Grothendieck Universes and Indefinite Extensibility

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Abstract

This essay endeavors to define the concept of indefinite extensibility in the setting of category theory. I argue that the generative property of indefinite extensibility for set-theoretic truths in the category of sets is identifiable with the elementary embeddings of large cardinal axioms. A modal coalgebraic automata’s mappings are further argued to account for both reinterpretations of quantifier domains as well as the ontological expansion effected by the elementary embeddings in the category of sets. The interaction between the interpretational and objective modalities of indefinite extensibility is defined via the epistemic interpretation of two-dimensional semantics. The semantics can be defined intensionally or hyperintensionally. By characterizing the modal profile of Ω-logical validity, and thus the generic invariance of mathematical truth, modal coalgebraic automata are further capable of capturing the notion of definiteness for set-theoretic truths, in order to yield a non-circular definition of indefinite extensibility.

1 Introduction

This essay endeavors to provide a characterization of the defining properties of indefinite extensibility for set-theoretic truths in the category of sets, i.e. generation and definiteness. The concept of indefinite extensibility is introduced by Dummett (1963/1978), in the setting of a discussion of the philosophical significance of Gödel’s (1931) first incompleteness theorem. Gödel’s first incompleteness theorem can be characterized as stating that – relative to a coding defined over the signature of first-order arithmetic, a predicate expressing the property of provability, and a fixed point construction – the formula can be defined as not satisfying the provability predicate. Dummett’s concern is with the conditions on our grasp of the concept of natural number, given that the latter figures in a formula whose truth appears to be satisfied despite the unprovability – and thus non-constructivist profile – thereof (186). His conclusion is that the concept of natural number ‘exhibits a particular variety of inherent vagueness, namely indefinite extensibility’, where a ‘concept is indefinitely extensible
if, for any definite characterisation of it, there is a natural extension of this characterisation, which yields a more inclusive concept; this extension will be made according to some general principle for generating such extensions, and, typically, the extended characterisation will be formulated by reference to the previous, unextended, characterisation’ (195-196). Elaborating on the notion of indefinite extensibility, Dummett (1996: 441) redefines the concept as follows: an ‘indefinitely extensible concept is one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it’. Subsequent approaches to the notion have endeavored to provide a more precise elucidation thereof, either by providing an explanation of the property which generalizes to an array of examples in number theory and set theory (cf. Wright and Shapiro, 2006), or by availing of modal notions in order to capture the properties of definiteness and extendability which are constitutive of the concept (cf. Fine, 2006; Linnebo, 2013; Uzquiano, 2015). However, the foregoing modal characterizations of indefinite extensibility have similarly been restricted to set-theoretic languages. Furthermore, the modal notions that the approaches avail of are taken to belong to a proprietary type which is irreducible to either the metaphysical or the logical interpretations of the operator.

The aim of this essay is to redress the foregoing, by providing a modal characterization of indefinite extensibility in the setting of category theory, rather than number or set theory. One virtue of the category-theoretic, modal definition of indefinite extensibility is that it provides for a robust account of the epistemological foundations of modal approaches to the ontology of mathematics. A second aspect of the philosophical significance of the examination is that it can serve to redress the lacuna noted in the appeal to an irreducible type of mathematical modality, which is argued (i) to be representational, (ii) still to bear on the ontological expansion of domains of sets, and yet (iii) not to range over metaphysical possibilities. By contrast to the latter approach, the category-theoretic characterization of indefinite extensibility is able to identify the mappings of coalgebraic non-deterministic automata with elementary embeddings and the modal properties of set-theoretic, Ω-logical consequence. Mappings receive their values relative to two parameters, the first ranging over epistemically possible worlds and the second ranging over objective, though not metaphysical, possible worlds. Mappings thus receive their values in an epistemic two-dimensional semantics.1

1Linnebo (2018b) discusses the differences between Putnam’s and Parsons’ accounts of the role of modality in mathematics. Berry (2022) also discusses the differences between the foregoing. Linnebo (op. cit.: 265-266) avails of two-dimensional indexing for the relation between interpretational and circumstantial modalities. In Linnebo (2018a), he characterizes the relation between interpretational and circumstantial modalities via a bimodal product logic, rather than a two-dimensional semantics. He countenances two commutativity principles – $\Box\Box\phi \iff \Box\Box\phi$ and $\Diamond\Diamond\phi \iff \Diamond\Diamond\phi$, with $\Box$ a circumstantial modality and $\Box$ an interpretational modality – although details a counterexample to which they are susceptible. The present approach occurs in the setting of epistemic two-dimensional semantics, such that there is a one-way dependence of metaphysical profiles on epistemic profiles. The question of whether there might perhaps be other fruitful interaction principles between epistemic and
In Section 2, I examine the extant approaches to explaining both the property and the understanding-conditions on the concept of indefinite extensibility. In Section 3, I outline the elements of the category theory of sets and define Grothendieck Universes. In Section 4, modal coalgebraic automata are availed of to model Grothendieck Universes, and I define the notion of indefinite extensibility in the category-theoretic setting. I argue that the category-theoretic definition of indefinite extensibility, via Grothendieck Universes as modal coalgebraic automata, yields an explanation of the generative property of indefinite extensibility for set-theoretic truths as well as of the notion of definiteness which figures in the definition. I argue that the generative property of indefinite extensibility can be captured with elementary embeddings. I argue, then, that the notion of definiteness for set-theoretic truths can be captured by the role of Grothendieck Universes-as-modal coalgebraic automata in characterizing the modal profile of Ω-logical consequence, where the latter accounts for the absoluteness of mathematical truths throughout the set-theoretic universe. The category-theoretic definition is shown to circumvent the issues faced by rival attempts to define indefinite extensibility via extensional and intensional notions within the setting of set theory. Section 5 provides concluding remarks.

2 Indefinite Extensibility in Set Theory: Modal and Extensional Approaches

Characterizations of indefinite extensibility have so far occurred in the language of set theory, and have availed of both extensional and intensional resources. In an attempt to define the notion of definiteness, Wright and Shapiro (op. cit.) argue, for example, that indefinite extensibility may be intuitively characterized as occurring when there is a function on a first-order concept, an application of the function on a sub-concept thereof is such that set-theoretic unions of the sub-concept and the function applied thereon are formed (266).

Formally, let Π be a higher-order concept of type τ. Let P be a first-order concept falling under Π of type τ. Let f be a function from entities to entities of the same type as P. Finally, let X be a sub-concept of P. P is indefinitely extensible if there is a function on P such that the union of the sub-concept and the function applied thereon is such that set-theoretic unions of the sub-concept and the function applied thereon are formed (266).

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extensible with respect to $\Pi$, if and only if:

\[ \epsilon(P) = f(X), \]
\[ \epsilon(X) = \neg[f(X)], \] and
\[ \exists X'[\Pi(X') = (X \cup \{f(X)\})] \] (op. cit.).

The notion of definiteness is then defined as the limitless preservation of ‘$\Pi$-hood’ by sub-concepts thereof ‘under iteration of the relevant operation’, $f$ (269).

The foregoing impresses as a necessary condition on the property of indefinite extensibility. Wright and Shapiro note, e.g., that the above formalization generalizes to an array of concepts countenanced in first-order number theory and analysis, including concepts of the finite ordinals (defined by iterations of the successor function); of countable ordinals (defined by countable order-types of well-orderings); of regular cardinals (defined as occurring when the cofinality of a cardinal, $\kappa$ – comprised of the unions of sets with cardinality less than $\kappa$ – is identical to $\kappa$); of large cardinals (defined by elementary embeddings from the universe of sets into proper subsets thereof, which specify critical points measured by the ordinals); of real numbers (defined as cuts of sets of rational numbers); and of Gödel numbers (defined as codings by natural numbers of symbols and formulas) (266-267).

As it stands, however, the definition might not be sufficient for the definition of indefinite extensibility, by being laconic about the reasons for which new sub-concepts – comprised as the union of preceding sub-concepts with a target operation defined thereon – are presumed interminably to generate. In response to the above desideratum, concerning the reasons for which indefinite extensibility might be engendered, philosophers have recently appealed to modal properties of the formation of sets. In his (2018a), Linnebo countenances both interpretational and metaphysical modalities, and he argues that the former also satisfy S4.2. Fine (2006) argues, e.g., that – in order to avoid the Russell property when quantifying over all sets – there are interpretational modalities which induce a reinterpretation of quantifier domains, and serve as a mechanism for tracking the ontological inflation of the hierarchy of sets via, e.g., the power-set operation (2007). Reinhardt (1974) and Williamson (2007) argue that modalities are inter-definable with counterfactuals. While Williamson (2016) argues both that imaginative exercises take the form of counterfactual presuppositions and that it is metaphysically possible to decide propositions which are undecidable relative to the current axioms of extensional mathematical languages such as ZF, Reinhardt (op. cit.) argues that large cardinal axioms and undecidable sentences in extensional ZF can similarly be imagined as obtaining via counterfactual presupposition.

In an examination of the iterative hierarchy of sets, Parsons (1977/1983) notes that the notion of potential infinity, as anticipated in Book 3, ch. 6 of Aristotle’s *Physics*, may be codified in a modal set theory by both a principle which is an instance of the Barcan formula (namely, for predicates $P$ and rigidifying predicates $Q$, $\forall x(Px \leftrightarrow Qx) \land \Box\{\forall x(\Box Qx \lor \Box \neg Qx) \land \forall R[\forall x \Box(Qx \rightarrow Rx) \rightarrow \Box \forall x(Qx \rightarrow Rx)]\}$ (fn. 24), as well as a principle for definable set-forming operations (e.g., unions) for Borel sets of reals $\square(\forall x)\diamond(\exists y)[y = x \cup \{x\}]$ (528).
The modal extension is argued to be a property of the imagination, or intuition, and to apply further to iterations of the successor function in an intensional variant of arithmetic (1979-1980).

Hellman (1990) develops the program intimated in Putnam (1967), and thus argues for an eliminativist, modal approach to mathematical structuralism as applied to second-order plural ZF. The possibilities at issue are taken to be logical – concerning both the consistency of a set of formulas as well as the possible satisfaction of existential formulas – and he specifies, further, an ‘extendability principle’, according to which ‘every natural model [of ZF] has a proper extension’ (421).

Extending Parsons’ project, Linnebo (2009, 2013) avails of a second-order, plural modal set theory in order to account for both the notion of potential infinity as well as the notion of definiteness. Similarly to Parsons’ use of the Barcan formula (i.e., $\forall \Box \phi \rightarrow \Box \forall \phi$), Linnebo’s principle for the foregoing is as follows: $\forall u(u \prec xx \rightarrow \Box \phi) \rightarrow \Box \forall u(u \prec xx \rightarrow \phi)$ (2013: 211). He argues, further, that the logic for the modal operator is S4.2, i.e. K $[\Box(\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)]$, T ($\Box \phi \rightarrow \phi$), 4 ($\Box \forall \phi \rightarrow \forall \Box \phi$), and G ($\Diamond \Box \phi \rightarrow \Box \Diamond \phi$). Studd (2013) examines the notion of indefinite extensibility by availing of a bimodal temporal logic. Uzquiano’s (2015) approach to defining the concept of indefinite extensibility argues that the height of the cumulative hierarchy is in fact fixed, and that indefinite extensibility can similarly be captured via the use of modal operators in second-order plural modal set theory. The modalities are taken to concern the possible reinterpretations of the intensions of the non-logical vocabulary – e.g., the set-membership relation – which figures in the augmentation of the theory with new axioms and the subsequent climb up the fixed hierarchy of sets (cf. Gödel, 1947/1964).

Author (ms1) proffers a novel epistemology of mathematics, based on an application of the epistemic interpretation of two-dimensional semantics in set-theoretic languages to the values of large cardinal axioms and undecidable sentences. Modulo logical constraints such as consistency and absoluteness in the extensions of ground models of the set-theoretic universe, the epistemic possibility that an undecidable proposition receives a value may serve, then, as a guide to the metaphysical possibility thereof. Finally, Author (blinded for review) argues that the modal profile of the consequence relation, in the $\Omega$-logic defined in Boolean-valued models of set-theory, can be captured by coalgebraic modal automata, and provides a necessary condition on the formal grasp of the concept of ‘set’.

The foregoing accounts of the metaphysics and epistemology of indefinite extensibility are each defined in the languages of number and set theory. In the following section, I examine the nature of indefinite extensibility in the setting of category theory, instead. One aspect of the philosophical significance of the examination is that it can serve to provide an analysis of the mathematical modality at issue. By contrast to Hellman’s approach, which takes the mathematical modality at issue to be logical (cf. Field, 1989: 37; Rayo, 2013), and Fine’s (op. cit.) approach, which takes the mathematical modality to be either interpretational or dynamic, I argue in the following sections that the mathe-
matical modality can be captured by mappings in coalgebraic modal automata, where the latter can model Grothendieck Universes, and the mappings are defined relative to two parameters, the first ranging over epistemically possible worlds and the second ranging over objective possible worlds.

The formal clauses for epistemic and objective mathematical modalities are as follows:

Let $C$ denote a set of epistemically possibilities, such that $\ell_\phi \subseteq C$; ($\phi$ is a formula encoding a state of information at an epistemically possible world).

- $\text{pri}(x) = \lambda c. \ell_x[c,c]$;
  (This is an epistemic intension, such that the two parameters relative to which $x$ – a propositional variable – obtains its value are epistemically possible worlds).

- $\text{sec}(x) = \lambda w. \ell_x[w,w]$;
  (This is a subjunctive intension, such that the two parameters relative to which $x$ obtains its value are objective possible worlds).

Epistemic mathematical modality is constrained by consistency, and the formal techniques of provability and forcing. A mathematical formula is metaphysically impossible, if it can be disproved or induces inconsistency in a model.

- $\text{2D}(x) = \lambda c \lambda w. \ell_x[c,w] = 1$.
  (This is a 2D intension. The intension determines a semantic value relative to two parameters, the first ranges over worlds from a first space and the second ranges over worlds from a distinct, second space. The value of the formula relative to the first parameter determines the value of the formula relative to the second.)

If one prefers hyperintensional semantics to possible worlds semantics – in order e.g. to avoid the situation in intensional semantics according to which all necessary formulas express the same proposition because they are true at all possible worlds – one can avail of the following epistemic two-dimensional truthmaker semantics, which specifies a notion of exact verification in a state space and where states are parts of whole worlds (Fine 2017a,b; Hawke and Özkün, forthcoming). According to truthmaker semantics for epistemic logic, a modalized state space model is a tuple $\langle S, P, \leq, v \rangle$, where $S$ is a non-empty set of states, i.e. parts of the elements in $A$ in the foregoing epistemic modal algebra $U$, $P$ is the subspace of possible states where states $s$ and $t$ comprise a fusion when $s \sqcup t \in P$, $\leq$ is a partial order, and $v$: $\text{Prop} \rightarrow (2^S \times 2^S)$ assigns a bilateral proposition $(p^+, p^-)$ to each atom $p \in \text{Prop}$ with $p^+$ and $p^-$ incompatible (Hawke and Özkün, forthcoming: 10-11). Exact verification $(\vdash)$ and exact falsification $(\dashv)$ are recursively defined as follows (Fine, 2017a: 19; Hawke and Özkün, forthcoming: 11):

$s \vdash p$ if $s \in \ell_p^+$
(s verifies $p$, if $s$ is a truthmaker for $p$ i.e. if $s$ is in $p$’s extension);

$s \vdash_{\lnot} p$ if $s \in \ell_p^-$
(s falsifies $p$, if $s$ is a falsifier for $p$ i.e. if $s$ is in $p$’s anti-extension);

$s \vdash p$ if $s \vdash_{\lnot} p$
(s verifies not $p$, if $s$ falsifies $p$);
s ⊬ p if s ⊨ p
(s falsifies not p, if s verifies p);

s ⊨ p ∧ q if ∃t,u, t ⊨ p, u ⊨ q, and s = t ∪ u
(s verifies p and q, if s is the union of states, t and u, t verifies p, and u verifies q);

s ⊩ p ∧ q if s ⊩ p or s ⊩ q
(s verifies p and q, if s is the fusion of states, t and u, t verifies p, and u verifies q);

s ⊣ p ∧ q if s ⊣ p or s ⊣ q
(s falsifies p and q, if s falsifies p or s falsifies q);

s ⊨ p ∨ q if s ⊨ p or s ⊨ q
(s verifies p or q, if s verifies p or s verifies q);

s ⊣ p ∨ q if ∃t,u, t ⊣ p, u ⊣ q, and s = t ∪ u
(s falsifies p or q, if s is the state overlapping the states, t and u, t falsifies p, and u falsifies q);

s ⊨ ∀x ϕ(x) if ∃s_1, ..., s_n, with s_1 ⊨ ϕ(a_1), ..., s_n ⊨ ϕ(a_n), and s = s_1 ∪ ... ∪ s_n
(s verifies ∀x ϕ(x) "if it is the fusion of verifiers of its instances ϕ(a_1), ..., ϕ(a_n)" (Fine, 2017c));

s ⊣ ∀x ϕ(x) if s ⊣ ϕ(a) for some individual a in a domain of individuals (op. cit.)
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(s falsifies ∃x ϕ(x) "if it is the fusion of falsifiers of its instances" (op. cit.));

s exactly verifies ϕ if and only if s ⊨ ϕ if s ∈ [ϕ];

s inexacty verifies ϕ if and only if s ⊢ ϕ if s ∈ [ϕ];

s loosely verifies ϕ if and only if, ∀t, s ⊤ t, s ⊤ t ⊨ ϕ, where ∩ is the relation of compatibility (35-36);

s ⊨ Aϕ if and only if for all t ∈ P there is a t' ∈ P such that t' ⊢ ϕ;

s ⊣ Aϕ if and only if there is a t ∈ P such that for all u ∈ P either t ⊤ u or u ⊤ t ⊬ ϕ, where Aϕ denotes the apriority of ϕ;

s ⊨ A(∀x ϕ(x)) if and only if for all u ∈ P there is a u' ∈ P such that u ⊨ [u' ⊨ ϕ(a_1), ..., ϕ(a_n), and u' = s_1 ∪ ... ∪ s_n];

s ⊣ A(∃x ϕ(x)) if and only if for all u ∈ P there is a u' ∈ P such that u ⊣ [u' ⊣ ϕ(a_1), ..., ϕ(a_n), and u' = s_1 ∪ ... ∪ s_n].

In order to account for two-dimensional indexing, we augment the model, M, with a second state space, S*, on which we define both a new parthood relation, ⊣*, and partial function, V*, which serves to map propositions in a domain, D, to pairs of subsets of S*, {1,0}, i.e. the verifier and falsifier of p, such that [P]^+ = 1 and [P]^− = 0. Thus, M = ⟨S, S*, D, ⊣*, ≤*, V, V*⟩. The two-dimensional hyperintensional profile of propositions may then be recorded by defining the value of p relative to two parameters, c.i: c ranges over subsets of S, and i ranges over subsets of S*. 
(*) $M, s \in S, s^* \in S^* \vdash p$ iff:

(i) $\exists_c \llbracket p \rrbracket^c \cdot c = 1$ if $s \in \llbracket p \rrbracket^+$; and

(ii) $\exists_i \llbracket p \rrbracket^c \cdot i = 1$ if $s^* \in \llbracket p \rrbracket^+$

(Distinct states, $s, s^*$, from distinct state spaces, $S, S^*$, provide a two-dimensional verification for a proposition, $p$, if the value of $p$ is provided a truthmaker by $s$. The value of $p$ as verified by $s$ determines the value of $p$ as verified by $s^*$).

We say that $p$ is hyper-rigid iff:

(**) $M, s \in S, s^* \in S^* \vdash p$ iff:

(i) $\forall c' \llbracket p \rrbracket^c \cdot c' = 1$ if $s \in \llbracket p \rrbracket^+$; and

(ii) $\forall i \llbracket p \rrbracket^c \cdot i = 1$ if $s^* \in \llbracket p \rrbracket^+$

The foregoing provides a two-dimensional hyperintensional semantic framework within which to interpret the values of a proposition. Further:

$s$ is a two-dimensional exact truthmaker of $p$ if and only if (*);

$s$ is a two-dimensional inexact truthmaker of $p$ if and only if $\exists s' \llbracket S, s \rightarrow s' \rrbracket$, $s' \vdash p$, such that $\exists c \llbracket s' \rrbracket^c \cdot c = 1$ if $s' \in \llbracket p \rrbracket^+$, and $\exists i \llbracket s' \rrbracket^c \cdot i = 1$ if $s^* \in \llbracket p \rrbracket^+$;

$s$ is a two-dimensional loose truthmaker of $p$ if and only if, $\exists t, s.t. s \sqcup t \vdash p$:

$\exists c \llbracket t \rrbracket^c \cdot c = 1$ if $s' \in \llbracket p \rrbracket^+$, and $\exists i \llbracket t \rrbracket^c \cdot i = 1$ if $s^* \in \llbracket p \rrbracket^+$.

Epistemic (primary), subjunctive (secondary), and 2D hyperintensions can be defined as follows, where hyperintensions are functions from states to extensions, and intensions are functions from worlds to extensions:

- **Epistemic Hyperintension**:
  \[
  \text{pri}(x) = \lambda s. \llbracket x \rrbracket^s \cdot r, \text{ with } s \text{ a state in the state space defined over the foregoing epistemic modal algebra, } U; \]

- **Subjunctive Hyperintension**:
  \[
  \text{sec}_{\text{vo}}(x) = \lambda i. \llbracket x \rrbracket^i \cdot q, \text{ with } i \text{ a state in objective state space } I; \]

- **2D-Hyperintension**:
  \[
  \text{2D}(x) = \lambda s \lambda w \llbracket x \rrbracket^w \cdot r = 1. \]

Following the presentation of topic models in Berto (2018; 2019), Canavotto et al (2020), and Berto and Hawke (2021), atomic topics comprising a set of topics, $T$, record the hyperintensional intentional content of atomic formulas, i.e. what the atomic formulas are about at a hyperintensional level. Topic fusion is a binary operation, such that for all $x, y, z \in T$, the following properties are satisfied: idempotence $(x \oplus x = x)$, commutativity $(x \oplus y = y \oplus x)$, and associativity $[(x \oplus y) \oplus z = x \oplus (y \oplus z)]$ (Berto, 2018: 5). Topic parthood is a partial order, $\leq$, defined as $\forall x, y \in T(x \leq y \iff x \oplus y = y)$ (op. cit.: 5-6).
Atomic topics are defined as follows: 
\[ \text{Atom}(x) \iff \neg \exists y < x, \text{ with } < \text{ a strict order.} \]

Topic parthood is thus a partial ordering such that, for all \( x, y, z \in T \), the following properties are satisfied: reflexivity (\( x \leq x \)), antisymmetry (\( x \leq y \land y \leq x \implies x = y \)), and transitivity (\( x \leq y \land y \leq z \implies x \leq z \)) (6). A topic frame can then be defined as \( \{ W, R, T, \oplus, t \} \), with \( t \) a function assigning atomic topics to atomic formulas. For formulas, \( \phi \), atomic formulas, \( p, q, r \) (\( p_1, p_2, \ldots \)), and a set of atomic topics, \( Ut(\phi) = \{ p_1, \ldots, p_n \} \), the topic of \( \phi \), \( t(\phi) = \bigoplus Ut(\phi) = t(p_1) \oplus \ldots \oplus t(p_n) \) (op. cit.). Topics are hyperintensional, though not as fine-grained as syntax. Thus \( t(\phi) = t(\neg \neg \phi) \), \( t(\phi) = t(\neg \phi) \), \( t(\phi \land \psi) = t(\phi) \oplus t(\psi) = t(\phi \lor \psi) \) (op. cit.).

The diamond and box operators can then be defined relative to topics:
\[
\langle M, w \rangle \Vdash \Diamond t(\phi) \iff \langle R_{w,t} \rangle(\phi) \\
\langle M, w \rangle \Vdash \Box t(\phi) \iff \langle R_{w,t} \rangle(\phi)
\]

We can then combine topics with truthmakers rather than worlds, thus countenancing doubly hyperintensional semantics, i.e. topic-sensitive epistemic two-dimensional truthmaker semantics:

- **Topic-sensitive Epistemic Hyperintension:**
  \[
  \text{pri}(x) = \lambda s \lambda t. [x]^{s_{\oplus t}, s_{\oplus t}}, \text{ with } s \text{ a truthmaker from an epistemic state space.}
  \]

- **Topic-sensitive Subjunctive Hyperintension:**
  \[
  \text{sec}(x) = \lambda w \lambda t. [x]^{t_{\oplus w}, t_{\oplus w}}, \text{ with } w \text{ a truthmaker from a metaphysical state space.}
  \]

- **Topic-sensitive 2D-Hyperintension:**
  \[
  2D(x) = \lambda s \lambda w \lambda t. [x]^{s_{\oplus t}, w_{\oplus t}} = 1.
  \]

In the subsequent section, I outline the axioms of category theory and define Grothendieck Universes. In the section that follows, I examine the properties of indefinite extensibility in the category-theoretic setting.

### 3 Grothendieck Universes

We work within a two-sorted language in which the Eilenberg - Mac Lane Axioms of category theory are specified. Types are labeled \( A, B, C \) for objects and \( x, y, z \) for arrows. The relevant operators are the domain operator, \( \text{Dom} \), which takes arrows to objects; the codomain operator, \( \text{Cod} \), which operates similarly, and the identity operator, \( 1_x \). Finally, a composition relation, \( C(x,y; z) \), is defined on arrows, where the open formula reads \( z \) is the composite of \( x \) and \( y \) (McLarty, 2008: 13). The Eilenberg - Mac Lane axioms can then be defined as follows:
• Axioms of Domain and Codomain:
  \(\forall f, g, h, \text{ if } C(f, g, h), \text{ then } \text{Dom } f = \text{Dom } h \text{ and } \text{Cod } f = \text{Dom } g \text{ and } \text{Cod } g = \text{Cod } h\)

• Axioms of Existence and Uniqueness of Composites:
  \(\forall f, g, \text{ if } \text{Cod } f = \text{Dom } g, \text{ then } \exists! h, \text{ s.t. } C(f, g; h)\)

• Axioms for Identity Arrows:
  \(\forall A, \text{Dom}_{1A} = \text{Cod}_{1A} = A\)
  \(\forall f, C(1_{\text{Dom } f}, f; f)\)
  \(\forall f, C(f, 1_{\text{Dom } f}; f)\)

• Axiom of Associativity of Composition:
  \(\forall f, g, h, i, j, k, \text{ if } C(f, g; i) \text{ and } C(g, h; j) \text{ and } C(f, j; k), \text{ then } C(i, h; k) \text{ (op. cit.)}\)

Categorical Set Theory is defined by augmenting the Eilenberg - Mac Lane axioms with the axioms of Lawvere’s Elementary Theory for the Category of Sets (ETCS) (op. cit.; Lawvere, 2005). Following McLarty, we define the singleton of a set as one for which ‘every set has exactly one function to it (op. cit.: 25). An element of a set A, \(x \in A\), is a function \(x: 1 \to A\) and ‘a set A has as exactly many elements \(x \in A\) as there are functions \(x: 1 \to A\) from a singleton 1 to A: each element x corresponds to the function x: 1 \to A taking the sole element of 1 to the element x of A, and vice versa’ (op. cit.). Composition occurs if and only if, for two arrows, f,g, and object x, \((gf)(x) = g(f(x))\) (26). Finally, an equalizer \(e: E \to A\) for a pair of functions f,g: A \to B is defined as ‘a universal solution to the equation fe = ge’ (29). The axioms are then defined as follows (op. cit.):

• Every pair of sets, A,B, has a product:
  \(\forall T, f, g, \text{ with } f: T \to A, g: T \to B, \exists! (f, g): T \to A \times B\)

• Every parallel pair of functions, f,g: A \to B, has an equalizer:
  \(\forall T, h, \text{ with } fh = gh, \exists! u: T \to E\)

• There is a function set from each set A to each set B:
  \(\forall C \text{ and } g: C \times A \to B, \exists! g': C \to B^A\)

• There is a truth value true: 1 \to 2:
  \(\forall A \text{ and monic } S \to A \text{ (where being monic means being an injective function), } \exists! \chi, \text{ with } \chi \text{ an arrow, such that } S \text{ is an equalizer}\)

• There is a natural number triple, \(N, 0, s\):
  \(\forall T \text{ and } x: 1 \to T \text{ and } f: T \to T, \exists! u: N \to T\)

• Extensionality
  \(\forall f \neq g: A \to B, \exists! x: 1 \to A, \text{ with } f(x) \neq g(x)\)
From the axioms of ETCS, only a version of the ZF separation axiom with bounded quantifiers can be recovered (37). The axiom of separation states that \( \forall \mathcal{F}, y. \exists z. \forall w. w \in z \iff w \in y \land A(w, \mathcal{F}) \), with \( \mathcal{F} \) a parameter. In order to redress the restriction to bounded quantifiers, we work within a stronger category theory for sets, i.e. the ‘category of categories as foundation’ (CCAF). The axioms of the CCAF build upon those of both ETCS and Eilenberg - Mac Lane category theory, by augmenting them with the following (53):

- Every category \( C \) has a unique functor, \( C \to 1 \)
- The higher category \( 2 \) has exactly two functors from \( 1 \) and \( 3 \) to itself\(^2\)
- Let a pushout be defined such that if \( f: A \to C \) and \( g: B \to C \), then \( a: C \to A \) and \( b: C \to B \) (Pettigrew, ms: 19). The category \( 3 \) is a pushout, and there is a functor \( \gamma: 2 \to 3 \), with \( \gamma_0 = a_0 \) and \( \gamma_1 = b_1 \).

2\(^{\text{Higher category theory countenances } k\text{-morphisms between } (k-1)\text{-morphisms for all } k \in \mathbb{N}, \text{by contrast to first order category theory which countenances morphisms between objects. See https://ncatlab.org/nlab/show/higher+category+theory}}\)

- Arrow Extensionality
  \( \forall F, G: A \to B, \text{if } F \neq G \text{ then } \exists f: 2 \to A \text{ with } Ff \neq Gf. \)

Grothendieck Universes can then be defined as follows.\(^3\)

An elementary topos is a category which

- has finite limits,
- is cartesian closed, and
- has a subobject classifier.\(^4\)

---

\(^2\)Higher category theory countenances \( k\)-morphisms between \((k-1)\)-morphisms for all \( k \in \mathbb{N} \), by contrast to first order category theory which countenances morphisms between objects. See https://ncatlab.org/nlab/show/higher+category+theory

\(^3\)All of the definitions below in this section follow the presentation in https://ncatlab.org/nlab/, with particular links for the provenance of the definitions cited in footnotes.

\(^4\)https://ncatlab.org/nlab/show/topos.
A subobject classifier is a monomorphism – where a monomorphism is an ‘injective map of sets from the category Set to arbitrary categories’ – ‘true: * → Ω out of the terminal object, such that for every monomorphism U → X in C there is a unique morphism ξ_U: X → Ω such that the following square commutes and is Cartesian (a pullback):

\[
\begin{array}{ccc}
U & \xrightarrow{pb} & * \\
\downarrow & & \downarrow \\
X & \xrightarrow{\xi_U} & Ω
\end{array}
\]

If it exists, the object Ω is also called the object of truth values, a global element F → Ω is called a truth value and the element true: * → Ω, with ↪ an injective monomorphism, is the truth value true, where all these terms allude to the internal logic of the category C'.

A Grothendieck Universe in a topos E is a morphism e: E → U. el: E → U is an U-indexed family of objects/sets and a morphism a: A → I (i.e. an I-indexed family of objects) is U-small if there exists a morphism f: I → U and a pullback square:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & E \\
\downarrow & & \downarrow \\
I & \xrightarrow{el} & U
\end{array}
\]

An object X is U-small if X → 1 is U-small. The axioms of a Grothendieck Universe are as follows:

1. (1) Every monomorphism is U-small.
2. (2) The composite of U-small morphisms is U-small.
3. (3) If f: A → I and g: B → A are U-small, then so is the dependent product \(\prod_f g\). \(\prod_f\) is the right adjoint to \(f^*\): E / I → E / A), where the slice category, C / c, of a category C over an object c ∈ C has objects that are all arrows f ∈ C such that cod(f) = c, and morphisms g: X → X' ∈ C from f: X → c to f': X' → c such that f' ∘ g = f.
4. (4) The subobject classifier Ω is U-small.

‘(1) corresponds to transitivity of a Grothendieck universe, (3) and (4) correspond to closure under power sets, and (2) corresponds to closure under indexed unions.'

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5https://ncatlab.org/nlab/show/monomorphism; https://ncatlab.org/nlab/show/subobject+classifier
6https://ncatlab.org/nlab/show/universe+in+a+topos
7https://ncatlab.org/nlab/show/over+category

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4 Modal Coalgebraic Automata and Indefinite Extensibility

This section examines, finally, the reasons for which category theory provides a more theoretically adequate setting in which to define indefinite extensibility than do competing approaches such as the Neo-Fregean epistemology of mathematics. According, e.g., to the Neo-Fregean program, concepts of number in arithmetic and analysis are definable via implicit definitions which take the form of abstraction principles. Abstraction principles specify biconditionals in which – on the left-hand side of the formula – an identity is taken to hold between numerical term-forming operators from entities of a type to abstract objects, and – on the right-hand side of the formula – an equivalence relation on such entities is assumed to hold.

In the case of cardinal numbers, the relevant abstraction principle is referred to as Hume's principle, and states that, for all x and y, the number of the x's is identical to the number of the y's if and only if the x's and the y's can be put into a one-to-one correspondence, i.e., there is a bijection from the x's onto the y's.

\[ \forall A \forall B [Nx: A = Nx: B \equiv \exists R[\forall x(Ax \rightarrow \exists y(By \wedge Rxz \rightarrow y = z)] \wedge \forall y(By \rightarrow \exists x(Ax \wedge Rxz \wedge \forall z(Az \wedge Rzy \rightarrow x = z))]]. \]

Abstraction principles for the concepts of other numbers have further been specified. Thus, e.g., Shapiro (2000: 337-340) specifies an abstraction principle for real numbers, which proceeds along the method of Dedekind’s definition of the reals (cf. Wright, 2007: 172). According to the latter method, one proceeds by specifying an abstraction principle which avails of the natural numbers, in order to define pairs of finite cardinals: 

\[ \forall x,y,z,w [\langle x,y \rangle = \langle z,w \rangle \iff x = z \wedge y = w]. \]

A second abstraction principle is defined which takes the differences of the foregoing pairs of cardinals, identifying the differences with integers: 

\[ \text{Diff}(\langle x,y \rangle) = \text{Diff}(\langle z,w \rangle) \iff x + w = y + z]. \]

One specifies, then, a principle for quotients of the integers, identifying them subsequently with the rational numbers: 

\[ Q(\langle m,n \rangle) = Q(\langle p,q \rangle) \iff n = 0 \wedge q = 0 \vee n \neq 0 \wedge q \neq 0 \wedge m x q = n x p]. \]

Finally, one specifies sets of rational numbers, i.e. the Dedekind cuts thereof, and identifies them with the reals: 

\[ \forall F,G[\text{Cut}(F) = \text{Cut}(G) \iff \forall r(F \leq r \iff G \leq r)]. \]

The abstractionist program faces several challenges, including whether conditions can be delineated for the abstraction principles, in order for the principles to avoid entraining inconsistency\(^8\); whether unions of abstraction principles can avoid the problem of generating more abstracts than concepts (Fine, 2002); and whether abstraction principles can be specified for mathematical entities in branches of mathematics beyond first and second-order arithmetic (cf. Boolos, 1997; Hale, 2000; Shapiro, op. cit.; and Wright, 2000). I will argue that the last issue – i.e., being able to countenance definitions for the entities and structures

\[^8\text{Cf. Hodes (1984); Hazen (1985); Boolos (1990); Heck (1992); Fine (2002); Weir (2003); Cook and Ebert (2005); Linnebo and Uzquiano (2009); Linnebo (2010); and Walsh (2016).} \]
in branches of mathematics beyond first and second-order arithmetic – is a crucial desideratum, the satisfaction of which remains elusive for the Neo-Fregean program while yet being satisfiable and thus adding in favor of the modal platonist approach that is outlined in what follows.

One issue for the attempt, along abstractionist lines, to provide an implicit definition for the concept of set is that doing so with an unrestricted comprehension principle yields a principle identical to Frege’s (1893/2013) Basic Law V; and thus – in virtue of Russell’s paradox – entrains inconsistency. However, two alternative formulas can be defined, in order to provide a suitable restriction to the inconsistent abstraction principle. The first, conditional principle states that
\[
\forall F,G (\text{Good}(F) \lor \text{Good}(G) \rightarrow [\{x|Fx\} = \{Gx\} \iff \forall x (Fx \iff Gx)]).
\]
The second principle is an unconditional version of the foregoing, and states that
\[
\forall F,G (\{x|Fx\} = \{Gx\} \iff [\text{Good}(F) \lor \text{Good}(G) \rightarrow \forall x (Fx \iff Gx)]).
\]
Following von Neumann’s (1925/1967: 401-402) suggestion that Russell’s paradox can be avoided with a restriction of the set comprehension principle to one which satisfies a constraint on the limitation of its size, Boolos (1997) suggests that the ‘Good’ predicate in the above principles is intensionally isomorphic to the notion of smallness in set size, and refers to the principle as New V. However, New V is insufficient for deriving all of the axioms of ZF set theory, precluding, in particular, both the axioms of infinity and the power-set axiom (cf. Wright and Hale, 2005: 193). Further, there are other branches of number theory for which it is unclear whether acceptable abstraction principles can be specified. Wiles’ proof of Fermat’s Last Theorem (i.e., that, save for when one of the variables is 0, the Diophantine equation, \(x^n = y^n = z^n\), has no solutions when \(n > 2\); cf. Hardy and Wright, 1979: 190) relies, e.g., on both invariants and Grothendieck Universes in cohomological number theory (cf. McLarty, 2009: 4).

The foregoing issues with regard to the definability of abstracta in number theory, algebraic geometry (McLarty, op. cit.: 6-8), set theory, et al., can be circumvented in the category-theoretic setting; and in particular by coalgebras. In the remainder of this section, I endeavor to demonstrate how modal coalgebraic automata are able to countenance two of the fundamental properties of indefinite extensibility. The first concerns the property of generation. The second includes the properties of intensional and extensional definiteness.

A labeled transition system is a tuple, \(\text{LTS} \equiv (M, V, R)\), comprised of a set of worlds, \(M\); a valuation, \(V\), from \(M\) to its powerset, \(\mathcal{P}(M)\); and a family of accessibility relations, \(R\). So \(\text{LTS} = (M, V, R)\) (cf. Venema, 2012: 7). A Kripke coalgebra combines \(V\) and \(R\) into a Kripke functor, \(\sigma_R\); i.e. the set of binary morphisms from \(M\) to \(\mathcal{P}(M)\) (op. cit.: 7-8). Thus, for an \(s \in M\), \(\sigma(s) := [\sigma_V(s), \sigma_R(s)]\) (op. cit.). \(\sigma(s)\) can be interpreted both epistemically and metaphysically. Thus, \(\sigma(s)^{s \in M, \wedge \Gamma, d}\). Satisfaction for the system is defined inductively as follows: For a formula \(\phi\) defined at a state, \(s\), in \(M\),
\[
\begin{align*}
\llbracket \phi \rrbracket^M_s &= V(s) \quad \text{(9)} \\
\llbracket \neg \phi \rrbracket^M_s &= S - V(s) \\
\llbracket \perp \rrbracket^M_s &= \emptyset
\end{align*}
\]

\(\text{Equivalently, } M, s \models \phi \text{ if } s \in V(\phi) \text{ (9)}.\)
Kripke coalgebras can be availed of in order to model Grothendieck Universes. In CCAF, the elementary embeddings which are jointly necessary and sufficient for positing the existence of large cardinals can further be identified with the transition functions of modal coalgebraic automata, defined relative to two parameters, with the first ranging over topic-sensitive epistemic truthmakers and the second ranging over metaphysical truthmakers. A coalgebraic model of deterministic automata can finally be thus defined (391). An automaton is a tuple, $A = \langle A, a_I, C, \Xi, F \rangle$, such that $A$ is the state space of the automaton $A$; $a_I \in A$ is the automaton’s initial state; $C$ is the coding for the automaton’s alphabet, mapping numerals to the natural numbers; $\Xi : A \times C \rightarrow A$ is a transition function defined relative to a parameter ranging over epistemically possible worlds or hyperintensional states which determines the value of the function relative to a second parameter ranging over objectively possible worlds or states, and $F \subseteq A$ is the collection of admissible states, where $F$ maps $A$ to $\{1,0\}$, such that $F : A \rightarrow 1$ if $a \in F$ and $A \rightarrow 0$ if $a \notin F$ (op. cit.).

Modal automata are defined over a modal one-step language (Venema, 2020: 7.2). With $A$ being a set of propositional variables the set $\text{Latt}(X)$ of lattice terms over $X$ has the following grammar:

$$\pi ::= \bot | \top | x | \pi \land \pi | \pi \lor \pi,$$

with $x \in X$ and $\pi \in \text{Latt}(A)$ (op. cit.).

A second way to capture the dependence of objective modalities or states on epistemic modalities or states is via interaction principles such as $\Box \land \Box \Leftrightarrow \Box \land \Box$, with $\Box$ an epistemic modality and $\Box$ an objective modality.\(^{11}\)

Hamkins and Linnebo (2022) argue that the modal logic of Grothendieck potentialism has a lower bound of S4.3 [\(\Box \land \Box \rightarrow \Box \lor (\Box \land \Box)\lor (\Box \land \Box)\lor (\Box \land \Box)\) and an upper bound of S5 (KTE; E: $\neg \Box \rightarrow \Box \neg \Box$). The idea of accounting for indefinite extensibility with regard to category-theoretic Grothendieck Universes came to mind in January 2016, and this chapter was written in that month. Hamkins and Linnebo’s paper was posted on Arxiv.org in 2017.\(^{11}\)

\(^{10}\)Hamkins and Linnebo (2022) argue that the modal logic of Grothendieck potentialism has a lower bound of S4.3 [\(\Box \land \Box \rightarrow \Box \lor (\Box \land \Box)\lor (\Box \land \Box)\) and an upper bound of S5 (KTE; E: $\neg \Box \rightarrow \Box \neg \Box$). The idea of accounting for indefinite extensibility with regard to category-theoretic Grothendieck Universes came to mind in January 2016, and this chapter was written in that month. Hamkins and Linnebo’s paper was posted on Arxiv.org in 2017.

\(^{11}\)Other such interaction principles might include Roberts (2019)’s bimodal version of the converse Barcan formula: $\Box \forall \phi \rightarrow \forall \Box \phi$ (1159); as well as interpretation-constrained circumstantial 5 axiom: $\Box \forall \phi \rightarrow \forall \Box \phi$ (1161).
The set, $1\text{ML}(A)$, of modal one-step formulas over $A$ has the following grammar:

$$\alpha \in A \ ::= \bot \ | \top \ | \lozenge \pi \ | \square \pi \ | \square \lozenge \pi \ \iff \square \square \pi \ | \alpha \land \alpha \ | \alpha \lor \alpha \ \text{(op. cit.)}.$$  

A modal P-automaton $A$ is a quadruple $(A, \Theta, a_I)$, with $A$ a non-empty finite set of states, $a_I \in A$ an initial state, and the transition map $\Theta: A \times \wp(P) \to 1\text{ML}(A)$ maps states to modal one-step formulas, with $\wp(P)$ the powerset of the set of proposition letters, $P$ (op. cit.: 7.3).

Finally, Kripke coalgebras are the dual representations of Boolean-valued models of the $\Omega$-logic of set theory (cf. Venema, 2007). Modal coalgebraic automata are able, then, to countenance the constitutive conditions of indefinite extensibility. Modal coalgebraic automata are capable, e.g., of defining both the generative property of indefinite extensibility, as well as the notion of definiteness which figures therein. Further, the category-theoretic definition of indefinite extensibility is arguably preferable to those advanced in the set-theoretic setting, because modal coalgebraic automata can account for both the modal profile and the epistemic tractability of $\Omega$-logical consequence.

The generative property of indefinite extensibility for set-theoretic truths is captured by elementary embeddings, $j: A \to B$, $\phi(a_1, \ldots, a_n) \in A$ if and only if $\phi(j(a_1), \ldots, j(a_n)) \in B$.

The notion of definiteness for set-theoretic truths is captured by the role of modal coalgebraic automata in characterizing the modal profile of $\Omega$-logical validity. $\Omega$-validity can be defined as follows:

$\text{Sent}$ denotes a set of sentences in a first-order language of set theory. For $T \cup \{\phi\} \subseteq \text{Sent}$,

$$T \models_{\Omega} \phi, \text{ if for all ordinals } \alpha \text{ and countable Boolean algebras } B, \text{ if } V^B\models T, \text{ then } V^B_\alpha \models \phi \text{ (Bagaria et al., 2006). The } \Omega\text{-Conjecture states that } V \models_{\Omega} \phi \iff V^B \models_{\Omega} \phi \text{ (Woodin, ms). Thus, } \Omega\text{-logical validity is invariant in all set-forcing extensions of ground models in the set-theoretic universe.}$$

Leach-Krouse (ms) defines the modal logic of $\Omega$-consequence as satisfying the following axioms:

- For a theory $T$ and with $\square \phi := T^B_\alpha \vDash \text{ZFC} \Rightarrow T^B_\alpha \vDash \phi$,
- $\text{ZFC} \vdash \phi \Rightarrow \text{ZFC} \vdash \square \phi$
- $\text{ZFC} \vdash \square(\phi \to \psi) \to (\square \phi \to \square \psi)$
- $\text{ZFC} \vdash \square \phi \to \phi \Rightarrow \text{ZFC} \vdash \phi$
- $\text{ZFC} \vdash \square \phi \to \square \square \phi$
- $\text{ZFC} \vdash \square(\square \phi \to \phi) \to \square \phi$
- $\square(\square \phi \to \psi) \lor \square(\square \psi \land \psi \to \phi)$, where this clause added to GL is the logic of 'true in all $V_\kappa$ for all $\kappa$ strongly inaccessible' in ZFC.

The absoluteness of set-theoretic truths in virtue of $\Omega$-logical validity corresponds to a type of objective – perhaps maximal, and thus metaphysical – necessity. This characterization of definiteness for set-theoretic truths would thus satisfy Linnebo (2018b)’s characterizations of both intensional and extensional definiteness. According to Linnebo (op. cit.: 203), a concept is intensionally
definite if it ‘has a sharp application condition’, and extensionally definite if ‘it has a fixed extension in all the circumstances in which the concept is available’. Linnebo (2018a: 210) argues that extensional definiteness is satisfied if set-membership is rigid: \( \exists x \forall u [u \prec xx \iff \phi(u)] \), with \( \prec \) a plurality membership relation. Our approach departs from this suggestion, by concerning the definiteness of set-theoretic truths rather than the concept of set more generally. With regard to extensional definiteness as secured via an absoluteness condition, Koelner (2010) writes of the invariance property of \( \Omega \)-logical consequence: ‘[T]he logical consequence relation is not perturbed by passing to a generic extension. If we think of the models \( V^B \) as possible worlds, then this is tantamount to saying that the logical consequence relation is invariant across the possible worlds’.

Whereas the Neo-Fregean approach to comprehension for the concept of set relies on an unprincipled restriction of the size of the universe in order to avoid inconsistency, and one according to which the axioms of ZF still cannot all be recovered, modal coalgebraic automata provide a natural means for defining the minimal conditions necessary for formal grasp of the concept set. The category-theoretic definition of indefinite extensibility is sufficient for uniquely capturing both the generative property as well as the notion of definiteness which are constitutive of the concept. The category-theoretic definition of indefinite extensibility avails of a notion of mathematical modality which captures both the epistemic property of possible interpretations of quantifiers, as well as the objective, circumstantial property of set-theoretic ontological expansion.

One objection to the two-dimensional characterization of indefinite extensibility might be that modality is itself indefinitely extensible, such that it would be circular to define indefinite extensibility via modal notions.\(^{12}\) In response, the modal characterization of indefinite extensibility can be illuminating, even if modality itself indefinitely extensible. The explanations required for each phenomenon are distinct. Thus, indefinite extensibility can have interpretational and objective modalities, expressing epistemically possible reinterpretations of quantifier domains and objectively possible ontological expansion. However, the explanation for modality being itself indefinitely extensible can proceed via, e.g., Fritz (2017)’s puzzle. Fritz proffers two modal principles which are inconsistent. The first states that \( \Box \uparrow_1 \Diamond \exists x (Ax \land \downarrow_1 \neg Ax) \) (op. cit.: 551). The second principle states that \( \Diamond \uparrow_1 \forall x (Ax \rightarrow \downarrow_1 Ax) \) (550). The first principle claims that possibilia are indefinitely extensible (557; see also Rayo, 2020).\(^{13}\) So the indefinite extensibility of modality would not have to be explained by further modalities, but is rather explained by a principle which shows that it is always possible to expand the domain of possibilia.

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\(^{12}\) Thanks to xx for the objection.

\(^{13}\) Fritz suggests that the notion of metaphysical possibility might not be in good standing, in light of the inconsistency of the two foregoing principles. By contrast, I take the modalities at issue to be objective, though not maximal and thus not metaphysical, where non-maximal objective modalities can be indefinitely extensible. See Author (ms) for further discussion.
5 Concluding Remarks

In this essay, I outlined a number of approaches to defining the notion of indefinite extensibility, each of which restricts the scope of their characterization to set-theoretic languages. I endeavored, then, to define indefinite extensibility in the setting of category-theoretic languages, and examined the benefits accruing to the approach, by contrast to the extensional and modal approaches pursued in ZF.

The extensional definition of indefinite extensibility in ZF was shown to be insufficient for characterizing the generative property in virtue of which number-theoretic concepts are indefinitely extensible. The generative property of indefinite extensibility for set-theoretic truths in the category-theoretic setting was argued, by contrast, to be identifiable with the elementary embeddings by which large cardinal axioms can be specified. The modal definitions of indefinite extensibility in ZF were argued to be independently problematic, in virtue of endeavoring simultaneously to account for the epistemic properties of indefinite extensibility – e.g., possible reinterpretations of quantifier domains and mathematical vocabulary – as well as the objective properties of indefinite extensibility – i.e., the ontological expansion of the target domains, without providing an account of how this might be achieved. Mappings in modal coalgebraic automata can secure these two dimensions, by having both epistemic and objective interpretations. The mappings are interpreted both epistemically and objectively, such that the mappings are defined relative to two parameters, the first ranging over epistemically possible worlds and the second ranging over objective possible worlds. The mappings thus receive their values in an epistemic two-dimensional semantics.

Finally, against the Neo-Fregean approach to defining concepts of number, and the limits thereof in the attempt to define concepts of mathematical objects in other branches of mathematics beyond arithmetic, I demonstrated how – by characterizing the modal profile of \( \Omega \)-logical validity and thus the generic invariance and absoluteness of mathematical truths concerning large cardinals throughout the set-theoretic universe – modal coalgebraic automata are capable of capturing the notion of definiteness within the concept of indefinite extensibility for set-theoretic truths in the category of sets.
References


Leach-Krouse, G. ms. Ω-Consequence Interpretations of Modal Logic.
Pettigrew, R. ms. An Introduction to Toposes.


Venema, Y. 2012. Lectures on the Modal $\mu$-Calculus.

Venema, Y. 2020. Lectures on the Modal $\mu$-Calculus.


Woodin, W.H. ms. The $\Omega$ Conjecture.
