Cognitivism about Epistemic Modality: Epistemic Modal Algebra, Homotopy Type Theory, and the Computational Theory of Mind

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Abstract

This paper aims to vindicate the thesis that cognitive computational properties are abstract objects implemented in physical systems. I avail of the equivalence relations countenanced in Homotopy Type Theory, in order to specify an abstraction principle for intensional, computational properties. The homotopic abstraction principle for intensional mental functions provides an epistemic conduit into our knowledge of cognitive algorithms as abstract objects. I examine, then, how intensional functions in Epistemic Modal Algebra are deployed as core models in the philosophy of mind, Bayesian perceptual psychology, the program of natural language semantics in linguistics, and in quantum information theory, and I argue that this provides abductive support for the truth of homotopic abstraction. Epistemic modality can thus be shown to be both a compelling and a materially adequate candidate for the fundamental structure of mental representational states, comprising a fragment of the Language of Thought.

1 Introduction

This essay aims to vindicate the thesis that cognitive computational properties are abstract objects implemented in physical systems.\(^1\) A recent approach to the foundations of mathematics is Homotopy Type Theory.\(^2\) In Homotopy Type Theory, homotopies can be defined as equivalence relations on intensional functions. In this essay, I argue that homotopies can thereby figure in abstraction

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\(^1\)Cf. Turing (1950); Putnam (1967); Newell (1973); Fodor (1975); and Pylyshyn (1978).

\(^2\)Cf. The Univalent Foundations Program (2013).
principles for intensional, cognitive computational properties. Homotopies for intensional functions thus comprise identity criteria for some cognitive mechanisms. The philosophical significance of the foregoing is twofold. First, the proposal demonstrates how epistemic modality is a viable candidate for a fragment of the Language of Thought. The identity of intensional functions and epistemic modal operators can be witnessed via a unique, epistemic interpretation of the algebraic semantics of modal logic. Second, the proposal serves to delineate one conduit for our epistemic access to cognitive algorithms as abstract objects.

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3For the first proposal to the effect that abstraction principles can be used to define abstracta such as cardinal number, see Frege (1884/1980: 68; 1893/2013: 20). For the locus classicus of the contemporary abstractionist program, see Hale and Wright (2001).

4See, by contrast, Stalnaker (1984), who argues that possible worlds model theory ought to model the pragmatics of intentional states, rather than internal representations of agents, and who argues against the existence of epistemic possibilities (2003: ch. 11). Given a metalanguage, a precedent to the current approach – which models thoughts and internal representations via possible worlds model theory – can be found in Wittgenstein (1921/1974: 2.15-2.151, 3-3.02).

5Bealer (1982) proffers a non-modal algebraic logic for intensional entities – i.e., properties, relations, and propositions – which avails of a $\lambda$-definable variable-binding abstraction operator (op. cit.: 46-48, 209-210). Bealer reduces modal notions to logically necessary conditions-cum-properties, as defined in his non-modal algebraic logic (207-209). The present approach differs from the foregoing by: (i) countenancing a modal algebra, on an epistemic interpretation thereof; (ii) treating the abstraction operator as a Fregean function from concepts to objects, rather than as a $\lambda$-operator; (iii) availing of the univalence axiom in Homotopy Type Theory – which collapses identity and isomorphism – in order to provide an equivalence relation for the abstraction principle pertinent to (ii); and (iv) demonstrating how the model is availed of in various branches of the cognitive sciences, such that Epistemic Modal Algebra may be considered a viable candidate for the Language of Thought.

Katz (1998) proffers a view of the epistemology of abstracta, according to which the propositions of the Language of Thought are abstract, and on which the syntax and the semantics for the propositions are innate (35). Katz suggests that the proposal is consistent with both a Fregean approach to propositions, according to which they are thoughts formed by the concatenation of concepts, and a Russellian approach, according to which they are structured tuples of non-conceptual entities (36). He endorses, ultimately, an approach to propositions which eschews Frege’s (1892/1997) distinction between sense and reference, yet according to which they are ‘abstract senses’, or thoughts, correlated to natural language sentence types (114-115). That propositions are abstract is argued, then, to suffice for knowledge of abstract entities (op. cit.). One difference between Katz’s proposal and the one here presented is that Katz rejects modal approaches to propositions, because the latter cannot countenance reductio proofs based on counterfactuals with impossible antecedents (38; cf. Lewis, 1973: I.6). Following Lewis (op. cit.), the present approach does not avail of impossible worlds, i.e., worlds at which true contradictions obtain; and thus counterfactuals with impossible antecedents are vacuously true. If so, then Katz’s argument against modal approaches to propositions can be circumvented. A second difference is that, on Katz’s approach, the necessity of mathematical truths is argued to consist in reductio proofs, such that the relevant formulas will be true on all interpretations, and thus true of logical necessity (39). However, the endeavor to reduce
In Section 2, I provide an abstraction principle for cognitive algorithms, by availing of the equivalence relations countenanced in Homotopy Type Theory. In Section 3, I define a topological boolean algebraic semantics for modal logic, where the modalities defined on the algebra range over states of information. In Section 4, I describe how models of Epistemic Modal Algebra are availed of when perceptual representational states are modeled in Bayesian perceptual psychology; when speech acts are modeled in natural language semantics; when transformations are defined in quantum information theory; and when knowledge, belief, intentional action, and rational intuition are modeled in philosophical approaches to the nature of propositional attitudes. This provides abductive support for the claim that Epistemic Modal Algebra is both a compelling and materially adequate candidate for a fragment of the Language of Thought. In Section 5, I argue that the proposal (i) resolves objections to the relevant abstraction principles advanced by both Dean (2016) and Linnebo and Pettigrew (2014), and (ii) reveals a commitment to in re platonism in the appeal to cognitive mechanisms in Azzouni’s (2013: 9.1-9.2 forthcoming) version of nominalism. Section 6 provides concluding remarks.

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6 The epistemic modality can be interpreted so as to target notions of conceivability (⋄) and apriority (□ ≡ ¬¬); belief and knowledge; and psychophysical possibilities. See Section 4, for further discussion.
2 An Abstraction Principle for Cognitive Algorithms

In this section, I specify a homotopic abstraction principle for intensional functions. Intensional isomorphism, as a jointly necessary and sufficient condition for the identity of intensions, is first proposed in Carnap (1947: §14). The isomorphism of two intensional structures is argued to consist in their logical, or L-, equivalence, where logical equivalence is co-extensive with the notions of both analyticity (§2) and synonymy (§15). Carnap writes that: 

"An expression in S is L-equivalent to an expression in S' if and only if the semantical rules of S and S' together, without the use of any knowledge about (extra-linguistic) facts, suffice to show that the two have the same extension" (p. 56), where semantical rules specify the intended interpretation of the constants and predicates of the languages (4).

The current approach differs from Carnap's by defining the equivalence relation necessary for an abstraction principle for epistemic intensions on Voevodsky’s (2006) Univalence Axiom, which collapses identity with isomorphism in the setting of intensional type theory.

In the following section, I define, then, a class of models for Epistemic Modal Algebra.

For criticism of Carnap's account of intensional isomorphism, based on Carnap's (1937: 17) 'Principle of Tolerance' to the effect that pragmatic desiderata are a permissible constraint on one's choice of logic, see Church (1954: 66-67).

Note further that, by contrast to Carnap's approach, epistemic intensions are here distinguished from linguistic intensions (cf. Essay 5, for further discussion), and the current work examines the philosophical significance of the convergence between epistemic intensions and formal, rather than natural, languages. For a translation from type theory to set theory – which is of interest to, inter alia, the definability of epistemic intensions in the setting of set theory (cf. Essays 15, 16, and 18, below) – see Linnebo and Rayo (2012). For topological Boolean-valued models of epistemic set theory – i.e., a variant of ZF with the axioms augmented by epistemic modal operators interpreted as informal provability and having a background logic satisfying S4 – see Scedrov (1985), Flagg (1985), and Goodman (1990).
Topological Semantics

In the topological semantics for modal logic, a frame is comprised of a set of points in topological space, \( X \), and an accessibility relation, \( R \):

\[
F = \langle X, R \rangle;
\]

\[
X = (X_x)_{x \in X}; \quad \text{and}
\]

\[
R = (R_{xy})_{x, y \in X} \text{ iff } R_x \subseteq X_x \times X_x, \text{ s.t. if } R_{xy}, \text{ then } \exists o \subseteq X, \text{ with } x \in o \text{ s.t. } \forall y \in (R_{xy}),
\]

where the set of points accessible from a privileged node in the space is said to be open.\(^9\) A model defined over the frame is a tuple, \( M = \langle F, V \rangle \), with \( V \) a valuation function from subsets of points in \( F \) to propositional variables taking the values 0 or 1. Necessity is interpreted as an interiority operator on the space:

\[
M, x \models \Box \phi \iff \exists o \subseteq X, \text{ with } x \in o, \text{ such that } \forall y \in (o), M, y \models \phi.
\]

Homotopy Theory

Homotopy Theorycountenances the following identity, inversion, and concatenation morphisms, which are identified as continuous paths in the topology. The formal clauses, in the remainder of this section, evince how homotopic morphisms satisfy the properties of an equivalence relation.\(^{10}\)

\[
p : [0,1] \to X, \text{ with } p(0) = x \text{ and } p(1) = y;
\]

\[
f : X_1 \to X_2;
\]

\[
g : X_1 \to X_2;
\]

\[
H : X_1 \times \{0,1\} \to X_2, \text{ with } H_{x,0} = f(x) \text{ and } H_{x,1} = g(x).
\]

\(^9\)In order to ensure that the Kripke semantics matches the topological semantics, \( X \) must further be Alexandrov; i.e., closed under arbitrary unions and intersections.

\(^{10}\)The definitions and proofs at issue can be found in the Univalent Foundations Program (op. cit.: ch. 2).
**Reflexivity**

\[ \forall x, y : A \forall p (p : x =_A y) : \tau(x,y,p), \text{ with } A \text{ and } \tau \text{ designating types, and } U \text{ designating a universe of elements, } e: \]

\[ \forall \alpha : A \exists e(\alpha) : \tau(\alpha, \alpha, \text{ref}_\alpha); \]

\[ p, q : (x =_A y) \]

\[ \exists r \in e : p =_{(x =_A y)} q \]

\[ \exists \mu : r =_{(p =_{(x =_A y)} q)} s. \]

**The Induction Principle**

If:

\[ \forall x, y : A \forall p (p : x =_A y) \exists \tau[\tau(x,y,p)] \land \forall \alpha : A \exists e(\alpha) : \tau(\alpha, \alpha, \text{ref}_\alpha) \]

Then:

\[ \forall x, y : A \exists p (p : x =_A y) \exists e[\text{ind}_{=_A}(\tau,e,x,y,p) : \tau(x,y,p), \text{ such that } \text{ind}_{=_A}(\tau,e,\alpha,\alpha,\text{ref}_\alpha) \equiv e(\alpha)]. \]

**Symmetry**

\[ \forall A \forall x, y : A \exists H_\Sigma(x = y \rightarrow y = x) \]

\[ H_\Sigma := p \rightarrow p^{-1}, \text{ such that } \]

\[ \forall x : A(\text{ref}_x \equiv \text{ref}_x^{-1}). \]

**Transitivity**

\[ \forall A \forall x, y : A \exists H_T(x = y \rightarrow y = z \rightarrow x = z) \]

\[ H_T := p \rightarrow q \rightarrow p \bullet q, \text{ such that } \]

\[ \forall x : A[\text{ref}_x \bullet \text{ref}_x \equiv \text{ref}_x]. \]
Homotopic Abstraction

For all type families A, B, there is a homotopy:

\[ H := [(f \sim g) \equiv \prod_{x \in A} (f(x) = g(x))], \]

where

\[ \prod_{f : A \to B} [(f \sim f) \land (f \sim g \to g \sim f) \land (f \sim g \to g \sim h \to f \sim h)], \]

such that, via Voevodsky’s (op. cit.) Univalence Axiom, for all type families A, B : U, there is a function:

\[ \text{idtoeqv} : (A =_U B) \to (A \simeq B), \]

which is itself an equivalence relation:

\[ (A =_U B) \simeq (A \simeq B). \]

Abstraction principles for intensional computational properties take, then, the form:

- \[ \forall A, B \exists f, g : \prod_{f : A \to B} A(f(x) = A g(x)) \simeq [f(x) \simeq g(x)], \]

with \( A \) an abstraction operator from the domain of functions to a domain of abstract objects.

3 Epistemic Modal Algebra

In Epistemic Modal Algebra, the topological boolean algebra, \( A \), is formed by taking the powerset of the topological space, X, defined above; i.e., \( A = P(X) \).

The domain of \( A \) is comprised of formula-terms – eliding propositions with names – assigned to elements of \( P(X) \), where the proposition-letters are interpreted as encoding states of information. The top element of the algebra is denoted ‘1’ and the bottom element is denoted ’0’. We interpret modal operators, \( f(x) \), – i.e., intensional functions in the algebra – as both concerning topological interiority, as well as reflecting epistemic possibilities. An Epistemic
Modal-valued Algebraic structure has the form, $F = \langle \mathcal{A}, D_{P(X)}, \rho \rangle$, where $\rho$ is a mapping from points in the topological space to elements or regions of the algebraic structure; i.e., $\rho : D_{P(X)} \times D_{P(X)} \to \mathcal{A}$. A model over the Epistemic-Modal Topological Boolean Algebraic structure has the form $M = \langle F, V \rangle$, where $V(a) \leq \rho(a)$ and $V(a, b) \land \rho(a, b) \leq V(b)$.\textsuperscript{11} For all $x, \phi, \psi \in \mathcal{A}$:

\begin{align*}
  f(1) &= 1; \\
  f(x) &\leq x; \\
  f(x \land y) &= f(x) \land f(y); \\
  f[f(x)] &= f(x); \\
  V(a, a) &> 0; \\
  V(a, a) &= 1; \\
  V(a, b) &= V(b, a); \\
  V(a, b) \land V(b, c) &\leq V(a, c); \\
  V(a = a) &= \rho(a, a); \\
  V(a, b) &\leq f[V(a, b)]; \\
  V(\neg \phi) &= \rho(\neg \phi) - f(\phi); \\
  V(\phi) &= \rho \phi - f[\neg V(\phi)]; \\
  V(\square \phi) &= f[V(\phi)] \quad \text{(cf. Lando, op. cit.)}.
\end{align*}

4 Examples in Philosophy, Cognitive Science, and Quantum Information Theory

The foregoing possible worlds model theory is availed of by a number of paradigms in contemporary empirical theorizing: the computational theory of mind, Bayesian perceptual psychology, natural language semantics, and the theory of quantum

\textsuperscript{11}See Lando (2015); McKinsey and Tarski (1944); and Rasiowa (1963), for further details.
Marcus (2001) argues that mental representations can be treated as algebraic rules characterizing the computation of operations on variables, where the values of a target domain for the variables are universally quantified over and the function is one-one, mapping a number of inputs to an equivalent number of outputs (35-36). Models of the above algebraic rules can be defined in both classical and weighted, connectionist systems: Both a single and multiple nodes can serve to represent the variables for a target domain (42-45). Temporal synchrony or dynamic variable-bindings are stored in short-term working memory (56-57), while information relevant to long-term variable-bindings are stored in registers (54-56). Examples of the foregoing algebraic rules on variable-binding include both the syntactic concatenation of morphemes and noun phrase reduplication in linguistics (37-39, 70-72), as well as learning algorithms (45-48). Conditions on variable-binding are further examined, including treating the binding relation between variables and values as tensor products – i.e., an application of a multiplicative axiom for variables and their values treated as vectors (53-54, 105-106). In order to account for recursively formed, complex representations, which he refers to as structured propositions, Marcus argues instead that the syntax and semantics of such representations can be modeled via an ordered set of registers, which he refers to as `treelets’ (108).

A strengthened version of the algebraic rules on variable-binding can be accommodated in models of epistemic modal algebras, when the latter are augmented by cylindrifications, i.e., operators on the algebra simulating the treatment of quantification, and diagonal elements.\[^{12}\] By contrast to Boolean Algebras with Operators, which are propositional, cylindric algebras define first-
order logics. Intuitively, valuation assignments for first-order variables are, in cylindric modal logics, treated as possible worlds of the model, while existential and universal quantifiers are replaced by, respectively, possibility and necessity operators (⋄ and □) (Venema, 2013: 249). For first-order variables, \{v_i | i < \alpha\} with \alpha an arbitrary, fixed ordinal, \(v_i = v_j\) is replaced by a modal constant \(d_{i,j}\) (op. cit: 250). The following clauses are valid, then, for a model, M, of cylindric modal logic, with \(E_{i,j}\) a monadic predicate and \(T_i\) for \(i,j < \alpha\) a dyadic predicate:

\[
\begin{align*}
M,w \models p \iff w \in V(p); \\
M,w \models d_{i,j} \iff w \in E_{i,j}; \\
M,w \models \diamond_i \psi \iff \text{there is a } v \text{ with } wT_i v \text{ and } M,v \vdash \psi. \quad (252).
\end{align*}
\]

Finally, a cylindric modal algebra of dimension \(\alpha\) is an algebra, \(A = \langle A, +, \cdot, -, 0, 1, \diamond_i, d_{i,j} \rangle_{i,j < \alpha}\), where \(\diamond_i\) is a unary operator which is normal (\(\diamond_i 0 = 0\)) and additive \([\diamond_i(x + y) = \diamond_i x + \diamond_i y]\) (257).

The philosophical interest of cylindric modal algebras to Marcus’ cognitive models of algebraic variable-binding is that variable substitution is treated in the modal algebras as a modal relation, while universal quantification is interpreted as necessitation. The interest of translating universal generalization into operations of epistemic necessitation is, finally, that – by identifying epistemic necessity with apriority – both the algebraic rules for variable-binding and the recursive formation of structured propositions can be seen as operations, the

\[13\] Cylindric frames need further to satisfy the following axioms (op. cit.: 254):
1. \(p \to \diamond p\)
2. \(p \to \Box \diamond p\)
3. \(\diamond \diamond p \to \diamond p\)
4. \(\diamond \diamond p \to \diamond \diamond p\)
5. \(d_{i,i}\)
6. \(\diamond(d_{i,j} \land p) \to \Box \diamond (d_{i,j} \to p)\)
[Translating the diagonal element and cylindric (modal) operator into, respectively, monadic and dyadic predicates and universal quantification: \(\forall y x y \equiv \forall x (y = y)\) (op. cit.)]
7. \(d_{i,j} \iff \diamond_k (d_{i,k} \land d_{k,j})\).
implicit knowledge of which is apriori.

In Bayesian perceptual psychology, the problem of underdetermination is resolved by availing of a gradational possible worlds model. The visual system is presented with a set of possibilities with regard, e.g., to the direction of a light source. So, for example, the direction of light might be originating from above, or it might be originating from below. The visual system computes the constancy, i.e. the likelihood that one of the possibilities is actual.\textsuperscript{14} The computation of the perceptual constancy is an unconscious statistical inference, as anticipated by Helmholtz’s (1878) conjecture.\textsuperscript{15} The constancy places, then, a condition on the accuracy of the attribution of properties – such as boundedness and volume – to distal particulars.\textsuperscript{16}

In the program of natural language semantics in empirical and philosophical linguistics, the common ground or ‘context set’ is the set of possibilities presupposed by a community of speakers.\textsuperscript{17} Kratzer (1979: 121) refers to cases in which the above possibilities are epistemic as an ‘epistemic conversational background’, where the epistemic possibilities are a subset of objective or circumstantial possibilities (op. cit.). Modal operators are then defined on the space, encoding the effects of various speech acts in entraining updates on the context set.\textsuperscript{18} So, e.g., assertion is argued to provide a truth-conditional update on the context set, whereas there are operator updates, the effects of which are

\textsuperscript{14}Cf. Mamassian et al. (2002).
\textsuperscript{15}For the history of the integration of algorithms and computational modeling into contemporary visual psychology, see Johnson-Laird (2004).
\textsuperscript{16}Cf. Burge (2010), and Rescorla (forthcoming), for further discussion. A distinction ought to be drawn between unconscious perceptual representational states – as targeted in Burge (op. cit.) – and the inquiry into whether the properties of phenomenal consciousness have accuracy-conditions – where phenomenal properties are broadly construed, so as to include, e.g., color-phenomenal properties, as well as the property of being aware of one’s perceptual states.
\textsuperscript{17}Cf. Stalnaker (1978).
\textsuperscript{18}Cf. Kratzer (op. cit.); Stalnaker (op. cit.); Lewis (1980); Heim (1992); Veltman (1996); von Fintel and Heim (2011); and Yalcin (2012).
not straightforwardly truth-conditional and whose semantic values must then be defined relative to an array of intensional parameters (including a context – agent, time, location, et al. – and a tuple of indices).

In quantum information theory, let a constructor be a computation defined over physical systems. Constructors entrain nomologically possible transformations from admissible input states to output states (cf. Deutsch, 2013). On this approach, information is defined in terms of constructors, i.e., intensional computational properties. The foregoing transformations, as induced by constructors, are referred to as tasks. Because constructors encode the counterfactual to the effect that, were an initial state to be computed over, then the output state would result, modal notions are thus constitutive of the definition of the tasks at issue. There are, further, both topological and algebraic aspects of the foregoing modal approach to quantum computation.\textsuperscript{19} The composition of tasks is formed by taking their union, where the union of tasks can be satisfiable while its component tasks might not be. Suppose, e.g., that the information states at issue concern the spin of a particle. A spin-state vector will be the sum of the probabilities that the particle is spinning either upward or downward. Suppose that there are two particles which can be spinning either upward or downward. Both particles can be spinning upward; spinning downward; particle-1 can be spinning upward while particle-2 spins downward; and vice versa. The state vector, $V$ which records the foregoing possibilities – i.e., the superposition of the states – will be equal to the product of the spin-state of particle-1 and the spin-state of particle-2. If the particles are both spinning upward or both spinning downward, then $V$ will be .5. However – relative to the value of each particle vector, referred to as its eigenvalue – the probability that particle-1 will

\textsuperscript{19}For an examination of the interaction between topos theory and an S4 modal axiomatization of computable functions, see Awodey et al. (2000).
be spinning upward is .5 and the probability that particle-2 will be spinning downward is .5, such that the probability that both will be spinning upward or downward = .5 x .5 = .25. Considered as the superposition of the two states, \( V \) will thus be unequal to the product of their eigenvalues, and is said to be entangled. If the indeterminacy evinced by entangled states is interpreted as inconsistency, then the computational properties at issue might further have to be hyperintensional.\(^{20}\)

Finally, Epistemic Modal Algebra, as a fragment of the Language of Thought, is able to delineate the fundamental structure of the propositional attitudes targeted in 20th century philosophy; notably knowledge, belief, intentional action, and rational propositional intuition. Author (ms\(^1\)) argues, e.g., that the types of intention – acting intentionally; referring to an intention as an explanation for one’s course of action; and intending to pursue a course of action in the future – can be modeled as modal operators, whose semantic values are defined relative to an array of intensional parameters. Field (2001: 85-86) argues that – because possible worlds models of intentional states entrain logical equivalence for contents that ought to be distinct – such models underdetermine the identification of the intentions subserving agents’ behavior. The objection can be answered, in virtue of there being unique arrays of intensional parameters relative to which each of the types of intention (modeled as modal operators) receive their value. E.g., an agent can be said to act intentionally iff her ‘intention-in-action’ receives a positive semantic value, where a necessary condition on the latter is that there is at least one world in her epistemic modal space at which – relative to a context of a particular time and location, which constrains the admissibility of her possible actions as defined at a first index, and which subsequently

\(^{20}\) The nature of the indeterminacy in question is examined in Hawthorne (2010). For a thorough examination of approaches to the ontology of quantum mechanics, see Arntzenius (2012: ch. 3).
constrains the outcome thereof as defined at a second index – the intention is realized:

\[ \llbracket \text{Intenton-in-Action}(\phi) \rrbracket_w = 1 \text{ only if } \exists w' \llbracket \phi \rrbracket_{w', \gamma(t, l), \alpha, o} = 1. \]

The agent’s intention to pursue a course of action at a future time – i.e., her ‘intention-for-the-future’ – can receive a positive value only if there is a possible world and a future time, relative to which the possibility that a state, \( \phi \), is realized can be defined. Thus:

\[ \llbracket \text{Intention-for-the-future}(\phi) \rrbracket_w = 1 \text{ only if } \exists w' \forall t \exists t'[t < t' \land \llbracket \phi \rrbracket_{w', t'} = 1]. \]

In the setting of epistemic logic, epistemic necessity can further be modeled in a relational semantics encoding the property of knowledge, whereas epistemic possibility might encode the property of belief (cf. Hintikka, 1962; Fagin et al., 1995; Meyer and van der Hoek, 1995; Williamson, 2009; Author, ms2). Finally, Author (ms3) treats Gödel’s (1953) conception of rational propositional intuition as a modal operator in the setting of a bimodal, dynamic provability logic, and demonstrates how – via correspondence theory – the notion of ‘intuition-of’, i.e. a property of awareness of one’s cognitive states, can be shown to be formally equivalent to the notion of ‘intuition-that’, i.e. a modal operator concerning the value of the propositional state at issue.\(^{21}\)

\(^{21}\)The correspondence results between modal propositional and first-order logic are advanced in van Benthem (1983; 1984/2003) and Janin and Walukiewicz (1996). Availing of correspondence theory in order to account for the relationship between the notions of ‘intuition-of’ and ‘intuition-that’ resolves an inquiry posed by Parsons (1993: 233). As a dynamic interpretational modality, rational intuition can further serve as a guide to possible reinterpretations both of quantifier domains (cf. Fine, 2005) and of the extensions of mathematical vocabulary such as the membership-relation (cf. Uzquiano, 2015). This provides an account of Gödel’s (op. cit.; 1961) suggestion that rational intuition can serve as a guide to conceptual elucidation.
5 Objections and Replies

Dean (2016) raises two issues for a proposal similar to the foregoing, namely that algorithms – broadly construed – can be defined via abstraction principles which specify equivalence relations between implementations of computational properties in isomorphic machines.\textsuperscript{22} Dean’s candidate abstraction principle for algorithms as abstracts is: that the algorithm implemented by $M_1 = \text{the algorithm implemented by } M_2$ iff $M_1 \simeq M_2$.\textsuperscript{23} Both issues target the uniqueness of the algorithm purported to be identified by the abstraction principle.

The first issue generalizes Benacerraf’s (1965) contention that, in the reduction of number theory to set theory, there must be, and is not, a principled reason for which to prefer the identification of natural numbers with von Neumann ordinals (e.g., $2 = \{\emptyset, \{\emptyset\}\}$), rather than with Zermelo ordinals (i.e., order-types of well-orderings).\textsuperscript{24} The issue is evinced by the choice of whether to define algorithms as isomorphic \textit{iterations} of state transition functions (cf. Gurevich, 1999), or to define them as isomorphic \textit{recursions} of functions which assign values to a partially ordered set of elements (cf. Moschovakis, op. cit.). Linnebo and Pettigrew (2014: 10) argue similarly that, for two 'non-rigid' structures which admit of non-trivial automorphisms, one can define a graph which belies their isomorphism. E.g., let an abstraction principle be defined for the isomorphism between $S$ and $S^*$, such that

\textsuperscript{22}Fodor (2000: 105, n.4) and Piccinini (2004) note that the identification of mental states with their functional roles ought to be distinguished from identifying those functional roles with abstract computations. Conversely, a computational theory of mind need not be committed to the identification of abstract, computational operations with the functional organization of a machine. Identifying abstract computational properties with the functional organization of a creature’s mental states is thus a choice point, in theories of the nature of mental representation.


\textsuperscript{24}Cf. Zermelo (1908/1967) and von Neumann (1923/1967). Well-orderings are irreflexive, transitive, binary relations on all non-empty sets, which define a least or distinguished element in the sets.
∀S,S*[\mathbf{A}S = \mathbf{AS}^* \text{ iff } \langle S, R_1 \ldots R_n \rangle \simeq \langle S^*, R^*_1 \ldots R^*_n \rangle].

However, if there is a graph, G, such that:

\[ S = \{v_1, v_2\}, \text{ and } R = \{\langle v_1, v_2\rangle, \langle v_2, v_1\rangle\}, \]

then one can define an automorphism, \( f : G \simeq G \), such that \( f(v_1) = v_2 \) and \( f(v_2) = v_1 \), such that \( S^* = \{v_1\} \) while \( R^* = \{\langle v^*_1, v^*_1\rangle\} \). Then \( S^* \) has one element via the automorphism, while \( S \) has two. So, \( S \) and \( S^* \) are not, after all, isomorphic.

The second issue is that complexity is crucial to the identity criteria of algorithms. Two algorithms might be isomorphic, while the decidability of one algorithm is proportional to a deterministic polynomial function of the size of its input – with \( k \) a member of the natural numbers, \( N \), and \( \text{TIME} \) referring to the relevant complexity class: \( \bigcup_{k \in N} \text{TIME}(n^k) \) – and the decidability of the second algorithm will be proportional to a deterministic exponential function of the size of its input – \( \bigcup_{k \in N} \text{TIME}(2^{n^k}) \). The deterministic polynomial time complexity class is a subclass of the deterministic exponential time complexity class. However, there are problems decidable by algorithms only in polynomial time (e.g., the problem of primality testing, such that, for any two natural numbers, the numbers possess a greatest common divisor equal to 1), and only in exponential time (familarly from logic, e.g., the problem of satisfiability – i.e., whether, for a given formula, there exists a model which can validate it – and the problem of validity – i.e. whether a satisfiable formula is valid).²⁵

Both issues can be treated by noting that Dean’s discussion targets abstraction principles for the very notion of a computable function, rather than for abstraction principles for cognitive computational properties. It is a virtue of homotopic abstraction principles for cognitive intensional functions that both

²⁵For further discussion, see Dean (2015).
the temporal complexity class to which the functions belong, and the applications of the model, are subject to variation. Variance in the cognitive roles, for which Epistemic Modal Algebra provides a model, will crucially bear on the nature of the representational properties unique to the interpretation of the intensional functions at issue. Thus, e.g., when the internal representations in the Language of Thought – as modeled by Epistemic Modal Algebra – subserve perceptual representational states, then their contents will be individuated by both the computational constancies at issue and the external, environmental properties – e.g., the properties of lightness and distance – of the perceiver. A further virtue of the foregoing is that variance in the coding of Epistemic Modal Algebras – i.e. in the types of information over which the intensional functions will be defined – by constraint to a restriction of the Language of Thought to mathematical languages such as Peano arithmetic, permits homotopic abstraction principles to circumvent the Burali-Forti paradox for implicit definitions based on isomorphism.

The examples of instances of Epistemic Modal Algebra – witnessed by the possible worlds models in Bayesian perceptual psychology, linguistics, quantum information theory, and philosophy of mind – provide abductive support for the existence of the intensional functions specified in homotopic abstraction principles. The philosophical significance of independent, abductive support for the existence of epistemic modalities in the philosophy of mind and cognitive science is that the latter permits a circumvention of the objections to the abstractionist foundations of number theory that have accrued since its contemporary founding (cf. Wright, 1983). Eklund (2006) suggests, e.g., that the existence of the ab-

\[\text{\textsuperscript{26}}\text{The computational properties at issue can also be defined over non-propositional information states, such as cognitive maps possessed of geometric rather than logical structure. See, e.g., O'Keefe and Nadel (1978); Camp (2007); and Rescorla (2009).}\]

\[\text{\textsuperscript{27}}\text{Cf. Burali-Forti (1897/1967). Hodes (1984) and Hazen (1985) note that abstraction principles based on isomorphism with unrestricted comprehension entail the paradox.}\]
abstract objects which are the referents of numerical term-forming operators might need to be secured, prior to assuming that the abstraction principle for cardinal number is true. While Hale and Wright (2009) maintain, in response, that the truth of the relevant principles will be prior to the inquiry into whether the terms defined therein refer, they provide a preliminary endorsement of an 'abundant' conception of properties, according to which identifying the sense of a predicate will be sufficient for predicate reference. One aspect of the significance of empirical and philosophical instances of models of Epistemic Modal Algebra is thus that, by providing independent, abductive support for the truth of the homotopic abstraction principles for cognitive algorithms, the proposal remains neutral on the status of 'sparse' versus 'abundant' conceptions of properties. Another aspect of the philosophical significance of possible worlds semantics being availed of in Bayesian vision science, empirical linguistics, and quantum information theory, is that it belies the purportedly naturalistic grounds for Quine's (1963/1976) scepticism of de re modality.

A final objection derives from the work of Azzouni (2013: 9.1-9.2; forthcoming). Azzouni (op. cit.) argues for a position which he refers to as 'quantifier neutralism', according to which the truth-conditions for quantifiers are sufficient for identifying the latter, without requiring reference to domains. Thus, reference to the purported domains over which quantifiers range is a superfluous and misleading metaphor. On this approach, the truth-conditions for quantifiers will be theory-relative. In the absence of objects comprising domains over which the quantifiers are purported to range, there is thus no reason to believe that the truth-conditions in mathematical and scientific theories are ontologically-

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28 For identity conditions on abundant properties – where the domain of properties, in the semantics of second-order logic, is a subset of the domain of objects, and the properties are definable in a metalanguage by predicates whose satisfaction-conditions have been fixed – see Hale (2013). For a generalization of the abundant conception, such that the domain of properties is isomorphic to the powerset of the domain of objects, see Cook (2014).
committing. He suggests, then, that the explanation of the belief that there are objects derives from the 'involuntary' perception thereof (forthcoming: 282). The experience as of objects is then attributed to 'internal cognitive mechanisms' (286; see also pp. 8-9, 264, and the remainder of section 7.5).\footnote{For an examination of the psychological mechanisms necessary for object-perception, see, inter alia, Treisman and Gelade (1980); Spelke and Kinzler (2007); and Scholl and Flombaum (2010).}

Azzouni’s suggestion that there are 'internal cognitive mechanisms' subserving apparent object perception might betray an ontological commitment to there being abstracta; namely, the intensional, cognitive computational properties countenanced in Bayesian perceptual psychology. Azzouni (op. cit.: 7.8 - 7.9) anticipates and endeavors to counter the foregoing reply, by noting that the optimality of theories proffered in logic, metaphysics, and the natural sciences requires no commitment to the existence of the entities mentioned therein.

Azzouni’s argument against the existence of objects – concreta and abstracta alike – hinges on the individuation-conditions for quantifiers. He argues, as noted, that objectual, first-order quantifiers can be defined via metalinguistic truth-conditions (op. cit.: 10). The metalinguistic truth-conditions take, e.g., the form: \(\exists x.(P x)\) iff \(\exists X.\delta X\) (op. cit.). However, the foregoing theory of quantification is unduly restrictive. For example, the truth-condition for probability quantifiers is given by the formula:

\[(P x \geq r) \phi x,\]

where \(x\) is a bound second-order variable; \(P(x)\) encodes the probability that the set \(\{x: \phi x\}\) is less than or equal to \(r\); \(r\) is an element of the intersection between a subset of hereditarily countable sets and the real interval, \([0,1]\); and the quantifier, \((P x \geq r)\), is closed under finite conjunctions and disjunctions [cf. Keisler (1985: 509-510); Barwise and Feferman (1985: 507)]. Crucially, the metalinguistic truth-condition for probability quantifiers makes explicit refer-
ence to the real numbers.\textsuperscript{30} The claim, then, that quantifiers can be defined without reference to domains of objects does not generalize. So, if ontological commitment is generally to be eschewed of, the argument must proceed via a different means.

6 Concluding Remarks

In this essay, the equivalence relations countenanced in Homotopy Type Theory were availed of, in order to specify an abstraction principle for intensional, computational properties. The homotopic abstraction principle for intensional mental functions provides an epistemic conduit into our knowledge of cognitive algorithms as abstract objects. Because intensional functions in Epistemic Modal Algebra are deployed as core models in the philosophy of mind, Bayesian visual psychology, natural language semantics, and quantum information theory, there is independent abductive support for the truth of homotopic abstraction. Epistemic modality may thereby be recognized as both a compelling and a materially adequate candidate for the fundamental structure of mental representational states, and as thus comprising a fragment of the Language of Thought.

\textsuperscript{30}Note that, by contrast to Field’s (1980/2016: 55-56) approach, Azzouni’s eschewal of ontological commitment to concrete objects would preclude redefining the reals as relations of betweenness and congruence on points of spacetime.
References


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