane such that all vertices lie on the boundary of a single unbounded face without intersecting edges. If G contained a subgraph homomorphic to K_4 or $K_{2,3}$, such subgraphs would inherently require intersecting edges in any planar embedding, contradicting the outerplanar property. Therefore, G contains no neutrosophic subgraph homomorphic to K_4 or $K_{2,3}$.

(Sufficiency) Conversely, assume G is a neutrosophic graph containing no subgraph homomorphic to K_4 or $K_{2,3}$. By Kuratowski's Theorem, since G does not have a K_4 or $K_{2,3}$ subgraph, G must be planar. Assume for contradiction that G is not a neutrosophic outerplanar graph.

Let G be embedded such that it is a cyclic block not satisfying the outerplanar property. Suppose the embedding is such that the exterior region contains the maximum number of vertices. Let C be the boundary cycle of this exterior region. Since G is not outerplanar, there exists at least one vertex d lying in the interior of C.

If there exists a vertex d in the interior connected by three mutually disjoint paths to three distinct vertices on C, then G contains a subgraph homomorphic to K_4. Alternatively, if d has two disjoint paths connecting it to two disjoint vertices a and e on C, and the edge ae is not on C, then G contains a subgraph homomorphic to $K_{2,3}$. Both cases lead to contradictions since we assumed G contained no such subgraphs.

Thus, G is a neutrosophic outerplanar graph, completing the proof. □

Theorem 162. *If ψ is a connected neutrosophic outerplanar graph, then it has a neutrosophic dual graph ψ^* such that there exists a vertex v for which $\psi^* - v$ contains no neutrosophic cycle.*

Proof. Let $\psi = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a connected neutrosophic outerplanar graph with no intersections of edges $E' \subseteq E$ within the graph. We can divide the neutrosophic graph ψ into a finite number of regions. Each of these regions corresponds to a neutrosophic face $F_1, F_2, F_3, \ldots, F_k$.

For each face F_j, its membership value is determined by

$$F_j = \min\left\{\frac{\mu_T(x,y)}{\sigma_T(x) \wedge \sigma_T(y)}, \frac{\mu_I(x,y)}{\sigma_I(x) \wedge \sigma_I(y)}, \frac{\mu_F(x,y)}{\sigma_F(x) \wedge \sigma_F(y)} \mid \text{All edges bounding the region}\right\}.$$

When dividing the neutrosophic graph into such regions with neutrosophic face values, we construct the neutrosophic dual graph $\psi^* = (V^*, \sigma^*, \mu^*)$ as follows:

- Each face of ψ corresponds to a vertex in ψ^*.
- Each edge of ψ corresponds to an edge in ψ^*.

Since ψ is an outerplanar graph, the outer face forms a neutrosophic cycle. Therefore, the neutrosophic dual graph ψ^* exists for ψ.

Now, consider the removal of a vertex v from ψ^*. This corresponds to removing a face of the original graph ψ. The remaining graph $\psi^* - v$ represents the connections among the other faces of ψ but without forming any cycles since the outer face is removed.

Thus, $\psi^* - v$ contains no neutrosophic cycle, completing the proof. □

Theorem 163. *A neutrosophic outerplanar graph ψ always has a vertex-deleted neutrosophic outerplanar subgraph that is also a vertex-deleted neutrosophic subgraph of ψ.*

Proof. Let ψ be a neutrosophic outerplanar graph, and let H be a subgraph obtained by deleting a vertex from ψ. Since ψ is outerplanar, all vertices lie on the outer face in the neutrosophic embedding of the graph. Thus, removing any vertex $v \in V(\psi)$ from ψ does not affect the outerplanarity property of the graph. Therefore, the resulting subgraph $\psi - v$ is still a neutrosophic outerplanar graph.

Furthermore, because $H = \psi - v$ retains all other vertices and edges apart from the deleted vertex, H is a valid vertex-deleted neutrosophic subgraph of ψ. Thus, every neutrosophic outerplanar graph has a vertex-deleted subgraph that maintains its outerplanarity property and is also a valid subgraph of the original graph ψ, as required. □

Definition 164. A *Neutrosophic Forest* is a collection of one or more disjoint neutrosophic trees. Formally, a neutrosophic forest F is defined as a pair $F = (V, E)$, where:

- V is a finite set of vertices.
- $E \subseteq V \times V$ is a set of edges.

Each vertex $v \in V$ and each edge $e \in E$ is associated with neutrosophic membership values:

- The vertex membership function $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$ assigns to each vertex v:
  - $\sigma_T(v) \in [0, 1]$: Truth membership degree.
  - $\sigma_I(v) \in [0, 1]$: Indeterminacy membership degree.
  - $\sigma_F(v) \in [0, 1]$: Falsity membership degree.
- The edge membership function $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$ assigns to each edge $e = (u, v) \in E$:
  - $\mu_T(e) \in [0, 1]$: Truth membership degree.
  - $\mu_I(e) \in [0, 1]$: Indeterminacy membership degree.
  - $\mu_F(e) \in [0, 1]$: Falsity membership degree.

A neutrosophic forest satisfies the following properties:

1. Each connected component is a neutrosophic tree with no cycles.
2. The forest consists of disjoint neutrosophic trees.
3. For any vertex v, $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 1$, and for any edge e, $\mu_T(e) + \mu_I(e) + \mu_F(e) \leq 1$.

Theorem 165. *A Neutrosophic Forest is a Neutrosophic graph.*

Proof. Obviously holds. □

Theorem 166. *A neutrosophic forest is a neutrosophic outerplanar graph.*

Proof. Let $F = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a neutrosophic forest, where:

- V is the set of vertices.
- E is the set of edges.
- $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$ represents the truth, indeterminacy, and falsity membership degrees for each vertex $v \in V$.
- $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$ represents the truth, indeterminacy, and falsity membership degrees for each edge $e \in E$.

By definition, a neutrosophic forest is a collection of one or more disjoint neutrosophic trees, where each tree is acyclic and connected. This means that the overall structure of F contains no cycles.

To prove that F is a neutrosophic outerplanar graph, we need to show that:

1. F can be embedded in the plane such that all vertices lie on the boundary of the exterior region.
2. No two edges intersect except at their endpoints.

Since F consists of disjoint trees, each tree can be embedded in the plane without any cycles. All vertices of each tree can be arranged along a straight line or a circle, ensuring that every vertex lies on the boundary of the exterior face. Thus, no edge crossings occur, and all vertices lie on the outer face.

Therefore, F satisfies the conditions for being a neutrosophic outerplanar graph. Hence, every neutrosophic forest is a neutrosophic outerplanar graph. □

Corollary 167. *A neutrosophic tree is a neutrosophic outerplanar graph.*

Proof. Obviously holds. □

Theorem 168. *Let ψ be a connected neutrosophic outerplanar graph such that $f_\psi \neq 1$, and let W be a subset of its vertices such that $W \subseteq V$. The neutrosophic outerplanar subgraph of ψ obtained by deleting the vertices in W is $\psi' = \psi - W$. Its neutrosophic dual graph is denoted as ψ''.*

Proof. Consider a connected neutrosophic outerplanar graph $\psi = (V, E, \sigma, \mu)$ and a subset of vertices $W \subseteq V$. Let $\psi - W$ be the subgraph obtained by deleting the vertices in W, which we denote as ψ'. By construction, ψ' is a vertex-deleted neutrosophic outerplanar subgraph of ψ.

Since ψ is a neutrosophic outerplanar graph, the removal of vertices in W does not introduce any intersections or additional complexity that would violate the outerplanar property. Hence, ψ' retains the neutrosophic outerplanarity.

Next, we divide the neutrosophic graph ψ' into a finite number of regions, where each region corresponds to a neutrosophic face. The membership value for each face F_j is determined by

$$F_j = \min\left\{\frac{\mu_T(x,y)}{\sigma_T(x) \wedge \sigma_T(y)}, \frac{\mu_I(x,y)}{\sigma_I(x) \wedge \sigma_I(y)}, \frac{\mu_F(x,y)}{\sigma_F(x) \wedge \sigma_F(y)} \mid \text{all edges bounding the region}\right\}.$$

Since each neutrosophic outerplanar graph with a planarity value $f_\psi \neq 1$ has a corresponding neutrosophic dual graph, there exists a dual graph ψ'' for the vertex-deleted subgraph ψ'. In this dual graph ψ'', each face of the original graph ψ' corresponds to a vertex, and each edge in ψ' corresponds to an edge in ψ''.

Therefore, the neutrosophic dual graph ψ'' exists for the vertex-deleted neutrosophic outerplanar subgraph $\psi' = \psi - W$. This completes the proof. □

4. Conclusion and Future Works

This section presents the conclusion and outlines future works.

4.1 Conclusion in this paper

In this paper, we investigated outerplanar graphs and aim to clarify the relationships between planar graphs, fuzzy graphs, and neutrosophic graphs.

Given the extensive and continuously expanding body of literature on fuzzy mathematics, it is inevitable that similar concepts may emerge independently across different journals and periods. Nevertheless, we believe that efforts to unify these concepts are essential and will greatly contribute to advancing the field. We aim to conduct further investigations in this direction in the future(cf.[235]).

4.2 Future works: planar hypergraphs

In the future, we will explore planar hypergraphs. A hypergraph is a generalization of a graph in which edges, known as hyperedges, can connect any number of vertices, not just two. The formal definition of a hypergraph is given below[127].

Definition 169. [127] A hypergraph is a pair $H = (V(H), E(H))$, consisting of a nonempty set $V(H)$ of vertices and a set $E(H)$ of subsets of $V(H)$, referred to as the hyperedges of H. In this context, we consider only finite hypergraphs.

This hypergraph structure is valuable for modeling complex relationships in various fields, such as computer science and biology [221, 266, 267, 437]. Research on databases[44, 217, 304, 541] using hypergraphs and Hypergraph neural networks [133, 155, 205, 222, 249, 313, 364] are also actively being conducted. A planar hypergraph can be seen as the hypergraph counterpart of a planar graph, and its definition is provided below. It is important to note that a planar graph is a special case of a planar hypergraph where all edges consist of exactly two vertices.

Definition 170. (cf.[180, 208, 347, 545]) Let $H = (V, E)$ be a hypergraph, where V is a set of vertices and E is a set of hyperedges (subsets of V).

The bipartite representation of H, denoted by $B(H)$, is a bipartite graph with a vertex set $V \cup E$. In $B(H)$, a vertex $v \in V$ is adjacent to a vertex $e \in E$ if and only if $v \in e$ in the original hypergraph H.

A hypergraph H is called a *planar hypergraph* if its bipartite representation $B(H)$ is a planar graph. In other words, H is a planar hypergraph if $B(H)$ can be drawn on a plane without any edge crossings.

Theorem 171. *A planar hypergraph $H = (V, E)$ can be transformed into a planar graph by representing it as its bipartite graph $B(H)$.*

Proof. The bipartite representation $B(H)$ of a planar hypergraph H is planar by definition. $B(H)$ is constructed by treating V and E as two disjoint vertex sets and connecting a vertex $v \in V$ to an edge vertex $e \in E$ if and only if $v \in e$. Since H is planar, $B(H)$ can be drawn on a plane without edge crossings, satisfying the planar graph property. □

Definition 172 (Outerplanar Hypergraph)**.** Let $H = (V, E)$ be a hypergraph, where V is a set of vertices and E is a set of hyperedges (subsets of V). The bipartite representation of H, denoted by $B(H)$, is a bipartite graph with vertex set $V \cup E$. In $B(H)$, a vertex $v \in V$ is adjacent to a vertex $e \in E$ if and only if $v \in e$ in the original hypergraph H.

A hypergraph H is called an *outerplanar hypergraph* if its bipartite representation $B(H)$ is an outerplanar graph. In other words, H is an outerplanar hypergraph if $B(H)$ can be embedded in the plane such that all vertices lie on the boundary of the exterior face, and no two edges intersect except at their endpoints.

Theorem 173. *An outerplanar hypergraph $H = (V, E)$ can be transformed into an outerplanar graph by representing it as its bipartite graph $B(H)$.*

Proof. By definition, an outerplanar hypergraph H has a bipartite representation $B(H)$ that is outerplanar. $B(H)$ can be embedded in the plane such that all vertices lie on the boundary of the outer face, and no two edges intersect except at their endpoints. Hence, $B(H)$ is an outerplanar graph. □

Definition 174 (Apex Hypergraph)**.** A hypergraph $H = (V, E)$ is called an *apex hypergraph* if it can be made planar by removing a single vertex $v \in V$ from H. Formally, H is an apex hypergraph if there exists a vertex $v \in V$ such that the bipartite representation $B(H \setminus \{v\})$, where $H \setminus \{v\}$ denotes the hypergraph obtained by removing v and all hyperedges containing v, is a planar graph.

The properties of an apex hypergraph include:

- If the hypergraph H itself is planar, then any vertex can be considered an apex.

Theorem 175. *An apex hypergraph $H = (V, E)$ can be transformed into a planar hypergraph by removing a single vertex $v \in V$.*

Proof. By definition, an apex hypergraph H is a hypergraph for which there exists a vertex $v \in V$ such that the bipartite representation $B(H \setminus \{v\})$ is planar. Removing the vertex v and all hyperedges containing v eliminates all crossings introduced by v. The resulting hypergraph $H \setminus \{v\}$ is planar by the property of its bipartite representation. Hence, $H \setminus \{v\}$ is a planar hypergraph. □

Theorem 176. *An apex hypergraph $H = (V, E)$ can be transformed into an apex graph by representing it as its bipartite graph $B(H)$.*

Proof. An apex hypergraph H is a hypergraph for which there exists a vertex $v \in V$ such that the hypergraph $H \setminus \{v\}$, obtained by removing v and all hyperedges incident to v, is a planar hypergraph. The bipartite representation $B(H)$ of H is a graph with vertex sets $V \cup E$, where a vertex $v \in V$ is adjacent to $e \in E$ if $v \in e$. Removing v from $B(H)$ reduces the bipartite graph to a representation of $H \setminus \{v\}$, which is a planar graph. Thus, $B(H)$ can be transformed into an apex graph by removing the vertex v. □

Definition 177 (Apex Outerplanar Hypergraph)**.** A hypergraph $H = (V, E)$ is called an *apex outerplanar hypergraph* if it can be made outerplanar by removing a single vertex $v \in V$ from H. Formally, H is an apex outerplanar hypergraph if there exists a vertex $v \in V$ such that the bipartite representation $B(H \setminus \{v\})$, where $H \setminus \{v\}$ denotes the hypergraph obtained by removing v and all hyperedges containing v, is an outerplanar graph.

The properties of an apex outerplanar hypergraph include:

- If the hypergraph H itself is outerplanar, then any vertex can be considered an apex.

Theorem 178. *An apex outerplanar hypergraph $H = (V, E)$ can be transformed into an outerplanar hypergraph by removing a single vertex $v \in V$.*

Proof. By definition, an apex outerplanar hypergraph H is a hypergraph for which there exists a vertex $v \in V$ such that the bipartite representation $B(H \setminus \{v\})$ is outerplanar. Removing v and its associated hyperedges reduces $B(H)$ to an outerplanar graph. The resulting hypergraph $H \setminus \{v\}$ retains the outerplanar property in its bipartite representation, making it an outerplanar hypergraph. □

Theorem 179. *An apex outerplanar hypergraph $H = (V, E)$ can be transformed into an apex outerplanar graph by representing it as its bipartite graph $B(H)$.*

Proof. By definition, an apex outerplanar hypergraph H is a hypergraph for which there exists a vertex $v \in V$ such that the hypergraph $H \setminus \{v\}$, obtained by removing v and all hyperedges incident to v, is an outerplanar hypergraph. The bipartite representation $B(H)$ of H is a graph with vertex sets $V \cup E$, where a vertex $v \in V$ is adjacent to $e \in E$ if $v \in e$. Removing v from $B(H)$ reduces the bipartite graph to a representation of $H \setminus \{v\}$, which is an outerplanar graph. Thus, $B(H)$ can be transformed into an apex outerplanar graph by removing the vertex v. □

I anticipate that future research will increasingly apply concepts such as planar hypergraphs, outerplanar hypergraphs, apex hypergraphs, and apex outerplanar hypergraphs to structures like fuzzy hypergraphs and neutrosophic hypergraphs. Furthermore, we plan to investigate planar hypergraph neural networks, which are hypergraph neural networks constructed based on planar hypergraphs. In addition, we intend to explore super-hypergraph neural networks by applying the concept of superhypergraphs to neural networks.

4.3 Future works: planar and outerplanar superhypergraph

We will also explore planar, outerplanar, apex, and apex outerplanar superhypergraphs. Superhypergraphs [236, 240, 254, 280, 281, 492–495, 498] are well-established graph classes that bridge the concepts of graphs and hypergraphs. Future studies aim to investigate the mathematical structures and practical applications of planar, outerplanar, apex, and apex outerplanar superhypergraphs in greater depth.

Definition 180 (Extended Bipartite Representation of a SuperHypergraph)**.** The *Extended Bipartite Representation* of a superhypergraph SHG, denoted $B(SHG)$, is a bipartite graph (U, F) where:

- $U = V \cup E$, treating hyperedges as vertices.
- An edge $(u, e) \in F$ exists if:
  - $u \in V$ and $u \in e$ in SHG.
  - $u \in E$ and there is a higher-order relationship between hyperedge u and hyperedge e.

Definition 181 (Planar SuperHypergraph)**.** A *Planar SuperHypergraph SHG* is a superhypergraph whose extended bipartite representation $B(SHG)$ is a planar graph.

Definition 182 (Outerplanar SuperHypergraph)**.** An *Outerplanar SuperHypergraph SHG* is a superhypergraph whose extended bipartite representation $B(SHG)$ is an outerplanar graph.

Theorem 183. *Planar superhypergraphs and outerplanar superhypergraphs generalize planar hypergraphs and outerplanar hypergraphs, respectively.*

Proof. We will demonstrate that:

1. *Every Planar Hypergraph is a Planar SuperHypergraph.*

   A hypergraph is a special case of a superhypergraph where hyperedges are solely subsets of V, without any higher-order connections. For a planar hypergraph $H = (V, E)$, its bipartite representation $B(H)$ is planar.

   Viewing H as a superhypergraph SHG without superedges, the extended bipartite representation $B(SHG)$ is identical to $B(H)$. Therefore, SHG is a planar superhypergraph.

2. *Planar SuperHypergraphs Generalize Planar Hypergraphs.*

   Superhypergraphs introduce additional structures, such as superedges connecting hyperedges or subsets thereof, which are not present in standard hypergraphs. Planar superhypergraphs include all planar hypergraphs and extend them with these complex relationships, while maintaining planarity in their extended bipartite representations. Thus, planar superhypergraphs generalize planar hypergraphs.

3. *Every Outerplanar Hypergraph is an Outerplanar SuperHypergraph.*

   Similarly, for an outerplanar hypergraph $H = (V, E)$, its bipartite representation $B(H)$ is outerplanar. By considering H as a superhypergraph SHG without superedges, $B(SHG) = B(H)$, which is outerplanar. Hence, SHG is an outerplanar superhypergraph.

4. *Outerplanar SuperHypergraphs Generalize Outerplanar Hypergraphs.*

   Outerplanar superhypergraphs encompass all outerplanar hypergraphs and allow for more complex hyperedge structures while preserving outerplanarity in their extended bipartite representations. Therefore, they generalize outerplanar hypergraphs.

□

4.4 Future works: Semi-Directed Planar graph

In the future, I plan to define concepts such as Planar Graphs and Outerplanar Graphs within the context of Semi-Directed Graphs, as well as within frameworks like Fuzzy Semi-Directed Graphs, Neutrosophic Semi-Directed Graphs, Turiyam Neutrosophic Semi-Directed Graphs, and Plithogenic Semi-Directed Graphs.

The definition of a Semi-Directed Graph is provided as follows [111, 112, 171, 471].

Definition 184. Let V be a nonempty set of elements, called vertices or nodes. Let $E_1 \subseteq V \times V$ be a set of unordered pairs of vertices, referred to as undirected edges, and let $E_2 \subseteq V \times V$ be a set of ordered pairs of vertices, referred to as directed edges. A *Semi-Directed Graph* $G = (V, E_1, E_2)$ is a graph where the edge set $E = E_1 \cup E_2$, consisting of both undirected edges from E_1 and directed edges from E_2.

- *Undirected edges*: An edge $(u, v) \in E_1$ implies that there is an undirected connection between vertices u and v, meaning $(u, v) = (v, u)$.
- *Directed edges*: An edge $(a, b) \in E_2$ implies a directed connection from vertex a to vertex b, where $(a, b) \neq (b, a)$.

Although still in the conceptual stage, I have outlined the definition of a Semi-Directed Planar Graph below. By incorporating additional conditions for fuzzy, Neutrosophic, or Plithogenic elements, it is possible to extend this definition to fuzzy, Neutrosophic, or Plithogenic Semi-Directed Planar Graphs.

Definition 185. Let $G = (V, E_1, E_2)$ be a *Semi-Directed Graph* where V is a nonempty set of vertices, $E_1 \subseteq V \times V$ is the set of undirected edges, and $E_2 \subseteq V \times V$ is the set of directed edges. The graph G is called a *Semi-Directed Planar Graph* if there exists a drawing of G on a plane such that:

1. *Planarity condition*: The graph G can be embedded in the plane such that no two edges (whether undirected from E_1 or directed from E_2) intersect except at their endpoints.
2. *Semi-Directed edges*: Both undirected edges $(u, v) \in E_1$ and directed edges $(a, b) \in E_2$ must be drawn in such a way that their connections respect the planarity condition. Specifically:
   - For undirected edges (u, v), the edge is drawn as a straight or curved line between u and v without crossing other edges.
   - For directed edges (a, b), an arrow from a to b is drawn in a planar manner, ensuring no crossings with other edges.

The *Semi-Directed Planar Graph* satisfies the conditions of planarity for both directed and undirected edges simultaneously in a planar representation.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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General Plithogenic Soft Rough Graphs and Some Related Graph Classes

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Abstract: This study introduces and explores new concepts of Turiyam Neutrosophic Soft Graphs and General Plithogenic Soft Graphs. It examines models of uncertain graphs, including Fuzzy, Intuitionistic Fuzzy, Neutrosophic, Turiyam, and Plithogenic Graphs, all designed to handle uncertainty in various contexts. Turiyam Neutrosophic Graphs use four values: truth, indeterminacy, falsity, and liberal state. The General Plithogenic Graph serves as a broader generalization of the Plithogenic Graph. Additionally, we consider Turiyam Neutrosophic Soft Rough Graphs and General Plithogenic Soft Rough Graphs, which integrate the principles of Soft Graphs and Rough Graphs.

Keywords: Neutrosophic graph, Fuzzy graph, Plithogenic graphs, Soft graph, Rough graph

1. Introduction

1.1 Uncertain Graphs

Graph Theory is a branch of mathematics that studies graphs, which are structures used to model relationships between objects via vertices and edges [92]. To represent and manage real-world uncertainties, several mathematical frameworks have been developed, including Fuzzy Sets [313] and Neutrosophic Sets [267, 268].

This paper focuses on various models of uncertain graphs, such as Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs. Collectively referred to as uncertain graphs, these models generalize classical graph theory by incorporating uncertainty in different forms [4, 9, 14, 110, 111, 116, 119–122, 124, 125, 241, 289].

In addition to uncertain graph models, the underlying mathematical concepts of Fuzzy Sets and Neutrosophic Sets are foundational and widely studied [33–37, 82, 95–97, 101, 194, 196, 227, 267, 313, 314].

1.2 Soft Graphs and Rough Graphs

A *Soft Set* [20,22,199,310] (or *Soft Graph* [12,158]) is a mathematical tool designed to model uncertainty by associating parameters with elements, enabling more flexible decision-making processes. Research on soft graphs and soft sets, along with their variants, has been extensively conducted [86, 104, 135, 180, 204, 211].

A *Rough Set* [148,223–226] (or *Rough Graph* [75,88,159,211,292]) is another mathematical framework that approximates uncertain or imprecise data by defining lower and upper bounds for sets.

This paper explores Fuzzy Soft Graphs and Neutrosophic Soft Graphs, which extend the ideas of Fuzzy Graphs and Neutrosophic Graphs [270, 283, 284] by incorporating the flexibility of soft sets. Additionally, we examine Soft Rough Graphs [2, 212], which merge the principles of Soft Graphs and Rough Graphs.

1.3 Our Contribution

This study introduces and analyzes new concepts, including Turiyam Neutrosophic Soft Graphs and General Plithogenic Soft Graphs. Turiyam Neutrosophic Graphs are characterized by four parameters: truth, indeterminacy, falsity, and liberal state [113, 116, 126, 262]. Related concepts, such as Turiyam Neutrosophic Sets and Turiyam Neutrosophic Rings, have been investigated in prior studies [42, 116, 127, 128, 258–261, 263]. The General Plithogenic Graph extends the standard Plithogenic Graph [113].

Turiyam Neutrosophic Soft Graphs and General Plithogenic Soft Graphs combine the concept of soft sets with Turiyam Neutrosophic Graphs and General Plithogenic Graphs, respectively. Notably, the Turiyam Neutrosophic Set is a specific case of the Quadruple Neutrosophic Set, where the ”Contradiction” value is replaced with the ”Liberal” state [266].

Furthermore, this study considers Turiyam Neutrosophic Soft Rough Graphs and General Plithogenic Soft Rough Graphs, which integrate the features of Soft Graphs and Rough Graphs, providing a more comprehensive framework for uncertainty modeling.

1.4 The Structure of the Paper

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2. Preliminaries and definitions

In this section, we present a brief overview of the definitions and notations used throughout this paper.

2.1 Basic Graph Concepts

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [90, 91, 91, 92, 149, 304].

Definition 1 (Graph)**.** [92] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2 (Degree)**.** [92] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 3 (Subgraph)**.** [92] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

2.2 Uncertain Graph

This paper addresses Fuzzy, Intuitionistic Fuzzy, Neutrosophic, Turiyam, and Plithogenic concepts within the framework of Unified Uncertain Graphs. Note that Turiyam Neutrosophic Set is actually a particular case of the Quadruple Neutrosophic Set, by replacing ”Contradiction” with ”Liberal” [266].

Definition 4 (Unified Uncertain Graphs Framework)**.** (cf.[117]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, falsity, and other nuanced measures of uncertainty.

1. *Fuzzy Graph [131, 139, 185, 239, 241, 287, 303]:*
   - Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$.
   - Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$.
2. *Intuitionistic Fuzzy Graph (IFG) [6, 47, 79, 172, 206, 290, 295, 316]:*

- Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0,1]$ (degree of membership) and $\nu_A(v) \in [0,1]$ (degree of non-membership), such that $\mu_A(v) + \nu_A(v) \leq 1$.
- Each edge $e = (u,v) \in E$ is assigned two values: $\mu_B(u,v) \in [0,1]$ and $\nu_B(u,v) \in [0,1]$, with $\mu_B(u,v) + \nu_B(u,v) \leq 1$.

3. *Neutrosophic Graph [10, 15, 63, 150, 164, 182, 247, 275, 283]:*
   - Each vertex $v \in V$ is assigned a triplet $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where $\sigma_T(v), \sigma_I(v), \sigma_F(v) \in [0,1]$ and $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
   - Each edge $e = (u,v) \in E$ is assigned a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$.
4. *Turiyam Neutrosophic Graph [126–128]:*
   - Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where each component is in $[0,1]$ and $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.
   - Each edge $e = (u,v) \in E$ is similarly assigned a quadruple.
5. *Vague Graph [7, 8, 53–55, 235, 236, 249]:*
   - Each vertex $v \in V$ is assigned a pair $(\tau(v), \phi(v))$, where $\tau(v) \in [0,1]$ is the degree of truth-membership and $\phi(v) \in [0,1]$ is the degree of false-membership, with $\tau(v) + \phi(v) \leq 1$.
   - The grade of membership is characterized by the interval $[\tau(v), 1 - \phi(v)]$.
   - Each edge $e = (u,v) \in E$ is assigned a pair $(\tau(e), \phi(e))$, satisfying:

$$\tau(e) \leq \min\{\tau(u), \tau(v)\}, \quad \phi(e) \geq \max\{\phi(u), \phi(v)\}.$$

6. *Hesitant Fuzzy Graph [41, 142, 216, 222, 309]:*
   - Each vertex $v \in V$ is assigned a hesitant fuzzy set $\sigma(v)$, represented by a finite subset of $[0,1]$, denoted $\sigma(v) \subseteq [0,1]$.
   - Each edge $e = (u,v) \in E$ is assigned a hesitant fuzzy set $\mu(e) \subseteq [0,1]$.
   - Operations on hesitant fuzzy sets (e.g., intersection, union) are defined to handle the hesitation in membership degrees.
7. *Single-Valued Pentapartitioned Neutrosophic Graph [80, 166, 167, 232]:*
   - Each vertex $v \in V$ is assigned a quintuple $\sigma(v) = (T(v), C(v), R(v), U(v), F(v))$, where:
     - $T(v) \in [0,1]$ is the truth-membership degree.
     - $C(v) \in [0,1]$ is the contradiction-membership degree.
     - $R(v) \in [0,1]$ is the ignorance-membership degree.
     - $U(v) \in [0,1]$ is the unknown-membership degree.
     - $F(v) \in [0,1]$ is the false-membership degree.
     - $T(v) + C(v) + R(v) + U(v) + F(v) \leq 5$.
   - Each edge $e = (u,v) \in E$ is assigned a quintuple $\mu(e) = (T(e), C(e), R(e), U(e), F(e))$, satisfying:

$$\begin{cases} T(e) \leq \min\{T(u), T(v)\}, \\ C(e) \leq \min\{C(u), C(v)\}, \\ R(e) \geq \max\{R(u), R(v)\}, \\ U(e) \geq \max\{U(u), U(v)\}, \\ F(e) \geq \max\{F(u), F(v)\}. \end{cases}$$

Definition 5. [141, 272, 273, 285, 289] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

1. *Plithogenic Vertex Set $PM = (M, l, Ml, adf, aCf)$:*

- $M \subseteq V$ is the set of vertices.
- l is an attribute associated with the vertices.
- Ml is the range of possible attribute values.
- $adf : M \times Ml \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
- $aCf : Ml \times Ml \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

2. *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:

   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
   - $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. *Edge Appurtenance Constraint*: For all $(x,a),(y,b) \in M \times Ml$:

$$bdf\,((xy),(a,b)) \leq \min\{adf(x,a), adf(y,b)\}$$

   where $xy \in N$ is an edge between vertices x and y, and $(a,b) \in Nm \times Nm$ are the corresponding attribute values.

2. *Contradiction Function Constraint*: For all $(a,b),(c,d) \in Nm \times Nm$:

$$bCf\,((a,b),(c,d)) \leq \min\{aCf(a,c), aCf(b,d)\}$$

3. *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} &aCf(a,a) = 0, && \forall a \in Ml \\ &aCf(a,b) = aCf(b,a), && \forall a,b \in Ml \\ &bCf(a,a) = 0, && \forall a \in Nm \\ &bCf(a,b) = bCf(b,a), && \forall a,b \in Nm \end{aligned}$$

Example 6. (cf.[113, 117]) The following examples are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic Turiyam Neutrosophic Graph*.

2.3 Fuzzy Soft Graph and Neutrosophic Soft Graph

This subsection provides an explanation of Fuzzy Soft Graph and Neutrosophic Soft Graph. Fuzzy Soft Graph [24, 27, 38, 74, 165, 265] and Neutrosophic Soft Graph [16, 58, 255, 293] are generalized extensions of the Soft Graph.

The relevant concepts and definitions are provided below.

Definition 7. [?] Let U be a non-empty finite set called the *universe of discourse*, and let E be a non-empty set of parameters. A *soft set* over U is defined as follows:

$$F = (F, A) \text{ over } U \text{ is an ordered pair, where } A \subseteq E \text{ and } F : A \to P(U),$$

where $F(a) \subseteq U$ for each $a \in A$, and $P(U)$ denotes the power set of U. The set of all soft sets over U is denoted by $S(U)$.

1. *Soft Subset:* Let $F = (F, A)$ and $G = (G, B)$ be two soft sets over the common universe U. F is a soft subset of G, denoted $F \subseteq G$, if:

- $A \subseteq B$,
- $F(a) \subseteq G(a)$ for all $a \in A$.

2. *Union of Soft Sets:* The union of two soft sets $F = (F, A)$ and $G = (G, B)$ over U is defined as $H = (H, C)$, where $C = A \cup B$ and

$$H(e) = \begin{cases} F(e), & e \in A \setminus B, \\ G(e), & e \in B \setminus A, \\ F(e) \cup G(e), & e \in A \cap B. \end{cases}$$

3. *Intersection of Soft Sets:* The intersection of two soft sets $F = (F, A)$ and $G = (G, B)$ is defined as $H = (H, C)$, where $C = A \cap B$ and

$$H(e) = F(e) \cap G(e), \quad \forall e \in C.$$

Definition 8. [12,158] Let $G = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges. Let A be a non-empty set of parameters, and let $R \subseteq A \times V$ be a relation between elements of A and V. Define a set-valued function $F : A \to P(V)$ by

$$F(x) = \{y \in V \mid xRy\}.$$

The pair (F, A) is a soft set over V.

A *Soft Graph* of G is defined as follows:

A soft set (F, A) over V is a soft graph of G if the subgraph $F(x)$

is a connected subgraph of G for all $x \in A$. The set of all soft graphs of G is denoted by $SG(G)$.

Definition 9. A *Fuzzy Soft Graph* is defined as a 4-tuple $G_e = (G^*, F, K, A)$, where:

- $G^* = (V, E)$ is a simple graph with a set of vertices V and a set of edges E.
- A is a non-empty set of parameters.
- (F, A) is a fuzzy soft set over V, where $F : A \to P(V)$ is the fuzzy approximate function of the fuzzy soft set.
- (K, A) is a fuzzy soft set over E, where $K : A \to P(E)$ is the fuzzy approximate function of the fuzzy soft set.

Definition 10. For each $a \in A$, the fuzzy approximate functions are defined as follows:

- $F(a) : V \to [0, 1]$, where $F(a)(x)$ represents the degree of membership of vertex $x \in V$.
- $K(a) : E \to [0, 1]$, where $K(a)(xy)$ represents the degree of membership of edge $xy \in E$.

Definition 11. For all $a \in A$ and $x, y \in V$, the fuzzy soft graph must satisfy:

$$K(a)(xy) \leq \min\{F(a)(x), F(a)(y)\}.$$

The pair $(F(a), K(a))$ forms a fuzzy (sub)graph of G^*, denoted as $H_e(a)$, for each parameter $a \in A$. Thus, a fuzzy soft graph is essentially a parameterized family of fuzzy graphs.

The class of all fuzzy soft graphs of G^* is denoted by $F(G^*)$.

Definition 12. [16,58,255,293] *Neutrosophic Soft Graph* is defined as a 4-tuple $G = (G^*, J, K, A)$, where:

- $G^* = (V, E)$ is a *neutrosophic graph*, where:
  - V is the set of vertices.
  - $E \subseteq V \times V$ is the set of edges.
  - For each vertex $x \in V$, there exist three functions:

$$T : V \to [0, 1], \quad I : V \to [0, 1], \quad F : V \to [0, 1],$$

  representing the truth-membership, indeterminacy-membership, and falsity-membership degrees, respectively.

– For each vertex $x \in V$, the constraint holds:

$$0 \leq T(x) + I(x) + F(x) \leq 3.$$

- A is a non-empty set of parameters.
- (J, A) is a *neutrosophic soft set* over the vertex set V, where $J : A \to \rho(V)$, and $\rho(V)$ denotes the set of all neutrosophic sets of V.
- (K, A) is a *neutrosophic soft set* over the edge set E, where $K : A \to \rho(E)$, and $\rho(E)$ denotes the set of all neutrosophic sets of E.

Definition 13. For each $e \in A$, the *neutrosophic soft graph* $H(e)$ corresponding to parameter e is defined as a pair $(J(e), K(e))$, where:

$$T_{K(e)}(xy) \leq \min\{T_{J(e)}(x), T_{J(e)}(y)\},$$
$$I_{K(e)}(xy) \leq \min\{I_{J(e)}(x), I_{J(e)}(y)\},$$
$$F_{K(e)}(xy) \geq \max\{F_{J(e)}(x), F_{J(e)}(y)\},$$

for all $x, y \in V$, and

$$0 \leq T_{K(e)}(xy) + I_{K(e)}(xy) + F_{K(e)}(xy) \leq 3.$$

3. Result in this paper

In this section, we present the results of this paper.

3.1 Uncertain Soft Graph

The Turiyam Neutrosophic Soft Graph is defined as follows.

Definition 14. A *Turiyam Neutrosophic Soft Graph* is defined as a 4-tuple $G = (G^*, T, K, A)$, where:

- $G^* = (V, E)$ is a *Turiyam Neutrosophic graph*, where:
  - V is the set of vertices.
  - $E \subseteq V \times V$ is the set of edges.
  - For each vertex $x \in V$, there exist four functions:

    $$t : V \to [0,1], \quad iv : V \to [0,1], \quad fv : V \to [0,1], \quad lv : V \to [0,1],$$

    representing the truth-membership, indeterminacy-membership, falsity-membership, and liberal state-membership degrees, respectively.
  - For each vertex $x \in V$, the constraint holds:

    $$0 \leq t(x) + iv(x) + fv(x) + lv(x) \leq 4.$$

- A is a non-empty set of parameters.
- (T, A) is a *Turiyam Neutrosophic soft set* over the vertex set V, where $T : A \to \rho(V)$, and $\rho(V)$ denotes the set of all Turiyam Neutrosophic sets of V.
- (K, A) is a *Turiyam Neutrosophic soft set* over the edge set E, where $K : A \to \rho(E)$, and $\rho(E)$ denotes the set of all Turiyam Neutrosophic sets of E.

Definition 15. For each $e \in A$, the *Turiyam Neutrosophic soft graph* $H(e)$ corresponding to parameter e is defined as a pair $(T(e), K(e))$, where:

$$t_{K(e)}(xy) \leq \min\{t_{T(e)}(x), t_{T(e)}(y)\},$$
$$iv_{K(e)}(xy) \leq \min\{iv_{T(e)}(x), iv_{T(e)}(y)\},$$
$$fv_{K(e)}(xy) \geq \max\{fv_{T(e)}(x), fv_{T(e)}(y)\},$$
$$lv_{K(e)}(xy) \geq \max\{lv_{T(e)}(x), lv_{T(e)}(y)\},$$

for all $x, y \in V$, and

$$0 \leq t_{K(e)}(xy) + iv_{K(e)}(xy) + fv_{K(e)}(xy) + lv_{K(e)}(xy) \leq 4.$$

A *General Plithogenic Graph* G^{GP} is an extension of the classical Plithogenic Graph that allows for a more flexible and independent treatment of vertices and edges (cf.[110, 113, 210])

Definition 16 (General Plithogenic Graph)**.** [113] Let $G = (V, E)$ be a classical graph, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of edges.

A General Plithogenic Graph $G^{GP} = (PM, PN)$ consists of:

1. *General Plithogenic Vertex Set* PM:

$$PM = (M, l, Ml, adf, aCf)$$

where:

- $M \subseteq V$: Set of vertices.
- l: Attribute associated with the vertices.
- Ml: Range of possible attribute values.
- $adf : M \times Ml \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for vertices.
- $aCf : Ml \times Ml \to [0, 1]^t$: Degree of Contradiction Function (DCF) for vertices.

2. *General Plithogenic Edge Set* PN:

$$PN = (N, m, Nm, bdf, bCf)$$

where:

- $N \subseteq E$: Set of edges.
- m: Attribute associated with the edges.
- Nm: Range of possible attribute values.
- $bdf : N \times Nm \to [0, 1]^s$: Degree of Appurtenance Function (DAF) for edges.
- $bCf : Nm \times Nm \to [0, 1]^t$: Degree of Contradiction Function (DCF) for edges.

The General Plithogenic Graph G^{GP} only needs to satisfy the following *Reflexivity and Symmetry* properties of the Contradiction Functions:

- Reflexivity and Symmetry of Contradiction Functions:

$$\begin{aligned} aCf(a, a) &= 0, & \forall a \in Ml \\ aCf(a, b) &= aCf(b, a), & \forall a, b \in Ml \\ bCf(a, a) &= 0, & \forall a \in Nm \\ bCf(a, b) &= bCf(b, a), & \forall a, b \in Nm \end{aligned}$$

The General Plithogenic Soft Graph is defined as follows.

Definition 17. A *General Plithogenic Soft Graph* is defined as a 4-tuple $G^{GP} = (G^*, PM, PN, A)$, where:

- $G^* = (V, E)$ is a *crisp graph*, where:
  - V is the set of vertices.
  - $E \subseteq V \times V$ is the set of edges.
- $PM = (M, l, Ml, adf, aCf)$ is the *General Plithogenic Vertex Set*, where:
  - $M \subseteq V$ is the set of vertices.
  - l is an attribute associated with the vertices.
  - Ml is the range of possible attribute values.
  - $adf : M \times Ml \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
  - $aCf : Ml \times Ml \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.
- $PN = (N, m, Nm, bdf, bCf)$ is the *General Plithogenic Edge Set*, where:

- $N \subseteq E$ is the set of edges.
- m is an attribute associated with the edges.
- Nm is the range of possible attribute values.
- $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
- $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

- A is a non-empty set of parameters.

Definition 18. The General Plithogenic Soft Graph G^{GP} must satisfy the following conditions:
Edge Appurtenance Constraint For all $(x,a),(y,b) \in M \times Ml$, we have:

$$bdf((xy),(a,b)) \le \min\{adf(x,a), adf(y,b)\},$$

where $xy \in N$ is an edge between vertices x and y, and $(a,b) \in Nm \times Nm$ are the corresponding attribute values.
Contradiction Function Constraint For all $(a,b),(c,d) \in Nm \times Nm$, we have:

$$bCf((a,b),(c,d)) \le \min\{aCf(a,c), aCf(b,d)\}.$$

Reflexivity and Symmetry of Contradiction Functions

$$\begin{aligned} aCf(a,a) &= 0, & \forall a \in Ml, \\ aCf(a,b) &= aCf(b,a), & \forall a,b \in Ml, \\ bCf(a,a) &= 0, & \forall a \in Nm, \\ bCf(a,b) &= bCf(b,a), & \forall a,b \in Nm. \end{aligned}$$

For each $e \in A$, the *General Plithogenic Soft Graph* corresponding to parameter e, denoted by $G^{GP}(e)$, is defined as:

$$G^{GP}(e) = ((adf_e, aCf_e), (bdf_e, bCf_e)),$$

where:

- $adf_e : M \to [0,1]^s$ represents the Degree of Appurtenance Function for vertices under parameter e.
- $aCf_e : Ml \times Ml \to [0,1]^t$ represents the Degree of Contradiction Function for vertices under parameter e.
- $bdf_e : N \to [0,1]^s$ represents the Degree of Appurtenance Function for edges under parameter e.
- $bCf_e : Nm \times Nm \to [0,1]^t$ represents the Degree of Contradiction Function for edges under parameter e.

Theorem 19. *Every General Plithogenic Soft Graph can be transformed into a Plithogenic Graph under appropriate conditions.*

Proof. Given a General Plithogenic Soft Graph $G^{GP} = (G^*, PM, PN, A)$, we can construct a Plithogenic Graph PG by selecting specific parameters $a \in A$ and considering the corresponding attribute values.

Since the General Plithogenic Soft Graph includes all the elements of a Plithogenic Graph, with additional flexibility due to the parameter set A, we can fix A to a singleton set or select a specific parameter to obtain a Plithogenic Graph.

Moreover, if the Edge Appurtenance Constraint and Contradiction Function Constraint are satisfied (which they are in the Plithogenic Graph definition), the General Plithogenic Soft Graph reduces to a Plithogenic Graph. □

Corollary 20. *Every Turiyam Neutrosophic Soft Graph can be transformed into a Turiyam Neutrosophic Graph.*

Proof. It can be proved in the same manner as above. □

Corollary 21. *Every Neutrosophic Soft Graph can be transformed into a Neutrosophic Graph.*

Proof. It can be proved in the same manner as above. □

Theorem 22. *Every General Plithogenic Soft Graph with $s = 4$ can be transformed into a Turiyam Neutrosophic Soft Graph.*

Proof. Let $G^{GP} = (G^*, PM, PN, A)$ be a General Plithogenic Soft Graph with $s = 4$. The DAFs adf and bdf map to $[0, 1]^4$, representing four components.

For each parameter $a \in A$, define the Turiyam Neutrosophic Soft Graph $H(a)$ as follows:

- For each vertex $v \in V$:
$$\sigma_{H(a)}(v) = adf(v, l_v),$$
where l_v is the attribute value of v.
- For each edge $e = (u, v) \in E$:
$$\mu_{H(a)}(e) = bdf(e, m_e),$$
where m_e is the attribute value of e.

Since adf and bdf provide quadruples in $[0, 1]^4$, they correspond to the truth, indeterminacy, falsity, and liberal state degrees in a Turiyam Neutrosophic Soft Graph.

Thus, G^{GP} can be transformed into a Turiyam Neutrosophic Soft Graph. □

Theorem 23. *Every General Plithogenic Soft Graph with $s = 4$ and A being a singleton set can be transformed into a Turiyam Neutrosophic Graph.*

Proof. If $A = \{a_0\}$, then the General Plithogenic Soft Graph G^{GP} reduces to a single graph corresponding to the parameter a_0.

Using the same mapping as in Theorem 22, we obtain a Turiyam Neutrosophic Graph. □

Theorem 24. *Every General Plithogenic Soft Graph with $s = 3$ can be transformed into a Neutrosophic Soft Graph.*

Proof. Similar to Theorem 22, but with $s = 3$, the DAFs adf and bdf map to $[0, 1]^3$, representing truth, indeterminacy, and falsity degrees.

We define the Neutrosophic Soft Graph $H(a)$ for each $a \in A$ using the components from adf and bdf. □

Theorem 25. *Every General Plithogenic Soft Graph with $s = 3$ and A being a singleton set can be transformed into a Neutrosophic Graph.*

Proof. By selecting $A = \{a_0\}$ and applying the mapping from Theorem 24, we obtain a Neutrosophic Graph. □

Theorem 26. *Every General Plithogenic Soft Graph with $s = 1$ can be transformed into a Fuzzy Soft Graph.*

Proof. When $s = 1$, the DAFs adf and bdf map to $[0, 1]$, representing single membership degrees.

For each $a \in A$, we define the Fuzzy Soft Graph $H(a)$ with membership functions $F(a) = adf(\cdot, l.)$ and $K(a) = bdf(\cdot, m.)$. □

Theorem 27. *Any induced subgraph of a General Plithogenic Soft Graph is also a General Plithogenic Soft Graph.*

Proof. Let $G^{GP} = (G^*, PM, PN, A)$ be a General Plithogenic Soft Graph, and let $V' \subseteq V$ be a subset of vertices. The induced subgraph G'^{GP} is obtained by restricting PM and PN to V' and the corresponding edges.

Since the definitions of adf and bdf remain valid on the subsets, G'^{GP} satisfies the conditions of a General Plithogenic Soft Graph. □

3.2 General Plithogenic Soft Rough Graph

The Neutrosophic Soft Rough Graph is an extension of the Soft Rough Graph concept, utilizing the framework of a Neutrosophic Graph. It combines the principles of a Neutrosophic Soft Graph with the concept of Rough Graphs. The definition is provided below.

Definition 28 (Neutrosophic Soft Rough Graph)**.** [5,11] Let V be a non-empty set representing vertices, and M be a set of parameters. A *Neutrosophic Soft Rough Graph (NSRG)* is defined as a 4-tuple:

$$G = (V, M, Q(A), S(B)),$$

where:

1. M is the set of parameters.
2. Q is a neutrosophic soft relation over $V \times M$, forming a *Neutrosophic Soft Approximation Space (NSAS)*.
3. S is a neutrosophic soft relation over $V' \times M'$, where $V' \subseteq V \times V$ and $M' \subseteq M \times M$, forming a *Neutrosophic Soft Rough Relation (NSRR)*.
4. $Q(A) = (Q_A, Q_A)$ is the *Neutrosophic Soft Rough Set (NSRS)* of a neutrosophic soft set $A \subseteq N(M)$ with respect to the approximation space.
5. $S(B) = (S_B, S_B)$ is the *Neutrosophic Soft Rough Set (NSRR)* of a neutrosophic soft set $B \subseteq N(M')$ with respect to the relation S.

The *Neutrosophic Soft Rough Graph (NSRG)*, denoted by G, is represented as:

$$G = (Q(A), S(B)) = ((Q_A, Q_A), (S_B, S_B)) .$$

Here, the *Lower Neutrosophic Approximate Graph (LNAG)* and the *Upper Neutrosophic Approximate Graph (UNAG)* of the NSRG are defined as:

$$\text{LNAG} = (Q_A, S_B), \quad \text{UNAG} = (Q_A, S_B).$$

Remark 29. *In the context of the NSRG, the neutrosophic soft rough approximations of a neutrosophic soft set A and a neutrosophic soft set B are defined as follows:*

1. *The Upper Neutrosophic Soft Rough Approximation (UNSRA) of A, denoted by $Q(A)$, is given by:*

$$Q(A) = \left\{ \left(u, T_{Q(A)}(u), I_{Q(A)}(u), F_{Q(A)}(u) \right) \mid u \in V \right\},$$

where:

$$T_{Q(A)}(u) = \bigwedge_{e \in M} \left(T_{Q(A)}(u, e) \wedge T_A(e) \right), \quad I_{Q(A)}(u) = \bigvee_{e \in M} \left(I_{Q(A)}(u, e) \vee I_A(e) \right),$$

$$F_{Q(A)}(u) = \bigvee_{e \in M} \left(F_{Q(A)}(u, e) \vee F_A(e) \right).$$

2. *The Lower Neutrosophic Soft Rough Approximation (LNSRA) of A, denoted by $Q(A)$, is given by:*

$$Q(A) = \left\{ \left(u, T_{Q(A)}(u), I_{Q(A)}(u), F_{Q(A)}(u) \right) \mid u \in V \right\},$$

where:

$$T_{Q(A)}(u) = \bigvee_{e \in M} \left(F_{Q(A)}(u, e) \vee T_A(e) \right), \quad I_{Q(A)}(u) = \bigwedge_{e \in M} \left((1 - I_{Q(A)}(u, e)) \wedge I_A(e) \right),$$

$$F_{Q(A)}(u) = \bigwedge_{e \in M} \left(T_{Q(A)}(u, e) \wedge F_A(e) \right).$$

3. *Similarly, the neutrosophic soft rough approximations of B with respect to the relation S are defined for both upper and lower approximations:*

$$S(B) = \left\{ \left(u_i u_j, T_{S(B)}(u_i u_j), I_{S(B)}(u_i u_j), F_{S(B)}(u_i u_j) \right) \mid u_i u_j \in V' \right\},$$

where:

$$T_{S(B)}(u_i u_j) = \bigwedge_{e_i e_j \in M'} \left(T_S(u_i u_j, e_i e_j) \wedge T_B(e_i e_j) \right),$$

$$I_{S(B)}(u_i u_j) = \bigvee_{e_i e_j \in M'} \left(I_S(u_i u_j, e_i e_j) \vee I_B(e_i e_j) \right),$$

$$F_{S(B)}(u_i u_j) = \bigvee_{e_i e_j \in M'} \left(F_S(u_i u_j, e_i e_j) \vee F_B(e_i e_j) \right).$$

Definition 30 (Turiyam Neutrosophic Soft Rough Graph**).** Let V be a non-empty set representing vertices, and M be a set of parameters. A *Turiyam Neutrosophic Soft Rough Graph (TSRG)* is defined as a 4-tuple:

$$G^{TSR} = (V, M, Q_T(A), S_T(B)),$$

where:

1. M is the set of parameters.
2. Q_T is a Turiyam Neutrosophic soft relation over $V \times M$, forming a *Turiyam Neutrosophic Soft Approximation Space (TSAS)*.
3. S_T is a Turiyam Neutrosophic soft relation over $V' \times M'$, where $V' \subseteq V \times V$ and $M' \subseteq M \times M$, forming a *Turiyam Neutrosophic Soft Rough Relation (TSRR)*.
4. $Q_T(A) = (Q_{T,A}, Q_{T,A})$ is the *Turiyam Neutrosophic Soft Rough Set (TSRS)* of a Turiyam Neutrosophic soft set $A \subseteq T(M)$ with respect to the approximation space.
5. $S_T(B) = (S_{T,B}, S_{T,B})$ is the *Turiyam Neutrosophic Soft Rough Set (TSRR)* of a Turiyam Neutrosophic soft set $B \subseteq T(M')$ with respect to the relation S_T.

The *Turiyam Neutrosophic Soft Rough Graph (TSRG)*, denoted by G^{TSR}, is represented as:

$$G^{TSR} = (Q_T(A), S_T(B)) = \left((Q_{T,A}, Q_{T,A}), (S_{T,B}, S_{T,B}) \right).$$

Here, the *Lower Turiyam Neutrosophic Approximate Graph (LTAG)* and the *Upper Turiyam Neutrosophic Approximate Graph (UTAG)* of the TSRG are defined as:

$$\text{LTAG} = (Q_{T,A}, S_{T,B}), \quad \text{UTAG} = (Q_{T,A}, S_{T,B}).$$

Remark 31. *In the context of the TSRG, the Turiyam Neutrosophic soft rough approximations of a Turiyam Neutrosophic soft set A and a Turiyam Neutrosophic soft set B are defined as follows:*

1. *The Upper Turiyam Neutrosophic Soft Rough Approximation (UTSRA) of A, denoted by $Q_T(A)$, is given by:*

$$Q_T(A) = \left\{ \left(u, t_{Q_T(A)}(u), iv_{Q_T(A)}(u), fv_{Q_T(A)}(u), lv_{Q_T(A)}(u) \right) \mid u \in V \right\},$$

where:

$$t_{Q_T(A)}(u) = \bigwedge_{e \in M} \left(t_{Q_T(A)}(u, e) \wedge t_A(e) \right), \quad iv_{Q_T(A)}(u) = \bigvee_{e \in M} \left(iv_{Q_T(A)}(u, e) \vee iv_A(e) \right),$$

$$fv_{Q_T(A)}(u) = \bigvee_{e \in M} \left(fv_{Q_T(A)}(u, e) \vee fv_A(e) \right), \quad lv_{Q_T(A)}(u) = \bigvee_{e \in M} \left(lv_{Q_T(A)}(u, e) \vee lv_A(e) \right).$$

2. *The Lower Turiyam Neutrosophic Soft Rough Approximation (LTSRA) of A, denoted by $Q_T(A)$, is given by:*

$$Q_T(A) = \left\{ \left(u, t_{Q_T(A)}(u), iv_{Q_T(A)}(u), fv_{Q_T(A)}(u), lv_{Q_T(A)}(u)\right) \mid u \in V \right\},$$

where:

$$t_{Q_T(A)}(u) = \bigvee_{e \in M} \left(lv_{Q_T(A)}(u,e) \vee t_A(e)\right), \quad iv_{Q_T(A)}(u) = \bigwedge_{e \in M} \left((1 - iv_{Q_T(A)}(u,e)) \wedge iv_A(e)\right),$$

$$fv_{Q_T(A)}(u) = \bigwedge_{e \in M} \left(t_{Q_T(A)}(u,e) \wedge fv_A(e)\right), \quad lv_{Q_T(A)}(u) = \bigwedge_{e \in M} \left(iv_{Q_T(A)}(u,e) \wedge lv_A(e)\right).$$

Theorem 32. *Every Turiyam Neutrosophic Soft Rough Graph can be transformed into a Neutrosophic Soft Rough Graph.*

Proof. Let $G^{TSR} = (V, M, Q_T(A), S_T(B))$ be a Turiyam Neutrosophic Soft Rough Graph. We aim to transform it into a Neutrosophic Soft Rough Graph $G^{NSR} = (V, M, Q_N(A), S_N(B))$.

The transformation is based on mapping the four membership components of the Turiyam Neutrosophic set (truth, indeterminacy, falsity, and liberal state) to the three components of the neutrosophic set (truth, indeterminacy, and falsity) as follows:

1. *Truth Mapping*: The truth-membership function in the Turiyam Neutrosophic Soft Rough Graph, denoted by $t_{Q_T(A)}(u)$ and $t_{S_T(B)}(u_i u_j)$, is directly mapped to the truth-membership function in the Neutrosophic Soft Rough Graph:

$$T_{Q_N(A)}(u) = t_{Q_T(A)}(u), \quad T_{S_N(B)}(u_i u_j) = t_{S_T(B)}(u_i u_j).$$

2. *Indeterminacy Mapping*: The indeterminacy-membership function in the Turiyam Neutrosophic Soft Rough Graph, denoted by $iv_{Q_T(A)}(u)$ and $iv_{S_T(B)}(u_i u_j)$, is mapped to the indeterminacy-membership function in the Neutrosophic Soft Rough Graph:

$$I_{Q_N(A)}(u) = iv_{Q_T(A)}(u), \quad I_{S_N(B)}(u_i u_j) = iv_{S_T(B)}(u_i u_j).$$

3. *Falsity Mapping*: The falsity-membership function in the Turiyam Neutrosophic Soft Rough Graph, denoted by $fv_{Q_T(A)}(u)$ and $fv_{S_T(B)}(u_i u_j)$, is directly mapped to the falsity-membership function in the Neutrosophic Soft Rough Graph:

$$F_{Q_N(A)}(u) = fv_{Q_T(A)}(u), \quad F_{S_N(B)}(u_i u_j) = fv_{S_T(B)}(u_i u_j).$$

4. *Liberal State Exclusion*: The liberal state-membership function, denoted by $lv_{Q_T(A)}(u)$ and $lv_{S_T(B)}(u_i u_j)$, does not have a direct counterpart in the neutrosophic set framework. To ensure a correct transformation, we define:

$$lv_{Q_T(A)}(u) = 0, \quad lv_{S_T(B)}(u_i u_j) = 0.$$

This exclusion aligns the Turiyam Neutrosophic membership functions to match the three components of the neutrosophic membership functions.

The transformed components now satisfy the definition of a Neutrosophic Soft Rough Graph, resulting in:

$$G^{NSR} = (V, M, Q_N(A), S_N(B)) = \left((Q_{N,A}, Q_{N,A}), (S_{N,B}, S_{N,B})\right).$$

Hence, a Turiyam Neutrosophic Soft Rough Graph can be converted into a Neutrosophic Soft Rough Graph. □

Definition 33 (General Plithogenic Soft Rough Graph)**.** Let V be a non-empty set representing vertices, and M be a set of parameters. A *General Plithogenic Soft Rough Graph (GPSRG)* is defined as a 4-tuple:

$$G^{GPSR} = (V, M, Q_G(A), S_G(B)),$$

where:

1. M is the set of parameters.

2. Q_G is a plithogenic soft relation over $V \times M$, forming a *General Plithogenic Soft Approximation Space (GPSAS)*.
3. S_G is a plithogenic soft relation over $V' \times M'$, where $V' \subseteq V \times V$ and $M' \subseteq M \times M$, forming a *General Plithogenic Soft Rough Relation (GPSRR)*.
4. $Q_G(A) = (Q_{G,A}, Q_{G,A})$ is the *General Plithogenic Soft Rough Set (GPSRS)* of a plithogenic soft set $A \subseteq GP(M)$ with respect to the approximation space.
5. $S_G(B) = (S_{G,B}, S_{G,B})$ is the *General Plithogenic Soft Rough Set (GPSRR)* of a plithogenic soft set $B \subseteq GP(M')$ with respect to the relation S_G.

The *General Plithogenic Soft Rough Graph (GPSRG)*, denoted by G^{GPSR}, is represented as:

$$G^{GPSR} = (Q_G(A), S_G(B)) = \left((Q_{G,A}, Q_{G,A}), (S_{G,B}, S_{G,B})\right).$$

Here, the *Lower General Plithogenic Approximate Graph (LGAG)* and the *Upper General Plithogenic Approximate Graph (UGAG)* of the GPSRG are defined as:

$$\text{LGAG} = (Q_{G,A}, S_{G,B}), \quad \text{UGAG} = (Q_{G,A}, S_{G,B}).$$

Remark 34. *In the context of the GPSRG, the general plithogenic soft rough approximations of a plithogenic soft set A and a plithogenic soft set B are defined as follows:*

1. *The Upper General Plithogenic Soft Rough Approximation (UGPSRA) of A, denoted by $Q_G(A)$, is given by:*
$$Q_G(A) = \left\{\left(u, adf_{Q_G(A)}(u), aCf_{Q_G(A)}(u), bdf_{Q_G(A)}(u), bCf_{Q_G(A)}(u)\right) \mid u \in V\right\},$$
where:
$$adf_{Q_G(A)}(u) = \bigwedge_{e \in M} \left(adf_{Q_G(A)}(u,e) \wedge adf_A(e)\right), \quad aCf_{Q_G(A)}(u) = \bigvee_{e \in M} \left(aCf_{Q_G(A)}(u,e) \vee aCf_A(e)\right),$$
$$bdf_{Q_G(A)}(u) = \bigvee_{e \in M} \left(bdf_{Q_G(A)}(u,e) \vee bdf_A(e)\right), \quad bCf_{Q_G(A)}(u) = \bigvee_{e \in M} \left(bCf_{Q_G(A)}(u,e) \vee bCf_A(e)\right).$$
2. *The Lower General Plithogenic Soft Rough Approximation (LGPSRA) of A, denoted by $Q_G(A)$, is given by:*
$$Q_G(A) = \left\{\left(u, adf_{Q_G(A)}(u), aCf_{Q_G(A)}(u), bdf_{Q_G(A)}(u), bCf_{Q_G(A)}(u)\right) \mid u \in V\right\},$$
where:
$$adf_{Q_G(A)}(u) = \bigvee_{e \in M} \left(bCf_{Q_G(A)}(u,e) \vee adf_A(e)\right), \quad aCf_{Q_G(A)}(u) = \bigwedge_{e \in M} \left((1 - aCf_{Q_G(A)}(u,e)) \wedge aCf_A(e)\right),$$
$$bdf_{Q_G(A)}(u) = \bigwedge_{e \in M} \left(adf_{Q_G(A)}(u,e) \wedge bdf_A(e)\right), \quad bCf_{Q_G(A)}(u) = \bigwedge_{e \in M} \left(aCf_{Q_G(A)}(u,e) \wedge bCf_A(e)\right).$$

Theorem 35. *Every General Plithogenic Soft Rough Graph can be transformed into a General Plithogenic Soft Graph.*

Proof. Given a GPSRG $G^{GPSR} = (V, M, Q_G(A), S_G(B))$, we can obtain a General Plithogenic Soft Graph G^{GPS} by considering only the upper approximations $Q_{G,A}$ and $S_{G,B}$. Specifically, we define:

$$G^{GPS} = (V, M, Q_{G,A}, S_{G,B}).$$

Since the upper approximations capture the maximum membership degrees, the resulting structure satisfies the definition of a General Plithogenic Soft Graph. □

Theorem 36. *Every General Plithogenic Soft Rough Graph can be transformed into a General Plithogenic Graph.*

Proof. By fixing the parameter set M and considering specific parameter values, we can reduce the GPSRG to a General Plithogenic Graph G^{GP}. Specifically, select a parameter $m_0 \in M$ and define:

$$G^{GP} = (V, E, adf, aCf, bdf, bCf),$$

where the functions adf, aCf, bdf, bCf are derived from Q_G and S_G under the fixed parameter m_0. The resulting graph satisfies the conditions of a General Plithogenic Graph. □

Theorem 37. *Every General Plithogenic Soft Rough Graph with $s = 4$ can be transformed into a Turiyam Neutrosophic Soft Rough Graph.*

Proof. When $s = 4$, the DAFs and DCFs map to $[0, 1]^4$, corresponding to the four components of a Turiyam Neutrosophic set (truth, indeterminacy, falsity, and liberal state). We define the Turiyam Neutrosophic Soft Rough Graph G^{TSR} by mapping the DAFs and DCFs from the GPSRG to the membership functions in the Turiyam Neutrosophic framework. The approximation operations in the GPSRG correspond to those in the Turiyam Neutrosophic Soft Rough Graph, ensuring a correct transformation. □

Theorem 38. *Every General Plithogenic Soft Rough Graph with $s = 4$ can be transformed into a Turiyam Neutrosophic Soft Graph.*

Proof. Using the same approach as in Theorem 37, but focusing on the upper approximations, we obtain a Turiyam Neutrosophic Soft Graph G^{TS}. By considering only the upper approximation $Q_{G,A}$ and $S_{G,B}$, we define:

$$G^{TS} = (V, M, Q_{T,A}, S_{T,B}),$$

where $Q_{T,A}$ and $S_{T,B}$ are derived from $Q_{G,A}$ and $S_{G,B}$ respectively. □

Theorem 39. *Every General Plithogenic Soft Rough Graph with $s = 4$ and a singleton parameter set $M = \{m_0\}$ can be transformed into a Turiyam Neutrosophic Graph.*

Proof. With $M = \{m_0\}$, the GPSRG reduces to a single parameter case. By extracting the membership degrees corresponding to m_0, we obtain a Turiyam Neutrosophic Graph G^T where the vertex and edge membership functions are derived from the DAFs and DCFs of the GPSRG. □

Theorem 40. *Every General Plithogenic Soft Rough Graph with $s = 3$ can be transformed into a Neutrosophic Soft Rough Graph.*

Proof. When $s = 3$, the DAFs and DCFs correspond to the three components of a neutrosophic set (truth, indeterminacy, and falsity). By mapping these functions appropriately, we can define a Neutrosophic Soft Rough Graph G^{NSR} where the neutrosophic membership functions are derived from the GPSRG's DAFs and DCFs. □

4. Future tasks

In this section, we outline the future prospects.

4.1 Future Tasks: Refined General Plithogenic Soft Rough Mixed graph

In this subsection, we briefly outline future prospects.

We aim to investigate the mathematical structure of the Refined General Plithogenic Soft Rough Graph, which integrates the characteristics of the General Plithogenic Soft Rough Graph and the Refined General Plithogenic Graph[113]. We also plan to extend these concepts to hypergraphs [66, 132, 143, 144] and superhypergraphs [114, 123, 153, 154, 274–278], creating the Refined General Plithogenic Soft Rough Hypergraph and Refined General Plithogenic Soft Rough Superhypergraph. Note that examples of Related Refined concepts of Refined General Plithogenic Graph are Refined Neutrosophic concepts [30, 85, 181, 251, 269, 294].

Additionally, we plan to explore the Mixed Graph variants [83, 93, 208], including the Refined General Plithogenic Soft Rough Mixed Graph, Refined General Plithogenic Soft Rough Mixed Hypergraph, and Refined General Plithogenic Soft Rough Mixed Superhypergraph. We also aim to investigate the mathematical properties of the Refined General Plithogenic Soft Rough Mixed n-Superhypergraph [274, 275], along with the potential incorporation of Hypersoft set[246, 271, 280, 281, 312] and superhypersoft set[138, 203, 279, 282] conditions into the General Plithogenic Soft Rough Graph. These ideas are currently at the conceptual stage, with more concrete definitions expected to be developed in the future.

In summary, we aim to identify suitable graph classes by combining these perspectives. While theoretical generalization is crucial in mathematics, it may not always translate immediately into practical applications.

From an applied mathematics standpoint, it is equally important to assess the practical utility of such concepts. Experimental approaches and algorithmic explorations are essential to complement theoretical development. Research often involves expanding or restricting graph classes based on the following aspects:

- *Classic Graph Properties:* Regular [131, 156, 308], Irregular [130, 214, 254], Complete [17, 46, 162], Perfect [1, 49, 129], claw-free[77, 77, 78, 106], Tree, Path[45, 183, 317], Forest[250], Planar [118], Linear [83, 98], OuterPlanar [171, 242], Median[43, 44, 173], Multigraph [32, 64, 65, 240].
- *Graph Using Operations:* Intersection Graph [116], Product Graph [18, 107, 179, 192, 296], Union Graph [25, 291].
- *Subgraph/Hypergraph Properties:* Supergraph [52], Hypergraph [66, 68, 108, 132, 143, 145], Superhypergraph [27, 136, 170, 207, 231, 274, 275, 282, 286], n-Superhypergraph [274, 276], Subgraph, Induced Subgraph [161, 193].
- *Graph Directionality:* Undirected, Directed, Mixed [244, 245], Bidirected [89, 133], Semi-directed [50, 51, 248].
- *Graph Partition:* Bipartite [31, 99, 102, 160, 195], Tripartite [39, 56, 147], n-partite [19, 155].
- *Uncertain Properties:* Fuzzy, Neutrosophic, Turiyam, Plithogenic, Rough [201,223,234,238,256], Vague [8,236,249], Soft [**?**,13,40,134,174,175,215],Hypersoft[246,271,280,281,312], Weighted [94,159,176, 186,220], Picture Fuzzy [188,189,200], Paraconsistent [109,116,301,302], Grey[152], N-Graph, Triangular Fuzzy[288], Z-Number[23,198,221,297,315], Multi-valued logic[26,113,217], Refined Plithogenic[113], q-Rung Orthopair Fuzzy[230,307], Ambiguous[257,264], Quadripartitioned Neutrosophic [168,233], Entropy[84], HyperFuzzy [137, 209], Type-2 Fuzzy [69, 71, 81, 184, 190, 202, 213]
- *Graph Dimensionality:* 2-dimensional, 3-dimensional [76, 219, 306], 4-dimensional[72, 163, 205], Multidimensional[48, 197, 243] etc.
- *Themes:* Graph Classes Hierarchy[57], Mathematical Structure of Graph Classes, Graph Parameters [115, 151, 157, 252], Algorithms [105], Computational Complexity [29, 218], Real-world Applications, Combinatorics [103, 140, 253].

4.2 Future tasks: Some Graph Class

As a future prospect, we plan to define a new class of "Some Uncertain Graphs" based on the perspectives described above and explore their mathematical properties. We also aim to consider possible extensions of these graphs (cf.[57]). In this subsection, we present an example of such a graph.

4.2.1 Future tasks: HyperFuzzy Graph

The HyperFuzzy set [137,177,178,209] has been studied in several papers. It is known as a generalization of the concepts of Fuzzy and interval-valued fuzzy sets. Here, we briefly consider the concept of extending these sets to graph theory. The definition is provided below.

Definition 41. [137] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0,1])$ is called a *hyperfuzzy set* over X, where $\tilde{P}([0,1])$ is the family of all non-empty subsets of $[0,1]$.

A Hyperfuzzy Graph is defined as an extension of the traditional Fuzzy Graph by incorporating hyperfuzzy sets. As a future prospect, we aim to clarify the mathematical structure of this graph.

Definition 42. A hyperfuzzy graph G_H is a structure given by:

$$G_H = (V, E, \tilde{\sigma}, \tilde{\mu}),$$

where:

- V is a non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges.
- $\tilde{\sigma} : V \to \tilde{P}([0,1])$ is a hyperfuzzy set that assigns each vertex a set of membership degrees in $[0,1]$.
- $\tilde{\mu} : E \to \tilde{P}([0,1])$ is a hyperfuzzy set that assigns each edge a set of membership degrees in $[0,1]$.

Theorem 43. *Every fuzzy graph is a special case of a hyperfuzzy graph.*

Proof. Consider a fuzzy graph $G_F = (V, E, \sigma, \mu)$.

To show that a fuzzy graph is a special case of a hyperfuzzy graph, define a mapping for a hyperfuzzy graph as follows:

- For each vertex $v \in V$, set $\tilde{\sigma}(v) = \{\sigma(v)\}$, where $\{\sigma(v)\}$ is a singleton set containing the membership degree of v in the fuzzy graph.
- For each edge $e = (u, v) \in E$, set $\tilde{\mu}(e) = \{\mu(u, v)\}$, where $\{\mu(u, v)\}$ is a singleton set containing the membership degree of the edge e in the fuzzy graph.

In this construction, the hyperfuzzy graph G_H reduces to the fuzzy graph G_F, as the hyperfuzzy sets $\tilde{\sigma}$ and $\tilde{\mu}$ contain only one element, corresponding to the original membership degrees in the fuzzy graph.

Hence, a fuzzy graph can be viewed as a hyperfuzzy graph where each hyperfuzzy set is a singleton, proving that hyperfuzzy graphs generalize fuzzy graphs. □

4.2.2 Future tasks: Shadowed Graph

Shadowed sets convert fuzzy sets into three distinct regions: fully included, fully excluded, and partially included, based on threshold parameters [67, 70, 228, 229, 305, 311]. Here, we briefly consider the concept of extending these sets to graph theory. The definition is provided below.

Definition 44. [228] Let A be a fuzzy set. A *Shadowed Set* S_A of A is defined as:

$$S_A(x) = \begin{cases} 1, & \text{if } \mu_A(x) \geq \alpha; \\ 0, & \text{if } \mu_A(x) \leq \beta; \\ [\beta, \alpha], & \text{if } \beta < \mu_A(x) < \alpha. \end{cases}$$

Here, $\mu_A(x)$ represents the membership grade of element x in the fuzzy set A, and α and β are threshold parameters that determine the boundaries for elevation to 1 and reduction to 0, respectively.

Definition 45. A *Shadowed Graph* is an extension of the Fuzzy Graph, incorporating the concept of shadowed sets to represent uncertainty more comprehensively. Formally, a shadowed graph G_S is defined as:

$$G_S = (V, E, S_\sigma, S_\mu),$$

where:

- V is a non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges.
- $S_\sigma : V \to [0, 1]$ is a shadowed set that assigns each vertex a shadowed membership grade.
- $S_\mu : E \to [0, 1]$ is a shadowed set that assigns each edge a shadowed membership grade.

For each vertex $v \in V$ and each edge $e \in E$, the shadowed membership grades are defined as:

$$S_\sigma(v) = \begin{cases} 1, & \text{if } \sigma(v) \geq \alpha; \\ 0, & \text{if } \sigma(v) \leq \beta; \\ [\beta, \alpha], & \text{if } \beta < \sigma(v) < \alpha, \end{cases}$$

$$S_\mu(e) = \begin{cases} 1, & \text{if } \mu(e) \geq \alpha; \\ 0, & \text{if } \mu(e) \leq \beta; \\ [\beta, \alpha], & \text{if } \beta < \mu(e) < \alpha, \end{cases}$$

where $\sigma(v)$ and $\mu(e)$ represent the membership degrees of vertices and edges in a fuzzy graph, respectively.

Theorem 46. *Every fuzzy graph is a special case of a shadowed graph.*

Proof. Consider a fuzzy graph $G_F = (V, E, \sigma, \mu)$, where:

- $\sigma : V \to [0, 1]$ assigns membership degrees to vertices.
- $\mu : E \to [0, 1]$ assigns membership degrees to edges.

To show that a fuzzy graph is a special case of a shadowed graph, set the threshold parameters $\alpha = 1$ and $\beta = 0$ in the definition of a shadowed graph. In this case:

- For each vertex $v \in V$, the shadowed membership grade $S_\sigma(v)$ reduces to $\sigma(v)$.
- For each edge $e \in E$, the shadowed membership grade $S_\mu(e)$ reduces to $\mu(e)$.

Therefore, the shadowed graph G_S becomes identical to the fuzzy graph G_F, as the shadowed regions disappear, leaving only the original membership grades of the fuzzy graph. Hence, the shadowed graph generalizes the fuzzy graph. □

4.2.3 Future tasks: Nonstandard Fuzzy Graph

A *Nonstandard Fuzzy Set* extends fuzzy sets by including infinitesimal and infinite membership degrees, represented within the nonstandard interval $*[0, 1]$. Here, we briefly consider the concept of extending these sets to graph theory. The definition is provided below.

Definition 47. Let X be a non-empty set. A *Nonstandard Fuzzy Set* on X is defined as an L-subset of X, where L is the totally ordered set $*[0, 1]$, the nonstandard extension of the unit interval $[0, 1]$.

Mathematically, a nonstandard fuzzy set f on X is a function:

$$f : X \to *[0, 1],$$

where $*[0, 1]$ represents the set of all nonstandard elements in the interval $[0, 1]$, including infinitesimal and infinite values.

Definition 48. Given two nonstandard fuzzy sets $f, g \in F(X, *[0, 1])$, where $F(X, *[0, 1])$ denotes the set of all nonstandard fuzzy sets on X, the following operations are defined:

1. *Union:*
$$(f \vee g)(x) = \max\{f(x), g(x)\} \quad \text{for all } x \in X.$$
2. *Intersection:*
$$(f \wedge g)(x) = \min\{f(x), g(x)\} \quad \text{for all } x \in X.$$
3. *Complement:*
$$(-f)(x) = 1 - f(x) \quad \text{for all } x \in X.$$

Here, the arithmetic operations are carried out in the nonstandard field $*\mathbb{R}$, and the values are bounded within the interval $*[0, 1]$.

Definition 49. A *Nonstandard Fuzzy Graph* is an extension of the Fuzzy Graph, incorporating nonstandard fuzzy sets to represent vertices and edges. Formally, a nonstandard fuzzy graph G_N is defined as:

$$G_N = (V, E, \sigma_N, \mu_N),$$

where:

- V is a non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges.
- $\sigma_N : V \to *[0, 1]$ is a nonstandard fuzzy set that assigns each vertex a membership degree in the nonstandard interval $*[0, 1]$.
- $\mu_N : E \to *[0, 1]$ is a nonstandard fuzzy set that assigns each edge a membership degree in the nonstandard interval $*[0, 1]$.

Theorem 50. *Every fuzzy graph is a special case of a nonstandard fuzzy graph.*

Proof. Consider a fuzzy graph $G_F = (V, E, \sigma, \mu)$, where:

- $\sigma : V \to [0, 1]$ assigns membership degrees to vertices.
- $\mu : E \to [0, 1]$ assigns membership degrees to edges.

To show that a fuzzy graph is a special case of a nonstandard fuzzy graph, define a mapping for the nonstandard fuzzy graph as follows:

- For each vertex $v \in V$, set $\sigma_N(v) = \sigma(v)$, where $\sigma_N(v)$ is the nonstandard membership degree that is identical to the standard membership degree of the fuzzy graph.
- For each edge $e \in E$, set $\mu_N(e) = \mu(e)$, where $\mu_N(e)$ is the nonstandard membership degree that matches the standard membership degree of the fuzzy graph.

In this construction, the nonstandard fuzzy graph G_N reduces to the fuzzy graph G_F, as the nonstandard membership degrees are equivalent to the standard membership degrees in the interval $[0, 1]$. Hence, a fuzzy graph can be viewed as a nonstandard fuzzy graph with standard membership degrees, demonstrating that nonstandard fuzzy graphs generalize fuzzy graphs. □

4.2.4 Future tasks: Contextual Fuzzy Graph

A Contextual Fuzzy Set (CFS) is a fuzzy set extension that incorporates context-dependent reliability, allowing more precise multi-criteria decision-making by considering both fuzzy evaluation and contextual reliability simultaneously[169]. Here, we briefly consider the concept of extending these sets to graph theory. The definition is provided below.

Definition 51 (Contextual Fuzzy Set, CFS)**.** Let X be a reference universe and C be a set of contexts (or relative viewpoints). A *Contextual Fuzzy Set* $\tilde{A}$ in X is defined as:

$$\tilde{A} = \left\{ ((x, c), \mu(x), \varphi(x, c)) \;\middle|\; x \in X,\ c \in C \right\},$$

where:

- $\mu : X \to [0, 1]$ represents the *membership degree* of element x in context c.
- $\varphi : X \times C \to [0, 1]$ represents the *reliability degree* of the assessment of x in context c.

The Contextual Fuzzy Set $\tilde{A}$ can be represented as a 2-tuple:

$$\tilde{A} = \langle \mu(x), \varphi(x, c) \rangle.$$

Definition 52 (Contextual Fuzzy Graph, CFG)**.** Let $G = (V, E)$ be a simple graph, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. Let C be a set of contexts. A *Contextual Fuzzy Graph* G_C is defined as:

$$G_C = \left\{ ((v, c), \mu_G(v), \varphi_G(v, c)) \;\middle|\; v \in V,\ c \in C \right\},$$

where:

- $\mu_G : V \to [0, 1]$ represents the *membership degree* of vertex v.
- $\varphi_G : V \times C \to [0, 1]$ represents the *reliability degree* of the membership of vertex v in the context c.

Similarly, for edges $(u, v) \in E$, the Contextual Fuzzy Graph G_C is defined as:

$$G_C = \left\{ (((u, v), c), \mu_G(u, v), \varphi_G(u, v, c)) \;\middle|\; (u, v) \in E,\ c \in C \right\},$$

where:

- $\mu_G : E \to [0, 1]$ represents the *membership degree* of edge (u, v).
- $\varphi_G : E \times C \to [0, 1]$ represents the *reliability degree* of the membership of edge (u, v) in the context c.

The Contextual Fuzzy Graph G_C can be represented as a set of 2-tuples:

$$G_C = \langle \mu_G(v), \varphi_G(v, c) \rangle, \quad \text{for vertices,}$$

and

$$G_C = \langle \mu_G(u, v), \varphi_G(u, v, c) \rangle, \quad \text{for edges.}$$

Theorem 53. *A Contextual Fuzzy Graph G_C is a generalization of a Fuzzy Graph.*

Proof. A Fuzzy Graph (FG) G_F is defined as follows:

$$G_F = \left\{ (v, \mu_G(v)) \;\middle|\; v \in V \right\} \quad \text{and} \quad G_F = \left\{ ((u, v), \mu_G(u, v)) \;\middle|\; (u, v) \in E \right\},$$

where:

- $\mu_G : V \to [0, 1]$ represents the membership degree of vertex v.
- $\mu_G : E \to [0, 1]$ represents the membership degree of edge (u, v).

In the case of a Contextual Fuzzy Graph (CFG), each vertex and edge is associated with an additional component, the *reliability degree*, which depends on the context c. Specifically:

$$G_C = \left\{ ((v, c), \mu_G(v), \varphi_G(v, c)) \;\middle|\; v \in V,\ c \in C \right\},$$

and

$$G_C = \left\{ (((u, v), c), \mu_G(u, v), \varphi_G(u, v, c)) \;\middle|\; (u, v) \in E,\ c \in C \right\}.$$

When the reliability degree φ_G is constant for all contexts c or is omitted, the definition of the Contextual Fuzzy Graph reduces to that of a Fuzzy Graph:

$$G_C = G_F, \quad \text{when} \quad \varphi_G(v, c) = 1\, \forall v \in V,\ c \in C,$$

and similarly for edges:

$$G_C = G_F, \quad \text{when} \quad \varphi_G(u, v, c) = 1\, \forall (u, v) \in E,\ c \in C.$$

Thus, the Contextual Fuzzy Graph generalizes the Fuzzy Graph by incorporating context-dependent reliability degrees, making it more flexible for modeling complex systems where the reliability of information varies with context. □

4.2.5 Future tasks: Quartic Fuzzy Graph and Quintic Fuzzy Graph

I aim to define the concepts of the Quartic Fuzzy Graph and Quintic Fuzzy Graph. Related concepts, such as Fermatean [59–62, 191, 237], Pythagorean [3, 73, 112, 146], and q-rung [21, 87, 112, 187, 230, 300], are already established in the literature. The definitions are presented below.

Definition 54 (Quartic Fuzzy Set)**.** [28] Let X be a universal set. A *Quartic Fuzzy Set A* in X is defined as:

$$A = \left\{ (x, \mu_A(x), \nu_A(x)) \;\middle|\; x \in X \right\},$$

where:

- $\mu_A(x) : X \to [0, 1]$ is called the *membership degree* of $x \in X$.
- $\nu_A(x) : X \to [0, 1]$ is called the *non-membership degree* of $x \in X$.

These degrees satisfy the following condition:

$$(\mu_A(x))^4 + (\nu_A(x))^4 \leq 1, \quad \forall x \in X.$$

The *indeterminacy degree* for any $x \in X$ in the Quartic Fuzzy Set A is defined as:

$$\pi(x) = \sqrt[4]{1 - \left((\mu_A(x))^4 + (\nu_A(x))^4 \right)}.$$

Definition 55 (Quintic Fuzzy Set)**.** [100] Let X be a universal set. A *Quintic Fuzzy Set Q* in X is defined as:

$$Q = \left\{ (x, \mu_Q(x), \nu_Q(x)) \;\middle|\; x \in X \right\},$$

where:

- $\mu_Q(x) : X \to [0, 1]$ is the *membership degree* of $x \in X$.
- $\nu_Q(x) : X \to [0, 1]$ is the *non-membership degree* of $x \in X$.

These degrees satisfy the following condition:

$$\left(\mu_Q(x)\right)^5 + \left(\nu_Q(x)\right)^5 \leq 1, \quad \forall x \in X.$$

The *hesitancy degree* for any $x \in X$ in the Quintic Fuzzy Set Q is defined as:

$$\pi_5(x) = 1 - \left(\left(\mu_Q(x)\right)^5 + \left(\nu_Q(x)\right)^5\right).$$

For simplicity, we denote the Quintic Fuzzy Set Q as $Q = \left(\mu_Q, \nu_Q\right)$.

Definition 56 (Quartic Neutrosophic Set, QNS)**.** Let X be a universal set. A *Quartic Neutrosophic Set* N in X is defined as:

$$N = \left\{(x, T_N(x), I_N(x), F_N(x)) \;\middle|\; x \in X\right\},$$

where:

- $T_N(x) : X \to [0, 1]$ is the *truth-membership degree* of $x \in X$.
- $I_N(x) : X \to [0, 1]$ is the *indeterminacy-membership degree* of $x \in X$.
- $F_N(x) : X \to [0, 1]$ is the *falsity-membership degree* of $x \in X$.

These degrees satisfy the following condition:

$$(T_N(x))^4 + (I_N(x))^4 + (F_N(x))^4 \leq 2, \quad \forall x \in X.$$

Definition 57 (Quintic Neutrosophic Set, QuNS)**.** Let X be a universal set. A *Quintic Neutrosophic Set* N in X is defined as:

$$N = \left\{(x, T_N(x), I_N(x), F_N(x)) \;\middle|\; x \in X\right\},$$

where:

- $T_N(x) : X \to [0, 1]$ is the *truth-membership degree* of $x \in X$.
- $I_N(x) : X \to [0, 1]$ is the *indeterminacy-membership degree* of $x \in X$.
- $F_N(x) : X \to [0, 1]$ is the *falsity-membership degree* of $x \in X$.

These degrees satisfy the following condition:

$$(T_N(x))^5 + (I_N(x))^5 + (F_N(x))^5 \leq 2, \quad \forall x \in X.$$

Definition 58 (Quartic Turiyam Neutrosophic Set, QTS)**.** Let X be a universal set. A *Quartic Turiyam Neutrosophic Set* T in X is defined as:

$$T = \left\{(x, t(x), iv(x), fv(x), lv(x)) \;\middle|\; x \in X\right\},$$

where:

- $t(x) : X \to [0, 1]$ is the *truth value* of $x \in X$.
- $iv(x) : X \to [0, 1]$ is the *indeterminacy value* of $x \in X$.
- $fv(x) : X \to [0, 1]$ is the *falsity value* of $x \in X$.
- $lv(x) : X \to [0, 1]$ is the *liberal state value* of $x \in X$.

These values satisfy the following condition:

$$(t(x))^4 + (iv(x))^4 + (fv(x))^4 + (lv(x))^4 \leq 3, \quad \forall x \in X.$$

Definition 59 (Quintic Turiyam Neutrosophic Set, QuTS)**.** Let X be a universal set. A *Quintic Turiyam Neutrosophic Set* T in X is defined as:

$$T = \left\{(x, t(x), iv(x), fv(x), lv(x)) \;\middle|\; x \in X\right\},$$

where:

- $t(x) : X \to [0,1]$ is the *truth value* of $x \in X$.
- $iv(x) : X \to [0,1]$ is the *indeterminacy value* of $x \in X$.
- $fv(x) : X \to [0,1]$ is the *falsity value* of $x \in X$.
- $lv(x) : X \to [0,1]$ is the *liberal state value* of $x \in X$.

These values satisfy the following condition:

$$(t(x))^5 + (iv(x))^5 + (fv(x))^5 + (lv(x))^5 \leq 3, \quad \forall x \in X.$$

Definition 60. (cf.[298, 299]) Let X be a universe of discourse. A *q-rung orthopair Neutrosophic set S* on X is defined as:

$$S = \left\{ (x, T_S(x), I_S(x), F_S(x)) \;\middle|\; x \in X \right\},$$

where:

- $T_S(x)$ is the truth-membership degree,
- $I_S(x)$ is the indeterminacy-membership degree,
- $F_S(x)$ is the falsity-membership degree.

These degrees satisfy the following constraint:

$$[T_S(x)]^q + [I_S(x)]^q + [F_S(x)]^q \leq 2, \quad \forall x \in X,$$

where $q \geq 1$ is a fixed positive integer, known as the *q-rung parameter*. The q-rung parameter controls the flexibility of the Neutrosophic set, allowing for more comprehensive modeling of uncertainty.

Theorem 61. *Let S be a q-rung orthopair Neutrosophic set defined on a universal set X, given by:*

$$S = \left\{ (x, T_S(x), I_S(x), F_S(x)) \;\middle|\; x \in X \right\},$$

where:

- $T_S(x)$ *is the truth-membership degree of* $x \in X$*,*
- $I_S(x)$ *is the indeterminacy-membership degree of* $x \in X$*,*
- $F_S(x)$ *is the falsity-membership degree of* $x \in X$*,*

satisfying the constraint:

$$(T_S(x))^q + (I_S(x))^q + (F_S(x))^q \leq 2, \quad \forall x \in X,$$

for some fixed positive integer $q \geq 1$*, known as the q-rung parameter.*

Then, S can be transformed into the following sets:

1. *Quartic Neutrosophic set by setting* $q = 4$.
2. *Quintic Neutrosophic set by setting* $q = 5$.
3. *Quartic Fuzzy set by setting* $q = 4$ *and assuming* $I_S(x) = 0$.
4. *Quintic Fuzzy set by setting* $q = 5$ *and assuming* $I_S(x) = 0$.

Proof. To prove that the q-rung orthopair Neutrosophic set can be transformed into each of these four sets, we consider each case separately:

1. *Transformation to Quartic Neutrosophic Set:*

   Let $q = 4$ in the definition of the q-rung orthopair Neutrosophic set. The condition becomes:

   $$(T_S(x))^4 + (I_S(x))^4 + (F_S(x))^4 \leq 2, \quad \forall x \in X.$$

   This matches the definition of a Quartic Neutrosophic set:

   $$N = \left\{ (x, T_N(x), I_N(x), F_N(x)) \;\middle|\; x \in X \right\},$$

   where:

   $$(T_N(x))^4 + (I_N(x))^4 + (F_N(x))^4 \leq 2, \quad \forall x \in X.$$

   Hence, a q-rung orthopair Neutrosophic set with $q = 4$ is equivalent to a Quartic Neutrosophic set.

2. *Transformation to Quintic Neutrosophic Set:*
   Let $q = 5$ in the definition of the q-rung orthopair Neutrosophic set. The condition becomes:

$$(T_S(x))^5 + (I_S(x))^5 + (F_S(x))^5 \leq 2, \quad \forall x \in X.$$

   This matches the definition of a Quintic Neutrosophic set:

$$N = \left\{(x, T_N(x), I_N(x), F_N(x)) \;\middle|\; x \in X\right\},$$

   where:

$$(T_N(x))^5 + (I_N(x))^5 + (F_N(x))^5 \leq 2, \quad \forall x \in X.$$

   Hence, a q-rung orthopair Neutrosophic set with $q = 5$ is equivalent to a Quintic Neutrosophic set.

3. *Transformation to Quartic Fuzzy Set:*
   Set $q = 4$ and assume $I_S(x) = 0$ in the q-rung orthopair Neutrosophic set. The condition simplifies to:

$$(T_S(x))^4 + (F_S(x))^4 \leq 1, \quad \forall x \in X.$$

   This matches the definition of a Quartic Fuzzy set:

$$A = \left\{(x, \mu_A(x), \nu_A(x)) \;\middle|\; x \in X\right\},$$

   where:

$$(\mu_A(x))^4 + (\nu_A(x))^4 \leq 1, \quad \forall x \in X.$$

   By identifying $T_S(x)$ as the membership degree $\mu_A(x)$ and $F_S(x)$ as the non-membership degree $\nu_A(x)$, we obtain the Quartic Fuzzy set.

4. *Transformation to Quintic Fuzzy Set:*
   Set $q = 5$ and assume $I_S(x) = 0$ in the q-rung orthopair Neutrosophic set. The condition simplifies to:

$$(T_S(x))^5 + (F_S(x))^5 \leq 1, \quad \forall x \in X.$$

   This matches the definition of a Quintic Fuzzy set:

$$Q = \left\{\left(x, \mu_Q(x), \nu_Q(x)\right) \;\middle|\; x \in X\right\},$$

   where:

$$\left(\mu_Q(x)\right)^5 + \left(\nu_Q(x)\right)^5 \leq 1, \quad \forall x \in X.$$

   By identifying $T_S(x)$ as the membership degree $\mu_Q(x)$ and $F_S(x)$ as the non-membership degree $\nu_Q(x)$, we obtain the Quintic Fuzzy set.

□

Definition 62. A *q-rung orthopair Turiyam Neutrosophic set (qROTS)* over a universe X is defined as:

$$S = \{(x, (t_S(x), iv_S(x), fv_S(x), lv_S(x))) \mid x \in X\},$$

where:

- $t_S(x) \in [0, 1]$ represents the degree of truth-membership of x in the set.
- $iv_S(x) \in [0, 1]$ represents the degree of indeterminacy-membership of x in the set.
- $fv_S(x) \in [0, 1]$ represents the degree of falsity-membership of x in the set.
- $lv_S(x) \in [0, 1]$ represents the degree of liberal state-membership of x in the set.

These membership degrees satisfy the following condition for each $x \in X$:

$$(t_S(x))^q + (iv_S(x))^q + (fv_S(x))^q + (lv_S(x))^q \leq 3,$$

where $q \geq 1$ is a fixed positive integer that determines the *q-rung* constraint.

Theorem 63. *Let S be a q-rung orthopair Turiyam Neutrosophic set (qROTS) defined over a universal set X, represented as:*

$$S = \left\{(x, (t_S(x), iv_S(x), fv_S(x), lv_S(x))) \;\middle|\; x \in X\right\},$$

where:

- $t_S(x) \in [0,1]$ *represents the degree of truth-membership,*
- $iv_S(x) \in [0,1]$ *represents the degree of indeterminacy-membership,*
- $fv_S(x) \in [0,1]$ *represents the degree of falsity-membership,*
- $lv_S(x) \in [0,1]$ *represents the degree of liberal state-membership.*

These degrees satisfy the following constraint:

$$(t_S(x))^q + (iv_S(x))^q + (fv_S(x))^q + (lv_S(x))^q \leq 3, \quad \forall x \in X,$$

where $q \geq 1$ is the q-rung parameter.

Then, S can be transformed into the following sets:

1. *Quartic Turiyam Neutrosophic set by setting $q = 4$.*
2. *Quintic Turiyam Neutrosophic set by setting $q = 5$.*
3. *Quartic Neutrosophic set by setting $q = 4$ and assuming $lv_S(x) = 0$.*
4. *Quintic Neutrosophic set by setting $q = 5$ and assuming $lv_S(x) = 0$.*

Proof. We prove that a q-rung orthopair Turiyam Neutrosophic set can be transformed into each of these four sets by considering each case separately:

1. *Transformation to Quartic Turiyam Neutrosophic Set:*
   Let $q = 4$ in the definition of the q-rung orthopair Turiyam Neutrosophic set. The condition becomes:

   $$(t_S(x))^4 + (iv_S(x))^4 + (fv_S(x))^4 + (lv_S(x))^4 \leq 3, \quad \forall x \in X.$$

   This matches the definition of a Quartic Turiyam Neutrosophic set:

   $$T = \left\{(x, t(x), iv(x), fv(x), lv(x)) \;\middle|\; x \in X\right\},$$

   where:
   $$(t(x))^4 + (iv(x))^4 + (fv(x))^4 + (lv(x))^4 \leq 3, \quad \forall x \in X.$$

   Hence, a q-rung orthopair Turiyam Neutrosophic set with $q = 4$ is equivalent to a Quartic Turiyam Neutrosophic set.

2. *Transformation to Quintic Turiyam Neutrosophic Set:*
   Let $q = 5$ in the definition of the q-rung orthopair Turiyam Neutrosophic set. The condition becomes:

   $$(t_S(x))^5 + (iv_S(x))^5 + (fv_S(x))^5 + (lv_S(x))^5 \leq 3, \quad \forall x \in X.$$

   This matches the definition of a Quintic Turiyam Neutrosophic set:

   $$T = \left\{(x, t(x), iv(x), fv(x), lv(x)) \;\middle|\; x \in X\right\},$$

   where:
   $$(t(x))^5 + (iv(x))^5 + (fv(x))^5 + (lv(x))^5 \leq 3, \quad \forall x \in X.$$

   Hence, a q-rung orthopair Turiyam Neutrosophic set with $q = 5$ is equivalent to a Quintic Turiyam Neutrosophic set.

3. *Transformation to Quartic Neutrosophic Set:*

   Set $q = 4$ and assume $lv_S(x) = 0$ in the q-rung orthopair Turiyam Neutrosophic set. The condition simplifies to:
   $$(t_S(x))^4 + (iv_S(x))^4 + (fv_S(x))^4 \leq 2, \quad \forall x \in X.$$
   This matches the definition of a Quartic Neutrosophic set:
   $$N = \left\{(x, T_N(x), I_N(x), F_N(x)) \mid x \in X\right\},$$
   where:
   $$(T_N(x))^4 + (I_N(x))^4 + (F_N(x))^4 \leq 2, \quad \forall x \in X.$$
   By identifying $t_S(x)$ as $T_N(x)$, $iv_S(x)$ as $I_N(x)$, and $fv_S(x)$ as $F_N(x)$, we obtain the Quartic Neutrosophic set.

4. *Transformation to Quintic Neutrosophic Set:*

   Set $q = 5$ and assume $lv_S(x) = 0$ in the q-rung orthopair Turiyam Neutrosophic set. The condition simplifies to:
   $$(t_S(x))^5 + (iv_S(x))^5 + (fv_S(x))^5 \leq 2, \quad \forall x \in X.$$
   This matches the definition of a Quintic Neutrosophic set:
   $$N = \left\{(x, T_N(x), I_N(x), F_N(x)) \mid x \in X\right\},$$
   where:
   $$(T_N(x))^5 + (I_N(x))^5 + (F_N(x))^5 \leq 2, \quad \forall x \in X.$$
   By identifying $t_S(x)$ as $T_N(x)$, $iv_S(x)$ as $I_N(x)$, and $fv_S(x)$ as $F_N(x)$, we obtain the Quintic Neutrosophic set.

□

Definition 64 (Quartic Fuzzy Graph, QFG)**.** Let $G = (V, E)$ be a simple graph, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. A *Quartic Fuzzy Graph G* is defined as:
$$G = \left\{(v, \mu_G(v), \nu_G(v)) \mid v \in V\right\},$$
where:

- $\mu_G(v) : V \to [0, 1]$ is the *membership degree* of vertex v.
- $\nu_G(v) : V \to [0, 1]$ is the *non-membership degree* of vertex v.

These degrees satisfy the following condition:
$$(\mu_G(v))^4 + (\nu_G(v))^4 \leq 1, \quad \forall v \in V.$$
The *indeterminacy degree* for any vertex $v \in V$ in the Quartic Fuzzy Graph G is defined as:
$$\pi(v) = \sqrt[4]{1 - \left((\mu_G(v))^4 + (\nu_G(v))^4\right)}.$$

Definition 65 (Quintic Fuzzy Graph, QuFG)**.** Let $G = (V, E)$ be a simple graph, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. A *Quintic Fuzzy Graph G* is defined as:
$$G = \left\{(v, \mu_G(v), \nu_G(v)) \mid v \in V\right\},$$
where:

- $\mu_G(v) : V \to [0, 1]$ is the *membership degree* of vertex v.
- $\nu_G(v) : V \to [0, 1]$ is the *non-membership degree* of vertex v.

These degrees satisfy the following condition:

$$(\mu_G(v))^5 + (\nu_G(v))^5 \leq 1, \quad \forall v \in V.$$

The *hesitancy degree* for any vertex $v \in V$ in the Quintic Fuzzy Graph G is defined as:

$$\pi_5(v) = 1 - \left((\mu_G(v))^5 + (\nu_G(v))^5\right).$$

Definition 66 (Quartic Neutrosophic Graph, QNG)**.** Let $G = (V, E)$ be a simple graph, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. A *Quartic Neutrosophic Graph* G is defined as:

$$G = \left\{(v, T_G(v), I_G(v), F_G(v)) \;\middle|\; v \in V\right\},$$

where:

- $T_G(v) : V \to [0, 1]$ is the *truth-membership degree* of vertex v.
- $I_G(v) : V \to [0, 1]$ is the *indeterminacy-membership degree* of vertex v.
- $F_G(v) : V \to [0, 1]$ is the *falsity-membership degree* of vertex v.

These degrees satisfy the following condition:

$$(T_G(v))^4 + (I_G(v))^4 + (F_G(v))^4 \leq 2, \quad \forall v \in V.$$

Definition 67 (Quintic Neutrosophic Graph, QuNG)**.** Let $G = (V, E)$ be a simple graph, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. A *Quintic Neutrosophic Graph* G is defined as:

$$G = \left\{(v, T_G(v), I_G(v), F_G(v)) \;\middle|\; v \in V\right\},$$

where:

- $T_G(v) : V \to [0, 1]$ is the *truth-membership degree* of vertex v.
- $I_G(v) : V \to [0, 1]$ is the *indeterminacy-membership degree* of vertex v.
- $F_G(v) : V \to [0, 1]$ is the *falsity-membership degree* of vertex v.

These degrees satisfy the following condition:

$$(T_G(v))^5 + (I_G(v))^5 + (F_G(v))^5 \leq 2, \quad \forall v \in V.$$

Definition 68 (Quartic Turiyam Neutrosophic Graph, QTG)**.** Let $G = (V, E)$ be a simple graph, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. A *Quartic Turiyam Neutrosophic Graph* G is defined as:

$$G = \left\{(v, t(v), iv(v), fv(v), lv(v)) \;\middle|\; v \in V\right\},$$

where:

- $t(v) : V \to [0, 1]$ is the *truth value* of vertex v.
- $iv(v) : V \to [0, 1]$ is the *indeterminacy value* of vertex v.
- $fv(v) : V \to [0, 1]$ is the *falsity value* of vertex v.
- $lv(v) : V \to [0, 1]$ is the *liberal state value* of vertex v.

These values satisfy the following condition:

$$(t(v))^4 + (iv(v))^4 + (fv(v))^4 + (lv(v))^4 \leq 3, \quad \forall v \in V.$$

Definition 69 (Quintic Turiyam Neutrosophic Graph, QuTG)**.** Let $G = (V, E)$ be a simple graph, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. A *Quintic Turiyam Neutrosophic Graph* G is defined as:

$$G = \left\{(v, t(v), iv(v), fv(v), lv(v)) \;\middle|\; v \in V\right\},$$

where:

- $t(v) : V \to [0,1]$ is the *truth value* of vertex v.
- $iv(v) : V \to [0,1]$ is the *indeterminacy value* of vertex v.
- $fv(v) : V \to [0,1]$ is the *falsity value* of vertex v.
- $lv(v) : V \to [0,1]$ is the *liberal state value* of vertex v.

These values satisfy the following condition:

$$(t(v))^5 + (iv(v))^5 + (fv(v))^5 + (lv(v))^5 \leq 3, \quad \forall v \in V.$$

Definition 70. A *q-rung orthopair Neutrosophic graph* is defined as a pair $G = (P, Q)$, where:

- P is a q-rung orthopair Neutrosophic set defined on the set of vertices V.
- Q is a q-rung orthopair Neutrosophic set defined on the set of edges $E \subseteq V \times V$.

  For each vertex $u \in V$, we have:

$$[T_P(u)]^q + [I_P(u)]^q + [F_P(u)]^q \leq 2, \quad \forall u \in V.$$

For each edge $(u, v) \in E$, we have:

$$\left[T_Q(u,v)\right]^q + \left[I_Q(u,v)\right]^q + \left[F_Q(u,v)\right]^q \leq 2, \quad \forall (u,v) \in E.$$

The relationship between the degrees of vertices and edges is defined as follows:

$$\begin{aligned} T_Q(u,v) &\leq \min\{T_P(u), T_P(v)\}, \\ I_Q(u,v) &\leq \min\{I_P(u), I_P(v)\}, \\ F_Q(u,v) &\geq \max\{F_P(u), F_P(v)\}. \end{aligned}$$

If $T_Q(u,v) = I_Q(u,v) = F_Q(u,v) = 0$, then the edge (u,v) does not exist in the graph, i.e., $(u,v) \notin E$.

Definition 71. A *q-rung orthopair Turiyam Neutrosophic graph (qROTG)* is defined as a pair $G = (V, E)$, where:

- V is a set of vertices, and $E \subseteq V \times V$ is the set of edges.
- Each vertex $v \in V$ is associated with a quadruple:

$$(t_V(v), iv_V(v), fv_V(v), lv_V(v)),$$

  where:

  - $t_V(v)$ is the truth-membership degree,
  - $iv_V(v)$ is the indeterminacy-membership degree,
  - $fv_V(v)$ is the falsity-membership degree,
  - $lv_V(v)$ is the liberal state-membership degree,
  - and they satisfy the condition:

$$(t_V(v))^q + (iv_V(v))^q + (fv_V(v))^q + (lv_V(v))^q \leq 3.$$

- Each edge $e = (u,v) \in E$ is associated with a quadruple:

$$(t_E(u,v), iv_E(u,v), fv_E(u,v), lv_E(u,v)),$$

  where:

  - $t_E(u,v)$ is the truth-membership degree of the edge,
  - $iv_E(u,v)$ is the indeterminacy-membership degree of the edge,
  - $fv_E(u,v)$ is the falsity-membership degree of the edge,
  - $lv_E(u,v)$ is the liberal state-membership degree of the edge,

– and they satisfy:

$$(t_E(u,v))^q + (iv_E(u,v))^q + (fv_E(u,v))^q + (lv_E(u,v))^q \leq 3.$$

Additionally, the following constraints hold for the vertices and edges:

1. For any vertices $u, v \in V$, if $(u, v) \in E$:

$$t_E(u,v) \leq \min\{t_V(u), t_V(v)\},$$

$$iv_E(u,v) \leq \min\{iv_V(u), iv_V(v)\},$$

$$fv_E(u,v) \geq \max\{fv_V(u), fv_V(v)\},$$

$$lv_E(u,v) \leq \min\{lv_V(u), lv_V(v)\}.$$

2. If $t_E(u,v) = iv_E(u,v) = fv_E(u,v) = lv_E(u,v) = 0$, then no edge exists between u and v.

The *q-rung orthopair Turiyam Neutrosophic graph* extends classical graph theory by incorporating multiple membership degrees with the flexibility of q-rung orthopairs, enabling a more nuanced representation of uncertainty, indeterminacy, and liberal states.

Theorem 72. *Let G be a q-rung orthopair Turiyam Neutrosophic graph (qROTG) defined as:*

$$G = (V, E),$$

where:

- *Each vertex $v \in V$ is associated with a quadruple $(t_V(v), iv_V(v), fv_V(v), lv_V(v))$,*
- *Each edge $e = (u, v) \in E$ is associated with a quadruple $(t_E(u,v), iv_E(u,v), fv_E(u,v), lv_E(u,v))$.*

The degrees satisfy the following condition:

$$(t_V(v))^q + (iv_V(v))^q + (fv_V(v))^q + (lv_V(v))^q \leq 3, \quad \forall v \in V,$$

$$(t_E(u,v))^q + (iv_E(u,v))^q + (fv_E(u,v))^q + (lv_E(u,v))^q \leq 3, \quad \forall (u,v) \in E,$$

where $q \geq 1$ is the q-rung parameter.

Then, G can be transformed into the following graphs:

1. *q-rung orthopair Neutrosophic graph by setting $lv_V(v) = 0$ and $lv_E(u,v) = 0$.*
2. *Quartic Turiyam Neutrosophic graph by setting $q = 4$.*
3. *Quintic Turiyam Neutrosophic graph by setting $q = 5$.*
4. *Quartic Neutrosophic graph by setting $q = 4$ and assuming $lv_V(v) = 0$ and $lv_E(u,v) = 0$.*
5. *Quintic Neutrosophic graph by setting $q = 5$ and assuming $lv_V(v) = 0$ and $lv_E(u,v) = 0$.*
6. *Quartic fuzzy graph by setting $q = 4$, $iv_V(v) = 0$, $fv_V(v) = 0$, $lv_V(v) = 0$, $iv_E(u,v) = 0$, $fv_E(u,v) = 0$, and $lv_E(u,v) = 0$.*
7. *Quintic fuzzy graph by setting $q = 5$, $iv_V(v) = 0$, $fv_V(v) = 0$, $lv_V(v) = 0$, $iv_E(u,v) = 0$, $fv_E(u,v) = 0$, and $lv_E(u,v) = 0$.*

Proof. We prove that a q-rung orthopair Turiyam Neutrosophic graph can be transformed into each of these specific graphs:

1. *Transformation to q-rung orthopair Neutrosophic graph:*
   By setting $lv_V(v) = 0$ and $lv_E(u,v) = 0$, the condition becomes:

$$(t_V(v))^q + (iv_V(v))^q + (fv_V(v))^q \leq 2, \quad \forall v \in V,$$

$$(t_E(u,v))^q + (iv_E(u,v))^q + (fv_E(u,v))^q \leq 2, \quad \forall (u,v) \in E.$$

   This matches the definition of a q-rung orthopair Neutrosophic graph.

2. *Transformation to Quartic Turiyam Neutrosophic graph:*
   Set $q = 4$. The condition becomes:
$$(t_V(v))^4 + (iv_V(v))^4 + (fv_V(v))^4 + (lv_V(v))^4 \leq 3, \quad \forall v \in V,$$
$$(t_E(u,v))^4 + (iv_E(u,v))^4 + (fv_E(u,v))^4 + (lv_E(u,v))^4 \leq 3, \quad \forall (u,v) \in E.$$
   This matches the definition of a Quartic Turiyam Neutrosophic graph.
3. *Transformation to Quintic Turiyam Neutrosophic graph:*
   Set $q = 5$. The condition becomes:
$$(t_V(v))^5 + (iv_V(v))^5 + (fv_V(v))^5 + (lv_V(v))^5 \leq 3, \quad \forall v \in V,$$
$$(t_E(u,v))^5 + (iv_E(u,v))^5 + (fv_E(u,v))^5 + (lv_E(u,v))^5 \leq 3, \quad \forall (u,v) \in E.$$
   This matches the definition of a Quintic Turiyam Neutrosophic graph.
4. *Transformation to Quartic Neutrosophic graph:*
   Set $q = 4$ and assume $lv_V(v) = 0$ and $lv_E(u,v) = 0$. The condition becomes:
$$(t_V(v))^4 + (iv_V(v))^4 + (fv_V(v))^4 \leq 2, \quad \forall v \in V,$$
$$(t_E(u,v))^4 + (iv_E(u,v))^4 + (fv_E(u,v))^4 \leq 2, \quad \forall (u,v) \in E.$$
   This matches the definition of a Quartic Neutrosophic graph.
5. *Transformation to Quintic Neutrosophic graph:*
   Set $q = 5$ and assume $lv_V(v) = 0$ and $lv_E(u,v) = 0$. The condition becomes:
$$(t_V(v))^5 + (iv_V(v))^5 + (fv_V(v))^5 \leq 2, \quad \forall v \in V,$$
$$(t_E(u,v))^5 + (iv_E(u,v))^5 + (fv_E(u,v))^5 \leq 2, \quad \forall (u,v) \in E.$$
   This matches the definition of a Quintic Neutrosophic graph.
6. *Transformation to Quartic fuzzy graph:*
   Set $q = 4$, $iv_V(v) = 0$, $fv_V(v) = 0$, $lv_V(v) = 0$, $iv_E(u,v) = 0$, $fv_E(u,v) = 0$, and $lv_E(u,v) = 0$. The condition becomes:
$$(t_V(v))^4 \leq 1, \quad \forall v \in V,$$
$$(t_E(u,v))^4 \leq 1, \quad \forall (u,v) \in E.$$
   This matches the definition of a Quartic fuzzy graph, where the membership and non-membership degrees are represented by $t_V(v)$ and $t_E(u,v)$.
7. *Transformation to Quintic fuzzy graph:*
   Set $q = 5$, $iv_V(v) = 0$, $fv_V(v) = 0$, $lv_V(v) = 0$, $iv_E(u,v) = 0$, $fv_E(u,v) = 0$, and $lv_E(u,v) = 0$. The condition becomes:
$$(t_V(v))^5 \leq 1, \quad \forall v \in V,$$
$$(t_E(u,v))^5 \leq 1, \quad \forall (u,v) \in E.$$
   This matches the definition of a Quintic fuzzy graph, where the membership and non-membership degrees are represented by $t_V(v)$ and $t_E(u,v)$.

□

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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Survey of Trees, Forests, and Paths in Fuzzy and Neutrosophic Graphs

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Abstract: In this paper, we conduct a comprehensive study of Trees, Forests, and Paths within the framework of Fuzzy and Neutrosophic Graphs. Graph theory, known for its wide-ranging applications across various fields, is extended here to account for uncertainties through the use of fuzzy and neutrosophic structures. These graphs introduce degrees of truth, indeterminacy, and falsity to better model real-world complexities. Our research delves into the classification and properties of graph structures like trees and paths in this context, aiming to offer deeper insights and contribute to ongoing developments in the field of graph theory.

Keywords: Neutrosophic graph, Turiyam Neutrosophic graph, Fuzzy graph, Tree

1. Introduction

1.1 Graph Theory

Graph theory, as many readers may already know, is a fundamental branch of mathematics that studies networks made up of nodes (vertices) and connections (edges). It focuses on analyzing the paths, structures, and properties of these networks [104]. One of the advantages of using graphs is their ability to visually and conceptually represent connections between real-world elements. Because of this, graph theory has become a critical tool in numerous fields and is widely applied across various disciplines [27, 53, 95, 124, 158, 261, 339].

For example, in artificial intelligence (AI), research on graph neural networks, which heavily rely on concepts from graph theory, is a rapidly growing area [120, 297, 349, 350, 366].

1.2 Graph Classes

One of the strengths of graph theory is its ability to classify graphs into distinct types based on shared properties or structural characteristics. These graph classes provide a foundation for designing efficient algorithms, simplifying problem-solving, and offering deeper insights into computational complexity (cf.[28, 54, 58, 244]).

Some well-known examples of graph classes include: Tree Graphs [322], Path Graphs [354], Complete Graphs [82], Trapezoid Graphs [89], Circle Graphs [57], Unit Disk Graphs [85], Circular-Arc Graphs [145], and Edge-Transitive Graphs [228, 229].

There has been extensive research on various graph structures. A *tree* is a connected, acyclic graph, meaning it contains no cycles and has exactly one path between any two vertices. A tree with n vertices always has $n - 1$ edges. In contrast, a *forest* is a collection of disjoint trees, each of which is an acyclic and connected subgraph. A *path* in a graph is a sequence of distinct vertices connected by edges. In an undirected graph, a path is simply a sequence of vertices where each consecutive pair is connected by an edge. Paths are often used to represent how one vertex can be reached from another.

These graph classes and structures have been widely studied due to their practical applications and ease of handling.

1.3 Fuzzy Graphs and Neutrosophic Graphs

To address the uncertainties present in the world, various types of graphs like fuzzy, neutrosophic, Turiyam, and plithogenic graphs have been studied.

A fuzzy graph assigns a membership degree between 0 and 1 to each edge and vertex, representing the level of uncertainty. Simply put, a fuzzy graph is a graphical representation of a fuzzy set (cf. [113, 201, 358, 359]). In real-world applications, fuzzy graphs are used in areas such as social networks, decision-making, and transportation systems to model relationships or connections that are not precise or are uncertain [237, 287]. This wide range of applicability has led to significant research interest in fuzzy graphs.

Recently, neutrosophic graphs [13,14,63,159,166,175,291,311,315] and neutrosophic hypergraphs [12, 15,99,215] have also gained attention within the framework of neutrosophic set theory [18,320]. "Neutrosophic" extends classical and fuzzy logic by introducing degrees of truth, indeterminacy, and falsity, offering a more comprehensive way to handle uncertainty.

Following this, the Turiyam Neutrosophic graph was introduced as a further extension of neutrosophic and fuzzy graphs. A Turiyam Neutrosophic graph stands out by assigning four attributes to each vertex and edge: truth, indeterminacy, falsity, and a liberal state, thus broadening the framework of neutrosophic and fuzzy graphs [140–142]. Note that the Turiyam Neutrosophic Graph is, in fact, a specific case of the Quadripartitioned Neutrosophic Graph, achieved by replacing "Contradiction" with "Liberal." (cf.[307, 312, 313]) Additionally, plithogenic graphs have emerged as a more generalized form and are currently a subject of active study [137, 176,304,324,325].

1.4 Our Contribution

This subsection explains our contributions in this paper. Research on Fuzzy Graphs, Neutrosophic Graphs, and the study of various graph classes and structures is of significant importance, though it cannot be said that these areas have been fully explored. In this paper, we conduct a comprehensive investigation into Trees, Forests, and Paths within the framework of Fuzzy and Neutrosophic Graphs. Additionally, we introduce concepts such as Quasi-Trees, Quasi-Forests, and Quasi-Paths. We hope that this work will contribute, even in a small way, to further advancements in graph theory research.

1.5 The Structure of the Paper

The format of this paper is described below.

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2. Preliminaries and definitions

In this section, we present a brief overview of the definitions and notations used throughout this paper. We will specifically cover fundamental concepts related to graphs, including fuzzy graphs, intuitionistic fuzzy graphs, Turiyam Neutrosophic graphs, neutrosophic graphs, and plithogenic graphs.

Additionally, please note that this paper may also incorporate concepts from set theory alongside graph theory. For a more comprehensive understanding of set theory, you may refer to the relevant surveys or notes [162, 174, 199].

2.1 Basic Graph Concepts

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [102, 103, 103, 104, 158, 346].

Definition 1 (Graph)**.** [104] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2 (Subgraph)**.** [104] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Example 3 (Example of a Subgraph)**.** Consider a graph $G = (V, E)$ with the vertex set $V = \{1, 2, 3, 4, 5\}$ and the edge set

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}.$$

This graph G can be visualized as having 5 vertices connected by 5 edges.

Now, let us define a subgraph G' by selecting the subset of vertices $V' = \{1, 2, 4\}$ and the subset of edges $E' = \{\{1, 2\}, \{2, 4\}\}$. Thus, the subgraph $G' = (V', E')$ has:

- The vertex set $V' = \{1, 2, 4\}$,
- The edge set $E' = \{\{1, 2\}, \{2, 4\}\}$.

In this example, G' is a valid subgraph of G because both $V' \subseteq V$ and $E' \subseteq E$, and the edges in E' only connect vertices within V'.

Definition 4 (Degree)**.** [104] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 5. (cf.[160, 360]) A graph $G = (V, E)$ is called a *connected graph* if for any two vertices $u, v \in V$, there exists a path between them. A path is a sequence of distinct vertices $v_1, v_2, \dots, v_k$ such that $(v_i, v_{i+1}) \in E$ for all $i = 1, \dots, k - 1$.

In other words, G is connected if, for any pair of vertices in V, there is a way to travel from one vertex to the other by following the edges of the graph.

Definition 6. (cf.[301,306,348]) A graph $G = (V, E)$ is called an *acyclic graph* if it contains no cycles. That is, for any sequence of vertices $v_1, v_2, \dots, v_k$ such that there is an edge between each consecutive pair (v_i, v_{i+1}) for $i = 1, \dots, k - 1$, and between v_k and v_1, the sequence does not form a cycle.

Formally, G is acyclic if for any subset of edges $E' \subseteq E$, there is no closed path that starts and ends at the same vertex.

Definition 7 (Complete Graph)**.** (cf.[35, 108]) A *complete graph* is a graph $G = (V, E)$ in which every pair of distinct vertices is connected by a unique edge. Formally, a graph $G = (V, E)$ is complete if for every pair of vertices $u, v \in V$ with $u \neq v$, there exists an edge $\{u, v\} \in E$.

The complete graph on n vertices is denoted by K_n, and it has the following properties:

- The number of vertices is $|V| = n$.
- The number of edges is $|E| = \binom{n}{2} = \frac{n(n-1)}{2}$.
- Each vertex has degree $\deg(v) = n - 1$ for all $v \in V$.

Definition 8. (cf.[55,106,119,164,347]) Two graphs $G = (V, E)$ and $H = (V', E')$ are said to be *homomorphic* if there exists a mapping $\phi : V \to V'$ such that for every edge $(u, v) \in E$, the image $(\phi(u), \phi(v))$ is an edge in E'. In other words, there is a structure-preserving mapping from G to H that maintains the adjacency relationships between vertices.

2.2 Classic class about Tree, Forest and path

We introduce some representative classical graph classes. These graph classes have also been studied in the contexts of fuzzy, neutrosophic, and plithogenic settings.

2.2.1 Tree

A tree graph is one of the most fundamental structures in graph theory. Its definition is provided below.

Definition 9 (Tree)**.** (cf.[363]) A tree is a connected, acyclic graph. In other words, a tree is a graph where there is exactly one path between any two vertices, and no cycles exist.

Example 10. Consider a graph with 4 vertices

$$V = \{V_A, V_B, V_C, V_D\}$$

and 3 edges

$$E = \{(V_A, V_B), (V_B, V_C), (V_C, V_D)\}$$

. This graph is connected, has no cycles, and there is exactly one path between any pair of vertices. Hence, this graph is a tree.

The related concepts and research fields involving trees are outlined as follows. Due to the tree's simplicity and versatility, it is widely used in various domains and theoretical concepts.

- Spanning-tree problem: A spanning tree is a subgraph that connects all vertices in a graph without any cycles, using the minimum edges. A minimal spanning tree (MST) is a subset of edges that connects all vertices in a weighted graph with the minimum possible total edge weight, forming a tree structure without any cycles. It ensures every vertex is reachable while minimizing the overall connection cost[81, 138, 156, 157, 163, 265, 355]. The maximum spanning tree problem has also been studied in a similar manner [29, 223]. As an applied concept, research has been conducted on Spanning Tree Protocol and related topics in the field of networking[153, 212, 220]. The spanning tree is highly applicable because it connects all vertices/concepts with the minimum cost or edges. As a result, there is extensive research on its practical applications.
- Tree-width: Tree-width is a graph parameter measuring how closely a graph resembles a tree. It represents the minimum width of a tree decomposition, indicating the graph's complexity in terms of connectivity. Given its effectiveness in capturing the structural complexity of graphs, tree-width has been widely studied, especially for its role in solving computational problems and algorithm design[48–51, 283]. Furthermore, numerous studies have explored other graph width parameters, such as rank-width, branch-width, path-width, proper-path-width, and linear-width, as well as related concepts beyond tree-width [5, 24, 83, 105, 127, 333].
- Decision tree: A decision tree is a flowchart-like model used for decision-making and classification. It splits data into branches based on feature values, leading to outcomes or predictions at the leaves[125, 193, 240, 272, 290, 319]. Decision trees are commonly used in decision-making[216, 271] and artificial intelligence[75] fields. Strictly speaking, decision trees are used in the field of logic rather than graph theory. However, they are introduced here as a reference.
- Binary tree: A binary tree is a type of data structure[101, 218]. It is a rooted tree structure where each node has at most two children, typically referred to as "left" and "right." For example, binary trees are commonly used to implement binary search algorithms[204] and binary heaps[114]. Related concepts, such as the extended binary tree[73] and the weakly binary tree[345], are well known. Additionally, related concepts to binary trees include the ternary tree[196, 225, 259], quad tree [221, 224, 318], d-ary tre [88, 110], and octree[37, 151], which are also well known.
- Hypertree: A hypertree is an extension of a tree structure to a hypergraph. Research topics include hypertree-width [4, 5, 154, 155] and the spanning-hypertree problem [16, 68].
- Self-balancing binary search tree: A self-balancing binary search tree automatically adjusts its structure during insertions and deletions to maintain a balanced height, ensuring efficient search, insertion, and deletion operations[25, 71]. Examples of self-balancing binary search trees include the AVL tree [123], red-black tree [40, 249], splay tree [39], scapegoat tree [139], and AA tree [26]. Self-balancing binary search trees are among the most efficient data structures for implementing abstract data types such as associative arrays [1] and sets.

[1] Associative arrays, also called maps or dictionaries, are data structures that store key-value pairs, allowing efficient lookup, insertion, and deletion by key [38, 191].

- B-tree: A B-tree is a self-balancing tree data structure that maintains sorted data and allows efficient insertion, deletion, and search operations, commonly used in databases and file systems[34]. B-trees are widely used in real-world systems, and many database management systems implement indexing using B-trees[285, 361]. Related concepts to B-trees include B$^+$ tree[340], AS B-tree[286], UB-tree[222], and H-tree.

Notation 11. In this paper, we define the term "Related graph class" as a graph class that either extends or restricts a corresponding graph class in some way.

Theorem 12. *The following are examples of graph classes or structures related to trees, including but not limited to:*

- *Starlike tree: A starlike tree is a tree with one central vertex of degree greater than 2, connected to multiple linear paths[242].*
- *Caterpillar tree: A caterpillar tree is a tree where all vertices are within distance 1 of a central path[22, 87, 117, 362]. The caterpillar tree is also used in chemistry and physics.*
- *Lobster tree: A lobster tree is a tree where removing the leaf nodes results in a caterpillar tree[247, 279].*
- *Regular tree: A regular tree is a balanced tree where each vertex has exactly k children, forming a consistent branching structure[263, 282, 337].*
- *Irregular tree: Irregular trees are trees where the degrees of the vertices are not all the same, differing from regular or balanced trees[109, 121].*
- *Hypertree:A hypertree is a generalization of a tree in hypergraphs, where hyperedges can connect multiple vertices simultaneously[4, 5, 154, 155].*
- *Superhypertree: A superhypertree is a generalization of a tree in superhypergraphs [130, 147]. It is worth noting that a superhypergraph is known as a generalization of a hypergraph[134, 310, 311].*
- *Multitree: A multitree is a directed acyclic graph where each node has a unique parent path, resembling a tree-like structure[6, 126].*
- *Polytree (also called directed tree, oriented tree, or singly connected network): A polytree is a directed acyclic graph (DAG) whose underlying undirected graph is a connected, acyclic tree[198].*
- *Directed tree: A directed tree is a tree where each edge has a direction, forming a hierarchy with a single root[78, 194].*
- *co-tree: Co-tree is the complement of a spanning tree, consisting of edges not in the spanning tree of a graph[45, 252, 277].*
- *Banana tree: A banana tree is formed by connecting one leaf from each of star graphs to a single, distinct root vertex[41, 253, 341].*
- *Bicentered tree: A bicenter tree is a tree with two central vertices, each having minimal eccentricity, and these vertices are adjacent[187].*
- *Cayley tree: A Cayley tree is a tree where each non-leaf vertex has a constant number of branches, often denoted as an n-Cayley tree[122, 254].*
- *Centered tree: A centered tree is a tree with exactly one central vertex, having the smallest eccentricity compared to other vertices[187].*
- *Weighted tree: A weighted tree is a tree where each edge is assigned a numerical weight, representing costs or distances between vertices[96, 112, 116, 186].*
- *Random spanning trees: A random spanning tree is a spanning tree chosen uniformly at random from all possible spanning trees of a graph[69].*
- *Spanning directed trees: Directed Version of Spanning trees[19, 90].*
- *Steiner tree: A Steiner tree is a minimum-weight tree that connects a given subset of vertices (terminals) in a graph, possibly using additional vertices [181, 190, 264, 269, 284].*
- *Directed Steiner tree: Directed Version of a Steiner tree[76, 288, 327].*
- *Pseudo tree: A pseudo tree is a connected graph with at most one cycle, including both trees and unicyclic graphs[198, 227].*
- *Oriented tree: An oriented tree is a directed graph that becomes a tree when its directed edges are replaced with undirected edges[2, 111, 192, 250].*

- *Ordered tree: An ordered tree is a tree where the children of each node have a defined order, typically represented by lists or numbered edges [214, 255, 258].*
- *Soft Tree and Hypersoft Tree: It represents a Tree in Soft Graphs and Hypersoft Graphs [289, 336]. Note that Soft Graphs and Hypersoft Graphs are graph concepts used to manage uncertainty[23, 132, 135, 167, 168, 305].*

Proof. Please refer to the respective papers listed in the references. □

In this paper, the concept of a spanning tree will also be considered as necessary. The definition is provided below.

Definition 13. (cf.[81, 157]) In graph theory, a *spanning tree* of a graph $G = (V, E)$ is a subgraph $T = (V, E_T)$ where $E_T \subseteq E$ such that:

- T includes all vertices of G (i.e., $V(T) = V(G)$).
- T is a tree, meaning it is connected and acyclic.

A spanning tree exists if and only if the graph G is connected. If G is disconnected, no spanning tree can be formed.

Example 14. Consider the following graph $G = (V, E)$, where:

$$V(G) = \{1, 2, 3, 4\}, \quad E(G) = \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}.$$

A *spanning tree* T of G can be constructed as follows:

$$V(T) = V(G) = \{1, 2, 3, 4\}, \quad E_T = \{(1, 2), (2, 4), (3, 4)\}.$$

This subgraph T includes all vertices of G and is acyclic and connected, fulfilling the requirements of a spanning tree.

Note that multiple spanning trees can exist for a graph. Another valid spanning tree for the same graph G could be:

$$V(T) = V(G), \quad E_T = \{(1, 3), (3, 4), (2, 4)\}.$$

Both are valid spanning trees because they include all the vertices of G and are connected and acyclic.

2.2.2 Forest

Next, the definition of a forest is provided below. Similar to trees, a forest is also one of the fundamental concepts in graph theory.

Definition 15. (cf.[205,234,266]) A Forest is an undirected graph that is acyclic, meaning it contains no cycles. It consists of one or more disjoint trees, where each tree is a connected subgraph with no cycles.

Formally, a graph $F = (V, E)$ is a forest if:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- Each connected component of F is a tree, i.e., there is no path in any component that returns to the starting vertex, which implies no cycles exist in F.

In simpler terms, a forest is a collection of trees, and if a forest consists of only one connected component, it is itself a tree.

Example 16. Consider a graph with 6 vertices

$$V = \{V_A, V_B, V_C, V_D, V_E, V_F\}$$

and 4 edges

$$E = \{(V_A, V_B), (V_B, V_C), (V_D, V_E), (V_E, V_F)\}$$

. The graph consists of two disjoint components: one component with vertices $\{V_A, V_B, V_C\}$ and edges

$$(V_A, V_B), (V_B, V_C)$$

, and another component with vertices $\{V_D, V_E, V_F\}$ and edges

$$(V_D, V_E), (V_E, V_F)$$

. Each component is a tree, and thus, the entire graph is a forest.

2.2.3 Path

The definition of a path is provided below.

Definition 17 (Path)**.** (cf.[363]) A path in a graph $G = (V, E)$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \ldots, k-1$. A path is represented as:

$$P = (v_1, v_2, \ldots, v_k),$$

where no vertex is repeated. The length of a path is the number of edges it contains, i.e., $k - 1$.

Example 18. Consider a graph $G = (V, E)$, where

$$V = \{v_1, v_2, v_3, v_4, v_5\}$$

and

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\}$$

. A path in G is:

$$P = (v_1, v_2, v_3, v_4, v_5),$$

with length 4, as there are 4 edges in the path.

The related concepts and research fields involving paths are outlined as follows. Due to the tree's simplicity and versatility, it is widely used in various domains and theoretical concepts.

- shortest path problem: The shortest path problem involves finding the minimum distance or path between two vertices in a graph, optimizing based on edge weights or travel costs [7,171,298,342,356]. Similarly, the longest path problem has also been studied[169, 178, 179]. In particular, the shortest path has a wide range of real-life applications, and extensive research has also been conducted on directed graphs[107,338].
- path-width: Pathwidth is a graph parameter that measures how closely a graph resembles a path. It represents the minimum width of a path decomposition, indicating the graph's linearity and complexity[97, 177, 330–333].
- hamiltonian path problem: A Hamiltonian path is a path in a graph that visits each vertex exactly once. It explores the entire graph without revisiting any vertices or forming cycles[172, 231, 256].
- induced path problem: An induced path in a graph is a sequence of vertices where each pair of consecutive vertices is connected by an edge, and no additional edges exist between any nonconsecutive vertices in the sequence[74, 226, 238]
- path partition problem: The path-partition problem in graph theory involves dividing the vertices of a graph into the fewest number of disjoint induced paths, optimizing based on constraints like connectivity or edge distribution[80, 219, 233].

2.2.4 Quasi-Tree,Forest,and path

Concepts closely related to trees, forests, and paths include the quasi-tree, quasi-forest, and quasi-path. Additionally, their generalizations, k-quasi-tree, k-quasi-forest, and k-quasi-path, are also well-known. These are sometimes referred to as apex-tree[9, 17, 189, 210, 211], apex-forest[52, 56, 165], and apex-path[165]. The definitions are provided below.

Definition 19. (cf.[170,202,209,270,326]) A connected graph $G = (V, E)$ is called a *quasi-tree* if there exists a vertex $v \in V(G)$ such that the graph $G - v$ is a tree. In other words, removing one vertex from the graph results in a tree structure.

A connected graph $G = (V, E)$ is called a *k-quasi-tree* if k is the smallest positive integer such that there exists a k-element subset $U \subseteq V(G)$ of vertices where $G - U$ is a tree. In other words, removing exactly k vertices from G results in a tree.

The following is clearly valid.

Proposition 20. *A quasi-tree or a k-quasi-tree can be transformed into a tree.*

Proof. **Case 1: Quasi-tree**

By definition, a quasi-tree $G = (V, E)$ is a connected graph such that there exists a vertex $v \in V(G)$ where $G - v$ is a tree. A tree is defined as a connected graph with no cycles and exactly $|V| - 1$ edges.

Removing v from G eliminates all cycles, leaving $G - v$ as a tree. To transform G into a tree:

- Remove the vertex v and all edges incident to v.
- The resulting graph $G - v$ is a tree.

Case 2: k-quasi-tree

By definition, a k-quasi-tree $G = (V, E)$ is a connected graph where there exists a k-element subset $U \subseteq V(G)$ such that $G - U$ is a tree.

Removing the vertices in U eliminates all cycles in G, resulting in $G - U$ being a tree. To transform G into a tree:

- Remove all k vertices in the subset U and their incident edges.
- The resulting graph $G - U$ is a tree.

Both quasi-trees and k-quasi-trees are connected graphs that can be reduced to a tree by removing a carefully chosen set of vertices. The distinction lies in the size of the subset of vertices removed:

- For quasi-trees, $k = 1$.
- For k-quasi-trees, k can be any positive integer.

This completes the proof. □

Definition 21. (cf.[205,206]) A graph $G = (V, E)$ is called a *quasi-forest* if there exists a vertex $v \in V(G)$ such that $G - v$ is a forest. That is, removing one vertex leaves a graph that is a collection of trees (a forest).

A graph $G = (V, E)$ is called a *k-quasi-forest* if k is the smallest positive integer such that there exists a k-element subset $U \subseteq V(G)$ of vertices where $G - U$ is a forest. This means removing exactly k vertices from G results in a collection of trees.

Definition 22. A graph $G = (V, E)$ is called a *quasi-path* if there exists a vertex $v \in V(G)$ such that $G - v$ is a path, meaning removing one vertex results in a simple, acyclic, and connected sequence of edges.

A graph $G = (V, E)$ is called a *k-quasi-path* if k is the smallest positive integer such that there exists a k-element subset $U \subseteq V(G)$ of vertices where $G - U$ is a path. Removing exactly k vertices results in a simple, acyclic, connected sequence of edges.

The following is clearly valid.

Proposition 23. *A quasi-forest and a k-quasi-forest can be transformed into a forest.*

Proof. By definition, a quasi-forest $G = (V, E)$ is a graph such that there exists a vertex $v \in V(G)$ where $G - v$ is a forest. Since removing v leaves a forest, the remaining connected components of $G - v$ consist of trees, satisfying the definition of a forest. Thus, G can be transformed into a forest by removing the single vertex v.

Similarly, for a k-quasi-forest $G = (V, E)$, there exists a k-element subset $U \subseteq V(G)$ such that $G - U$ is a forest. Removing U eliminates any cycles or additional connections within G, leaving a collection of trees. Hence, G can be transformed into a forest by removing the k vertices in U. □

Proposition 24. *A quasi-path and a k-quasi-path can be transformed into a path.*

Proof. By definition, a quasi-path $G = (V, E)$ is a graph such that there exists a vertex $v \in V(G)$ where $G - v$ is a path. Removing v results in a simple, acyclic, and connected graph that satisfies the definition of a path. Thus, G can be transformed into a path by removing the single vertex v.

For a k-quasi-path $G = (V, E)$, there exists a k-element subset $U \subseteq V(G)$ such that $G - U$ is a path. Removing U reduces the graph to a simple, acyclic sequence of connected edges, satisfying the definition of a path. Therefore, G can be transformed into a path by removing the k vertices in U. □

In this paper, while apex graphs and quasi-graphs focus on vertex deletion, we define an ”Almost graph” based on edge deletion as described below.

Definition 25. A connected graph $G = (V, E)$ is called an *almost tree* if there exists an edge $e \in E(G)$ such that $G - e$ is a tree. This means removing a single edge from the graph results in a tree structure.

A connected graph $G = (V, E)$ is called a *k-almost tree* if k is the smallest positive integer such that there exists a k-element subset $F \subseteq E(G)$ of edges where $G - F$ is a tree. This means removing exactly k edges results in a tree structure.

Proposition 26. *An almost tree and a k-almost tree can be transformed into a tree.*

Proof. By definition:

1. An *almost tree* $G = (V, E)$ satisfies $G - e$ is a tree for some edge $e \in E$. Removing e eliminates any cycle, leaving $G - e$ connected and acyclic, which is the definition of a tree.

2. A *k-almost tree* $G = (V, E)$ satisfies $G - F$ is a tree for some k-element subset $F \subseteq E$. Removing all edges in F eliminates k cycles while preserving connectivity, transforming G into a tree.

□

Definition 27. A graph $G = (V, E)$ is called an *almost forest* if there exists an edge $e \in E(G)$ such that $G - e$ is a forest. In other words, removing one edge results in a collection of trees.

A graph $G = (V, E)$ is called a *k-almost forest* if k is the smallest positive integer such that there exists a k-element subset $F \subseteq E(G)$ of edges where $G - F$ is a forest. Removing exactly k edges results in a collection of trees.

Proposition 28. *An almost forest and a k-almost forest can be transformed into a forest.*

Proof. By definition:

1. An *almost forest* $G = (V, E)$ satisfies $G - e$ is a forest for some edge $e \in E$. Removing e splits one connected component or removes a cycle, leaving a collection of acyclic components (a forest).

2. A *k-almost forest* $G = (V, E)$ satisfies $G - F$ is a forest for some k-element subset $F \subseteq E$. Removing k edges removes up to k cycles or splits components, leaving a collection of acyclic, disjoint trees, which defines a forest.

□

Definition 29. A graph $G = (V, E)$ is called an *almost path* if there exists an edge $e \in E(G)$ such that $G - e$ is a path. Removing a single edge results in a simple linear sequence of connected vertices.

A graph $G = (V, E)$ is called a *k-almost path* if k is the smallest positive integer such that there exists a k-element subset $F \subseteq E(G)$ of edges where $G - F$ is a path. Removing exactly k edges results in a simple linear sequence of connected vertices.

Proposition 30. *An almost path and a k-almost path can be transformed into a path.*

Proof. By definition:

1. An *almost path* $G = (V, E)$ satisfies $G - e$ is a path for some edge $e \in E$. Removing e eliminates a cycle or redundant connection, leaving $G - e$ as a simple linear sequence of connected vertices (a path).

2. A *k-almost path* $G = (V, E)$ satisfies $G - F$ is a path for some k-element subset $F \subseteq E$. Removing k edges removes all cycles or extra connections, leaving $G - F$ as a connected, acyclic sequence of vertices, which defines a path.

□

Definition 31. A graph $G = (V, E)$ is called an *almost spanning tree* if there exists an edge $e \in E(G)$ such that $G - e$ is a spanning tree. In other words, removing a single edge from G results in a subgraph that connects all vertices without any cycles.

A graph $G = (V, E)$ is called a *k-almost spanning tree* if k is the smallest positive integer such that there exists a k-element subset $F \subseteq E(G)$ of edges where $G - F$ is a spanning tree. Removing exactly k edges from G results in a subgraph that connects all vertices without any cycles.

Proposition 32. *An almost spanning tree and a k-almost spanning tree can be transformed into a spanning tree.*

Proof. By definition:

1. An *almost spanning tree* $G = (V, E)$ satisfies $G - e$ is a spanning tree for some edge $e \in E$. Removing e reduces redundancy without disconnecting the graph, leaving $G - e$ connected and acyclic, covering all vertices (a spanning tree).

2. A *k-almost spanning tree* $G = (V, E)$ satisfies $G - F$ is a spanning tree for some k-element subset $F \subseteq E$. Removing k edges removes redundancy and extra connections while preserving connectivity, resulting in a subgraph that connects all vertices without cycles, which defines a spanning tree.

□

2.3 Tree, Forest, and Path in Fuzzy Graphs

In recent years, inspired by tree, the concepts of fuzzy tree have been introduced. To begin, the definition of fuzzy graph is provided below. A fuzzy graph is an extension of graph theory that incorporates the principles of fuzzy sets [21,67,86,94,148,213,335,351,358]. Extensive research has been conducted on both fuzzy graphs [43,91,132,150,235,257,281,287,323] and fuzzy planar graphs [11,149,267,268,295].

Definition 33. [287] A *fuzzy graph* $\psi = (V, \sigma, \mu)$ is defined as follows:

- V is a set of vertices.
- $\sigma : V \to [0,1]$ is a function that assigns a membership degree to each vertex $v \in V$, indicating the degree of membership of v in the fuzzy graph.
- $\mu : V \times V \to [0,1]$ is a fuzzy relation that represents the strength of the connection between each pair of vertices $(u, v) \in V \times V$, such that $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

  In this definition, the following properties hold:

- The fuzzy function μ is symmetric, meaning $\mu(u, v) = \mu(v, u)$ for all $u, v \in V$.
- Additionally, $\mu(v, v) = 0$ for all $v \in V$, meaning that there is no self-loop in the fuzzy graph.

The fuzzy graph ψ allows for the representation of uncertainty in the presence or strength of connections between vertices, making it a valuable tool for modeling complex systems with ambiguous or imprecise relationships.

The definitions of Fuzzy Path[235, 251, 262], Fuzzy Tree[42, 44, 235, 236, 241, 296, 321], and Fuzzy Forest[203,235,236,296,321] are provided below. Related concepts, such as the Fuzzy Spanning Tree [93, 98, 182, 243, 276, 344, 365], Fuzzy decision tree[72, 79, 173, 299, 352, 357], and Fuzzy Shortest Path [84, 185, 188, 207, 208, 232, 248], have also been explored in research.

Definition 34 (Fuzzy Tree)**.** (cf.[236, 241, 321]) A *fuzzy tree* is a connected fuzzy graph $G = (V, \sigma, \mu)$ with a fuzzy spanning subgraph $F = (V, \sigma, \mu_F)$, where:

- V is a set of vertices.
- $\sigma : V \to [0,1]$ assigns membership degrees to vertices.
- $\mu_F : V \times V \to [0,1]$ is a fuzzy relation representing the connection strength between vertices, satisfying:
  - $\mu_F(u, v) > 0$ for every $u, v \in V$, and μ_F forms a tree.
  - For any edge $(u, v) \in E(G)$ not in F, there is a path in F connecting u and v with a strength greater than $\mu(u, v)$.

Theorem 35. *The following are examples of graph classes or structures related to fuzzy trees, including but not limited to:*

- *fuzzy Event-tree: A fuzzy event-tree incorporates fuzzy logic to handle uncertainty in event-tree analysis, using qualitative terms for probabilities and outcomes[184].*
- *fuzzy fault-tree: A fuzzy fault-tree uses fuzzy logic to estimate system failure, accounting for uncertain component failure probabilities with fuzzy sets[328, 334].*

Proof. Please refer to the respective papers listed in the references. □

Definition 36 (Fuzzy Path)**.** (cf.[235, 251, 262]) A *fuzzy path* of length n in a fuzzy graph $G = (V, \sigma, \mu)$ is a sequence of distinct vertices $v_0, v_1, \ldots, v_n$, such that:

- $\mu(v_{i-1}, v_i) > 0$ for all $i = 1, \ldots, n$.
- The degree of membership of the weakest edge in the path defines the strength of the path.

Definition 37 (Fuzzy Forest)**.** A *fuzzy forest* is a fuzzy graph $G = (V, \sigma, \mu)$ where:

- $\sigma : V \to [0,1]$ assigns membership degrees to vertices.
- $\mu : V \times V \to [0,1]$ is a fuzzy relation, such that:

– The graph contains no cycles (i.e., no fuzzy cycle).
– It may consist of multiple disconnected fuzzy trees, forming a forest structure.

Theorem 38. *A fuzzy tree can be transformed into a standard tree.*

Proof. A fuzzy tree $G = (V, \sigma, \mu)$ consists of vertices and edges with membership degrees in $[0, 1]$. To transform it into a standard tree:

1. For each vertex $v \in V$, include it in the standard tree if $\sigma(v) > 0$.
2. For each edge $(u, v) \in E$, include it in the standard tree if $\mu(u, v) > 0$, ensuring that no cycles are formed.
3. The resulting graph G' is connected and acyclic, satisfying the definition of a standard tree.

□

Theorem 39. *A fuzzy path can be transformed into a standard path.*

Proof. A fuzzy path $G = (V, \sigma, \mu)$ is defined as a sequence of vertices $v_0, v_1, \ldots, v_n$ with positive edge membership $\mu(v_{i-1}, v_i) > 0$. To transform it into a standard path:

1. Include all vertices $v_i \in V$ for which $\sigma(v_i) > 0$.
2. Include all edges (v_{i-1}, v_i) where $\mu(v_{i-1}, v_i) > 0$.
3. Arrange the vertices linearly according to their sequence, ensuring no cycles or branching.
4. The resulting graph is a simple, connected, acyclic sequence of vertices, satisfying the definition of a standard path.

□

Theorem 40. *A fuzzy forest can be transformed into a standard forest.*

Proof. A fuzzy forest $G = (V, \sigma, \mu)$ is a fuzzy graph with no cycles. To transform it into a standard forest:

1. Include all vertices $v \in V$ for which $\sigma(v) > 0$.
2. Include all edges $(u, v) \in E$ where $\mu(u, v) > 0$, ensuring no cycles are formed.
3. The resulting graph G' consists of multiple disconnected components, each of which is a tree.
4. Therefore, G' satisfies the definition of a standard forest.

□

Definition 41 (Fuzzy Quasi-Tree)**.** A *fuzzy quasi-tree* is a connected fuzzy graph $G = (V, \sigma, \mu)$ such that there exists a vertex $v \in V$ where the fuzzy subgraph $G - v$ is a fuzzy tree.

A *fuzzy k-quasi-tree* is a connected fuzzy graph $G = (V, \sigma, \mu)$ where k is the smallest positive integer such that there exists a k-element subset $U \subseteq V$, and the fuzzy subgraph $G - U$ is a fuzzy tree.

Definition 42 (Fuzzy Quasi-Forest)**.** A *fuzzy quasi-forest* is a fuzzy graph $G = (V, \sigma, \mu)$ such that there exists a vertex $v \in V$ where the fuzzy subgraph $G - v$ is a fuzzy forest.

A *fuzzy k-quasi-forest* is a fuzzy graph $G = (V, \sigma, \mu)$ where k is the smallest positive integer such that there exists a k-element subset $U \subseteq V$, and the fuzzy subgraph $G - U$ is a fuzzy forest.

Definition 43 (Fuzzy Quasi-Path)**.** A *fuzzy quasi-path* is a fuzzy graph $G = (V, \sigma, \mu)$ such that there exists a vertex $v \in V$, and the fuzzy subgraph $G - v$ is a fuzzy path.

A *fuzzy k-quasi-path* is a fuzzy graph $G = (V, \sigma, \mu)$ where k is the smallest positive integer such that there exists a k-element subset $U \subseteq V$, and the fuzzy subgraph $G - U$ is a fuzzy path.

Theorem 44. *A fuzzy quasi-tree can be transformed into a quasi-tree.*

Proof. Let $G = (V, \sigma, \mu)$ be a fuzzy quasi-tree. By definition, there exists a vertex $v \in V$ such that the fuzzy subgraph $G - v$ is a fuzzy tree.

1. Transform the fuzzy tree $G - v$ into a standard tree by including vertices $u \in V(G - v)$ and edges $(u, w) \in E(G - v)$ with positive membership degrees $\sigma(u) > 0$ and $\mu(u, w) > 0$.
2. Reintroduce the vertex v to G, assigning it a binary state ($\sigma(v) = 1$ if present, $\sigma(v) = 0$ otherwise).
3. The resulting graph G' is a quasi-tree, as removing v leaves a tree.

Thus, G can be transformed into a quasi-tree. □

Theorem 45. *A fuzzy quasi-forest can be transformed into a quasi-forest.*

Proof. Let $G = (V, \sigma, \mu)$ be a fuzzy quasi-forest. By definition, there exists a vertex $v \in V$ such that the fuzzy subgraph $G - v$ is a fuzzy forest.

1. Transform each fuzzy tree in the fuzzy forest $G - v$ into a standard tree by including vertices and edges with positive membership degrees.
2. Reintroduce the vertex v to G, assigning it a binary state ($\sigma(v) = 1$ if present, $\sigma(v) = 0$ otherwise).
3. The resulting graph G' is a quasi-forest, as removing v leaves a forest.

Thus, G can be transformed into a quasi-forest. □

Theorem 46. *A fuzzy quasi-path can be transformed into a quasi-path.*

Proof. Let $G = (V, \sigma, \mu)$ be a fuzzy quasi-path. By definition, there exists a vertex $v \in V$ such that the fuzzy subgraph $G - v$ is a fuzzy path.

1. Transform the fuzzy path $G - v$ into a standard path by including vertices and edges with positive membership degrees.
2. Reintroduce the vertex v to G, assigning it a binary state ($\sigma(v) = 1$ if present, $\sigma(v) = 0$ otherwise).
3. The resulting graph G' is a quasi-path, as removing v leaves a path.

Thus, G can be transformed into a quasi-path. □

2.4 Tree, Forest, and Path in Intuitionistic fuzzy Graphs

Intuitionistic fuzzy graphs are an extended version of fuzzy graphs and have been the subject of extensive study for over 15 years [143, 183, 260, 274]. Intuitionistic fuzzy planar graphs [20] are, in turn, an extension of fuzzy planar graphs. The concept of intuitionistic fuzzy sets [30–32, 92, 115, 200, 329], which is well-known in set theory, is closely related to intuitionistic fuzzy graphs. The definitions of intuitionistic fuzzy graphs and intuitionistic fuzzy planar graphs are provided below.

Definition 47 (Intuitionistic Fuzzy Graph (IFG))**.** [260] Let $G = (V, E)$ be a classical graph where V denotes the set of vertices and E denotes the set of edges. An *Intuitionistic Fuzzy Graph* (IFG) on G, denoted $G_{IF} = (A, B)$, is defined as follows:

1. (μ_A, ν_A) is an *Intuitionistic Fuzzy Set (IFS)* on the vertex set V. For each vertex $x \in V$, the degree of membership $\mu_A(x) \in [0, 1]$ and the degree of non-membership $\nu_A(x) \in [0, 1]$ satisfy:

$$\mu_A(x) + \nu_A(x) \leq 1$$

   The value $1 - \mu_A(x) - \nu_A(x)$ represents the hesitancy or uncertainty regarding the membership of x in the set.

2. (μ_B, ν_B) is an *Intuitionistic Fuzzy Relation (IFR)* on the edge set E. For each edge $(x, y) \in E$, the degree of membership $\mu_B(x, y) \in [0, 1]$ and the degree of non-membership $\nu_B(x, y) \in [0, 1]$ satisfy:

$$\mu_B(x, y) + \nu_B(x, y) \leq 1$$

   Additionally, the following constraints must hold for all $x, y \in V$:

$$\mu_B(x, y) \leq \mu_A(x) \wedge \mu_A(y)$$

$$\nu_B(x, y) \leq \nu_A(x) \vee \nu_A(y)$$

In this definition:

- $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and non-membership of the vertex x, respectively.
- $\mu_B(x, y)$ and $\nu_B(x, y)$ represent the degree of membership and non-membership of the edge (x, y), respectively.
- If $\nu_A(x) = 0$ and $\nu_B(x, y) = 0$ for all $x \in V$ and $(x, y) \in E$, then the Intuitionistic Fuzzy Graph reduces to a Fuzzy Graph.

The definitions of Intuitionistic Fuzzy Path[180], Connected Intuitionistic Fuzzy Graph, Intuitionistic Fuzzy Spanning Subgraph, Intuitionistic Fuzzy Tree[273], and Intuitionistic Fuzzy Forest [300] are provided below. Related concepts, such as the Intuitionistic Fuzzy Spanning Tree[66, 230, 246, 364], Intuitionistic Fuzzy decision tree[64, 65, 278], and Intuitionistic Fuzzy Shortest Path [33, 46, 47, 144, 239], have also been explored in research.

Definition 48 (Intuitionistic Fuzzy Path)**.** [180] An *intuitionistic fuzzy path* in an intuitionistic fuzzy graph $G = (A, B)$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ such that for each consecutive pair of vertices (v_i, v_{i+1}) in the sequence, one of the following conditions holds:

1. $\mu_B(v_i, v_{i+1}) > 0$ and $\nu_B(v_i, v_{i+1}) > 0$ (i.e., the membership and non-membership degrees of the edge are both positive).
2. $\mu_B(v_i, v_{i+1}) > 0$ and $\nu_B(v_i, v_{i+1}) = 0$ (i.e., the edge has positive membership and zero non-membership).
3. $\mu_B(v_i, v_{i+1}) = 0$ and $\nu_B(v_i, v_{i+1}) > 0$ (i.e., the edge has zero membership and positive non-membership).

If $\mu_B(v_i, v_{i+1}) = 0$ and $\nu_B(v_i, v_{i+1}) = 0$, there is no edge between v_i and v_{i+1}, meaning that no path exists between these two vertices.

Definition 49 (Connected Intuitionistic Fuzzy Graph)**.** [180] An *intuitionistic fuzzy graph* $G = (A, B)$ is said to be *connected* if for every pair of distinct vertices x and y in G, there exists an intuitionistic fuzzy path connecting x and y. In other words, any two vertices in the graph are reachable from one another through a series of edges where the membership and non-membership conditions of the path are satisfied.

Definition 50 (Intuitionistic Fuzzy Spanning Subgraph)**.** [10, 273] An *intuitionistic fuzzy spanning subgraph* $H = (A, C)$ of an intuitionistic fuzzy graph $G = (A, B)$ is a subgraph such that:

- $V(H) = V(G)$ (the vertex set of H is the same as the vertex set of G).
- $E(H) \subseteq E(G)$ (the edge set of H is a subset of the edge set of G).
- For every edge $(x, y) \in E(H)$, the membership and non-membership degrees in H satisfy:

  $$\mu_C(x, y) \leq \mu_B(x, y), \quad \nu_C(x, y) \geq \nu_B(x, y)$$

  where μ_B and ν_B are the membership and non-membership functions of the original graph G, and μ_C and ν_C are the membership and non-membership functions of the subgraph H.

Definition 51 (Intuitionistic Fuzzy Tree)**.** [10,273,292,293] An *intuitionistic fuzzy tree* is a connected intuitionistic fuzzy graph $G = (A, B)$ where there exists an intuitionistic fuzzy spanning subgraph $H = (A, C)$ that is a tree. Additionally, for every edge $(x, y) \in E(G) \setminus E(H)$, the membership and non-membership degrees of the edge (x, y) in G must satisfy:

$$\mu_B(x, y) < \mu_C(x, y), \quad \nu_B(x, y) > \nu_C(x, y)$$

where μ_B and ν_B represent the membership and non-membership functions of G, and μ_C and ν_C represent those of the subgraph H.

Definition 52 (Intuitionistic Fuzzy Forest)**.** [10,273] An *intuitionistic fuzzy forest* is an intuitionistic fuzzy graph $G = (A, B)$ where there exists an intuitionistic fuzzy spanning subgraph $H = (A, C)$ that is a forest. For all edges $(x, y) \in E(G) \setminus E(H)$, the membership and non-membership degrees of (x, y) in G satisfy:

$$\mu_B(x, y) < \mu_C(x, y), \quad \nu_B(x, y) > \nu_C(x, y)$$

where μ_B and ν_B are the membership and non-membership functions of G, and μ_C and ν_C are those of the subgraph H.

Theorem 53. *An Intuitionistic Fuzzy Tree can be transformed into a Fuzzy Tree.*

Proof. Let $G_{IF} = (A, B)$ be an Intuitionistic Fuzzy Tree. By definition, G_{IF} is a connected intuitionistic fuzzy graph where there exists an intuitionistic fuzzy spanning subgraph $H = (A, C)$ that is a tree. The membership and non-membership degrees satisfy:

$$\mu_B(x, y) < \mu_C(x, y), \quad \nu_B(x, y) > \nu_C(x, y),$$

for all edges $(x, y) \in E(G_{IF}) \setminus E(H)$.

To transform G_{IF} into a Fuzzy Tree:

1. Ignore the non-membership degree $\nu_B(x, y)$ for all $(x, y) \in E(G_{IF})$.
2. Retain only the membership degree $\mu_B(x, y)$ for each edge (x, y), which now defines a fuzzy graph.
3. By construction, the spanning subgraph H remains a tree under the membership function $\mu_C(x, y)$.

The resulting fuzzy graph $G_F = (A, \mu_B)$ is a Fuzzy Tree. □

Theorem 54. *An Intuitionistic Fuzzy Path can be transformed into a Fuzzy Path.*

Proof. Let $G_{IF} = (A, B)$ be an Intuitionistic Fuzzy Path. By definition, G_{IF} is a sequence of distinct vertices $v_1, v_2, \dots, v_n$ where each consecutive pair satisfies:

$$\mu_B(v_i, v_{i+1}) + \nu_B(v_i, v_{i+1}) \leq 1.$$

To transform G_{IF} into a Fuzzy Path:

1. Ignore the non-membership degree $\nu_B(v_i, v_{i+1})$ for all edges (v_i, v_{i+1}).
2. Retain only the membership degree $\mu_B(v_i, v_{i+1})$, which defines the fuzzy relation for edges.
3. The sequence of vertices and edges remains a path under the membership function μ_B.

The resulting fuzzy graph $G_F = (A, \mu_B)$ is a Fuzzy Path. □

Theorem 55. *An Intuitionistic Fuzzy Forest can be transformed into a Fuzzy Forest.*

Proof. Let $G_{IF} = (A, B)$ be an Intuitionistic Fuzzy Forest. By definition, G_{IF} contains an intuitionistic fuzzy spanning subgraph $H = (A, C)$ that is a forest. For all edges $(x, y) \in E(G_{IF}) \setminus E(H)$, the membership and non-membership degrees satisfy:

$$\mu_B(x, y) < \mu_C(x, y), \quad \nu_B(x, y) > \nu_C(x, y).$$

To transform G_{IF} into a Fuzzy Forest:

1. Ignore the non-membership degree $\nu_B(x, y)$ for all edges (x, y).
2. Retain only the membership degree $\mu_B(x, y)$, which defines the fuzzy relation for edges.
3. The spanning subgraph H, under the membership function $\mu_C(x, y)$, remains a forest.

The resulting fuzzy graph $G_F = (A, \mu_B)$ is a Fuzzy Forest. □

Definition 56 (Intuitionistic Fuzzy Quasi-Tree)**.** An *intuitionistic fuzzy quasi-tree* is an intuitionistic fuzzy graph $G_{IF} = (A, B)$ where there exists a vertex $v \in V$ such that the intuitionistic fuzzy subgraph $G - v$ is an intuitionistic fuzzy tree.

A *k-quasi-tree* is an intuitionistic fuzzy graph where removing k vertices results in an intuitionistic fuzzy tree.

Definition 57 (Intuitionistic Fuzzy Quasi-Forest)**.** An *intuitionistic fuzzy quasi-forest* is an intuitionistic fuzzy graph $G_{IF} = (A, B)$ where there exists a vertex $v \in V$ such that the intuitionistic fuzzy subgraph $G - v$ is an intuitionistic fuzzy forest.

A *k-quasi-forest* is an intuitionistic fuzzy graph where removing k vertices results in an intuitionistic fuzzy forest.

Definition 58 (Intuitionistic Fuzzy Quasi-Path)**.** An *intuitionistic fuzzy quasi-path* is an intuitionistic fuzzy graph $G_{IF} = (A, B)$ where there exists a vertex $v \in V$ such that the intuitionistic fuzzy subgraph $G-v$ is an intuitionistic fuzzy path.

A *intuitionistic fuzzy k-quasi-path* is an intuitionistic fuzzy graph where removing k vertices results in an intuitionistic fuzzy path.

Theorem 59. *An Intuitionistic Fuzzy Quasi-Tree can be transformed into a Fuzzy Quasi-Tree.*

Proof. Let $G_{IF} = (A, B)$ be an Intuitionistic Fuzzy Quasi-Tree. By definition, there exists a vertex $v \in V$ such that the intuitionistic fuzzy subgraph $G_{IF} - v$ is an Intuitionistic Fuzzy Tree. To transform G_{IF} into a Fuzzy Quasi-Tree:

1. Ignore the non-membership degree $\nu_A(x)$ for all $x \in V$ and $\nu_B(x, y)$ for all $(x, y) \in E$.
2. Retain only the membership degree $\mu_A(x)$ for vertices and $\mu_B(x, y)$ for edges.
3. The resulting fuzzy subgraph $G - v$ becomes a Fuzzy Tree because the membership degrees satisfy the conditions of a fuzzy graph.

Thus, G_{IF} transforms into a Fuzzy Quasi-Tree. □

Theorem 60. *An Intuitionistic Fuzzy Quasi-Path can be transformed into a Fuzzy Quasi-Path.*

Proof. Let $G_{IF} = (A, B)$ be an Intuitionistic Fuzzy Quasi-Path. By definition, there exists a vertex $v \in V$ such that the intuitionistic fuzzy subgraph $G_{IF} - v$ is an Intuitionistic Fuzzy Path. To transform G_{IF} into a Fuzzy Quasi-Path:

1. Ignore the non-membership degree $\nu_A(x)$ for all $x \in V$ and $\nu_B(x, y)$ for all $(x, y) \in E$.
2. Retain only the membership degree $\mu_A(x)$ for vertices and $\mu_B(x, y)$ for edges.
3. The resulting fuzzy subgraph $G - v$ becomes a Fuzzy Path because the membership degrees satisfy the conditions of a fuzzy path.

Thus, G_{IF} transforms into a Fuzzy Quasi-Path. □

Theorem 61. *An Intuitionistic Fuzzy Quasi-Forest can be transformed into a Fuzzy Quasi-Forest.*

Proof. Let $G_{IF} = (A, B)$ be an Intuitionistic Fuzzy Quasi-Forest. By definition, there exists a vertex $v \in V$ such that the intuitionistic fuzzy subgraph $G_{IF} - v$ is an Intuitionistic Fuzzy Forest. To transform G_{IF} into a Fuzzy Quasi-Forest:

1. Ignore the non-membership degree $\nu_A(x)$ for all $x \in V$ and $\nu_B(x, y)$ for all $(x, y) \in E$.
2. Retain only the membership degree $\mu_A(x)$ for vertices and $\mu_B(x, y)$ for edges.
3. The resulting fuzzy subgraph $G - v$ becomes a Fuzzy Forest because the membership degrees satisfy the conditions of a fuzzy forest.

Thus, G_{IF} transforms into a Fuzzy Quasi-Forest. □

2.5 Tree, Forest, and Path in Neutrosophic Graphs

First, the definition of a neutrosophic graph is provided. Similar to fuzzy graphs, Neutrosophic graphs have been the focus of extensive research [13, 14, 63, 131, 159, 166, 175, 291, 311, 315]. Additionally, the concept of neutrosophic sets[59, 118, 217, 308, 316, 317, 343] in set theory is closely related to neutrosophic graphs. The definition is given below[315].

Definition 62. [315] A neutrosophic graph $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ is a graph where:

- $\sigma : V \to [0, 1]^3$ assigns a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees to each vertex $v \in V$.
- $\mu : E \to [0, 1]^3$ assigns a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$ representing the truth, indeterminacy, and falsity membership degrees to each edge $e \in E$.

- For every edge $e = v_i v_j \in E$, the following condition holds:

$$\mu_T(e) \leq \min(\sigma_T(v_i), \sigma_T(v_j)).$$

The definitions of Neutrosophic Path [8, 161, 197], Neutrosophic Tree [146, 161], and Neutrosophic Forest[161] are presented below. For instance, related concepts such as the Neutrosophic Shortest Path [61, 62, 70, 197, 275, 353], and spanning trees [3, 60, 100, 195, 245] have also been studied.

Definition 63 (Neutrosophic Path)**.** [8, 161, 197] A *neutrosophic path* in a neutrosophic graph

$$G = (V, E, \sigma_T, \sigma_I, \sigma_F, \mu_T, \mu_I, \mu_F)$$

is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ such that:

- Each pair of consecutive vertices v_i and v_{i+1} is connected by an edge $e_i = (v_i, v_{i+1}) \in E$.
- The truth membership degree $\mu_T(e_i)$, indeterminacy membership degree $\mu_I(e_i)$, and falsity membership degree $\mu_F(e_i)$ of each edge e_i satisfy the following conditions:

$$\mu_T(e_i) > 0, \quad \mu_I(e_i) > 0, \quad \mu_F(e_i) > 0 \quad \forall i = 1, 2, \ldots, n-1.$$

- The total membership degree of each edge e_i, representing the sum of the truth, indeterminacy, and falsity values, satisfies the following constraint:

$$0 \leq \mu_T(e_i) + \mu_I(e_i) + \mu_F(e_i) \leq 3.$$

- The vertices in the path $P = (v_1, v_2, \ldots, v_n)$ are distinct, meaning the path does not repeat any vertices, ensuring that the path is simple.
- The neutrosophic membership degrees of the vertices along the path also satisfy:

$$0 \leq \sigma_T(v_i) + \sigma_I(v_i) + \sigma_F(v_i) \leq 3 \quad \forall i = 1, 2, \ldots, n.$$

- For every pair of consecutive vertices $v_i, v_{i+1} \in V$, the neutrosophic membership degrees satisfy:

$$\mu_T(e_i) \leq \min(\sigma_T(v_i), \sigma_T(v_{i+1})),$$

$$\mu_I(e_i) \geq \max(\sigma_I(v_i), \sigma_I(v_{i+1})),$$

$$\mu_F(e_i) \geq \max(\sigma_F(v_i), \sigma_F(v_{i+1})).$$

Definition 64 (Neutrosophic Tree)**.** [161] Let $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a neutrosophic graph, where V is the set of vertices and E is the set of edges. G is called a *neutrosophic tree* if it satisfies the following conditions:

1. $G^* = (V, E)$, the crisp version of G, is a tree, meaning that:
   - G^* is acyclic, i.e., it contains no cycles.
   - G^* is connected, i.e., for every pair of vertices $u, v \in V$, there exists a unique path between u and v.
2. G has a *neutrosophic spanning subgraph* $T = (V, E_T, \sigma, \mu_T)$, where $E_T \subseteq E$, such that for all edges $e = uv \in E_T$, the following conditions hold:

$$\mu_T(e) \leq \min(\sigma_T(u), \sigma_T(v)), \quad \mu_I(e) \geq \max(\sigma_I(u), \sigma_I(v)), \quad \mu_F(e) \geq \max(\sigma_F(u), \sigma_F(v)).$$

3. For every edge $e = uv \in E_T$, the combined truthness, indeterminacy, and falsity values must satisfy:

$$0 \leq \mu_T(e) + \mu_I(e) + \mu_F(e) \leq 3.$$

4. For every edge $e = uv \in E - E_T$ (edges not in the spanning subgraph), the following holds:

$$\mu_T(e) < T_C^\infty(e), \quad \mu_I(e) > I_C^\infty(e), \quad \mu_F(e) > F_C^\infty(e),$$

   where $T_C^\infty(e), I_C^\infty(e), F_C^\infty(e)$ are the maximum neutrosophic truth, indeterminacy, and falsity values over all spanning trees in G.

Definition 65 (Neutrosophic Forest)**.** [161] Let $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a neutrosophic graph. G is called a *neutrosophic forest* if it satisfies the following conditions:

1. $G^* = (V, E)$, the crisp version of G, is a forest, meaning that:
   - G^* consists of a collection of disjoint trees (connected components that are trees).
   - G^* contains no cycles.
2. G has a *neutrosophic spanning subgraph* $F = (V, E_F, \sigma, \mu_F)$, where $E_F \subseteq E$ is the set of edges forming the neutrosophic forest, such that for each connected component (tree) in F, the following conditions hold for every edge $e = uv \in E_F$:
$$\mu_T(e) \leq \min(\sigma_T(u), \sigma_T(v)), \quad \mu_I(e) \geq \max(\sigma_I(u), \sigma_I(v)), \quad \mu_F(e) \geq \max(\sigma_F(u), \sigma_F(v)).$$
3. For every edge $e = uv \in E_F$, the combined truthness, indeterminacy, and falsity values must satisfy:
$$0 \leq \mu_T(e) + \mu_I(e) + \mu_F(e) \leq 3.$$
4. For every edge $e = uv \in E - E_F$, the following holds:
$$\mu_T(e) < T_C^\infty(e), \quad \mu_I(e) > I_C^\infty(e), \quad \mu_F(e) > F_C^\infty(e),$$
where $T_C^\infty(e), I_C^\infty(e), F_C^\infty(e)$ are the maximum neutrosophic truth, indeterminacy, and falsity values over all spanning forests in G.
5. *Connectedness of Components:* Each connected component of G must adhere to the above neutrosophic constraints, ensuring that all components are neutrosophic trees.

Theorem 66. *A Neutrosophic Tree can be transformed into an Intuitionistic Fuzzy Tree, and subsequently into a Fuzzy Tree.*

Proof. Let $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a Neutrosophic Tree. To transform G into an Intuitionistic Fuzzy Tree:

1. Define the Intuitionistic Fuzzy membership degrees for each vertex $v \in V$ as:
$$\mu_A(v) = \sigma_T(v), \quad \nu_A(v) = \sigma_F(v), \quad 1 - \mu_A(v) - \nu_A(v) = \sigma_I(v).$$
2. Similarly, define the Intuitionistic Fuzzy membership degrees for each edge $e \in E$ as:
$$\mu_B(e) = \mu_T(e), \quad \nu_B(e) = \mu_F(e), \quad 1 - \mu_B(e) - \nu_B(e) = \mu_I(e).$$
3. The resulting Intuitionistic Fuzzy Graph $G_{IF} = (A, B)$ satisfies the properties of an Intuitionistic Fuzzy Tree since $G^* = (V, E)$, the crisp version of G, is acyclic and connected.

To transform G_{IF} into a Fuzzy Tree:

1. Disregard the non-membership $\nu_A(v)$ and $\nu_B(e)$ for all vertices $v \in V$ and edges $e \in E$.
2. Retain only the membership degrees $\mu_A(v)$ and $\mu_B(e)$, which correspond to $\sigma_T(v)$ and $\mu_T(e)$, respectively.
3. The resulting fuzzy graph satisfies the properties of a Fuzzy Tree.

Thus, a Neutrosophic Tree can be transformed into a Fuzzy Tree via an intermediate Intuitionistic Fuzzy Tree. □

Theorem 67. *A Neutrosophic Path can be transformed into an Intuitionistic Fuzzy Path, and subsequently into a Fuzzy Path.*

Proof. Let $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a Neutrosophic Path. To transform G into an Intuitionistic Fuzzy Path:

1. Define the Intuitionistic Fuzzy membership degrees for each vertex $v \in V$ as:
$$\mu_A(v) = \sigma_T(v), \quad \nu_A(v) = \sigma_F(v), \quad 1 - \mu_A(v) - \nu_A(v) = \sigma_I(v).$$

2. Similarly, define the Intuitionistic Fuzzy membership degrees for each edge $e \in E$ as:

$$\mu_B(e) = \mu_T(e), \quad \nu_B(e) = \mu_F(e), \quad 1 - \mu_B(e) - \nu_B(e) = \mu_I(e).$$

3. The resulting Intuitionistic Fuzzy Graph $G_{IF} = (A, B)$ satisfies the properties of an Intuitionistic Fuzzy Path since $G^* = (V, E)$, the crisp version of G, is a simple, connected, acyclic sequence of edges.

To transform G_{IF} into a Fuzzy Path:

1. Disregard the non-membership $\nu_A(v)$ and $\nu_B(e)$ for all vertices $v \in V$ and edges $e \in E$.
2. Retain only the membership degrees $\mu_A(v)$ and $\mu_B(e)$, which correspond to $\sigma_T(v)$ and $\mu_T(e)$, respectively.
3. The resulting fuzzy graph satisfies the properties of a Fuzzy Path.

Thus, a Neutrosophic Path can be transformed into a Fuzzy Path via an intermediate Intuitionistic Fuzzy Path. □

Theorem 68. *A Neutrosophic Forest can be transformed into an Intuitionistic Fuzzy Forest, and subsequently into a Fuzzy Forest.*

Proof. Let $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a Neutrosophic Forest. To transform G into an Intuitionistic Fuzzy Forest:

1. Define the Intuitionistic Fuzzy membership degrees for each vertex $v \in V$ as:

$$\mu_A(v) = \sigma_T(v), \quad \nu_A(v) = \sigma_F(v), \quad 1 - \mu_A(v) - \nu_A(v) = \sigma_I(v).$$

2. Similarly, define the Intuitionistic Fuzzy membership degrees for each edge $e \in E$ as:

$$\mu_B(e) = \mu_T(e), \quad \nu_B(e) = \mu_F(e), \quad 1 - \mu_B(e) - \nu_B(e) = \mu_I(e).$$

3. The resulting Intuitionistic Fuzzy Graph $G_{IF} = (A, B)$ satisfies the properties of an Intuitionistic Fuzzy Forest since $G^* = (V, E)$, the crisp version of G, is a collection of disjoint trees.

To transform G_{IF} into a Fuzzy Forest:

1. Disregard the non-membership $\nu_A(v)$ and $\nu_B(e)$ for all vertices $v \in V$ and edges $e \in E$.
2. Retain only the membership degrees $\mu_A(v)$ and $\mu_B(e)$, which correspond to $\sigma_T(v)$ and $\mu_T(e)$, respectively.
3. The resulting fuzzy graph satisfies the properties of a Fuzzy Forest.

Thus, a Neutrosophic Forest can be transformed into a Fuzzy Forest via an intermediate Intuitionistic Fuzzy Forest. □

Definition 69 (Neutrosophic Quasi-Tree)**.** A *neutrosophic quasi-tree* is a connected neutrosophic graph $G = (V, E, \sigma, \mu)$ such that there exists a vertex $v \in V$ and the neutrosophic subgraph $G - v$ is a neutrosophic tree.

A *k-quasi-tree* is a neutrosophic graph where removing k vertices results in a neutrosophic tree.

Definition 70 (Neutrosophic Quasi-Forest)**.** A *neutrosophic quasi-forest* is a neutrosophic graph $G = (V, E, \sigma, \mu)$ where there exists a vertex $v \in V$ and the neutrosophic subgraph $G - v$ is a neutrosophic forest.

A *k-quasi-forest* is a neutrosophic graph where removing k vertices results in a neutrosophic forest.

Definition 71 (Neutrosophic Quasi-Path)**.** A *neutrosophic quasi-path* is a neutrosophic graph $G = (V, E, \sigma, \mu)$ such that there exists a vertex $v \in V$ and the neutrosophic subgraph $G - v$ is a neutrosophic path.

A *k-quasi-path* is a neutrosophic graph where removing k vertices results in a neutrosophic path.

Theorem 72. *A Neutrosophic Quasi-Tree can be transformed into an Intuitionistic Fuzzy Quasi-Tree, and subsequently into a Fuzzy Quasi-Tree.*

Proof. Let $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a Neutrosophic Quasi-Tree. By definition, there exists a vertex $v \in V$ such that $G - v$ is a Neutrosophic Tree.

To transform G into an Intuitionistic Fuzzy Quasi-Tree:

1. Define the Intuitionistic Fuzzy membership degrees for each vertex $u \in V$ as:

$$\mu_A(u) = \sigma_T(u), \quad \nu_A(u) = \sigma_F(u), \quad 1 - \mu_A(u) - \nu_A(u) = \sigma_I(u).$$

2. Similarly, define the Intuitionistic Fuzzy membership degrees for each edge $e \in E$ as:

$$\mu_B(e) = \mu_T(e), \quad \nu_B(e) = \mu_F(e), \quad 1 - \mu_B(e) - \nu_B(e) = \mu_I(e).$$

3. The resulting Intuitionistic Fuzzy Graph $G_{IF} = (A, B)$ satisfies the properties of an Intuitionistic Fuzzy Quasi-Tree, since $G - v$ is an Intuitionistic Fuzzy Tree by the same transformation process described for trees.

To transform G_{IF} into a Fuzzy Quasi-Tree:

1. Disregard the non-membership $\nu_A(u)$ and $\nu_B(e)$ for all vertices $u \in V$ and edges $e \in E$.
2. Retain only the membership degrees $\mu_A(u)$ and $\mu_B(e)$, which correspond to $\sigma_T(u)$ and $\mu_T(e)$, respectively.
3. The resulting fuzzy graph satisfies the properties of a Fuzzy Quasi-Tree, as $G - v$ becomes a Fuzzy Tree.

Thus, a Neutrosophic Quasi-Tree can be transformed into a Fuzzy Quasi-Tree via an intermediate Intuitionistic Fuzzy Quasi-Tree. □

Theorem 73. *A Neutrosophic Quasi-Path can be transformed into an Intuitionistic Fuzzy Quasi-Path, and subsequently into a Fuzzy Quasi-Path.*

Proof. Let $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a Neutrosophic Quasi-Path. By definition, there exists a vertex $v \in V$ such that $G - v$ is a Neutrosophic Path.

To transform G into an Intuitionistic Fuzzy Quasi-Path:

1. Define the Intuitionistic Fuzzy membership degrees for each vertex $u \in V$ as:

$$\mu_A(u) = \sigma_T(u), \quad \nu_A(u) = \sigma_F(u), \quad 1 - \mu_A(u) - \nu_A(u) = \sigma_I(u).$$

2. Similarly, define the Intuitionistic Fuzzy membership degrees for each edge $e \in E$ as:

$$\mu_B(e) = \mu_T(e), \quad \nu_B(e) = \mu_F(e), \quad 1 - \mu_B(e) - \nu_B(e) = \mu_I(e).$$

3. The resulting Intuitionistic Fuzzy Graph $G_{IF} = (A, B)$ satisfies the properties of an Intuitionistic Fuzzy Quasi-Path, since $G - v$ is an Intuitionistic Fuzzy Path by the same transformation process described for paths.

To transform G_{IF} into a Fuzzy Quasi-Path:

1. Disregard the non-membership $\nu_A(u)$ and $\nu_B(e)$ for all vertices $u \in V$ and edges $e \in E$.
2. Retain only the membership degrees $\mu_A(u)$ and $\mu_B(e)$, which correspond to $\sigma_T(u)$ and $\mu_T(e)$, respectively.
3. The resulting fuzzy graph satisfies the properties of a Fuzzy Quasi-Path, as $G - v$ becomes a Fuzzy Path.

Thus, a Neutrosophic Quasi-Path can be transformed into a Fuzzy Quasi-Path via an intermediate Intuitionistic Fuzzy Quasi-Path. □

Theorem 74. *A Neutrosophic Quasi-Forest can be transformed into an Intuitionistic Fuzzy Quasi-Forest, and subsequently into a Fuzzy Quasi-Forest.*

Proof. Let $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a Neutrosophic Quasi-Forest. By definition, there exists a vertex $v \in V$ such that $G - v$ is a Neutrosophic Forest.

To transform G into an Intuitionistic Fuzzy Quasi-Forest:

1. Define the Intuitionistic Fuzzy membership degrees for each vertex $u \in V$ as:

$$\mu_A(u) = \sigma_T(u), \quad \nu_A(u) = \sigma_F(u), \quad 1 - \mu_A(u) - \nu_A(u) = \sigma_I(u).$$

2. Similarly, define the Intuitionistic Fuzzy membership degrees for each edge $e \in E$ as:

$$\mu_B(e) = \mu_T(e), \quad \nu_B(e) = \mu_F(e), \quad 1 - \mu_B(e) - \nu_B(e) = \mu_I(e).$$

3. The resulting Intuitionistic Fuzzy Graph $G_{IF} = (A, B)$ satisfies the properties of an Intuitionistic Fuzzy Quasi-Forest, since $G - v$ is an Intuitionistic Fuzzy Forest by the same transformation process described for forests.

To transform G_{IF} into a Fuzzy Quasi-Forest:

1. Disregard the non-membership $\nu_A(u)$ and $\nu_B(e)$ for all vertices $u \in V$ and edges $e \in E$.
2. Retain only the membership degrees $\mu_A(u)$ and $\mu_B(e)$, which correspond to $\sigma_T(u)$ and $\mu_T(e)$, respectively.
3. The resulting fuzzy graph satisfies the properties of a Fuzzy Quasi-Forest, as $G - v$ becomes a Fuzzy Forest.

Thus, a Neutrosophic Quasi-Forest can be transformed into a Fuzzy Quasi-Forest via an intermediate Intuitionistic Fuzzy Quasi-Forest. □

2.6 Tree, Forest, and Path in Turiyam Neutrosophic Graph

Research on Turiyam Neutrosophic Graphs, which incorporate parameters into Neutrosophic Graphs, is currently being conducted [128, 133, 140–142]. These graphs are a graphical representation of the Turiyam Neutrosophic Set [142, 302]. Similar concepts include four-valued logic [36, 77]. The definition is provided below.Note that the Turiyam Neutrosophic Graph is, in fact, a specific case of the Quadripartitioned Neutrosophic Graph, achieved by replacing "Contradiction" with "Liberal." (cf.[307, 312, 313])

Definition 75 (Turiyam Neutrosophic Graph)**.** [140–142] Let $G = (V, E)$ be a classical graph with a finite set of vertices $V = \{v_i : i = 1, 2, \ldots, n\}$ and edges $E = \{(v_i, v_j) : i, j = 1, 2, \ldots, n\}$. A *Turiyam Neutrosophic Graph* of G, denoted $G^T = (V^T, E^T)$, is defined as follows:

1. *Turiyam Neutrosophic Vertex Set*: For each vertex $v_i \in V$, the Turiyam Neutrosophic graph assigns the following mappings:
$$t(v_i), iv(v_i), fv(v_i), lv(v_i) : V \to [0, 1],$$
where:
   - $t(v_i)$ is the truth value (tv) of the vertex v_i,
   - $iv(v_i)$ is the indeterminacy value (iv) of v_i,
   - $fv(v_i)$ is the falsity value (fv) of v_i,
   - $lv(v_i)$ is the Turiyam Neutrosophic state (or liberal value) (lv) of v_i,

   for all $v_i \in V$, such that the following condition holds for each vertex:
$$0 \le t(v_i) + iv(v_i) + fv(v_i) + lv(v_i) \le 4.$$

2. *Turiyam Neutrosophic Edge Set*: For each edge $(v_i, v_j) \in E$, the Turiyam Neutrosophic graph assigns the following mappings:
$$t(v_i, v_j), iv(v_i, v_j), fv(v_i, v_j), lv(v_i, v_j) : E \to [0, 1],$$
where:
   - $t(v_i, v_j)$ is the truth value of the edge (v_i, v_j),
   - $iv(v_i, v_j)$ is the indeterminacy value of (v_i, v_j),
   - $fv(v_i, v_j)$ is the falsity value of (v_i, v_j),
   - $lv(v_i, v_j)$ is the Turiyam Neutrosophic state (or liberal value) of (v_i, v_j),

   for all $(v_i, v_j) \in E$, such that the following condition holds for each edge:
$$0 \le t(v_i, v_j) + iv(v_i, v_j) + fv(v_i, v_j) + lv(v_i, v_j) \le 4.$$

In this case, V^T represents the Turiyam Neutrosophic vertex set of the graph G^T, and E^T represents the Turiyam Neutrosophic edge set of G^T.

Considering the above, the definitions of Turiyam Neutrosophic Path, Tree, and Forest are described below.

Definition 76 (Turiyam Neutrosophic Path)**.** A *Turiyam Neutrosophic Path* in a Turiyam Neutrosophic Graph $G^T = (V, E, T, I, F, L)$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ such that:

1. For each $i = 1, 2, \ldots, n-1$, the edge $e_i = (v_i, v_{i+1}) \in E$.
2. The membership degrees of the edges satisfy:
$$T(e_i) + L(e_i) > 0$$
$$0 \leq T(e_i) + I(e_i) + F(e_i) + L(e_i) \leq 1$$
3. The membership degrees of the vertices satisfy:
$$0 \leq T(v_i) + I(v_i) + F(v_i) + L(v_i) \leq 1 \quad \forall i = 1, 2, \ldots, n$$
4. The path is continuous, meaning the edges connect the vertices in sequence.

If for any i, $T(e_i) + L(e_i) = 0$, then there is no Turiyam Neutrosophic Path between v_i and v_{i+1}.

Definition 77 (Turiyam Neutrosophic Spanning Subgraph)**.** A *Turiyam Neutrosophic Spanning Subgraph* $H^T = (V, E', T', I', F', L')$ of a Turiyam Neutrosophic Graph $G^T = (V, E, T, I, F, L)$ is a subgraph where:

- $V(H^T) = V(G^T)$
- $E' \subseteq E$
- The membership degrees satisfy:
$$\begin{aligned} &T'(v) = T(v), \quad I'(v) = I(v), \quad F'(v) = F(v), \quad L'(v) = L(v) \quad \forall v \in V \\ &T'(e) \leq T(e), \quad I'(e) \geq I(e), \quad F'(e) \geq F(e), \quad L'(e) \geq L(e) \quad \forall e \in E' \\ &\quad 0 \leq T'(e) + I'(e) + F'(e) + L'(e) \leq 1 \end{aligned}$$

Definition 78 (Turiyam Neutrosophic Tree)**.** A *Turiyam Neutrosophic Tree* is a connected Turiyam Neutrosophic Graph $G^T = (V, E, T, I, F, L)$ that satisfies the following conditions:

1. There exists a Turiyam Neutrosophic Spanning Subgraph $H^T = (V, E', T', I', F', L')$ such that the underlying classical graph (V, E') is a tree (connected and acyclic).
2. For every edge $e \in E'$, we have:
$$T'(e) = T(e), \quad I'(e) = I(e), \quad F'(e) = F(e), \quad L'(e) = L(e)$$
3. For every edge $e \in E \setminus E'$:
$$T(e) < T'(e), \quad I(e) > I'(e), \quad F(e) > F'(e), \quad L(e) > L'(e)$$
4. Edge membership constraints:
$$T(e) \leq \min\{T(u), T(v)\}, \quad I(e) \geq \max\{I(u), I(v)\}$$
$$F(e) \geq \max\{F(u), F(v)\}, \quad L(e) \geq \max\{L(u), L(v)\}$$
5. Total membership degrees:
$$0 \leq T(v) + I(v) + F(v) + L(v) \leq 1 \quad \forall v \in V$$
$$0 \leq T(e) + I(e) + F(e) + L(e) \leq 1 \quad \forall e \in E$$

6. Between any two vertices $u, v \in V$, there exists a unique Turiyam Neutrosophic Path in H^T.

Definition 79 (Turiyam Neutrosophic Forest)**.** A *Turiyam Neutrosophic Forest* is a Turiyam Neutrosophic Graph $G^T = (V, E, T, I, F, L)$ that satisfies the following conditions:

1. There exists a Turiyam Neutrosophic Spanning Subgraph $H^T = (V, E', T', I', F', L')$ such that the underlying classical graph (V, E') is a forest (a collection of disjoint trees).
2. For every edge $e \in E'$:
$$T'(e) = T(e), \quad I'(e) = I(e), \quad F'(e) = F(e), \quad L'(e) = L(e)$$
3. For every edge $e \in E \setminus E'$:
$$T(e) < T'(e), \quad I(e) > I'(e), \quad F(e) > F'(e), \quad L(e) > L'(e)$$
4. Edge membership constraints:
$$T(e) \leq \min\{T(u), T(v)\}, \quad I(e) \geq \max\{I(u), I(v)\}$$
$$F(e) \geq \max\{F(u), F(v)\}, \quad L(e) \geq \max\{L(u), L(v)\}$$
5. Total membership degrees:
$$0 \leq T(v) + I(v) + F(v) + L(v) \leq 1 \quad \forall v \in V$$
$$0 \leq T(e) + I(e) + F(e) + L(e) \leq 1 \quad \forall e \in E$$

Theorem 80. *A Turiyam Neutrosophic Tree can be transformed into a Neutrosophic Tree.*

Proof. Let $G^T = (V, E, T, I, F, L)$ be a Turiyam Neutrosophic Tree. By definition, there exists a Turiyam Neutrosophic spanning subgraph $H^T = (V, E', T', I', F', L')$, where (V, E') is a tree and the membership degrees satisfy:
$$0 \leq T(e) + I(e) + F(e) + L(e) \leq 1 \quad \forall e \in E.$$

To transform G^T into a Neutrosophic Tree:

1. Disregard the liberal value $L(v)$ for all vertices $v \in V$ and $L(e)$ for all edges $e \in E$.
2. Normalize the remaining membership degrees $T(v), I(v), F(v)$ and $T(e), I(e), F(e)$ as follows:
$$T^N(v) = \frac{T(v)}{T(v) + I(v) + F(v)}, \quad I^N(v) = \frac{I(v)}{T(v) + I(v) + F(v)}, \quad F^N(v) = \frac{F(v)}{T(v) + I(v) + F(v)},$$
$$T^N(e) = \frac{T(e)}{T(e) + I(e) + F(e)}, \quad I^N(e) = \frac{I(e)}{T(e) + I(e) + F(e)}, \quad F^N(e) = \frac{F(e)}{T(e) + I(e) + F(e)}.$$
3. The resulting graph $G^N = (V, E, T^N, I^N, F^N)$ satisfies the definition of a Neutrosophic Tree, as the underlying crisp graph (V, E') is a tree, and the membership degrees T^N, I^N, F^N meet the requirements for a Neutrosophic Tree.

Thus, a Turiyam Neutrosophic Tree can be transformed into a Neutrosophic Tree. □

Theorem 81. *A Turiyam Neutrosophic Path can be transformed into a Neutrosophic Path.*

Proof. Let $G^T = (V, E, T, I, F, L)$ be a Turiyam Neutrosophic Path. By definition, there exists a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ such that the edges $e_i = (v_i, v_{i+1}) \in E$ satisfy:
$$0 \leq T(e_i) + I(e_i) + F(e_i) + L(e_i) \leq 1, \quad T(e_i) + L(e_i) > 0.$$

To transform G^T into a Neutrosophic Path:

1. Disregard the liberal value $L(v)$ for all vertices $v \in V$ and $L(e)$ for all edges $e \in E$.

2. Normalize the remaining membership degrees $T(v), I(v), F(v)$ and $T(e), I(e), F(e)$ using the same process as in the proof for Turiyam Neutrosophic Trees.
3. The resulting graph $G^N = (V, E, T^N, I^N, F^N)$ satisfies the definition of a Neutrosophic Path, as the underlying crisp graph forms a simple path, and the membership degrees T^N, I^N, F^N are well-defined.

Thus, a Turiyam Neutrosophic Path can be transformed into a Neutrosophic Path. □

Theorem 82. *A Turiyam Neutrosophic Forest can be transformed into a Neutrosophic Forest.*

Proof. Let $G^T = (V, E, T, I, F, L)$ be a Turiyam Neutrosophic Forest. By definition, there exists a Turiyam Neutrosophic spanning subgraph $H^T = (V, E', T', I', F', L')$, where (V, E') is a forest and the membership degrees satisfy:

$$0 \leq T(e) + I(e) + F(e) + L(e) \leq 1 \quad \forall e \in E.$$

To transform G^T into a Neutrosophic Forest:

1. Disregard the liberal value $L(v)$ for all vertices $v \in V$ and $L(e)$ for all edges $e \in E$.
2. Normalize the remaining membership degrees $T(v), I(v), F(v)$ and $T(e), I(e), F(e)$ as in the previous proofs.
3. The resulting graph $G^N = (V, E, T^N, I^N, F^N)$ satisfies the definition of a Neutrosophic Forest, as the underlying crisp graph (V, E') forms a collection of disjoint trees, and the membership degrees T^N, I^N, F^N are well-defined.

Thus, a Turiyam Neutrosophic Forest can be transformed into a Neutrosophic Forest. □

Definition 83 (Turiyam Neutrosophic Quasi-Tree)**.** A *Turiyam Neutrosophic quasi-tree* is a connected Turiyam Neutrosophic graph $G^T = (V^T, E^T)$ such that there exists a vertex $v \in V$ where the subgraph $G^T - v$ is a Turiyam Neutrosophic tree.

A *k-quasi-tree* is a Turiyam Neutrosophic graph where removing k vertices results in a Turiyam Neutrosophic tree.

Definition 84 (Turiyam Neutrosophic Quasi-Forest)**.** A *Turiyam Neutrosophic quasi-forest* is a Turiyam Neutrosophic graph $G^T = (V^T, E^T)$ such that there exists a vertex $v \in V$ where the subgraph $G^T - v$ is a Turiyam Neutrosophic forest.

A *k-quasi-forest* is a Turiyam Neutrosophic graph where removing k vertices results in a Turiyam Neutrosophic forest.

Definition 85 (Turiyam Neutrosophic Quasi-Path)**.** A *Turiyam Neutrosophic quasi-path* is a Turiyam Neutrosophic graph $G^T = (V^T, E^T)$ such that there exists a vertex $v \in V$ where the subgraph $G^T - v$ is a Turiyam Neutrosophic path.

A *k-quasi-path* is a Turiyam Neutrosophic graph where removing k vertices results in a Turiyam Neutrosophic path.

Theorem 86. *A Turiyam Neutrosophic Quasi-Tree can be transformed into a Neutrosophic Quasi-Tree.*

Proof. Let $G^T = (V, E, T, I, F, L)$ be a Turiyam Neutrosophic Quasi-Tree. By definition, there exists a vertex $v \in V$ such that the Turiyam Neutrosophic subgraph $G^T - v$ is a Turiyam Neutrosophic Tree.

To transform G^T into a Neutrosophic Quasi-Tree:

1. Disregard the liberal value $L(v)$ for all vertices $v \in V$ and $L(e)$ for all edges $e \in E$.
2. Normalize the remaining membership degrees $T(v), I(v), F(v)$ and $T(e), I(e), F(e)$ using the following normalization:

$$T^N(v) = \frac{T(v)}{T(v) + I(v) + F(v)}, \quad I^N(v) = \frac{I(v)}{T(v) + I(v) + F(v)}, \quad F^N(v) = \frac{F(v)}{T(v) + I(v) + F(v)},$$

$$T^N(e) = \frac{T(e)}{T(e) + I(e) + F(e)}, \quad I^N(e) = \frac{I(e)}{T(e) + I(e) + F(e)}, \quad F^N(e) = \frac{F(e)}{T(e) + I(e) + F(e)}.$$

3. The resulting graph $G^N = (V, E, T^N, I^N, F^N)$ satisfies the definition of a Neutrosophic Quasi-Tree, as $G^N - v$ is a Neutrosophic Tree by construction.

Thus, a Turiyam Neutrosophic Quasi-Tree can be transformed into a Neutrosophic Quasi-Tree. □

Theorem 87. *A Turiyam Neutrosophic Quasi-Path can be transformed into a Neutrosophic Quasi-Path.*

Proof. Let $G^T = (V, E, T, I, F, L)$ be a Turiyam Neutrosophic Quasi-Path. By definition, there exists a vertex $v \in V$ such that the Turiyam Neutrosophic subgraph $G^T - v$ is a Turiyam Neutrosophic Path.

To transform G^T into a Neutrosophic Quasi-Path:

1. Disregard the liberal value $L(v)$ for all vertices $v \in V$ and $L(e)$ for all edges $e \in E$.
2. Normalize the remaining membership degrees $T(v), I(v), F(v)$ and $T(e), I(e), F(e)$ using the same process as in the proof for Turiyam Neutrosophic Quasi-Trees.
3. The resulting graph $G^N = (V, E, T^N, I^N, F^N)$ satisfies the definition of a Neutrosophic Quasi-Path, as $G^N - v$ is a Neutrosophic Path by construction.

Thus, a Turiyam Neutrosophic Quasi-Path can be transformed into a Neutrosophic Quasi-Path. □

Theorem 88. *A Turiyam Neutrosophic Quasi-Forest can be transformed into a Neutrosophic Quasi-Forest.*

Proof. Let $G^T = (V, E, T, I, F, L)$ be a Turiyam Neutrosophic Quasi-Forest. By definition, there exists a vertex $v \in V$ such that the Turiyam Neutrosophic subgraph $G^T - v$ is a Turiyam Neutrosophic Forest.

To transform G^T into a Neutrosophic Quasi-Forest:

1. Disregard the liberal value $L(v)$ for all vertices $v \in V$ and $L(e)$ for all edges $e \in E$.
2. Normalize the remaining membership degrees $T(v), I(v), F(v)$ and $T(e), I(e), F(e)$ as in the previous proofs.
3. The resulting graph $G^N = (V, E, T^N, I^N, F^N)$ satisfies the definition of a Neutrosophic Quasi-Forest, as $G^N - v$ is a Neutrosophic Forest by construction.

Thus, a Turiyam Neutrosophic Quasi-Forest can be transformed into a Neutrosophic Quasi-Forest. □

2.7 Tree, Forest, and Path in Plithogenic Graph

Recently, Plithogenic Graphs have been proposed as a generalization of Fuzzy Graphs and Turiyam Neutrosophic Graphs, as well as a graphical representation of Plithogenic Sets [1, 152, 280, 303, 309, 314, 324]. Plithogenic Graphs have been developed and are currently being actively studied [136, 176, 304, 324, 325] The definition is provided below.

Definition 89. [325] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

1. *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:
   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the range of possible attribute values.
   - $adf : M \times Ml \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
   - $aCf : Ml \times Ml \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.
2. *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
   - $bCf : Nm \times Nm \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. *Edge Appurtenance Constraint*: For all $(x, a), (y, b) \in M \times Ml$:

$$bdf\ ((xy), (a, b)) \leq \min\{adf(x, a), adf(y, b)\}$$

   where $xy \in N$ is an edge between vertices x and y, and $(a, b) \in Nm \times Nm$ are the corresponding attribute values.

2. *Contradiction Function Constraint*: For all $(a, b), (c, d) \in Nm \times Nm$:

$$bCf\ ((a, b), (c, d)) \leq \min\{aCf(a, c), aCf(b, d)\}$$

3. *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} aCf(a, a) &= 0, & \forall a \in Ml \\ aCf(a, b) &= aCf(b, a), & \forall a, b \in Ml \\ bCf(a, a) &= 0, & \forall a \in Nm \\ bCf(a, b) &= bCf(b, a), & \forall a, b \in Nm \end{aligned}$$

Example 90. The following is an example of a Plithogenic Graph.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic Turiyam Neutrosophic Graph*.

The definitions of Path, Tree, and Forest in a Plithogenic Graph are provided below.

Definition 91 (Plithogenic Path)**.** In a Plithogenic Graph $PG = (PM, PN)$, a *Plithogenic Path* between vertices v_1 and v_n is a sequence of distinct vertices $v_1, v_2, \ldots, v_n \in M$ such that:

1. For each $i = 1, 2, \ldots, n - 1$, there exists an edge $e_i = v_i v_{i+1} \in N$.
2. The *Net Degree of Appurtenance (NDA)* for each vertex and edge is positive:

$$\begin{aligned} \text{NDA}(v_i) &= \text{DAF}_V(v_i, l(v_i)) \times (1 - \text{DCF}_V(l(v_i), c_l)) > 0, & \forall i = 1, \ldots, n \\ \text{NDA}(e_i) &= \text{DAF}_E(e_i, m(e_i)) \times (1 - \text{DCF}_E(m(e_i), c_m)) > 0, & \forall i = 1, \ldots, n-1 \end{aligned}$$

   where $c_l \in Ml$ and $c_m \in Nm$ are reference attribute values.

3. The *Overall Net Degree of Appurtenance* of the path is defined as:

$$\text{NDA}_{\text{path}} = \min\{\text{NDA}(v_i), \text{NDA}(e_i) \mid i = 1, \ldots, n - 1\}$$

4. The path is *simple*, meaning no vertices are repeated.

Definition 92 (Plithogenic Tree)**.** A *Plithogenic Tree* is a connected Plithogenic Graph $PG = (PM, PN)$ that satisfies the following conditions:

1. The underlying crisp graph $G^* = (M, N)$ is a *tree*; that is, it is connected and acyclic.
2. For every vertex $v \in M$:

$$\text{NDA}(v) = \text{DAF}_V(v, l(v)) \times (1 - \text{DCF}_V(l(v), c_l)) > 0$$

3. For every edge $e = uv \in N$:

$$\text{NDA}(e) = \text{DAF}_E(e, m(e)) \times (1 - \text{DCF}_E(m(e), c_m)) > 0$$

4. The *Edge Appurtenance Constraint* holds:

$$\mathrm{DAF}_E(e, m(e)) \leq \min\{\mathrm{DAF}_V(u, l(u)), \mathrm{DAF}_V(v, l(v))\}$$

5. The *Overall Net Degree of Appurtenance* of the tree is defined as:

$$\mathrm{NDA}_{\mathrm{tree}} = \min\{\mathrm{NDA}(v), \mathrm{NDA}(e) \mid v \in M, e \in N\}$$

Definition 93 (Plithogenic Forest)**.** A *Plithogenic Forest* is a Plithogenic Graph $PG = (PM, PN)$ that satisfies the following conditions:

1. The underlying crisp graph $G^* = (M, N)$ is a *forest*; that is, it consists of a collection of disjoint trees.
2. For every vertex $v \in M$:

$$\mathrm{NDA}(v) = \mathrm{DAF}_V(v, l(v)) \times (1 - \mathrm{DCF}_V(l(v), c_l)) > 0$$

3. For every edge $e = uv \in N$:

$$\mathrm{NDA}(e) = \mathrm{DAF}_E(e, m(e)) \times (1 - \mathrm{DCF}_E(m(e), c_m)) > 0$$

4. The *Edge Appurtenance Constraint* holds for all edges in N.
5. Each connected component (tree) in the forest satisfies the conditions of a Plithogenic Tree.
6. The *Overall Net Degree of Appurtenance* of the forest is defined as:

$$\mathrm{NDA}_{\mathrm{forest}} = \min\{\mathrm{NDA}(v), \mathrm{NDA}(e) \mid v \in M, e \in N\}$$

Definition 94 (Plithogenic Quasi-Tree)**.** A *Plithogenic Quasi-Tree* is a connected Plithogenic graph PG such that there exists a vertex $v \in V$ where the subgraph $PG - v$ is a Plithogenic tree.

A *k-Quasi-Tree* is a Plithogenic graph where removing k vertices results in a Plithogenic tree.

Definition 95 (Plithogenic Quasi-Forest)**.** A *Plithogenic Quasi-Forest* is a Plithogenic graph PG such that there exists a vertex $v \in V$ where the subgraph $PG - v$ is a Plithogenic forest.

A *k-Quasi-Forest* is a Plithogenic graph where removing k vertices results in a Plithogenic forest.

Definition 96 (Plithogenic Quasi-Path)**.** A *Plithogenic Quasi-Path* is a Plithogenic graph PG such that there exists a vertex $v \in V$ where the subgraph $PG - v$ is a Plithogenic path.

A *k-Quasi-Path* is a Plithogenic graph where removing k vertices results in a Plithogenic path.

3. Result in this paper

In this section, we present the results of this paper.

3.1 Some property of Plithogenic tree, path, and forest

We consider the properties of the Turiyam Neutrosophic tree. These properties also hold similarly for fuzzy graphs, intuitionistic fuzzy graphs, neutrosophic graphs, and Turiyam Neutrosophic graphs.

Theorem 97. *A Plithogenic tree, path, or forest is a Plithogenic graph.*

Proof. This result follows directly from the definition of a Plithogenic graph, where a Plithogenic tree, path, or forest adheres to the structure and properties defined for such graphs. □

Theorem 98. *A Plithogenic tree, path, forest, quasi-tree, quasi-path, or quasi-forest can be transformed into a fuzzy, neutrosophic, or Turiyam Neutrosophic tree, path, forest, quasi-tree, quasi-path, or quasi-forest.*

Proof. This result holds because Plithogenic graphs generalize fuzzy, neutrosophic, and Turiyam Neutrosophic graphs, and their corresponding transformations can be made by adjusting the appurtenance, contradiction, and other related parameters to align with the specific framework (fuzzy, neutrosophic, or Turiyam) being considered. □

Theorem 99. *Every Plithogenic Tree has a Plithogenic Path.*

Proof. A Plithogenic Tree is defined as a connected Plithogenic graph whose underlying classical graph is a tree. The properties of Plithogenic Trees extend those of classical trees by incorporating truth, indeterminacy, and falsity values for both vertices and edges. However, the fundamental properties of trees remain unchanged: they are connected, acyclic structures.

We aim to prove that every Plithogenic Tree has a Plithogenic Path.

Let $T = (V, E)$ be a Plithogenic Tree. By definition, the underlying classical graph of T, denoted $T^* = (V, E)$, is a connected, acyclic graph. In a tree, for any two distinct vertices $u, v \in V$, there exists a unique path in T^* that connects u and v.

Now, since T is a Plithogenic Tree, the edges and vertices of T are assigned membership degrees of truth (T), indeterminacy (I), and falsity (F). Additionally, these membership degrees satisfy the following constraints for each vertex $v \in V$ and each edge $e \in E$:

$$0 \le T(v) + I(v) + F(v) \le 1 \quad \text{and} \quad 0 \le T(e) + I(e) + F(e) \le 1.$$

Given any pair of vertices $u, v \in V$, the path connecting u and v in the underlying classical tree T^* must also exist in the Plithogenic Tree T, provided that the membership degrees of the vertices and edges along this path satisfy the constraints for Plithogenic graphs. Specifically, for the path to be considered a *Plithogenic Path*, the following conditions must hold:

$$T(e_i) + I(e_i) + F(e_i) \le 1 \quad \text{for each edge } e_i \text{ in the path,}$$

$$T(v_i) + I(v_i) + F(v_i) \le 1 \quad \text{for each vertex } v_i \text{ in the path.}$$

Since T is a Plithogenic Tree, all edges and vertices satisfy these constraints. Therefore, the unique path that exists in the underlying classical tree T^* is also a valid Plithogenic Path in T, as it adheres to the required membership conditions.

Thus, we conclude that every Plithogenic Tree has a Plithogenic Path. □

Theorem 100. *Every Plithogenic Tree with at least one edge has at least two leaves.*

Proof. A Plithogenic tree is defined as a connected Plithogenic graph whose underlying classical structure is a tree. Since Plithogenic graphs extend classical graph theory by incorporating parameters such as truth, indeterminacy, and falsity values, the basic properties of trees, including their acyclic and connected nature, remain valid.

Let $T = (V, E)$ be a Plithogenic tree with at least one edge. By definition, the underlying classical graph of T is also a tree, meaning it is connected and acyclic.

Now, consider a path $P = (v_1, v_2, \dots, v_k)$ of maximum length in the underlying classical tree of T. We will show that both endpoints of this path, v_1 and v_k, are leaves, i.e., vertices with degree 1.

1. Suppose, for the sake of contradiction, that one of the endpoints, say v_1, has a degree greater than 1. This means that v_1 is connected not only to v_2 (part of the path P) but also to some other vertex $w \in V$.

2. If w is not part of the path P, then we can extend the path P by adding w, which contradicts the assumption that P is the longest path in the tree.

3. If w is part of the path P, then adding w creates a cycle in the graph, contradicting the acyclic nature of the tree.

Thus, the assumption that v_1 has a degree greater than 1 leads to a contradiction. Therefore, v_1 must be a leaf, i.e., it has degree 1. By the same reasoning, the other endpoint v_k must also be a leaf.

Since the Plithogenic tree inherits the basic structural properties of its underlying classical tree, we conclude that a Plithogenic tree with at least one edge must have at least two leaves. □

Theorem 101. *Every Plithogenic Tree with n Plithogenic edges has $n + 1$ Plithogenic vertices.*

Proof. We will prove this theorem by induction on the number of edges n in the Plithogenic tree.

Base Case ($n = 1$): Consider a Plithogenic tree with $n = 1$ edge. By the definition of a tree (whether Plithogenic or classical), a tree with one edge must connect exactly two vertices.

Thus, for $n = 1$, the tree has $n + 1 = 1 + 1 = 2$ vertices. The theorem holds for the base case.

Inductive Step: Now, assume that the theorem holds for any Plithogenic tree with k edges. That is, a Plithogenic tree with k edges has $k + 1$ vertices.

We will prove that a Plithogenic tree with $k + 1$ edges has $(k + 1) + 1 = k + 2$ vertices.

Consider a Plithogenic tree T_k with k edges and $k+1$ vertices by the inductive hypothesis. Now, add one more Plithogenic edge to this tree, say e_{k+1}, which connects a new vertex v_{k+2} to one of the existing vertices of T_k. Since trees are connected and acyclic, the addition of the new edge e_{k+1} must introduce exactly one new vertex to maintain the acyclic property.

Thus, the Plithogenic tree with $k+1$ edges will have exactly $(k+1)+1 = k+2$ vertices, as required.

By mathematical induction, the theorem holds for all $n \geq 1$. Therefore, every Plithogenic tree with n Plithogenic edges has $n+1$ Plithogenic vertices. □

Theorem 102. *A Plithogenic tree is connected, but it would become disconnected if any single Plithogenic edge is removed from the Plithogenic tree.*

Proof. Let $T = (V, E, PM, PN)$ be a Plithogenic tree, where V is the set of vertices, E is the set of edges, PM represents the Plithogenic vertex set, and PN represents the Plithogenic edge set.

By definition, a Plithogenic tree is a connected Plithogenic graph whose underlying classical structure is a tree. This means that the underlying graph of T, denoted $T^* = (V, E^*)$, is a classical tree, which is acyclic and connected.

Since T^* is a classical tree, it is connected. This implies that for any two vertices $u, v \in V$, there exists a path in T^* that connects them. The Plithogenic tree inherits this property because the Plithogenic edge set PN maintains the relationships between the vertices. Therefore, the Plithogenic tree T is connected.

Now, assume we remove a single Plithogenic edge $e \in PN$ from the Plithogenic tree. The corresponding edge in the underlying classical graph T^* is also removed. Since T^* is a classical tree, it contains no cycles, and removing any edge from a tree breaks the connectivity of the graph. Specifically, removing the edge e from T^* disconnects the graph into two disjoint components, say T_1^* and T_2^*.

In the Plithogenic tree T, removing the edge $e \in PN$ similarly breaks the Plithogenic graph into two disjoint components, each corresponding to a subset of vertices from T_1^* and T_2^*. Since there are no Plithogenic edges connecting these two components anymore, the Plithogenic tree T becomes disconnected.

Thus, removing any single Plithogenic edge from the Plithogenic tree results in a disconnected graph. □

4. Conclusion and Future works

In this section, we present the conclusion and outline potential future work based on the findings of this paper.

4.1 Conclusion in this paper

In this paper, we have explored the concepts of Trees, Forests, and Paths within the framework of Plithogenic Graphs. By extending traditional graph theory with plithogenic characteristics—incorporating degrees of appurtenance, contradiction, and multiple attributes—we have developed a more generalized and flexible approach to model complex systems. This work serves as a bridge between classical graph theory and advanced graph structures such as fuzzy, neutrosophic, and Turiyam Neutrosophic graphs.

Given the extensive and continuously expanding body of literature on fuzzy mathematics, it is inevitable that similar concepts may emerge independently across different journals and periods. Nevertheless, we believe that efforts to unify these concepts are essential and will greatly contribute to advancing the field. We aim to conduct further investigations in this direction in the future(cf.[129]).

4.2 Future works

We explore the concepts of Hypertrees and Superhypertrees within the frameworks of fuzzy, neutrosophic, and Turiyam Neutrosophic graphs(cf.[310]). Additionally, we examine Semi-Directed Trees in these same graph structures (cf.[294]).

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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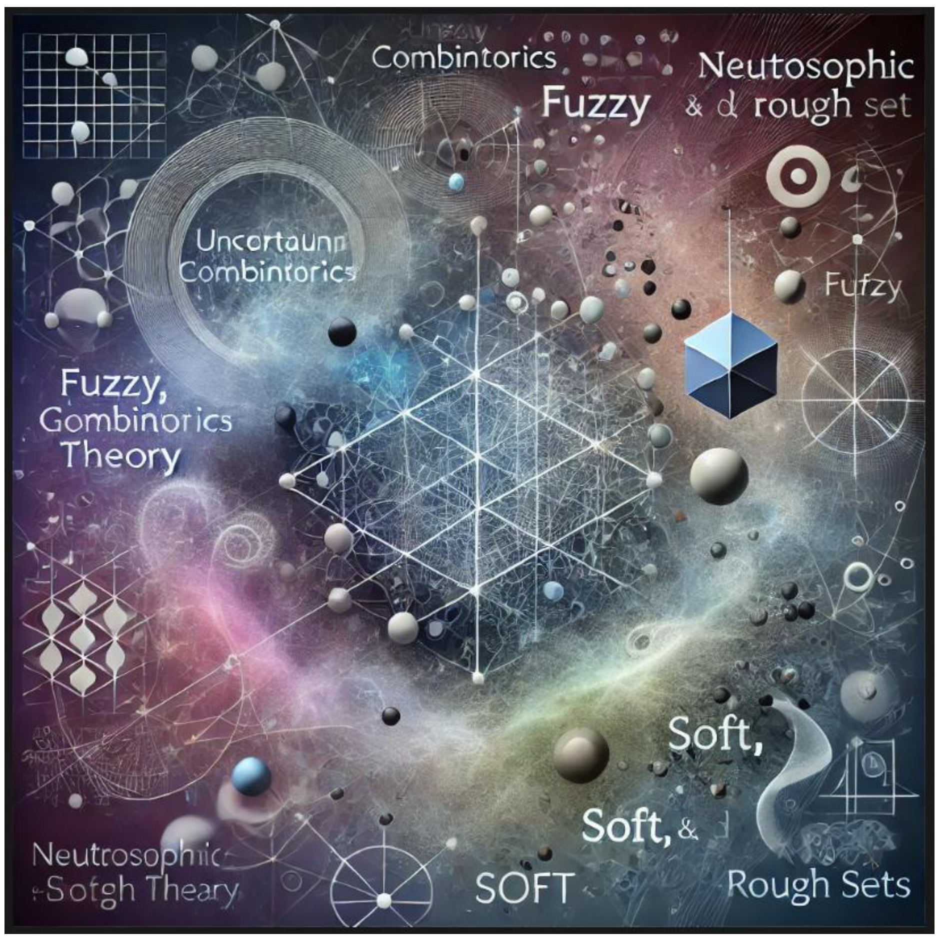
The second volume of “Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond” presents a deep exploration of the progress in uncertain combinatorics through innovative methodologies like graphization, hyperization, and uncertainization. This volume integrates foundational concepts from fuzzy, neutrosophic, soft, and rough set theory, among others, to further advance the field. Combinatorics and set theory, two central pillars of mathematics, focus on counting, arrangement, and the study of collections under defined rules. Combinatorics excels in handling uncertainty, while set theory has evolved with concepts such as fuzzy and neutrosophic sets, which enable the modeling of complex real-world uncertainties by addressing truth, indeterminacy, and falsehood. These advancements, when combined with graph theory, give rise to novel forms of uncertain sets in "graphized" structures, including hypergraphs and superhypergraphs. Innovations such as Neutrosophic Oversets, Undersets, and Offsets, as well as the Nonstandard Real Set, build upon traditional graph concepts, pushing both theoretical and practical boundaries. The synthesis of combinatorics, set theory, and graph theory in this volume provides a robust framework for addressing the complexities and uncertainties inherent in both mathematical and real-world systems, paving the way for future research and application.



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