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Faithfulness for Naive Validity*

Ulf Hlobil

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Abstract Nontransitive responses to the validity Curry paradox face a dilemma that was recently formulated by Barrio, Rosenblatt and Tajer. It seems that, in the nontransitive logic ST enriched with a validity predicate, either you cannot prove that all derivable metarules preserve validity, or you can prove that instances of Cut that are not admissible in the logic preserve validity. I respond on behalf of the nontransitive approach. The paper argues, first, that we should reject the detachment principle for naive validity. Secondly, I show how to add a validity predicate to ST while avoiding the dilemma.

Keywords naive validity · nontransitive logic · v-Curry paradox · substructural approaches to paradox

1 Introduction

Adding a validity predicate to a language that allows for self-reference can easily lead to triviality via the validity-Curry or v-Curry Paradox (Beall and Murzi, 2013). David Ripley and others have suggested the nontransitive logic ST as a solution to the v-Curry and other semantic paradoxes (Ripley, 2013, 2012; Cobreros et al., 2013, 2012).[1] Recently, this nontransitive approach has

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U. Hlobil
Department of Philosophy, Concordia University, 2145 MacKay, Montreal, Quebec, Canada.
Affiliated faculty at the University of Hradec Králové, Rokitanského 62, 500 03 Hradec Králové, Czechia.
Tel.: +1-514-848-2424
E-mail: ulf.hlobil@concordia.ca

[1] See Fig. 1 below for a formulation of ST. The logic is called "ST" because its semantics can be described as "strict-tolerant" in a three-valued setting. It says that an argument is

been confronted with a dilemma (Barrio et al., 2017; Rosenblatt, 2017).[2] Either ST enriched with a validity predicate (henceforth STV) doesn't prove of all instances of its derivable metarules that they preserve validity, or STV proves of instances of Cut that are not admissible in STV that they preserve validity. Either way, the validity predicate of STV doesn't seem to express validity-in-STV, i.e. the consequence relation of STV.

In this paper, I respond to the dilemma on behalf of the nontransitive approach. I argue that we should reject the idea that we can detach from validity statements. And I show how to add a validity predicate to ST while avoiding the dilemma.

The paper is structured as follows: In Section 1, I will provide some context and explain the dilemma for the nontransitive approach. The dilemma rests on the requirement that the validity predicate must capture all derivable metarules. I call that requirement "faithfulness." In Section 2, I argue that if advocates of the nontransitive approach accept faithfulness, they should reject the idea that we can detach from validity statements; i.e., they should say that sometimes B doesn't follow from A and *The argument from A to B is valid.* In Section 3, I show how we can add a faithful validity predicate to ST. And I end, in Section 4, by discussing an objection.

2 Naive Validity, v-Curry, and Faithfulness

Let me begin by providing some background about naive validity and the v-Curry paradox. I will then explain the dilemma raised by Barrio, Rosenblatt and Tajer.

2.1 Naïveté About Validity

One goal of solving the semantic paradoxes is to formulate a correct semantic theory of a language in that language itself and not in an essentially richer meta-language. If *truth* is the basic semantic notion, this probably requires a naive truth predicate, i.e., a predicate T such that $T\langle A\rangle \Leftrightarrow A$, where $\langle A\rangle$ is a name for the sentence A. If, however, semantic inferentialism is correct and *validity* (appropriately interpreted) is the basic semantic notion, we need a validity predicate that applies to arguments in which it occurs. After all, we want to be able to say in our language which arguments in our language are valid. Let's call a validity predicate that allows us to do that "naive."

valid iff, in every model in which all premises are strictly true (have truth value 1), at least one conclusion is at least tolerantly true (has truth value $\frac{1}{2}$ or 1).

[2] Advocates of other approaches to the semantic paradoxes have voiced various worries about the nontransitive approach (e.g. Shapiro, 2013; Zardini, 2013). Others have argued that the v-Curry isn't as troubling as it may seem (Field, 2017; Cook, 2014; Ketland, 2012). I will not discuss any of these issues here.

To set the stage, suppose our object language contains canonical names of multisets of sentences, and $\langle \Gamma \rangle$ is the canonical name of the multiset Γ.[3] We want to introduce a two-place validity predicate, *Val*, such that $Val(\langle \Gamma \rangle, \langle \Delta \rangle)$ expresses that the argument from premises Γ to conclusions Δ is valid.

It seems to many that naive validity predicates should obey so-called validity detachment (VD) and validity proof (VP) (Beall and Murzi, 2013):

$$\frac{}{\Gamma, Val(\langle \Gamma \rangle, \langle \Delta \rangle) \vdash \Delta}\ \text{VD} \qquad \frac{\Gamma \vdash \Delta}{\vdash Val(\langle \Gamma \rangle, \langle \Delta \rangle)}\ \text{VP}$$

Here $\vdash$ denotes the kind of validity that our favorite inferentialism takes to be the basic semantic notion, including the theory of validity. Hence, $\vdash$ won't be *logical* validity.[4] The goal is that our validity predicate captures the relation expressed by $\vdash$. We can think of $\vdash$ as standing for the implicit inferential rules of our discursive practice and of *Val* as our way to talk about these rules. The inferentialist talks about these rules in order to explain what expressions mean. And in order to have a semantic theory that applies to itself, she must be able to give an account of what we do when we talk about these rules.

VP is plausible because it says that *Val* captures all arguments that are actually valid (in the broad sense of $\vdash$). That is, if the rules of our discursive practice underwrite an argument, then they also underwrite the claim that the argument is valid. VD says that we can detach from validity. That is, if we assume that an argument is valid and that the premises hold, it follows that at least one of the conclusions holds. That seems *prima facie* plausible.[5] I will argue below, however, that we should reject VD.

2.2 Validity-Curry

As is well known, VP and VD easily lead to triviality, via the v-Curry paradox (Beall and Murzi, 2013). The v-Curry paradox can arise in languages in which we can formulate so-called v-Curry sentences, π, that are intersubstitutable, without loss of validity, with sentences of the form $Val(\langle \pi \rangle, \langle A \rangle)$. So π is a sentence like "The sentence *'A'* follows from this sentence." From now on, I will simply assume that our object language allows us to formulate such sentences.[6]

[3] If Γ is a singleton, like $\{A\}$, I will write $\langle A \rangle$ instead of $\langle \{A\} \rangle$ to avoid clutter.

[4] Some have pointed out that the v-Curry paradox doesn't arise for *logical* validity (see Cook, 2014; Ketland, 2012). For a semantic inferentialist, this is cold comfort because logical validity can at best explain the meanings of logical constants.

[5] Here is some intuition pumping in favor of VD: If $\Gamma \vdash \Delta$, then $\Gamma, Val(\langle \Gamma \rangle, \langle \Delta \rangle) \vdash \Delta$ follows by weakening. If $\Gamma \nvdash \Delta$, then either $Val(\langle \Gamma \rangle, \langle \Delta \rangle)$ is incoherent and we get $\Gamma, Val(\langle \Gamma \rangle, \langle \Delta \rangle) \vdash \Delta$ by explosion or $Val(\langle \Gamma \rangle, \langle \Delta \rangle)$ holds only in worlds (models, contexts, or what have you) in which either one element of Γ is false or one element of Δ is true.

[6] I will follow Ripley and deal with this issue by fiat. Our meta-language terms "π" and "$Val(\langle \pi \rangle, \langle A \rangle)$" pick out the same formula in the object language, i.e., $\pi = Val(\langle \pi \rangle, \langle A \rangle)$. I will assume that "$\vdash$" is extensional and, hence, that π and $Val(\langle \pi \rangle, \langle A \rangle)$ are intersubstitutable on both sides of the turnstile. For a discussion of genuine self-reference in formal

Let κ be a v-Curry sentence that is intersubstitutable with $Val(\langle\kappa\rangle\ ,\langle\bot\rangle)$,[7] and call such a substitution "κ-Def." Moreover, let's suppose that VD, VP, Contraction and Cut hold. We can now argue as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\kappa, Val(\langle\kappa\rangle, \langle\bot\rangle) \vdash \bot}\ \text{VD}}{\kappa, \kappa \vdash \bot}\ \kappa\text{-Def}}{\kappa \vdash \bot}\ \text{Contraction}}{\vdash Val(\langle\kappa\rangle, \langle\bot\rangle)}\ \text{VP}}{\vdash \kappa}\ \kappa\text{-Def} \qquad \dfrac{\dfrac{\dfrac{}{\kappa, Val(\langle\kappa\rangle, \langle\bot\rangle) \vdash \bot}\ \text{VD}}{\kappa, \kappa \vdash \bot}\ \kappa\text{-Def}}{\kappa \vdash \bot}\ \text{Contraction}}{\vdash \bot}\ \text{Cut}$$

Since this proof goes through with $\bot$ replaced by any sentence we like, we have reached triviality, even without assuming Weakening. Something has to give. But what?

The idea of nontransitive responses to the v-Curry Paradox, like Ripley's (2013), is that we should reject the application of Cut. According to this view, $\kappa \vdash \bot$ and $\vdash \kappa$ both hold, but $\vdash \bot$ doesn't follow from them.

Ripley makes sense of this idea in light of his bilateralist interpretation of the turnstile. According to bilateralism, $\Gamma \vdash \Delta$ means, roughly, that it is incoherent to assert everything in Γ and deny everything in Δ. So, $\kappa \vdash \bot$ means that we cannot coherently assert κ, and $\vdash \kappa$ means that we cannot coherently deny κ (Ripley, 2013, p. 154). Rejecting Cut amounts to the claim that our position, i.e. a collection of assertions and denials, may be coherent even if there is something that we can neither coherently assert nor coherently deny. Paradoxical sentences like κ are best simply left alone.

I want to defend the nontransitive approach against criticism. As will become clear in due course, however, the situation is complicated. While I like the nontransitive approach in general, I will argue that the response to the v-Curry just rehearsed must be modified. It is not enough to give up Cut; we should also give up VD.[8] In order to see this, we must look at a recent criticism of the nontransitive approach by Barrio, Rosenblatt and Tajer.

2.3 Internalization and Faithfulness

Barrio, Rosenblatt and Tajer (2017) and Rosenblatt (2017) have argued that a predicate can obey VP and VD while failing to capture important aspects

languages see (Heck, 2007). We could also add Peano Arithmetic (PA) (see Cobreros et al., 2013) or Robinson arithmetic (by using a technique from Negri and von Plato, 1998) to get the diagonal lemma. It is not obvious, however, that in a nontransitive setting, the diagonal lemma suffices to guarantee the intersubstitutability of π and $Val(\langle\pi\rangle\ ,\langle A\rangle)$. I am ignoring such issues here.

[7] I assume that we have $\bot$ and $\top$ in the object language. We could also use the name of the empty multiset to play the roles of $\langle\bot\rangle$ and $\langle\top\rangle$.

[8] As an anonymous referee points out, one might worry that this means that the nontransitive approach's solutions to paradoxes aren't as uniform as sometimes advertised. I will discuss some worries regarding the rejection of VD in the next section.

of validity. In particular, they point out that, with a validity predicate, we can not only say that particular arguments are valid, but we can also infer that an argument is valid, $Val(\langle\Gamma\rangle, \langle\Delta\rangle)$, from a sentence saying that another argument is valid, $Val(\langle\Theta\rangle, \langle\Xi\rangle)$. Such an inference may be underwritten by our logic or not, i.e., either $Val(\langle\Theta\rangle, \langle\Xi\rangle) \vdash Val(\langle\Gamma\rangle, \langle\Delta\rangle)$ or not. Hence, a logic with a validity predicate, as it were, "has an opinion" about instances of its own metarules.

Barrio, Rosenblatt and Tajer suggest that a naive theory of validity should not only show of all and only the valid arguments that they are valid, it should also show of all and only the right metarules that they preserve validity. They express this idea by saying that our logic should *internalize* the right metarules.

What are the "right" metarules? Two candidates immediately suggest themselves: all admissible metarules or all derivable metarules. As I argue elsewhere (Hlobil, ms), it is not plausible to require that a logic must internalize all its admissible metarules.[9] So I will assume that a validity predicate should internalize all and only the derivable metarules.[10] Let's be clear about what "derivable metarule" means.

Definition 1 (Derivable metarule) A metarule $\mathcal{R}$ of the form

$$\frac{\Gamma_1 \vdash \Delta_1 \qquad ... \qquad \Gamma_n \vdash \Delta_n}{\Theta \vdash \Lambda}\ \mathcal{R}$$

is derivable in a sequent calculus SC just in case, for every instance of the metarule, there is a proof-tree of applications of SC's rules with some of the instances of $\Gamma_1 \vdash \Delta_1$, ..., $\Gamma_n \vdash \Delta_n$ and, optionally, some axioms of SC as its leaves and the instance of $\Theta \vdash \Lambda$ as its root.

Rosenblatt (2017) defines internalization of metarules as follows:

[9] Internalizing all admissible metarules would force us to go non-classical in our meta-theory. To see this, suppose we had an object language predicate Adm such that $\vdash Adm(\langle\Theta \succ \Xi\rangle, \langle\Gamma \succ \Delta\rangle)$—where $\langle\Gamma \succ \Delta\rangle$ is an object language name of the sequent $\Gamma \vdash \Delta$—just in case if $\Theta \vdash \Xi$, then $\Gamma \vdash \Delta$, i.e., just in case the move from $\Theta \vdash \Xi$ to $\Gamma \vdash \Delta$ is admissible. Call this biconditional ADM. Now, let $\kappa = Adm(\langle\succ\kappa\rangle, \langle\succ A\rangle)$, and call this identity CU. That trivializes our consequence relation if our meta-theory is classical. To see this, suppose that $\vdash \kappa$. By CU, $\vdash Adm(\langle\succ\kappa\rangle, \langle\succ A\rangle)$. By ADM, if $\vdash \kappa$, then $\vdash A$. By modus ponens, in the metalanguage, $\vdash A$. Discharging our assumption by conditional proof in the metalanguage, if $\vdash \kappa$, then $\vdash A$. By ADM, $\vdash Adm(\langle\succ\kappa\rangle, \langle\succ A\rangle)$. By CU, $\vdash \kappa$. By modus ponens in the metalanguage, $\vdash A$. Since A was arbitrary, that means that everything is provable. This is a variant of what Wansing and Priest (2015) call an "external Curry." All we need to trivialize our consequence relation is a classical metalanguage, the internalization of all and only the admissible metarules (given by ADM), and self-reference (given by CU). Supposing that rejecting CU is not an option, this means that internalizing admissible metarules requires going non-classical in the meta-theory. While I am in principle open to this possibility, discussing it would lead us too far afield.

[10] In order to be as ambitious as I can, I will aim to make the derivability relation pretty strong. That is why I used double-line rules in my formulation of ST below. These double-line rules make many metarules derivable that would be merely admissible if we used single-line rules. For example, the rule that allows us to move from $\Gamma, \neg A \vdash \Delta$ to $\Gamma \vdash A, \Delta$ is derivable with the double-line rules. With single-line rules, it would be admissible but not derivable in ST.

Definition 2 (Internalization) A logic internalizes a metarule $\mathcal{R}$, of the form given above, just in case it proves every instance of
$Val(\langle \Gamma_1 \rangle , \langle \Delta_1 \rangle), ..., Val(\langle \Gamma_n \rangle , \langle \Delta_n \rangle) \vdash Val(\langle \Theta \rangle , \langle \Lambda \rangle)$.[11]

With this notion of internalization in hand, we can turn to the dilemma formulated by Barrio, Rosenblatt and Tajer. If we use the above stated rules VP and VD as our rules for the validity predicate, STV doesn't internalize all of STV's derivable metarules. Such a logic doesn't internalize, e.g., the left-rule for negation. After all, we cannot derive $Val(\langle \Gamma \rangle , \langle \Delta \cup \{A\} \rangle) \vdash Val(\langle \Gamma \cup \{\neg A\} \rangle , \langle \Delta \rangle)$. That would have to come by VP or VD, but neither rule can conclude that sequent (see Barrio et al., 2017).

If we choose stronger rules for the validity predicate (e.g. by allowing left contexts consisting of validity statements in VP), the resulting logic will internalize instances of Cut that do not hold in STV. With such strong rules, we can prove sequents like $Val(\langle \top \rangle , \langle \lambda \rangle), Val(\langle \lambda \rangle , \langle \bot \rangle) \vdash Val(\langle \top \rangle , \langle \bot \rangle)$, where λ is the Liar sentence (see Barrio et al., 2017). It is characteristic of ST, however, that $\top \vdash \lambda$ and $\lambda \vdash \bot$ both hold but $\top \vdash \bot$ fails. So either STV doesn't capture all of its derivable metarules, or it proves a sequent that says something false about its own metarules.

We can formulate Barrio, Rosenblatt and Tajer's requirement by saying that our validity predicate ought to be faithful, where by "faithful" we mean the following:

Definition 3 (Faithfulness) A validity predicate, *Val*, is faithful just in case $Val(\langle \Gamma_1 \rangle , \langle \Delta_1 \rangle), ..., Val(\langle \Gamma_n \rangle , \langle \Delta_n \rangle) \vdash Val(\langle \Theta \rangle , \langle \Lambda \rangle)$ is provable iff $\Theta \vdash \Lambda$ follows from $\Gamma_1 \vdash \Delta_1$,...,$\Gamma_n \vdash \Delta_n$ via a derivable metarule.

Faithfulness implies that the validity predicate obeys VP and its converse, i.e., $\Theta \vdash \Lambda$ iff $\vdash Val(\langle \Theta \rangle , \langle \Lambda \rangle)$. It is independent, however, of VD.

Faithfulness is a plausible requirement. After all, if we want to formulate our logic within that logic itself, it should probably be able to talk about derivable metarules. And we want to be able to infer a sequent from other sequents, in the object language (where they appear as validity statements), just in case a corresponding metarule is derivable. It seems to me that we should avoid the view that it is impossible to have such a validity predicate, if we can at all do so. Hence, I suggest that we accept the faithfulness requirement Barrio, Rosenblatt and Tajer are advocating.[12] From now on, I will assume that naive validity predicates must be faithful. I want to investigate what follows from that for the nontransitive approach.

[11] I will call instances of that sequent schema the internalization of the corresponding instance of the metarule.

[12] Ripley has responded to Barrio, Rosenblatt and Tajer in talks and in an unpublished manuscript (Ripley, 2017). Barrio, Rosenblatt and Tajer argue that if we use rules for validity that are strong enough to internalize all derivable metarules, we can also derive: $Val(\langle \top \rangle , \langle \lambda \rangle), Val(\langle \lambda \rangle , \langle \bot \rangle) \vdash Val(\langle \top \rangle , \langle \bot \rangle)$. According to Ripley's bilateralist interpretation of the turnstile, this sequent tells us that it is incoherent to assert $\top \vdash \lambda$ and $\lambda \vdash \bot$ while also denying that $\top \vdash \bot$. But it can seem that this is precisely what Ripley actually does by endorsing ST. He seems to assert that we cannot assert the Liar and that we cannot deny the Liar, but he also seems to deny that all positions are incoherent.

3 v-Curry and Detachment

Ripley (2013) claims that the nontransitive approach solves the v-Curry paradox. In this section, I argue that giving up Cut isn't enough to solve the v-Curry if we want to have a faithful validity predicate. Some step before the application of Cut in the proof above must be rejected.[13] I argue that we should reject VD.

3.1 Rejecting VD

I will presuppose that rejecting Contraction isn't an option. As Ripley (2015) has argued, giving up Contraction doesn't make much sense on his bilateralist view. After all, how many times you assert or deny something shouldn't make any difference to whether your overall position is coherent. Moreover, it seems to me that Contraction is a selling-point of nontransitive approaches. It would be a serious blow against nontransitive approaches if they incurred strictly higher costs than noncontractive approaches.[14] Hence, I want to investigate what happens if we hold Contraction fixed.

Notice that according to the usual nontransitive response to the v-Curry, the v-Curry sentence is provable ($\vdash \kappa$) and something absurd follows from it ($\kappa \vdash \bot$). What blocks triviality is that these facts don't imply that an absurdity is provable ($\vdash \bot$). That is how rejecting Cut saves us from triviality. Hence, it seems fair to assume, at least for a start, that v-Curry sentences are provable, i.e., that $\vdash \kappa$ if κ is of the form $Val(\langle\kappa\rangle, \langle A\rangle)$.

Unfortunately, if our validity predicate is faithful and v-Curry sentences are provable, triviality results. To see this, let δ be a v-Curry sentence that is intersubstitutable with $Val(\langle\delta\rangle, \langle Val(\langle\top\rangle, \langle\bot\rangle)\rangle)$.

Ripley's solution is to distinguish between strict and tolerant assertion and denial. What is tolerant assertion? It is coherent to assert A tolerantly just in case A cannot be coherently strictly denied. Similarly for denial. Now, Ripley says that his assertions of $\top \vdash \lambda$ and $\lambda \vdash \bot$ are not strict assertions but tolerant assertions. His denial of $\top \vdash \bot$ is a strict denial. If, as his collateral view, Ripley asserts everything in Γ and denies everything in Δ, then we can formulate his view on the liar as: $\Gamma \vdash \Delta, Val(\langle\top\rangle, \langle\lambda\rangle)$ and $\Gamma \vdash \Delta, Val(\langle\lambda\rangle, \langle\bot\rangle)$ but $\Gamma \nvdash \Delta, Val(\langle\top\rangle, \langle\bot\rangle)$. Thus, when Ripley formulates ST, what he expresses by the line in the sequent calculus is the preservation of undeniability among statements about positions. I think it is fair, at this point, to ask Ripley whether we can express this relation in the object language. After all, we would like to formulate our sequent calculus within our object language, in order to show that our semantic theory offers a treatment of the concepts it uses. I am skeptical about that, but I won't investigate here whether preservation of undeniabilty can be expressed in an extension of ST. That is because Ripley's response strikes me as radical. I would like to strictly assert my preferred semantic theory.

13 This conclusion is similar to Shapiro's (2013) view that every solution of the v-Curry must block the derivation of $\kappa \vdash \bot$. But my reasons are very different.

14 Ripley argued in a 2014 talk that all plausible arguments for transitivity also speak in favor of Contraction. Hence, he thinks that giving up Contraction without giving up Cut is not an option (see http://davewripley.rocks/docs/whynot-slides-silfs.pdf, accessed on April 13, 2017). It would be disappointing, from the nontransitive perspective, if the converse also held and giving up Cut while keeping Contraction wasn't a stable position.

Proposition 1 *If a logic with a faithful validity predicate proves the v-Curry sentence* δ *that is intersubstitutable with* $Val(\langle\delta\rangle, \langle Val(\langle\top\rangle, \langle\bot\rangle)\rangle)$*, then the logic proves* $\top\vdash\bot$.

Proof Suppose our validity predicate is faithful and we have $\vdash \delta$. By the definition of δ, the last assumption implies that $\vdash Val(\langle\delta\rangle, \langle Val(\langle\top\rangle, \langle\bot\rangle)\rangle)$. By faithfulness this means that $\delta \vdash Val(\langle\top\rangle, \langle\bot\rangle)$. Given the definition of δ, it follows that $Val(\langle\delta\rangle, \langle Val(\langle\top\rangle, \langle\bot\rangle)\rangle) \vdash Val(\langle\top\rangle, \langle\bot\rangle)$. By faithfulness, this last sequent implies that $\top\vdash\bot$ follows, via a derivable metarule, from $\delta \vdash Val(\langle\top\rangle, \langle\bot\rangle)$. But we already know that $\delta \vdash Val(\langle\top\rangle, \langle\bot\rangle)$. Hence, we can conclude that $\top\vdash\bot$. □

Assuming that our validity predicate is faithful and that we can formulate v-Curry sentences, it follows that advocates of the nontransitive approach must reject the provability of v-Curry sentences.[15]

At this point, we could go back and reexamine the assumption of faithfulness or the assumption that we can formulate v-Curry sentences. To give up faithfulness, however, amounts to granting the point Barrio, Rosenblatt and Tajer are making. And if we say that v-Curry sentences cannot be formulated, we set unattractive limits on the expressive power of our language and it seems that we are trying to avoid, rather than solve, the original problem. So let us accept that we must block the proof of the v-Curry sentence.

We can turn the argument that we must block the proof of the v-Curry sentence into an argument against VD. Given Contraction, it follows from faithfulness and the assumption that we can formulate v-Curry sentences that we must reject VD.[16]

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[15] For bilateralists like Ripley, this means that the v-Curry sentence is not undeniable. Nor is it unassertable. After all, $\kappa\vdash\bot$ implies $\vdash\kappa$ via faithfulness if $\kappa = Val(\langle\kappa\rangle, \langle\bot\rangle)$.

[16] As Field (2017) has pointed out, giving up VD is what Peano Arithmetic (PA), as it were, does naturally with respect to the predicate that expresses that A is derivable in PA from premise B, i.e. $Deriv_{PA}(\ulcorner B\urcorner, \ulcorner A\urcorner)$, where $\ulcorner\phi\urcorner$ is ϕ's Gödel number. For, by Gödel's second incompleteness theorem, for every sentence A whose negation is provable in PA, if PA is consistent, then it doesn't prove that $Deriv_{PA}(\ulcorner 1=1\urcorner, \ulcorner A\urcorner) \vdash_{PA} A$. So it doesn't prove $1=1, Deriv_{PA}(\ulcorner 1=1\urcorner, \ulcorner A\urcorner) \vdash_{PA} A$ (otherwise we would get $Deriv_{PA}(\ulcorner 1=1\urcorner, \ulcorner A\urcorner) \vdash_{PA} A$ by Cut). In effect, I argue in the text that the nontransitive approach should follow PA's example.
Despite the similarity, there is also an important difference between $Deriv_{PA}$ and *Val*, namely that $Deriv_{PA}$ distributes over the conditional but *Val* doesn't. That is, we have $Deriv_{PA}(\ulcorner A\urcorner, \ulcorner B\urcorner) \vdash_{PA} Deriv_{PA}(\ulcorner\top\urcorner, \ulcorner A\urcorner) \to Deriv_{PA}(\ulcorner\top\urcorner, \ulcorner B\urcorner)$. But we don't generally have $Val(\langle A\rangle, \langle B\rangle) \vdash Val(\langle\top\rangle, \langle A\rangle) \to Val(\langle\top\rangle, \langle B\rangle)$. That is a reflection of the non-transitivity of the consequence relation *Val* expresses. After all, if the conditional obeys the deduction theorem, this means that $Val(\langle A\rangle, \langle B\rangle), Val(\langle\top\rangle, \langle A\rangle) \vdash Val(\langle\top\rangle, \langle B\rangle)$ can fail. That is how it should be. For, given faithfulness, that sequent expresses that the sequent $\top\vdash B$ follows via a derivable metarule from $A\vdash B$ and $\top\vdash A$. But that is false. In moving from $A\vdash B$ and $\top\vdash A$ to $\top\vdash B$ we are cutting on A. It is essential to the nontransitive approach that this fails, for example, when A is a Curry sentence and B is an absurdity.
It is a consequence of this dissimilarity that we cannot give a proof that parallels the proof of Gödel's second incompleteness theorem for *Val*. It seems to me, however, that even though we cannot give a proof that parallels a proof of the second incompleteness theorem, the situation in PA should nevertheless make us reluctant to say that obeying VD is essential to

Proposition 2 *Contraction, faithfulness and the assumption that δ is a v-Curry sentence that is intersubstitutable with $Val(\langle\delta\rangle, \langle Val(\langle\top\rangle, \langle\bot\rangle)\rangle)$ jointly imply that VD fails.*

Proof Suppose for reductio that VD holds. Hence, $\delta, Val(\langle\delta\rangle, \langle Val(\langle\top\rangle, \langle\bot\rangle)\rangle) \vdash Val(\langle\top\rangle, \langle\bot\rangle)$. By the intersubstitutability of δ and Contraction, it follows that $Val(\langle\delta\rangle, \langle Val(\langle\top\rangle, \langle\bot\rangle)\rangle) \vdash Val(\langle\top\rangle, \langle\bot\rangle)$. By another application of intersubstitution of δ it follows that $\delta \vdash Val(\langle\top\rangle, \langle\bot\rangle)$. By faithfulness, the last two sequents imply that $\top \vdash \bot$. By reductio, VD fails. □

The upshot of this section is that if we want to reply to Barrio, Rosenblatt and Tajer, we must give up VD. The only alternatives (short of going non-classical in the meta-theory) are rejecting Contraction or denying that we can formulate v-Curry sentences. Both options are non-starters for advocates of the nontransitive approach.

3.2 Worries About Rejecting VD

VD can seem very plausible, and giving it up has costs that some might find too high. So before I move on, I want to address three potential worries about rejecting VD.

First, an opponent might worry that if the nontransitive approach gives up VD, then the nontransitive approach cannot deal with the Liar, the traditional Curry paradox, and the v-Curry in a uniform way. It suffices to reject Cut to solve the Liar and the traditional Curry paradox, but, assuming faithfulness, we must also reject VD in order to deal with the v-Curry. It can seem, at this point, that the real culprit is Contraction. Once we accept faithfulness and allow for self-referential sentences, however, VD is implausible for reasons that are independent of the metarule of Contraction. To see this let $\alpha = Val(\langle\alpha\rangle, \langle Val(\langle\Gamma\rangle, \langle\Delta\rangle)\rangle)$. Then $\alpha, \alpha \vdash Val(\langle\Gamma\rangle, \langle\Delta\rangle)$ is an instance of VD. By faithfulness, this sequent implies that the following metarule is derivable,

$$\frac{\alpha \vdash Val(\langle\Gamma\rangle, \langle\Delta\rangle) \qquad \alpha \vdash Val(\langle\Gamma\rangle, \langle\Delta\rangle)}{\Gamma \vdash \Delta}\ \text{R1}$$

While it is standard to reject the metarule of Contraction, it is less common to keep track of the number of times you use a sequent in a proof-tree. If we can freely reuse sequents in proof-trees and R1 is a derivable metarule, then R2 is also a derivable metarule.

$$\frac{\alpha \vdash Val(\langle\Gamma\rangle, \langle\Delta\rangle)}{\Gamma \vdash \Delta}\ \text{R2}$$

any predicate that captures what follows from what. After all, $Deriv_{PA}$ captures, in some respectable sense, what follows from what in PA. Otherwise, the significance of the second incompleteness theorem would be unclear. Thanks to an anonymous referee for raising this issue.

That, however, leads to disaster. For if R2 is derivable, then, by faithfulness, we have $\alpha \vdash Val(\langle \Gamma \rangle, \langle \Delta \rangle)$ and hence $\Gamma \vdash \Delta$. So even if we reject the metarule of Contraction, faithfulness is still in tension with VD as long as we hold that we can freely reuse sequents in derivable metarules. Thus, responding to Proposition 2 by rejecting Contraction only works if we are willing to use a non-classical notion of "derivable metarule" for which Contraction fails. Hence, if we want our validity predicate to faithfully capture derivable metarules, in a classical sense, then VD is already ruled out by this requirement alone. If this is correct, it doesn't seem problematic to blame the v-Curry on VD, while blaming Cut for the other paradoxes. For, neither faithfulness nor VD play a role in any of the other paradoxes. The advocate of the nontransitive approach can see the situation as follows: VD and faithfulness both put constraints on the behavior of *Val* on the left of the turnstile. What the argument featuring R1 and R2 brings out is that, given self-reference and free reuse of initial sequents in proof-trees, these constraints conflict. This has nothing to do with the usual paradoxes. It merely reminds us that if we stipulate how a piece of vocabulary functions in premises, our stipulations should fit together. In the case of VD and faithfulness, they don't.

Second, an opponent may object that it is implausible to reject inferences like $Val(\langle \top \rangle, \langle A \rangle) \vdash A$ on a bilateralist reading. According to bilateralism, such a rejection amounts to the claim that it is sometimes coherent to deny A while also asserting that it is incoherent to deny A. But that looks like a Moorean absurdity. In response, I reject the claim that denying A while asserting that it is incoherent to deny A is incoherent in the relevant sense. As Ripley argues in an unpublished manuscript entitled "Bilateralism, Coherence, Warrant," in response to Rumfitt (2008, p. 80), the kind of incoherence that matters for bilateralism is the kind of incoherence that can occur inside supposition contexts. Now, Moorean absurdities don't occur inside supposition contexts. It is not absurd to suppose that it is raining but I don't believe it. That allows bilateralists to deny that "I believe it is raining" follows from "It is raining." Similarly, it is at least not obvious that it is incoherent to deny A, under a supposition, and to assert, under the same supposition, that it is incoherent to deny A. We can see an example if we contrapose across the turnstile: Gödel's second incompleteness theorem has made us aware of the coherence of the position that asserts that $1 + 1 \neq 5$ but denies that "$1 + 1 = 5$" is not provable.

Third, an opponent may worry that, once we reject VD and its variants, we cannot prove any sequent of the form $\vdash \neg Val(\Gamma, \Delta)$. In parallel to what happens with the derivability predicate in PA, our logic will not be able to prove that any argument is invalid. It seems to me that this isn't as implausible as it might at first seem. The notion of validity we want to capture is tied to a particular sequent calculus. After all, we are trying to capture the derivable metarules of a particular calculus. Now, the kind of sequent calculus in which ST is usually formulated doesn't contain any anti-sequents. We cannot express in the sequent calculus that an argument is invalid. Hence, if our goal is to internalize what we do in such a sequent calculus, it shouldn't surprise us that we cannot capture invalidities. It is meta-meta-theoretical reasoning about the sequent calculus that allows us to say, e.g., that $\top \vdash \bot$ doesn't hold. We could try to internalize such reasoning by adding anti-sequents to ST and have a rule that allows us to derive $\vdash \neg Val(\Gamma, \Delta)$ from $\nvdash Val(\Gamma, \Delta)$. This is an admirable project, but it is, as it were, one level up from the metarules that I am concerned with here. An investigation of this issue will have to wait for another occasion.[17]

It is not my goal here to present conclusive arguments for the claim that the right reaction to Proposition 2 is to give up VD. It suffices if I convinced you that rejecting VD in response to Proposition 2 is at least an option and, in particular, one of the most attractive options for advocates of the nontransitive approach.[18] From now on, I will assume that VD is not a desideratum for validity predicates. Can the advocate of the nontransitive approach introduce a faithful validity predicate into ST if she rejects VD? In the next section, I will suggest a positive answer to this question.

4 Naturalizing Gentzen

If we reject VD, we no longer know how the validity predicate behaves on the left of the turnstile. We must ask: What does a validity statement on the left of a turnstile mean? It is commonplace to think of the left of the turnstile as the place for assumptions. So, we may understand our question as asking: What does it mean to assume a validity statement? I will suggest an answer to this question, and I'll give rules such that all and only derivable metarules are internalized. The calculus I am going to suggest is a cross-breed between a sequent calculus and a natural deduction system.

Before turning to my suggestion, however, let's be explicit about the formulation of ST that I'm going to use. It is stated in Fig. 1. I build in Con-

[17] Moving up to meta-meta-theoretic reasoning may suggest that we should try to create a higher-order sequent calculus in the spirit of von Kutschera (1968), similar to what Wansing and Priest (2015) have presented for the noncontractive approach. A higher-order sequent calculus that internalizes all its derivable rules while rejecting Cut would indeed allow the nontransitive approach to give a more general and powerful reply to Barrio et al.'s dilemma than what I can offer here. Unfortunately, I must confess that I am unable to construct such a system. As far as I can see, Wansing and Priest are going non-classical in their meta-theory (e.g. by disallowing multiple uses of an initial sequent in proof-trees, e.g., for their $\vdash_l$). The issue is complex and I cannot pursue it here. But I admit that if such a solution can be worked out, it may be superior to what I am offering here.
It is worth noting, however, that as long as we focus only on derivable rules, the idea that we must internalize meta-meta-rules cannot get any grip. The reason is that there simply are no derivable meta-meta-rules in ST. In order to derive a meta-meta-rule, we would need primitive rules that operate on metarules, and ST doesn't contain any such rules.

[18] Obviously, if the noncontractive theorist can present strong cases for faithfulness and VD, Proposition 2 can be used as an argument against Contraction. It would be interesting to pursue this line of thought. Here, however, I want to stick to the perspective of the nontransitive approach.

Axioms of ST

$$\text{ID: } A \vdash A$$

Rules of ST

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{LK} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ \text{RK}$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \text{LN} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \text{RN}$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \text{Lv} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \text{Rv}$$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, Tr\langle A \rangle \vdash \Delta}\ \text{LT} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash Tr\langle A \rangle, \Delta}\ \text{RT}$$

Fig. 1 Logic ST

traction and Permutation by working with sets.[19] As is well known, ST is supra-classical, consistent, and contains a transparent truth predicate. Cut fails, e.g., for the Liar and the traditional Curry sentence.

4.1 Defining NG

In this subsection, I will add three rules to ST. These rules govern the language of ST enriched with a newly added two-place validity predicate, *Val*. I call the resulting logic NG for "Naturalized Gentzen." The basic idea is that we allow ourselves to assume and discharge sequents. The first rule we add is VLR. It governs the validity predicate.

$$\frac{\dfrac{\begin{array}{ccc} \dfrac{{}^{1:}\ \Gamma_1 \vdash \Delta_1}{\vdots} & \dots & \dfrac{{}^{m:}\ \Gamma_m \vdash \Delta_m}{\vdots} \end{array}}{\Gamma \vdash \Delta}\ n \text{ applications of rules of NG}}{Val(\langle \Gamma_1 \rangle, \langle \Delta_1 \rangle), ..., Val(\langle \Gamma_m \rangle, \langle \Delta_m \rangle) \vdash Val(\langle \Gamma \rangle, \langle \Delta \rangle)}\ \text{VLR, } [1,...,m]$$

VLR allows us to assume and discharge sequents in much the way we can assume and discharge formulae in a natural deduction system. The superscripts that number the assumed sequents are not part of those sequents; they are merely devices that help us to keep track of our assumptions. We allow vacuous discharges, i.e., we allow to discharge assumptions of sequents that were

[19] From now on, $\langle \Gamma \rangle$ will be the object language name of the set Γ. As before, I omit set brackets for singletons.

not used in the proof-tree. Moreover, an application of VLR can discharge several assumptions at once, but it need not discharge all open assumptions. If you choose to discharge an assumption, you must encode it in a validity statement on the left. The proof-tree above $\Gamma \vdash \Delta$ can include leaves that are not assumptions but axioms.

VLR is a generalization of VP. To see this, notice that if $\Gamma \vdash \Delta$ is derivable from axioms of NG, then $\vdash Val(\langle\Gamma\rangle, \langle\Delta\rangle)$ follows by VLR. So VP holds in NG. In order to make sure that NG does not only internalize VP but also its converse, let us add the converse of VP as a primitive rule.

$$\frac{\vdash Val(\Gamma, \Delta)}{\Gamma \vdash \Delta}\ \text{VP-converse}$$

Adding this rule has the effect that the converse of VP is not just admissible but derivable. Hence, if we succeed in defining a faithful validity predicate, we will be able to prove in the object language that VP and its converse preserve validity.

As a third additional rule, we add a repeat rule that allows us to derive a sequent under the assumption of that very sequent.

$$\frac{^{1:}\ \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ \text{Repeat}$$

This rule is needed for technical reasons that will become apparent in the proof of Lemma 1 below.

We say that a proof-tree of NG is closed iff all undischarged sequents are axioms of NG. A sequent is provable in NG iff there is a closed proof-tree of NG that has the sequent as its root.

Adding a validity predicate to the language of ST and adding the rules VLR, VP-converse, and Repeat extends ST conservatively. To see this, notice that all ST derivations are NG derivations. Moreover, if a sequent in which *Val* doesn't occur (including inside name forming brackets) is derivable in NG, then it is derivable in ST. This is so because, given Lemma 1 below, if *Val* doesn't occur in the bottom sequent of VP-converse, then this VP-converse application is admissible in NG', which is NG without VP-converse. And it is easy to see that *Val*-free sequents that are derivable in NG' are derivable in ST. So NG conservatively adds a validity predicate to ST by extending the metarules and the language of ST.

4.2 Validity in NG

In this subsection, I will show that the validity predicate of NG internalizes all and only its derivable metarules. In order to show this, we first need a lemma:

Lemma 1 *If* $Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle), ..., Val(\langle\Gamma_n\rangle, \langle\Delta_n\rangle) \vdash Val(\langle\Theta\rangle, \langle\Lambda\rangle)$ *is provable in NG, then it is provable in a proof-tree where the root comes by VLR.*

Proof We argue by induction on proof-height. If our sequent is an axiom, it has the form $Val(\langle\Theta\rangle, \langle\Lambda\rangle) \vdash Val(\langle\Theta\rangle, \langle\Lambda\rangle)$. This is provable via VLR with the help of Repeat. Assume that the lemma holds for sequents derivable in proof-trees of height strictly less than n and that our target sequent is provable in a tree of height n. Since the principal connectives of all formulae in our sequent are *Val*, it must come by LK, VP-converse, or a bottom-to-top application of one of the other rules. If it comes by LK, the premise is of the form $Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle), ..., Val(\langle\Gamma_{n-1}\rangle, \langle\Delta_{n-1}\rangle) \vdash Val(\langle\Theta\rangle, \langle\Lambda\rangle)$. By hypothesis this is derivable via VLR. We can get the desired sequent by adding a vacuous discharge to the application of VLR.

If it comes by VP-converse, the premise has the form $\vdash Val(\langle Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle), ..., Val(\langle\Gamma_n\rangle, \langle\Delta_n\rangle)\rangle, \langle Val(\langle\Theta\rangle, \langle\Lambda\rangle)\rangle)$. By our hypothesis, this is derivable via VLR. The premise must be $Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle), ..., Val(\langle\Gamma_n\rangle, \langle\Delta_n\rangle) \vdash Val(\langle\Theta\rangle, \langle\Lambda\rangle)$ without any open assumptions. By hypothesis, this is derivable via VLR.

Suppose $Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle), ..., Val(\langle\Gamma_n\rangle, \langle\Delta_n\rangle) \vdash Val(\langle\Theta\rangle, \langle\Lambda\rangle)$ comes by a bottom-to-top application of one of the double-line rules. Let's look at the part of the proof-tree between the last application of VLR (exclusive, if there is a VLR application, otherwise the axioms) in each branch and the root. This fragment of the proof-tree doesn't contain any applications of VLR or Repeat. In such fragments of proof-trees the bottom-to-top applications of ST rules can be eliminated, i.e., they are admissible even if we delete them from our primitive rules. This can be verified by induction on the height of such fragments. I will just do one case to exemplify the strategy: Suppose $\Gamma \vdash A, \Delta$ comes from $\Gamma, \neg A \vdash \Delta$ via a bottom-to-top application of LN. By hypothesis, $\Gamma, \neg A \vdash \Delta$ must come by some top-to-bottom rule or be an axiom. If it is an axiom $\Gamma = \emptyset$ and $\Delta = \{\neg A\}$, then we get $\vdash A, \neg A$ from $A \vdash A$ via RN. If $\Gamma, \neg A \vdash \Delta$ comes by a top-to-bottom application of a rule, that is either LN applied to A and we are done, or it is some other rule. If it is LK weakening with $\neg A$, we get $\Gamma \vdash A, \Delta$ by RK. In all other cases, $\neg A$ is already on the left of the premise sequent. Hence, we can apply our hypothesis, then apply the rule in question, and we get $\Gamma \vdash A, \Delta$. The other cases are analogous. Since the top-to-bottom rules cannot yield our target sequent, the whole part of the proof-tree we are looking at is redundant or an instance of ID. Hence, we can get our desired sequent by an application of VLR. □

With this lemma in hand, we can show that NG internalizes all and only its derivable metarules.

Proposition 3 $Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle), ..., Val(\langle\Gamma_n\rangle, \langle\Delta_n\rangle) \vdash Val(\langle\Theta\rangle, \langle\Lambda\rangle)$ *is provable in NG iff there is a proof-tree applying only rules of NG with some of* $\Gamma_1 \vdash \Delta_1$,..., $\Gamma_n \vdash \Delta_n$ *and, optionally, some axioms of NG as its leaves and* $\Theta \vdash \Lambda$ *as its root.*

Proof Right-to-left: Immediate by VLR. Left-to-right: Suppose that we can derive $Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle), ..., Val(\langle\Gamma_n\rangle, \langle\Delta_n\rangle) \vdash Val(\langle\Theta\rangle, \langle\Lambda\rangle)$ in NG. By Lemma 1, it can be derived via VLR. Hence, the premise is $\Theta \vdash \Xi$ as the root of a proof-tree with the open assumptions being some of $\Gamma_1 \vdash \Delta_1, ..., \Gamma_n \vdash \Delta_n$ plus, perhaps, some axioms of NG. □

This means that our validity predicate is faithful. We can prove the internalization of an instance of a metarule just in case that instance can be derived in NG.

It is easy to derive the internalizations of applications of primitive rules other than VLR. Just assume the premise(s), derive the conclusion, and apply VLR. Internalizing VLR is also easy. Suppose we assume $\Gamma_1 \vdash \Delta_1$,..., $\Gamma_n \vdash \Delta_n$, derive $\Theta \vdash \Lambda$, and then get $Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle), ..., Val(\langle\Gamma_n\rangle, \langle\Delta_n\rangle) \vdash Val(\langle\Theta\rangle, \langle\Lambda\rangle)$ via VLR. The internalization of this rule application is:

VLR-Int $\quad Val(\langle Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle), ..., Val(\langle\Gamma_n\rangle, \langle\Delta_n\rangle)\rangle, \langle Val(\langle\Theta\rangle, \langle\Lambda\rangle)\rangle) \vdash$
$\quad Val(\langle Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle), ..., Val(\langle\Gamma_n\rangle, \langle\Delta_n\rangle)\rangle, \langle Val(\langle\Theta\rangle, \langle\Lambda\rangle)\rangle)$

Notice that the internalization of VLR can be interpreted in different ways. It captures, simultaneously, that Repeat is a primitive rule, and that VLR is a primitive rule. That is, applications of these two rules share one and the same internalization. Both rules codify reflexivity at different levels. As NG basically flattens everything to one level, it shouldn't come as a surprise that these rules (as applied to validity statements) are captured in the same sequents.

As we have shown, NG also internalizes all other derivable rules that depend on VLR. Let's illustrate this with the example of the following derivable rule:

$$\frac{\Gamma, A \vdash \Delta}{Val(\langle\Gamma \cup \{B\}\rangle, \langle\Delta\rangle) \vdash Val(\langle\Gamma \cup \{A \vee B\}\rangle, \langle\Delta\rangle)}\ \text{derivable rule}$$

The internalization of this rule can be derived as follows:

$$\dfrac{\dfrac{\dfrac{{}^{1:}\ \Gamma, A \vdash \Delta \qquad {}^{2:}\ \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \text{L}\vee}{Val(\langle\Gamma \cup \{B\}\rangle, \langle\Delta\rangle) \vdash Val(\langle\Gamma \cup \{A \vee B\}\rangle, \langle\Delta\rangle)}\ \text{VLR, [2]}}{Val(\langle\Gamma \cup \{A\}\rangle, \langle\Delta\rangle) \vdash Val(\langle Val(\langle\Gamma \cup \{B\}\rangle, \langle\Delta\rangle)\rangle, \langle Val(\langle\Gamma \cup \{A \vee B\}\rangle, \langle\Delta\rangle)\rangle)}\ \text{VLR, [1]}$$

Because VP and its converse are both derivable rules of NG, we can prove in NG that a validity claim is provable if and only if it holds. That is, we can prove the following in NG:

$$Val(\langle\top\rangle, \langle Val(\langle\Gamma\rangle, \langle\Delta\rangle)\rangle) \dashv\vdash Val(\langle\Gamma\rangle, \langle\Delta\rangle)$$

Let's take stock. We can add a faithful validity predicate to ST. Our validity predicate obeys VP, and it extends ST conservatively. It is overdetermined in NG that the v-Curry paradox doesn't arise. Not only does Cut fail, but VD also fails. We have a transparent truth predicate, and our logic is supra-classical. The usual nontransitive solutions to the Liar and other paradoxes can stay unchanged. That suffices to meet the challenge posed by Barrio, Rosenblatt and Tajer.

Let me end my exposition by pointing out that the strategy for introducing a faithful validity predicate that I am suggesting here can be applied outside of the nontransitive approach. The proof of Proposition 3 goes through, if we add VLR to other logics, as long as every derivable sequent of the form $Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle)$, ..., $Val(\langle\Gamma_n\rangle, \langle\Delta_n\rangle) \vdash Val(\langle\Theta\rangle, \langle\Lambda\rangle)$ is derivable via VLR. This will hold, e.g., for any sequent calculus without rules that reduce the complexity of sequents and where VLR is the only rule that introduces *Val*. Hence, VLR introduces a faithful validity predicate for a wide variety of Cut-free sequent calculi. If our goal is to capture, in the object language, the derivability relation among sequents in a sequent calculus, then VLR is a simple and widely applicable way to do that.

5 An Objection

Someone might object that, in NG, validity statements that are embedded under validity on the left don't express what they intuitively ought to express. Consider, e.g., the following sequent:

OBJ $\quad Val(\langle Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle)\rangle, \langle Val(\langle\Gamma_2\rangle, \langle\Delta_2\rangle)\rangle) \vdash Val(\langle\Gamma_3\rangle, \langle\Delta_3\rangle)$

Intuitively, this sequent could be interpreted as saying that under the assumption that a metarule that brings us from $\Gamma_1 \vdash \Delta_1$ to $\Gamma_2 \vdash \Delta_2$ is derivable, we can infer that the sequent $\Gamma_3 \vdash \Delta_3$ is provable. In NG, however, the sequent OBJ must be derivable via VLR from $\Gamma_3 \vdash \Delta_3$ being derived under the assumption of $Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle) \vdash Val(\langle\Gamma_2\rangle, \langle\Delta_2\rangle)$, i.e., under the assumption of a sequent and not under the assumption of the corresponding metarule. Since Cut fails in NG, it can happen that we assume $Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle) \vdash Val(\langle\Gamma_2\rangle, \langle\Delta_2\rangle)$, and $\Gamma_1 \vdash \Delta_1$ is provable in NG, but even under our assumption we cannot derive $\Gamma_2 \vdash \Delta_2$. Hence, assuming the sequent $Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle) \vdash Val(\langle\Gamma_2\rangle, \langle\Delta_2\rangle)$ is very different from assuming that the inference from $\Gamma_1 \vdash \Delta_1$ to $\Gamma_2 \vdash \Delta_2$ is an instance of a derivable metarule. So it may seem that statements like $Val(\langle Val(\langle\Gamma_1\rangle, \langle\Delta_1\rangle)\rangle, \langle Val(\langle\Gamma_2\rangle, \langle\Delta_2\rangle)\rangle)$ don't really express what they ought to express when they occur on the left of the turnstile in NG.

In response to this worry, I want to point out that in order to have a proper treatment of assumptions of metarules, we would need to introduce machinery to assume that something is an instance of a derivable metarule. This would imply that we make assumptions regarding primitive sequent rules. We would need meta-metarules that allow us to assume and discharge metarules. While this may be a worthwhile project, I don't see why an advocate of the nontransitive approach is committed to carrying it out. The challenge was to internalize all and only the derivable metarules in a nontransitive logic of truth and validity. NG does that.

The kernel of truth in this objection is that we must be careful when interpreting sequents of NG with nested validity predicates. The occurrence of a sentence like $Val(\langle Val(\langle \Gamma_1 \rangle, \langle \Delta_1 \rangle) \rangle, \langle Val(\langle \Gamma_2 \rangle, \langle \Delta_2 \rangle) \rangle)$ on the left doesn't amount to an assumption about metarules. Rather, it must be read as the assumption of the sequent $Val(\langle \Gamma_1 \rangle, \langle \Delta_1 \rangle) \vdash Val(\langle \Gamma_2 \rangle, \langle \Delta_2 \rangle)$. Once we are careful to interpret sequents of NG in the right way, it is not a problem that there is also another tempting but unintended interpretation.

6 Conclusion

Let me sum up. I have formulated a faithfulness requirement for validity predicates. If we want a faithful validity predicate in a nontransitive approach to paradox, we should reject VD. In other words, validity predicates in nontransitive logics shouldn't obey a detachment rule. In my argument for this claim, I presuppose Contraction and that v-Curry sentences are available in the object language. It seems to me that every advocate of the nontransitive approach worth her salt should accept these assumptions.

With VD out of the way, I have added a faithful validity predicate to ST, i.e., a validity predicate that internalizes all and only the derivable metarules. This gives the advocate of the nontransitive approach to paradox what she needs in order to respond to Barrio, Rosenblatt and Tajer. The crucial idea is to introduce machinery that allows us to assume and discharge sequents.

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