Grothendieck Universes and Indefinite Extensibility

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Abstract

This essay endeavors to define the concept of indefinite extensibility in the setting of category theory. I argue that the generative property of indefinite extensibility in the category-theoretic setting is identifiable with the Kripke functors of modal coalgebraic automata, where the automata model Grothendieck Universes and the functors are further inter-definable with the elementary embeddings of large cardinal axioms. The Kripke functors definable in Grothendieck universes are argued to account for the ontological expansion effected by the elementary embeddings in the category of sets. By characterizing the modal profile of Ω-logical validity, and thus the generic invariance of mathematical truth, modal coalgebraic automata are further capable of capturing the notion of definiteness, in order to yield a non-circular definition of indefinite extensibility.

1 Introduction

This essay endeavors to provide a characterization of the notion of definiteness, in order to provide a non-circular definition of the concept of indefinite extensibility. The concept of indefinite extensibility is introduced by Dummett (1963/1978), in the setting of a discussion of the philosophical significance of Gödel's (1931) first incompleteness theorem. Gödel's theorem can be characterized as stating that – relative to a coding defined over the signature of first-order arithmetic, a predicate expressing the property of provability, and a fixed point construction which is non-trivial, such that the formula in which the above predicate figures precludes interpretations such as '0=1' – the formula can be defined as not satisfying the provability predicate. Dummett's concern is with the conditions on our grasp of the concept of natural number, given that the latter figures in a formula whose truth appears to be satisfied despite the unprovability – and thus non-constructivist profile – thereof (186). His conclusion is that the concept of natural number 'exhibits a particular variety of inherent vagueness, namely indefinite extensibility', where a 'concept is indefinitely extensible if, for any definite characterisation of it, there is a natural extension of this characterisation, which yields a more inclusive concept; this extension will be made according to some general principle for generating such extensions, and, typically, the extended characterisation will be formulated by reference to the previous, unextended, characterisation' (195-196). Elaborating on the notion of indefinite extensibility, Dummett (1996: 441) redefines the concept as follows: an 'indefinitely extensible concept is one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it'. Subsequent approaches to the notion have endeavored to provide a more precise elucidation thereof, either by providing an explanation of the property which generalizes to an array of examples in number theory and set theory (cf. Wright and Shapiro, 2006), or by availing of modal notions in order to capture the properties of definiteness and extendability which are constitutive of the concept (cf. Fine, 2006; Linnebo, 2013; Uzquiano, 2015). However, the foregoing modal characterizations of indefinite extensibility have similarly been restricted to set-theoretic languages. Furthermore, the modal notions that the approaches avail of are taken to belong to a proprietary type which is irreducible to either the metaphysical or the logical interpretations of the operator.

The aim of this essay is to redress the foregoing, by providing a modal characterization of indefinite extensibility in the setting of category theory, rather than number or set theory. One virtue of the category-theoretic, modal definition of indefinite extensibility is that it provides for a robust account of the epistemological foundations of modal-structuralist approaches to the ontology of mathematics. A second aspect of the philosophical significance of the examination is that it can serve to redress the lacuna noted in the appeal to an irreducible type of mathematical modality, which is argued (i) to be representational, (ii) still to bear on the ontological expansion of domains of sets, and yet (iii) not to range over metaphysical possibilities. By contrast to the latter approach, the category-theoretic characterization of indefinite extensibility is able to identify the functors of coalgebraic non-deterministic automata with elementary embeddings and the modal properties of set-theoretic, Ω-logical consequence.

In Section **2**, I examine the extant approaches to explaining both the property and the understanding-conditions on the concept of indefinite extensibility. In Section **3**, I outline the elements of the category theory of sets and define Grothendieck Universes. In Section **4**, I identify Grothendieck Universes with modal coalgebraic automata, and define the notion of indefinite extensibility in the category-theoretic setting. I argue that the category-theoretic definition of indefinite extensibility, via Grothendieck Universes as modal coalgebraic automata, yields an explanation of the generative property of indefinite extensibility, as well as of the notion of definiteness which figures in the definition. I argue that the generative property of indefinite extensibility can be captured by identifying Kripke functors of colagebras with elementary embeddings. I argue, then, that the notion of definiteness can be captured by the role of Grothendieck

Universes-as-modal coalgebraic automata in characterizing the modal profile of Ω-logical consequence, where the latter accounts for the absoluteness of mathematical truths throughout the set-theoretic multiverse. The category-theoretic definition is shown to circumvent the issues faced by rival attempts to define indefinite extensibility via extensional and intensional notions within the setting of set theory. Section **5** provides concluding remarks.

2 Indefinite Extensbility in Set Theory: Modal and Extensional Approaches

Characterizations of indefinite extensibility have so far occurred in the language of set theory, and have availed of both extensional and intensional resources. In an attempt to define the notion of definiteness, Wright and Shapiro (op. cit.) argue, for example, that indefinite extensibility may be intuitively characterized as occurring when there is a function which falls under a first-order concept; for a sub-concept of the first-order concept, an application of the function on the sub-concept does not fall within that sub-concept's range; however, a new sub-concept can be formed, and defined as the set-theoretic union of the initial sub-concept and the function applied thereon (266).

Formally, let Π be a higher-order concept of type τ. Let P be a first-order concept falling under Π of type τ. Let f be a function from entities to entities of the same type as P. Finally, let X be a sub-concept of P. P is indefinitely extensible with respect to Π, if and only if:

$\epsilon(\mathrm{P}) = f(\mathrm{X})$,

$\epsilon(\mathrm{X}) = \neg[f(\mathrm{X})]$, and

$\exists \mathrm{X'}[\Pi(\mathrm{X'}) = (\mathrm{X} \cup \{f\mathrm{X}\})]$ (op. cit.).

The notion of definiteness is then defined as the limitless preservation of 'Π-hood' by sub-concepts thereof 'under iteration of the relevant operation', f (269).

The foregoing impresses as a necessary condition on the property of indefinite extensibility. Wright and Shapiro note, e.g., that the above formalization generalizes to an array of concepts countenanced in first-order number theory and analysis, including concepts of the finite ordinals (defined by iterations of the successor function); of countable ordinals (defined by countable order-types of well-orderings); of regular cardinals (defined as occurring when the cofinality of a cardinal, κ – comprised of the unions of sets with cardinality less than κ – is identical to κ); of large cardinals (defined by elementary embeddings from the universe of sets into proper subsets thereof, which specify critical points measured by the ordinals); of real numbers (defined as cuts of sets of rational numbers); and of Gödel numbers (defined as natural numbers of a sequence of recursively enumerable truths of arithmetic) (266-267).

As it stands, however, the definition might not be sufficient for the definition of indefinite extensibility, by being laconic about the reasons for which new sub-concepts – comprised as the union of preceding sub-concepts with a target operation defined thereon – are presumed interminably to generate. In response to the above desideratum, concerning the reasons for which indefinite extensibility might be engendered, philosophers have recently appealed to modal properties of the formation of sets. Fine (2006) argues, e.g., that – in order to avoid the Russell property when quantifying over all sets – there are interpretational modalities which induce a reinterpretation of quantifier domains, and serve as a mechanism for tracking the ontological inflation of the hieararchy of sets via, e.g., the power-set operation (2007). Fine (2005) suggests that the interpretational modality at issue might be a species of dynamic modality, which defines modalities as concerning the information entrained by program executions. Reinhardt (1974) and Williamson (2007) argue that modalities are inter-definable with counterfactuals. While Williamson (2016) argues both that imaginative exercises take the form of counterfactual presuppositions and that it is metaphysically possible to decide propositions which are undecidable relative to the current axioms of extensional mathematical languages such as ZF – Reinhardt (op. cit.) argues that large cardinal axioms and undecidable sentences in extensional ZF can similarly be imagined as obtaining via counterfactual presupposition. In an examination of the iterative hierarchy of sets, Parsons (1977/1983) notes that the notion of potential infinity, as anticipated in Book 3, ch. 6 of Aristotle's *Physics*, may be codified in a modal set theory by both a principle which is an instance of the Barcan formula (namely, for predicates P and rigidifying predicates Q, $\forall x(Px \iff Qx) \wedge \Box\{\forall x(\Box Qx \vee \Box\neg Qx) \wedge \forall R[\forall x\Box(Qx \rightarrow Rx) \rightarrow \Box\forall x(Qx \rightarrow Rx)]\}$ (fn. 24), as well as a principle for definable set-forming operations (e.g., unions) for Borel sets of reals $\Box(\forall x)\Diamond(\exists y)[y{=}x \cup \{x\})$ (528). The modal extension is argued to be a property of the imagination, or intuition, and to apply further to iterations of the successor function in an intensional variant of arithmetic (1979-1980).

Hellman (1990) develops the program intimated in Putnam (1967), and thus argues for an eliminativist, modal approach to mathematical structuralism as applied to second-order plural ZF. The possibilities at issue are taken to be logical – concerning both the consistency of a set of formulas as well as the possible satisfiaction of existential formulas – and he specifies, further, an 'extendability principle', according to which 'every natural model [of ZF] has a proper extension' (421).

Extending Parsons' and Fine's projects, Linnebo (2009, 2013) avails of a second-order, plural modal set theory in order to account for both the notion of potential infinity as well as the notion of definiteness. Similarly to Parsons' use of the Barcan formula (i.e., $\Box\forall\phi \rightarrow \forall\Box\phi$), Linnebo's principle for the foregoing is as follows: $\forall$u(u $\prec$ xx $\rightarrow \Box\phi) \rightarrow \Box\forall$u(u $\prec$ $\rightarrow \phi$) (2013: 211). He argues, further, that the logic for the modal operator is S4.2, i.e. K $[\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)]$, T $(\Box\phi \rightarrow \phi)$, 4 $((\Box\phi \rightarrow \Box\Box\phi)$, and G $(\Diamond\Box\phi \rightarrow \Box\Diamond\phi)$. Studd (2013) examines the notion of indefinite extensibility by availing of a bimodal temporal logic. Uzquiano's (2015) approach to defining the concept of indefinite extensibility argues that the height of the cumulative hierarchy is in fact fixed, and that indefinite extensibility can similarly be captured via the use of modal operators in second-order plural modal set theory. The modalities are taken to concern the possible reinterpretations of the intensions of the non-logical vocabulary – e.g., the set-membership relation – which figures in the augmentation of the theory with new axioms and the subsequent climb up the fixed hierarchy of sets (cf. Gödel, 1947/1964).

Khudairi (ms_1) proffers a novel epistemology of mathematics, based on an application of the epistemic interpretation of multi-dimensional intensional semantics in set-theoretic languages to the values of large cardinal axioms and undecidable sentences. Modulo logical constraints such as consistency and generic absoluteness in the extensions of ground models of the set-theoretic multiverse, the epistemic possibility that an undecidable proposition receives a value may serve, then, as a guide to the metaphysical possibility thereof. Finally, Khudairi (ms_2) argues that the modal profile of the consequence relation, in the Ω-logic defined in Boolean-valued models of set-theory, can be captured by coalgebraic modal automata, and provides a necessary condition on the formal grasp of the concept of ’set’.

The foregoing accounts of the metaphysics and epistemology of indefinite extensibility are each defined in the languages of number and set theory. In the following section, I examine the nature of indefinite extensibility in the setting of category theory, instead. One aspect of the philosophical significance of the examination is that it can serve to provide an analysis of the mathematical modality at issue, by availing only of model-theoretic resources. By contrast to Hellman’s approach, which takes the mathematical modality at issue to be logical (cf. Field, 1989: 37; Rayo, 2013), and Fine’s (op. cit.) approach, which takes the mathematical modality to be dynamic, I argue in the following sections that the mathematical modality can be captured by the functors of coalgebraic modal automata, where the latter are identifiable with Grothendieck Universes.

3 Grothendieck Universes

We work within a two-sorted language in which the Eilenberg - Mac Lane Axioms of category theory are specified. Types are labeled A,B,C for objects and x,y,z for arrows. The relevant operators are the domain operator, $\mathtt{Dom}$, which takes arrows to objects; the codomain operator, $\mathtt{Cod}$, which operates similarly, and the identity operator, $\mathtt{1}_\alpha$, which takes objects to arrows. Finally, a composition relation, C(x,y; z), is defined on arrows, where the open formula reads z is the composite of x and y (McLarty, 2008: 13). The Eilenberg - Mac Lane axioms can then be defined as follows:

- Axioms of Domain and Codomain:

  $\forall$f,g,h, if C(f,g,h), then $\mathtt{Dom}$f = $\mathtt{Dom}$h and $\mathtt{Cod}$f = $\mathtt{Dom}$g and $\mathtt{Cod}$g = $\mathtt{Cod}$h

- Axioms of Existence and Uniqueness of Composites:

$\forall$f,g, if $\texttt{Codf} = \texttt{Domg}$, then $\exists!$h, s.t. C(f,g; h)

- Axioms for Identity Arrows:

  $\forall$A, $\texttt{Dom1}_A = \texttt{Cod1}_A = A$

  $\forall$f, C($1_{(\texttt{Dom}f)}$,f; f)

  $\forall$f, C(f,$1_{(\texttt{Dom}f)}$; f)

- Axiom of Associativity of Composition:

  $\forall$f,g,h,i,j,k, if C(f,g; i) and C(g,h; j) and C(f,j; k), then C(i,h; k) (op. cit.).

Categorical Set Theory is defined by augmenting the Eilenberg - Mac Lane axioms with the axioms of Lawvere's Elementary Theory for the Category of Sets (ECTS) (op. cit.; Lawvere, 2005). Following McLarty, we define the singleton of a set as one for which 'every set has exactly one function to it' (op. cit.: 25). An element of a set A, x$\in$A, is a function x: $1 \rightarrow A$ (op. cit.). Composition occurs if and only if, for two arrows, f,g, and object x, (gf)(x) = g(f(x)) (26). Finally, an equalizer e: $E \rightarrow A$ for a pair of functions f,g: $A \rightarrow B$ is defined as 'a universal solution to the equaltion fe = ge' (29). The axioms are then defined as follows (op. cit.):

- Every pair of sets, A,B, has a product:

  $\forall$T,f,g, with f: $T \rightarrow A$, g: $T \rightarrow B$, $\exists!\langle f,g \rangle$: $T \rightarrow AxB$

- Every parallel pair of functions, f,g: $A \rightarrow B$, has an equalizer:

  $\forall$T,h, with fh = gh, $\exists!$ u: $T \rightarrow E$

- There is a function set from each set A to each set B:

  $\forall$C and g: CxA $\rightarrow$ B, $\exists!$g': $C \rightarrow B^A$

- There is a truth value $true$: 1 → 2:

  ∀A and monic S ↦A, ∃!χ_i, such that S is an equalizer

- There is a natural number triple, ℕ, 0, s:

  ∀T and x: 1 → T and f: T → T, ∃! u: ⋉ → T

- Extensionality

  ∀f ≠ g: A → B, ∃x: 1 → A, with f(x) ≠ g(x)

- Non-triviality

  $\exists false$: 1 → 2, s.t. $false \neq true$

- Choice

  ∀ onto functions f: A → B, ∃h: B → A, s.t. fh = 1_A.

From the axioms of ETCS, only a version of the ZF separation axiom with bounded quantifiers can be recovered (37). The axiom of separation states that ∃x∀u[u∈x ⟺ u∈a ∧ ϕ(u)]. In order to redress the restriction to bounded quantifiers, we work within a stronger category theory for sets, i.e. the 'category of categories as foundation' (CCAF). The axioms of the CCAF build upon those of both ETCS and Eilenberg - Mac Lane category theory, by augmenting them with the following (53):

- Every category C has a unique functor, C → 1
- The category 2 has exactly two functors from 1 and 3 to itself
- Let a pushout be defined such that if f: A → C and g: B → C, then a: C → A and b: C → B (Pettigrew, ms: 19). The category 3 is a pushout, and there is a functor γ: 2 → 3, with $\gamma_0 = \alpha_0$ and $\gamma_1 = \beta_1$

- Arrow Extensionality

  $\forall$F,G: A $\rightarrow$ B, if F $\neq$ G then $\exists$f: 2 $\rightarrow$ A with Ff $\neq$ Gf.

A Grothendieck Universe may finally be defined as a set, U, which satisfies the axioms of ZF set theory without choice, yet as augmented by at least strongly inaccessible large cardinals. The axioms of ZF are:

- Empty set:

  $\exists x \forall u (u \notin x)$

- Extensionality:

  $x = y \iff \forall u (u \in x \iff u \in y)$

- Pairing:

  $\exists x \forall u (u \in x \iff u = a \vee u = b)$

- Union:

  $\exists x \forall u [u \in x \iff \exists v (u \in v \wedge v \in a)]$

- Separation:

  $\exists x \forall u [u \in x \iff u \in a \wedge \phi(u)]$

- Power Set:

  $\exists x \forall u (u \in x \iff u \subseteq a)$

- Infinity:

  $\exists x \varnothing \in x \wedge \forall u (u \in x \rightarrow \{u\} \in x)$

- Replacement:

  $\forall u \exists ! v \psi(u,v) \rightarrow \forall x \exists y (\forall u \in x)(\exists v \in y) \psi(u,v)$.

Large cardinal axioms are defined by elementary embeddings.[1] Elementary embeddings can be defined thus. For models A,B, and conditions ϕ, j: A $\rightarrow$ B, $\phi\langle a_1, \ldots, a_n\rangle$ in A if and only if $\phi\langle j(a_1), \ldots, j(a_n)\rangle$ in B (Kanamori, 2012: 363). A cardinal κ is regular if the cofinality of κ – comprised of the unions of sets with cardinality less than κ – is identical to κ (op. cit.: 360). Uncountable regular limit cardinals are weakly inaccessible (op. cit.). A strongly inaccessible cardinal is regular and has a strong limit, such that if $\lambda < \kappa$, then $2^\lambda < \kappa$ (op. cit.).

By augmenting languages of the theory of CCAF with Grothendiek Universes, U, CCAF proves thereby that:

CCAF $\vdash \forall n \in \mathbb{N}, \exists\{\aleph_0, \aleph_1, \ldots, \aleph_n\}$, in the category of Sets, U-Set (37-38).

4 Modal Coalgebraic Automata and Indefinite Extensibility

This section examines, finally, the reasons for which Grothendieck Universes provide a more theoretically adequate model of the understanding-conditions for mathematical concepts than do competing approaches such as the Neo-Fregean epistemology of mathematics. According, e.g., to the Neo-Fregean program, concepts of number in arithmetic and analysis are definable via implicit definitions which take the form of abstraction principles. Abstraction principles specify biconditionals in which – on the left-hand side of the formula – an identity is taken to hold between numerical term-forming operators from second-order entities to abstract objects, and – on the right-hand side of the formula – an equivalence relation on lower-order entities is assumed to hold.

[1] Cf. Koellner and Woodin (2010); Woodin (2010).

In the case of cardinal numbers, the relevant abstraction principle is referred to as Hume's principle, and states that, for all x and y, the number of the x's is identical to the number of the y's if and only if the x's and the y's can be put into a one-to-one correspondence, i.e., there is a bijection from the x's onto the y's. Abstraction principles for the concepts of other numbers have further been specified. Thus, e.g., Shapiro (2000: 337-340) specifies an abstraction principle for real numbers, which proceeds along the method of Dedekind's definition of the reals (cf. Wright, 2007: 172). According to the latter method, one proceeds by specifying an abstraction principle which avails of the natural numbers, in order to define pairs of finite cardinals: '$\forall$x,y,z,w[$\langle$x,y$\rangle$ = $\langle$z,w$\rangle$ $\iff$ x = z $\wedge$ y = w]. A second abstraction principle is defined which takes the differences of the foregoing pairs of cardinals, identifying the differences with integers: [Diff($\langle$x,y$\rangle$) = Diff($\langle$z,w$\rangle$) $\iff$ x + w = y + z]. One specifies, then, a principle for quotients of the integers, identifying them subsequently with the rational numbers: [Q$\langle$m,n$\rangle$ = Q$\langle$p,q$\rangle$ $\iff$ n = 0 $\wedge$ q = 0 $\vee$ n $\neq$ 0 $\wedge$ q $\neq$ 0 $\wedge$ m x q = n x p]. Finally, one specifies sets of rational numbers, i.e. the Dedekind cuts thereof, and identifies them with the reals: $\forall$F,G[Cut(F) = Cut(G) $\iff$ $\forall$r(F$\leq$r $\iff$ G$\leq$r)].

The abstractionist program faces several challenges, including whether conditions can be delineated for the abstraction principles, in order for the principles to avoid entraining inconsistency[2]; whether unions of abstraction principles can avoid the problem of generating more abstracts than concepts (Fine, 2002); and whether abstraction principles can be specified for mathematical entities in branches of mathematics beyond first and second-order arithmetic (cf. Boolos, 1997; Hale, 2000; Shapiro, op. cit.; and Wright, 2000). I will argue that the last

[2]Cf. Hodes (1984); Hazen (1985); Boolos (1990); Heck (1992); Fine (2002); Weir (2003); Cook and Ebert (2005); Linnebo and Uzquiano (2009); Linnebo (2010); and Walsh (2016).

issue – i.e., being able to countenance definitions for the entities and structures in branches of mathematics beyond first and second-order arithmetic – is a crucial desideratum, the satisfaction of which remains elusive for the Neo-Fregean program while yet being satisfiable and thus adducing in favor of the modal platonist approach that is outlined in what follows.

One issue for the attempt, along abstractionist lines, to provide an implicit definition for the concept of set is that doing so with an unrestricted comprehension principle yields a principle identical to Frege's (1893/2013) Basic Law V; and thus – in virtue of Russell's paradox – entrains inconsistency. However, two alternative formulas can be defined, in order to provide a suitable restriction to the inconsistent abstraction principle. The first, conditional principle states that $\forall$F,G[[Good(F) $\vee$ Good(G)] $\rightarrow$ [{x|Fx} = {Gx} $\iff$ $\forall$x(Fx $\iff$ Gx)]]. The second principle is an unconditional version of the foregoing, and states that $\forall$F,G[{x|Fx} = {Gx} $\iff$ [Good(F) $\vee$ Good(G) $\rightarrow$ $\forall$x(Fx $\iff$ Gx)]]. Following von Neumann's (1925/1967: 401-402) suggestion that Russell's paradox can be avoided with a restriction of the set comprehension principle to one which satisfies a constraint on the limitation of its size, Boolos (1997) suggests that the 'Good' predicate in the above principles is intensionally isomorphic to the notion of smallness in set size, and refers to the principle as New V. However, New V is insufficient for deriving all of the axioms of ZF set theory, precluding, in particular, both the axioms of infinity and the power-set axiom (cf. Wright and Hale, 2005: 193). Further, there are other branches of number theory for which it is unclear whether acceptable abstraction principles can be specified. Wiles' proof of Fermat's Last Theorem (i.e., that, save for when one of the variables is 0, the Diophantine equation, $x^n = y^n = z^n$, has no solutions when n >2; cf. Hardy and Wright, 1979: 190) relies, e.g., on both invariants and

Grothendieck universes in cohomological number theory (cf. McLarty, 2009: 4).

The foregoing issues with regard to the definability of abstracta in number theory, algebraic geometry (McLarty, op. cit.: 6-8), set theory, et al., can be circumvented in the category-theoretic setting; and in particular by colagebras. In the remainder of this section, I endeavor to demonstrate how modal coalgebraic automata are able to countenance two, fundamental mathematical concepts. The first is the target concept in this essay, namely indefinite extensibility. The second concerns the epistemic and modal properties of the concept of logical consequence, in the Ω-logic in axiomatic set theory.

A labeled transition system is a tuple, LTS, comprised of a set of worlds, M; a valuation, V, from M to its powerset, P(M); and a family of accessibility relations, R. So LTS = ⟨M,V,R⟩ (cf. Venema, 2012: 7). A Kripke coalgebra combines V and R into a Kripke functor, σ_R; i.e. the set of binary morphisms from M to P(M) (op. cit.: 7-8). Thus, for an s∈M, σ(s) := [σ_V(s), σ_R(s)] (op. cit.). Satisfaction for the system is defined inductively as follows: For a formula ϕ defined at a state, s, in M,

$[\![\phi]\!]^M$ = V(s) [3]

$[\![\neg\phi]\!]^M$ = S – V(s)

$[\![\bot]\!]^M = \emptyset$

$[\![T]\!]^M$ = M

$[\![\phi \vee \psi]\!]^M = [\![\phi]\!]^M \cup [\![\psi]\!]^M$

$[\![\phi \wedge \psi]\!]^M = [\![\phi]\!]^M \cap [\![\psi]\!]^M$

$[\![\diamond_s\phi]\!]^M = \langle R_s\rangle[\![\phi]\!]^M$

$[\![\Box_s\phi]\!]^M = [R_s][\![\phi]\!]^M$, with

$\langle R_s\rangle(\phi)$:= {s'∈S | R_s[s'] ∩ $\phi \neq \emptyset$} and

[3] Equivalently, M,s ⊩ ϕ if s∈V(ϕ) (9).

$[R_s](\phi) := \{s' \in S \mid R_s[s'] \subseteq \phi\}$ (9).

Kripke Coalgebras can be identified with Grothendieck Universes. In CCAF, the elementary embeddings which are jointly necessary and sufficient for positing the existence of large cardinals can further be identified with the functors, i.e. transition functions, in modal coalgebraic automata. Finally, Kripke coalgebras are the dual representations of Boolean-valued models of the Ω-logic of set theory (cf. Venema, 2007). When identified with Grothendieck Universes, modal coalgebraic automata are able, then, to countenance the consitutive conditions of indefinite extensibility. Modal coalgebraic automata are capable, e.g., of defining both the generative property of indefinite extensibility, as well as the notion of definiteness which figures therein. Further, the category-theoretic definition of indefinite extensibility is arguably preferable to those advanced in the set-theoretic setting, because modal coalgebraic automata can account for both the modal profile and the epistemic tractability of Ω-logical consequence.

The *generative* property of indefinite extensibility is captured by the foregoing Kripke functor, σ_R – i.e., the morphism mapping a Boolean-valued model to the powerset thereof – and which we have identified with elementary embeddings, j: $A \rightarrow B$, $\phi\langle a_1, \ldots, a_n\rangle$ in A if and only if $\phi\langle j(a_1), \ldots, j(a_n)\rangle$ in B.

The notion of *definiteness* is captured by the role of modal coalgebraic automata in characterizing the modal profile of Ω-logical validity. Ω-logical validity can be defined as follows:

For $T \cup \{\phi\} \subseteq Sent$,

$T \models_\Omega \phi$, if for all ordinals a and countable Boolean algebras $\mathbb{B}$, if $V^{\mathbb{B}}{}_a \models T$, then $V^{\mathbb{B}}{}_a \models \phi$ (Bagaria et al., 2006). The Ω-Conjecture states that $V \models_\Omega \phi$ iff $V^{\mathbb{B}} \models_\Omega \phi$ (Woodin, ms). Thus, Ω-logical validity is invariant in all set-forcing extensions of ground models in the set-theoretic multiverse.

The invariance property of Ω-logical consequence can then be characterized by modal coalgebraic automata. Mathematical truths are thus said to be definite in virtue of holding of necessity, as recorded by the functors of the modal colagebraic automata which are dually isomorphic to the Boolean-valued algebraic models for the Ω-logic of set theory.

Thus, whereas the Neo-Fregean approach to comprehension for the concept of set relies on an unprincipled restriction of the size of the universe in order to avoid inconsistency, and one according to which the axioms of ZF still cannot all be recovered, modal coalgebraic automata provide a natural means for defining the minimal conditions necessary for formal grasp of the concept set. The category-theoretic definition of indefinite extensibility is sufficient for uniquely capturing both the generative property as well as the notion of definiteness which are constitutive of the concept. Finally, a further point adducing in favor of the category-theoretic definition of indefinite extensibility is that it requires no appeal to a notion of mathematical modality which problematically endeavors to capture both the epistemic property of possible interpretations of quantifiers, as well as the metaphysical property of set-theoretic ontological expansion.

5 Concluding Remarks

In this essay, I outlined a number of approaches to defining the notion of indefinite extensibility, each of which restricts the scope of their characterization to set-theoretic languages. I endeavored, then, to define indefinite extensibility in the setting of category-theoretic languages, and examined the benefits accruing to the approach, by contrast to the extensional and modal approaches pursued in ZF.

The extensional definition of indefinite extensibility in ZF was shown to be insufficient for characterizing the generative property in virtue of which number-theoretic concepts are indefinitely extensible. The generative property of indefinite extensibility in the category-theoretic setting was argued, by contrast, to be identifiable with the Kripke functors of modal coalgebraic automata, where the automata model Grothendieck Universes, and Kripke functors are further identifiable with the elementary embeddings by which large cardinal axioms can be specified. The modal definitions of indefinite extensibility in ZF were argued to be independently problematic, in virtue of endeavoring simultaneously to account for the epistemic properties of indefinite extensibility – e.g., possible reinterpretations of quantifier domains and mathematical vocabulary – as well as the metaphysical properties of indefinite extensibility – i.e., the ontological expansion of the target domains. By contrast, the Kripke functors definable in Grothendieck universes-as-modal coalgebraic automata were argued to circumvent the foregoing conflation, accounting just for the ontological expansion effected by elementary embeddings in the category of sets.

Finally, against the Neo-Fregean approach to defining concepts of number, and the limits thereof in the attempt to define concepts of mathematical objects in other branches of mathematics beyond arithmetic, I demonstrated how – by characterizing the modal profile of Ω-logical validity and thus the generic invariance and absoluteness of mathematical truths concerning large cardinals throughout the set-theoretic multiverse – modal coalgebraic automata are capable of capturing the notion of definiteness within the concept of indefinite extensibility.

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