

Throwing Darts, Time, and the Infinite

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Received: 23 August 2011 / Accepted: 2 March 2012
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Abstract In this paper, I present a puzzle involving special relativity and the random selection of real numbers. In a manner to be specified, darts thrown later hit reals further into a fixed well-ordering than darts thrown earlier. Special relativity is then invoked to create a puzzle. I consider four ways of responding to this puzzle which, I suggest, fail. I then propose a resolution to the puzzle, which relies on the distinction between the potential infinite and the actual infinite. I suggest that certain structures, such as a well-ordering of the reals, or the natural numbers, are examples of the potential infinite, whereas infinite integers in a nonstandard model of arithmetic are examples of the actual infinite.

Let us throw darts at $[0, 1]$. We work in ZFC and assume the continuum hypothesis, CH. Then let $<$ be a well-ordering of $[0, 1]$ of length \aleph_1 . There is a well-ordering because we are working in ZFC. There is a well-ordering of length \aleph_1 because we are assuming CH. Throughout this paper, any reference to order refers to this fixed well-ordering. Then throw a dart at $[0, 1]$ and let r_1 be the real hit by this first dart. Note that the set of reals less than or equal to r_1 , being countable, is of Lebesgue measure zero, and so the set of reals greater than r_1 is of measure one. Thus with probability one¹ we find that $r_2 > r_1$, where r_2 is the real hit by a second dart thrown at $[0, 1]$. Put loosely, r_2 hits a real greater than r_1 in the well-ordering because that is where almost all of the reals are. And as we continue to throw darts, the reals hit by

¹ I discuss potential problems with this notion of probability below.

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the darts move further into the well-ordering. Darts thrown later hit reals further into the well-ordering than darts thrown earlier.²

This reasoning, though it seems correct, has puzzling consequences. Indeed a consideration of special relativity leads to the following puzzle. Imagine two darts, d_1 and d_2 , and two observers, O_1 and O_2 . The two darts are thrown. Due to relative motion, in O_1 's frame of reference d_1 lands first, hitting r_1 . In O_2 's frame of reference d_2 lands first, hitting r_2 . With probability one O_1 finds that $r_1 < r_2$ because r_1 landed first. With probability one O_2 finds that $r_2 < r_1$ because r_2 landed first.³ But it cannot be the case that $r_1 < r_2$ and $r_2 < r_1$. How should we respond to this puzzle?

First, we could reject CH or some axiom of ZFC (most likely the axiom of choice). Besides the drastic nature of this response, it fails for another reason, namely, a similar puzzle can be created without using either CH or ZFC. I (Gwiazda 2006) considered two random⁴ selections from the positive integers, and noted that random selections from the positive integers get larger through time. Let m be the first positive integer selected. Then, as there are only finitely many positive integers less than or equal to m , the probability of a second selection's being less than or equal to m is either 0 or infinitesimal.⁵ That is, a second selection from the positive integers is almost certainly larger than the first selection, m . Special relativity is then invoked to create a puzzle. Given that this similar puzzle can be created in the absence of ZFC and CH, rejecting either of them based on the puzzle above would be unwarranted. I suggest that the puzzle is more problematic in the uncountable case presented above than in the countable case. One way to respond to the puzzle in the countable case is to deny that there can be uniform selection from a countable set such as the positive integers, as it is well known that countable additivity fails. This response is not available for the puzzle involving selections from the uncountable set $[0, 1]$. That is, there is a clear response available in the countable case, which is not available in the case where the selections are from $[0, 1]$.

Having suggested that rejecting ZFC or CH is not sensible, are any other responses available? A second potential response simply forbids any discussion of time and random selections. Many people are drawn to this response; those who see mathematics as a timeless, unchanging realm often wish to forbid sully

² Chris Freiling (1986, p. 190) presents "a simple 'philosophical' proof of the negation of Cantor's continuum hypothesis (CH)." He imagines that two darts are sequentially thrown at $[0, 1]$. A crucial assumption that Freiling (pp. 192, 199) makes is that "...the real number line does not really know which dart was thrown first or second..." and again "[t]he real number line cannot tell the order of the darts." Freiling argues: $[0, 1]$ cannot tell the order of the darts, along with other assumptions, proves not CH. In this paper, I argue: if CH, then $[0, 1]$ can tell the order of the darts. And so we can respond to Freiling's argument by defending CH and accepting that $[0, 1]$ can tell the order of the darts. Of course, I go on to create a puzzle if $[0, 1]$ can tell the order of the darts. But I still suggest that $[0, 1]$ can tell the order of the darts.

³ To make the example work in terms of the relative motion, it may be necessary to employ two 'dartboards', that is, two copies of $[0, 1]$, and have one dart thrown at each. The conclusion (puzzle) still follows.

⁴ 'Random' means that any single positive integer has the same chance of selection as any other, from which it follows that the chance of selecting a fixed positive integer is either 0 or infinitesimal.

⁵ Due to finite additivity and the fact that each singleton's chance of selection is either 0 or infinitesimal, as noted in footnote 4.

mathematics with any temporal considerations. And yet this reply is again too drastic and, also, ad hoc. Unless there are other compelling reasons, in addition to this puzzle, to forbid discussion of time and mathematics, enacting such a strong restriction to respond to this puzzle does not make sense. Note that it seems logically possible (it is not obviously logically impossible) that there exists a possible world where people can throw darts at $[0, 1]$ and find that exactly one real is hit.

Third, we could argue that a consideration of the measure involved provides a way out of the puzzle. However, there appears to be consensus in the literature that the Lebesgue measure provides the proper notion of probability in this context. Freiling (1986, p. 193) defends the position that “Lebesgue measure zero does ... have a strong justification for being the correct extension of countable” in the context of selecting a real in $[0, 1]$ via a random dart. He continues, “So the reason a random dart will miss a predetermined countable set is not because a countable set has cardinality less than the reals, but because a countable set is null, i.e., it has Lebesgue measure 0.” Brown (2004, p. 1135) writes, “The measure of any countable set is 0. So, according to standard probability theory, the probability of landing on a point [in a countable set] is 0. While logically possible, this sort of thing is almost never the case.”⁶ Norton (2004, p. 1147) also discusses the link between measure and probability in some detail, eventually writing, “Let us say that the ‘dart principle’ assures us that there is a probability zero of choosing a real from a measure-zero set.” Given this consensus, I do not think that a consideration of the measure involved resolves the puzzle. Finally, and as discussed in reply to the first objection, since a similar puzzle arises in the countable case, and since a consideration of measure does not dissolve that countable puzzle, this provides some evidence that the measure is not the way to resolve this puzzle.

A fourth way to resolve the puzzle would be to argue that selections do not get larger through time. As discussed in footnote 2, Freiling (1986, p. 192) writes that “the real number line does not really know which dart was thrown first or second....” I have suggested that the real number line does know the order of the darts. Imagine that a first dart is thrown on Monday, selecting a real in the well-ordering. A second dart will be thrown on Friday. It is currently Wednesday, and we must choose which dart will hit a real further into the well-ordering. Following the reasoning in the first paragraph, it seems that we must choose the second dart thrown on Friday.⁷

Before concluding, let me discuss what I believe the puzzle shows. I do not believe that special relativity is playing a crucial role. Rather, the oddity of the puzzle comes from the fact that selections grow through time, and special relativity is simply used to highlight this oddity. Indeed, I believe that selections do grow

⁶ In reply to the worry that events of probability 0 can occur, it may also be helpful to think of repeated trials. That is, imagine running the experiment (involving two darts throws and special relativity) many times, e.g., 100. One player must win a minimum of 50 trials (in the case where each player wins 50 trials). It would be odd to suggest that, in repeated trials, an event of probability 0 occurs half of the time.

⁷ To my knowledge, Norton (2004, pp. 1147–1148) comes closest in the literature to endorsing the position that darts get larger through time. Norton does not couch his discussion in temporal terms. Instead he uses the terms “direct property” and “inverse property,” but the point seems very much the same.

through time, and I believe that this “oddity” is demonstrating something interesting about the distinction between the potential and the actual infinite. The *potential infinite* is given over time and is capable of increase beyond any finite value. There may be a sense in which it is undetermined and variable. The *actual infinite*, by contrast, is present at one time, fixed, determined, and constant. Aristotle is most commonly associated with the potential infinite; Cantor with the actual infinite. Moore (2001, p. 39) discusses Aristotle’s position on the infinite, writing, “If something is potentially infinite ... then it is not even possible for it to be actually infinite.” Hallett (1988, p. 7) discusses Cantor’s position on the infinite, and argues that a key principle of Cantor’s thought was “Cantor’s principle of actual infinity, or the domain principle: Any potential infinity presupposes a corresponding actual infinity.” A simple question arises: Who was correct, Aristotle or Cantor? I believe that the considerations above provide evidence that Aristotle was correct when it comes to certain structures, such as a well-ordering of the reals (under the assumptions made above), or the natural numbers. These structures are potentially infinite, not actually infinite. How do the considerations above support this position? Because if something is actually infinite, that is, if it is fixed, determinate, constant, and present all at once, then random selections from it would not get larger through time. If something is actually infinite, then selections do not grow through time. So by the contrapositive, if selections grow through time, then we are dealing with a potential infinity. Because selections grow through time, a well-ordering of the reals is not an actual infinity. Similarly the natural numbers are not an actual infinity. Any random selection, from either structure, is singularly and preposterously low, (almost) certain to be bested by subsequent selections. In this sense, random selections from a potential infinity are problematic, because any selection is biased towards the “front end” of the potential infinity. An interesting question that then arises is: Are there any examples of actual infinities? I believe that there are. An infinite integer in a nonstandard model of arithmetic is an example of the actual infinite. Note that random selections from an infinite integer do not progress through time, but rather “bounce around,” just as random selections from a finite integer do. And so I suggest that Cantor was wrong in claiming that ω is an actual infinity. Rather it is an example of the potential infinite, as Aristotle held. But Aristotle was wrong in suggesting that there is no actual infinite. There is, where infinite integers are an example. I also believe that such infinite integers are the correct extension of the concept of finite, natural number into the infinite, and so for example, when one says, “I performed infinitely many actions,” this should mean infinitely many in the sense of an infinite integer.⁸ At this point a difficult question arises. I have suggested

⁸ I believe that recognizing this correct conception of infinite number dissolves many paradoxes of the infinite, not only the puzzle presented in this paper. For example, consider Thomson’s Lamp (Thomson 1954). If a lamp button is pressed infinitely many times, and if “infinitely many” is an infinite integer in a nonstandard model of arithmetic, then there is no paradox. Any infinite integer is either even or odd. If the button was pressed an even number of times, then the lamp is in its starting state. If the button was pressed an odd number of times, then the lamp is in the opposite state from its starting state. But why, it may be asked, can’t we ask about button presses of the structure ω , that is, a super-task as presently conceived? Simply put, because ω is only potentially infinite—it is never complete and actual; it is impossible to complete a task of structure ω . See Gwiazda’s (2012a) “A Proof of the Impossibility of Completing Infinitely Many Tasks” for an argument to this conclusion.

that ω is a potential infinity. I have also suggested that an infinite integer is an actual infinity. And yet, an infinite integer is comprised of nothing but ω 's and ω^* 's. How can it be that many potential infinities are able to comprise an actual infinity? The answer to this question lies beyond the scope of this paper, but see (Gwiazda 2012b) for one possible answer to this question. We live in a Cantorian age, which is simply to say that most philosophers and mathematicians accept the positions of Cantor. The challenge, then, for those who reject my non-Cantorian position outlined above, is to find a way out of the puzzle.

We began by throwing darts at the reals and quickly ran into the following problem. It seems that darts must hit reals further into a fixed well-ordering as time passes. But the passage of time (the order of events, e.g., darts landing) need not be the same for all observers. As argued above, this leads to the conclusion that two observers find that $r_1 < r_2$ and $r_2 < r_1$. How is this puzzle best resolved?

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