Modal Cognitivism and Modal Expressivism

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Abstract

This paper aims to provide a mathematically tractable background against which to model both modal cognitivism and modal expressivism. I argue that epistemic modal algebras comprise a materially adequate fragment of the language of thought, and endeavor to show how such algebras provide the resources necessary to resolve Russell's paradox of propositions. I demonstrate, then, how modal expressivism can be regimented by modal coalgebraic automata, to which the above epistemic modal algebras are dually isomorphic. I examine, in particular, the virtues unique to the modal expressivist approach here proffered in the setting of the foundations of mathematics, by contrast to competing approaches based upon both the inferentialist approach to concept-individuation and the codification of speech acts via intensional semantics.

1 Introduction

This essay endeavors to reconcile two approaches to the intensional foundations of thought: modal cognitivism and modal expressivism. The novel contribution of the paper is its argument for a reconciliation between the two positions, by providing a hybrid account in which both internal cognitive architecture, on the model of epistemic possibilities, as well as modal automata, are accommodated, while retaining what is supposed to be their unique and inconsistent roles.

Modal cognitivism is the proposal that the internal representations comprising the language of thought can be modeled via a possible world semantics. Modal expressivism has, in turn, been delineated in two ways. On the first approach, the presuppositions shared by a community of speakers have been modeled as circumstantial possibilities (cf. Kratzer, 1979; Stalnaker, 1978, 1984).

Speech acts have in turn been modeled as modal operators, the semantic values of which are then defined relative to an array of intensional parameters (Stalnaker, op. cit.; Veltman, 1996; Yalcin, 2007). On the second approach, the content of concepts is supposed to be individuated via the ability to draw inferences, and the pragmatic abilities of individuals have been modeled as automata comprised of two transition functions. A counterfactual transition functional – encoding the recognition of distal properties – determines the range of admissible values for another transition function encoding the individual's actions (cf. Brandom, 2008). Inferential conditions constitutive of concept possession are then taken to have the same counterfactual form as the foregoing functions (Brandom, 2014), while truth-evaluable descriptions of the automata are specified in a metalanguage (Brandom, 2008). Both the modal approach to shared information and the speech acts which serve to update the latter, and the modal-inferential approach to concept-individuation – are thus consistent with mental states having semantic values or truth-conditional characterizations.[1]

So defined, the modal cognitivist and modal expressivist approaches have been assumed to be in constitutive opposition. While the cognitivist proposal avails of modal resources in order to model the internal representations comprising an abstract language of thought, the expressivist proposal targets informational properties which extend beyond the remit of internal cognitive archi-

[1]The notions of cognitivism and expressivism here targeted concern the role of internal – rather than external – factors in countenancing the nature of thought and information (cf. Fodor, 1975; Haugeland, 1978). Possible worlds semantics is taken then to provide the most descriptively adequate means of countenancing the structure of the foregoing. Delineating cognitivism and expressivism by whether the positions avail of internal representations is thus orthogonal to the eponymous dispute between realists and antirealists with regard to whether mental states are truth-apt, i.e., have a representational function, rather than being non-representational and non-factive, even if real (cf. Dummett, 1959; Blackburn, 1984; Price, 2013). Whereas the type of modal cognitivism examined here assumes that thoughts and information take exclusively the form of internal representations, the target modal expressivist proposals assume that information states are exhaustively individuated by both linguistic behavior and conditions external to the cognitive architecture of agents.

tecture: both the form and the parameters relevant to determining the semantic values of linguistic utterances, where the informational common ground is taken to be reducible to circumstantial possibilities; and the individuation of the contents of concepts on the basis of inferential behavior.

In this paper, I provide a background mathematical theory, in order to account for the reconciliation of the cognitivist and expressivist proposals. I avail, in particular, of the dual isomorphism between Boolean-valued models of epistemic modal algebras and coalgebras; i.e., labeled transition systems defined in the setting of category theory.[2] The functors of coalgebras permit of flexible interpretations, such that they are able to characterize both modal logics as well as discrete-state automata. I argue that the correspondence between epistemic modal algebras and modal coalgebraic automata is sufficient then for the provision of a mathematically tractable, modal foundation for thought and action.

In Section **2**, I provide the background mathematical theory, in order to account for the reconciliation of the cognitivist and expressivist proposals.

In Section **3**, I provide reasons adducing favor of modal cognitivism, and argue for the materially adequacy of epistemic modal algebras as a fragment of the language of thought.[3] I argue that a further virtue of the proposal is

[2]For an algebraic characterization of dynamic-epistemic logic, see Kurz and Palmigiano (2013). Baltag (2003) develops a colagebraic semantics for dynamic-epistemic logic, where coalgebraic functors are intended to record the informational dynamics of single- and multi-agent systems. The current approach differs from the foregoing by examining the dual isomorphism between static epistemic modal algebras and coalgebraic automata, where the functors in the latter are interpreted as expressive of the epistemic states comprising the modal algebras.

[3]The proposal that possible worlds semantics comprises the model for thoughts and propositions is anticipated by Wittgenstein (1921/1974: 2.15-2.151, 3-3.02); Chalmers (2011); and Jackson (2011). Their approaches depart, however, from the one here examined in the following respects. (i) Wittgenstein (op. cit.: 1-1.1) has been interpreted as endorsing an identity theory of propositions, which does not distinguish between internal thoughts and external propositions (cf. McDowell, 1994: 27; and Hornsby, 1997: 1-3). How the identity theory of propositions is able to accommodate Wittgenstein's suggestion that a typed hierarchy of propositions can be generated – only if the class of propositions has a general form and the sense of propositions over which operations range is invariant by being individuated by the

that the possible worlds semantics for thought, in which the semantics is modeled algebraically, provides a novel resolution to Russell's (1903) paradox of propositions. The correspondence, in particular, between modal operators on formulas, in the algebra, and monotonic, intensional functions on terms, permits materially equivalent classes of possibilities to be possessed of distinct, hyperintensional arrays of intensions or senses. The application to Russell's paradox provides further support for the adequacy of the modal algebraic approach to countenancing the structure of thought.

Modal coalgebraic automata are argued, finally, to be preferred as models of modal expressivism, by contrast to the speech-act and inferentialist approaches, in virtue of the advantages accruing to the model in the philosophy of mathematics. The interest in modal coalgebraic automata consists, in particular, in the range of mathematical properties that can be recovered on the basis thereof.[4] By contrast to the above competing approaches to modal expressivism, the functors of modal coalgebraic automata are able both to model and explain elementary embeddings in the category of sets; the intensions of mathematical terms; as well as the modal profile of Ω-logical consequence.

Section **4** provides concluding remarks.

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possibilities figuring as their truth and falsity conditions (cf. Wittgenstein, 1979: 21/11/16, 23/11/16, 7/11/17; and Potter, 2009: 283-285 for detailed discussion) – is an open question. Wittgenstein (1921/1974: 5.5561) writes that 'Hierarchies are and must be independent of reality', although provides no account of how the independence can be effected. (ii) Jackson (2008: 48-50) distinguishes between personal and subpersonal theories by the role of neural science in individuating representational states (cf. Shea, 2013, for further discussion), and argues in favor of a 'personal-level implicit theory' for the possible worlds semantics of mental representations. (iii) Chalmers (2012: 462-463) argues for a hybrid cognitivist-expressivist approach, according to which epistemic intensions – i.e. functions from epistemically possible worlds to extensions – are individuated by their inferential roles.

[4] See Wittgenstein (2001: IV, 4-6, 11, 30-31), for a prescient expressivist approach to the modal profile of mathematical formulas.

2 The Hybrid Proposal

2.1 Epistemic Modal Algebra

In Epistemic Modal Algebra, a topological Boolean algebra, A, is formed by taking the powerset of a topological space, X; A = P(X). The domain of A is comprised of formula-terms – eliding propositions with names – assigned to elements of P(X), where the formula-terms encode *epistemic possibilities*; i.e., possibilities defined relative to states of information. The top element of the algebra is denoted '1' and the bottom element is denoted '0'. We interpret the modal operators, f(x), – i.e., monotonic intensional functions in the algebra – as both equivalent to concepts, as well as countenancing the property of topological interiority. An Epistemic Modal-valued Algebraic structure has the form, F = $\langle A$, $D_{P(X)}$, $\rho\rangle$, where ρ is a mapping from points in the topological space to elements or regions of the algebraic structure; i.e., ρ : $D_{P(X)}$ x $D_{P(X)} \rightarrow A$. A model over the Epistemic-Modal Topological Boolean Algebraic structure has the form M = ⟨F, V⟩, where V(a) ≤ ρ(a) and V(a,b) ∧ ρ(a, b) ≤ V(b).[5] For all $x_{x/a,\phi}$,y∈A:

f(1) = 1;

f(x) ≤ x;

f(x ∧ y) = f(x) ∧ f(y);

f[f(x)] = f(x);

V(a, a) > 0;

V(a, a) = 1;

V(a, b) = V(b, a);

V(a, b) ∧ V(b, c) ≤ V(a, c);

[5]See Lando (2015); McKinsey and Tarski (1944); and Rasiowa (1963), for further details.

$V(a = a) = \rho(a, a)$;

$V(a, b) \leq f[V(a, b)]$;

$V(\neg\phi) = \rho(\neg\phi) - f(\phi)$;

$V(\diamond\phi) = \rho\phi - f[- V(\phi)]$;

$V(\Box\phi) = f[V(\phi)]$ (cf. Lando, op. cit.).

An abstraction principle for epistemic intensions can be defined as follows:

For all types, A,B, there is a homotopy:

$H := [(f \sim g) :\equiv \prod_{x:A}(f(x) = g(x)]$, where

$\prod_{f:A \to B}[(f \sim f) \wedge (f \sim g \to g \sim f) \wedge (f \sim g \to g \sim h \to f \sim h)]$,

such that, via Voevodsky's (2006) Univalence Axiom, for all type families

A,B:U, there is a function:

$\texttt{idtoeqv} : (A =_U B) \to (A \simeq B)$,

which is itself an equivalence relation:

$(A =_U B) \simeq (A \simeq B)$.

Abstraction principles for intensional computational properties take, then, the form:

- $\forall A,B \exists f,g[\prod_{f:A \to B} \mathbf{A}f(x) = \mathbf{A}g(x)] \simeq [f(x) \simeq g(x)]$,

with **A** an abstraction operator from the domain of functions to a domain of abstract objects.

2.2 Modal Coalgebraic Automata

Modal coalgebraic automata can be thus characterized. Let a category C be comprised of a class Ob(C) of objects and a family of arrows for each pair of objects C(A,B) (Venema, 2007: 421). A functor from a category C to a category

D, **E**: C $\rightarrow$ D, is an operation mapping objects and arrows of C to objects and arrows of D (422). An endofunctor on C is a functor, **E**: C $\rightarrow$ C (op. cit.).

A **E**-coalgebra is a pair $\mathbb{A}$ = (A, μ), with A an object of C referred to as the carrier of $\mathbb{A}$, and μ: A $\rightarrow$ **E**(A) is an arrow in C, referred to as the transition map of $\mathbb{A}$ (390).

As, further, a coalgebraic model of modal logic, $\mathbb{A}$ can be defined as follows (407):

For a set of formulas, Φ, let $\nabla\Phi := \Box \bigvee \Phi \wedge \bigwedge \diamond\Phi$, where $\diamond\Phi$ denotes the set $\{\diamond\phi \mid \phi{\in}\Phi\}$ (op. cit.). Then,

$\diamond\phi \equiv \nabla\{\phi, \mathrm{T}\}$,

$\Box\phi \equiv \nabla\varnothing \vee \nabla\phi$ (op. cit.).

Let, then, an **E**-coalgebraic modal model, $\mathbb{A} = \langle \mathrm{S}, \lambda, \mathrm{R}[.] \rangle$, be such that $\mathbb{S}$,s $\Vdash$ $\nabla\Phi$ if and only if, for all (some) successors σ of s$\in$S, [Φ,σ(s)$\in$**E**($\Vdash_{\mathbb{A}}$)] (op. cit.).

A coalgebraic model of deterministic automata can finally be thus defined (391). An automaton is a tuple, $\mathbb{A} = \langle \mathrm{A}, \mathrm{a}_I, \mathrm{C}, \delta, \mathrm{F} \rangle$, such that A is the state space of the automaton $\mathbb{A}$; $\mathrm{a}_I \in \mathrm{A}$ is the automaton's initial state; C is the coding for the automaton's alphabet, mapping numerals to properties of the natural numbers; δ: A X C $\rightarrow$ A is a transition function, and F$\subseteq$ A is the collection of admissible states, where F maps A to {1,0}, such that F: A $\rightarrow$ 1 if a$\in$F and A $\rightarrow$ 0 if a$\notin$F (op. cit.).

The crux of the reconciliation between algebraic models of cognitivism and the formal foundations of modal expressivism is based on the dual isomorphism between categories of algebras and coalgebras: $\mathbb{A} = \langle \mathrm{A}, \alpha{:}\mathrm{A} \rightarrow \mathbf{E}(\mathrm{A}) \rangle$ is dually isomorphic to the category of algebras over the functor α (417-418). For a category C, object A, and endofunctor **E**, define a new arrow, α, s.t. α:**E**A $\rightarrow$ A. A homomorphism, f, can further be defined between algebras $\langle \mathrm{A}, \alpha \rangle$, and

$\langle$B, $\beta\rangle$. Then, for the category of algebras, the following commutative square can be defined: (i) $\mathbf{E}$A $\rightarrow$ $\mathbf{E}$B ($\mathbf{E}f$); (ii) $\mathbf{E}$A $\rightarrow$ A (α); (iii) $\mathbf{E}$B $\rightarrow$ B (β); and (iv) A $\rightarrow$ B (f) (cf. Hughes, 2001: 7-8). The same commutative square holds for the category of coalgebras, such that the latter are defined by inverting the direction of the morphisms in both (ii) [A $\rightarrow$ $\mathbf{E}$A (α)], and (iii) [B $\rightarrow$ $\mathbf{E}$B (β)] (op. cit.).[6]

The significance of the foregoing is twofold. First and foremost, the above demonstrates how a formal correspondence can be effected between algebraic models of cognition and coalgebraic models which provide a natural setting for modal logics and automata. The second aspect of the philosophical significance of modal colagebraic automata is that – as a model of modal expressivism – the proposal is able to countenance fundamental properties in the foundations of mathematics, and circumvent the issues accruing to the attempt so to do by the competing expressivist approaches.

3 Discussion

3.1 Material Adequacy

The material adequacy of epistemic modal algebras as a fragment of the representational theory of mind is witnessed by the prevalence of possible worlds semantics – the model theory for which is algebraic (cf. Blackburn et al., 2001: ch. 5) – in cognitive psychology, and the convergences thereof with theoretical physics.

In Bayesian perceptual psychology, e.g., the visual system is presented with

[6]Note that the theory for the model will be a topological coalgebra – cf. Takeuchi, 1985 for further discussion – and that, unlike isomorphism, the property of being dually isomorphic is not one-one (cf. Hughes, op. cit.: 13). Dual isomorphisms are thus more akin to bisimulations (cf. Blackburn et al., 2001: 64-66).

a prior distribution of possibilities concerning the direction of a source of light. The set of possibilities is pointed, as the visual system calculates the likelihood that one of the possibilities is actual, and places a condition thereby on the accuracy of the attribution of properties – such as boundedness and volume – to distal particulars (cf. Mamassian et al., 2000).

In the philosophy of physics, modal notions play, further, an ineliminable role in physical ontology and in the theoretical and observational characterizations thereof. Transition functions in quantum information theory can, for example, be interpreted counterfactually, in order to countenance the possible transformations of systems, the inputs of which include the expectation values of spin-state vectors (Deutsch, 2013). In the interpretation of physical theories, Ruetsche (2011: 9) argues, further, that '[T]o interpret a physical theory is to characterize the worlds possible according to that theory'. She notes that the interpretation of a physical theory falls into two phases, both of which constitutively involve modal notions. The first phase consists in the specification of the possible states of the models at issue; the possible values taken with regard to the observable kinematics thereof, e.g., the expectation values of Hilbert operators; and then the laws comprising the dynamic transformations therein (op. cit.). The second phase targets the nomologically possible worlds in which the properties of the first phase of interpretation are satisfied (op. cit.). In quantum field theory, unital algebras are availed of, as well, where the algebras are closed under commutativity and associativity; scalar multiplication by complex numbers, such that for all vectors A,B, and complex numbers c_n: $c_1(A + B) = c_1A + c_1B$; $(c_1 + c_2)A = c_1A + c_2A$; $c_1(c_2A) = (c_1c_2)A$; $(c_1A)B$ - $A(c_1B) = c_1(AB)$; and multiplicative identity, i.e., for an element I, AI = IA = A (74). Finally, Belot (2011) argues that – in the dispute between relationalist and substantivalist conceptions of spatial geometry – a relationalist proposal can be proffered, according to which – by analogy with nomological possibilities – the different metric relations that can be grounded in the intrinsic points comprising spatial regions are geometric possibilities.

In the next subsection, I endeavor to avail of the formal properties of epistemic modal algebras, in order to resolve Russell's (op. cit.) paradox of propositions. I argue that the paradox can be resolved in virtue of the formal correspondence, in the modal algebras characterizing the language of thought, between modal operators on formulas and monotonic intensional functions on terms.

3.2 The Paradox of Propositions

Russell's paradox is as follows. Let there be two classes, m and n, of equal cardinality, where classes are aggregations of non-sets. Define a proposition, ζ, such that ζ says of m that all the propositions defined therein are true. Then ζ is true of m, but not of n; so $|m| \neq |n|$ (§500, 2; cf. Uzquiano, 2015a: 288).

In response, Russell (op. cit.: 500, 3) suggests that the logical product of the propositions comprising m should itself be codified by the proposition, η, i.e. that 'All the propositions in m are true'. η and ζ are, according to the proposal, thus materially equivalent; and thus cardinalities of m and n would subsequently remain equivalent, as well. However, Russell's objection to the purported equivalence of the above propositions is the difference exhibited by their senses (§500, 4).

The objection to the resolution can be readily addressed. In Boolean Algebras with Operators, operators in the algebras are functions of rank > 0, satisfying conditions on normality, additivity, and monotonicity (cf. Blackburn et al., op. cit.: 277-278). For term variables, x,y, normality states that f(x) =

x, while additivity states that f(x) + f(y) = f(x + y) (277). Monotonicity states that, if a $\leq$ b, then θa $\leq$ θb. Because 'a $\leq$ b' iff 'a $\bullet$ b = a' iff 'a + b = b', then f(a) + f(b) = f(a +b) = f(b); so, f(a) $\leq$ f(b) (278). The intensional properties of the Boolean functions are witnessed by their correspondence to the following modal formulas:

$f(0) = 0 \mapsto \diamond\bot \iff \bot$;

$f(x) + f(y) = f(x + y) \mapsto \diamond(p \vee q) \iff \diamond p \vee \diamond q$; and,

$\vdash p \rightarrow q \mapsto \vdash \diamond p \rightarrow \diamond q$ (277-288).

The correspondence between modal operators on formulas and intensional functions on terms permits interpretations of the former – e.g., that it is conceivable that a situation obtains – to be equivalent to an intensional function – i.e., a concept – which can itself figure within, and be concatenated so as to form, distinct propositions. Then, while η and ζ are materially equivalent possibilities, or propositions, their equivalence to monotonic algebraic functions permits them to be reconfigured as intensions or concepts, which can themselves figure as the components of sense in arrays for which the variance in sequential composition will entrain unique classes of propositions. So, η and ζ can be materially equivalent propositions as possibilities, while – in virtue of their modal-algebraic functional analogues – having different values in hyperintensional contexts.[7]

A generalized version of the paradox of propositions can be similarly treated

[7] Whittle (2009) provides two versions of Russell's paradox of propositions, in a setting of a discussion of the interaction between propositions and epistemic modality. The first variation corresponds to the presentation above (op. cit.: 273-274); and can thus be similarly treated. The second variation is engendered by considering two classes of worlds, where a proposition unique to each class can be countenanced either via being known apriori (275-276); being centered yet not known apriori, where a centered world is an epistemic possibility whose value is defined relative to a context ranging over an individual and time (277-278; see Quine (1968: 17) for the introduction of centered worlds); or by – as on Kaplan's (1995: 43) paradox – there being an agent who bears an attitude toward a particular world in one of the two classes (278-279). Whittle notes that it is indeterminate which propositions are ruled out apriori (276). In the latter two variations on the paradox, it is unclear, however, why designating a world in one of the two classes, either via centering or bearing an attitude such as querying it, ought to bear on the cardinality of the class, and thus on whether the cardinality of the two classes are equivalent.

(Uzquiano, op. cit.: 290). Let a class, τ, be maximal, if and only if for all propositions α in τ, either 'α' := p or 'α' := ¬p. For a maximal class, M, of propositions, let a subclass, m, therein have cardinality k. Let there be a distinct subclass, n, of equal cardinality, k. Define, then, a proposition, ϕ, in m which states that – for a distinct proposition ψ in n – either 'ψ' := p or 'ψ' := ¬p. Then, given that ϕ is unique to m, $|m| \neq |n|$.

In response, a proposition, ϑ, can be added to n, where ϑ states that there is a proposition in m which characterizes the maximality condition. Then, the addition of the propositions, ϕ and ϑ, unique to each of m and n, would ensure that m and n remain possessed of the same cardinality. The response to Russell's objection can then be simulated as above. While ϕ and ϑ are equivalent propositions which codify the thought that the maximality condition obtains in the corresponding model, the status thereof as possibilities permits the modalities to be interpreted as intensional functions, the compositions of which entraining unique hyperintensional ingredient senses.

In the the remainder of the paper, I endeavor to demonstrate the advantages accruing to the present approach to countenancing modal expressivism via modal colagebraic automata, via a comparison of the theoretical strength of the proposal when applied to characterizing the fundamental properties of the foundations of mathematics, by contrast to the competing approaches to modal expressivism and the limits of their applications thereto.

3.3 Modal Expressivism and the Philosophy of Mathematics

When modal expressivism is modeled via speech acts on a common ground of presuppositions, the application thereof to the foundations of mathematics is limited both (i) by the manner in which necessary propositions are characterized, as well as (ii) by the metalinguistc – rather than, e.g., epistemic – interpretation of the semantics.[8]

Because for example a proposition is taken, according to the proposal, to be identical to a set of possible worlds, all necessarily true mathematical formulas can only express a single proposition; namely, the set of all possible worlds (cf. Stalnaker, 1978; 2003: 51). Thus, although distinct set-forming operations will be codified by distinct axioms of a language of set theory, the axioms will be assumed to express the same proposition: The axiom of Pairing in set theory – which states that a unique set can be formed by combining an element from each of two extant sets: $\exists x \forall u(u \in x \iff u = a \vee u = b)$ – will be supposed to express the same proposition as the Power Set axiom – which states that a set can be formed by taking the set of all subsets of an extant set: $\exists x \forall u(u \in x \iff u \subseteq a)$. However, that distinct operations – i.e., the formation of a set by selecting elements from two extant sets, by contrast to forming a set by collecting all of the subsets of a single extant set – are characterized by the different axioms is readily apparent.

Stalnaker endeavors to redress the objection by availing of the metasemantic interpretation of multi-dimensional intensional semantics, according to which the necessarily true propositions comprising the common ground are yet consistent with contingently valued speech acts – e.g., assertions – about those

[8]See Author (ms) for an application of the epistemic interpretation of multi-dimensional intensional semantics to account for the modal profile of Orey sentences; i.e. mathematical propositions that are undecidable relative to the axioms of a given language. (For the origins of multi-dimensional intensional semantics, see Kamp, 1967; Vlach, 1973; and Segerberg, 1973.) The distinction between epistemic and metaphysical possibilities, as they pertain to the values of mathematical formulas, is anticipated by Gödel's (1951: 11-12) distinction between mathematics in its subjective and objective senses, where the former concerns decidable formulas, and the latter records the values of formulas defined, owing to the incompleteness theorems, in a variant of the language augmented by stronger axioms of infinity. Axioms of infinity take the form, '$\exists x \varnothing \in x \wedge \forall u(u \in x \rightarrow \{u\} \in x)$', and record closure on the singleton of a real in the cumulative hierarchy of sets, where the real can be a large cardinal.

propositions (2003: 54). Although it does record a difference in the epistemic status of agents with regard to whether they know or believe the propositions at issue, the proposal is yet not itself sufficient to distinguish between the senses of distinct, albeit necessarily true formulas.

Further, the metalinguistic interpretation of the semantics places a restriction on the expressive capacity of the mathematical languages at issue. By contrast to the objectual interpretation of the quantifiers, the metalinguistc interpretation of multi-dimensional intensional semantics – by restricting itself to the remit of natural language semantics – is limited to substitutional quantification; i.e. to quantification over only those objects for which a name, in the natural language, has been specified (op. cit.). Given, however, the above restriction, substitutional quantification cannot account for impredicative comprehension; i.e., for the specification of extensions of terms, with reference to the totality of items of the type that the terms are intended to designate.

By contrast to the limits of Stalnaker's approach to modal expressivism, differences in the senses of formulas can, as noted, be recorded by the correspondence, in modal algebras, between modal operators and intensional functions, such that the senses of materially equivalent propositions-as-possibilities can shift when the possibilities are translated as intensions and figure within hyperintensional contexts. Further, both epistemic modal algebras and their dually corresponding modal coalgebraic automata permit of unrestricted objectual quantification, and are thus unconfined to the expressive limitations of the substitutional quantifiers of natural language.

A variation of Stalnaker's semantics is proffered in Yalcin (2016). Yalcin (op. cit.) argues that concepts and beliefs can be modeled in the manner of subject matters (cf. Lewis, 1998; Yablo, 2014), where the latter are interrogative updates on a background set of epistemically possible worlds, and the inquiry whether ϕ induces a focus on a subset of worlds which answer whether ϕ. In the setting of philosophy of mathematics, however, it is unclear how identifying subject matters with fragments of an epistemic modal space induced by interrogative updates can model the nature of, inter alia, Orey sentences such as the generalized continuum hypothesis (i.e., ${\aleph_\alpha}^{\aleph_\alpha} = \aleph_\alpha + 1$),

whose values are indeterminate relative to the present axioms of ZF set theory with choice, and the reduction in the incompleteness of which depends upon augmenting the relevant mathematical languages with stronger axioms of infinity (cf. Woodin, 2010). In order to determine the truth-values of Orey sentences, the space of epistemic modality might require an expanding domain by way of which the new axioms and their corresponding theorems can be accommodated. It is thus unclear how a treatment of subject matters as an induced *restriction* on the space of epistemic possibilities – i.e., as interrogative-induced operations from a set of possibilities to any particular subset thereof – could begin to account for the required expansion in epistemic states.

Thomasson (2007) argues for a version of modal expressivism which she refers to as 'modal normativism', according to which alethic modalities are to be replaced by deontic modalities taking the form of object-language, modal indicative conditionals (op. cit.: 136, 138, 141). The modal indicative conditionals serve to express constitutive rules pertaining, e.g., to ontological dependencies which state that: 'Necessarily, if an entity satisfying a property exists then a distinct entity satisfying a property exists' (143-144), and generalizes to other expressions, such as analytic conditionals which state, e.g., that: 'Necessarily, if an entity satisfies a property, such as being a bachelor, then the entity satisfies a distinct yet co-extensive property, such as being unmarried' (148). A virtue of

Thomasson's interpretation of modal indicative conditionals as expressing both analytic and ontological dependencies is that it would appear to converge with the 'If-thenist' proposal in the philosophy of mathematics. 'If-thenism' is an approach according to which, if an axiomatized mathematical language is consistent, then (i) one can either bear epistemic attitudes, such as fictive acceptance, toward the target system (cf. Leng, 2010: 180) or (ii) the system (possibly) exists [cf. Russell (op. cit.: §1); Hilbert (1899/1980: 39); Menger (1930/1979: 57); Putnam (1967); Shapiro (2000: 95); Chihara (2004: Ch. 10); and Awodey (2004: 60-61)].[9] However, there are at least two issues for the modal normativist approach in the setting of the philosophy of mathematics. One general issue for the proposal is that the treatment of quantification remains unaddressed, given that there are translations from modal operators, such as figure in modal indicatives, into existential and universal quantifiers.[10] A second issue for the normative indicative conditional approach is that Thomasson's normative modalities are unimodal. They are thus not sufficiently fine-grained to capture distinctions such as Gödel's (op. cit.) between mathematics in its subjective and objective

[9]See Leng (2009), for further discussion. Field (1980/2016: 11-21; 1989: 54-65, 240-241) argues in favor of the stronger notion of conservativeness, according to which consistent mathematical theories must be satisfiable by internally consistent theories of physics. More generally, for a class of assertions, A, comprising a theory of fundamental physics, and a class of sentences comprising a mathematical language, M, any sentences derivable from A+M ought to be derivable from A alone. Another variation on the 'If-thenist' proposal is witnessed in Field (2001: 333-338), who argues that the existence of consistent forcing extensions of set-theoretic ground models adduces in favor of there being a set-theoretic pluriverse, and thus entrains indeterminacy in the truth-values of undecidable sentences. For a similar proposal, which emphasizes the epistemic role of examining how instances of undecidable sentences obtain and fail so to do relative to forcing extensions in the set-theoretic pluriverse, see Hamkins (2012: §7).

[10]The formal correspondence between modalities and quantifiers is anticipated by Aristotle (*De Interpretatione*, 9; *De Caelo*, I.12), who defines the metaphysical necessity of a proposition as its being true at all times. For detailed discussion of Aristotle's theory, see Waterlow (1982). For a contemporary account of the multi-modal logic for metaphysical and temporal modalities, see Dorr and Goodman (ms). For contemporary accounts of the correspondence between modal operators and quantifiers see von Wright (1952/1957), where von Wright anticipates Kripke's (1963) relational semantics for modal logic; Montague (1960/1974: 75); Lewis (1975/1998; 1981/1998); Kratzer (op. cit.; 1981/2012); and Kuhn (1980).

senses.[11] Further distinctions between the types of mathematical modality can be delineated which permit epistemic types of mathematical possibility to serve as a guide as to whether a formula is metaphysically mathematically possible.[12] The convergence between epistemic and metaphysical mathematical modalities can be countenanced via a multi-dimensional intensional semantics. Thus, by eschewing alethic modalities for unimodal, normative indicatives, the normative modalities are unable to account for the relation between the alethic interpretation of modality and, e.g., logical mathematical modalities treated as consistency operators on languages (cf. Field, 1989: 249-250, 257-260; Leng: 2007; 2010: 258), or for the convergence between epistemic possibilities concerning decidability and their bearing on the metaphysical modal status of undecidable sentences.

According, finally, to Brandom's (op. cit.) modal expressivist approach, terms are individuated by their rules of inference, where the rules are taken to have a modal profile translatable into the counterfactual forms taken by the transition functions of automata (cf. Brandom, 2008: 142). In order to countenance the metasemantic truth-conditions for the object-level, pragmatic abilities captured by the automata's counterfactual transition states, Brandom augments a first-order language comprised of a stock of atomic formulas with an incompatibility function (141). An incompatibility function, I, is defined as the incoherence of the union of two sentences, where incoherence is a generalization of the notion of inconsistency to nonlogical vocabulary.

$\mathrm{x} \cup \mathrm{y} \in Inc \iff \mathrm{x} \in I(\mathrm{y})$ (141-142).

[11] See footnote 8.

[12] See Author (ms), for further discussion. A precedent is Reinhardt (1974: 199-200), who proposes the use of imaginary sets, classes, and projections, as 'imaginary experiments' (204), in order to ascertain the consequences of accepting new axioms for ZF which might account for the reduction of the incompleteness of Orey sentences. See Maddy (1988,b), for critical discussion.

Incompatibility is supposed to be a modal notion, such that the union of the two sentences is incompossible (126). A sentence, β is an incompatibility-consequence, $\Vdash_I$, of a sentence, α, iff there is no sequence of sentences, $<\gamma_1, \ldots, \gamma_n>$, such that it can be the case that $\alpha \Vdash_I <\gamma_1, \ldots, \gamma_n>$, yet not be the case that $\beta \Vdash_I <\gamma_1, \ldots, \gamma_n>$ (125). To be incompatible with a necessary formula is to be compatible with everything that does not entail the formula (129-130). Dually, to be incompatible with a possible formula is to be incompatible with everything compatible with something compatible with the formula (op. cit.).

There are at least two, general issues for the application of Brandom's modal expressivism to the foundations of mathematics.

The first issue is that the mathematical vocabulary – e.g., the set-membership relation, $\in$ – is axiomatically defined. I.e., the membership relation is defined by, inter alia, the Pairing and Power Set axioms of set-theoretic languages. Thus, mathematical terms have their extensions individuated by the axioms of the language, rather than via a set of inference rules that can be specified in the absence of the mention of truth values. Even, furthermore, if one were to avail of modal notions in order to countenance the intensions of the mathematical vocabulary at issue – i.e., functions from terms in intensional contexts to their extensions – the modal profile of the intensions is orthogonal to the properties encoded by the incompatibility function. Fine (2006) avails, e.g., of a dynamic logic in order to countenance the possibility of reinterpreting the intensions at issue, and of thus accounting for variance in the range of the domains of quantifier expressions. The dynamic possibilities are specified as operational conditions on tracking increases in the size of the cardinality of the universe (Fine, 2005). Uzquiano (2015b) argues that it is always possible to reinterpret the intensions of non-logical vocabulary, as one augments one's language with stronger axioms of infinity and climbs thereby farther up the cumulative hierarchy of sets. The reinterpretations of, e.g., the concept of set are effected by the addition of new large cardinal axioms, which stipulate the existence of larger inaccessible cardinals. However, it is unclear how the incompatibility function – i.e., a modal operator defined via Boolean negation and a generalized condition on inconsistency – might similarly be able to model the intensions pertaining to the ontological expansion of the cumulative hierarchy.

The second issue is that the truth-conditional formulas in the metalanguage of Brandom's modal expressivist semantics are not compositional. I.e., it is not the case of the clauses in the metalanguage that their truth-conditions are formed by the composition of the semantic values of their component expressions. While the formulas are recursively formed – because the decomposition of complex formulas into atomic formulas is decidable although computationally infeasible – formulas in the language are not compositional, because they fail to satisfy the subformula property to the effect that the value of a logically complex formula is calculated as a function of the values of the component logical connectives applied to subformulas therein (135).[13]

By contrast to the limits of Brandom's approach to modal expressivism, modal coalgebraic automata can circumvent both of the issues mentioned in the foregoing. In response to the first issue, concerning the axiomatic individuation and intensional profiles of mathematical terms, functors of modal coalgebraic automata can be interpreted in order to provide a precise delineation of the intensions of the target vocabulary [cf. Author (ms). In response, finally, to the second of the above issues, the values taken by modal coalgebraic automata are both decidable and computationally feasible, while the dual isomorphism of co-

[13]Let a decision problem be a propositional function which is feasibly decidable, if it is a member of the polynomial time complexity class; i.e., if it can be calculated as a polynomial function of the size of the formula's input (cf. Dean, 2015).

lagebras to Boolean-valued models of modal algebras ensures that the formulas therein retain their compositionality. The decidability of colagebraic automata can further be witnessed by the role of modal coalgebras in countenancing the modal profile of Ω-logical consequence, where – given a proper class of Woodin cardinals – the values of mathematical formulas can remain invariant throughout extensions of the ground models comprising the set-theoretic pluriverse (cf. Woodin, 2010; Author, ms). The individuation of large cardinals can further be characterized by the functors of modal coalgebraic automata, when the latter are interpreted so as to countenance the elementary embeddings constitutive of large cardinal axioms in the category of sets.

4 Concluding Remarks

In this essay, I have endeavored to account for a mathematically tractable background against which to model both modal cognitivism and modal expressivism. I availed, to that end, of the dual isomorphism between epistemic modal algebras and modal coalgebraic automata. Epistemic modal algebras were shown to comprise a materially adequate fragment of the language of thought, given that models thereof figure in both the cognitive, as well as in the interpretations of the physical, sciences. A further virtue of such algebras is that they provide the resources necessary to resolve Russell's paradox of propositions. The elision of the elements in the algebra permits epistemic modal operators on formulas to correspond to monotonic intensional functions as concepts, where the latter can then be concatenated in hyperintensional arrays of intensions in order to yield new propositions. It was then shown how the approach to modal expressivism here proffered, as regimented by the modal coalgebraic automata to which the epistemic modal algebras are isomorphic, avoids the pitfalls attending the competing modal expressivist approaches based upon both the inferentialist approach to concept-individuation and the approach to codifying the speech acts in natural language via intensional semantics. The present modal expressivist approach was shown, e.g., to avoid the limits of the foregoing in the philosophy of language, as they concerned the status of necessary propositions; substitutional quantification; the inapplicability of inferentialist-individuation to mathematical vocabulary; and failures of compositionality. Countenancing modal expressivism via modal coalgebraic automata was shown, then, to be able to account for both the intensions of mathematical terms and possible reinterpretations thereof; for the modal profile of Ω-logical consequence in the category of sets; and for the elementary embeddings constitutive of large cardinal axioms in set-theoretic languages.

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