An alternative to the Schwarzschild metric.

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Abstract.

The Schwarzschild metric is for a spherical static central mass, and is the simplest solution in General Relativity (GTR). But here we introduce an alternative spherical metric, which we call the *K-metric*, and argue that it may be considered as a possible *alternative metric law*, and should be empirically tested. It is the analytic continuation of the Schwarzschild metric, and the two are almost indistinguishable in the solar system. But they have different forms, they can be tested in the solar system, and the result has strong consequences for black holes and cosmology.

We first show the K-metric is a consistent GTR metric, deriving the stress-energy tensor to produce it, working backwards through the GTR equation to find the mass distribution. We then observe that this can be taken as an alternative "gravitational tensor", in which "gravitational mass" is dispersed around the inertial mass of a particle. This gives an alternative theory of gravity, with a linear superposition principle for fields, which we briefly outline. We then describe how a test may be carried out.

The main point is that this appears as *the natural solution for gravity* for a class of models in which gravity is produced as a scale-symmetric continuous spatial distortion, without discontinuities. The proposition is that this represents *a plausible testable variation of GTR*, and is perhaps the most radical variation that remains untested today.

An alternative to the Schwarzschild metric.

> "The chief attraction of the theory lies in its logical completeness. If a single one of the conclusions drawn from it proves wrong, it must be given up; to modify it without destroying the whole structure seems to be impossible." Albert Einstein *(1919).*

The Schwarzschild metric is for a spherical static central mass, and is the simplest solution in General Relativity (GTR). Birkhoff's theorem[1] tells us it is unique, i.e. that the exterior solution of a spherical, nonrotating body must be given by the Schwarzschild metric. It is the basis for all experimental studies of GTR, and for the theory of black holes. But here we introduce an alternative spherical metric, which we call the *K-metric,* and argue that it may be considered as a possible *alternative metric law,* and should be empirically tested. It is the analytic continuation of the Schwarzschild metric, and the two are almost indistinguishable in the solar system. But they have different forms, they can be tested in the solar system, and the result has strong consequences for black holes and cosmology.

We first show the K-metric is a consistent GTR metric, deriving the stress-energy tensor to produce it, working backwards through the GTR equation to find the mass distribution. We then observe that this can be taken as an alternative "gravitational tensor", in which "gravitational mass" is dispersed around the inertial mass of a particle. This gives an alternative theory of gravity, with a linear superposition principle for fields, which we briefly outline.

The main point is that this appears as *the natural solution for gravity* for a class of models in which gravity is produced as a scale-symmetric continuous spatial distortion, without discontinuities. The proposition is that this represents *a plausible testable variation of GTR,* and perhaps the most radical untested variation.

Many GTR theorists would probably only consider a variation of GTR "plausible" if it *retained* the Schwarzschild solution – to *retain* fundamental assumptions. But we take a different approach: instead of taking the *forms of general equations* as the point to generalize from, we may instead consider the *special solutions that exemplify the force or potential,* as the distinctive pattern that nature is providing us; and consider if there is a theory that gives generalized *solutions*.[2]

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[1] [Birkhoff, 1923]. First published 1921 by [Jørg Tofte Jebsen, 1921]. See [Deser and Franklin, 2005].

[2] This is how Newton's law of gravity was found: by generalizing from special *solutions* noticed earlier by Kepler & co, to a more general *theory*. Same with Maxwell's equations, starting with Coulomb's electrostatic law. We cannot generalize GTR from the classical Newtonian *equations of motion,* because they are non-relativistic; but the classical central mass *solution* still provides the major clue. The more general idea is that we do not reason inductively from *data to theory* directly, but rather in two steps, from *data* → *natural patterns* → *theory.* In this case, we argue that the K-metric is the real pattern hiding behind the Schwarzschild metric, and from this *solution* we can reconstruct a theory.

So *starting with* the Schwarzschild metric as the function embodying a natural pattern that we want to *generalize from,* we can ask what a plausible variation within the tested bounds of GTR are possible. The K-metric appears as the primary candidate.

We spend some time showing the consistency of such a solution, in two steps, first by analyzing the K-metric in GTR, and then by showing there is a consistent superposition principle. It corresponds physical to a kind of *continuous elastic behavior of space* around mass. It is governed an exponential strain function, instead of the quadratic function in the Schwarzschild metric, which has a discontinuity.

The surprising thing perhaps is that the difference between them cannot presently be decided empirically. The gravitational data is not quite good enough, and no one has done an experiment. The difference between the two metrics has almost been tested by accident – but through only one experiment, and the results are inconclusive. So at present no one knows which metric holds.

But it is testable in the solar system with a relatively simple experiment. If the K-metric proved correct, black holes and GTR would become simpler, and there would be significant changes in cosmology. If the Schwarzschild metric proved correct, ordinary GTR would pass another test, this time of the form of its *stress-energy tensor law,* and another possible alternative theory (from a shrinking list) would be rejected.

The key difference between them is that the stress-energy tensor for the K-metric never becomes zero "outside" the region of the central mass, but remains finite at all distances. Setting $T_{\mu\nu} = 0$ outside the central mass is the key assumption of the Schwarzschild solution. This has the effect of dividing space into one region where: $T_{\mu\nu} \neq 0$ and another region where: $T_{\mu\nu} = 0$. This *analytic discontinuity* is ultimately the cause of the singularity at the event horizon. It disappears with the K-metric, which is the smooth analytic continuation of the Schwarzschild metric, differing from it only in second and higher order terms of $1/r$.

Requiring the tensor $T_{\mu\nu}$ used in the GTR equation to extend outside the central point-mass of a particle may be seen as analogous to quantum mechanics, where the *position wave function* is extended across space. An underlying model for this is briefly explained, in which mass is identified with a continuous spatial field, viz. a *strain field,* which corresponds directly to the distortion of the space metric. The K-metric is the natural solution for gravity, and it has a simple superposition principle for multiple masses. The propagation is proposed to be like the electric field, based on a retarded scalar potential. This allows it to be conceived as a real natural law.

Hence, the primary argument is that it is realistically plausible to interpret the *K-metric* as the natural physical lawlike solution for gravity. It is a theory that still conforms to the Einstein equation, but involves reinterpreting the stress-energy tensor. A primary appeal is from the symmetries of *K* function. It is the natural *exponential elastic function,* compared to the k-function, which is a *quadratic elastic function*.

Part 1. The stress-energy tensor for K-metric.

1. The K-metric.

We give the *K-metric* analysis from first principles, side-by-side with the *Schwarzschild metric.*[3] The *Schwarzschild metric* is written in polar coordinates:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2/k^2 - k^2 dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \qquad (1)$$

$$k(M,r) = \left(1 - \frac{2MG}{c^2 r}\right)^{-1/2}$$

We will analyze the alternative metric:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2/K^2 - K^2 dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \qquad (2)$$

obtained by substituting *K ('big K')* for *k*, defined:

$$K(M,r) = e^{\left(\frac{MG}{c^2 r}\right)} \qquad (3)$$

We refer to (2) as the *K-metric* and (1) as the *k-metric* or *Schwarzschild metric.*

They converge for large *r*. E.g. comparing $1/k^2$ with $1/K^2$:

$$1/k^2 = 1 - 2MG/c^2 r$$

$$1/K^2 = 1 - 2MG/c^2 r + (2MG/c^2 r)^2 (1/2!) - (2MG/c^2 r)^3 (1/3!) + \dots$$

These differ by the second-order term: $2(MG/c^2 r)^2$ and higher. $1/K^2$ is the analytic continuation of $1/k^2$. Comparing *k* and *K* directly:

$k = 1 + MG/c^2 r + (3/2)(MG/c^2 r)^2 + \dots$	Binomial expansion.
$K = 1 + MG/c^2 r + (1/2)(MG/c^2 r)^2 + \dots$	Exponential expansion.
$k - K = (MG/c^2 r)^2 + \dots$	For $r >> MG/c^2$

Hence for large *r*, $k \approx K + \left(\frac{MG}{c^2}\right)^2 \frac{1}{r^2}$. Schwarzschild gravity has stronger fields than K-gravity for the same source *M*, everywhere outside the event horizon: $r > 2MG/c^2$. We illustrate with normalized units for *r* on the graphs, setting: $MG/c^2 = 1$. So: $r = 2$ represents the normal black hole radius for this mass *M*, at: $r = 2MG/c^2$.

[3] There are more general techniques, e.g. [Suvorov, 2021], but the analysis for a single mass is simple. See any standard text, e.g. [Landau, 1975], [Misner, 1973], [Spivak, 1979], [Wald, 1984], [Kay, 1976], [Kobayashi, 1963]). [Oas, 2014] or [Vojinovic, 2010] are convenient to follow the derivations here.

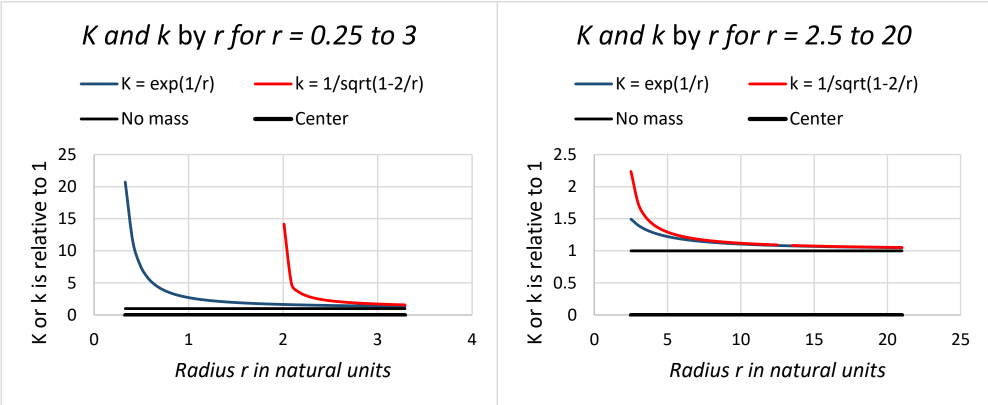

Fig. 1. Left. *k (red)* has a singularity at $r = 2$ where: $k(2) \rightarrow \infty$. *K (blue)* has only a central singularity at $r = 0$. At around $r=2$, the slope: $\partial K/\partial r$, changes from vertical to horizontal. Right. Outside the black hole at $r = 2$, they converge: $k \rightarrow K \rightarrow 1$.

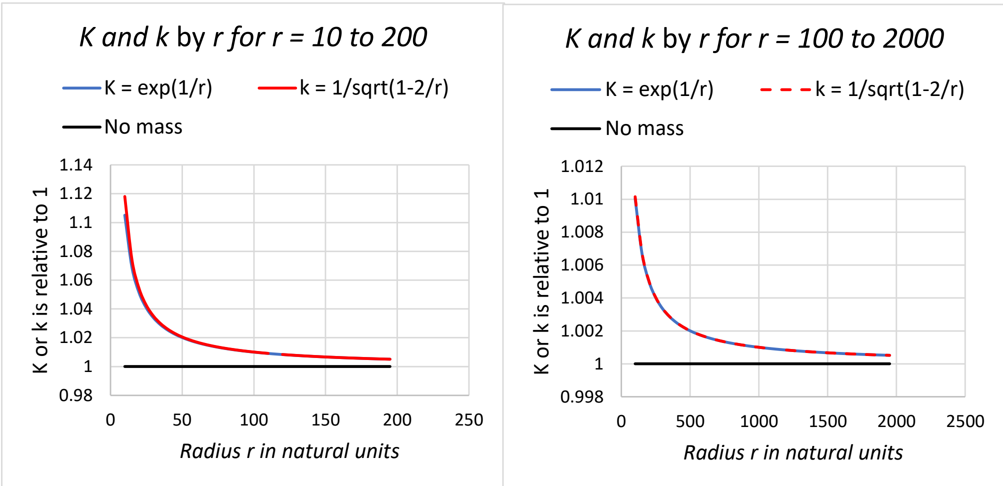

Fig. 2. Left. Outside about $r = 100$, K and k become indistinguishable on the graph, and they approach each other much faster than their asymptote at 1. Right. The curve has the uncanny property of retaining its shape with changing *r or m*, when axes are rescaled. This is a gauge symmetry. But there is an absolute scale, defined by 1.

2. Derivation of tensor components.

We now analyze the K-gravity metric as a solution to Einstein's equation:

$$G_{\mu\nu} = R_{\mu\nu} + \tfrac{1}{2} g_{\mu\nu} R = (8\pi G/c^4) T_{\mu\nu} \qquad (4)$$

$G_{\mu\nu}$ is determined by the K-gravity metric, and we work out the stress-energy tensor, $T_{\mu\nu}$, and corresponding mass-energy distribution ρ, required to produce the metric.

Whereas the Schwarzschild solution corresponds to a symmetric mass M at a central region in otherwise empty space, the K-gravity metric will correspond to the same mass M, but distributed radially in increasingly fine shells.

We start by writing both metrics in the form:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = U(r)dt^2 - V(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \qquad (5)$$

where U and V are spatial functions of r alone. This means:

$$g_{00} = U,\ \ g_{11} = -V,\ \ g_{22} = -r^2,\ \ g_{33} = -r^2 \sin^2\theta \qquad (6)$$

K-gravity and Schwarzschild gravity are defined by alternative choices of U and V, and they are distinct metrics: they cannot be transformed into each other by any coordinate transformation. However they are both spherically symmetric, and because of this, we can use the form (5) for the K-gravity metric, and obtain solutions for Christoffel symbols, Ricci tensors and Ricci scalar in terms of U, V, as commonly done for the Schwarzschild metric.

2.1 Useful Identities.

We start with some identities for k, K, U, V, and their differentials. (7)

K-gravity	**Schwarzschild gravity**
$K(M,r) = e^{\left(\frac{MG}{c^2 r}\right)}$	$k(M,r) = \left(1 - \frac{2MG}{c^2 r}\right)^{-1/2}$
$ln(K) = MG/c^2 r$	
Definitions of *U* and *V*.	
$U = c^2/K^2 = c^2 exp(-2MG/c^2 r)$	$U = c^2/k^2 = c^2(1-2MG/c^2 r)$
$V = -K^2 = -exp(2MG/c^2 r)$	$V = -k^2 = -1/(1-2MG/c^2 r)$
$U = -c^2/V$	$U = -c^2/V$
$V = -c^2/U$	$V = -c^2/U$
$UV = -c^2$	$UV = -c^2$
$U/V = -c^2/K^4$	$U/V = -c^2/k^4$
Partial derivatives by *r*.	
$K = exp(MG/c^2 r)$	$k = (1-2MG/c^2 r)^{-½}$
$K' = \partial K/\partial r = -(MG/c^2 r^2)K$	$k' = \partial k/\partial r = -(MG/c^2 r^2)k^3$
$K^{2}{}' = -(2MG/c^2 r^2)K^2$	$k^{2}{}' = -(2MG/c^2 r^2)k^4$
$K^{-1}{}' = (MG/c^2 r^2)K^{-1}$	$k^{-1}{}' = (MG/c^2 r^2)k$
$K^{-2}{}' = (2MG/c^2 r^2)K^{-2}$	$k^{-2}{}' = (2MG/c^2 r^2)$
Second partial derivatives by *r*.	
$K'' = (2MG/c^2 r^3)K+(MG/c^2 r^2)^2 K$	$k'' = (2MG/c^2 r^3)k^3+(MG/c^2 r^2)^2 3k^5$
Gradient vector field.	
$\nabla K = -(MG/c^2 r^2)K\ \underline{\mathbf{r}}$	$\nabla k = -(MG/c^2 r^2)k^3\ \underline{\mathbf{r}}$

3D Laplacian in spherical coordinates.

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial f}{\partial\varphi^2} \qquad \text{General}$$

$$\nabla^2 K = \nabla.\nabla K = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-MG}{c^2}K\right) = \left(\frac{MG}{c^2}\right)^2 \frac{K}{r^4} \qquad \textit{K-function}$$

$$\nabla^2 k = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-MG}{c^2}k^3\right) = \left(\frac{MG}{c^2}\right)^2 \frac{3k^5}{r^4} \qquad \textit{k-function}$$

Derivatives of *U* and *V* in terms of *K* and *k*:

$U' = -2c^2K'/K^3 = (2MG/r^2)/K^2$	$U' = -2c^2k'/k^3 = 2MG/r^2$
$V' = -2KK' = (2MG/c^2r^2)K^2$	$V' = -2kk' = (2MG/c^2r^2)k^4$
$U'' = 4(MG/r^2K)^2 - (4MG/r^3K^2)$	$U'' = -4MG/r^3$
$V'' = -(4MGK^2/c^2r^3)(1 + MG/c^2r)$	$V'' = -(4MGk^4/c^2r^3)(1 + 2MGk^2/c^2r)$

Derivatives of *U* in terms of *U*:

$U' = (2MG/c^2r^2)U$	$U' = 2MG/r^2$
$U'' = -(4MG/c^2r^3)(1-MG/c^2r)U$	$U'' = -4MG/r^3$
$= 4(MG/c^2r^2)^2U - (4MG/c^2r^3)U$	

Derivatives of *V* in terms of *V*:

$V' = -(2MG/c^2r^2)V$	$V' = -(2MG/c^2r^2)V^2$
$V'' = 4(MG/c^2r^2)^2V + (4MG/c^2r^3)V$	$V'' = 8(MG/c^2r^2)^2V^3 + 4(MG/c^2r^3)V^2$
$= (4MG/c^2r^3)(1 + MG/c^2r)V$	$= (4MG/c^2r^3)(1 + 2VMG/c^2r)V^2$

2.2 Christoffel symbols.

Non-vanishing Christoffel symbols in terms of *U* and *V* are for a general spherically symmetric metric:[4]

Christoffel symbols written in U, V (8)

$\Gamma^0{}_{01} = \Gamma^0{}_{10} = U'/2U$

$\Gamma^1{}_{00} = U'/2V, \quad \Gamma^1{}_{11} = V'/2V$

$\Gamma^1{}_{22} = -r/V, \quad \Gamma^1{}_{33} = -r \sin^2\theta/V, \quad \Gamma^2{}_{12} = \Gamma^2{}_{21} = 1/r, \quad \Gamma^2{}_{33} = -\cos\theta \sin\theta$

$\Gamma^3{}_{13} = \Gamma^3{}_{31} = 1/r, \quad \Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot\theta$

Other terms are zero.

Substituting the coordinate functions for *U, V* we obtain:

Christoffel symbols in coordinate functions (9)

K-gravity:	Schwarzschild gravity:
$\Gamma^0{}_{01} = \Gamma^0{}_{10} = MG/c^2r^2$	$\Gamma^0{}_{01} = \Gamma^0{}_{10} = k^2MG/c^2r^2$
$\Gamma^1{}_{00} = -MG/r^2K^4$	$\Gamma^1{}_{00} = -MG/r^2k^2$
$\Gamma^1{}_{11} = MG/c^2r^2$	$\Gamma^1{}_{11} = k^2MG/c^2r^2$
$\Gamma^1{}_{22} = r/K^2$	$\Gamma^1{}_{22} = r/k^2$
$\Gamma^1{}_{33} = r \sin^2\theta/K^2$	$\Gamma^1{}_{33} = r \sin^2\theta/k^2$

Other terms are zero.

[4] See (Oas 2014), p. 3-4.

2.3 Ricci tensor and scalar.

The Christoffel symbols determine the Ricci tensor, which has four non-zero terms.

Ricci Tensor written in U,V (10)

$R_{00} = -U''/2V + U'V'/4V^2 + U'^2/4UV - U'/rV$

$R_{11} = U''/2U - U'^2/4U^2 - U'V'/4UV - V'/Vr$

$R_{22} = rU'/2UV + 1/V - rV'/2V^2 + 1$

$R_{33} = R_{22} \sin^2\theta$

Ricci tensor components written in coordinate variables (11)

K-gravity:	Schwarzschild gravity:
$R_{00} = 2(MG/cr^2)^2/K^4$	$R_{00} = 0$
$R_{11} = -4MG/c^2r^3$	$R_{11} = 0$
$R_{22} = 1-1/K^2$	$R_{22} = 0$
$R_{33} = R_{22} \sin^2\theta$	$R_{33} = 0$
$R_{00}/R_{11} = -(MG/2r)/K^4$	

Ricci Scalar written in U and V (12)

$R = R^{\mu}{}_{\nu} = g^{\mu\nu}R_{\mu\nu}$

$= g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}$

$= g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}\sin^2\theta\, R_{22}$

$= R_{00}/U - R_{11}/V - R_{22}/r^2 - \sin^2\theta\, R_{22}/(r^2\sin^2\theta)$

$= R_{00}/U - R_{11}/V - 2R_{22}/r^2$

Substituting for $R_{\mu\nu}$ and simplifying:

$$R = -U''/UV + U'V'/2UV^2 + U'^2/2VU^2 - 2U'/rUV + 2V'/V^2r - 2/r^2(1+1/V) \quad (13)$$

Then substitute for U, V, U', V', U'' to obtain:

Ricci Scalar in coordinate functions (14)

K-gravity:	Schwarzschild gravity:
$R = (2/r^2K^2)(K^2 - 1 - (2MG/c^2r))$	$R = 0$
$R \approx (4M^2G^2/c^4r^4K^2)$	

For *K-gravity*, R is positive and proportional to $1/r^4$ in its highest term.

3. Stress-Energy tensor.

Using the Einstein equation, we can now determine the $T_{\mu\nu}$ components directly. Only diagonal terms can be non-zero, and we obtain three independent equations:

Field Equations written in U, V (15)

$(8\pi G/c^4)T_{00} = R_{00} + ½g_{00}$ $R = UV'/rV^2 - (U/r^2)(1+1/V)$

$(8\pi G/c^4)T_{11} = R_{11} + ½g_{11}$ $R = -U'/rU - (V/r^2)(1+1/V)$

$(8\pi G/c^4)T_{22} = R_{22} + ½g_{22}$ $R = (r/2V)(-U'/U + V'/V - rU''/U + rU'V'/2UV + rU'^2/2U^2)$

The fourth equation, for T_{33}, is equivalent to the third.

In the Schwarzschild derivation these are set to zero, and this leads to the solutions: $V = k^2$ and $U = c^2/k^2$. We now use these to solve $T_{\mu\nu}$ for K-gravity. The solutions are given below in coordinate functions and series in $1/r$, and as approximations from above and below.

T_{00} for K-Gravity (16)

$$T_{00} = Mc^4/4\pi r^3K^4 + c^6/8\pi Gr^2K^4 - c^6/8\pi Gr^2K^2$$
$$= (Mc^4/4\pi r^3K^4) + (c^6/8\pi Gr^2K^4)(1-K^2)$$
$$= (c^6/8\pi Gr^2K^4)(1+(2MG/c^2r)- K^2)$$
$$= -(c^6/8\pi Gr^2K^4)((2MG/c^2r)^2/2!+(2MG/c^2r)^3/3! + (2MG/c^2r)^4/4! + ...)$$
$$= -(M^2Gc^2/4\pi r^4K^4)(1 + 2(2MG/c^2r)/3! + 2(2MG/c^2r)^2/4! ...)$$
$$= -(M^2Gc^2/4\pi r^4K^4) - (c^6/8\pi Gr^2K^4)((2MG/c^2r)^3/3! + (2MG/c^2r)^4/4! + ...)$$

Hence for large r:

$$-(M^2Gc^2/4\pi r^4K^2) <\approx T_{00} <\approx -(M^2Gc^2/4\pi r^4K^4) \quad \text{for large } r$$

T_{11} for K-Gravity (17)

$$T_{11} = -T_{00}\,K^4/c^2 = T_{00}\,g_{11}/g_{00}$$
$$= -Mc^2/4\pi r^3 - (1-K^2)(c^4/8\pi Gr^2)$$
$$= (M^2G/4\pi r^4) + (c^4/8\pi Gr^2)((2MG/c^2r)^3/3! + (2MG/c^2r)^4/4! + ...)$$

Hence for large r:

$$(M^2G/4\pi r^4) \approx< T_{11} \approx< (M^2GK^2/4\pi r^4) \quad \text{for large } r$$

T_{22} for K-Gravity (18)

$$T_{22} = T_{11}\,r^2/K^4 = T_{11}\,g_{22}/g_{11}$$

Note that:

$$T_{11} = -T_{00}\,K^4/c^2 = T_{00}V/U = T_{00}\,g_{11}/g_{00}$$
$$T_{22} = -T_{00}\,g_{22}/g_{00}, \quad T_{33} = -T_{00}\,g_{33}/g_{00}$$

I.e.: $T_{\mu\mu} = T_{\nu\nu}\,g_{\mu\mu}/g_{\nu\nu}$ (no summation). T_{00} is negative, T_{11}, T_{22}, T_{33} are positive.

4. Pressure-Density in K-gravity.

In K-gravity the gravitational mass, M, is like a fluid, and we now determine the distribution. We can follow Vojinovic (2010) p.7.[5]

> "The stress-energy tensor of a fluid element with density ρ, pressure p, and 4-velocity u^μ, is: $T_{\mu\nu} = (\rho+p)u_\mu u_\nu + pg_{\mu\nu}$. We wish to describe the static fluid ($u_1 = u_2 = u_3 = 0$). So the stress-energy has the form:
>
> $$T_{00} = \rho u_0 u_0 + p(u_0 u_0 + g_{00}), \quad T_{11} = pg_{11}, \quad T_{22} = pg_{22}, \quad T_{33} = pg_{33}$$
>
> while other components vanish. Next the 4-velocity vector must be normalized, $u_\mu u_\nu g^{\mu\nu} = -1$, which means that $u_0 u_0 = -g_{00}$." Vojinovic (2010) p.7.

Applying this to the K-metric gives four equations:

K-Gravity: Pressure-Density Tensor Equations

$$T_{00} = -\rho c^2/K^2, \quad T_{11} = pK^2, \quad T_{22} = pr^2, \quad T_{33} = -pg_{33} \qquad (19)$$

Or inversely:

$$\rho = -T_{00} K^2/c^2, \quad p = T_{11}/K^2, \quad p = T_{22}/r^2 \qquad (20)$$

For the Schwarzschild solution, these are all zero: $T_{\mu\nu} = 0$ so $p=0$ and $\rho = 0$.

We now calculate p and ρ for K-gravity. Since from above: $T_{\mu\mu} = T_{\nu\nu} g_{\mu\mu}/g_{\nu\nu}$, there is only one equation to solve, and: $p = \rho$. We will solve for ρ. Substituting T_{00} from Equation (16) gives:

$$\rho = -T_{00} K^2/c^2 = -Mc^2/4\pi r^3 K^2 - c^4/8\pi Gr^2 K^2 + c^4/8\pi Gr^2 \qquad (21)$$

$$= (c^4/8\pi Gr^2)(1 - 1/K^2 - 2GM/c^2 rK^2)$$

$$= (c^4/8\pi Gr^2)(1 - 1 + 2GM/c^2 r - (2GM/c^2 r)^2/2! + (2GM/c^2 r)^3/3! \ldots$$

$$-2GM/c^2 r + (2GM/c^2 r)^2 - (2GM/c^2 r)^3/2! + \ldots)$$

$$= (c^4/8\pi Gr^2)(½(2GM/c^2 r)^2 - (1/3)(2GM/c^2 r)^3 + (1/8)(2GM/c^2 r)^4 \ldots$$

$$\ldots + (-1)^n (2GM/c^2 r)^n ((n-1)/n!)) \ldots$$

Or expanded as a series in $1/r$:

$$\rho = (GM^2/4\pi r^4) - (G^2 M^3/3\pi c^2 r^5) + (G^3 M^4/16\pi c^4 r^6) \ldots \qquad (22)$$

$$= (GM^2/4\pi r^4)K(1 - (1/3)(GM/c^2 r) + (1/4)(GM/c^2 r)^2 \ldots)$$

Approximations from below and above, for large r, are:

$$(M^2 G/4\pi r^4 K^2) \approx< \rho \approx< (M^2 G/4\pi r^4) \qquad \textit{for large } r \qquad (23)$$

ρ is constrained between these two limits, and: $\rho \rightarrow (M^2 G/4\pi r^4 K)$ for large r.

Hence ρ varies with M^2/r^4 in the lowest order term, for $r >> MG/c^2$. The first-order variation with M^2 may seem odd: but when we integrate ρ (next) we find the full mass integral is proportional to M. But this integral is dependent on the behavior at small r, i.e. where $r < MG/c^2$, and higher-order terms in $1/r$ and M dominate. Note from the

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[5] [Vojinovic, 2010] use the reverse metric signature, so we must reverse signs when we apply this.

Laplacian earlier: $\nabla^2 K = \left(\frac{MG}{c^2}\right)^2 \frac{K}{r^4} = 4\pi G\rho \frac{K^2}{c^2}$. Compare with classical gravitational systems where: $\nabla^2 \emptyset = 4\pi G\rho$. As $r \rightarrow 0$, $\rho \rightarrow \infty$, and there is a central naked singularity. But we see when we integrate for the mass that there is no event horizon.

5. Integrating the mass-energy density.

We now verify that the total mass-energy adds up to Mc^2, by integrating ρ over the spatial volume. This is required to match the Newtonian and Schwarzschild solutions in the limit. We first find the indefinite integral:

The mass-energy integral

$$I = \int (\rho)(4\pi r^2 dr) \tag{24}$$

Substituting from (21):

$$I = \int (-Mc^2/4\pi r^3K^2 - c^4/8\pi Gr^2K^2 + c^4/8\pi Gr^2)(4\pi r^2 dr) \tag{25}$$

$$= \int (-Mc^2/rK^2 - c^4/2GK^2 + c^4/2G)dr$$

This has the exact solution:

The mass-energy integral solution

$$I = -rc^4/2GK^2 + rc^4/2G + E \tag{26}$$

$$= (rc^4/2G)(1-1/K^2) + E$$

where E is an arbitrary constant of integration. To verify this calculate:

$$d/dr(rc^4/2GK^2) = c^4/2GK^2 + (-2rc^4/2GK^3)(dK/dr) = c^4/2GK^2 + (Mc^2/rK^2)$$

And:

$$d/dr(rc^4/2G) = c^4/2G$$

Next we obtain the limit of I as: $r \rightarrow \infty$. We expand the solution in terms of r.

$$I = (rc^4/2G)(1-1/K^2) + E \tag{27}$$

$$= (rc^4/2G)(1-1+2MG/c^2r -(2MG/c^2r)^2/2! + (2MG/c^2r)^3/3!) - \ldots) + E$$

$$= Mc^2 - M^2G/r + 2M^3G^2/3c^2r^2 - \ldots + E$$

As we limit $r \rightarrow \infty$ all terms in r disappear and only constant terms remain:

$$I_\infty = Mc^2 + E$$

We will set the constant E equal to 0,[6] so the indefinite integral is Mc^2 at $r = \infty$. Hence the indefinite integral is:

$$I = (rc^4/2G)(1-1/K^2) \tag{28}$$

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[6] For an empty universe. But in any realistic universe model there is a lot of background mass-energy that has to be included. However it is just the differentials of I that matter for the metric in GTR.

We next obtain the limit of I as $r \rightarrow 0$. To simplify, we can define: $r = \alpha 2MG/c^2$, i.e. r is defined as a multiple α of the fundamental distance: $2MG/c^2$. Thus: $\alpha \rightarrow 0$ as $r \rightarrow 0$, and: $\lim_{r\rightarrow 0} (I) = \lim_{\alpha\rightarrow 0} (I)$. Terms reduce to: $1/K^2 = exp(-2MG/c^2r) = exp(-1/\alpha)$, and: $rc^4/2G = \alpha Mc^2$. Substituting in I:

$$I = (Mc^2)(\alpha(1-exp(-1/\alpha)))$$

We need the value of: $\alpha(1-exp(-1/\alpha))$ as: $\alpha \rightarrow 0$. This goes 0, because:

$$exp(-1/\alpha) = 1/exp(1/\alpha) \text{ and: } exp(1/\alpha) \rightarrow \infty \text{ as: } \alpha \rightarrow 0, \text{ so: } exp(-1/\alpha) \rightarrow 0$$

So: $\alpha(1-exp(-1/\alpha)) \rightarrow \alpha \rightarrow 0$. Hence:

$$I(0) = 0 \quad \text{and:} \quad I(\infty) = Mc^2 \tag{29}$$

Hence the definite integral over the whole volume of space is:

The total mass-energy integral.

$$I_{0 \text{ to } \infty} = {}_{r=0 \text{ to } \infty}\int (\rho)(4\pi r^2 dr) = [rc^4/2G)(1-1/K^2)]_0^\infty \tag{30}$$
$$= Mc^2$$

The total mass-energy of the system is Mc^2. Note the mass-energy within a radius r is:

$$I_{0 \text{ to } r} = {}_{0 \text{ to } r}\int (\rho)(4\pi r^2 dr) \tag{31}$$
$$= [rc^4/2G)(1-1/K^2)]_0^r = (rc^4/2G)(1-1/K^2)$$
$$= (rc^4/2G)(2MG/c^2r)(1 - (2MG/c^2r)/2! + (2MG/c^2r)^2/3! - \ldots)$$
$$= Mc^2(1 - (2MG/c^2r)/2! + (2MG/c^2r)^2/3! - \ldots)$$

The factor on the right is larger than $1/K^2$ and smaller than $1/K$ from the region: $r>2MG/c^2$, hence:

$$Mc^2/K \;>\approx\; I_{0 \text{ to } r} \;>\approx\; Mc^2/K^2 \qquad \text{for } r >> 2MG/c^2 \tag{32}$$

For $r >> MG/c^2$, the total amount of gravitational mass outside the spherical shell of r is closely approximated by: M^2G/c^2r. Conversely $M(1–MG/c^2r) \approx M/K$ is approximately the gravitational mass within the sphere of radius r. The overall effect on proper acceleration at r is very similar to adopting a reduced central mass: M/K^2 in the Schwarzschild solution (Section 12). The effect of the reduced mass within the shell at r adds to the effect of the external mass outside the shell at r, and it weakens the effective mass by M/K^2 (not just M/K). Two further results help confirm the physical consistency of this solution.

Black hole radius is consistent. Although the mass-density increases indefinitely as we approach the center, the Schwarzschild (black hole) radius r_s for the central mass within r is always smaller than r, so there is no conventional 'black hole' event horizon formed inside. The Schwarzschild radius is: $r_s = 2MG/c^2$. The mass within a radius r is: $M = (rc^2/2G)(1-1/K^2)$. Substituting for M we get: $r_s = (2G/c^2)(rc^2/2G)(1-1/K^2) = r(1-1/K^2)$, or: $r_s/r = (1-1/K^2) < 1$

Hence the Schwarzschild radius, r_s, is always smaller than the enclosed-mass radius, r. The mass distribution appears consistent, and no problems of singularities arise, except the central (naked) singularity, which also appears in conventional GTR.

Pressure is consistent with a quasi-Newtonian force. Note if we differentiate the mass integral at r by r we get a force term, and this is exactly equal to: $dI/dr = 4\pi r^2 p$. Since $4\pi r^2$ is the surface area at r, this can be interpreted as meaning that the *total internal force of the mass distribution over the surface at r* generates the pressure term, p. Note (differentiating the series (27)) that this is like a gravitational self-attraction: $F = M^2G/r^2 - 4M^3G^2/3c^2r^3 + \ldots \approx M^2G/r^2$ for large r, as if the mass M was attracting to itself at a distance of r by a quasi-Newtonian force law. This begins to reduce at small r, e.g. at the point where: $r = 4MG/3c^2$, the Newtonian term cancels with higher order terms as: $M^2G/r^2 - 4M^3G^2/3c^2r^3 = 0$.

We conclude that the mass-energy distribution for ρ in (21) generates the K-gravity metric, and is a consistent solution in GTR.

Part 2. K-gravity as a physical theory.

6. K-gravity as an alternative physical theory.

The K-gravity metric is the conventional solution for a special mass distribution in GTR. Hence it is consistent if we imagine tiny test masses moving in this potential, except at the central (naked) singularity, and it has no event horizon. The mass is symmetric and finite but is not constrained within a finite boundary. It is extended indefinitely like a fine dust or gas.

This does not seem very applicable to physical systems at first, because we would not expect to encounter such a distribution of *matter* as a natural phenomenon, and if we did we would hardly expect it to be stable. But we now propose to consider this as *the physical solution for gravity for any inertial center-of-mass particle or system.*

The K-metric contradicts one of the two key assumptions in the normal derivation of the Schwarzschild metric:

(A) *Symmetric distribution assumption:* the inertial mass distribution is spherically symmetric, and static and of finite magnitude.

(B) *Stress-energy tensor assumption:* $T_{\mu\nu} = 0$ for empty space, i.e. all space outside the central mass.

These determine the usual Schwarzschild solution uniquely from GTR. The symmetry assumption (A) defines the type of system being analyzed, and is not questioned. But we are proposing to replace (B) with (B*).

(B*) *K-gravity Assumption:* $T_{\mu\nu} \neq 0$ for empty space. Instead $T_{\mu\nu}$ for a central mass is smooth, continuous and corresponds to the K-gravity metric.[7]

This is proposed as an alternative principle governing the *gravitational stress-energy tensor for a central mass M*, and what we will now refer to as *K-gravity*.

This requires us to reinterpret the *stress-energy tensor* normally assumed in the GTR equation. It means a localized inertial mass *M* produces an *extended gravitational-mass-density source field* throughout space around it. We note that this may not be feasible for an infinite flat (STR) universe, but it appears consistent for a finite curved universe. However we do not try to solve a full theory here. Rather we argue that it can be plausibly generalized to an alternative theory, where the *K-metric* replaces the Schwarzschild metric as the natural approximation for central systems, such as the sun in the solar system. This is primarily to support the case that the *K-metric* should be tested against the Schwarzschild metric.

This proposal to modify the *stress-energy tensor* does not immediately contradict GTR, because the form of the stress-energy tensor is not determined by the GTR equation, but by theories of particles and fields. (It is determined in turn by mass-momentum and charge-current distributions.) The main effect of the modified tensor is to remove discontinuities in what we may call *gravitational mass distributions*. Note the conventional assumption that $T_{\mu\nu}$ *is strictly zero outside a central mass boundary* already introduces a discontinuity in the fields. This consequently shows up in the event horizon singularity.[8]

We may also question the discontinuity in the context of quantum mechanics. Hawking radiation is generated at the event horizon, in apparent contradiction of the GTR discontinuity – because space has background fields in quantum field theory, and matter has position-momentum uncertainties. In QM, there appear generally to be no strict spatial boundaries to fields or wave-functions, and no strictly point-like classical particles.

GTR postulates a $g_{\mu\nu}$ characterizing *space-time,* and a $T_{\mu\nu}$ characterizing a *mass-energy distribution,* and a precise connection between these through the Einstein equation. However the specification of $T_{\mu\nu}$ is not determined within GTR: it comes from other branches of physics: particle and field theories. It is interpreted from the mass and field distributions around a point. So what makes us assume that: $T_{\mu\nu} = 0$ for ‘empty space’ around a mass? Primarily the classical approximation that mass-energy is strictly localized within particles.[9] This is what is being questioned here, not the

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[7] $T_{\mu\nu}$ for complex mass distributions requires a superposition principle consistent with the K-gravity metric in the limit of a single mass, as in the following section.

[8] If a (non-zero) field is analytic and continuous everywhere, it cannot become identically zero over a finite region. Hence the classical Schwarzschild mass-density function is not analytic.

[9] The relationship between the *metric tensor* and the *stress-energy tensor* and the *mass-energy distribution* is really far from clear anyway, e.g. see Lehmkuhl (2010).

GTR equation itself. In terms of laws of motion, we can continue to use the theory of geodesics in GTR once we have determined the metric tensor.

So we can take K-gravity as a GTR model with a non-standard stress-energy tensor, having a contribution from an extended “gravitational mass” not recognized in ordinary particle physics, which deals only with inertial mass. To summarize, we can view the *K-gravity* metric in two ways. First as a purely conventional solution for a particular mass density distribution. Second, in a novel interpretation, as the stress-energy distribution *for a single center-of-mass, M*.

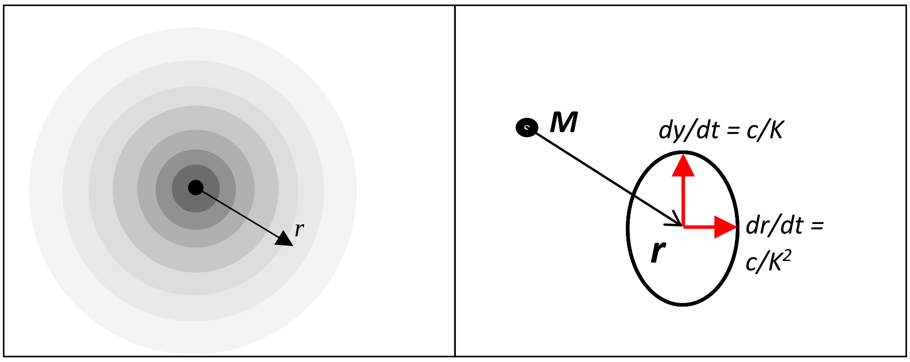

Fig. 3. Left. A conventional fluid with total mass M, thinning in space as r increases. Right. A single inertial mass M produces a metric field $g_{\mu\nu}$.

On the left, conventional GTR relates $g_{\mu\nu}(r)$ to the mass-energy density-pressure function, $\rho(r)$, via the tensor $T_{\mu\nu}(r)$. The K-gravity solution is a special case for $\rho(r)$. On the right, we switch to taking K-gravity as a theory for a single central mass, with $g_{\mu\nu}$ corresponding to the solution for the tensor $T_{\mu\nu}(r)$. The ‘gravitational mass’ of a single particle thus becomes a *density field in space,* around the inertial center-of-mass at a point.

This is somewhat analogous to the quantization of particle particles. A classical particle has a mass with a precise location and trajectory, but in quantum mechanics it has a position wave function ‘smeared out’ across space. QM thus required a radical change in the conception of particles (still not resolved after a century). K-gravity proposes analogously that the ‘gravitational mass’ of “point-like” particles is smeared across space.[10]

We must distinguish between *inertial mass,* which we conceive as the centralized mass of a localized body, and *gravitational mass,* which now becomes a *mass density field across space* corresponding “classically” to the stress-energy tensor. The conceptual distinction between inertial and gravitational mass was emphasized by Einstein, e.g. his [1919] and played an important role in his development of GTR.

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[10] GTR cannot deal consistently with quantum wave functions in the most fundamental respect: viz. the quantum distribution of matter is given by *superpositions of position states,* but when these undergo wave function collapse, there is no concept in GTR for the metric tensor to undergo collapse. GTR is deterministic, QM is probabilistic. This is part of the failure to unify GTR and quantum theory.

‘Inertial mass’ in this sense has a rest-mass, and a trajectory, and carries energy and momentum, and is what is accelerated by forces. ‘Gravitational mass’ is like the ‘charge distribution’ that the inertial mass provides for the gravitational field. Einstein recognized the importance of *identifying them,* in Newtonian and GTR theories. But this also introduces the conceptual possibility of distinguishing them.

We have analyzed the K-metric for a single mass, but for a viable theory, we need to show that there is a viable *superposition principle* for determining the metric for general distributions. We do this next for static mass distributions. The further generalization to dynamic systems then requires a principle similar to electrodynamics where charges (inertial masses in our case) generate retarded potentials which determine the fields (the metric tensor in our case). So we now turn to a superposition principle.

Before we start, we can conceptualize this in terms of a background *strain field on space,* which we briefly illustrate first. In this model, we imagine that we have *an elastic theory of space,* with mass-energy stored in the “stretching of space”, for which we introduce an underlying strain variable, *W.* This acts exactly like an additional dimension of space, and is regarded as creating a circular “surface”.

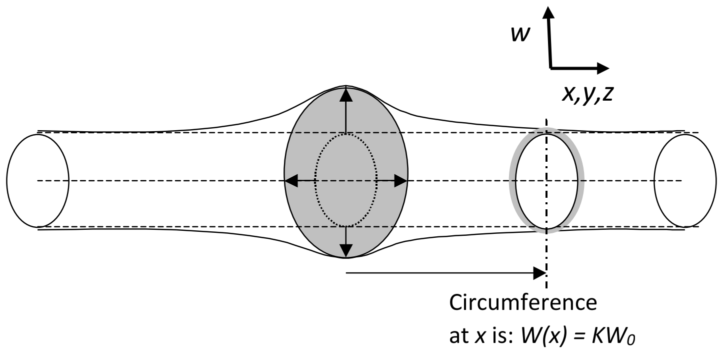

Fig. 4. We postulate a strain function, $W = W_0K(M,r)$, as a model. With the exponential K function, it goes to an asymptote: $K(r) \rightarrow \infty$ as $r \rightarrow 0$.

We postulate that the natural law for elasticity (strain) in this model for a single mass is: $W = W_0K$, and postulate functions relating strain to the metric that make this consistent with the K-metric solution. This strain has two effects: it changes the speed of light for particles or waves, and it induces curvature on space. By adding three dimensions of space in a 6D spatial model, we can use Whitney’s theorem/s [Whitney 1936/1996], to show that we can make an extrinsically curved space with metric homomorphic to any intrinsically curved GTR solution (including global curvature for a finite close universe). We then postulate linear superposition of *strain functions for multiple masses* (as in an elastic theory of space), and we obtain functions for the metric field directly from this. This strain can be used as an underlying variable to substitute for ρ, and obtain the stress-energy tensor. The metric is then a function of the strain and its gradient and divergence. But when we add multiple masses, the strain functions multiply linearly to give the full strain function, and we find a linear function for the metric directly from this, so we do not have to go through the stress-energy tensor.

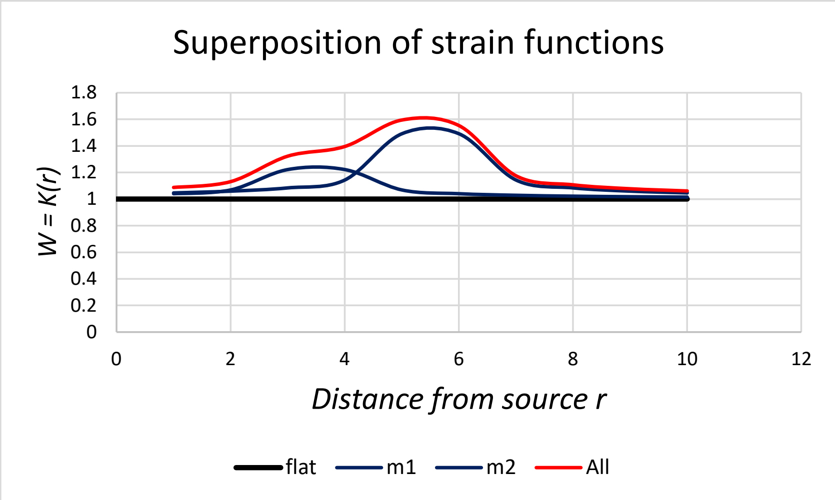

Fig. 5. If we add two masses (*m1, m2*) at different points in space, the combined strain function is the linear product. Note the functions should have singularities at the mass centers, they are shown smoothed to a finite displacement for the diagram.

To see how this gives Minkowski and Schwarzschild metrics in the limit, imagine a small test-mass *m* at a large distance, *r,* from a central mass, *M.* The *background speed of light* is now *c/K*. We see this by rearranging (2), first as locally Cartesian coordinates where $dr = dx$.

$$ds^2 = c^2dt^2/K^2 - K^2dr^2 - dy^2 - dz^2$$

We define an alternative variable: $dw = ds$, as a spatial variable affected by the strain *K*. Then the metric is the same as this 4D velocity equation:

$$\sqrt{(dw^2 + dy^2 + dz^2 + K^2dr^2)}/dt = c/K$$

Or: $\sqrt{(v_w^2 + v_y^2 + v_z^2 + K^2v_r^2)} = c/K$

This is the equation for a particle or wave, travelling in a 4D space *(w,r,y,z)* with an elliptical speed function. In *(w,y,z)-directions* the wave speed is *c/K*, while in *r* it is c/K^2. (Instead of *c/k* and c/k^2 for the Schwarzschild metric).

7. Consistency for the single central mass field.

We first observe that the single mass K-metric is a conservative energy field exactly like usual Schwarzschild solution because it is spherically symmetric, and the curl is zero. It gives rise to a universal acceleration field on masses (equivalence principle), which appears like a force field. It accelerates test particles towards a central mass. It gives a conservative force, because it must be spherically symmetric: $\boldsymbol{F} = F(r)\underline{\mathbf{r}}$.

Note the classical force is the gradient of a spherical *potential energy* function $c^2ln(K)$, giving: $\boldsymbol{f} = \nabla(c^2ln(K)) = c^2 \nabla K/K$. The classical force field is the gradient ratio: $(\partial K/\partial r)(1/K) \equiv \nabla(ln(K))$. But the relativistic force is different (later). Note also if we rotate (or reflect) a beam of light from motion in the (y,z) plane into r, its speed reduces, but its frequency remains the same. Hence the wavelength reduces: $\lambda = c_0/Kf$, or: $f = c/\lambda = c_0/\lambda_0$. This means the quantum wave energy remains the same under rotation: $E = hf$, with gravitational red shift like the usual Schwarzschild solution. More generally, potential energy is only a function of *position in the field* - rotation does not change energy.

In any case, the full dynamics are determined by the metric tensor and the GTR geodesic principle, so as long as it has a consistent superposition principle for creating the fields, it should have a consistent mechanics. Now we show the superposition law for adding multiple masses at a single point is consistent.

Linear superposition of mass holds for K for two masses imposed at the same point.

$$K(m_1,r)K(m_2,r) = K(m_1+m_2,r) \quad (34)$$

This is because K is exponential: $exp(m_1G/c^2r)exp(m_2G/c^2r) = exp((m_1+m_2)G/c^2r)$.

So the linear superposition of two fields: $K(m_1,r)K(m_2,r)$ acts exactly like K for a single point mass: $K(m_1+m_2) = K(M)$. This corresponds to a physical superposition property: treating a single mass M as two component masses, M_1+M_2.

- The effect of imposing an aggregate mass of: (M_1+M_2) on empty space at a given point ($K = exp((M_1+M_2)G/c^2r)$) is identical to imposing M_1 on empty space first ($K_1 = exp(M_1G/c^2r)$), and then imposing M_2 linearly on the resulting space at the same point ($K_{12} = K_1K_2 = exp(M_1G/c^2r)exp(M_2G/c^2r)$).

This means we can define a more general *K-scalar field* as the linear product of *K-fields* of separate masses: $K = exp(M_1/r_1 + M_2/r_2 + \ldots + M_N/r_N)$, for N masses. This general K is the exponential of the classical gravitational potential energy functions, the usual 3D spherical harmonic functions in (M/r). Note we could also add a constant, but this is just equivalent to multiplying K by a constant. K is normalized with an absolute scale of 1 for empty space. Multiplying by an arbitrary constant appears like a *gauge symmetry*, but we should be wary of this, because there is actually a fixed gauge when we consider the universe as a whole, with $K=1$.

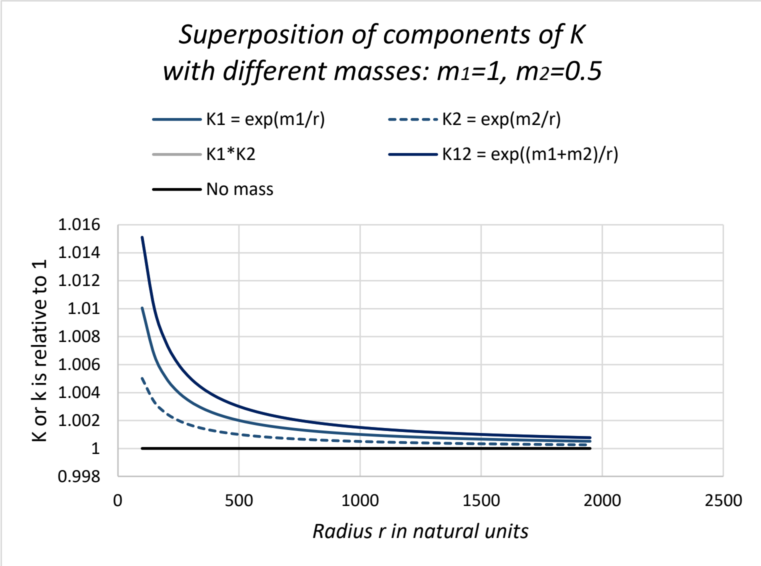

Fig. 6. Superposition of masses at the same point. Note rescaling of the axes.

Since superposition of two or masses at the same point gives the same as the sum of the masses this is consistent. Note also that this is quite different to the Schwarzschild metric, where linear superposition does not hold for *k*, because e.g.:

$$k(m_1+m_2,r) = (1-2(m_1+m_2)G/c^2r)^{-½}$$

While:

$$k(m_1,r)k(m_2,r) = (1-2(m_1+m_2)(G/c^2r)+4m_1m_2(G/c^2r)^2)^{-½}$$

So:

$$k(m_1,r)k(m_2,r) < k(m_1+m_2,r)$$[11]

We now define a more general metric from the scalar field *K* and its gradient.

8. General superposition of K-functions.

To generalize we need a law for the metric tensor at a field point, arising from multiple masses at different distances in different directions. We now propose a *superposition principle* for obtaining solutions for $g_{\mu\nu}$.

Note we cannot do this through any direct superposition of the mass-energy density functions ρ (Section 8) for individual masses, because they are not suitably linear. Normally if we combine two distinct mass distributions in space, we can add their masses together to get a combined mass distribution. This is how we superimpose classical mass distributions. But the K-gravity *gravitational mass* functions ρ, or

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[11] *k* also has an interesting property of converging to *K* if we take the linear product: k^N of many smaller masses. We divide a mass *M* into *N* masses of *m*, so: $m = M/N$, and take the superposition function *k(m)* repeated *N* times: $k(m,r)^N = ((1-2mG/c^2r)^N)^{-½}$. In the limit of $N \rightarrow \infty$, $(1-x/N)^N = e^{-x}$. So let: $x = 2mNG/c^2r = 2MG/c^2r$, and obtain: $k(m,r)^N \rightarrow (exp(2mNG/c^2r))^{-½} = exp(MG/c^2r)$. Thus in the limit of $N \rightarrow \infty$: $k(m,r)^N \rightarrow K(M,r) = K(mN,r) = K(m,r)^N$.

equivalently the $T_{\nu\mu}$, are not linear. We cannot take the solution (21) and sum $\rho_{(n)}$'s for a collection of N masses to get a total: $\rho_{total} = \rho_{(n)}$ for the whole system of masses, and then use this to derive $T_{\nu\mu}$ for the whole system, and subsequently derive $g_{\nu\mu}$ from the Einstein equation. The reason is that ρ, or equivalently $T_{\nu\mu}$, for the K-gravity central mass is not linear with mass. E.g. using the approximation: $T_{11} \approx M^2G/4\pi r^4$, which is accurate for large r, and defining M as a composite mass: $M = M_1+M_2$, we see:

$$T_{11}(M_1+M_2) \approx (M_1+M_2)^2G/4\pi r^4 \qquad (33)$$

$$= M_1^2G/4\pi r^4 + M_2^2G/4\pi r^4 + 2M_1M_2G/4\pi r^4$$

$$\approx T_{11}(M_1) + T_{11}(M_2) + 2M_1M_2G/4\pi r^4$$

Hence: $T_{11}(M_1+M_2) > T_{11}(M_1) + T_{11}(M_2)$. E.g. when: $M_1 = M_2 = M/2$, we have: $T_{11}(M_1+M_2) \approx 2T_{11}(M_1) + 2T_{11}(M_2)$, not: $T_{11}(M_1+M_2) = T_{11}(M_1) + T_{11}(M_2)$.

So we cannot write a superposition principle directly in terms of ρ or $T_{\mu\nu}$.

(C.f. the *k*-metric is not linear with mass, and has no linear superposition principle.)

Instead, we can define $g_{\mu\nu}$ *directly from the inertial mass-energy distributions.*

The linearity of K means first that there is a well-defined function generalizing K for multiple masses (at multiple positions), called the *K scalar field.* This is defined over all the masses, N, in the space. It is just the product of all the individual $K(M_n,r_{(n)})$'s for the individual masses. At any field point K is defined:

The K scalar field.

$$K \equiv K(M_1,r_{(1)})K(M_2,r_{(2)})\ldots K(M_N,r_{(N)}) \qquad (35)$$

The $r_{(i)}$'s are the distances from the field point O to the masses M_n. We may write this:

$$K \equiv K(M_1/r_{(1)}+ \ldots +M_N/r_{(N)}) = \exp((G/c^2)(\textstyle\sum_{n=1 \text{ to } N}(M_n/r_{(n)}))) \qquad (36)$$

This magnitude depends only the masses M_n and their distances: $r_{(n)} = |\mathbf{r}_{(n)}|$ from the field point. I.e. it is independent of the relative directions of the masses. The *gradient field of K* is defined in local rectangular coordinates for the empty space at a field point O as usual, with $i = 1,2,3$ and $\underline{\mathbf{x}}_i$ the basis vectors for coordinates: x^i.

The K gradient field.

$$\mathbf{K} = \nabla(K) = (\partial K/\partial x^i)\underline{\mathbf{x}}_i \qquad (37)$$

To differentiate note: $\nabla(r_{(n)}) = \underline{\mathbf{r}}_{(n)}$ and: $\nabla(M_n/r_{(n)}) = -(M_n/r_{(n)}^2)\underline{\mathbf{r}}_{(n)}$. Because K is linear, these are linear. So summing over the masses n:

$$\nabla(\mathrm{K}) = \nabla(\exp((\mathrm{G}/\mathrm{c}^2)\textstyle\sum(\mathrm{M_n}/\mathrm{r_{(n)}}))) \qquad (38)$$

$$= -(\mathrm{KG}/\mathrm{c}^2)\textstyle\sum((\mathrm{M_n}/\mathrm{r_{(n)}}^2)\underline{\mathbf{r}}_{(n)})$$

This is simply related to the Newtonian acceleration field, as illustrated.

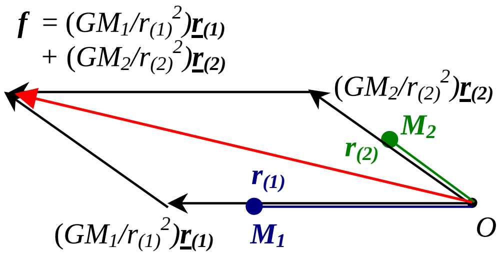

Fig. 7. The vector sum ***f*** of Newtonian acceleration fields, illustrated for two masses. The K-gradient field is closely related.

The K-gradient field is just K/c^2 times the Newtonian force vector field. The latter will be denoted ***f***, defined by linear supposition:

Newtonian acceleration field

$$\mathbf{f} = -G\sum((M_n/r_{(n)}{}^2)\underline{\mathbf{r}}_{(n)}) \tag{39}$$

Thus:

$$c^2\,\nabla K/K = \mathbf{f} \tag{40}$$

$$\nabla K = K\mathbf{f}/c^2 \qquad \text{Same}$$

So ∇K and ***f*** are closely related vector fields. Since: $\mathbf{f} = \nabla(\phi)$ and: $\nabla^2(\phi) = 0$, we have: $c^2\,\nabla^2(K)/K = c^2\,\nabla K^2/K^2$ or: $\nabla^2(K) = \nabla K^2/K$. These have magnitudes:

$$f = |\mathbf{f}| = (\mathbf{f}.\mathbf{f})^{1/2} \tag{41}$$

We only need to work with one or other of ***f*** or $\nabla c^2\,\nabla K/K$, so we will use ***f***.

We now state three rules to determine $g_{\mu\nu}$ directly for multiple source masses, and combine them in equation (46). We state this initially *in the special local rectangular coordinate system, at the field point O, with x^1 chosen in the direction of* ***f***. The $g_{\mu\nu}$ representation may be locally diagonalized in this coordinate system. We develop the rules first for simple systems where we know what the result must be (e.g. the single particle K-metric), and then we can generalize by taking coordinate transformations.

The first two rules (for static systems) are:

- First, the off-diagonal terms with a time component are zero:

$$g_{01} = g_{02} = g_{03} = g_{10} = g_{20} = g_{30} = 0 \tag{42}$$

- Second, the *t-t* component g_{00} is:

$$g_{00} = c^2/K \tag{43}$$

The third rule is given for the case where ***f*** is in the first coordinate direction $\underline{\mathbf{x}}_1$:

- Third, in our special rectangular coordinates with $x = x^1$ chosen in the direction of ***f*** the spatial components are:

$$g_{11} = -1-(\mathbf{f}.\underline{\mathbf{x}}_1/f)^2(K^2-1),$$

$$All\ other\ g_{ij} = -\delta_{ij} \tag{44}$$

Hence *for this special case* the full metric tensor is:

Coordinates: $t=x^0$, $x=x^1$, $y=x^2$, $z=x^3$ (45)

$$[g_{\mu\nu}] = \begin{pmatrix} c^2/K^2, & 0, & 0, & 0 \\ 0, & -1-(\mathbf{f.\underline{x}}_1/f)^2(K^2-1), & 0, & 0 \\ 0, & 0, & -1 & 0 \\ 0, & 0, & 0, & -1 \end{pmatrix}$$

Note first that the 'time dilation' component, viz. $g_{00}/c^2 = (\partial\tau/\partial t)^2$, is always given by $1/K^2$. All masses M_n contribute to this by the factor: $exp(GM_n/c^2r_{(n)})$, independent of their direction from the field point. This is in conformity with ordinary GTR, within the scale factor of: K^2/k^2.

Now for the space components, which determine the accelerations, in this special case: $(\boldsymbol{f}.\underline{\boldsymbol{x}}_1/f)^2 = 1$, because $\boldsymbol{f}$ is chosen in the direction of $\underline{\boldsymbol{x}}_1$ and $\boldsymbol{f}.\underline{\boldsymbol{x}}_1 = f$. So we can just write: $g_{11} = -K^2$.

It is written in the functional form above to compare with the form of the more general case, which is:

$$g_{ij} = -\delta_{ij} - (\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_i/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_j/f)(K^2-1) \tag{46}$$

This is the third rule generalized for rectangular coordinates (x^i) rotated with respect to $\boldsymbol{f}$ in a plane of $\boldsymbol{f}$ by an angle θ. The dot product: $\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_i$ gives the magnitude of $\boldsymbol{f}$ in the $\underline{\boldsymbol{x}}_i$ direction, and we may write this as: $\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_i = f_i$.

For consistency as a tensor relation, this metric (46) in rotated spatial coordinates in the *x-y* plane of $\boldsymbol{f}$, by an angle θ, must match that obtained through the usual coordinate transformation rule: $g_{\mu\nu}' = g_{kl}\,(\partial x^k/\partial x'^\mu)(\partial x^l/\partial x'^\nu)$, with the Jacobian $(\partial x^k/\partial x'^\mu)$ for rotation defined:

$$\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & \cos\theta, & -\sin\theta, & 0 \\ 0, & \sin\theta, & \cos\theta, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix} \tag{47}$$

This transforms the simple K-gravity metric (2) or (45) to:

K-Gravity metric tensor for M in rotated Cartesian coordinates (48)

$$[g_{(1)\mu\nu}]' = \begin{pmatrix} c^2/K^2, & 0, & 0, & 0 \\ 0, & -1-\cos^2\theta(K^2-1), & -\cos\theta\sin\theta(K^2-1), & 0 \\ 0, & -\cos\theta\sin\theta(K^2-1), & -1-\sin^2\theta(K^2-1), & 0 \\ 0, & 0, & 0, & -1 \end{pmatrix}$$

We can confirm we get the same result by using our general rule (46) to assign the components in a rotated frame. First consider the simple central mass case, where: $\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_i = f$, and use (46) to assign components in a rotated frame. In the simple frame, $x = x^1$ is chosen in the direction $\boldsymbol{f}$, so: $\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1 = f$, and $\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2 = \boldsymbol{f}\bullet\underline{\boldsymbol{x}}_3 = 0$. Now suppose we rotate in the x-y plane by θ, as in the transformation (35). We find that:

$$\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1 = f\cos\theta, \quad \boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2 = f\sin\theta \qquad (49)$$

This is simply the vector geometry of rotating $\boldsymbol{f}$. Thus we find the components directly from (46) as:

$$g_{11} = -\delta_{11} - (f\cos\theta/f)(f\cos\theta/f)(K^2-1) = -1 - \cos^2\theta(K^2-1) \qquad (50)$$

$$g_{22} = -\delta_{22} - (f\sin\theta/f)(f\sin\theta/f)(K^2-1) = -1 - \sin^2\theta(K^2-1)$$

$$g_{12} = -\delta_{12} - (f\cos\theta/f)(f\sin\theta/f)(K^2-1) = -\cos\theta\sin\theta(K^2-1)$$

$$g_{21} = -\delta_{21} - (f\sin\theta/f)(f\cos\theta/f)(K^2-1) = -\cos\theta\sin\theta(K^2-1)$$

This confirms we get the same result using (46) directly as by transforming the diagonalized metric to the rotated coordinate system (48). This holds generally when $\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_i = f_i < f$, because f_i/f acts as a constant when we differentiate the g_{ij}. We may write the rule (46) in a generalized matrix form:

K-metric for a static system in rectangular coordinates (51)

$$\begin{pmatrix} c^2/K^2, & 0, & 0, & 0 \\ 0, & -1-(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1/f)^2(K^2-1), & -(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2/f)(K^2-1), & -(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_3/f)(K^2-1) \\ 0, & -(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1/f)(K^2-1), & -1-(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2/f)^2(K^2-1), & -(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_3/f)(K^2-1) \\ 0, & -(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_3/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1/f)(K^2-1), & -(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_3/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2/f)(K^2-1), & -1-(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_3/f)^2(K^2-1) \end{pmatrix}$$

and this is consistent and symmetric with general coordinate rotations in space. This is the metric represented in *orthogonal rectangular coordinates* (in any direction).

Of course this is only for a static system (so the space-time terms are zero). To generalize we should take the individual *K-functions* as retarded sources, as we in electrodynamics. It is more difficult as the propagation speed now varies with the metric (c/K), and we leave a full treatment for another discussion.

To verify this is physically realistic, and to prepare to discuss empirical tests in the subsequent section we examine accelerations next.

9. Acceleration of a stationary test particle.

We will use: U^μ for the velocity 4-vector and A^μ for the acceleration 4-vector. These are the differentials of the x^μ w.r.t. proper time, $d\tau$. Thus for a *stationary test particle* at the field point: $U^0 = dt/d\tau = c/\sqrt{g_{00}} = K$, and $U^i = dU^i/d\tau = 0$ for the spatial velocities. The general tensor relationship for acceleration is:

$$A^\kappa = U^\lambda \nabla_\lambda U^\kappa \tag{52}$$

$$= U^\lambda(\partial U^\kappa/\partial x^\lambda + \Gamma^\kappa{}_{\lambda\mu}U^\mu)$$

For the stationary particle, only $U^0 \neq 0$, and this simplifies to:

$$A^\kappa = U^0(\partial U^\kappa/\partial x^0 + \Gamma^\kappa{}_{00}U^0) \tag{53}$$

$$= (U^0)^2\Gamma^\kappa{}_{00}$$

For a Schwarzschild-type metric, the only non-vanishing Christoffel symbol is $\Gamma^1{}_{00}$. So the proper-time acceleration: $d^2x/d\tau^2$ of a stationary particle at a field point $\boldsymbol{O}$ is:

$$A^1 = (c^2/g_{00})\Gamma^1{}_{00} = (c^2/g_{00})(\partial g_{00}/\partial x)(1/2g_{11}) \tag{54}$$

This is then equal to:

$$A^1 = (c^2/g_{00})\Gamma^1{}_{00} = -(c^2K^2/c^2)(c^2\partial K^{-2}/\partial x)(1/2K^2) \tag{55}$$

$$= -\tfrac{1}{2}c^2(\partial K^{-2}/\partial x)$$

In the case of the single-mass K, the differential is simply: $\partial K^{-2}/\partial x = 2MG/c^2r^2K^2$, and the result is: $A^1 = -MG/r^2K^2$. C.*f.* the Schwarzschild result is: $A^1 = -MG/r^2$. Thus the Schwarzschild acceleration is greater by a factor of K^2. This is the acceleration with respect to proper time. The acceleration w.r.t. real time t for a stationary test particle with: $dr/dt = 0$ is then: $a = d^2r/dt^2 = A^1(d\tau/dt)^2 = -MG/r^2K^4$. c.f. the Schwarzschild result is: $a = -MG/r^2k^2$.

In the more general case of multiple masses, we do not have a spherically symmetric metric, the Christoffel symbols $\Gamma^\kappa{}_{00}$ other than $\Gamma^1{}_{00}$ are not generally vanishing, and we have to go back to the more general equation (53). However we can use the special assumption at the field point $\boldsymbol{O}$, that we have chosen $x = x^1$ in the direction of $\boldsymbol{f}$. I.e. $\boldsymbol{f} = f\,\underline{\mathbf{x}}$. The partial differentials of K in other directions are zero *at this point*, and *for this point* the Christoffel symbols $\Gamma^\kappa{}_{00}$ do vanish except for $\Gamma^1{}_{00}$.

For the generalized *K scalar field*, the differential: $\partial K^{-2}/\partial x$ is given through the gradient function: $\partial K/\partial x = \nabla(K).\underline{\mathbf{x}} = K\boldsymbol{f}.\underline{\mathbf{x}}/c^2$. We have: $\partial K^{-2}/\partial x = -2\boldsymbol{f}.\underline{\mathbf{x}}/c^2K^2$. Thus the result of calculating A^1 is more generally:

$$A^1 = (c^2/g_{00})\Gamma^1{}_{00} = -\tfrac{1}{2}c^2(\partial K^{-2}/\partial x) = \boldsymbol{f}.\underline{\mathbf{x}}/K^2 \tag{56}$$

This diverges from the Newtonian acceleration by the factor $1/K^2$ (as $\boldsymbol{f}.\underline{\mathbf{x}}$ is just the Newtonian acceleration).

We can give a simple example to illustrate. Take a field-point $\boldsymbol{O}$ half-way between two masses of magnitude M and $2M$ respectively, at a distance r_0 from each.

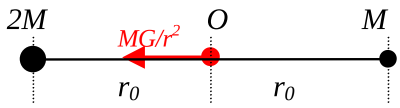

Fig. 8. Field point halfway between two unequal masses.

The resultant Newtonian acceleration is towards the larger mass with: $f = MG/r^2$. The function K in x is:

$$K = exp((G/c^2)(2M/(r_0\text{-}x)+M/(r_0+x)) \qquad (57)$$

The magnitude at the field point, where $x = 0$, is simply: $K = exp((G/c^2)(3M/r_0)$. However notice the signs of the variable x are different in the two denominators in K, so when we differentiate we get:

$$\partial K/\partial x = ((G/c^2)(2M/(r_0\text{-}x)^2)\text{-} (G/c^2)(M/(r_0 +x)^2))K \qquad (58)$$

The value at the field point where $x = 0$ is: $\partial K/\partial x_{|x=0} = (GM/c^2r_0)^2K = Kf/c^2$. And this is what should give the correct acceleration.

(56) is not a general expression for K of course, because it does not show the general dependence on the other two coordinates, y and z. The general expression is rather given by defining radial distance variables:

$$r_{(1)} = \sqrt{((r_0+x)^2+y^2+z^2)},\ r_{(2)} = \sqrt{((r_0\text{-}x)^2+y^2+z^2)} \qquad (59)$$

and then writing:

$$K = exp((G/c^2)(M/r_{(1)} +2M/r_{(2)})) \qquad (60)$$

When we differentiate this w.r.t. x, y and z we get the same result.[12] We conclude this discussion of a static superposition principle for K-gravity here.

Of course it is a further problem to generalize for *dynamic systems*. Source masses in motion must be treated as retarded sources, like moving electric charges in electrodynamics. This is beyond the scope here. This development is primarily meant to justify raising the question of the empirical difference between (1) and (2), by showing that it has a plausible generalization to a more general theory for multiple masses, with the general character of a natural law.

There may be various choices to fully generalize it, but there is a clear path to empirically testing it. We do not have to establish a full theory for this. This was also the situation when GTR was first developed: it was subject to tests against Newtonian

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[12] The result is that because we have chosen <u>***x***</u> in the direction ***f*** at the field point **O**, only the differential w.r.t. x is non-zero at that point. This is why the matrix is diagonal at the point **O** in this coordinate system. However when we have multiple masses, the differentials in all directions are involved.

gravity without fully understanding its theoretical implications. Similarly, K-gravity may be immediately tested against Schwarzschild gravity.

Before we move on to empirical tests, we note one more important theoretical point, which helps to motivate the concept.

Part 3. Testing K-gravity empirically.

10. Empirical tests.

We now consider how K-gravity compares empirically against Schwarzschild gravity. The main testing domain is solar system gravity, with the sun acting as an approximately spherical central mass for the major gravitational effect.[13] This is a weak gravity domain, and the two metrics (1) and (2) give very similar predictions for this, because (a) the functional form of the metrics are very similar, predicting very similar qualitative effects in weak gravity, and (b) k and K are very close in these weak fields, giving very similar quantitative effects. The critical term: MG/c^2r is about 10^{-8} for the gravity of the sun at roughly 1 AU.[14] Hence the terms k and K from the sun for inner planetary orbits typically differ by about: $(k - K) \approx (MG/c^2r)^2 \approx 10^{-16}$. This is not directly detectible in itself. Rather, the key difference is for *accelerations* of slow-moving bodies, which differ by the factor: $K^2 \approx 1+2MG/c^2r$.

Accelerations by the sun at orbits around 1 AU calculated with the Schwarzschild solution will be about $1+10^{-8}$ times larger than those calculated using K-gravity using the same M_{sun}/r. The accuracy to which we can measure $M_{sun}G$ is a critical limiting factor for testing this.[15] The relative uncertainty in $M_{sun}G$ is currently claimed to be around 10^{-11} [Pitjeva, 2015]. This precision would make predicted differences well-measurable in principle. However this accuracy is obtained from averaging over hundreds of thousands of measurements of planets and space probes *at different orbits*, taken over decades (Pitjeva 2015), and with averages *modelled on the assumption of ordinary GTR*. But to test K-gravity directly through accelerations we need precise measurements made at definite orbits. The relative error of 10^{-11} in $M_{sun}G$ that is claimed for averaged results is not applicable, and measurement error in single experiments is much poorer than this.

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[13] There are implications for cosmology, galaxy or star formation, black holes, etc, but these cannot be used for direct tests. If K-gravity was directly confirmed it might be supported by further observations in these domains, but there is a lot of theory-dependant modelling required.

[14] While MG/c^2r is only about 7×10^{-10} for the gravity of the Earth at the surface.

[15] The CODATA (2014) recommended value of the gravitational constant G alone has a relative uncertainty of 4.7×10^{-5}, which is poor. This is the uncertainty provided by laboratory-scale experiments, which cannot provide a test of K-gravity. Hence there is about the same error in estimates of solar or planetary masses. The accuracy of $M_{sun}G$ is much better; see Pitjeva (2015).

And it is also not a simple matter of measuring acceleration at a single orbit: we have to compare measurements of acceleration *at two different orbits.* We need to investigate whether gravitational experiments at single orbits can be made sufficiently accurately to decide between the two theories.

Before we look at this, it should be emphasized that the classic tests of GTR against Newtonian gravity do not distinguish the Schwarzschild solution from K-gravity. The relativistic phenomena of K-gravity are *qualitatively* identical to those of Schwarzschild gravity in weak gravity: bending of light, gravitational red shift, time dilation and orbital precession all work almost identically.[16] These phenomenon represent distinctive *mechanisms* in GTR that are absent from Newtonian theory, and differentiate those two theories. But there are no such qualitative differences between Schwarzschild gravity and K-gravity in weak fields, there are only fine quantitative differences. No classic tests of Schwarzschild gravity against Newtonian gravity are sensitive enough to distinguish Schwarzschild gravity from K-gravity.

We are aware of only one set of observations which is potentially precise enough, viz. the Pioneer 11 spacecraft trajectories.[17] (This was reinforced by a similar anomaly in Pioneer 10, but the data is much weaker, although it confirmed an anomaly.) This initially promised to give a sensitive quantitative measurement of gravitational acceleration over a large radial trajectory. This appears to be the only direct measurement to date of sufficient precision to directly test between the two metrics.

If the Pioneer data had unambiguously confirmed Schwarzschild gravity, K-gravity would be rejected. But instead, famously, anomalies appeared in the Pioneer data, inconsistent with Schwarzschild gravity. In an earlier study[18] it was found that these anomalies are consistent with K-gravity. So at first it appeared to confirm K-gravity.

But this evidence is now weak, because the experiment is in doubt. NASA researchers and others spent many years searching for a conventional explanation, to reconcile with GTR. It was eventually claimed by the NASA-based study [Turyshev, 2012] that the anomalies are due to anisotropic radiation from the spacecraft. Probably most GTR theorists accept this since it confirms their expectations.

But this proposed explanation is complicated, theoretical, and untested. There is no independent confirmation or experimental replication to prove the cause. It is proposed that a new experiment is the only way to decisively test the matter. First however we look briefly at the basic concept of testing the theories through measurement of accelerations, and then point out the most practical method.

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[16] In strong fields the theories diverge, e.g. there is no event horizon in K-gravity. But there is no experimental confirmation of the existence of the Schwarzschild black hole event horizon yet.

[17] E.g. [Musser, 1998], [Mbelek, 2004] [Trencevski, 2004], [Ranada, 2004 a,b], [Nieto, 2004].

[18] Unpublished preprint, 2005.

11. Acceleration differences.

The conceptual starting point is that the predicted difference in accelerations, for slowly moving test particles in weak gravity, is K^2. For Earth orbit (1 AU), K^2 is about $1+10^{-8}$. For Saturn orbit (10 AU), it is reduced to about $1+10^{-9}$.[19]

Now for experiments *at a fixed orbit*, adopting K-gravity instead of Schwarzschild gravity is essentially the same as recalibrating the estimated magnitude of $M_{sun}G$ for the sun by the factor K^2 at that orbit. Testability at first sight seems to depend upon whether acceleration measurements are made accurately enough to detect the difference between $M_{sun}G$ and $M_{sun}GK^2$. But measuring absolute accelerations at one orbit is no good because these are what we use to *determine* $M_{sun}G$ in the first place - using the assumption of the Schwarzschild metric. Any such observation at a single orbit is equally consistent with K-gravity: we would just recalibrate $M_{sun}G$ by the factor K^2. Instead we must compare *accelerations across different orbits.*[20] This point is critical and needs a brief analysis.

$M_{sun}G$ can be measured quite accurately, currently to an uncertainty of around 10^{-9} - 10^{-10}, using space probes *at a single orbit* (over a period of several orbits, i.e. several years for 1 AU).[21] So it might seem a difference of K^2 could be immediately detected in absolute accelerations. But to repeat the point above, this is wrong. $M_{sun}G$ at a single orbit may be calculated from measuring *acceleration* (of orbiting bodies or space probes), and then using the assumption of Schwarzschild gravity to infer $M_{sun}G$. But if we assumed K-gravity instead, we would just infer that $M_{sun}G$ is larger by K^2, using K for the orbit where we measured the acceleration. Note because K-gravity is weaker, we infer a *larger* $M_{sun}G$ from the same observed acceleration. $M_{sun}G$ inferred from Schwarzschild gravity from a single orbit will correspond by definition to $M_{sun}GK^2$ inferred from K-gravity.

To use acceleration measurements, we need to measure accelerations at two different orbits, and compare their values. Suppose we first determine (proper) *accelerations:* $M_{sun}G/r_1^2$ and $M_{sun}G/r_2^2$ at two different orbits, using the assumption of Schwarzschild gravity to infer $M_{sun}G$ in both cases. Their difference is:

$$\Delta_N = (M_{sun}G/r_1^2 - M_{sun}G/r_2^2) = (M_{sun}G)(1/r_1^2 - 1/r_2^2) \qquad (61)$$

We can measure this accurately to the sum of relative uncertainties in the terms. This uncertainty involves the r terms as well as $M_{sun}G$. Let us define this uncertainty (to one standard error) as: $\pm\varepsilon M_{sun}G/r_1^2$. Now this depends on the measurements at both orbits. If we do a very careful measurement at a primary orbit r_1 we may get a small

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[19] Note accelerations can be measured directly by tracking positions, in the case of space-craft, or from orbital periods and radius, in the case of orbiting bodies.

[20] The same applies to time dilation or red shift effects, but red shift effects are measured to relative error of only about 10^{-6} [Will 2014 p.13-15] and are not sensitive enough. Measurement of the precession of the perihelion of Mercury is less accurate again.

[21] [Hofmann, 2018], [Pavlov, 2016], [Pitjeva, 2015].

error, but for a good comparison we need a similarly careful measurement at r_2 and as we see next, we need r_2 to be in a suitable range to maximize the differences.

To see the predictions of K-gravity, we can *recalibrate* $M_{sun}G$ to the value: $M_{sun}GK_1^2$ at r_1, and then use this value for $M_{sun}G$ at r_2. The accelerations predicted by K-gravity will then be very close to: $M_{sun}GK_1^2/r_1^2K_1^2 = M_{sun}G/r_1^2$ and $M_{sun}GK_1^2/K_2^2r_2^2$. Their difference will then be predicted as:

$$\Delta_K = (M_{sun}G/r_1^2 - M_{sun}GK_1^2/K_2^2r_2^2) = (M_{sun}G)(1/r_1^2 - K_1^2/K_2^2r_2^2) \qquad (62)$$

Expanding the K term, this is approximately:

$$\Delta_K \approx (M_{sun}G)(1/r_1^2 - 1/r_2^2 - (1/r_2^2)(2M_{sun}G/c^2)(1/r_1 - 1/r_2)) \qquad (63)$$

Then the absolute difference: $\Delta_K - \Delta_N$ is:

$$\Delta_K - \Delta_N = -(M_{sun}G)(1/r_2^2)(2M_{sun}G/c^2)(1/r_1 - 1/r_2) \qquad (64)$$

This is the difference between the two theories for the accelerations predicted at r_2.

Define $\sigma = r_1/r_2$, so this is:

$$\Delta_K - \Delta_N = -(M_{sun}G\sigma^2/r_1^2)(2M_{sun}G/c^2r_1)(1-\sigma) \qquad (65)$$

Now we need to compare this magnitude to the error term: $\pm\varepsilon M_{sun}G/r_1^2$.

Dividing gives:

$$(\Delta_K - \Delta_N)/(\varepsilon M_{sun}G/r_1^2) = -2\sigma^2(1-\sigma)(M_{sun}G/c^2r_1)(1/\varepsilon) \qquad (66)$$

Effects become conclusively detectible when this is substantially greater than 1, say on the scale of 10.[22] Let us use this to define a *conclusively detectible limit,* so:

$\sigma^2(1-\sigma)(M_{sun}G/c^2r_1) = \pm 10\varepsilon$ Conclusively detectible limit for ε.

Now we can put in approximate numbers, for $r_1 = 1\ AU$ as: $M_{sun}G/c^2r_1 \approx 10^{-8}$, so:

$$\varepsilon \approx \pm\sigma^2(1-\sigma)10^{-9}$$

This tells us the maximum limit of ε required at different choices of $\sigma = r_1/r_2$ to achieve a clear detection of the K-gravity effect.

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[22] Theoretically 4-5 standard errors is sufficient; but the likelihood of *systematic error* through miscalculation of small effects, like radiation pressure, dust collisions, planetary effects, solar asymmetry, which are larger in the inner solar system than at the 10-80 AU orbit of the Pioneer 11, means we need a much better precision to decisively confirm or disconfirm an effect.

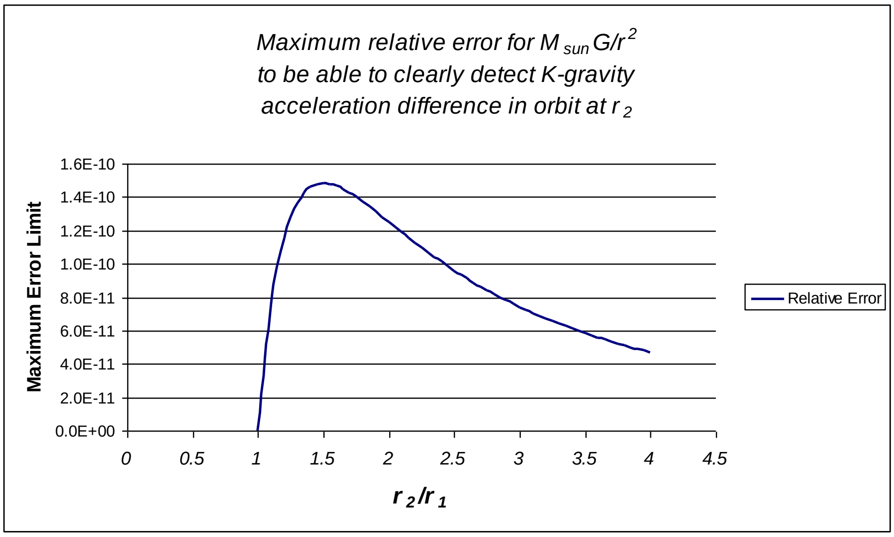

Fig. 9. Graph of ε against r_2/r_1, where the radii are two different orbits at which acceleration is measured.

This illustrates minimal precisions required in acceleration measurements to achieve experimental precision of the z-score = *±20,* for a range of the second orbit radius. The best precision is found when r_2/r_1 is 1.5. (Or inversely, *0.66*). I.e. assuming the closest orbit is at r_1 = 1 AU, the second orbit would need to be around 1.5 AU for the most sensitive experiment. Orbits from about 1.3 – 2.5 AU will be good enough if we can achieve relative measurement error better than about 10^{-10} for the acceleration measurements at both orbits. Note this error must include compensation terms for forces other than the solar gravity component, e.g. solar radiation, solar wind, dust or small particle collisions, planetary tugs, possible EM forces, oblateness of the sun, and velocity of the probe, as well as measurement error of *r* itself, and the period.

Since relative measurement error of 10^{-10} is about the present limit for measuring accelerations of space probes, it is possible for this experiment to be done, but it would take many years. Data for such a test is not available from previous experiments. It requires high-precision measurements at two appropriate distances from the sun, but these have not been done. (Without a theory to test GTR against there is no way to guess the appropriate distances for such an experiment.) There is no indication that any analysis of data from gravitational experiments has been undertaken to test this. Any inconsistencies present in current gravitational data that might confirm K-gravity have gone unexplained. Current analyses do not envisage a possibility like K-gravity, where the gravitational field *changes shape significantly compared to Schwarzschild gravity with radial distance.*[23]

[23] Variations conceived in the framework of the Paramistised Post-Newtonian formalism, e.g. [Will, 2014] do not allow for such differences, and neither do popular alternative theories of gravity.

12. A practical experiment.

There is a much better way to do the experiment than by measuring accelerations directly at two orbits; viz. by carefully observing the trajectory of a probe in radial free-fall. The optimal experiment is a probe launched at an optimal speed, that sees it traveling radially from Earth to about Jupiter orbit, over several years. Such an experiment has not been done. The tracking of the Pioneer spacecraft at more distant orbits (traveling from around 10 AU to 80 AU) provided a similar experiment in principle. However the Pioneer test is far from optimal: it is in much weaker gravity, and the speed is not optimal. But as far we know this is the only data of sufficient precision available to potentially test the hypothesis of K-gravity.

Note there are two reasons the Pioneer experiment provides a more accurate test of acceleration than any space-craft experiments at closer orbits. First it was taken over a long period of time (several decades), so that small differences in acceleration accumulate. Second, it involves a free-fall trajectory over a substantial range of *r*, from about 10 AU (when the spacecraft left Saturn's orbit) to around 80 AU. This second point is most important. The functions *K* and *k* which determine accelerations *change shape over changes of r.* It is easier to detect the predicted anomalies in radial trajectories observed over an appropriate range of *r* than to detect differences in absolute accelerations measured at two orbits directly.

This also leads to a critical realization when analyzing the effects on radial trajectories. Because K gravity is *weaker* than Schwarzschild gravity, for the same assumed $M_{sun}G/r$, we intuitively expect it to predict that probes traveling in free-fall outwards to large *r* will travel *faster* under the K-gravity metric. "*The Pioneers have been slowing down faster than predicted … some tiny extra force … must be acting on the probes, braking their outward motion."* [Musser 1998]. This is true if the cause of slowing was a non-gravitational force. But if the cause is a modified form of gravity, the opposite is the case: a weaker rather than stronger gravity is required. This is because $M_{sun}G/r$ is initially *calibrated* from the inner solar system, on the assumption of Schwarzschild gravity, and from the point of view of K-gravity this leads us to *underestimate* the magnitude of $M_{sun}G/r$ (by $1/K^2$). If K-gravity is correct, then we should *increase* the conventional magnitude of $M_{sun}G/r$ by this factor, i.e. K^2. In weak gravity, the differences between *K* and *k* are very small, and we will get almost the right acceleration predictions for K-gravity from the conventional Schwarzschild analysis - but by applying it with the larger value: $M_{sun}GK^2/r$ instead of $M_{sun}G/r$. This is what we saw in the analysis above.

So if K-gravity is correct, we should notice the spacecraft slowing down more than expected on the basis of the Schwarzschild solution. There should be an increasing delay in the expected position, exactly as first observed with the Pioneer spacecraft. Anomalies of about a 16 seconds delay in the expected journey to around 80 AU appeared, and we found a similar magnitude of difference (predicting about 12 - 18 seconds delay, the range due to uncertainties in parameters).

However as mentioned above, the Pioneer evidence has subsequently become unclear. Turyshev *et alia* [2012] claim a faint source of anisotropic heat radiation from the Pioneer spacecraft is the cause of slowing. But if such a tiny factor is able to be overlooked for 20-30 years, who knows if there are further tiny factors also overlooked? Tiny effects are amplified over a long period of time, and there are multiple possible effects to calculate, e.g. radiation and particle pressure from the sun, small planetary pulls, 'dark matter', heat anisotropy, dust collisions, so-called 'frame-dragging' effects, possible tiny EM forces, and even the Hubble expansion of the universe are on roughly the same scale. The upshot is that the analysis of the Pioneer trajectories is vulnerable to too many possible uncertainties which are *precisely in the magnitude of anomalies predicted by K-gravity* to provide any conclusive test of K-gravity or confirmation of GTR.

Experimental replication would be the only real way to resolve the question of the Pioneer anomalies. But replication of the original experiment is not feasible. However we do not have to replicate experiments exactly*:* we replicate to test for possible alternative *causes* of phenomenon. This is where having an alternative theory to test against is necessary. It lets us design variations of the original experiment, calculated to enhance anomalous effects on the hypothesis of a specific alternative cause.

A decisive experiment to test K-gravity, and simultaneously try to replicate the Pioneer phenomenon on the hypothesis that K-gravity, may be done in a timeframe of around 3-4 years, by precisely tracking a probe in free-fall traveling from roughly Earth to Jupiter orbit. The speed is optimized to amplify the anomaly predicted by K-gravity. This is a more efficient and robust experiment than trying to measure accelerations of probes at orbits of 1 AU and 1.5 AU to high precision. Calculation of optimal initial trajectory speeds and predicted effect can be easily obtained using the basic theory above.

13. Summary.

The *K*-metric is the analytic continuation of the Schwarzschild metric, and is a consistent solution in ordinary GTR, for a mass-density extending indefinitely with spherical symmetry. Such distributions of *matter* do not form in the real world, but it appears as a natural type of solution for an alternative GTR-type theory. The main argument is that this should be regarded as a real possibility, and tested empirically.

The metric in the context of GTR requires us to distinguish the point-like *inertial center of mass* from a distributed *gravitational mass*. This is somewhat analogous to quantum mechanics, where classical point-particles become spatially extended waves. Opinions will differ over whether this is a plausible concept. But unless it is shown to be positively inconsistent (e.g. by failing to have consistent definitions of momentum or energy), it cannot be decided by theoretical principles alone. It has such similar solutions to the standard theory that they can barely be distinguished. And it has strong symmetries of its own, although they will ultimately differ from the current theory. Theoretical intuitions are useful heuristic guides to theory development, and theorists become highly attached to their favored “symmetry principles”. Theorists often take the success of theories to *prove their symmetries* as general principles of nature. But we cannot decide such matters by arguing about the symmetries we prefer. Only empirical tests can decide objectively between two similar such theories.

There is also a motivation today to look for *alternatives* to ordinary GTR, and a strong suspicion that it will have to be modified in some way, because of its inconsistency with quantum mechanics, as reflected in the failure to find a unified theory. Something must change in either GTR or QM, or more likely in both, to allow a unified theory. GTR also has difficulties with singularities or discontinuities. K-gravity removes the event horizon singularity (a discontinuity), and it provides alternatives to deal with the central singularity. We also note that it only appears viable in a closed finite universe, although we did not discuss that here. When applied to a finite universe, it also leads to distinctive differences with standard cosmology – but in areas that are fraught with anomalies, so this is difficult to evaluate. But it has very significant implications beyond gravity alone.

Although the K-metric is quite different to the Schwarzschild metric, the difference does not appear to be empirically decidable from current experimental data. This may be surprising to those who think GTR is so well confirmed that there is now little possibility of it failing. But it is a mistake to think we can simply confirm a general theory like GTR *in isolation*, by doing more and more precise measurements. We cannot test a theory *unless we have some idea of where it might fail.* We have to test theories against *alternative theories* that predict precise differences that we might expect. Ordinary GTR has never been tested against K-gravity, and at present there is no way to tell which is empirically correct.

K-gravity can be tested with a fairly simple solar system experiment. This represents perhaps *the largest plausible divergence from ordinary GTR* that remains untested at present. A test would set a new limit to the accuracy of GTR. This also falls outside the main program for testing GTR, through the *parameterized post-Newtonian formalism.* ([Will, 2014]). The PPN is a model of variations of GTR, and includes tests of some well-known theories, e.g. Brans–Dicke theory, string theory or quantum loop gravity; but nothing as radical as the *K-metric*. Such a test might reject K-gravity, and confirm a new limit of the accuracy for GTR; or it might confirm K-gravity, and force us to reconstruct the fundamental theory; or it might contradict both, and reveal something quite unexpected again.

To conclude, referring to the quotation from Einstein at the start, GTR is a freakishly accurate theory, and this leads many physicists to conclude that it is irrevocable; but there may be more theoretical possibilities than first appears possible. We should distinguish the *metric theory,* which appears robust as a general description of motion once we have obtained a metric tensor, from the connection to the *stress-energy tensor* proposed in Einstein’s equation, which may have an alternative model at a more fundamental level.

Statements and declarations. Competing interests.

There are no competing interests. No funding was obtained for this study.

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