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Fragmenting modal logic

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ABSTRACT
Fragmentalism allows incompatible facts to constitute reality in an absolute manner, provided that they fail to obtain together. In recent years, the view has been extensively discussed, with a focus on its formalisation in model-theoretic terms. This paper focuses on three formalisations: Lipman's approach, the subvaluationist interpretation, and a novel view that has been so far overlooked. The aim of the paper is to explore the application of these formalisations to the alethic modal case. This logical exploration will allow us to study (i) cases of metaphysical incompatibility between *modal* facts and (ii) cases of *modal* dialetheias. In turn, this will enrich our understanding of the role of impossibility in the fragmentalist framework.

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1. Dialetheias and metaphysical incompatibility

This paper is devoted to fragmentalism, a view originally introduced by Kit Fine (2005, 2006) as a novel version of tense realism. Fragmentalism allows incompatible facts to constitute reality in an absolute manner (that is, not relative to a given standpoint), provided that they fail to obtain *together*. The only facts that are allowed to obtain together are those that are jointly compatible. By obtaining together, the latter assemble themselves into maximal coherent collections of facts, thus dividing up the world into 'fragments' of reality. While each fragment is internally coherent, facts across fragments are jointly incompatible. When understood as a theory of time, as in Fine's original application, the view describes the temporal dimension as constituted–in an absolute

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manner–by jointly incompatible tensed facts that never obtain together. But in what sense, precisely, are these facts incompatible? Before answering, let us introduce some key notions.

We will use the term '*dialetheia*' to refer to a sentence ϕ that is both true and false. Assuming that ϕ is a dialetheia, if one adopts both standard negation (ϕ is false if and only if $\neg\phi$ is true) and standard conjunction (ϕ and ψ are true if and only if $\phi \wedge \psi$ is true), then one can derive the conjunction of ϕ and $\neg\phi$, which is a *true contradiction*. True contradictions are incompatible with the *Law of Non-Contradiction*. While the exact definition of this law is a matter of debate (Priest, Berto, and Weber 2022), we will stick to the following definition:

(LNC) For every ϕ, $\phi \wedge \neg\phi$ is false.

Note that LNC is technically compatible with the existence of dialetheias, for it only prohibits the existence of true contradictions. As we will see in the next sections, once the notion of conjunction is properly devised, one can accept dialetheias while rejecting true contradictions.

Let us now return to our question: how should we understand the notion of incompatibility at work in the fragmentalist framework? Suppose that Aristotle is seated and then standing, and bear in mind that, in a nutshell, fragmentalism maintains (i) that incompatible facts can constitute reality in an absolute manner, and (ii) that reality is locally coherent, that is, no fragment of reality contains incompatible facts. Two different readings of fragmentalism are possible. The *strong* reading allows the temporal dimension to contain facts that are *logically* incompatible, in the form of dialetheias. Reality can thus be constituted by both the tensed fact that Aristotle is seated and the tensed fact that Aristotle is not seated or, equivalently, by both the tensed fact that Aristotle is not standing and the tensed fact that Aristotle is standing. Although these facts constitute reality in an absolute manner, by hypothesis no fragment is able to host both of them. This means that no contradictory fact, like the fact that Aristotle is both seated and not seated, is allowed to obtain. As a consequence, no contradiction like 'Aristotle is both seated and not seated' is allowed to be true, in accordance with LNC (Fine 2005, 282). So, according to this interpretation of fragmentalism, reality is able to host *logical impossibilities*, in the specific sense that there can be dialetheias, but not true contradictions. A little more formally, the fact that ϕ and the fact that $\neg\phi$ can constitute reality in an absolute manner, but the fact that $\phi \wedge \neg\phi$ cannot.

The *weak* reading of fragmentalism allows facts that are incompatible only in a mere *metaphysical* sense. Given that metaphysical incompatibilities do not engender logical incompatibilities, reality can be constituted by both the tensed fact that Aristotle is seated and the tensed fact that Aristotle is standing, from which one cannot infer that it is also constituted by the tensed fact that Aristotle is not seated and the tensed fact that he is not standing. So, according to this version of fragmentalism, reality is unable to host logical impossibilities, even in the form of mere dialetheias. However, there can be *metaphysical* impossibilities, meaning that there can be facts that constitute reality in an absolute manner despite their being metaphysically incompatible. A little more formally, the fact that ϕ and the fact that ψ can constitute reality in an absolute manner, even when the fact that ϕ and the fact that ψ are metaphysically incompatible.

In recent years, fragmentalism has been thoroughly discussed,[1] with a focus on pinning down Fine's original insight in formal terms. In what follows, we will limit our attention to three logical approaches. Two of them are the most prominent in the formal literature on fragmentalism: Lipman's approach and the subvaluationist interpretation.[2] Lipman's approach embraces the weak reading: *metaphysically* incompatible facts can constitute reality, but no dialetheias are possible. This view prevents reality from containing logical impossibilities, even in the form of dialetheias, by means of a properly devised negation. By contrast, subvaluationists embrace the strong reading: reality is constituted not only by metaphysically incompatible but also by *logically* incompatible facts. However, in the subvalutationist framework conjunction does not obey the rule of adjunction, so preventing true contradictions from arising. The third formalisation can be obtained from Lipman's one by properly modifying the behaviour of negation and conjunction, and has not been investigated before. This view adopts a subvaluationist approach to negation, while interpreting conjunction in a quantificational way. For reasons that will be clear in a while, we will call it the *hybrid* view.

[1]See, among others, Correia and Rosenkranz (2012, 2020), Deng (2013), Frischhut (2013), Merlo (2013, 2023), Pooley (2013), Tallant (2013), Lipman (2015, 2016, 2018, 2020), Savitt (2016), De Florio and Frigerio (2017, 2019, Chap. 6), Hofweber and Lange (2017, 2020), Loss (2017, 2023), Simon (2018), Iaquinto (2019, 2020), Slavov (2020), Torrengo and Iaquinto (2019, 2020), Fine (2020), Iaquinto and Calosi (2021), Zhan (2021), Eker (2022), Iaquinto and Torrengo (2022, Chap. 2), and Read (2022).

[2]*Smooth fragmentalism*, proposed by Simon (2018), is another notable and widely discussed interpretation of fragmentalism, according to which ϕ and $\neg\phi$ are never both true, not even in distinct fragments. As Simon did not delve into the formal details of his view, focussing instead on its application to the metaphysics of quantum mechanics, we will leave it to another occasion.

The aim of this paper is to explore the application of the three approaches to the alethic modal case. The idea of a modal analogue of fragmentalism is not new. It has been mentioned by Fine himself in introducing the view (2005: 284–285) and then preliminarily explored in more recent works (Iaquinto 2020; Zhan 2021). Modal fragmentalism–as the view is sometimes labelled–divides the modal dimension into the fragmentalist analogues of possible worlds, that is, maximal collections of worldly facts, all these modal fragments being ontologically on a par. In analogy to the temporal case, two different readings of the view are possible, depending on what notion of incompatibility is adopted. The strong reading allows the modal dimension to contain worldly facts that are logically incompatible, provided that they do not obtain together. While reality can be constituted by both the worldly fact that Aristotle is from Stagira and the worldly fact that Aristotle is not from Stagira, no contradictory fact, like the fact that Aristotle is and is not from Stagira, is allowed to obtain, so preserving LNC. The weak reading allows the modal dimension to contain metaphysically, but not logically incompatible worldly facts, like the fact that Aristotle is from Stagira and the fact that Aristotle is from New York.

The logical exploration we are about to offer, aside from its intrinsic theoretical interest, will be crucial in enriching our understanding of which metaphysical and logical impossibilities can be allowed in the fragmentalist framework. In particular, it will offer the tools to study (i) cases of metaphysical incompatibility between *modal* facts and (ii) cases of *modal* dialetheias. Although several of the topics we will cover raise delicate metaphysical questions, our main focus will be on *logical* issues, with further investigations into genuinely metaphysical matters left to another occasion. In other words, our aim is not to argue *in favour* of the fragmentation of the modal dimension. Instead, we seek to demonstrate that, whether or not the modal dimension is a fragmented place, there are at least three coherent ways to formally articulate this idea. The paper is structured as follows. In Section 2, we will focus on Lipman's approach, with particular attention to his notion of negation and conjunction. In Section 2.1, we will extend his framework to the modal case. This will allow us to capture cases of metaphysical incompatibility between *modal* facts. In Section 3, we will see how to modify Lipman's formalisation to articulate the hybrid view. The latter can be considered an intermediate position between Lipman's and the subvaluationist approach. Unlike Lipman's approach, it regards reality as hosting dialetheias. Unlike the subvaluationist approach, it allows dialetheias where ϕ is both necessarily true and not necessarily true, or both possibly true and impossibly true, while ruling out cases where ϕ is both necessarily true and necessarily not true. Section 4 will be devoted to the subvaluationist approach, which encompasses these latter cases as well. Section 5 concludes with a brief comparison of the three approaches.

The three modal languages we will explore promise to deepen our comprehension of the fragmentalist's reality. But it is important to stress that their interest goes beyond the interpretation of fragmentalism. Indeed, they can enrich our conception of alethic modal notions themselves, in particular of necessity and impossibility. We commonly conceive alethic necessity and impossibility as ranging over reality *as a whole*. If the fact that it is necessary that ϕ constitutes reality, then nowhere in reality it is the case that $\neg\phi$. Similarly, if the fact that it is impossible that ϕ constitutes reality, then nowhere in reality it is the case that ϕ. This view of necessity and impossibility is unproblematic when reality is viewed as a unified whole. However, once the idea that reality is a fragmented place is taken seriously, it becomes possible to articulate a *local* understanding of these notions. The fact that it is necessary that ϕ is allowed to constitute reality even if, somewhere, it is the case that $\neg\phi$. Likewise, the fact that it is impossible that ϕ is allowed to constitute reality even if, somewhere, it is the case that ϕ. The paper will investigate how this conception of modality impacts on the fragmentalist understanding of reality.

2. Lipman's formalisation

Let us begin with the formalisation proposed by Martin Lipman (2015, 2016, 2018), which only allows *metaphysical* impossibilities, thus excluding logical ones, even in the form of dialetheias. Lipman's idea is to understand facts that obtain together as being related by a primitive relation of *co-obtainment*. The latter can be informally understood as a 'conjunction' whose range is only limited to those facts that, by being metaphysically compatible, 'form a unified qualitative manifestation of the relevant objects, one single bit of world within which the things are a certain way' (Lipman 2015, 3127). While both the fact that Aristotle is seated and the fact that Aristotle is standing are allowed to constitute reality in an absolute manner, there is no way for them to form a 'unified qualitative manifestation' of Aristotle as *both* seated and standing; a 'single bit of world' within which Aristotle is both seated and standing cannot be the case.

Let us see how Lipman captures this idea by means of model-theoretic tools. Let $\mathcal{L}^\circ$ be the language of a standard propositional calculus, containing negation and conjunction, enriched with the co-obtainment operator $\circ$: formulae of the form $\ulcorner\phi \circ \psi\urcorner$ informally read 'ϕ insofar as ψ'. Let a model $\mathcal{M}$ be a pair $\langle\mathcal{F}, v\rangle$ where $\mathcal{F}$ is a non-empty set of fragments $f_n, f_m, \ldots$, while v is a valuation function that, given a fragment $f_i \in \mathcal{F}$, assigns to each atomic formula of $\mathcal{L}^\circ$ a truth value in {T, F}. The valutation v for atomic formulae is recursively extended to a valuation for all formulae of $\mathcal{L}^\circ$ as follows.

(1) $v_{f_i}(\neg\phi) = \mathrm{T}$ iff $v_{f_i}(\phi) = \mathrm{F}$
(2) $v_{f_i}(\phi \wedge \psi) = \mathrm{T}$ iff $v_{f_i}(\phi) = v_{f_i}(\psi) = \mathrm{T}$
(3) $v_{f_i}(\phi \circ \psi) = \mathrm{T}$ iff $v_{f_i}(\phi) = v_{f_i}(\psi) = \mathrm{T}$

Note that, by clause (2) and (3), formulae of form $\ulcorner\phi \wedge \psi\urcorner$ and $\ulcorner\phi \circ \psi\urcorner$ are assigned the same truth-conditions. We will return to the relationship between co-obtainment and conjunction in a moment, after defining the notion of truth in a model.

As we stressed in the previous section, the fragmentalist takes facts to constitute reality in an *absolute* manner, not relative to a given standpoint. The valuation function is thus inadequate to formalise the notion of constitution, for it assigns truth-values only relative to fragments, which are intuitively understood as temporal standpoints. To properly articulate the idea that constitution is absolute, a notion of *truth simpliciter*, understood as truth in a model $\mathcal{M}$, will be recursively defined as follows.

(4) $\mathcal{M} \vDash p$ iff for some fragment f_i, $v_{f_i}(p) = \mathrm{T}$, provided that p is atomic
(5) $\mathcal{M} \vDash \neg\phi$ iff $\mathcal{M} \nvDash \phi$
(6) $\mathcal{M} \vDash \phi \wedge \psi$ iff $\mathcal{M} \vDash \phi$ and $\mathcal{M} \vDash \psi$
(7) $\mathcal{M} \vDash \phi \circ \psi$ iff for some fragment f_i, $v_{f_i}(\phi \circ \psi) = \mathrm{T}$

Logical truth and logical consequence are defined in the following way.

(8) $\vDash \phi$ iff for any model $\mathcal{M}$, $\mathcal{M} \vDash \phi$
(9) $\Sigma \vDash \phi$ iff for any model $\mathcal{M}$, if $\mathcal{M} \vDash \Sigma$, then $\mathcal{M} \vDash \phi$

Clause (4) says that it is true simpliciter that p – where p is atomic – if and only if there is at least one fragment where it is true that p or, alternatively, if and only if it is *sometimes* true that p. The clause has thus a subvaluationist flavour, since it allows an atomic formula that is true in a given fragment to be true simpliciter regardless of its truth-value in the other fragments. Metaphysically speaking, the idea conveyed by this clause is that the fact that p – where p is atomic – constitutes reality in an absolute manner if and only if there is at least one fragment where the fact that p obtains.

Clause (5) says that a negation $\neg\phi$ is true simpliciter if and only if ϕ is false simpliciter. It is important to notice that, given an atomic formula p, $\neg p$ is true simpliciter if and only if there is *no* fragment where it is true that p or, alternatively, if and only if it is *always* false that p. Metaphysically speaking, this means that the fact that $\neg p$ constitutes reality in an absolute manner if and only if the fact that p obtains nowhere in reality.[3]

The reason why atomic formulae and their negations are treated differently is to prevent true contradictions. To see how the latter could arise if both a formula and its negation were allowed to be true simpliciter, let us turn our attention to clause (6). The clause says that a conjunction $\phi \wedge \psi$ is true simpliciter if and only if both ϕ and ψ are true simpliciter. Notice that in order for ϕ and ψ to be true simpliciter, they are not required to be true in the *same* fragment. Put differently, conjunction is conceived as able to 'bridge' different fragments of reality. As an important consequence of this understanding of the notion, a conjunction can be true simpliciter even when it expresses a metaphysical impossibility: 'Aristotle is seated and Aristotle is standing' is allowed to be true simpliciter, provided that 'Aristotle is seated' and 'Aristotle is standing' are true simpliciter as well. In metaphysical terms, the fact that $\phi \wedge \psi$ constitutes reality in an absolute sense if and only if the fact that ϕ and the fact that ψ constitute reality in an absolute sense, regardless of whether they belong to the same fragment, and thus regardless of their metaphysical compatibility. So in this framework, the conjunction operator no longer adheres to the intuitive meaning of conjunction, but rather serves as a tool to describe (an incoherent) reality in its entirety. As we will see shortly, the connective that comes closest to the intuitive conception of conjunction is now that of co-obtainment. Now, if both an atomic formula p and its negation $\neg p$ were allowed to be true simpliciter, nothing would prevent the contradiction $p \wedge \neg p$ from being true simpliciter as well, in violation of LNC. Clause (5) is thus adopted to avoid the truth simpliciter of a contradiction like 'Aristotle is seated and Aristotle is not seated' or 'Aristotle is standing

[3]For a metaphysical critique of this approach, see Simon (2018, 129–130) and Iaquinto and Torrengo (2022, Ch. 2).

and Aristotle is not standing', so preserving LNC: $\vDash \neg(\phi \wedge \neg\phi)$ (Lipman 2015, 3131).

Clause (7) is devoted to the co-obtainment operator. Informally, it says that it is true simpliciter that $\phi \circ \psi$ if and only if there is at least one fragment where both ϕ and ψ are true. From a metaphysical point of view: the fact that $\phi \circ \psi$ constitutes reality in an absolute manner if and only if the fact that ϕ and the fact that ψ belong to the same fragment. Co-obtainment can thus be understood–as anticipated above–as a special case of conjunction, that is, the case where the conjuncts are true in the *same* fragment. But notice that the truth simpliciter of $\phi \circ \psi$ does not guarantee the truth simpliciter of $\phi \wedge \psi$: $\phi \circ \psi \nvDash \phi \wedge \psi$. The reason is that, by clause (5), the truth of a formula ϕ in a given fragment does not guarantee its truth simpliciter, as it is clear when ϕ represents a negation. It follows that co-obtainment does not satisfy simplification: $\phi \circ \psi \nvDash \phi$ (Lipman 2015, 3130), and thus, *a fortiori*, it does not entail the conjunction of ϕ and ψ. Also, the truth simpliciter of $\phi \wedge \psi$ does not guarantee the truth simpliciter of $\phi \circ \psi$: $\phi \wedge \psi \nvDash \phi \circ \psi$. In this framework, the rule of adjunction applies to conjunction, but not to co-obtainment: $\phi, \psi \vDash \phi \wedge \psi$; $\phi, \psi \nvDash \phi \circ \psi$. Co-obtainment is thus regarded – borrowing the terminology of Calosi, Iaquinto, and Loss (*ms*) – as a *local* operator, as it focuses only on part of reality, unlike conjunction which is a *global* operator that looks at reality as a whole. The failure of adjunction for co-obtainment vindicates the idea that two incompatible facts can constitute reality absolutely speaking *without* obtaining together (Lipman 2015, 3130).

2.1. The extension to the modal case

Let us now see how to extend this framework to the modal case. First, we enrich $\mathcal{L}^{\circ}$ with the modal operator $\Box$, which stands for 'it is necessary that'. The modal operator $\Diamond$, which stands for 'it is possible that', can be defined from $\neg$ and $\Box$ is the usual way: $\Diamond\phi =_{df} \neg\Box\neg\phi$. Let us call the resulting language $\mathcal{L}^{\circ\Box}$. Second, we define the model $\mathcal{M}$ as a triple $\langle \mathcal{F}, R, v\rangle$. $\mathcal{F}$ is now understood as a non-empty set of *modal* fragments $f_n, f_m, \ldots$, that is, the fragmentalist analogues of possible worlds. Like the latter, modal fragments can be regarded as complete and coherent descriptions of the ways the world could be. They can be instantaneous, that is, spatially complete descriptions of a possible present state of the world, or temporally extended, covering both past and future. One can also stipulate that they exhaust the plenitude of possibility, meaning they cover *all* the ways the world could be. We will remain neutral on these choices, as they have substantially no impact on the formalism. R is an accessibility relation, which we characterise, as is customary, as a subset of the Cartesian product $\mathcal{F} \times \mathcal{F}$. Since we aim to model alethic modality, we assume that R is (at least) reflexive: for every fragment $f_i \in \mathcal{F}$, $f_i R f_i$. The valuation function v is defined as above, with the obvious difference that v is now taken to assign truth-values to atomic formulae in $\mathcal{L}^{\circ\Box}$. Third, we recursively extend the valutation v for atomic formulae to a valuation for all formulae of $\mathcal{L}^{\circ\Box}$ by simply adding the following clause.

(10) $v_{f_i}(\Box\phi) = \mathrm{T}$ iff for every fragment f_n such that $f_i R f_n$, $v_{f_n}(\phi) = \mathrm{T}$

Clause (10) is quite standard. It captures the idea that $\Box\phi$ is true in a fragment f_i if and only if ϕ is true in all the fragments f_n that are accessible from f_i. As is customary, one can derive the clause for the diamond operator from clause (10). This clause states that a formula $\Diamond\ \phi$ is true in a fragment f_i if and only if there exists a fragment f_n accessible from f_i such that ϕ is true in f_n:

(11) $v_{f_i}(\Diamond\phi) = \mathrm{T}$ iff for some fragment f_n such that $f_i R f_n$, $v_{f_n}(\phi) = \mathrm{T}$

Assuming that logical truth and logical consequence are defined as before, the fourth and last step is to extend the definition of truth in a model $\mathcal{M}$ to the modal operators. As anticipated in the introduction, we commonly understand alethic necessity as a *global* notion, namely, as applying to reality *in its entirety*: the fact that it is necessary that ϕ cannot constitute reality if somewhere in reality it is the case that $\neg\phi$. However, once the assumption that reality is unitary is dropped, a more nuanced understanding of necessity becomes available, where one can distinguish a global conception and a local one. According to the *local* conception, it can be true simpliciter that it is necessary that ϕ even if $\neg\phi$ is true in *some* fragment of reality. Within Lipman's framework, the local conception can be articulated as follows:[4]

(12) $\mathcal{M} \vDash \Box p$ iff for some fragment f_i, $v_{f_i}(\Box p) = \mathrm{T}$, provided that p is atomic

[4]Given the exploratory nature of this paper, we will leave a discussion of the clauses for formulae containing iterated modal operators to another occasion.

(13) $\mathcal{M} \vDash \Box\neg\phi$ iff for every fragment f_i, $v_{f_i}(\Box\neg\phi) = \mathrm{T}$
(14) $\mathcal{M} \vDash \Box(\phi \wedge \psi)$ iff $\mathcal{M} \vDash \Box\phi$ and $\mathcal{M} \vDash \Box\psi$
(15) $\mathcal{M} \vDash \Box(\phi \circ \psi)$ iff for some fragment f_i, $v_{f_i}(\Box(\phi \circ \psi)) = \mathrm{T}$

Clauses (12) and (15) capture the idea that, when ϕ stands for an atomic or a co-obtainment formula, $\Box\phi$ is true simpliciter if and only if $\Box\phi$ is true in at least one fragment, regardless of its value in the other ones. Metaphysically put, the fact that $\Box\phi$ constitutes reality in an absolute manner if and only if the fact that $\Box\phi$ obtains in at least one fragment. Clause (13) says that $\Box\neg\phi$ is true simpliciter if and only if $\Box\neg\phi$ is true in all fragments. Clause (14) says that $\Box(\phi \wedge \psi)$ is true simpliciter if and only if both $\Box\phi$ and $\Box\psi$ are true simpliciter. The reason why negations and conjunctions are treated differently from atomic and co-obtainment formulae is to ensure the validity of the rule $\Box\phi \vDash \phi$, which expresses a *defining* feature of alethic necessity. Within this framework, clauses like:

(13*) $\mathcal{M} \vDash \Box\neg\phi$ iff for some fragment f_i, $v_{f_i}(\Box\neg\phi) = \mathrm{T}$
(14*) $\mathcal{M} \vDash \Box(\phi \wedge \psi)$ iff for some fragment f_i, $v_{f_i}(\Box(\phi \wedge \psi)) = \mathrm{T}$

would be unable to validate this rule. To see why, consider a model where we have (i) a fragment f_n at which $\neg p$ is true (where p is atomic), (ii) a fragment f_m at which p is true, and (iii) the accessibility relation $R = \{\langle f_n, f_n\rangle, \langle f_m, f_m\rangle\}$. By clause (13*), $\Box\neg p$ is true simpliciter. However, by clause (5), $\neg p$ is false simpliciter. As for conjunction, consider a model where we have (i) a fragment f_n at which $p \wedge \neg q$ is true (where p and q are atomic), (ii) a fragment f_m where q is true, and (iii) the accessibility relation $R = \{\langle f_n, f_n\rangle, \langle f_m, f_m\rangle\}$. By clause (14*), $\Box(p \wedge \neg q)$ is true simpliciter. However, by clause (5), $\neg q$ is false simpliciter, and thus, by clause (6), $p \wedge \neg q$ is false simpliciter as well.

As an interesting feature of this framework, clause (6) allows for conjunctions of incompatible *modal* claims. Given two sentences, ϕ and ψ, describing two facts that are metaphysically incompatible, there can be a model where $\Box\phi \wedge \Box\psi$ is true simpliciter. However, given clause (5), a formula of form $\ulcorner\Box\phi \wedge \neg\Box\phi\urcorner$ can never be true simpliciter.

Let us now focus on the possibility operator. Within Lipman's framework, the following clauses can be offered:

(16) $\mathcal{M} \vDash \Diamond p$ iff for some fragment f_i, $v_{f_i}(\Diamond p) = \mathrm{T}$, provided that p is atomic
(17) $\mathcal{M} \vDash \Diamond\neg\phi$ iff for some fragment f_i, $v_{f_i}(\Diamond\neg\phi) = \mathrm{T}$

(18) $\mathcal{M} \vDash \Diamond(\phi \wedge \psi)$ iff $\mathcal{M} \vDash \Diamond\phi$ and $\mathcal{M} \vDash \Diamond\psi$
(19) $\mathcal{M} \vDash \Diamond(\phi \circ \psi)$ iff for some fragment f_i, $v_{f_i}(\Diamond(\phi \circ \psi)) = \mathrm{T}$

Clauses (16), (17), and (19) express the idea that, when ϕ is an atomic formula, a negation or a co-obtainment formula, $\Diamond\phi$ is true simpliciter if and only if there is at least one fragment where $\Diamond\phi$ is true. In metaphysical terms: the fact that $\Diamond\phi$ constitutes reality in an absolute manner if and only if there is at least one fragment where the fact that $\Diamond\phi$ obtains. Clause (18), instead, captures the idea that $\Diamond(\phi \wedge \psi)$ is true simpliciter if and only if both $\Diamond\phi$ and $\Diamond\psi$ are true simpliciter. The reason why conjunction is treated differently is to preserve the rule $\phi \vDash \Diamond\phi$. Analogously to the rule $\Box\phi \vDash \phi$, this rule is a non-negotiable constraint, as it captures a *defining* feature of alethic possibility. A clause like:

(18*) $\mathcal{M} \vDash \Diamond(\phi \wedge \psi)$ iff for some fragment f_i, $v_{f_i}(\Diamond(\phi \wedge \psi)) = \mathrm{T}$

would be unable to vindicate it. The proof is easy. Consider a model where we have (i) a fragment f_n at which p and $\neg q$ are true (where p and q are atomic), (ii) a fragment f_m at which $\neg p$ and q are true, and (iii) the accessibility relation $R = \{\langle f_n, f_n\rangle, \langle f_m, f_m\rangle\}$. By clause (6), $p \wedge q$ is true simpliciter. However, by clause (18*), $\Diamond(p \wedge q)$ is false simpliciter.

It is important to notice that, while alethic necessity is treated as a local notion, alethic impossibility remains *global*: if it is true simpliciter that $\neg\Diamond\phi$, then, given the reflexivity of the accessibility relation R, there is *no* fragment where it is true that ϕ.

A crucial question is whether the interdefinability of modal operators, as encoded by the following principles, applies at the level of truth simpliciter:

(P_1) $\Box\phi \vDash \neg\Diamond\neg\phi$
(P_2) $\neg\Diamond\neg\phi \vDash \Box\phi$
(P_3) $\Diamond\phi \vDash \neg\Box\neg\phi$
(P_4) $\neg\Box\neg\phi \vDash \Diamond\phi$

Principles P_1-P_4 articulate two main intuitions governing possibility and necessity. First, that a proposition is necessarily true if and only if it is impossible for it to be false (P_1-P_2). Second, that a proposition is possibly true if and only if it is not necessary for it to be false (P_3-P_4). It is thus interesting to note that, in the framework we are exploring, P_1 and P_3 are invalid. It follows that necessity cannot be defined in terms of possibility, and possibility cannot be defined in terms of necessity.

Proof.

Consider P_1 and take a model where we have (i) a fragment f_n at which the atomic formula p is true, (ii) a fragment f_m at which $\neg p$ is true, and (iii) the relation $R = \{\langle f_n, f_n\rangle, \langle f_m, f_m\rangle\}$. By clause (12), $\Box p$ is true simpliciter. Given clause (5), $\neg\Diamond\neg p$ is true simpliciter if and only if $\Diamond\neg p$ is false simpliciter. By clause (17), $\Diamond\neg p$ is false simpliciter if and only if there is no fragment where $\Diamond\neg p$ is true, that is, if and only if $\neg\Diamond\neg p$ is true in both f_n and f_m. But $\Diamond\neg p$ is true in f_m. Therefore, $\neg\Diamond\neg p$ is false simpliciter.

As for P_3, consider a model where we have (i) a fragment f_n where p and $\neg q$ are true (where p and q are atomic), (ii) a fragment f_m where q and $\neg p$ are true, and (iii) the accessibility relation $R = \{\langle f_n, f_n\rangle, \langle f_m, f_m\rangle\}$. By clause (16), $\Diamond p$ and $\Diamond q$ are true simpliciter, and then, by clause (18), $\Diamond(p \wedge q)$ is true simpliciter as well. Given clause (5), $\neg\Box\neg(p \wedge q)$ is true simpliciter if and only if $\Box\neg(p \wedge q)$ is false simpliciter. By clause (13), $\Box\neg(p \wedge q)$ is false simpliciter if and only if there is at least one fragment where $\Diamond(p \wedge q)$ is true. But $\Diamond(p \wedge q)$ is false both in f_n and in f_m, and therefore $\neg\Box\neg(p \wedge q)$ is false simpliciter.

The invalidity of P_1 and P_3 is a strong departure from the canonical conception of alethic modality. The interdefinability of possibility and necessity is deeply intuitive, to the point that one might think of P_1 and P_3 as no less negotiable than the rules $\Box\phi \vDash \phi$ and $\phi \vDash \Diamond\phi$. The failure of these principles – one might insist – suggests that $\Box$ and $\Diamond$, as characterised in this formalisation, fall short of capturing necessity and possibility.[5] In response, proponents of this formalisation would arguably defend a revisionary approach to the ordinary conception of alethic modality: rejecting P_1 and P_3 is counterintuitive, but it becomes acceptable when the metaphysics that motivates the framework is given due priority over our intuitions. In this exploratory paper, at any rate, we will show that fragmentalism is not necessarily incompatible with principles P_1-P_4: while the hybrid view rejects P_1 and P_2 (as we will prove in Section 3), the subvaluationist approach validates them all (Section 4). Readers who want to preserve the interdefinability of possibility and necessity might view this result as a reason to favour the subvaluationist approach.

[5]We would like to thank an anonymous reviewer for pushing us to develop this point.

3. The hybrid view

Let us now discuss the hybrid view. The language, model, and clauses for extending the valuation to all formulae, as well as logical truth and consequence, remain the same as in Section 2.1. We also stick to clause (4) for the truth simpliciter of atomic formulae and to clause (7) for the truth simpliciter of co-obtainment. Let us now focus on clauses (5) and (6), which we repeat below for reader's convenience:

(5) $\mathcal{M} \vDash \neg\phi$ iff $\mathcal{M} \nvDash \phi$
(6) $\mathcal{M} \vDash \phi \wedge \psi$ iff $\mathcal{M} \vDash \phi$ and $\mathcal{M} \vDash \psi$

As previously discussed, by clause (5), the negation of an atomic formula p is true simpliciter if and only if there is no fragment where p is true. The purpose of this clause – as we saw – is to prevent true contradictions from arising when clause (6) is adopted.

Now, suppose that the mere existence of a fragment where $\neg\phi$ is true is necessary and sufficient to conclude that it is true simpliciter that $\neg\phi$:

(5*) $\mathcal{M} \vDash \neg\phi$ iff for some fragment f_i, $v_{f_i}(\neg\phi) = \mathrm{T}$

If clauses (5*) were adopted, reality would be able to contain dialetheias. But given clause (6), the existence of dialetheias would lead to true contradictions, in violation of LNC. So the only way to maintain clause (5*) and uphold LNC is by defining conjunction in a manner that excludes true contradictions. There are two possible approaches to achieve this. The first approach is to consider $\phi \wedge \psi$ as true simpliciter if and only if there exists at least one fragment where $\phi \wedge \psi$ is true. In the next section, we will see that this is the option embraced by the subvaluationist. The second approach, the one we would like to explore in this section, is to consider $\phi \wedge \psi$ as true simpliciter if and only if $\phi \wedge \psi$ is true in all fragments:

(6*) $\mathcal{M} \vDash \phi \wedge \psi$ iff for every fragment f_i, $v_{f_i}(\phi \wedge \psi) = \mathrm{T}$

Clause (6*) conveys a quantificational understanding of conjunction, meaning that it involves universal quantification over fragments. This approach is an interesting combination of Lipman's and the subvaluationist formalisation: clause (5*) treats negation in the same way as the subvaluationist approach (as we will see in the next section), while preserving Lipman's co-obtainment operator, which is usually dropped by the subvaluationist. This is the reason why we labelled it 'hybrid view'.

Clause (6*) prevents any conjunction whose conjuncts are (logically or metaphysically) incompatible from being true, for if a conjunct is true in a fragment, then the other conjunct must also be true in that fragment. In other words, in contrast to Lipman's account, conjunction does not obey the rule of adjunction: $\phi, \psi \nvDash \phi \wedge \psi$. Notice also that, with the adoption of clause (5*), the co-obtainment operator vindicates the rule of simplification: $\phi \circ \psi \vDash \phi, \psi$, a rule that is not validated in Lipman's framework, as we have seen.

As for the necessity operator, we will keep clauses (12) and (15), while replacing clauses (13) and (14) with (13**) and (14**):

(12) $\mathcal{M} \vDash \Box p$ iff for some fragment f_i, $v_{f_i}(\Box p) = \text{T}$, provided that p is atomic
(13**) $\mathcal{M} \vDash \Box\neg\phi$ iff $\mathcal{M} \nvDash \Box\phi$
(14**) $\mathcal{M} \vDash \Box(\phi \wedge \psi)$ iff for every fragment f_i, $v_{f_i}(\Box(\phi \wedge \psi)) = \text{T}$
(15) $\mathcal{M} \vDash \Box(\phi \circ \psi)$ iff for some fragment f_i, $v_{f_i}(\Box(\phi \circ \psi)) = \text{T}$

Once clause (5*) is adopted, the rule $\Box\phi \vDash \phi$ can be preserved without the strict requirement, as per clause (13), that $\Box\neg\phi$ be true in every fragment. To impose a weaker condition, one option is to adopt clause (13*), which instead requires the existence of at least one fragment where $\Box\neg\phi$. As we will see in next section, this is how $\Box\neg\phi$ is treated by the subvaluationist. Another option, the one we will now explore, is to adopt clause (13**), which requires $\Box\phi$ to be false simpliciter. Also the adoption of clause (14**) has to do with the rule $\Box\phi \vDash \phi$. It is easy to see that, within this framework, clause (14) would lead to its failure. Consider a model where we have (i) a fragment f_n at which p and $\neg q$ are true (where p and q are atomic), (ii) a fragment f_m at which q is true, and (iii) the accessibility relation $R = \{\langle f_n, f_n\rangle, \langle f_m, f_m\rangle\}$. By clause (12), $\Box p$ and $\Box q$ are both true simpliciter, and then, by clause (14), $\Box(p \wedge q)$ is true simpliciter as well. By clause (6*), $p \wedge q$ is true if and only if there is no fragment where it is false. But $p \wedge q$ is false in f_n, and therefore it is false simpliciter.

Let us now turn to the possibility operator. We will maintain clauses (16), (17), and (19), which we repeat below for reader's convenience:

(16) $\mathcal{M} \vDash \Diamond p$ iff for some fragment f_i, $v_{f_i}(\Diamond p) = \text{T}$, provided that p is atomic
(17) $\mathcal{M} \vDash \Diamond\neg\phi$ iff for some fragment f_i, $v_{f_i}(\Diamond\neg\phi) = \text{T}$

(19) $\mathcal{M} \vDash \Diamond(\phi \circ \psi)$ iff for some fragment f_i, $v_{f_i}(\Diamond(\phi \circ \psi)) = \mathrm{T}$

Now that, by clause (6*), a conjunction cannot be true simpliciter without being true in all fragments, there is little use in keeping clause (18). To preserve the rule $\phi \vDash \Diamond\phi$, it is enough to adopt a clause analogous in spirit to clauses (16), (17), and (19). In place of clause (18), will will then adopt the following clause:

(18*) $\mathcal{M} \vDash \Diamond(\phi \wedge \psi)$ iff for some fragment f_i, $v_{f_i}(\Diamond(\phi \wedge \psi)) = \mathrm{T}$

The clause says that $\Diamond(\phi \wedge \psi)$ is true simpliciter if and only if there is at least one fragment where $\Diamond(\phi \wedge \psi)$ is true. Once clause (18*) is adopted, all the clauses for the possibility operator can be replaced by the following one, which applies to every formula ϕ, whether atomic or not:

(20) $\mathcal{M} \vDash \Diamond\phi$ iff for some fragment f_i, $v_{f_i}(\Diamond\phi) = \mathrm{T}$

It is important to note that the truth simpliciter of $\neg\Diamond\phi$ does *not* exclude the existence of a fragment where ϕ is true. Thus, this framework treats not only alethic necessity but also alethic impossibility as a local notion.

Note also that, as anticipated in the previous section, this framework invalidates principles P_1 and P_2.

Proof.

Consider P_1 and take a model where we have (i) a fragment f_n at which p is true (where p is atomic), (ii) a fragment f_m at which $\neg p$ is true, and (iii) the accessibility relation $R = \{\langle f_n, f_n\rangle, \langle f_n, f_m\rangle, \langle f_m, f_n\rangle, \langle f_m, f_m\rangle\}$. The formula $\Diamond\neg p$ is true both in f_n and in f_m. Thus, by clause (13**), $\Box\neg p$ is true simpliciter. The formula $\neg\Diamond\neg\neg p$ is true simpliciter if and only if there is at least one fragment where $\Box\neg p$ is true. But $\Box\neg p$ is false both in f_n and in f_m. Therefore, $\neg\Diamond\neg\neg p$ is false simpliciter.

As for P_2, take a model where we have (i) a fragment f_n at which $\neg p$ is true (where p is atomic), (ii) a fragment f_m at which p is true, and (iii) the accessibility relation $R = \{\langle f_n, f_n\rangle, \langle f_m, f_m\rangle\}$. The formula $\neg\Diamond\neg\neg p$ is true in f_n and thus, by clause (5*), is true simpliciter. By clause (13**), $\Box\neg p$ is true simpliciter if and only if $\Box p$ is false in every fragment. But $\Box p$ is true in f_m, and therefore $\Box\neg p$ is false simpliciter.

Interestingly, if clauses (12), (13**), (14**), (15), and (20) are adopted, then reality is allowed to contain *modal* dialetheias. This means that, for some ϕ, it can be true simpliciter both that $\neg\Diamond\phi$ and that $\Diamond\phi$, or true simpliciter both that $\Box\phi$ and that $\neg\Box\phi$, although it cannot be true simpliciter that $\neg\Diamond\phi \wedge \Diamond\phi$ or that $\Box\phi \wedge \neg\Box\phi$. It is even possible a model where it is true simpliciter both that $\Box\phi$ and that $\neg\Diamond\phi$, although it cannot be true simpliciter that $\Box\phi \wedge \neg\Diamond\phi$. This is certainly far from our intuitive view of reality. While focussing on formal details, however, it is crucial to keep an eye on the bigger picture behind the hybrid view. This approach attempts to combine the thesis that the modal dimension is a fragmented place with the idea that reality can contain facts that are not only metaphysically but also logically incompatible. It is the combination of these two tenets that guides the hybrid view toward a radical form of fragmentation. But however radical, given clause (13**), there can be *no* model where it is true simpliciter both that $\Box\phi$ and $\Box\neg\phi$. The framework could accommodate their truth only by replacing clause (13**) with clause (13*), which would lead to the equivalence between $\Box\neg\phi$ and $\neg\Diamond\phi$. As we will see in the next section, a fully developed subvaluationist approach is bound to treat $\Box\neg\phi$ in accordance with clause (13*), and it is thus unable to prevent models where both $\Box\phi$ and $\Box\neg\phi$ are true simpliciter.

4. Going fully subvaluationist

This section is devoted to yet another way to articulate the view that reality can host dialetheias, while avoiding true contradictions. In short, the idea is to opt for a full-blown version of subvaluationism (Iaquinto and Torrengo 2022; Loss 2017; Torrengo and Iaquinto 2020). This means that any formula ϕ, be it atomic or not, is true simpliciter if and only if there is at least one fragment where ϕ is true.

The language of this view is simpler compared to the previous ones. This is because when conjunction is given a subvaluationist clause, it becomes able to perform the same function as the co-obtainment operator. For this reason, as is customary in the literature, the latter will be now replaced by the former. More precisely, let us call $\mathcal{L}$ the language of a standard propositional calculus, containing negation and conjunction. Let a model $\mathcal{M}$ be a pair $\langle \mathcal{F}, v \rangle$ where $\mathcal{F}$ is defined as in Section 2, and v is a valuation function that, given a fragment in $\mathcal{F}$, assigns to each atomic formula in $\mathcal{L}$ a truth value in {T, F}. As for the extension of the valuation v for atomic formulae of $\mathcal{L}$ to a valuation for all formulae of $\mathcal{L}$, only clauses (1) and (2) are needed.

Truth in a model $\mathcal{M}$ is non-recursively defined as follows.

(21) $\mathcal{M} \vDash \phi$ iff for some fragment f_i, $v_{f_i}(\phi) = \mathrm{T}$

Logical truth and logical consequence are defined as in the previous sections. Clause (21) captures the idea that, in order for the fact that ϕ to constitute reality absolutely speaking, it is necessary and sufficient that there is at least one fragment where the fact that ϕ obtains, so allowing reality to be constituted by both the fact that ϕ and the fact that $\neg\phi$. Once again, the idea that reality can contain logical impossibilities is vindicated by means of dialetheias. However, there is no way to evaluate a contradiction as true, for conjunction does not obey the rule of adjunction: $\phi, \psi \nvDash \phi \wedge \psi$. Indeed, similarly to the hybrid view, reality is unable to contain any conjunction with incompatible conjuncts, regardless of whether they are logically or metaphysically incompatible. This is crucial in vindicating the idea that reality is a fragmented place. Even if the fact that ϕ and the fact that ψ constitute reality absolutely speaking, there is no guarantee that the fact that $\phi \wedge \psi$ constitutes reality as well. But the two facts obtain together only if the fact that $\phi \wedge \psi$ constitutes reality. It follows that, even if the fact that ϕ and the fact that ψ constitute reality absolutely speaking, there is no guarantee that the fact that ϕ and the fact that ψ obtain together. It is worth noting that, in this framework, conjunction obeys the rule of simplification: $\phi \wedge \psi \vDash \phi, \psi$, a rule that fails – as we saw in Section 2 – for Lipman's co-obtainment operator.

The extension to the modal case is plain and simple. First, we enrich $\mathcal{L}$ with the necessity operator $\Box$, from which one derives the possibility operator $\Diamond$ in the usual way. Let us call the resulting language $\mathcal{L}^{\Box}$. Second, we let a model $\mathcal{M}$ be a triple $\langle \mathcal{F}, R, v \rangle$ whose members are defined as in the previous sections, with the sole, obvious exception that the evaluation function v is now taken to assign truth-values to atomic formulae in $\mathcal{L}^{\Box}$. As for the extension of the valuation v for atomic formulae of $\mathcal{L}^{\Box}$ to a valuation for all formulae of $\mathcal{L}^{\Box}$, we add clause (10), from which one can derive clause (11). Truth in a model $\mathcal{M}$, logical truth, and logical consequence remain the same.

Clause (21) validates all principles P_1-P_4, thus allowing both the definition of necessity in terms of possibility and the definition of possibility in terms of necessity. As anticipated in Section 2, one might regard this feature of the subvaluationist approach as an important advantage over the others, as principles P_1-P_4 articulate a strong intuition on alethic modality. But there is another notable feature. Clause (21) not only allows for models where both $\neg\Diamond\phi$ and $\Diamond\phi$, or $\Box\phi$ and $\neg\Box\phi$, or even $\Box\phi$ and $\neg\Diamond\phi$ are true simpliciter, without their conjunction being true as well, but it also leads to models where both $\Box\phi$ and $\Box\neg\phi$ are true simpliciter, without their conjunction being true as well. As we saw, the hybrid view rules out models where both $\Box\phi$ and $\Box\neg\phi$ are true simpliciter, as it relies on clause (13**). Still, it could potentially accommodate such models if it were to adopt clause (13*). In contrast, once clause (21) is adopted, accepting the existence of these models becomes the *only* available option. For the sake of a little drama: the subvaluationist approach takes the fragmentation of reality to its extreme.

5. Taking stock

Now that we have presented all the relevant features of the three formalisations, let us proceed to a final, brief comparison. One way to understand the fragmentation of the modal dimension is to say that the latter cannot contain any logical impossibilities, including dialetheias, but it can host metaphysical impossibilities, in the sense that there can be true conjunctions, like 'Aristotle is from Stagira and Aristotle is from New York', whose conjuncts formalise metaphysically incompatible facts. When two or more facts are metaphysically compatible, they form collections of co-obtaining facts, which fragment reality into jointly incompatible portions. To capture this view, the modal version of Lipman's approach resorts to: (i) a co-obtainment connective, which acts as a conjunction whose 'range' is only limited to a given fragment, (ii) a conjunction that can combine formulae from distinct fragments, and (iii) a negation that, aptly devised, prevents reality from containing true contradictions. Among the metaphysical impossibilities that reality is able to host, one can find true conjunctions of incompatible *modal* claims: when ϕ and ψ formalise metaphysically incompatible facts, there can be a model where $\Box\phi \wedge \Box\psi$ is true simpliciter. However, by clause (5), there is no way to evaluate $\Box\phi \wedge \neg\Box\phi$ as true simpliciter.

As seen in presenting the hybrid view, an alternative view is also possible, where reality can contain logical impossibilities in the form of dialetheias, like 'Aristotle is from Stagira' and 'Aristotle is not from Stagira'. One way to articulate this idea is to adopt, in addition to the co-obtainment connective, (i) a subvaluationist negation and (ii) a conjunction that, by being quantificational, prevents reality from containing true contradictions. When this approach is extended to the modal case, in virtue of clause (5*), it allows *modal* dialetheias like $\Box\phi$ and $\neg\Box\phi$, or $\Diamond\phi$ and $\neg\Diamond\phi$. It is even possible that both $\Box\phi$ and $\neg\Diamond\phi$ are true simpliciter.

However, as per clause (13**), it forbids models where both $\Box\phi$ and $\Box\neg\phi$ are true simpliciter.

The other way to articulate the idea that reality can host dialetheias is to opt for a fully subvaluationist approach. Within this framework, (i) negation behaves as in the hybrid view, while (ii) the co-obtainment connective is replaced by a conjunction that, by being non-adjunctive, forbids true contradictions. The adoption of clause (21) leads to the most radical picture of reality: in addition to modal dialetheias and cases where both $\Box\phi$ and $\neg\Diamond\phi$ are true simpliciter, the modal dimension is now allowed to contain both the fact that $\Box\phi$ and the fact that $\Box\neg\phi$.

Which approach should be preferred is a question that cannot be settled on purely logical grounds. Rather, it crucially depends on one's *metaphysical* stance on impossibility. Specifically, it is the idea that reality can accommodate metaphysical impossibilities but not logical impossibilities that drives the acceptance of clause (5). The adoption of clause (5*), as in the hybrid view, or clause (21), as in the subvaluationist approach, is analogously driven by the idea that reality can host modal dialetheias. The existence of such entities is a substantive question, and it is beyond the scope of our exploration. Metaphysics will have the final say about whether to allow them in the realm of existence. In the meantime, we hope we convinced the reader that the fragmentation of the modal dimension can be coherently articulated in at least three ways.

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