*Logic and Limits of Knowledge and Truth**

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Though my ultimate concern is with issues in epistemology and metaphysics, let me phrase the central question I will pursue in terms evocative of philosophy of religion:

What are the implications of our logic—in particular, of Cantor and Gödel—for the possibility of omniscience?[1]

The attempt to draw philosophical lessons from metalogical texts is a notoriously perilous business.[2] With that in mind let me frame what follows as a *suggestion*, or offer it as an argument worthy of consideration, rather than trumpet it as a proof. What I want to suggest is that within any logic we have—in particular, in terms of systems and sets—omniscience appears to be simply impossible. In that sense Cantor and Gödel offer at least a suggestive case against the possibility of omniscience.

The path to this conclusion will be somewhat more circuitous than this introductory statement might suggest, however. In the first section that follows I consider the standard Gödel result, and in the second some intriguing nonconstructive extensions, neither of which suggests so definitive a negative conclusion regarding omniscience. In section III, however, I offer a more general argument for 'expressive' incompleteness of relevant systems, supplemented in section IV by a general argument for 'internal' incompleteness closer in spirit to Gödel. Each form of incompleteness appears again in section V as the basis of a first argument against omniscience.

The work of the paper to this point concerns formal *systems* as analogues of omniscience.[3] In section VI, I offer a more direct Cantorian argument against a *set* of all truths and hence against omniscience, disgressing slightly to consider implications for possible worlds. Here alternative set theories may seem promising as

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a way out, however, and these are briefly considered—with negative results—in section VII.

Perhaps in the end there really *cannot* be any totality of truths and really *cannot* be any omniscience. Plantinga and Wittgenstein are used in a final section to summarize major epistemological and metaphysical suggestions.

I. OMNISCIENCE AND THE STANDARD GÖDEL RESULT

Many a body of knowledge may be thought of—at least ideally—on the model of an interpreted formal system. To the admissible formulae of the system, on this analogy, correspond all statements in the general domain of the body of knowledge, true or false, known or unknown. Formulae interpreted as *basic* truths or *basic* items of knowledge are chosen as axioms, and formulae interpreted as *derivative* truths—given appropriate transformation rules—will appear as theorems.[4] Axiomatic geometries are prime examples here, of course. But given a liberal enough attitude towards sets of axioms, types of transformation rules, and the like, it appears that *any* body of knowledge might ideally be conceived on the model of a formal system.[5]

If bodies of knowledge might be so conceived, might not *knowers*? Here we need not suggest that processes of knowing or patterns of epistemic justification must somehow correspond to demonstrations within a formal system. But might not at least *what* a knower knows—the mere content of his knowledge—be conceived on the model of a formal system?

There is a major obstacle here, at least for familiar types of knowers and familiar types of systems. A formal system contains as a theorem every formula obtainable from its axioms by its specified transformation rules. If the transformation rules of such a system correspond to standard patterns of logical inference, then, a corresponding knower would have to know—as derivative truths—every truth derivable from his basic items of knowledge. None of us is such a knower.[6]

This is precisely the difficulty which seems to arise concerning Hintikka's work in *Knowledge and Belief*. Once rules which appear indispensable from a formal point of view are adopted, we seem to be saddled with the following result. Given any '$p \supset q$' valid in ordinary propositional logic, we appear to be committed to:

$$Kap \supset Kaq,$$

where '*Kap*' is initially glossed as '*a* knows that *p*' (Hintikka, 1962).[7]

But this does *not* appear to hold for ordinary knowers, who may well *not* know all that what they know entails. Hintikka originally responded to the problem as follows:

> Our results are not directly applicable to what is true or false in the actual world of ours. They tell us something definite about the truth and falsity of statements only in a world in which everybody follows the consequences of what he knows as far as they lead him. (Hintikka, 1962, p. 36)[8]

On this approach, then, Hintikka's knowers are *ideal* knowers. So too are those knowers envisaged above, modelable on formal systems transformation rules of which correspond to standard patterns of logical inference. This limitation to ideal knowers is not the worrisome constraint for our purposes that it is commonly held to be for Hintikka's, however. For here it *is* ideal knowers—in fact divine knowers—that are at issue.[9]

Let us thus construct in imagination an ideal knower whose knowledge *can* be conceived on the model of a formal system G. The transformation rules of G, we suppose, do include on interpretation all standard rules of logical inference, so our ideal knower *will* know all that follows from what he knows. We will also suppose that his rules of inference operate only on workably finite sets of premises, that formulae of G are kept finite as well, and that the alphabet of G is kept manageably denumerable.

What of the *basic* knowledge of our ideal knower, corresponding to the axioms of G? Here we might insist that the axioms of G be finite as well, but will instead impose only the weaker stipulation that they be recursively enumerable.

If an ideal knower so constructed is to have even a pretense of *divine* knowledge, of course, he must have at least a working knowledge of number theory. We will then suppose all statements of number theory to be expressible in his corresponding system, and will expect to find among the axioms of G the mere handful required for predicate calculus with identity and the five Peano postulates. But of course we can also build in much more: a googolplex of basic propositions of biochemistry, perhaps, an infinite (though recursively enumerable) set of basic propositions of physics, and all seven true propositions of macroeconomics.

What Gödel's standard incompleteness result shows, however, is that no matter what other basic knowledge we imagine building into such an ideal knower—no matter what else is included among the recursively enumerable axioms of G—such an ideal knower *cannot* be omniscient. If, as specified, he knows enough to handle basic number theory, in fact—no matter what *else* he knows—he already knows too *much* to know everything.[10]

Gödel numbering for G is assured by the fact that formulae of G are finite and the axioms of G are adequate for number theory. Given Gödel numbering and these axioms a substitution predicate can be introduced, and—since the axioms of G are recursively enumerable and its rules of inference operate only on finite sets of premises—a proof predicate as well. These are in essence all we need to construct an undecidable sentence for G: a formula which, if G is omega-consistent,[11] demonstrably cannot appear as a theorem of G—and so cannot represent anything our ideal knower knows—and yet *does* represent a number-theoretical truth.

A similar incompleteness result will hold, moreover, for any improved model of an ideal knower we attempt to construct within the confines of the basic conditions above. It is tempting, for example, to try adding as an axiom to G the formula for G's missing truth. When this in turn gives us an incomplete system it is tempting to try adding an infinite series of missing truths, or an infinite series of infinite series of missing truths. As long as what we add remains recursively enumerable, however, any improved system we build will still be incomplete, and for basically the same reasons.[12] All systems within the basic conditions above are *essentially* incomplete, and the ideal knowers to which they correspond are *essentially* non-omniscient.[13]

II. BEYOND STANDARD SYSTEMS

The ideal knowers considered above are analogous to formal systems adequate at least for the general purposes of number theory and which have (1) recursively enumerable axioms, (2) formulae of finite length and a denumerable alphabet, and (3) rules of inference from only finitely many premises.

On the grounds of Gödel's standard incompleteness result, no such ideal knower can be omniscient. But God *is* standardly conceived as omniscient. So God—if there be such a being—must not be an ideal knower of this kind.

Should this be considered a negative theological conclusion? Not necessarily. The work of the preceding section might instead be considered a positive theological contribution in the spirit of the *via negativa*—an approach to God by way of an understanding of what He is not. God's knowledge is quite standardly said, for example, to be infinite. But this would clearly be inadequate as a characterization of omniscience, since knowledge of many a mere ideal knower of the lowly sort considered above is literally infinite. *God's* knowledge would have to be much *more* than merely infinite, essentially incapable of being captured at all within the systematic confines laid down above. Some have argued in effect that those confines are our confines as well (see for example Benacerraf, 1967). If so, our work to this point might be welcomed with open arms in theological circles as a particularly precise vindication of the doctrine that a divine mind must be humanly incomprehensible.

But what lies *beyond* the type of system considered above? What happens if we weaken one or more of the constraints imposed above on systems and on corresponding knowers?

That is a question for which no general and exhaustive answer can be said to exist. The systematic constraints outlined above are essentially the limits of constructive methods, and to go beyond them is to leave constructive methods behind. Beyond such constraints 'formal systems' cease to be genuinely 'formal' at all, and conceptions of 'proof' and 'demonstration' must change at the border. Beyond lies not logic in the familiar sense but what Geoffrey Hellman has not inappropriately termed 'theologic' (Hellman, 1981).[14]

Important attempts to cross over have been made, however. Among the most promising for our purposes are the following.[15]

Barkley Rosser was the first to propose relaxing that condition which limits rules of inference to finite premises, introducing a form of transfinite induction instead. Rosser considered only systems allowing up to ω^2 uses of a non-constructive rule of inference from $f(0)$, $f(1), f(2), \ldots$ to $(x)f(x)$, however, and within that limitation systems still prove incomplete in the standard ways: each system still contains an undecidable sentence and a consistency formula unprovable in the system (Rosser, 1937).

Transfinite induction is taken further in a system S_∞ developed in various forms by G. Gentzen, W. Ackerman, P. Lorenzen, K. Schutte, and I. Hlodovskii.[16] 'Proof' within S_∞ is redefined in terms of proof trees. To each formula of a proof tree an ordinal is assigned—the result of applying a weak rule in the system is given the same ordinal as its premise, but the result of applying a strong rule or cut is given an ordinal greater than that of its premises.

Restriction on the ordinals assignable to the formulae of proof trees restricts the notion of proof accordingly. But if *no* restriction is placed on the class of ordinals which can be attached to proofs, we get a system S_∞ that is both ω-consistent and complete (Mendelson, 1964, 270).

This may make it appear as if omniscience can escape the curse of Gödel on the wings of transfinite induction. But here some important limitations of S_∞ should be noted. The proof of S_∞'s consistency, first of all, is not formalizable internally; as Gentzen himself showed, transfinite induction up to ϵ_0 cannot be formalized in S_∞ (see Mendelson, 1964, p. 270; Wang, 1964, pp. 369-370; and

Webb, 1968, p. 177). A second difficulty is perhaps more crucial. S_∞, obtained from a more standard system S for number theory by the addition of a rule of inference permitting transfinite induction, is still capable of dealing only with finite sets. For systems dealing with infinite sets as well, an obvious desideratum in any system intended to mirror omniscience, even transfinite induction will not be enough—those systems will *still* be incomplete.[17]

A somewhat different non-constructive approach appears in the work of Solomon Feferman (Feferman, 1962; see also Feferman, 1960 and Feferman and Spector, 1962). In 1939 A. M. Turing dealt with collections of axiom systems under the name *ordinal logics*. Feferman's work, although related, is extended to transfinite *sequences* of recursively enumerable axiom systems. Building on an initial axiom system A_0, we construct a progression of systems; for each successor ordinal we add a formula asserting the consistency of the preceding system, taking unions at each limit ordinal. Consider then the theorems of an entire transfinite progression of axiom systems of this sort. Might not *these* offer a promise of completeness?

So it might seem. Feferman notes that a general incompleteness result for such progressions would have been dramatic proof of the far-reaching extent of incompleteness phenomena. "However, the situation has not turned out this way" (Feferman, 1962, p. 261). For progressions based on a particular reflection principle, all true statements of elementary number theory *are* provable in the progression. It *is* possible, moreover, to select a path through the ordinals along which all theorems of the progression are provable.

But here again it would be rash to think that crucial limitations had finally been overcome.

As Feferman emphasizes, the construction of progressions at issue is intensional in character. This gives us a peculiar non-uniqueness result: two systems A_d and $A_{d'}$ may yield radically different theorems even though they are associated with the same progression function and even though $|d| = |d'|$ (Feferman, 1962, pp. 261-262, 286). It is the intensional character of Feferman's progressions that allows for proofs of consistency and the appearance of completeness. These rely, however, on what Michael Resnik has termed 'pathological' consistency predicates (Resnik, 1974). As R. G. Jeroslow notes,

> The issue is that a non-standard designator s(w) may so mysteriously describe S that S can prove consistent whatever s(w) may designate, not "knowing" that s(w) designates S itself. (Jeroslow, 1971, p. 25)[18]

Those features of such progressions which seem initially to transcend Gödelian limits, then, rest ultimately on the progressions' basic ignorance—a strongly presumptive disqualification for any system intended to model omniscience.

Even the initially attractive features of such progressions, moreover, are lost for sequences based on higher than first-order calculi. Here Feferman does demonstate a quite general incompleteness result: For any consistent progression based on at least the second-order calculus, either there is a true Π^1_1 sentence or there is a true Σ^1_1 sentence which is not provable from $\cup_{d\in O} A_d$ (Feferman, 1962, p. 314).

Feferman's work has been incorporated and extended in R. G. Jeroslow's 'experimental logics' (Jeroslow, 1975; see also Hajek, 1977). These logics transcend the limits of standard systems in being in effect dynamic rather than static, progressively building by trial and error. As such a system develops, axioms and even rules of inference can be withdrawn or supplemented. Thus experimental logics model not merely ideal knowers but ideal learners.

As might be expected from the link with Feferman's work, there *are* experimental logics capable of proving their own consistency. But this is not enough to offer much hope for modelling omniscience.

A logic of this sort is termed *convergent* if its recurring formulae do not vacillate indefinitely—if eventually "the conceptual superstructure settles down" (Jeroslow, 1975, p. 256).[19] To model an (eventually) *omniscient* being, then, we would need a system which converged on all truth—on at least, say, all truths of the form $(\forall x)R(x)$ for recursive predicates R. But this does not appear to be possible. For systems at issue Jeroslow has shown that joint requirements of consistency, convergence, and closure under reasoning are in fact inconsistent with the goal of obtaining all truths $(\forall x)R(x)$ (Jeroslow, 1975, pp. 257, 264-265).

None of the non-constructive attempts we've considered, then, seems to offer an acceptable model for omniscience. Is that enough, with the results of section I, to show that there simply *is* no such model?

Certainly not. The standard Gödel result stops at the limit of standard systems, and there are options for non-constructive systems that have not yet been developed.[20] In the next section we will consider a more general negative result, the first of a series which does seem to suggest the genuine impossibility of omniscience.

III. EXPRESSIVE INCOMPLETENESS

The knowledge of an omniscient being can correspond to no system yet considered. But we have not yet shown that it can correspond to no system at all.

A quite simple but powerful Cantorian argument seems to show just that. For at least a particular type of incompleteness—what we will term 'expressive incompleteness'—*any* system meeting certain minimal conditions will prove incomplete. Those minimal conditions would seem clearly necessary in any system intended to model omniscience. But no system which meets those conditions can be complete, and no *in*complete system can model omniscience. To the knowledge of an omniscient being, it appears, will correspond no system at all.

It should perhaps be noted that what follows is not a form of Gödel's argument, and that expressive incompleteness is not the familiar form that incompleteness takes in his work. A broad but more strictly Gödelian treatment will be left to the following section.

We have offered above a very general statement of the conclusion of the argument. But let us begin with a proof of expressive incompleteness for a particular and particularly familiar type of system.

Consider the standard systems of number theory to which Gödel's theorems apply. In such systems, basic strings of formulae correspond on interpretation to the natural numbers, and it is these that such systems are taken to be *about*. Fairly intuitively, then, the natural numbers are what we will term the *objects* of such systems.

Within such systems appear formulae of various kinds, among which are formulae of one variable. These, in accord with common parlance, we will call *predicates*.[21]

Suppose now such a system with the following characteristics:

(1) It can, first of all, take each predicate expressible in its language as an object. A system with this capacity we will call *self-reflective*, for fairly obvious reasons. The systems immediately at issue are of course self-reflective by virtue of gödel numbering: to each predicate corresponds a gödel number which can be taken as an object of the system.

(2) Secondly, the system is designed to be capable of at least expressing all properties of its individual objects, the natural numbers.

Here's the rub: conditions (1) and (2) cannot both be satisfied for any system of the sort at issue. For consider the following questions: How many objects would be at issue for such a system? How many properties of individual objects? How many expressible predicates?

There must first of all be at least as many objects of such a system as there are predicates within it, since each predicate can be taken as an object. The device of gödel numbering assigns a distinct number to each open formula of one variable, so there must be at least as many numbers—the objects of such a system—as there are predicates within it.

But there must also be more *properties* of individual objects of the system than there are objects. For consider the set of objects of the system—numbers—which we might envisage as the set O:

$$O = \{o_1, o_2, o_3, \ldots\}.$$

If we treat properties purely extensionally, possession of distinct properties will amount to membership in distinct sets. The properties of objects at issue, then, will correspond to subsets of the set of objects—to elements of the power set $\mathcal{P}O$ of the set of objects at issue. A complete listing of such properties we might envisage as follows:

P^1,	corresponding to	$\emptyset$
P^2,	corresponding to	$\{o_1\}$
P^3,	corresponding to	$\{o_2\}$
P^4,	corresponding to	$\{o_3\}$
	. . .	
P'^1,	corresponding to	$\{o_1, o_2\}$
P'^2,	corresponding to	$\{o_1, o_3\}$
	. . .	
P''^1,	corresponding to	$\{o_1, o_2, o_3\}$
	. . .	

Intensionally construed, of course, a number of coextensional properties may correspond to each set.

There will then be as many properties applicable to individual objects as there are elements in the power set of objects. But by Cantor's power set theorem we know that the power set of any set is larger than the set itself.[22] Thus there must be more properties applicable to individual objects than there are objects.

Let us sum up. By the first part of the argument, for any system of the sort specified, there are as many objects of the system—numbers—as expressible predicates—open formulae of one variable. But by the second part of the argument there must be more *properties* of individual objects than there are objects.

For any such system, then, there will be more properties of individual objects of the system than there will be appropriate predicates with which to express them. Properties will outnumber corresponding predicates. Some genuine property of the objects of a self-reflective system must thus go unexpressed, and so condition (2) above—that each property of its objects be at least expressible in the system—cannot be satisfied.

Correspondingly, of course, some *truth* regarding an object of the system—that it does (or does not) have a particular property—will be incapable even of *expression* in the system. All such systems will be expressively incomplete.[23]

As presented above, the argument for expressive incompleteness is tied to particular features of familiar systems; numbers are taken as *objects* in the argument, open formulae of one variable as *predicates*, and it is by the device of gödel numbering that such systems can take their own predicates as objects.

The argument will hold, however, for any system interpreted as applying to a domain of objects—those things the system is taken to be about—and including a range of predicates applicable within the system. If any such system is self-reflective—if each of its predicates can also be taken as an object to which the system applies—it will have at least as many objects as predicates. But taking properties purely extensionally, by Cantor's power set theorem there will be more properties of individual objects of the system than corresponding predicates with which to express them. Some genuine property of some object of the system—and thus some truth—will be inexpressible, and any such system will prove expressively incomplete.

Expressive incompleteness, then, is by no means limited to the standard systems of section I: it will apply for any system which meets the basic condition of self-reflection.

At least no *system*, it appears, can achieve omniscience. For any system intended to model omniscience would surely have to include its own predicates among the objects it knows things about—it would have to be self-reflective in the sense outlined above. But any self-reflective system, we've seen, will be expressively incomplete: some property of its objects, and thus some truth, will be incapable even of expression within the system.

Note also how very thin a notion of 'system' is in fact required in the basic argument above. Nothing has been said to indicate that any system at issue must be formal or axiomatic, that it must generate theorems by means of demonstrations or even that it must contain a category of assertions or asserted theses.

All the argument requires, in fact, is a system of *expression*—in a word, a *language*. This first result might then be put as follows. Given even minimal requirements of expressible self-reflection, it appears, any system of expression must prove expressively incomplete. In that sense there could not even be a language adequate for the expression of all truths.

IV. GÖDEL GENERALIZED

Expressive incompleteness, because it does appear to hold for every self-reflective system, may pose real difficulties for omniscience.

As noted above, however, this is not Gödel's argument and expressive incompleteness is not the form that incompleteness takes in his work. *Expressive* incompleteness is a pervasive limitation on what can even be expressed within certain systems. What Gödel shows is that for a wide range of systems, on the assumption of consistency,[24] some formula which *is* expressible in the system and represents a truth on interpretation nonetheless cannot be captured as a theorem of the system. This more familiar form we might term *internal* incompleteness.

These two forms of incompleteness are not co-extensive; internal incompleteness holds only for a somewhat more restricted class of systems than does expressive incompleteness. In the basic spirit of the preceding section, however, we can also offer a fairly general argument for internal incompleteness—an argument general enough to indicate that even internal incompleteness applies far beyond its usual association with the standard systems of section I.[25]

Let us start, as before, with a self-reflective system, capable of taking each of its expressible predicates as a particular object of the system.

For any self-reflective system, consider the set P of expressible predicates:

$$P = \{P_1, P_2, P_3, \ldots\}$$

and the corresponding set P° of *predicate objects*—those objects of the system which are expressible predicates taken as objects:

$$P^\circ = \{P^\circ_1, P^\circ_2, P^\circ_3, \ldots\}$$[26]

Clearly these two sets will be the same size. A one-to-one function f will be possible between them, then, which assigns each predicate object P° to an expressible predicate $f(P^\circ)$ and such that to each expressible predicate of the system some predicate object is assigned. An obvious candidate for such a function, of course, would be that which assigns each predicate object to the predicate of which it *is* the object. Within the specified conditions there will be many other possibilities for f as well, however.

For any such f, now, consider any individual predicate object P° and its associated predicate by f, namely $f(P^\circ)$. That predicate may or may not in fact apply to the object at issue. The predicate

$f(P^\circ)$ applied to the object P°, in other words—giving us the formula $f(P^\circ)P^\circ$—may or may not represent a *truth* on the intended innterpretation. $f(P^\circ)P^\circ$ may also, or may not, appear as a *theorem* within the system at issue.

For any choice of f, then, consider the following set:

$$P^{\circ\prime} = \{P^\circ : f(P^\circ)P^\circ \text{ is not a theorem}\}.$$

Here P° is any predicate object, and $f(P^\circ)$ its associated predicate by our chosen function f. $P^{\circ\prime}$, then, is the set of those predicate objects to which the corresponding predicates $f(P^\circ)$ do not apply as theorems.

Note however that P° is explicitly just a set of objects of the system, and in that regard might seem a plausible candidate for the extension of a predicate. Such a predicate would apply to precisely those objects which are members of $P^{\circ\prime}$: to all and only predicate objects P° to which the associated predicate $f(P^\circ)$ does not apply as a theorem.

The crucial question here is this: Is such a predicate—a predicate of which this is the extension—*expressible* in the system?

If not, of course, the system is expressively impoverished in certain respects. But if such a predicate *is* expressible, for *any* f of the sort indicated, and if the system at issue is also consistent, then it must be internally incomplete. Some truth expressible within the system will not be captured as a theorem.

For suppose that such a predicate, for some appropriate f, *is* expressible in the system. f, it will be remembered, has been chosen as a function mapping some predicate object P° onto each predicate expressible in the system. If *this* predicate is expressible, f must then also map some P° onto *it*.

Consider then the predicate at issue and any P° which our chosen f assigns to it. Does the predicate at issue in fact apply to *that* P° or not? We have two options:

Let us suppose first that the predicate at issue will *not* apply to its correlated object. Here we will bring in our final assumption—of consistency—in the following form: that it is only truths on the intended interpretation that are taken as theorems of the system.[27]

We are supposing that the predicate at issue will *not* apply to its associated P°. Since only truths are captured as theorems, then, $f(P^\circ)$ applied to P° in this case—the formula $f(P^\circ)P^\circ$—will not be a theorem. The predicate at issue, however, is specified as having $P^{\circ\prime}$ as its extension—as applying to *every* P° for which $f(P^\circ)P^\circ$ is not a theorem. Contrary to our intitial negative supposition, then, we are forced to conclude that the predicate at issue *will* apply to its associated object.

The only option left here is the second: that the predicate at issue *does* apply to that P° with which it is correlated.

It is then true in this case that P°'s correlate—$f(P^\circ)$—in fact applies to P°; $f(P^\circ)P^\circ$ represents a truth. The predicate at issue, however, has been specified as applying only to those objects P° for which $f(P^\circ)P^\circ$ is not a theorem. Since the predicate at issue does apply to its corresponding P° in this case—since $f(P^\circ)P^\circ$ is true—it is also true that $f(P^\circ)P^\circ$ is not a theorem of the system.

At least one truth expressible within the system, then, is not captured as a theorem; any such system must be internally incomplete.

The argument can be repeated, of course, for any choice of a one-to-one function f which assigns to each expressible predicate a corresponding object. For any such f there will be a predicate which if expressible in the system will give us the same result.

Note also that although the argument concerns systems conceived as containing expressible predicates, objects, and theorems, little else has been said to constrain that class of systems for which the argument will apply. Nothing has been said, in particular, to limit relevant systems to those meeting the formal contraints imposed in section I.

Where then does this leave us? Despite the surface complexities of the argument above, the assumptions we have made regarding any system at issue have been genuinely minimal: that it is self-reflective, consistent, and capable of expressing at least one of a range of predicates that we have specified in terms of their extensions. For any system which satisfies these basic conditions the argument above can be repeated, and thus any such system will prove internally incomplete.

How general then is the phenomenon of incompleteness? Expressive incompleteness, we've seen, will hold for any self-reflective system. Internal incompleteness will hold for any self-reflective and consistent system capable of expressing any of a range of particular predicates.

Here we can also say a bit more, however, about precisely how little expressive capacity is actually required for internal incompleteness. Just two elements will basically suffice: (1) that a system be capable of expressing theoremhood within the system—that a formula is or is not a theorem—as a predicate, and (2) that it be capable of expressing at least one function f which assigns an object P° of the system to each expressible predicate $f(P^\circ)$. Given essentially these two elements a predicate can be constructed with extension $P^{\circ\prime}$:

$$P^{\circ\prime} = \{P^\circ : f(P^\circ)P^\circ \text{ is not a theorem}\},$$

and with such a predicate expressible any self-reflective and consistent system will also prove internally incomplete.

V. FIRST ARGUMENTS AGAINST OMNISCIENCE

Let us return to the analogy between systems and knowers, and in particular to systems and the notion of an omniscient knower.

Do the incompleteness results of the preceding sections offer an argument that omniscience is impossible?

Consider first an argument which follows the pattern of expressive incompleteness. Here we'll speak of conceptions of properties instead of predicates, and will use 'objects of knowledge' somewhat irregularly to indicate those things a knower knows something *about*.

Any omniscient mind would surely be self-reflective in at least the following sense: among its objects of knowledge—among those things it knows something about—would be its own conceptions of properties. But here the argument of section III can be rephrased to show that the knowledge of no such mind can be complete. It will have at least as many objects of knowledge as conceptions of properties, since each of the latter is also an object of knowledge. But by Cantor's argument there will be more actual properties of its objects than objects themselves. Actual properties will outnumber its conceptions of properties, and thus some genuine property of its objects of knowledge—and so some truth—will remain *inconceivable* for such a being.

Any omniscient being, so the argument goes, would have to be self-reflective in the sense specified. But no self-reflective being can be omniscient. There can be no omniscient being.

Whatever our final verdict, I think, the argument from expressive incompleteness is an elegantly simple one. Although significantly more awkward, we can also offer an argument from internal incompleteness:

An omniscient mind, we've suggested above, must be self-reflective in at least the sense of being able to take its conceptions of properties as objects of knowledge. But we might also argue that a genuinely omniscient mind would have to be self-aware in deeper senses as well. Among the things that such a being will know, of course, is *that* it knows certain things, and thus '. . . is known by me' or the like will be among its conceivable properties. Such a mind, we might insist, will surely also be cognizant of obvious aspects of its own conceptual structure—it will for example be aware of one-to-one mappings between its conceptions of properties and these taken self-reflectively as objects of knowledge.

Omniscience, then,—so the argument goes—has formal features analogous to those outlined for systems in the preceding section: self-reflectivity, expressible theoremhood, and the expressibility of some one-to-one mapping *f* from predicate objects to predicates of the system. The knowledge of any omniscient being would of course also be consistent. But any *system* with these formal features will be internally incomplete: some truth expressible in the system will not be captured as a theorem. For the same reasons, it appears, the knowledge of any being proposed as omniscient must be correspondingly incomplete: there will be some truth which *is* expressible or conceivable by such a being and yet will *not* appear among those things it knows.

Or so the argument goes.

How good are these arguments as, say, genuine disproofs of omniscience?

The argument from expressive incompleteness seems by far the more persuasive of the two, if only because it is significantly simpler and more direct. The argument from internal incompleteness demands more points of comparison between systems and knowers, and here conviction may fade as the analogy begins to show the strain.

Both arguments presented rely on *some* points of analogy between knowers and systems, however. An objector might then take the following tack: '*If* analogous to a system in the sense required, the knowledge of any being proposed as omniscient *would* be demonstrably incomplete. But perhaps that merely indicates that omniscience is *not* analogous to any system. Perhaps the knowledge of an omniscient being not only cannot be conceived of on the model of a standard system such as those of section I, but cannot be conceived of in terms of any *system* at all.'

Full vindication of the arguments above against such a reply would call for further work. One option here would be to carefully strengthen, strand by strand, the relevant analogy between systems and knowers—to emphasize how little is really required by the notion of 'system' at issue, how a set of propositions known *will* have the crucial formal properties of a set of theorems, and so forth. Another option would be to rephrase the incompleteness arguments of the preceding sections entirely in terms of knowers and what they know, thereby avoiding talk of 'systems' entirely.

With patience, I think, even the more complex second argument above could be defended in one of these ways. But perhaps none of this is necessary. A short and direct Cantorian argument, offered below, seems to give us the same conclusion while avoiding the complications of systems altogether.

Consider also a related objection. Even where not tied to formal systems, an objector might claim, the arguments above are at least tied to formal *languages* in some way. 'Perhaps the arguments above indicate only that there can be no language adequate for the representation of all truths, or indicate only that there can be no divine *language* in the relevant sense.'[28]

This, I think, would be a mistake. What the arguments above require is not formal languages but merely certain features analogous to those of formal languages—features which may themselves be quite natural and *non*-linguistic features of knowers, minds, or sets of things known. The first argument against omniscience above, for example, is phrased entirely in terms of just objects of knowledge—things about which something is known—and conceptions of properties.

We can, at any rate, sidestep this second objection in the same way as the first. The Cantorian argument of the following section seems to cut quite neatly through complications of either systems or languages.

VI. THERE IS NO SET OF ALL TRUTHS

There is no set of all truths.

For suppose there *were* a set $\mathfrak{T}$ of all truths, and consider all subsets of $\mathfrak{T}$, elements of the power set $\mathcal{P}\mathfrak{T}$.

To each element of this power set will correspond a truth. To each set of the power set, for example, a particular truth T_1 either will or will not belong as a member. In either case we will have a truth: that T_1 is a member of that set, or that it is not.[30]

There will then be at least as many truths as there are elements of the power set $\mathcal{P}\mathfrak{T}$. But by Cantor's power set theorem the power set of any set will be larger than the original. There will then be *more* truths than there are members of $\mathfrak{T}$, and for *any* set of truths $\mathfrak{T}$ there will be some truth left out.

There can be no set of all truths.

One thing this gives us is a short and sweet Cantorian argument against omniscience, uncomplicated by systems or formal languages:

Were there an omniscient being, what that being would know would constitute a set of all truths. But there can be no set of all truths, and so can be no omniscient being.[31]

Let me digress slightly in order to mention some further implications of the argument as well, however. One victim of such an argument, it appears, is a common approach to the notion of *possible worlds* and in particular to the notion of an *actual* world.

Possible worlds are often introduced as maximal consistent sets of propositions—proposition-saturated sets to which no further proposition can be added without precipitating inconsistency—or as some sort of fleshed-out correlates to such sets. The *actual* world, on such an account, is that maximal consistent set of propositions all members of which actually obtain—a maximal and consistent set of all and only *truths*—or is an appropriately fleshed-out correlate to such a set (see Adams, 1974, Plantinga, 1974, and Plantinga, 1980).[32]

By the argument above, however, there is not and cannot be any set of all truths. *Any* set of true propositions will leave some true proposition out, and thus there can be no maximal set of truths. In this sense of 'actual world', then, there is and can be no actual world.[33]

It should perhaps not be too surprising that the notion of an actual world outlined above faces difficulties similar to those that can be raised against omniscience. These are, after all, largely correlative notions. That which would be known in omniscience is that which would obtain in such an actual world—omniscience is the epistemic correlate to this metaphysical conception of the actual world.

It should also be noted that the argument above can be applied against the existence of some 'smaller' sets as well. Consider for example the set not of all truths but merely of all *metamathematical* truths. *Is* there such a set?

Here we first have to answer a clarificatory question. Does each truth regarding the membership of a set of metamathematical truths itself quality as a metamathematical truth? If so, by an argument perfectly analogous to that offered above, there will be no set of all metamathematical truths; each set of metamathematical truths will leave out some metamathematical truth.[34]

The same will hold for any set of truths of a type Θ, where truths regarding membership in sets of truths of type Θ themselves qualify as truths of that type. For no such type of truths will there be a set of *all* truths of such a type.

VII. ALTERNATIVE SET THEORIES: A POSSIBLE WAY OUT?

The simple Cantorian argument runs as follows:

Were there an omniscient being, what that being would know would constitute a set of all truths. But there can be no set of all truths, and so can be no omniscient being.

Might we not give up the idea of a *set* of all truths, however, or of what an omniscient being knows as a *set* of things known, and substitute something else here instead? Perhaps there is no set of all truths, but there *is* a *class* of all truths (or *proper class* or *ultimate class*) in the sense of alternative set theories.

Will this offer a way out?

The short answer, I think, is 'no'.[35]

All axiomatic set theory, standard or alternative, is essentially a response to two paradoxes: Cantor's paradox regarding a set of all sets and Russell's paradox regarding a set of all non-self-membered sets. By the Aussonderung axiom of standard ZF set theory, of course, there simply *are* no such sets. What we've suggested in preceding sections, in effect, is that a 'set of all truths' leads to similar difficulties and should be similarly abandoned.

In some alternative set theories something like Cantor's and Russell's sets do appear, however—though in the guise of 'classes' or 'ultimate classes' or 'proper classes', carefully distinguished from sets and for which different principles hold. But in one way or another all such alternatives seem to come to grief.

Quine's "New Foundations" and the von Neumann-Bernays system, for example, both avoid paradox by effectively crippling the mechanism of Cantor's theorem, and in that sense may seem to offer hope for something like a class of all truths. NF and VNB also share one crucial and quite exhorbitant cost, however: both entail a sacrifice of general mathematical induction.[36] As Quine concludes with respect to NF,

> the fact remains that mathematical induction of unstratified conditions is not generally provided for . . . This omission seems needless and arbitrary. It hints that the standards of class existence . . . approximate insufficiently, after all, to the considerations that are really central to the paradoxes and their avoidance. (Quine, 1963, p. 199)[37]

Quine's system in *Mathematical Logic* and a modified VNB he suggests, on the other hand, both manage to remedy this glaring inadequacy with respect to mathematical induction. In order to do so, however, both restore the basic mechanism of the Cantorian argument just enough to dash any hopes for either a class of all classes or a class of all truths.

If these are any sample, then, alternative set theories do not seem a very promising route of escape. For our purposes, moreover, these technical problems are only one of the marks against them.

Ultimate classes in general, in whatever alternative system, are introduced as classes which are not members of further classes. But a 'class of all truths' would surely not qualify as ultimate in that sense. Wouldn't it be a member of the class of classes of propositions? Wouldn't it form an ordered pair with the class of all false propositions?

Consider also the class of classes of things known by existent beings—wouldn't what God knows be a member of *that* class?

In the end, I think, the ultimate classes of alternative set theory turn out to be an unacceptable option even on simple intuitive grounds. This last difficulty is closely related to Quine's general objection:

> [VNB modified] shares a serious drawback with ML, and with von Neumann's unextended system, and with any other system that invokes ultimate classes . . . We want to be able to form finite classes, in all ways, of all things there are assumed to be . . . and the trouble is that ultimate classes will not belong. (Quine, 1963, p. 312)

VIII. VERSUS PLANTINGA AND WITTENGSTEIN: CONCLUSION

I have attempted above to use Cantor and Gödel in suggesting an argument, or group of arguments, against omniscience—that there can in principle be no being that knows everything. There can in fact be no set of all truths, and alternative set theory offers little hope for even an ultimate *class* of all truths as an alternative.

Does the work above actually show that omniscience is impossible?

As noted in introduction, philosophical speculation regarding metalogical results is a notoriously risky business. With that in mind I have confined myself throughout to *suggesting* Cantorian and Gödelian arguments against omniscience, offering these as arguments worthy of consideration, but without trumpeting them as proofs.

Nonetheless, philosophical speculation—however risky—also has its place. In that spirit let me stick my neck out at least this far:

Is omniscience impossible?

Within any logic we have, I think, the answer is 'yes'.

What I mean is this. In terms of either systems or sets, on the basis of work presented above, omniscience appears to be simply incoherent. All the logic we have, however, is essentially a matter of systems and sets. Within any logic we have there appears to be no coherent notion of omniscience.

The theist, of course, can be expected to pounce on the crucial qualifying phrase above—'within any logic we have.' 'Perhaps our logic is merely inadequate to do justice to the notion of omniscience. Perhaps some other logic, *not* a matter of mere systems and sets, eventually within our grasp or forever beyond our grasp, *would* allow us a coherent notion of omniscience.'

What gives this response plausibility is the fact that new logics *have* been developed to serve special needs. And perhaps it could be done again. Perhaps despite appearances it would be possible to specify a 'bunch' or 'gob' that *would* coherently collect all truths in a way that neither sets nor their alternative classes can. Perhaps. As things stand, however, the theist's invocation of that 'perhaps' is merely a promisory note on a debt of coherence—a second or third or fourth mortgage on omniscience.

Note also that the same response could be made in behalf of *any* position, however ludicrous, and in the face of *any* argument, however rigorous. Perhaps our logic is merely inadequate to do justice to the notion of circular squares or married bachelors.[38] As it stands, then, the theist's response does nothing to distinguish omniscience from any of various incoherent notions that fall victim to logical argument.

Epistemological and metaphysical aspects of the work above have of course been mentioned throughout. Let me summarize these suggestions, however, by way of points regarding Plantinga and Wittgenstein respectively.

Gaunilo, a contemporary of Anselm's, parodied Anselm's ontological argument for the existence of God by constructing a parallel argument for a greatest possible island. In defending a form of Anselm's argument for Anselm's God, Plantinga attempts to avoid Gaunilo's argument for Gaunilo's island. He does so by insisting that the great-making characteristics of islands, unlike those of God, are without *intrinsic maxima*:

> The idea of an island than which it's not possible that there be a greater is like the idea of a natural number than which it's not possible that there be a greater . . . There neither is nor could be a greatest possible natural number; indeed, there isn't a greatest *actual* number, let alone a greatest possible. And the same goes for islands. No matter how great an island is, no matter how many Nubian maidens and dancing girls adorn it, there could always be a greater—one with twice as many, for example. The qualities that make for greatness in islands—numbers of palm trees, amount and quality of coconuts, for example—most of these qualities have no *intrinsic maximum*. That is, there is no degree of productivity or number of palm trees (or of dancing girls) such that it is impossible that an island display more of that quality. So the idea of a greatest possible island is an inconsistent or incoherent idea; it's not possible that there be such a thing. . . .
>
> But doesn't Anselm's argument founder on the same rock? If the idea of a greatest possible island is inconsistent, won't the same hold for the idea of a greatest possible being? Perhaps not. . . . Anselm clearly has in mind such properties as wisdom, knowledge, power, and moral excellence or moral perfection. And certainly knowledge, for example, does have an intrinsic maximum . . . (Plantinga, 1980, pp. 90-91)[39]

What the argument of the preceding sections suggests, however, is that knowledge does *not* have an intrinsic maximum. The case of a 'greatest possible number' would in fact be perfectly analogous here. For any natural number, there is a greater. What the Cantorian argument suggests is that for any body of knowledge—that possessed by any particular being, for example—there is some truth it leaves out, and so some body of knowledge beyond it.

If knowledge has no intrinsic maximum, of course, the notion of an omniscient being itself becomes "an inconsistent or incoherent idea; it's not possible that there be such a thing."

Metaphysical aspects of the work above can be summarized using Wittgenstein.

The opening lines of the *Tractatus* run as follows:

1* The world is all that is the case.
1.1 The world is the totality of facts, not of things.
1.11 The world is determined by the facts, and by their being *all* the facts. (Wittgenstein, 1961, p.7)

What the arguments of the preceding sections suggest is that these famous lines must be dead wrong.[40] Given Cantor and Gödel, it appears, there simply *is* no totality of facts or of all that is the case. The universe itself, on such a view—like any knowledge or description of it—is essentially open and incomplete.

Notes

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I would like to dedicate this paper to the memory of my father.

[1]Defining omniscience is harder than it looks. In the first section of Grim, 1983, I argue that definitions offered by Peter Geach, A. N. Prior, Richard Swinburne, James F. Ross, and William E. Mann are inadequate. As a replacement I there suggest the following:

x is *omniscient* $=_{df.}$ for all p, p is true IFF x believes that p,
AND x believes that p IFF x knows that p.

For the purposes of this paper, however, all that is crucial is that any omniscient being will believe all and only truths.

[2]Witness for example the checkered history of Lucas, 1961. Among its many replies see esp. Webb, 1968, and Benacerraf, 1967.

[3]The notion of a 'formal system' is stretched well beyond familiar limits in the course of the discussion, however.

[4]Which are the *basic* truths of a body of knowledge, of course, may be relative to the choice of transformation rules. Even given a particular set of transformation rules, moreover, there may be alternative sets of truths any of which might be taken as basic.

[5]Judson Webb notes that

> whether or not a discipline regarding a given subject matter can be deductively systematized is simply the question whether or not the set T of true sentences about the subject matter is recursively enumerable . . . (Webb, 1968, p. 167)

If requirements on 'systems' are relaxed beyond recursive enumerability, even this will not restrict those bodies of knowledge which might be captured as 'systems'.

[6]It might be thought that the following is an additional obstacle to any comparison between knowers and formal systems: a (standard) formal system, if it is to exclude any formula as a non-theorem, must be consistent. Knowers, on the other hand, are rarely if ever perfectly consistent.

There are unusual systems in which inconsistency does not result in the inclusion of everything as a theorem—see for example Rescher, 1979; Priest and Routley, forthcoming; Routley, 1984; and DaCosta, 1974. But at any rate inconsistency cannot pose a problem if we limit ourselves to formal systems analogous to merely *what a knower knows*. For no matter how inconsistent I—a knower—may be in my beliefs, *what I know* must be perfectly consistent. It must be consistent simply because what I know must all be true.

[7]The rules at issue here are the following:

(A. ~K) If λ is consistent and if "~K*ap*" ∈ λ, then λ + {"*Pa*~*p*"} is also consistent.

(A. ~P) If λ is consistent and if "~*Pap*" ∈ λ, then λ + {"*Ka*~*p*"} is also consistent. (Hintikka, 1962, p. 29)

[8]Alternatively, still in Hintikka, 1962, Hintikka proposes a reinterpretation of his operators; that "*Kap*" should perhaps be read not as "*a* knows that *p*" but "it follows from what *a* knows that *p*" (Hintikka, 1962, p. 38).

But Hintikka has since changed his tune. He now emphasizes that the difficulty above arises only if we insist that every epistemically possible world is logically possible. See Taylor, 1983, and Hintikka, 1975.

[9]This point actually applies to bodies of knowledge as well—it is only *ideal* bodies of knowledge that are to be captured by standard systems.

We might be able to simulate *non*-ideal knowers, and *non*-ideal bodies of knowledge, by means of crippled transformation rules. This is in fact one way of characterizing Nicholas Rescher and Robert Brandom's intriguing work on belief in esp. chapter 19 of Rescher and Brandom, 1979. Here, however, it *is* ideal knowers and bodies of knowledge that are at issue.

[10]A more standard statement of the standard Gödel incompleteness result is the following. Consider any formal system with recursively recognizable formulae and axioms, rules of inference only from finite sets of premises, and which is adequate at least for the purposes of number theory. If any such system is omega-consistent, it is unavoidably incomplete; something will be left out. Syntactically put: for some formula expressible in the system, neither that formula nor its negation will appear as a theorem. Semantically put: some truth of number theory will not be captured as a theorem in the system.

I do not consider my purpose here to be that of a general introduction to Gödel—that has been done wonderfully elsewhere by others. I have in mind particularly Nagel and Newman, 1956, and of course Hofstadter, 1979.

[11]In Rosser's extension of Gödel's theorems, Gödel's stronger hypothesis of omega-consistency is replaced with the weaker hypothesis of mere consistency. See Rosser, 1936, pp. 87-91.

[12]The attempt to 'fill in' incompleteness holes in such a manner eventually leads one to a progression which corresponds to that of the constructive ordinals. But by a result due to Alonzo Church and Stephen C. Kleene (Church and Kleene, 1936), there is no recursively-related notation system adequate even for *naming* each of the constructive ordinals.

[13]On essential undecidability in this sense see Tarski, Mostowski, and Robinson, 1968, and Goodstein, 1963.

[14]Similar comments on the peculiarity of non-constructive methods appear in Hofstadter, 1979, p. 470, and Wang, 1964, pp. 318-319.

[15]I have not here included work involving non-denumerable alphabets or formulae of infinite length, each of which seems to fizzle out at the level of first-order predicate calculus.

Non-denumerably many symbols appeared in a system Leon Henkin used to show completeness for first-order functional calculus (Henkin, 1949). The limitations of the system even in that context are comparable to those of S_∞ in Gentzen's consistency proof for first-order number theory, considered below.

Henkin originally considered three ways in which infinite formulae might be introduced: (1) by means of infinitary predicate symbols and hence infinitely long primitive formulae, (2) by means of infinitely long conjunctions and disjunctions together with quantification over infinitely many variables, and (3) infinitely alternating quantifiers of a peculiar type (Henkin, 1961). It is the second of these that has been most developed, in particular in the work of Carol R. Karp (see esp. Karp, 1964).

For some such predicate systems $L_{\alpha\beta}$, in which conjunctions of fewer than α formulae and quantifications of fewer than β variables are permitted, completeness can be proven. A fairly uninteresting case here is $L_{\omega\omega}$, which is simply the standard predicate calculus without extension to infinite formulae. Where genuinely infinite formulae are at issue, completeness holds only for those predicate systems in which an ability to handle conjunctions outstrips an ability to handle quantifications. Jon Barwise, G. Kreisel, and Dana Scott have expressed doubts about any such system admitting infinite quantifiers (Barwise, 1969, p. 227). But at any rate no definable system in which $\alpha = \beta = \gamma^+$, where γ is infinite—even if the underlying system has only one two-place predicate in addition to equality—will be complete (see Karp, 1964, pp. 166-174).

[16]The standard use of S_∞—to prove consistency for first order numbers theory—is not here of much importance. In my sketch of S_∞ I follow Mendelson, 1964, pp. 258-27. But see also Wang, 1964, pp. 362-375.

[17]A result attributed to Rosser in Wang, 1964, p. 45.

[18]In Jeroslow, 1971, Jeroslow also shows that consistency statements which differ from Feferman's can be proven in extraordinarily weak systems.

[19]The systems at issue, however, are still in some sense mechanical. See esp. Jeroslow, 1975, p. 255.

[20]The most promising candidates here would seem to be systems with non-recursively enumerable sets of axioms.

[21]As specified here these include only one-place predicates, for the sake of simplicity. The basic structure of the argument would be the same, however, if all *n*-ary predicates were included.

[22]There are of course many standard presentations. See for example Copi, 1979, pp. 185-190. My treatment below follows Copi's closely.

[23]This argument is related to an incompleteness argument for finitary formal systems presented in Hunter, 1971, pp. 28-30, and to some wonderful work by Hans Herzberger in Herzberger, 1970, Herzberger, 1981, and Herzberger and Herzberger, 1981. The form of the argument offered here, however, is perhaps most similar to Johannes Baagoe's in Baagoe, 1975. Baagoe's is a marvelous piece of work to which I owe a very great debt.

[24]See note 11.

[25]That form of Gödel's proof that the argument of this section most closely resembles, perhaps, is Gödel's own less formal presentation in the opening pages of Gödel, 1931. Gödel himself notes a resemblance to Richard's paradox, itself but a step away from some of the Cantorian techniques employed here.

[26]Here and throughout the argument, for noble motives of simplicity, I will unabashedly exploit a particular ambituity: P° will sometimes be referred to as an object of the system—that to which a predicate on interpretation *applies*—and yet will also appear as a term *for* such an object in formulae such as $f(P^\circ)P^\circ$. A similar ambiguity appears in many informal presentations of Gödel, and with good reason: the attempt to avoid it adds merely one more subtlety for the reader to try to keep track of. As a corrective for this type of subterfuge, however, see Fitzpatrick, 1966.

[27]Strictly speaking this is a somewhat stronger assumption than mere consistency, but such a simplification is fairly standard in informal presentations of Gödel and seems harmless in the present context.

[28]The expressive incompleteness argument of section III was of course characterized as showing that no language can be adequate for the expression of all truths. The objector's suggestion here is that perhaps that is all that any of the arguments really show.

[29]There is no need here to treat 'truths' as linguistic entities in any sense. The argument would be the same against any supposed set of all true propositions or of all facts—at least in the ordinary sense of 'fact' in which it's a fact that 7 + 5 = 12.

With regard to truths and linguistic entities another Cantorian argument should also be noted. In Castañeda, 1975, p. 34 ff. Castañeda uses a Cantorian argument to show that propositions are not reducible to classes of sentences.

[30]There is of course nothing special about T_1 here—we could have used any particular truth in its place. There are also myriad other ways of constructing a truth for each element $\mathcal{PT}$.

For a slightly expanded form of the argument see Grim, 1984.

[31]In personal correspondence J. H. Sobel has outlined a very similar Cantorian argument against omniscience, developed independently. Sobel has also pointed out that such an argument can be constructed against even a *non*-omniscient being of a certain type.

Consider any being which, although perhaps *not* omniscient, does know *itself* very well: it knows (*de re*, let us say), for each set that contains only propositions that it knows, *that* that set contains only propositions that it knows. Consider now the set of all propositions that that being knows, and the power set of that set. To each set of the power set will correspond a proposition that our being, as specified, knows—that that set consists only of propositions it knows. But by Cantor's theorem there are more elements of the power set, and thus more propositions our being knows, than in the original set—the set of *all* propositions that the being knows. There can be no such being, then, and even 'Know thyself' has Cantorian limits.

David L. Boyer has pointed out that the argument against omniscience can also be presented without explicit mention of sets of truths:

Let us assume there is an omniscient God. Consider all that such a being would know, and consider further what would be known by each of a chorus of archangels meeting the following conditions:

Each archangel knows something, no two archangels know precisely the same thing, and for each archangel there is something that God knows and it does not.

Let us also add two more 'archangels,' in a somewhat extended sense: an archangel who knows absolutely nothing, and God himself. (This is not entirely without theological precedent, by the way: Aquinas claims that to each degree of being there corresponds a being.)

Now for each of the archangels envisaged there would be something that an omniscient being would know: that it is possible that such a being exists, perhaps, or that it is not possible; that the knowledge of that archangel would include the fact that seven is prime, perhaps, or that it would not. There will then be at least as many things God knows as envisaged archangels.

By the basic mechanisms of Cantor's power set theorem, however, there will be *more* archangels than things God knows. Our initial assumption leads to contradiction, then, and so must be rejected: there is no omniscient God.

[32]This is not, however, the only way that possible worlds have been introduced. In Lewis, 1973, for example, possible worlds are ways things might have been. In Slote, 1975, they are possible histories of *the* world. Whether possible worlds in these senses must be analogously incomplete is a question I leave to others or to another paper.

[33]For a similar argument against possible worlds using a variation on the paradox of the Liar, see Grim, 1983.

[34]Does every conjunction of mathematical truths, even if transfinite, correspond to a mathematical truth? If so, there is not even a set of all *mathematical* truths. Here the argument would be the same as the above except that to each element of the power set of a supposed set of all mathematical truths would correspond that mathematical truth represented by the conjunction of all members of that set.

[35]For a longer answer see especially Quine, 1963; Fraenkel, Bar-Hillel, and Levy, 1973; and Kuratowski and Mostowski, 1966.

I have not included here a section on many-valued set theories. But these don't seem to offer a plausible way out either; many-valued logics exhibit many-valued forms of the Liar and of Russell's paradox, and for essentially the same reasons can be expected to exhibit many-valued forms of the Cantorian argument above as well. In this regard see Rescher, 1969, esp pp. 87-90 and 206-212.

[36]NF has other difficulties as well. In Rosser and Wang, 1950, J. B. Rosser and Hao Wang initially showed that no model of NF—no interpretation of '$\in$' compatible with the axioms—could make well-orderings of both the lesser-to-greater relation among ordinals and that among finite cardinals, except by interpreting '$=$' as something other than identity. In Specker, 1953, Ernest Specker went on to show that those sets of NF which are non-Cantorian cause the relations of lesser to greater among cardinals to fail of being a well-ordering, and thereby produced a disproof of the axiom of choice within NF.

[37]In a similar spirit Fraenkel, Bar-Hillel, and Levy note, drawing on work by Mostowski:

> A particularly embarrassing fact about VNB is that in VNB . . . one cannot prove all instances of the induction schema, "If 0 fulfils the condition $\mathbf{B}(x)$ and for every

natural number *n*, if *n* fulfils $\mathfrak{B}(x)$ then $n + 1$ fulfils $\mathfrak{B}(x)$ too, then every number fulfils $\mathfrak{B}(x)$." (Fraenkel, Bar-Hillel, and Levy, 1973, p. 139).

[38]With regard to circular squares serious work in a Meinongian tradition should perhaps be mentioned, including Parsons, 1980, Rapaport, 1978, Rapaport, 1979, Routely, 1980, Zalta, 1983, and Castañeda's guise theory (see esp. Alvin Planinga's "Guise Theory," and Castañeda's reply, in Tomberlin 1983). Regarding paraconsistent logics, designed to incorporate carefully quarantined contradictions, see esp. Da Costa, 1974, and refs.; Priest and Routley, forthcoming; Routley, 1984; and related work in Rescher and Brandom, 1979.

[39]For further cricial work on Plantinga's treatment of Gaunilo see Grim, 1979 and Grim 1982.

[40]To quote Wittgenstein is to risk contradiction by Wittgensteinian scholars, however. Evan W. Conyers has argued in personal correspondence that 'facts' appear in the *Tractatus* in a technical sense that does not include '7 + 5 = 12' or truths regarding set membership such as those relied on in the arguments above.

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