A Nonmonotonic Sequent Calculus for Inferentialist Expressivists

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Abstract: I am presenting a sequent calculus that extends a nonmonotonic consequence relation over an atomic language to a logically complex language. The system is in line with two guiding philosophical ideas: (i) logical inferentialism and (ii) logical expressivism. The extension defined by the sequent rules is conservative. The conditional tracks the consequence relation and negation tracks incoherence. Besides the ordinary propositional connectives, the sequent calculus introduces a new kind of modal operator that marks implications that hold monotonically. Transitivity fails, but for good reasons. Intuitionism and classical logic can easily be recovered from the system.

Keywords: nonmonotonic logic, sequent calculus, logical inferentialism, logical expressivism, material consequence relation

1 Philosophical motivation

What follows is motivated by two big philosophical ideas: logical inferentialism and logical expressivism. Logical inferentialism is a view about the meaning of logical vocabulary. Very roughly, it says that the meaning of logical vocabulary is settled by its inferential role, i.e., by what implies and is implied by sentences in which such vocabulary occurs. Logical expressivism is a view about the expressive function of logical vocabulary, i.e., a view about what such vocabulary is for, what it allows us to do. Very roughly, the view is that logical vocabulary allows us to explicitly undertake commitments regarding inferential goodness and incoherence by asserting logically complex sentences, whereas without logical vocabulary we could undertake such commitments only implicitly by reasoning or arguing in certain ways. It is part of this idea that we can introduce logical vocabulary

[1]The work I am presenting here comes out of joint work with Robert Brandom and his research group on nonmonotonic logic. So my debt to Robert Brandom and the other members of the group can hardly be overestimated. I also want to thank the participants of the Logica 2015 conference for invaluable comments and discussion.

purely in terms of a material consequence relation and incoherence property over a language that does not include logical vocabulary. I shall present a logical system that exemplifies logical inferentialism and logical expressivism. The system introduces logical vocabulary in terms of its inferential role, and it does so on the basis of material consequence and incoherence. The perhaps biggest challenge for such a project is that material, nonlogical consequence and incoherence are virtually always nonmonotonic. Nonmonotonicity, however, is notoriously difficult to deal with in formal systems. In this section, I want to explain the basic ideas just mentioned.

1.1 Logical inferentialism

Let's begin with logical inferentialism. This is the view that the meaning of logical vocabulary is a matter of its inferential role (for a recent exposition and defense see Peregrin, 2014). Gentzen (1934, p. 189) formulated a version of the idea when he famously said that the introduction rules of a bit of logical vocabulary constitute, "as it were, the 'definitions' of the symbols concerned." The version of the idea that will be relevant here, however, is closer to Dummett's (1991, p. 247) view that the "meaning of [a] logical constant [. . .] can be completely determined by laying down the fundamental logical laws governing it" (see also Kneale, 1956, pp. 254–55). For our current purposes, we can think of logical inferentialism as the idea that the meaning of a bit of logical vocabulary is fully determined by the full set of implications or good arguments in which it occurs. Hence, we can introduce such vocabulary into a language by giving rules that determine the consequence relation over the logically extended language. Below I will provide such rules in the form of a sequent calculus.

Logical inferentialism has been criticized in various ways. Entering such debates here would take us too far afield. The only point that will matter for me is so-called "conservativeness." Prior (1960) famously pointed out that one can introduce connectives, like his "tonk," that trivialize a consequence relation by laying down introduction and elimination rules. Supposing that such connectives are meaningless, this can seem to undermine inferentialism because it shows that not all rules that specify an inferential role specify a meaning. In response to this worry, most inferentialists follow Belnap (1962) and say that the rules by which we introduce a new bit of logical vocabulary must extend the consequence relation we start with in a conservative manner. That is, an implication that does not contain the new bit of vocabulary holds in the extended consequence relation just in case it already held in the unextended consequence relation. I accept this as a restriction on the rules we can use to introduce logical vocabulary. Many more such restrictions have been proposed in the literature, such as various versions of harmony and separability. However, I will ignore such further restrictions here and shall be content with the safeguard that conservativeness provides against 'tonk-like' connectives.

1.2 Logical expressivism

I am taking the idea of logical expressivism from Robert Brandom (together with whom I have developed the ideas presented here). Brandom builds on Frege's idea that his "concept-script is a formal language for the explicit codification of conceptual contents" (Brandom, 2000, p. 58). If one is (with Brandom) an inferentialist across the board and not just regarding logical vocabulary, one believes that all (non-logical) conceptual contents are a matter of material consequence and incoherence. On this view, Frege's idea is that the concept-script is a formal language for the explicit codification of material consequence and incoherence. Hence, the expressive function of a formal language is to let us talk 'about' material implication relations and incoherence properties.[2] Brandom sometimes puts this view in a slogan by saying that logic "is the organ of semantic self-consciousness" (Brandom, 2009, p. 11).

Logical expressivism would need a lot of unpacking, but for our purposes, we can simplify the idea to the claim that, for any well-behaved language, logical vocabulary can be introduced solely in terms of the material consequence relation and incoherence property of the unextended language, and the so introduced logical vocabulary allows us to make explicit this consequence relation and incoherence property within the object language.

Definition 1 *Logical expressivism is the thesis that (i) logical vocabulary can be introduced into any language with a well-behaved material consequence relation and incoherence property solely in terms of this consequence relation and incoherence property, and (ii) the thus introduced vocabulary allows us to form sentences that make explicit facts about the underlying (and also the extended) consequence relation and incoherence property.*

[2]Notice that, given logical inferentialism, the "about" here must not (or not primarily) be understood in representationalist terms.

For us, the first point concerns the raw materials that we start with: a material consequence relation and incoherence property defined over a language without logical vocabulary.

The second point is more difficult to understand. It concerns what we want to built from the basic material: we want to build logical expressions that fulfill their expressive job of making explicit consequence and incoherence. Now, when can a bit of vocabulary count as "making explicit" the material consequence relation and incoherence property? This is easiest to grasp for the two logical expressions that I take to be paradigmatic: the conditional and negation. In the system I will present below, the conditional makes explicit—or tracks—consequence, and the negation makes explicit—or tracks—incoherence. For the conditional, this means that a conditional $A \to B$ is implied by a premise-set just in case B is implied by the result of adding A to this premise-set, i.e., a deduction theorem holds. For negation, it means that the negation $\neg A$ is implied by a premise-set just in case adding A to this premise-set results in something incoherent. So logical expressivism puts constraints on the conditional and negation that are acceptable for us.

1.3 Nonmonotonicity

Brandom and Aker have already provided a system in which logical vocabulary is introduced solely in terms of a material incoherence property over sets of atomic sentences (Brandom, 2008, pp. 141–175). One crucial limitation of this so-called "incompatibility semantics" is that it's consequence relation is monotonic, i.e., if a set of sentences implies, say, the sentence A then so do all its supersets (see lemma 2.1 on p. 143 of Brandom, 2008).

This is a limitation because, according to inferentialist expressivism regarding logic, material inferences are not just enthymematic formal inferences. If we take such inferences at face value, however, it is hard to see how their nonmonotonicity could be merely apparent. Moreover, paradigmatic material implications, such as implications in legal matters, medicine, or morality, are virtually always defeasible. And the same holds for material incoherence. Sets of commitments that don't fit together can become jointly acceptable once we add another commitment into the mix.

Is there perhaps another off-the-shelf logic that suits the inferentialist expressivists as an exemplification of her ideas? Unfortunately, it does not seem so. There are many nonmonotonic logics on offer today (for an introductory overview see Antonelli, 2008). But, as far as I know, none of them uses a material consequence relation and incoherence property as their starting point. In fact, many nonmonotonic logics treat some logic—often classical logic—and its vocabulary as given and freely available in the new logic. Moreover, most nonmonotonic logics obey some version of Cut. As we will see below, this means that they cannot have a conditional that is in line with logical expressivism, i.e., they cannot have a deduction theorem.

If logical inferentialism and logical expressivism are good ideas and we take the nonmonotonicity of material consequences seriously, there should be formal systems that exemplify these ideas in a paradigmatic way. Thus, we want a way of conservatively extending a nonmonotonic material consequence relation and incoherence properties such that the conditional and negation track consequence and incoherence, respectively.

2 The basic setup

As I explained in the previous section, our motivating philosophical ideas are, firstly, that the meaning of logical vocabulary is determined by its inferential role and, secondly, that logical vocabulary makes explicit features of an underlying, nonmonotonic, material consequence relation and incoherence property. So we must start with a material consequence relation and incoherence property over a language that does not contain logical vocabulary. Call this language $\mathscr{L}_{0-}$. We can think of $\mathscr{L}_{0-}$ as a set of atomic sentences, $\{p_1, \ldots, p_n\}$. Some subsets of $\mathscr{L}_{0-}$ materially imply some sentences in $\mathscr{L}_{0-}$. And some subsets of $\mathscr{L}_{0-}$ are materially incoherent. So the structures that we start with are triples of (a) an atomic language, (b) a (single conclusion) consequence relation over it, and (c) an incoherence property defined over sets of atoms.

In order to express incoherence and consequence in a unified way, we introduce the constant "$\bot$." Let $\mathscr{L}_0 = \mathscr{L}_{0-} \cup \{\bot\}$. Let $\mid\!\sim_0$ be the relation over $\mathscr{L}_0$ such that $\{p_k, \ldots, p_l\} \mid\!\sim_0 p_i$ iff $\{p_k, \ldots, p_l\}$ materially implies p_i and, moreover, such that $\Gamma \mid\!\sim_0 \bot$ iff Γ is materially incoherent.[3] The constant $\bot$ cannot occur on the left of the snake-turnstile and it cannot be embedded. We used it merely to encode the incoherence property into the "underlying consequence relation;" so $\mid\!\sim_0 \subseteq \mathcal{P}(\mathscr{L}_{0-}) \times \mathscr{L}_0$. We say that a consequence relation $\mid\!\sim_0$ is proper just in case (a) the whole atomic language is incoherent ($\mathscr{L}_{0-} \mid\!\sim_0 \bot$), (b) the empty set is coherent ($\emptyset \not\mid\!\sim_0 \bot$), (c) $\mid\!\sim_0$ is

[3] I use capital Greek letters for sets of sentences, lower case Latin letters for atomic sentences, and upper case Latin letters for arbitrary sentences. I will omit the set-brackets on the left of the snake-turnstile if no confusion can arise.

reflexive ($\forall \Gamma \subseteq \mathscr{L}_0 \quad (p \in \Gamma \Rightarrow \Gamma \mid\!\sim_0 p)$), and (d) $\mid\!\sim_0$ obeys what we call "Ex Falso Fixo Quodlibet" (ExFF):

ExFF For any atom p, if $\forall \Delta \subseteq \mathscr{L}_0 \,.\, (\Gamma, \Delta \mid\!\sim_0 \bot)$, then $\Gamma \mid\!\sim_0 p$.

This principle is a variant of *ex falso quodlibet*, i.e., explosion. Notice that the difference between the traditional version of *ex falso* and ExFF only shows up in a nonmonotonic context. After all, $\Gamma \mid\!\sim_0 \bot$ guarantees $\forall \Delta \subseteq \mathscr{L}_{0-} (\Gamma, \Delta \mid\!\sim_0 \bot)$ if monotonicity holds for $\mid\!\sim_0$.

Let's sum up our starting point in two definitions.

Definition 2 ***Base Structure:*** *A base structure is a pair* $\langle \mathscr{L}_0, \mid\!\sim_0 \rangle$ *such that (i)* $\mathscr{L}_0$ *is a set of atomic sentences,* $\{p_1, \ldots, p_n\} = \mathscr{L}_{0-}$, *and the symbol* $\bot$, *and (ii)* $\mid\!\sim_0$ *is a material consequence relation that also encodes an incoherence property,* $\mid\!\sim_0 \subseteq \mathcal{P}(\mathscr{L}_{0-}) \times \mathscr{L}_0$.

Definition 3 ***Proper Base Structure:*** *A base structure is proper iff its underlying consequence relation,* $\mid\!\sim_0$, *is proper, i.e., if it satisfies the following conditions: (a)* $\mathscr{L}_{0-} \mid\!\sim_0 \bot$, *(b)* $\emptyset \not\mid\!\sim_0 \bot$, *(c)* $\mid\!\sim_0$ *is reflexive, and (d) ExFF.*

All base structures I will talk about are proper base structures. Our goal is to extend arbitrary proper base structures to structures with a language, $\mathscr{L}$, that contains logical vocabulary and a consequence relation, $\mid\!\sim$, over this extended language. Moreover, this extension should be such that, firstly, we introduce logical expressions by giving rules that determine their roles in the extended consequence relation. This is the logical inferentialism. And secondly, the so introduced logical vocabulary should make explicit features of the consequence relation into which it is introduced. That is the logical expressivism. As already intimated, for our purposes, the second point amounts to two desiderata: the extended consequence relation should satisfy, the Deduction Theorem (DT) and what I shall call the "Negation Theorem" (NT):

DT $\Gamma \mid\!\sim A \rightarrow B \iff \Gamma, A \mid\!\sim B.$

NT $\Gamma \mid\!\sim \neg A \iff \Gamma, A \mid\!\sim \bot.$

If DT holds, the conditional is tracking the consequence relation. Such a conditional allows us to not only practically acknowledge that B follows from A in the context of Γ by inferring B from A in the context of Γ, but to assert something on the basis of Γ that commits us to B following from A, in the context of Γ. Similarly, if NT holds, the negation is tracking the incoherence property. Such a negation allows us to assert something on the basis of Γ, that commits us to Γ being incompatible with A (i.e., they being jointly incoherent). That is the sense in which such a conditional and negation make explicit the consequence relation and incoherence property of the language in which they occur.

There are two further desiderata for the extension of the underlying consequence relation. First, as explained above, we want the extension to be conservative, i.e., if $\Gamma \subseteq \mathscr{L}_{0-}$ and $A \in \mathscr{L}_0$, then $\Gamma \mid\!\sim A$ iff $\Gamma \mid\!\sim_0 A$. Second, we want the extension to preserve reflexivity, i.e., if the base consequence relation is reflexive, the extended one must be so, too.

Let's sum up the goal that I shall pursue in the reminder of this paper:

GOAL We want to find a way to extend any proper base structure in such a way that the extension, $\langle \mathscr{L}, \mid\!\sim \rangle$, is conservative, preserves reflexivity, and obeys DT and NT.

Notice that the conservativeness of the extension means that the extension must not force monotonicity. After all, a nonmonotonic consequence relation cannot be extended conservatively to a monotonic consequence relation.

As a bonus, I will also introduce a new modal operator, $\Box$. This operator marks consequences that hold monotonically. In order to see what this means, notice that even in a consequence relation where monotonicity fails as a global property, there can be sets of sentences, Γ, such that Γ and every superset of it imply a certain sentence A, i.e., $\forall \Delta \subseteq \mathscr{L}_{-} (\Delta, \Gamma \mid\!\sim A)$. Thus, the implication $\Gamma \mid\!\sim A$ behaves monotonically. I will introduce an operator that tracks this property of implications in the object language. More precisely, the operator will obey the following principle.

BOX $\Gamma \mid\!\sim \Box A$ iff $\forall \Delta \subseteq \mathscr{L}_{-} (\Delta, \Gamma \mid\!\sim A)$.

I will sometimes call this operator the "monotonicity-box." Having such an operator is not necessary for a logical system that exemplifies the ideas of logical inferentialism and logical expressivism in a nonmonotonic setting. In so far as regions where monotonicity holds locally are of interest, however, having such an operator is desirable.

3 The construction

In the previous section, I explained that we want to extend a material consequence relation to a language, $\mathscr{L}$, with logical vocabulary. The extended language I shall use includes negation, a conditional, conjunction, disjunction, and the new kind of modal operator mentioned in the previous section. The syntax of the language without $\bot$ is straightforward.

Syntax of $\mathscr{L}_-$: $\varphi ::= p \mid \neg\varphi \mid \varphi \to \varphi \mid \varphi \& \varphi \mid \varphi \vee \varphi \mid \Box\varphi$

And p is an atomic sentence of $\mathscr{L}_-$ iff $p \in \mathscr{L}_{0-}$. We now define the extended language as $\mathscr{L} = \mathscr{L}_- \cup \{\bot\}$.

Extending the consequence relation to $\mid\!\sim\, \subseteq \mathcal{P}(\mathscr{L}_-) \times \mathscr{L}$ is more tricky. We do this by way of a sequent calculus in which the material implications serve as axioms. I call it the Non-Monotonic Modal sequent calculus (NMM).

We start with the straightforward idea that whatever is in the underlying consequence relation, $\mid\!\sim_0$, is an axiom of the sequent calculus. However, there is a complication that has to do with our monotonicity-box. Our sequent calculus does not only have one kind of turnstile but $|\mathcal{P}(\mathcal{P}(\mathscr{L}_0))|$ many turnstiles. The idea is that, for every subset X of $\mathcal{P}(\mathscr{L}_0)$, we want to have $\Gamma \mid\!\sim^{\uparrow X} A$ just in case $\forall \Delta \in X\, (\Delta, \Gamma \mid\!\sim A)$. We stipulate this for the axioms of our sequent calculus and, hence, get axioms with different kinds of turnstiles.

Here is how we construct the extended consequence relation, $\mid\!\sim$. First, we have two clauses that provide us with axioms of our sequent calculus.

Axioms of NMM:
Ax1: If $\Gamma \mid\!\sim_0 A$, then $\Gamma \mid\!\sim A$ is an axiom.
Ax2: If $X \subseteq \mathcal{P}(\mathscr{L}_0)$ and $\forall \Delta \in X\, (\Delta, \Gamma \mid\!\sim_0 A)$, then $\Gamma \mid\!\sim^{\uparrow X} A$ is an axiom.
Convention: If $X = \mathcal{P}(\mathscr{L}_0)$, we can write $\Gamma \mid\!\sim^{\uparrow X} A$ as $\Gamma \mid\!\sim^{\uparrow} A$.

A sequent is in $\mid\!\sim$ just in case it can be derived from these axioms in a proof-tree using only the following sequent rules:

Rules of NMM:
Note on the notation: What is on the left of the turnstile are sets of formulae. The comma is to be read as set-union with flanking individual formulae being read as in set brackets; e.g., "Γ, A" on the left means "$\Gamma \cup \{A\}$". Upward arrows and formulae in square brackets are optional. That is, both, the sequent with and the sequent without the bracketed upward arrow, are derivable via the rule. Some rules are presented as involving ordinary sequents,

i.e. $\mid\!\sim$-type sequents, but they also apply to quantified sequents, i.e. $\mid\!\sim^{\uparrow X}$-type sequents. That is, they should be read as systematically ambiguous in the following way. Ordinary sequents can be replaced by quanitified sequents in unified ways. I.e. the rule can be applied if all the $\mid\!\sim$-type turnstiles in the premises and the conclusion are replaced by $\mid\!\sim^{\uparrow X}$-type turnstiles with the same X in all of these premises and the conclusion.

$$\frac{\Gamma, A \mid\!\sim B}{\Gamma \mid\!\sim A \to B}\ \text{CP} \qquad \frac{\Gamma \mid\!\sim A \to B}{\Gamma, A \mid\!\sim B}\ \text{CCP}$$

$$\frac{\Gamma, A_1, \ldots, A_n \mid\!\sim^{\uparrow} B \qquad \Gamma, B \mid\!\sim^{\uparrow} A_1 \quad \ldots \quad \Gamma, B \mid\!\sim^{\uparrow} A_n \qquad \Gamma, C \mid\!\sim D}{\Gamma, A_1, \ldots, A_n, B \to C \mid\!\sim D}\ \text{LC}$$

$$\frac{\Gamma, A \mid\!\sim \bot}{\Gamma \mid\!\sim \neg A}\ \text{RN} \qquad \frac{\Gamma \mid\!\sim A}{\Gamma, \neg A \mid\!\sim \bot}\ \text{LN}$$

$$\frac{\Gamma, A, B \mid\!\sim C}{\Gamma, A\&B \mid\!\sim C}\ \text{L\&} \qquad \frac{\Gamma \mid\!\sim A \qquad \Gamma \mid\!\sim B}{\Gamma \mid\!\sim A\&B}\ \text{R\&}$$

$$\frac{\Gamma, A \mid\!\sim C \qquad \Gamma, B \mid\!\sim C}{\Gamma, A \vee B, [A], [B] \mid\!\sim C}\ \text{L}\vee \qquad \frac{\Gamma \mid\!\sim A}{\Gamma \mid\!\sim A \vee B}\ \text{R}\vee 1 \qquad \frac{\Gamma \mid\!\sim B}{\Gamma \mid\!\sim A \vee B}\ \text{R}\vee 2$$

$$\frac{\Gamma \mid\!\sim {}^{\uparrow}A}{\Gamma \mid\!\sim \Box A}\ \text{RB} \qquad \frac{\Gamma, A \mid\!\sim B}{\Gamma, \Box A \mid\!\sim B}\ \text{LB}$$

$$\frac{\Gamma \mid\!\sim {}^{\uparrow}A}{\Gamma, B \to C \mid\!\sim {}^{[\uparrow]}A}\ \text{CK} \qquad \frac{\Gamma \mid\!\sim {}^{\uparrow}A}{\Gamma, \neg B \mid\!\sim {}^{[\uparrow]}A}\ \text{NK} \qquad \frac{\Gamma \mid\!\sim {}^{\uparrow} \bot}{\Gamma \mid\!\sim {}^{[\uparrow]}A}\ \text{ExFF}$$

$$\frac{\Gamma \mid\!\sim {}^{\uparrow X}A \qquad \Gamma \mid\!\sim {}^{\uparrow Y}A}{\Gamma \mid\!\sim {}^{\uparrow X \cup Y}A}\ \text{UN} \qquad \frac{\Gamma, p_1 \ldots p_n \mid\!\sim A}{\Gamma \mid\!\sim {}^{\uparrow\{\{p_1 \ldots p_n\}\}}A}\ \text{PushUp}$$

These sequent rules define a consequence relation $\mid\!\sim \subseteq \mathcal{P}(\mathscr{L}_{-}) \times \mathscr{L}$. They also define many "quantified consequence relations" of the $\mid\!\sim^{\uparrow X}$ type.

The purpose of the latter ones is merely auxiliary. They allow us to introduce the monotonicity-box, to use ExFF as a rule, and to use rules like LC, CK and NK for our conditional and negation. In this way, quantified sequents influence the extension of $\mid\!\sim$ indirectly.

This construction gives us an extension of base structures: $\langle \mathscr{L}, \mid\!\sim \rangle$. We now have to show that this extension satisfies the requirements set out in GOAL and BOX above.

4 Properties of the extension

Given GOAL and BOX above, we want the extended consequence relation to have the following properties:

1. $\mid\!\sim$ is well defined.

2. $\mid\!\sim$ is reflexive—i.e. $\forall \Gamma \subseteq \mathscr{L}_{-}$ $(A \in \Gamma \Rightarrow \Gamma \mid\!\sim A)$—if $\mid\!\sim_0$ is reflexive.

3. $\mid\!\sim$ is a conservative extension of $\mid\!\sim_0$, i.e., for all $A \in \mathscr{L}_0$ and $\Gamma \subseteq \mathscr{L}_0$, $\Gamma \mid\!\sim_0 A$ iff $\Gamma \mid\!\sim A$.

4. DT holds, i.e., $\Gamma \mid\!\sim A \to B$ iff $\Gamma, A \mid\!\sim B$.

5. NT holds, i.e., $\Gamma \mid\!\sim \neg A$ iff $\Gamma, A \mid\!\sim \bot$.

6. BOX should hold, i.e., $\Gamma \mid\!\sim \Box A$ iff $\forall \Delta \subseteq \mathscr{L}_{-}$ $(\Gamma, \Delta \mid\!\sim A)$.

Two remarks are in order: first, I restrict all these claims to finite premise sets; and I will assume that the base language is finite. I will not worry about compactness. This is a restriction of the current approach that I hope can be lifted for future descendants of it. Second, due to limitations of space I can only sketch the proofs of these properties. And sometimes I will omit proofs entirely.[4]

Restricting ourselves to finite premise sets, the first of these claims can easily be seen to be true because we only add sequents to our consequence relation. Since we never explicitly require something to *not* be in the relation, we cannot contradict ourselves. If we can show that our extension is conservative, this will also show that our consequence relation is not trivial, i.e., that it does not hold between every premise set and every formula. So let's look that conservativeness and the preservation of reflexivity.

[4] Contact me for detailed versions of the proofs.

4.1 Preservation of reflexivity and conservativeness

In order to show that reflexivity is preserved and that the extension is conservative, we first need some lemmas.

Lemma 1 *If* $p_1, \ldots, p_n, \Gamma \mid\!\sim A$, *then* $\Gamma \mid\!\sim^{\uparrow\{\{p_1 \ldots p_n\}\}} A$.

Proof. Immediate from PushUp. □

Lemma 2 *If* $\forall\Delta \subseteq \mathscr{L}_- (\Delta, \Gamma \mid\!\sim A)$, *then* $\Gamma \mid\!\sim^{\uparrow} A$.

Proof. Suppose that $\forall\Delta\,(\Delta, \Gamma \mid\!\sim A)$. This implies $\forall\Delta \subseteq \mathscr{L}_{0-} (\Delta, \Gamma \mid\!\sim A)$. So, for every subset of our atoms, $\{p_1 \ldots p_m\}$, we have $p_1 \ldots p_m, \Gamma \mid\!\sim A$. So, by lemma 1, $\Gamma \mid\!\sim^{\uparrow\{\{p_1 \ldots p_m\}\}} A$. By $2^{|\mathscr{L}_0|}$ applications of UN, we get $\Gamma \mid\!\sim^{\uparrow} A$. □

Next, we need a lemma that says that if we can weaken a sequent with arbitrary sets of atoms, then we can weaken it with arbitrary sets of formulae.

Lemma 3 *If* $\forall\Delta \subseteq \mathscr{L}_0\ (\Delta, \Gamma \mid\!\sim A)$, *then* $\forall\Delta \subseteq \mathscr{L}_- (\Delta, \Gamma \mid\!\sim A)$.

Proof. By induction on the complexity of the most complex formulae in Δ, where complexity is the number of connectives in a formula. The base case is immediate. For the induction step, take an arbitrary set, Θ, with the maximally complex formulae in it being of complexity n+1. We divide Θ into the following sets: *N* is the set of formulae of complexity $\leq n$, *C* is the set of conditionals of complexity n+1, *NEG* is the set of negations of complexity n+1, *CON* is the set of conjunctions of complexity n+1, *D* is the set of disjunctions of complexity n+1, and *B* is the set of necessitations of complexity n+1. So, $\Theta = N \cup C \cup NEG \cup CON \cup D \cup B$. Looking at the proof of lemma 2 again, we know that the antecedent of our conditional gives us $\Gamma \mid\!\sim^{\uparrow} A$. So we can easily weaken with *N*, *C*, and *NEG*. We can also weaken with the embedded formulae of conjunctions, disjunctions and necessitations of complexity n. From this we can derive the conjunctions and necessitations via L& and LB. So the only potential difficulty is weakening with disjunctions of complexity n. In order to do this, we make a list of all the formulae that are the disjuncts of the k elements of *D*: $d_{1.1}$, $d_{1.2}$, $d_{2.1} \ldots d_{k-1.2}$, $d_{k.1}$, $d_{k.2}$, where the first index indicates the number of the disjunction from which the formula stems and the second index indicates whether it is the first or the second disjunct. We take the 2^k different subsets from this list for which: for each $1 \leq n < k$ exactly one of $d_{n.1}$ or $d_{n.2}$ is in the set and nothing else is in the set. Call these sets $\Xi_1 \ldots \Xi_{2^k}$. Let $\Pi = N \cup C \cup NEG \cup CON \cup B \cup \Gamma$. Thus, for each $1 \leq m \leq 2^k$, we get $\Xi_m, \Pi \mid\!\sim A$. We now construct our proof-tree in the following way:

$$\dfrac{\dfrac{\ldots \mid\!\sim \ldots}{d_{1.1} \vee d_{1.2} \ldots} \quad \dfrac{\ldots \mid\!\sim \ldots}{d_{k.1}, \Pi \mid\!\sim A}\,\text{L}\vee \qquad \dfrac{\ldots \mid\!\sim \ldots \qquad \ldots \mid\!\sim \ldots}{d_{1.1} \vee d_{1.2} \ldots d_{k.2}, \Pi \mid\!\sim A}\,\text{L}\vee}{d_{1.1} \vee d_{1.2} \ldots d_{k.1} \vee d_{k.2}, \Pi \mid\!\sim A}\,\text{L}\vee$$

Since Θ was arbitrary, we have $\forall \Delta\, (\Delta, \Gamma \mid\!\sim A)$ for Δs of complexity $n+1$. □

Proposition 1 *The extension preserves reflexivity.*

Proof. We assume that $\mid\!\sim_0$ is reflexive. First, we show, by induction on the complexity of α, that $\forall \Delta \subseteq \mathscr{L}_{0-}\, (\Delta, \alpha \mid\!\sim \alpha)$. And by lemma 3, this implies that $\forall \Delta \subseteq \mathscr{L}_{-}\, (\Delta, \alpha \mid\!\sim \alpha)$. □

We now know that the first two of the six points above hold. Before we turn to conservativeness, we need two more lemmas.

Lemma 4 *If* $\Gamma \mid\!\sim^{\uparrow X} A$, *then* $\forall \Delta \in X\, (\Delta, \Gamma \mid\!\sim A)$.

Proof. By induction on proof height, i.e., the number of rule-applications in the longest branch of the proof-tree. The proof is, for the most part, straightforward. I will leave some minor complications with LC, UN and PushUp as an exercise for the reader. □

Lemmas 3 and 4, when applied to the case where $X = \mathcal{P}(\mathscr{L}_0)$, imply the following:

Lemma 5 *If* $\Gamma \mid\!\sim^{\uparrow} A$, *then* $\forall \Delta \subseteq \mathscr{L}_{-}\, (\Delta, \Gamma \mid\!\sim A)$.

With these lemmas in hand, we can show that the extension is conservative for any underlying consequence relation that obeys ExFF.

Proposition 2 *The extension is a conservative extension of any non-monotonic material consequence relation that obeys ExFF; i.e., for all* $A \in \mathscr{L}_0$ *and* $\Gamma \subseteq \mathscr{L}_0$, $\Gamma \mid\!\sim_0 A$ *iff* $\Gamma \mid\!\sim A$.

Proof. The left-to-right direction is immediate from Ax1. So we only have to show that our construction does not add any sequent that can be formulated in the base language and is not already in $\mid\!\sim_0$. We argue by *reductio*, and we look at the (or a) shortest possible proof of a given violation of conservativeness (where length is the number of rule applications in a proof-tree). If NMM allows a violation of conservativeness, the last step is either an application of CCP or of ExFF. After all, the other rules have logical connectives in the conclusion-sequent; or else they apply only to quantified sequents. So we have two cases:

(Case 1) Assume that the violation, $\Gamma \mid\!\sim p$, comes by ExFF. The premise is $\Gamma \mid\!\sim^{\uparrow} \bot$. This must come by Ax2 or by UN. If it comes by Ax2, we have $\forall \Delta\, (\Delta, \Gamma \mid\!\sim_0 \bot)$. But $\mid\!\sim_0$ obeys ExFF by stipulation. So $\Gamma \mid\!\sim p$ cannot violate conservativeness. Hence, $\Gamma \mid\!\sim p$ must come by UN. The premises are $\Gamma \mid\!\sim^{\uparrow X} \bot$ and $\Gamma \mid\!\sim^{\uparrow Y} \bot$ and $X \cup Y = \mathcal{P}(\mathscr{L}_{0-})$. It can be shown by induction on proof height that if Γ contains only atoms and $\Gamma \mid\!\sim^{\uparrow X} \bot$, then $\forall \Delta \in X\, (\Delta, \Gamma \mid\!\sim_0 \bot)$. Thus, we get $\forall \Delta \in \mathcal{P}(\mathscr{L}_{0-})\, (\Delta, \Gamma \mid\!\sim_0 \bot)$. And by ExFF for the underlying consequence relation we have $\Gamma \mid\!\sim_0 p$.

(Case 2) Assume that the violation comes by CCP. The premise is $\Gamma \mid\!\sim A \to B$. This must come by CP or CCP or ExFF. If it comes by CP, the premise is $\Gamma, A \mid\!\sim B$. This violates our assumption that there is no shorter proof of $\Gamma, A \mid\!\sim B$. So, it must come by CCP or ExFF.

(Case 2.a) If $\Gamma \mid\!\sim A \to B$ comes by CCP, the premise is $\Gamma \setminus \{C\} \mid\!\sim C \to (A \to B)$. Since $\Gamma \setminus \{C\}$ contains only atoms, we are in the same situation again: either it must come by CP, CCP, or ExFF. If it comes by CP, we are back at $\Gamma \mid\!\sim A \to B$. If it comes by CCP, the premise is $\Gamma \setminus \{C, D\} \mid\!\sim D \to (C \to (A \to B))$. The same question arises again. If we continue like that, we are launched on an infinite regress of CCP applications. So at some point the conditional must come by ExFF. But if one of these conditionals comes by ExFF some subset, Θ, of Γ must be persistently incoherent, i.e., $\Theta \mid\!\sim^{\uparrow} \bot$. By lemma 5, $\forall \Delta \subseteq \mathscr{L}_{-}\, (\Delta, \Theta \mid\!\sim \bot)$. Since ExFF cannot conclude a violation of conservativeness (see Case 1) and everything in Θ is atomic, we have $\forall \Delta\, (\Delta, \Theta \mid\!\sim_0 \bot)$. Hence, $\forall \Delta\, (\Delta, \Gamma, A \mid\!\sim_0 \bot)$. But then ExFF for $\sim_0$ applies.

(Case 2.b) $\Gamma \mid\!\sim A \to B$ comes by ExFF. The same reasoning as in the previous subcase applies. □

We now know that our extension is well-defined, conservative and that it preserves reflexivity. So we can now turn to the last three properties listed at the beginning of this section.

4.2 Behavior of the conditional, negation, and box

We want the conditional to express the consequence relation, the negation to express incoherence, and the box to express monotonicity. What this comes to, for our purposes, is that DT, NT, and BOX hold.

It is immediate that DT holds. After all, CP gives us the right-to-left direction, and CCP gives us the left-to-right-direction. Parenthetically, it is worth pointing out the CCP is a simplifying rule. This would lead to problems if we wanted to prove Cut-elimination. As I will explain below, however, we don't want to do that.

Regarding negation, we want NT to hold, i.e., we want:

Proposition 3 $\Gamma, A \mathrel{|\!\sim} \bot \Leftrightarrow \Gamma \mathrel{|\!\sim} \neg A$.

Proof. The left-to-right direction is immediate because we have RN. So we must show that if $\Gamma \mathrel{|\!\sim} \neg A$, then $\Gamma, A \mathrel{|\!\sim} \bot$. We argue by induction on the height of a shortest proof of $\Gamma \mathrel{|\!\sim} \neg A$. Base case: $\Gamma \mathrel{|\!\sim} \neg A$ comes by the application of just one rule. It must come by RN or ExFF. In either case, we have $\Gamma, A \mathrel{|\!\sim} \bot$. Induction step: our hypothesis is that if $\Gamma \mathrel{|\!\sim} \neg A$ can be derived in a proof of height n, then $\Gamma, A \mathrel{|\!\sim} \bot$. For a proof of height n+1, the last rule applied can be: CCP, RN, L&, Lv, CK, NK, LC, LB, or ExFF. It is easy to see that in the cases of RN, ExFF, L&, Lv, CK, NK, and LB we get $\Gamma, A \mathrel{|\!\sim} \bot$ in one or two steps. For, either the premise itself is $\Gamma, A \mathrel{|\!\sim} \bot$, or we apply the hypothesis to the premise and derive $\Gamma, A \mathrel{|\!\sim} \bot$ with the same rule, or we get it by ExFF. So we are left with two cases.

(Case 1) the last rule applied is CCP. The premise is $\Gamma \setminus \{B\} \mathrel{|\!\sim} B \to \neg A$. If this comes by CP or ExFF, $\Gamma, A \mathrel{|\!\sim} \bot$ is immediate. If it comes by CCP, L&, Lv, CK, NK, LC, or LB, this also gives us what we want. As an example, suppose it comes by L&. The premise is $\Gamma \setminus \{B, C\&D\}, C, D \mathrel{|\!\sim} B \to \neg A$. We can argue thus:

$$\dfrac{\dfrac{\dfrac{\Gamma \setminus \{B, C\&D\}, C, D \mathrel{|\!\sim} B \to \neg A}{\Gamma \setminus \{C\&D\}, C, D \mathrel{|\!\sim} \neg A}\,\text{CCP}}{\Gamma \setminus \{C\&D\}, C, D, A \mathrel{|\!\sim} \bot}\,\text{Hyp}}{\Gamma, A \mathrel{|\!\sim} \bot}\,\text{L\&}$$

Lv, CK, NK, and LB work in an analogous way.

Next suppose $\Gamma \setminus \{B\} \mathrel{|\!\sim} B \to \neg A$ comes by LC. The right premise is $\Gamma \setminus \{C_1, \dots, C_n, D \to E\}, E \mathrel{|\!\sim} B \to \neg A$. So, by our hypothesis, $A, B, \Gamma \setminus \{C_1, \dots, C_n, D \to E\}, E \mathrel{|\!\sim} \bot$. The other premises are: $\Gamma \setminus \{D \to$

$E\}, E \mid\!\sim^{\uparrow} D$, and $\Gamma \setminus \{D \rightarrow E\}, D \mid\!\sim^{\uparrow} C_1$, and …, and $\Gamma \setminus \{D \rightarrow E\}, D \mid\!\sim^{\uparrow} C_n$. By lemma 5, the upward arrow implies that we can weaken with $\{A, B\}$. Hence, $A, B, \Gamma \setminus \{D \rightarrow E\}, E \mid\!\sim^{\uparrow} D$, and $A, B, \Gamma \setminus \{D \rightarrow E\}, D \mid\!\sim^{\uparrow} C_1$, and …, and $A, B, \Gamma \setminus \{D \rightarrow E\}, D \mid\!\sim^{\uparrow} C_n$. So by LC, $\Gamma, A \mid\!\sim \bot$.

Suppose $\Gamma \setminus \{B\} \mid\!\sim B \rightarrow \neg A$ comes by CCP. The premise is $\Gamma \setminus \{B, C\} \mid\!\sim C \rightarrow (B \rightarrow \neg A)$. The reasoning we just went through applies again. So CCP cannot conclude a sequent that contradicts our proposition.

(Case 2) the last rule applied is LC. We apply the same reasoning that we applied in the LC subcase of (Case 1). □

Finally, we must show that BOX holds. We divide BOX into two parts:

- $\Gamma \mid\!\sim \Box A$ iff $\Gamma \mid\!\sim^{\uparrow} A$.
- $\Gamma \mid\!\sim^{\uparrow} A$ iff $\forall\Delta\, (\Delta, \Gamma \mid\!\sim A)$.

Regarding the second part of BOX, notice that we have already proven both directions of this principle as lemmas 2 and 5. So we already know that:

Proposition 4 $\forall\Delta\, (\Delta, \Gamma \mid\!\sim A)$ *iff* $\Gamma \mid\!\sim^{\uparrow} A$.

Hence, it is just the first part of BOX that still needs to be proven. In order to do so, we again first need a lemma.

Lemma 6 *If* $\Gamma \mid\!\sim B_1 \rightarrow (B_2 \ldots \rightarrow (B_k \rightarrow \Box A))$, *then* $\Gamma, B_1 \ldots B_k \mid\!\sim^{\uparrow} A$.

Proof. By induction on proof height of $\Gamma \mid\!\sim B_1 \rightarrow (B_2 \ldots \rightarrow (B_k \rightarrow \Box A))$. The only tricky case is the induction step for LC. It goes as follows: The premises are $\Gamma \setminus \{D \rightarrow E\} \mid\!\sim^{\uparrow} D$, and $\Gamma \setminus \{C_1, \ldots, C_n, D \rightarrow E\}, D \mid\!\sim^{\uparrow} C_1, \ldots, \Gamma \setminus \{C_1, \ldots, C_n, D \rightarrow E\}, D \mid\!\sim^{\uparrow} C_n$, and $\Gamma \setminus \{C_1, \ldots, C_n, D \rightarrow E\}, E \mid\!\sim B_1 \rightarrow (B_2 \ldots \rightarrow (B_k \rightarrow \Box A))$. By our hypothesis, $\Gamma \setminus \{C_1, \ldots, C_n, D \rightarrow E\}, E, B_1 \ldots B_k \mid\!\sim^{\uparrow} A$. By a couple of CP application, this gives us $\Gamma \setminus \{C_1, \ldots, C_n, D \rightarrow E\}, E \mid\!\sim^{\uparrow} B_1 \rightarrow (B_2 \ldots \rightarrow (B_k \rightarrow A))$. Together with the other premises, LC allows us derive: $\Gamma \mid\!\sim^{\uparrow} B_1 \rightarrow (B_2 \ldots \rightarrow (B_k \rightarrow A))$. And by iterated CCP, $\Gamma, B_1 \ldots B_k \mid\!\sim^{\uparrow} A$. □

We can now prove the first part of our BOX-principle.

Proposition 5 $\Gamma \mid\!\sim \Box A$ *iff* $\Gamma \mid\!\sim^{\uparrow} A$.

Proof. First, the left-to-right direction. We argue by induction on proof height. Base case: The shortest proof-tree of such a sequent is RB or ExFF and both guarantee that $\Gamma \mid\!\sim^{\uparrow} A$. For the induction step, notice that the last step in a proof-tree for $\Gamma \mid\!\sim \Box A$ can be CCP, L&, L∨, CK, NK, LC, RB, LB, or ExFF. Lemma 6 gives us the induction step for CCP. The others are straightforward and I'll leave them as an exercise for the reader. The right-to-left direction is immediate because of RB. □

From propositions 5 and 4 the desired BOX follows immediately. Thus, we have shown that the extension defined by our sequent rules has all the six properties we want it to have. Hence, we have a sequent calculus that is in line with logical inferentialism and logical expressivism.

4.3 Why does cut fail?

Before I move on to the relation between NMM and intuitionism, I want to point out a feature of the system that might seem problematic: the consequence relation $\mid\!\sim$ is not transitive. That is, Cut is not only not provable but it actually fails. Monotonicity, transitivity, and reflexivity are often considered essential to anything being a consequence relation. Of course, we already abandoned that idea when we started to do nonmonotonic logic. But that we are also giving up transitivity might seem like a problem. I don't think it is a problem.[5] Rather, it is an insight that if you want to have a conditional that obeys a deduction theorem in a nonmonotonic setting, you need to give up transitivity.

To see this, take a mixed context version of Cut (Cut-MC):

$$\frac{\Gamma, A \mid\!\sim B \qquad \Delta \mid\!\sim A}{\Gamma, \Delta \mid\!\sim B}\text{ Cut-MC}$$

Proposition 6 *Cut-MC together with reflexivity implies that if* $\Gamma \mid\!\sim A$, *then* $\Gamma, \Delta \mid\!\sim A$.

Proof. We argue thus:

$$\frac{\Gamma, \Delta, A \mid\!\sim A \qquad \Gamma \mid\!\sim A}{\Gamma, \Delta \mid\!\sim A}\text{ Cut-MC}$$

The left premise is an instance of reflexivity and, hence, can be derived. □

[5] Dave Ripley has provided some independent motivation to be skeptical about Cut (Ripley, 2013, 2015; see also Schroeder-Heister, 2004).

So you cannot have a mixed context version of Cut in a nonmonotonic system with a reflexive consequence relation.

One might move to a shared context version of Cut (Cut-SC) to get rid of this problem.

$$\frac{\Gamma, A \mathrel{|\!\sim} B \qquad \Gamma \mathrel{|\!\sim} A}{\Gamma \mathrel{|\!\sim} B}\ \text{Cut-SC}$$

However, if we have a deduction theorem, we can run a similar argument for monotonicity with Cut-SC:

$$\frac{\dfrac{\dfrac{\Gamma, A, B \mathrel{|\!\sim} A}{\Gamma, A \mathrel{|\!\sim} B \to A}\ \text{CP} \qquad \Gamma \mathrel{|\!\sim} A}{\Gamma \mathrel{|\!\sim} B \to A}\ \text{Cut-SC}}{\Gamma, B \mathrel{|\!\sim} A}\ \text{CCP}$$

Hence, if you want a nonmonotonic, reflexive consequence relation with a conditional that obeys a deduction theorem, you need to give up Cut—even the shared context version. Of course, you can reason by modus tollens at this point (see Morgan, 2000). I think, however, that given the plausibility of nonmonotonicity and reflexivity and the logical expressivist motivation for DT, there is good reason to at least investigate systems in which Cut fails along with monotonicity.

There may be particularly well behaved regions of logical space in which transitivity holds. And in the fullness of time, we hope to study such regions systematically and perhaps even to introduce an object language operator that lets us mark such regions. Here I just want to point out that the failure of Cut is not an unmotivated quirk of the NMM system. It is entailed by the properties that I require the system to have.

5 Relation of NMM to intuitionistic and classical logic

I want to briefly describe the surprisingly straightforward relation between NMM and intuitionistic and classical logic. Due to limitations of space, I will omit the proofs of the results I am presenting.

I have already pointed out that Cut-SC fails in NMM. However, if we add Cut-SC to our sequent rules, the NMM rules are equivalent to Gentzen's sequent rules for intuitionistic logic, LJ, modulo the rules governing the box (which is pointless in a monotonic system). Call the system that results from adding Cut-SC to the NMM rules the "Cut-System." Moreover, read a sequent with $\bot$ on the right in the Cut-System as meaning the same as a sequent with an empty right side in Gentzen's LJ. Translate all other sequents in the obvious way. It can be shown that, under this translation, the following holds:

Proposition 7 *Translations of all rules of Gentzen's LJ system can be derived in the Cut-System, and translations of all rules of the Cut-System that don't use the box or sequents quantifying over less than $\mathcal{P}(\mathscr{L}_0)$ can be derived in Gentzen's LJ.*

In effect, the Cut-System without the apparatus governing the box is equivalent to Gentzen's LJ. This does not only hold at the level of sequent rules, but also at the level of theorems. All the theorems of intuitionistic logic are theorems of the Cut-System, i.e., they are implied by the empty set. Given these facts, it is easy to see that the following holds:

Proposition 8 *If the underlying material consequence relation contains all and only instances of reflexivity and we ignore the box (by deleting RB and LB), the (non-quantified) consequence relation of the Cut-System coincides with the intuitionistic consequence relation.*

Since the Cut-System gives us intuitionism, it is clear that adding double negation elimiation to the Cut-System will give us classical logic. It is easy to add sequent rules that give us double negation elimination. Hence, classical logic can be recovered by adding Cut and such further sequent rules to NMM. In this sense, the system I have presented can be viewed as a "mother-logic" that gives rise to intuitionism or classical logic under special circumstances.

6 Conclusion

I have presented a way of extending a nonmonotonic material consequence relation over an atomic base language to a consequence relation over a logically complex language. The extension is conservative; it preserves reflexivity; the conditional tracks the consequence relation; the negation tracks the incoherence property; and a new kind of modal operator tracks local regions of monotonicity. Thus, I have presented a logical system that does justice to the philosophical ideas of logical inferentialism and logical expressivism.

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