

# The repugnant conclusion can be avoided with moral intuitions intact: A lesson in order.

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## **Abstract**

The repugnant conclusion poses a conundrum in population ethics that has evaded satisfactory solution for four decades. In this article, I show that the repugnant conclusion can be avoided without sacrificing any moral intuitions. The resulting framework satisfies Parfit's requirements for 'theory X'. This is achieved using non-Archimedean orders, which admit the possibility of pairs of goods for which no amount of one is better than a single unit of the other. I show that with minimal assumptions, not only are such goods sensible, they are compulsory. I show that utilitarianism and expected utility theory in their canonical forms are not in general suitable in this setting, and using these tools naively can lead to ethical errors that are arbitrarily serious. Multi-dimensional lexicographic expected utility representations may be required. I use fuzzy sets to show that there needn't be a clear boundary separating goods that are not Archimedean equivalent. This may be unavoidable due to intrinsic physical limits on the ability to discriminate between different goods arising from e.g. quantum mechanics. Limited discriminatory power causes preferences to be non-transitive, which can resolve problems related to so-called 'fanaticism' and 'recklessness' wherein 'rational' decision-makers counterintuitively prefer arbitrary long shot bets to a guaranteed lesser outcome.

# 1 Introduction

The repugnant conclusion is the idea that for any finite population of lives of arbitrarily high quality, there is a larger population consisting of lives that are barely worth living that is ethically preferable. It has been argued that this conclusion follows if one accepts that a small reduction in the quality of life for a given population may be ‘compensated’ by adding more people. It is claimed that this reasoning can be applied recursively to make ethical improvements at each step, ending with a larger population of lives that are each barely worth living.[1, 2, 3]

Many attempts have been made to avoid the repugnant conclusion, but they all appear to require accepting ethical positions that are often perceived to be as counterintuitive or undesirable as the repugnant conclusion itself. For a summary of such attempts and their issues, see [3]. This state of affairs has inspired a number of ‘impossibility theorems’ that purport to prove there is no theory of population ethics satisfying an intuitively desirable set of axioms that includes avoiding the repugnant conclusion.[4, 5, 6, 7, 8]

Previous work has shown that non-Archimedean orders can evade these impossibility theorems.[9, 10] This entails the existence of a pair of goods  $x, y$  such that  $y$  is better than any number of  $x$ . In the literature,  $y$  is sometimes referred to as ‘superior’, or ‘lexically superior’, to  $x$ .

However, this leaves open the questions of whether non-Archimedean orders necessarily violate other desirable conditions that make them unsuitable for population ethics, and whether a concrete example of a non-Archimedean theory can be provided that avoids violating any moral intuitions including

avoiding the repugnant conclusion. In this article, I explicitly construct such a theory. I also show that if moral preferences between two populations are unchanged by scaling both up by the same factor, or by adding another identical population to both, non-Archimedean equivalent goods must necessarily exist. Critical to these insights is a proper formal mathematical treatment of the subject.

First, I introduce an appropriate mathematical structure for population ethics - ordered real vector spaces. I show that any totally ordered real vector space of dimension  $> 1$  is non-Archimedean. The result is that there are as many distinct Archimedean classes as there are levels of life quality. The conclusion is that not only are non-Archimedean goods sensible, they are compulsory.

Next, I show that non-Archimedean total orders on real vector spaces cannot be represented on the real numbers, with the immediate consequence that utilitarianism and expected-utility theory in their canonical forms are not generally suitable in population ethics. Generalisations of the von Neumann Morgenstern theorem that weaken the continuity/Archimedean axiom lead to multi-dimensional lexicographic expected utility representations.

I demonstrate that the repugnant conclusion can be avoided with moral intuitions intact using non-Archimedean orders. In particular, there need not be pairs of goods that are adjacent to each other in the ordering that are in different Archimedean classes. This result does not rely on any particular interpretation of concepts that may not have a precise definition, such as life quality or a life barely worth living, and nor does it rely on any appeal to 'vagueness'. The key innovation compared to previous work is an appropriate

handling of orders on infinite sets. In particular, generalising to transfinite sequences that are indexed by ordinal numbers as opposed to natural numbers allows for the possibility of elements that do not have a predecessor. The entire set of possible populations is ordered across multiple Archimedean classes without any two adjacent elements being non-Archimedean equivalent.

I use a fuzzy set construction to allow for the possibility that there may not be a precise apparent border between non-Archimedean equivalent goods. Such limitations are unavoidable if there are intrinsic bounds on the precision with which physical quantities can be measured stemming from e.g. uncertainty principles in quantum mechanics. In a multi-dimensional context, these limitations lead to non-transitive preferences. This can resolve problems related to so-called ‘fanaticism’ and ‘recklessness’, in which a decision-maker acting in accordance with a seemingly reasonable set of axioms may counterintuitively prefer bets that give a sufficiently large payoff with arbitrarily small probability to a lesser, sure payoff.[11, 12, 13]

## 2 The mathematics of population ethics

**Definition 1. Partial/total order.** Consider the following properties of a binary relation  $\geq$  on a set  $S$  that hold  $\forall x, y, z \in S$ ,

$$x \geq x \qquad \text{Reflexive} \qquad (2.1)$$

$$y \geq x \text{ and } x \geq y \Rightarrow y = x \qquad \text{Antisymmetric} \qquad (2.2)$$

$$z \geq y \text{ and } y \geq x \Rightarrow z \geq x \qquad \text{Transitive} \qquad (2.3)$$

$$x \geq y \text{ or } y \geq x \qquad \text{Complete.} \qquad (2.4)$$

The relation  $\geq$  is called a partial order if it satisfies 2.1-2.3, and a total order if it satisfies 2.1-2.4.

Let  $V$  be a real vector space, that may be finite- or infinite-dimensional. Let  $B$  be a basis for  $V$ . We may think of each element of  $B$  as corresponding to a life year with different levels of quality. For any  $n \geq 0$  and  $v \in B$ ,  $nv$  will correspond to  $n$  life years of quality  $v$ .

**Definition 2. Ordered real vector space.** An ordered real vector space is a pair  $(V, \geq)$  where  $V$  is a real vector space and  $\geq$  is a partial order on  $V$ , such that  $\forall x, y, z \in V, \lambda > 0$ ,

$$y \geq x \Rightarrow y + z \geq x + z \qquad (2.5)$$

$$y \geq x \Rightarrow \lambda y \geq \lambda x. \qquad (2.6)$$

$(V, \geq)$  will be called a totally ordered real vector space if  $\geq$  is a total order.

This captures the idea for any two populations, adding some other population or  $\lambda$ -fold replication does not affect the ordering.

Note that the structure of an ordered real vector space would not be expedient in e.g. consumer choice theory. One can imagine a pair of goods  $x$  and  $z$  (tea bags and milk) that complement each other, along with another pair of goods  $y$  and  $z$  (orange juice and milk) that don't, in a way that could lead to violations of 2.5. In the population ethics setting, however, it is useful for any inter-personal or inter-population complementarities to be subsumed under quality of life. This can be achieved by e.g. interpreting a basis vector as a year of life of a given quality spent in a virtual reality machine that is indistinguishable from reality. Then, for example, a year of life enhanced by the company of one's family needn't require vectors representing the lives of family members. In this way, any complementarities between lives can be captured through an appropriate interpretation of the basis vectors. Further, this interpretation maps any possible population with any profile of life qualities onto an element of the vector space. Putting a total order on this space can then allow any population ethics question in principle to be answered.

2.6 implies scale invariance. In the consumer choice setting, one can imagine two goods  $y$  and  $x$  (cake and bread) such that one unit of  $y$  is preferred to  $x$ , but there is some  $\lambda > 1$  such that  $\lambda$  units of  $x$  is preferred to  $\lambda$  units of  $y$ . In the population ethics setting, however, it is reasonable to take it as axiomatic that if life quality  $v_2$  is better than life quality  $v_1$ , then  $\lambda$  life years of quality  $v_2$  is better than  $\lambda$  life years of quality  $v_1$ .

Translation invariance 2.5 allows us to interpret a positive number of lives as a gain, and a negative number of lives as a loss, relative to some status quo. Informally speaking, for any prospective loss we wish to consider, we may imagine a population large enough that such a loss is possible, and

assign it the element 0. This is possible because the ordering is preserved by translations. More formally, we may consider the affine space that  $V$  is associated to.

Finally, the antisymmetry property of partial orders 2.2 precludes indifference between two distinct elements. This requirement can be relaxed. Consider two distinct elements  $x, y$  such that  $y \geq x$  and  $x \geq y$ . We will say that  $x \simeq y$ . Then 2.5 and 2.6 imply that  $\simeq$  is preserved by scaling and translation.  $x, y$  can be taken to be basis vectors, and the equivalence relation  $\simeq$  can be used to ‘collapse’  $x$  and  $y$  into each other so that the order is once again antisymmetric. Elements of an ordered real vector space may therefore be thought of as canonical representatives of a class of vectors that are equivalent under  $\simeq$ .

**Definition 3. Proper cone.** A proper cone  $C$  in a real vector space  $V$  is a subset  $C \subseteq V$  such that,

$$C + C \subseteq C \tag{2.7}$$

$$\lambda C \subseteq C \quad \forall \lambda > 0 \tag{2.8}$$

$$C \cap -C = \{0\}. \tag{2.9}$$

Here, addition and multiplication for subsets of a vector space are defined by  $S_1 + S_2 = \{s_1 + s_2 \mid s_1 \in S_1, s_2 \in S_2\}$  and  $\lambda S = \{\lambda s \mid s \in S\}$ . Proper cones must have their vertex at 0. Examples of non-proper and proper cones in  $\mathbb{R}^3$  are shown in Figure 1.

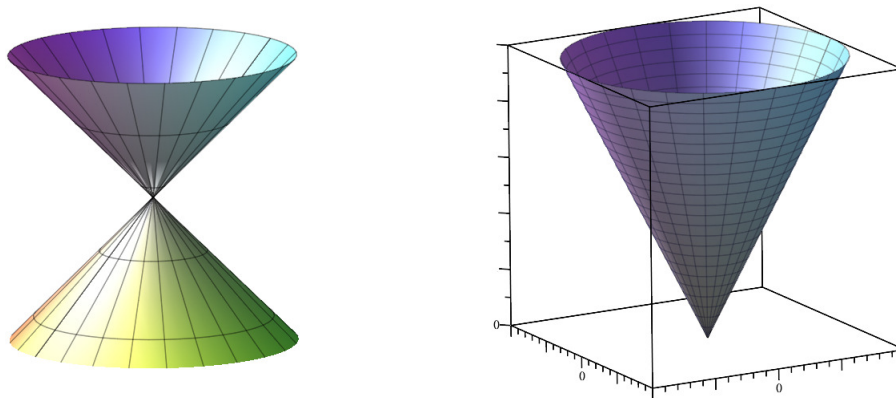


Figure 1: **Left:** A non-proper cone in  $\mathbb{R}^3$ . **Right:** A proper cone in  $\mathbb{R}^3$ .

There is a 1 – 1 correspondence between proper cones  $C$  in a real vector space  $V$ , and ordered real vector spaces  $(V, \geq)$ , given by defining  $y \geq x$  iff  $(y - x) \in C$ .  $C$  will be called the positive cone of  $(V, \geq)$ , and elements in  $C$  will be called positive.

**Definition 4. Maximal proper cone.** A proper cone  $C$  in a real vector space  $V$  is called maximal if there is no proper cone  $C' \subseteq V$  such that  $C \subset C'$ .

**Definition 5. Archimedean equivalence.** Two elements  $x, y \in V$  of an ordered real vector space  $(V, \geq)$  will be said to be Archimedean equivalent if there exists  $n, m \in \mathbb{N}$  such that

$$n|x| \geq |y| \tag{2.10}$$

$$m|y| \geq |x|, \tag{2.11}$$

where  $|x| := \max(x, -x)$ . I will use the notation  $y \gg x$  if  $y \geq x$  and  $x, y$



are not Archimedean equivalent.  $(V, \geq)$  will be called Archimedean if all its elements are Archimedean equivalent.

**Proposition 1.** Let  $x, y \in V$  be elements of an ordered real vector space  $(V, \geq)$ . If  $y \gg x$ , then  $ny \gg x \forall n > 0$ .

*Proof.* From the definition 5 of Archimedean equivalence, we have  $|y| \geq m|x| \forall m \in \mathbb{N}$ . Assume that there exists  $n > 0, m \in \mathbb{N}$  such that  $m|x| \geq |ny|$ . Then  $\frac{m}{n}|x| \geq |y|$ , which is a contradiction.  $\square$

This implies that if one unit of  $y$  is better than  $x$  and  $y$  is not Archimedean equivalent to  $x$ , then any strictly positive number of units of  $y$  is better than  $x$ .

**Proposition 2.** There is a 1 – 1 correspondence between maximal proper cones in a real vector space  $V$ , and totally ordered real vector spaces  $(V, \geq)$ .

*Proof.* Let  $(V, \geq)$  be an ordered real vector space, where  $\geq$  is defined by  $y \geq x$  iff  $(y - x) \in C$ , and  $C$  is a maximal proper cone in  $V$ . Assume that  $x$  and  $y$  are two elements that are not related, i.e. for which  $\geq$  is not defined. This implies that  $(y - x), (x - y) \notin C$ . Then the convex hull of the set  $C \cup \{\lambda(y - x) \mid \lambda > 0\}$  is a proper cone with  $C$  as a proper subset, which is a contradiction. Every totally ordered vector space on  $V$  has a maximal proper cone  $C$  consisting of all elements  $\geq 0$ . From 2, this total order is such that  $y \geq x$  iff  $(y - x) \in C$ .  $\square$

**Proposition 3.** Every totally ordered real vector space of dimension  $> 1$  is non-Archimedean.

*Proof.* Every totally ordered real vector space  $(V, \geq)$  of dimension  $> 1$  with proper cone  $C$  has a two-dimensional subspace  $V_2$  with an induced order given by the maximal positive cone  $C_2 = C \cap V_2$ . We can choose basis vectors  $e_1, e_2$  for  $V_2$  such that  $C_2$  is of the form

$$\{a_1e_1 + a_2e_2 \mid (a_1 \geq 0 \text{ and } a_2 \geq 0) \text{ or } (a_1 \leq 0 \text{ and } a_2 > 0)\}. \quad (2.12)$$

This positive cone can be visualised in the graph on the right hand side of Figure 2. It is clear that there exists  $y \in V_2$  such that  $y \neq 0$  and  $(y - ne_1) \geq 0 \forall n \in \mathbb{N}$  (for example,  $y = e_2$ ). This implies that there is no totally ordered real vector space of dimension  $> 1$  that is Archimedean.  $\square$

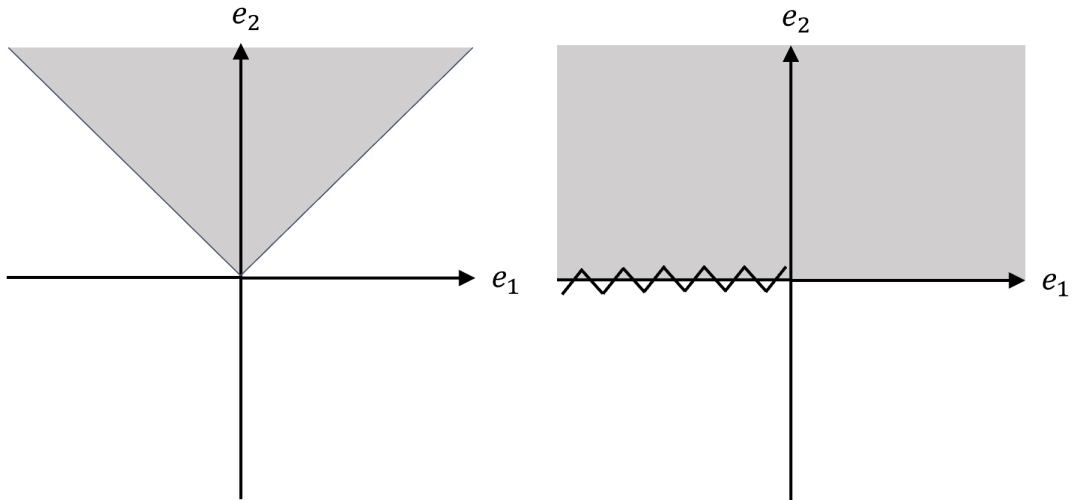


Figure 2: **Left:** A non-maximal proper cone in  $\mathbb{R}^2$ . **Right:** A maximal proper cone in  $\mathbb{R}^2$ . The zigzag line indicates that the strictly negative half-line in the  $e_1$  direction is not included in the shaded cone.

Looking at the positive cone on the right hand side of Figure 2, it should

be clear that if the vertical component of vector  $y$  is greater than or equal to the vertical component of vector  $x$ , then  $y \geq x$ . Therefore  $e_2$  is the greatest unit vector in  $V_2$ , and  $e_1$  is the least unit vector in  $V_2$  that is positive. In the  $(e_1, e_2)$  basis, we have that  $(a_1, a_2) \geq (b_1, b_2)$  iff  $a_2 > b_2$  or  $(a_2 = b_2$  and  $a_1 \geq b_1)$ , which is just the reverse lexicographic ordering. For any positive vector  $v$  that is not a scalar multiple of  $e_1$ ,  $v \gg e_1$ . All other vectors excluding multiples of  $e_1$  are Archimedean equivalent to each other.

While no two distinct vectors (populations) are equivalent to each other, there are populations that are arbitrarily close in the sense that their difference can be as ‘near to zero’ as one desires. This provides an intuition for the necessity of non-Archimedean equivalent goods. Given any two linearly independent vectors  $v_1, v_2$  corresponding to life years of different qualities, we can always find a linear combination  $d$  whose vertical component is zero. Then  $d$  is not Archimedean equivalent to at least one of  $v_1$  and  $v_2$ . For example, if  $v_1$  and  $v_2$  correspond to life qualities that are in the same Archimedean class, then  $d = a_1v_1 - a_2v_2$  represents a gain of  $a_1$  life years at quality  $v_1$ , and a loss of  $a_2$  life years at quality  $v_2$ . This difference can be as ‘small’ as one wants, leading to a good that is in a lower Archimedean class.

**Proposition 4.** The lexicographic order on  $\mathbb{R}^2$  cannot be represented on  $\mathbb{R}$ .

*Proof.* Let  $\geq$  denote the lexicographic order on  $\mathbb{R}^2$ . Assume there is a function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $u(y) \geq u(x) \Leftrightarrow y \geq x$ . Consider the map

$$f : \alpha \rightarrow [u((\alpha, 0)), u((\alpha, 1))]. \quad (2.13)$$

The Archimedean property of the real numbers can be used to show that every non-empty interval in the real numbers contains a rational number. I will use  $\phi$  to denote a function that selects a rational number from a non-empty interval given as its argument. The function  $\phi \circ f : \mathbb{R} \rightarrow \mathbb{Q}$  is an injection, since for  $\alpha \neq \beta$  we have that either  $u((\alpha, 0)) > u((\beta, 1))$ , or  $u((\beta, 0)) > u((\alpha, 1))$ . This in turn implies that  $f(\alpha) \cap f(\beta) = \emptyset$ , and  $\phi(f(\alpha)) \neq \phi(f(\beta))$  for  $\alpha \neq \beta$ . Then the cardinality of the rationals must be greater than or equal to the cardinality of the reals, which is a contradiction.  $\square$

Proposition 4 has the immediate consequence that there does not exist a real-valued function of the expected utility form that can represent an ordering of probability measures over an ordered real vector space  $(V, \geq)$  of dimension  $> 1$ . This is because there doesn't even exist a real-valued function that can represent an ordering on the subset of such probability measures that are degenerate on a single event (i.e. sure outcomes in  $V$ ). The axioms of the von Neumann Morgenstern utility theorem are not satisfied.[14] Note that this is not due to any 'irrationality' on our part regarding our evaluation of uncertain outcomes. The failure occurs when only considering outcomes that happen with probability 1, and is due to the fact there aren't enough real numbers to represent the order.

This has profound implications for the application of utilitarianism and expected utility theory in population ethics, and any setting involving non-Archimedean ordered real vector spaces or lexicographic orders. Any line of reasoning that implicitly or explicitly attempts to represent such orders using real numbers is unsound. If we proceeded ignoring the above, it would

be possible to come to ethical conclusions that are **arbitrarily wrong**. This can occur, for example, by mistakenly taking two goods to be Archimedean equivalent when they are not.

The conflict is with the continuity/Archimedean axiom of the von Neumann Morgenstern utility theorem. If this axiom is weakened, an ordering of probability measures may be represented with multi-dimensional lexicographic expected utilities.[15, 16] Let  $\{e_i\}_{i \in I}$  be a totally ordered basis for  $V$  where any finite subset lexicographically orders the subspace that it spans. Each dimension of  $V$  corresponds to a distinct Archimedean class. Let  $P, Q$  be probability measures defined on a sigma algebra over  $V$ . A lexicographic expected utility representation consists of real-valued functions  $u_i$  on  $V$  such that

$$P \geq Q \Leftrightarrow \mathbb{E}_P[u] \geq \mathbb{E}_Q[u]. \quad (2.14)$$

Here,  $\mathbb{E}_P[u]$  is the expectation with respect to the measure  $P$  of the vector  $u$  whose components are  $u_i$ . The comparison of  $P$  and  $Q$  is accomplished using the lexicographic order  $\geq$  on a finite dimensional subspace of  $V$ , since  $\mathbb{E}_P[u], \mathbb{E}_Q[u]$  are elements of  $V$  that can be written as finite linear combinations of basis vectors. Uncertain prospects are thus ordered lexicographically according to expected utility within Archimedean classes. For more detail on the conditions for such a representation to exist, see Chapter 1-4 of [16].

As an example of multi-dimensional lexicographic expected utility, consider the bets in Table 1 over quantities of two non-Archimedean equivalent

goods denoted by  $H$  ('higher') and  $L$  ('lower').

Table 1: Multi-dimensional expected utility

	<b>Bet 1</b>	<b>Bet 2</b>	<b>Bet 3</b>
<b>Probability distribution</b>	$P(H=10) = \frac{1}{2}$ $P(H=0) = \frac{1}{2}$ $P(L = 2) = 1$	$P(H=1) = 1$ $P(L = 3) = 1$	$P(H=2) = \frac{1}{2}$ $P(H=0) = \frac{1}{2}$ $P(L = 7) = 1$
<b>Expected utility</b>	(5 2)	(1 3)	(1 7)

Bet 1 is preferred to the other two bets because the first component of its expected utility vector is largest. Bet 3 is preferred to Bet 2 because the first components of their expected utility are equal, but the second component for Bet 3 exceeds that of Bet 2.

Our only assumptions so far are the axioms for an ordered real vector space  $\mathbb{R}^2$ . However, we are forced to conclude that there are pairs of goods that are not Archimedean equivalent. Although non-Archimedean life qualities have previously been suggested as a way of avoiding the repugnant conclusion, it has been claimed that if such a pair of life qualities exists, then other extremely counterintuitive results follow. In particular, that there necessarily must exist a pair of quality levels that are only incrementally different from each other that are not Archimedean equivalent. I will next show that this conclusion is false, after introducing some preliminaries on ordinal numbers.

### 3 An informal introduction to ordinal numbers

Alice likes apples and oranges. She always prefers a larger number of apples/oranges to a smaller number. However, she likes apples more than oranges, to the extent that there is no number of oranges that she would prefer to even a single apple.

Let us try to represent her preferences using numbers. We will assign the number 1 to the bundle consisting of 1 orange, the number 2 to 2 oranges, etc. Then the  $\geq$  operator on the natural numbers represents Alice's preferences over oranges.

However, if we limit ourselves to the natural numbers, we immediately run into a problem. The bundle consisting of a single apple cannot be assigned a natural number in a way that respects Alice's preferences. Whichever number it is assigned, there is always a larger natural number available, which would incorrectly imply that there is some number of oranges Alice would prefer to one apple.

In a sense, we have 'run out' of numbers that we can use to order the available bundles. This is a general issue encountered when naively attempting to define total orders on infinite sets. The theory of ordinal numbers was constructed to handle this more than a century ago by Cantor, and developed further by von Neumann and others.[17, 18] We give a brief whistlestop tour of this theory here.

There is nothing stopping us from simply defining an abstract number, which we will call  $\omega$ , and extending the relation  $\geq$  by assigning  $\omega$  to be

greater than any natural number.  $\omega$  can then be used to represent the bundle consisting of one apple for Alice.

We can take this further and consider bundles consisting of some number of both apples and oranges. Remembering that Alice always prefers more fruit to less, we may define a new element denoted by  $\omega + 1$  that is greater than  $\omega$ , corresponding to 1 apple and 1 orange. This can be repeated for  $\omega + 2$ ,  $\omega + 3$  etc. For 2 apples, we symbolically assign the element  $\omega 2$ .

$\omega$  is called the first infinite ordinal number. An arithmetic of ordinal numbers can be constructed recursively using disjoint unions of sets. A well-ordered set is a totally-ordered set in which every subset has a least element. Given two well-ordered sets  $X, Y$ , we may define  $X + Y$  as the set obtained by taking their disjoint union, and assigning every element of  $Y$  to be greater than every element of  $X$ , but otherwise preserving the ordering within  $X$  and  $Y$ . For example,

$$\{0, 1, 2\} + \{0', 1'\} := \{0, 1, 2, 0', 1'\}, \quad (3.1)$$

where sets are written so that the elements increase from left to right. We can map the set  $\{0, 1, 2, 0', 1'\}$  onto  $\{0, 1, 2, 3, 4\}$  while preserving the ordering by



$$0 \rightarrow 0, \quad (3.2)$$

$$1 \rightarrow 1, \quad (3.3)$$

$$2 \rightarrow 2, \quad (3.4)$$

$$0' \rightarrow 3, \quad (3.5)$$

$$1' \rightarrow 4. \quad (3.6)$$

We can go further and identify  $\{0, 1, 2, 3, 4\}$  as the ordinal number 5. This is because every element of a well-ordering is uniquely determined by the set of elements that it is larger than. Similarly,  $\omega$  can be identified with the set of natural numbers. In this scheme, ordinal numbers are simply canonical representatives of a set of well-orderings that are all equivalent to each other, with the different ordinal numbers providing labels for every possible distinct well-ordered set. The ordinal number associated with a well-ordered set is called its order type.

Note that the above addition operation is commutative for finite ordinal numbers, i.e.  $X + Y = Y + X \forall X, Y < \omega$ . This however, ceases to be the case when considering infinite ordinals. For example,

$$2 + \omega = \{0', 1', 0, 1, 2, \dots\}. \quad (3.7)$$

This ordering is equivalent to  $\omega$ , since we can simply relabel as follows

$$0' \rightarrow 0, \tag{3.8}$$

$$1' \rightarrow 1, \tag{3.9}$$

$$0 \rightarrow 2, \tag{3.10}$$

$$1 \rightarrow 3, \tag{3.11}$$

$$\vdots \tag{3.12}$$

On the other hand,

$$\omega + 2 = \{0, 1, 2, \dots, \omega, \omega + 1\}. \tag{3.13}$$

This is not equivalent to  $\omega$ , since there are two elements in  $\omega + 2$  that are greater than all the natural numbers, as opposed to none in  $\omega$ . Thus  $\omega + 2 \neq 2 + \omega$ .

Multiplication and exponentiation operations can also be defined for ordinal numbers, but I will not develop this here. I refer the reader to some of the many excellent texts on set theory.[19, 20, 21]

## 4 Ordinal numbers and the repugnant conclusion

Results in [22, 23] imply that in any finite increasing sequence for which the first and last elements are not Archimedean equivalent to each other, there

must be a pair of successive elements that are not Archimedean equivalent. This can be used as an argument against the existence of goods that are not Archimedean equivalent, based on the idea that one should be able to construct such a sequence where each element is only marginally better than its predecessor. This would then imply that there are a pair of life qualities that are near-identical, but there is no number of the marginally worse life years that is better than just one of the marginally better life years. The argument can be extended to infinite sequences, using transitivity of Archimedean equivalence to obtain the result that in any sequence where every two successive elements are Archimedean equivalent, all elements are Archimedean equivalent.

The problem can be remedied by generalising to transfinite sequences. A transfinite sequence is a collection of set elements indexed by ordinal numbers (rather than the natural numbers). We may, for example, take an ordered real vector space that is infinite dimensional, and construct an increasing transfinite sequence of vectors  $v_1, v_2, \dots, v_\omega$  where every pair of successive elements is Archimedean equivalent, but  $v_\omega$  is not Archimedean equivalent to any  $v_n$  for  $n \in \mathbb{N}$ . This is possible because  $v_\omega$  **does not have a predecessor**. That is, there is no natural number that has  $\omega$  as its successor, because one can always find a larger natural number. Thus it is perfectly possible to have vectors  $v_1, v_\omega$  such that  $v_\omega$  can only be ‘reached’ from  $v_1$  by a countably infinite number of marginal increments. If that is not enough, there are ordinal numbers that are uncountable when considered as a set, so one can construct an increasing transfinite sequence of vectors where at least an uncountable number of increments is required to move from some element

to another. If one wishes to have a set of vectors that contains arbitrarily large ascending and descending chains, one can index them using e.g. surreal numbers. Thus the entire set of possible populations, covering multiple distinct Archimedean classes, is ordered without any pair of successive elements being non-Archimedean equivalent.

## 5 Uncertainty and vagueness

The above formalism may be viewed as a theoretical framework for population ethics in circumstances of complete information, sharply demarcated boundaries and ethical preferences that are sensitive to arbitrarily small changes. However, these conditions may not exist in reality. For example, it may not be clear what constitutes a life that is barely worth living, or to give the concept a precise definition. One way of generalising to accommodate uncertainty/vagueness is to allow the positive cone to be a fuzzy set.

**Definition 6. Fuzzy set.** A fuzzy set is map  $m : V \rightarrow [0, 1]$ , where  $V$  is a set.  $m$  is called the membership function.

In the current setting, the value of the membership function may be interpreted as a frequentist or Bayesian probability that a given element of the ordered real vector space  $(V, \geq)$  is in the positive cone, and therefore greater than or equal to 0. That is, there really is a sharp boundary between elements that are  $\geq 0$  and elements that are not, but we do not know exactly where it lies. The uncertainty may originate at least partly from our inability to detect differences below a certain threshold.

Alternatively, the membership function may be interpreted as the ‘degree’ to which a given vector is greater than or equal to zero. In this view, the ‘positive cone’ need not have a sharp boundary, similar to the way there is no sharp distinction between e.g. a sunny day and a cloudy day. This interpretation is fundamentally distinct from a probabilistic interpretation, in which there is uncertainty about the answer to a well defined question. By contrast, fuzzy sets allow for the possibility of a question that is not well defined, and answers with truth values ranging between 0 (false) and 1 (true).

With either interpretation, there is no longer a precise apparent border between elements that are not Archimedean equivalent to each other. This is consonant with the intuition that there is no clear dividing line between life qualities that are qualitatively different. For example, we might hold the position that one completely blissful life is better than any number of lives that are barely worth living, without having a clear idea of where exactly the boundary lies between lives that are Archimedean equivalent to the single blissful life, and those that are not.

This induces a new preference relation  $\succeq$  characterised by a decision rule with threshold  $\alpha \in [0, 1]$ , defined by

$$y \succeq x \quad \text{if} \quad m(y - x) \geq \alpha \tag{5.1}$$

$$y \sim x \quad \text{if} \quad m(y - x) < \alpha, \tag{5.2}$$

where  $y \sim x \iff (y \succeq x \text{ and } x \succeq y)$ .  $\succeq$  may be non-transitive. For example,  $m$  and  $\alpha$  may be such that  $\succeq$  restricted to a finite-dimensional

subspace is the lexicographic semiorder.[24] This can be understood as the lexicographic ordering with imperfect discriminatory power, such that only differences above a certain threshold in a given dimension are detectable. Non-transitivity of the strict preference  $\succ$  can also occur as in examples given by Ng.[25] Non-transitivity of indifference  $\sim$  can occur along the lines of the example given in seminal work by Luce [26] of an individual who strictly prefers a coffee with one sugar over a coffee with five sugars, but is pairwise indifferent between a series of intermediate coffees that differ by tiny increments in the quantity of sugar.

If there are physical limitations on the ability to discriminate between different goods stemming from e.g. uncertainty principles in quantum mechanics, then non-transitive preferences may be unavoidable. Non-transitivity arising from limited discriminatory power can resolve problems associated with so-called ‘recklessness’ and ‘fanaticism’, wherein a decision-maker acting in accordance with a seemingly reasonable set of axioms counterintuitively prefers bets involving arbitrarily small probabilities of sufficiently large or ‘infinite’ payoffs to any given sure outcome.[11, 12, 13] This is because sufficiently small probabilities in long shot bets may be indistinguishable from zero, regardless of the magnitude of the payoff.

As an example, consider the problem of determining whether a coin is fair with probability 0.5 of landing on heads, or slightly biased, with probability  $(0.5 + \epsilon)$  of landing on heads. In order to do this, one might seek to make sure the coin is completely symmetric about its axis of rotation. However, for small enough  $\epsilon$  it may not be physically possible to distinguish between these two possibilities with sufficient certainty due to quantum indeterminacy.

The transitivity axiom of expected utility theorems can be weakened to accommodate non-transitivity of  $\sim$  and  $\succ$  while maintaining the less stringent condition that  $\succ$  is acyclic. This leads to a ‘one-way’ expected utility representation, where

$$P \succ Q \Rightarrow \mathbb{E}_P[u] > \mathbb{E}_Q[u], \quad (5.3)$$

but the converse is not true. Given complete preferences, this means that

$$\mathbb{E}_P[u] \geq \mathbb{E}_Q[u] \Rightarrow P \succeq Q \quad (5.4)$$

However, we may have  $P \sim Q$  even if  $\mathbb{E}_P[u] \neq \mathbb{E}_Q[u]$ . The conditions for such a representation are given in Chapter 5 of [16].

Further dropping the acyclicity condition can lead to representations of the form

$$P \succ Q \Leftrightarrow \phi(\mathbb{E}_P[u], \mathbb{E}_Q[u]) > 0, \quad (5.5)$$

for a function  $\phi$ . Some conditions for the existence of such a representation are given in [27, 28].

## 6 Conclusion

Applying informal logic to mathematical problems can lead to errors, particularly when dealing with infinities. It is also important to use appropriate mathematical structures for the problem at hand. In this paper, I have introduced ordered real vector spaces, transfinite sequences and fuzzy sets as suitable mathematical tools in population ethics. I have shown that in this setting, not only is the existence of non-Archimedean equivalent goods (i.e. goods for which no amount of one is better than a single unit of the other) sensible, it is compulsory. I note that utilitarianism and expected utility theory in their canonical form fail in this setting, as does any attempt to use real numbers to represent orders in population ethics. This is because non-Archimedean orders cannot generally be represented on Archimedean fields. Generalisations of the von Neumann Morgenstern utility theorem that weaken the continuity/Archimedean axiom lead to multi-dimensional lexicographic expected utility representations. I show that counterintuitive conclusions that have previously been thought to follow as a consequence of the existence of non-Archimedean equivalent goods, do not obtain. In particular, there need not be pairs of successively ordered goods that are not Archimedean equivalent. This is demonstrated using an appropriate handling of orders on infinite sets. I use a fuzzy set construction to show that there needn't be a sharp boundary between pairs of goods that are non-Archimedean equivalent. This may be an inevitable consequence of intrinsic physical limitations on the ability to discriminate between different goods arising from e.g. quantum mechanics. These limitations cause preferences to



be non-transitive, which can in turn resolve problems related to ‘fanaticism’ and ‘recklessness’.

These results are general and do not rely on any particular interpretation of concepts that may not have a precise definition, such as life quality or a life barely worth living. That is, they remain true regardless of any such interpretation that is made. The main implication of this work is that the repugnant conclusion can be avoided without contradicting key moral intuitions.

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