The Logic of Hyperlogic
Part A: Foundations

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This supplement to Part A of "The Logic of Hyperlogic" provides axiomatic derivations of the theorems and derivable rules from Tables A3–A4 and Table A7.[1]

All conventions adopted in Part A still apply. Some further conventions: First, I assume newly introduced variables are chosen so as to not occur anywhere in the relevant formulas. Second, once RE is derived, I generally suppress mention of it. Finally, if I cite multiple axioms/rules on one line, they are applied in the order they're written. (Common combinations include $\text{Gen}_{\downarrow}$ + $\text{Idle}_{\downarrow}$ + $\text{Vac}_{\downarrow}$ and $\text{Gen}_{@}$ + Red + Ref.)

Derivations in **H**

Tables S1–S2 include the theorems and derivable rules from Tables A3–A4 as well as "auxiliary" theorems and rules that are useful for deriving them. Note: the derivations in what follows are not presented in the order of the theorems and rules as they appear in these tables (the order of their derivation is not the most natural way to organize them). It is most convenient to derive the theorems and derivable rules from Table S2 before deriving those in Table S1.

H$_{\vdash}$ (theorems and derivable rules)	
Theorems	
$\text{Dist}_{@}$	$\iota \in cl \vdash (@_{\iota} ☆ \phi \leftrightarrow ☆ @_{\iota} \phi)$
	$\iota \in cl \vdash (@_{\iota}(\phi \circ \psi) \leftrightarrow (@_{\iota} \phi \circ @_{\iota} \psi))$
ClIntro	$☆ @_{\iota} \phi \dashv\vdash @_{cl} ☆ @_{\iota} \phi$
	$(@_{\iota} \phi \circ @_{\iota} \psi) \dashv\vdash @_{cl}(@_{\iota} \phi \circ @_{\iota} \psi)$
Bool	$\bigstar \phi \dashv\vdash ☆ \phi$
	$(\phi \bullet \psi) \dashv\vdash (\phi \circ \psi)$
	$(\phi = \psi) \dashv\vdash \Box(\phi \leftrightarrow \psi)$

[1] A version of this document was included in the original submission sent to RSL. It was requested that it not be included in the published version due to length considerations.

Rigid	$\iota \vdash \Box\, \iota$
	$\neg @_\iota\, \kappa \vdash \Box \neg @_\iota\, \kappa$
	$\lvert\iota\rvert_1 \vdash \Box \lvert\iota\rvert_1$
Rep	$\lvert\iota\rvert_1, \Box(@_\iota\, \phi \leftrightarrow @_\iota\, \psi) \vdash \Box(@_\iota \,\text{☆}\, \phi \leftrightarrow @_\iota \,\text{☆}\, \psi)$
	$\lvert\iota\rvert_1, \Box(@_\iota\, \phi \leftrightarrow @_\iota\, \phi'), \Box(@_\iota\, \psi \leftrightarrow @_\iota\, \psi') \vdash \Box(@_\iota(\phi \circ \psi) \leftrightarrow @_\iota(\phi' \circ \psi'))$
Derivable Rules	
RK	if $\phi_1, \ldots, \phi_n \vdash \psi$, then $\Box\, \phi_1, \ldots, \Box\, \phi_n \vdash \Box\, \psi$
U2C+	if $cl, \phi_1, \ldots, \phi_n \Vdash \psi$, then $\phi_1, \ldots, \phi_n \vdash \psi$
$\text{Gen}_\downarrow$	if $\lvert i\rvert_1, i, \phi_1, \ldots, \phi_n \vdash \psi$, then $\downarrow i.\phi_1, \ldots, \downarrow i.\phi_n \vdash \downarrow i.\psi$

Table S1: Some useful theorems and derivable rules for $\vdash$ in **H**

$\mathbf{H}_\Vdash$ (theorems and derivable rules)	
Theorems	
S5	$\Vdash \phi$ where ϕ is a substitution instance of an **S5**-theorem whose connectives are replaced with their rigidly classical counterparts
$\text{Subset}_@$	$@_\iota\, \kappa, @_\kappa\, \phi \Vdash @_\iota\, \phi$
$\text{Intro}_=$	$@_\iota\, \kappa, \lvert\kappa\rvert_1 \Vdash (\iota = \kappa)$
$\text{Intro}_@$	$\iota, \lvert\iota\rvert_1, \phi \Vdash @_\iota\, \phi$
$\text{DA}_@$	$\lvert\iota\rvert_1, @_\iota \downarrow i.\phi \Vdash @_\iota\, \phi[\iota/i]$ where ι is free for i in ϕ
	$\lvert\iota\rvert_1, @_\iota\, \phi[\iota/i] \Vdash @_\iota \downarrow i.\phi$ where ι is free for i in ϕ
$\text{DA}_\downarrow$	$\downarrow i. \downarrow j.\phi \dashv\Vdash \downarrow i.\phi[i/j]$ where i is free for j in ϕ
$\text{VE}_\downarrow$	$\downarrow i.\phi \dashv\Vdash \downarrow j.\phi[j/i]$ where j is free for i in ϕ and j is not free in $\downarrow i.\phi$
$\text{Dist}_@$	$\lvert\iota\rvert_1 \Vdash @_\iota \bigstar \phi \equiv \bigstar @_\iota\, \phi$
	$\lvert\iota\rvert_1 \Vdash @_\iota(\phi \bullet \psi) \equiv (@_\iota\, \phi \bullet @_\iota\, \psi)$
$\text{Dist}_\downarrow$	$\downarrow i. \bigstar \phi \dashv\Vdash \bigstar \downarrow i.\phi$
	$\downarrow i.(\phi \bullet \psi) \dashv\Vdash (\downarrow i.\phi \bullet \downarrow i.\psi)$
$\text{Dist}^+_@$	$@_\iota \blacksquare \phi \dashv\Vdash \blacksquare @_\iota\, \phi$
	$@_\iota(\phi \,\&\, \psi) \dashv\Vdash (@_\iota\, \phi \,\&\, @_\iota\, \psi)$
$\text{VDist}_@$	$@_\kappa(@_\iota\, \phi \bullet \psi) \dashv\Vdash (@_\iota\, \phi \bullet @_\kappa\, \psi)$
$\text{Intro}_\&$	$\phi, \psi \Vdash (\phi \,\&\, \psi)$
$\text{Elim}_\&$	$(\phi \,\&\, \psi) \Vdash \phi$ and $(\phi \,\&\, \psi) \Vdash \psi$
Bool	$@_\iota\, cl \Vdash @_\iota \,\text{☆}\, \phi \equiv @_\iota \bigstar \phi$
	$@_\iota\, cl \Vdash @_\iota(\phi \circ \psi) \equiv @_\iota(\phi \bullet \psi)$
Rigid	$@_\iota\, \kappa \Vdash \blacksquare @_\iota\, \kappa$
	$\iota \Vdash \blacksquare\, \iota$
	$\lvert\iota\rvert_1 \Vdash \blacksquare \lvert\iota\rvert_1$

Derivable Rules	
C2U+	if $\phi_1, \ldots, \phi_n \vdash \psi$, then $@_{cl}\, \phi_1, \ldots, @_{cl}\, \phi_n \Vdash @_{cl}\, \psi$
Ded	$\phi_1, \ldots, \phi_n, \phi \Vdash \psi$ iff $\phi_1, \ldots, \phi_n \Vdash \phi \supset \psi$
Nec	if $\Vdash \phi$, then $\Vdash \blacksquare\, \phi$
$\text{Gen}^{+}_{@}$	if $\phi_1, \ldots, \phi_n, i \in \iota \Vdash @_i\, \psi$ where i is not free in $\phi_1, \ldots, \phi_n$, or ψ, then $\phi_1, \ldots, \phi_n \Vdash @_\iota\, \psi$
RE	if $\phi \dashv\Vdash \phi'$, then $\psi \dashv\Vdash \psi'$ where ψ' is the result of replacing some occurrences of ϕ with ϕ' in ψ

Table S2: Some useful theorems and derivable rules for $\Vdash$ in **H**

Derivations for Table S2

- C2U+:

$$\begin{array}{rl} \phi_1, \ldots, \phi_n \vdash \psi & \text{premise} \\ cl, \phi_1, \ldots, \phi_n \Vdash \psi & \text{C2U} \\ @_{cl}\, cl, @_{cl}\, \phi_1, \ldots, @_{cl}\, \phi_n \Vdash @_{cl}\, \psi & \text{Gen}_{@} \\ @_{cl}\, \phi_1, \ldots, @_{cl}\, \phi_n \Vdash @_{cl}\, \psi & \text{Ref.} \end{array}$$

- $\text{Subset}_{@}$:

$$\begin{array}{rl} \kappa, @_\kappa\, \phi \Vdash \phi & \text{Elim}_{@} \\ @_\iota\, \kappa, @_\iota\, @_\kappa\, \phi \Vdash @_\iota\, \phi & \text{Gen}_{@} \\ @_\iota\, \kappa, @_\kappa\, \phi \Vdash @_\iota\, \phi & \text{Red.} \end{array}$$

- $\text{Intro}_{@}$:

$$\begin{array}{rl} @_\iota\, i, @_i\, \phi \Vdash @_\iota\, \phi & \text{Subset}_{@} \\ \downarrow i.\, @_\iota\, i, \phi \Vdash @_\iota\, \phi & \text{Gen}_{\downarrow}, \text{Idle}_{\downarrow}, \text{Vac}_{\downarrow} \\ \iota, @_\iota \downarrow i.\, @_\iota\, i, \phi \Vdash @_\iota\, \phi & \text{Elim}_{@} \\ \iota, |\iota|_1, \phi \Vdash @_\iota\, \phi & \text{def. of } |\iota|_1. \end{array}$$

- $\text{Intro}_{\&}$:

$$\begin{array}{rl} @_i\, \phi, @_i\, \psi \vdash (@_i\, \phi \wedge @_i\, \psi) & \text{S5} \\ @_i\, \phi, @_i\, \psi \Vdash @_{cl}(@_i\, \phi \wedge @_i\, \psi) & \text{C2U+, Red} \\ \phi, \psi \Vdash \downarrow i.\, @_{cl}(@_i\, \phi \wedge @_i\, \psi) & \text{Gen}_{\downarrow}, \text{Idle}_{\downarrow} \\ \phi, \psi \Vdash \phi\, \&\, \psi & \text{def. of \&.} \end{array}$$

- $\text{Elim}_{\&}$: similar to $\text{Intro}_{\&}$.
- $\text{DA}_{@}$: I just prove the first direction, since the other is similar.

$$\begin{array}{rl}
|\iota|_1, \iota, i \Vdash @_\iota\, i & \text{Intro}_{@} \\
|i|_1, \iota, i \Vdash @_i\, \iota & \text{Intro}_{@} \\
|\iota|_1, \iota, |i|_1, i, \phi \Vdash i = \iota & \text{Intro}_{\&} \\
|\iota|_1, \iota, |i|_1, i, \phi \Vdash \phi[\iota/i] & \text{SubId} \\
|\iota|_1, \iota, \downarrow i.\phi \Vdash \phi[\iota/i] & \text{Gen}_{\downarrow}, \text{Vac}_{\downarrow} \\
|\iota|_1, @_\iota \downarrow i.\phi \Vdash @_\iota\, \phi[\iota/i] & \text{Gen}_{@}, \text{Red, Ref.}
\end{array}$$

- Ded: Let $\Gamma \coloneqq \{\gamma_1, \ldots, \gamma_n\}$ and let $@_i\, \Gamma = \{@_i\, \gamma_1, \ldots, @_i\, \gamma_n\}$.
  Left-to-right:

$$\begin{array}{rl}
\Gamma, \phi \Vdash \psi & \\
@_i\, \Gamma, @_i\, \phi \Vdash @_i\, \psi & \text{Gen}_{@} \\
@_i\, \Gamma, @_i\, \phi \vdash @_i\, \psi & \text{U2C} \\
@_{cl}\, @_i\, \Gamma \Vdash @_{cl}(@_i\, \phi \to @_i\, \psi) & \text{C2U+} \\
@_i\, \Gamma \Vdash @_{cl}(@_i\, \phi \to @_i\, \psi) & \text{Red} \\
\Gamma \Vdash \downarrow i.\, @_{cl}(@_i\, \phi \to @_i\, \psi) & \text{Gen}_{\downarrow}, \text{Idle}_{\downarrow} \\
\Gamma \Vdash \phi \supset \psi & \text{def. of } \supset.
\end{array}$$

  Right-to-Left:

$$\begin{array}{rl}
\Gamma \Vdash \phi \supset \psi & \text{premise} \\
\Gamma \Vdash \downarrow i.\, @_{cl}(@_i\, \phi \to @_i\, \psi) & \text{def. of } \supset \\
@_i\, \Gamma \Vdash @_i \downarrow i.\, @_{cl}(@_i\, \phi \to @_i\, \psi) & \text{Gen}_{@} \\
|i|_1, @_i\, \Gamma \Vdash @_i\, @_{cl}(@_i\, \phi \to @_i\, \psi) & \text{DA}_{@} \\
|i|_1, @_i\, \Gamma \Vdash @_{cl}(@_i\, \phi \to @_i\, \psi) & \text{Red} \\
|i|_1, @_i\, \Gamma \vdash @_{cl}(@_i\, \phi \to @_i\, \psi) & \text{U2C} \\
|i|_1, @_i\, \Gamma \vdash (@_i\, \phi \to @_i\, \psi) & \text{Cl, Elim}_{@} \\
|i|_1, @_i\, \Gamma, @_i\, \phi \Vdash @_i\, \psi & \text{C2U+, Red} \\
\Gamma, \phi \Vdash \psi & \text{Gen}_{\downarrow}, \text{Idle}_{\downarrow}.
\end{array}$$

- Nec:

$$\begin{array}{rl}
\Vdash \phi & \text{premise} \\
\Vdash @_i\, \phi & \text{Gen}_{@}
\end{array}$$

$\vdash @_i\, \phi$	U2C
$\vdash \Box @_i\, \phi$	Nec (for $\vdash$)
$\Vdash @_{cl} \Box @_i\, \phi$	C2U+
$\Vdash \downarrow i.\, @_{cl} \Box @_i\, \phi$	Gen$_\downarrow$
$\Vdash \blacksquare\, \phi$	def. of $\blacksquare$.

- $\text{Gen}^+_@$: Assume first ι isn't i.

$\phi_1, \ldots, \phi_n, i \in \iota \Vdash @_i\, \psi$	premise
$\phi_1, \ldots, \phi_n, \lvert i \rvert_1, @_i\, \iota \Vdash @_i\, \psi$	Intro$_\&$
$@_j\, \phi_1, \ldots, @_j\, \phi_n, \lvert i \rvert_1, @_i\, \iota \Vdash @_i\, \psi$	Gen$_@$, Red
$\downarrow i.\, @_j\, \phi_1, \ldots, \downarrow i.\, @_j\, \phi_n, \downarrow i.\, @_i\, \iota \Vdash \downarrow i.\, @_i\, \psi$	Gen$_\downarrow$
$@_j\, \phi_1, \ldots, @_j\, \phi_n, \iota \Vdash \psi$	Idle$_\downarrow$, Vac$_\downarrow$
$@_j\, \phi_1, \ldots, @_j\, \phi_n \Vdash @_\iota\, \psi$	Gen$_@$, Ref, Red
$\phi_1, \ldots, \phi_n \Vdash @_\iota\, \psi$	Gen$_\downarrow$, Idle$_\downarrow$, Vac$_\downarrow$.

If ι is i, then the proof is the following:

$\phi_1, \ldots, \phi_n, i \in i \Vdash @_i\, \psi$	premise
$@_j\, \phi_1, \ldots, @_j\, \phi_n, \lvert i \rvert_1, @_i\, i \Vdash @_i\, \psi$	Intro$_\&$, Gen$_@$, Red
$@_j\, \phi_1, \ldots, @_j\, \phi_n, \lvert i \rvert_1 \Vdash @_i\, \psi$	Ref
$@_j\, \phi_1, \ldots, @_j\, \phi_n \Vdash \psi$	Gen$_\downarrow$, Idle$_\downarrow$, Vac$_\downarrow$
$@_j\, \phi_1, \ldots, @_j\, \phi_n \Vdash @_i\, \psi$	Gen$_@$, Red
$\phi_1, \ldots, \phi_n \Vdash @_i\, \psi$	Gen$_\downarrow$, Idle$_\downarrow$, Vac$_\downarrow$.

- RE: By induction on the structure of ψ. The base cases are straightforward. The @- and $\downarrow$-cases follow from Gen$_@$ and Gen$_\downarrow$. For the connectives, I'll just do the $\neg$-case. Observe that $\blacksquare\, \phi \Vdash \phi$:

$\Box @_i\, \phi \vdash @_i\, \phi$	S5
$@_{cl} \Box @_i\, \phi \Vdash @_i\, \phi$	C2U+, Red
$\downarrow i.\, @_{cl} \Box @_i\, \phi \Vdash \phi$	Gen$_\downarrow$, Idle$_\downarrow$
$\blacksquare\, \phi \Vdash \phi$	def. of $\blacksquare$.

Now here's the derivation of the $\neg$-case.

$\psi \dashv\Vdash \psi'$	premise
$\Vdash \psi \equiv \psi'$	Ded, Intro$_\&$

$$\Vdash \blacksquare(\psi \equiv \psi') \qquad \text{Nec}$$
$$\Vdash \blacksquare(\neg\,\psi \equiv \neg\,\psi') \qquad \text{Rep}$$
$$\Vdash \neg\,\psi \equiv \neg\,\psi' \qquad \text{T}$$
$$\neg\,\psi \dashVdash \neg\,\psi' \qquad \text{Elim}_{\&}\text{, Ded.}$$

- S5: Let $\alpha \in \mathcal{L}^0$ be an **S5**-theorem. What we need to show is that $\Vdash \alpha^+$, where $(\cdot)^+$ is defined recursively as follows:

$$p^+ = p$$
$$(\star\,\alpha)^+ = \downarrow i.\,@_{cl} \star @_i\,\alpha^+ \quad (\text{i.e., } \bigstar\,\alpha^+)$$
$$(\alpha \circ \beta)^+ = \downarrow i.\,@_{cl}(@_i\,\alpha^+ \circ @_i\,\beta^+) \quad (\text{i.e., } (\alpha^+ \bullet \beta^+)).$$

Our desired conclusion then follows from Lemma A3.2.[2] The strategy will be as follows. Let $q_1, \ldots, q_n$ be the free propositional variables in α, and let $\alpha' := \alpha[@_i\,q_1/q_1, \ldots, @_i\,q_n/q_n]$. By S5 (for $\vdash$), $\vdash \alpha'$. So by C2U+ and Gen$_\downarrow$, $\Vdash \downarrow i.\,@_{cl}\,\alpha'$. We will show that $\alpha^+ \dashVdash \downarrow i.\,@_{cl}\,\alpha'$. We'll do this in two steps. First, define $(\cdot)^*$ recursively as follows:

$$p^* = @_i\,p$$
$$(\star\,\alpha)^* = @_{cl} \star @_i\,\alpha^*$$
$$(\alpha \circ \beta)^* = @_{cl}(@_i\,\alpha^* \circ @_i\,\beta^*).$$

Below, we'll prove (1) $\alpha^* \dashv\vdash \alpha'$. From this, $\downarrow i.\,@_{cl}\,\alpha^* \dashVdash \downarrow i.\,@_{cl}\,\alpha'$ follows by C2U+ and Gen$_\downarrow$. So we'll also prove (2) $\alpha^+ \dashVdash \downarrow i.\,@_{cl}\,\alpha^*$. Thus, $\alpha^+ \dashVdash \downarrow i.\,@_{cl}\,\alpha'$, as desired.

To establish (1), we proceed by induction. The base case is trivial since $p^* = @_i\,p = p'$. I'll illustrate the inductive steps with the $\neg$-case. Suppose $\alpha^* \dashv\vdash \alpha'$. Then $\neg\,\alpha^* \dashv\vdash \neg\,\alpha' = (\neg\,\alpha)'$. So it suffices to show that $\neg\,\alpha^* \dashv\vdash (\neg\,\alpha)^* = @_{cl} \neg @_i\,\alpha^*$. By Red, $\alpha^* \dashv\vdash @_i\,\alpha^*$ given how α^* is defined. So $\neg\,\alpha^* \dashv\vdash \neg @_i\,\alpha^*$. But now:

$$\neg @_i\,\alpha^* \dashv\vdash \sim @_i\,\alpha^* \qquad \text{Bool}$$
$$\dashv\vdash \downarrow j.\,@_{cl} \neg @_j\,@_i\,\alpha^* \qquad \text{def. of } \sim$$
$$\dashv\vdash \downarrow j.\,@_{cl} \neg @_i\,\alpha^* \qquad \text{Red}$$
$$\dashv\vdash @_{cl} \neg @_i\,\alpha^* \qquad \text{Vac}_\downarrow.$$

[2] Note that while Lemma A3.2 does not hold in **QH**, we do still have the following: if $\Vdash_{\mathbf{H}} \phi$ and $\psi \in \mathcal{L}^{\mathsf{QH}}$ is free for p in ϕ, then $\Vdash_{\mathbf{QH}} \phi[\psi/p]$. In other words, so long as $\Vdash \phi$ is derivable in **H**, then $\Vdash \phi[\psi/p]$ is derivable in **QH** even if $\psi \in \mathcal{L}^{\mathsf{QH}}$. This ensures that the derivation of S5 (for $\Vdash$) in **H** carries over to **QH** as well.

Hence, $\neg\,\alpha^* \dashv\Vdash (\neg\,\alpha)^*$.

To establish (2), we again proceed by induction. For the base case, by Red and Idle$_\downarrow$:

$$\downarrow i.\,@_{cl}\, p^* = \downarrow i.\,@_{cl}\, @_i\, p \dashv\Vdash \downarrow i.\,@_i\, p \dashv\Vdash p, \text{i.e., } p^+.$$

Again, I'll just illustrate the inductive steps with the $\neg$-case. Suppose $\downarrow i.\,@_{cl}\,\alpha^* \dashv\Vdash \alpha^+$. By Red, $@_i\,\alpha^* \dashv\Vdash \alpha^* \dashv\Vdash @_{cl}\,\alpha^*$ given how α^* is defined. Now, $(\neg\,\alpha)^* = @_{cl}\,\neg\,@_i\,\alpha^*$ and $(\neg\,\alpha)^+ = \downarrow i.\,@_{cl}\,\neg\,@_i\,\alpha^+$. Thus:

$$\begin{array}{rll}
\downarrow i.\,@_{cl}(\neg\,\alpha)^* & \dashv\Vdash \downarrow i.\,@_{cl}\,\neg\,@_i\,\alpha^* & \text{Red}\\
& \dashv\Vdash \downarrow i.\,@_{cl}\,\neg\,@_{cl}\,\alpha^* & @_i\,\alpha^* \dashv\Vdash @_{cl}\,\alpha^*\\
& \dashv\Vdash \downarrow i.\,@_{cl}\,\neg\,@_i\,@_{cl}\,\alpha^* & \text{Red}\\
& \dashv\Vdash \downarrow i.\,@_{cl}\,\neg\,@_i \downarrow i.\,@_{cl}\,\alpha^* & \text{DA}_@,\ \text{Gen}_\downarrow\\
& \dashv\Vdash \downarrow i.\,@_{cl}\,\neg\,@_i\,\alpha^+ & \text{IH}\\
& \dashv\Vdash (\neg\,\alpha)^+ & \text{def. of } (\neg\,\alpha)^+.
\end{array}$$

- Intro$_=$: Observe first that $@_\iota\,\kappa, |\kappa|_1 \Vdash |\iota|_1$:

$$\begin{array}{rl}
@_\iota\,\kappa, @_\kappa\, i \Vdash @_\iota\, i & \text{Subset}_@\\
@_\iota\,\kappa, \downarrow i.\,@_\kappa\, i \Vdash \downarrow i.\,@_\iota\, i & \text{Gen}_\downarrow,\ \text{Vac}_\downarrow\\
@_\iota\,\kappa, @_\kappa \downarrow i.\,@_\kappa\, i \Vdash @_\kappa \downarrow i.\,@_\iota\, i & \text{Gen}_@,\ \text{Red}\\
@_\iota\,\kappa, @_\kappa \downarrow i.\,@_\kappa\, i \Vdash @_\iota \downarrow i.\,@_\iota\, i & \text{Subset}_@\\
@_\iota\,\kappa, |\kappa|_1 \Vdash |\iota|_1 & \text{def. of } |\kappa|_1 \text{ and } |\iota|_1.
\end{array}$$

Thus:

$$\begin{array}{ll}
@_\iota\,\kappa, @_\kappa \downarrow i.\,@_\kappa\, i \Vdash @_\iota \downarrow i.\,@_\kappa\, i & \text{Subset}_@\\
@_\iota\,\kappa, @_\kappa \downarrow i.\,@_\kappa\, i \Vdash @_\iota\, @_\kappa\, \iota & \text{DA}_@ \text{ (since } @_\iota\,\kappa, |\kappa|_1 \Vdash |\iota|_1)\\
@_\iota\,\kappa, @_\kappa \downarrow i.\,@_\kappa\, i \Vdash @_\kappa\, \iota & \text{Red}\\
@_\iota\,\kappa, @_\kappa \downarrow i.\,@_\kappa\, i \Vdash (\iota = \kappa) & \text{Intro}_\&.
\end{array}$$

- DA$_\downarrow$:

$$\begin{array}{rl}
|i|_1 \Vdash @_i \downarrow j.\phi \equiv @_i\,\phi[i/j] & \text{DA}_@,\ \text{Ded}\\
i, |i|_1 \Vdash \downarrow j.\phi \equiv \phi[i/j] & \text{Ded, Intro}_@,\ \text{Elim}_@\\
\downarrow i. \downarrow j.\phi \dashv\Vdash \downarrow i.\phi[i/j] & \text{Ded, Gen}_\downarrow.
\end{array}$$

- VE$_\downarrow$: Immediate by DA$_\downarrow$ and Vac$_\downarrow$.

- $\mathrm{Dist}_{@}$: Here's the $\bigstar$-case for illustration:

$$\begin{array}{rll}
|\iota|_1 \Vdash @_\iota \bigstar \phi \equiv @_\iota @_{cl} \,☆\, @_\iota \phi & & \mathrm{DA}_{@} \\
|\iota|_1 \Vdash @_\iota \bigstar \phi \equiv @_{cl} \,☆\, @_\iota \phi & & \mathrm{Red} \\
|\iota|_1 \Vdash @_\iota \bigstar \phi \equiv @_{cl} \,☆\, @_i @_\iota \phi & & \mathrm{Red} \\
\downarrow i.\, |\iota|_1 \Vdash \downarrow i.\, @_\iota \bigstar \phi \equiv \downarrow i.\, @_{cl} \,☆\, @_i @_\iota \phi & & \mathrm{Elim}_{\&}, \mathrm{Ded}, \mathrm{Gen}_{\downarrow}, \mathrm{Intro}_{\&} \\
|\iota|_1 \Vdash @_\iota \bigstar \phi \equiv \bigstar @_\iota \phi & & \mathrm{Vac}_{\downarrow}, \text{def. of } \bigstar.
\end{array}$$

- $\mathrm{Dist}_{\downarrow}$: Here's the $\bigstar$-case for illustration:

$$\begin{array}{rll}
\downarrow i.\, \bigstar \phi & \dashv\Vdash \downarrow i.\, \downarrow j.\, @_{cl} \,☆\, @_j \phi & \text{def. of } \bigstar \\
& \dashv\Vdash \downarrow i.\, @_{cl} \,☆\, @_i \phi & \mathrm{DA}_{\downarrow} \\
& \dashv\Vdash \downarrow i.\, @_{cl} \,☆\, @_i \downarrow i.\phi & \mathrm{DA}_{@} \\
& \dashv\Vdash \bigstar \downarrow i.\phi & \mathrm{VE}_{\downarrow}, \text{def. of } \bigstar.
\end{array}$$

- Rigid ($@_\iota \kappa \Vdash \blacksquare @_\iota \kappa$):

$$\begin{array}{ll}
@_\iota \kappa \vdash \blacksquare @_\iota \kappa & \text{Rigid (for } \vdash\text{), Bool} \\
@_\iota \kappa \Vdash @_{cl} \blacksquare @_\iota \kappa & \text{C2U+, Red} \\
@_\iota \kappa \Vdash @_{cl} \downarrow i.\, @_{cl} \square @_i @_\iota \kappa & \text{def. of } \blacksquare \\
@_\iota \kappa \Vdash @_{cl} \downarrow i.\, @_{cl} \square @_\iota \kappa & \text{Red} \\
@_\iota \kappa \Vdash @_{cl} \square @_\iota \kappa & \mathrm{Vac}_{\downarrow}, \text{Red} \\
@_\iota \kappa \Vdash @_{cl} \square @_i @_\iota \kappa & \text{Red} \\
@_\iota \kappa \Vdash \downarrow i.\, @_{cl} \square @_i @_\iota \kappa & \mathrm{Gen}_{\downarrow}, \mathrm{Vac}_{\downarrow} \\
@_\iota \kappa \Vdash \blacksquare @_\iota \kappa & \text{def. of } \blacksquare.
\end{array}$$

- Rigid ($\iota \Vdash \blacksquare \iota$):

$$\begin{array}{rl}
@_i \iota \vdash \square @_i \iota & \text{Rigid (for } \vdash\text{)} \\
@_i \iota \Vdash @_{cl} \square @_i \iota & \text{C2U+, Red} \\
\iota \Vdash \downarrow i.\, @_{cl} \square @_i \iota & \mathrm{Gen}_{\downarrow}, \mathrm{Idle}_{\downarrow} \\
\iota \Vdash \blacksquare \iota & \text{def. of } \blacksquare.
\end{array}$$

- Rigid ($|\iota|_1 \Vdash \blacksquare |\iota|_1$):

$$\begin{array}{cl}
@_\iota i \Vdash \blacksquare @_\iota i & \text{Rigid (above)} \\
@_\iota \downarrow i.\, @_\iota i \Vdash @_\iota \downarrow i.\, \blacksquare @_\iota i & \mathrm{Gen}_{\downarrow}, \mathrm{Gen}_{@} \\
@_\iota \downarrow i.\, @_\iota i \Vdash \blacksquare @_\iota \downarrow i.\, @_\iota i & \mathrm{Dist}_{\downarrow}, \mathrm{Dist}_{@} \\
|\iota|_1 \Vdash \blacksquare |\iota|_1 & \text{def. of } |\iota|_1.
\end{array}$$

- $\text{Dist}^+_@$ (with $\blacksquare$): Right-to-left:

$$\begin{array}{rl}
\iota, @_\iota\, \phi \Vdash \phi & \text{Elim}_@ \\
\blacksquare\, \iota, \blacksquare\, @_\iota\, \phi \Vdash \blacksquare\, \phi & \text{Nec, S5} \\
\iota, \blacksquare\, @_\iota\, \phi \Vdash \blacksquare\, \phi & \text{Rigid } (\iota \Vdash \blacksquare\, \iota) \\
@_\iota \blacksquare\, @_\iota\, \phi \Vdash @_\iota \blacksquare\, \phi & \text{Gen}_@\text{, Ref} \\
@_\iota \downarrow i.\, @_{cl} \Box\, @_i\, @_\iota\, \phi \Vdash @_\iota \blacksquare\, \phi & \text{def. of } \blacksquare \\
@_\iota \downarrow i.\, @_{cl} \Box\, @_\iota\, \phi \Vdash @_\iota \blacksquare\, \phi & \text{Red} \\
@_\iota\, @_{cl} \Box\, @_\iota\, \phi \Vdash @_\iota \blacksquare\, \phi & \text{Vac}_\downarrow \\
@_{cl} \Box\, @_\iota\, \phi \Vdash @_\iota \blacksquare\, \phi & \text{Red} \\
@_{cl} \Box\, @_i\, @_\iota\, \phi \Vdash @_\iota \blacksquare\, \phi & \text{Red} \\
\downarrow i.\, @_{cl} \Box\, @_i\, @_\iota\, \phi \Vdash @_\iota \blacksquare\, \phi & \text{Gen}_\downarrow\text{, Vac}_\downarrow \\
\blacksquare\, @_\iota\, \phi \Vdash @_\iota \blacksquare\, \phi & \text{def. of } \blacksquare.
\end{array}$$

  Left-to-right: By $\text{Intro}_\&$, Rigid ($@_i\, \iota \Vdash \blacksquare\, @_i\, \iota$ and $|i|_1 \Vdash \blacksquare\, |i|_1$), and S5, $\blacklozenge(i \in \iota) \Vdash i \in \iota$. Hence:

$$\begin{array}{rl}
i \in \iota, @_\iota \blacksquare\, \phi \Vdash @_i \blacksquare\, \phi & \text{Subset}_@\text{, Elim}_\& \\
i \in \iota, @_\iota \blacksquare\, \phi \Vdash \blacksquare\, @_i\, \phi & \text{Dist}_@ \text{ (since } i \in \iota \Vdash |i|_1) \\
\blacklozenge(i \in \iota), @_\iota \blacksquare\, \phi \Vdash \blacksquare\, @_i\, \phi & \blacklozenge(i \in \iota) \Vdash i \in \iota \\
@_\iota \blacksquare\, \phi \Vdash \blacksquare(i \in \iota \supset @_i\, \phi) & \text{S5} \\
\blacklozenge\, @_\iota \blacksquare\, \phi \Vdash \blacksquare(i \in \iota \supset @_i\, \phi) & \text{S5} \\
\blacklozenge\, @_\iota \blacksquare\, \phi, i \in \iota \Vdash @_i\, \phi & \text{S5} \\
\blacklozenge\, @_\iota \blacksquare\, \phi \Vdash @_\iota\, \phi & \text{Gen}^+_@ \\
\blacksquare\, \blacklozenge\, @_\iota \blacksquare\, \phi \Vdash \blacksquare\, @_\iota\, \phi & \text{Nec, S5} \\
@_\iota \blacksquare\, \phi \Vdash \blacksquare\, @_\iota\, \phi & \text{S5.}
\end{array}$$

- $\text{Dist}^+_@$ (with &): Left-to-right:

$$\begin{array}{rl}
\phi \,\&\, \psi \Vdash \phi & \text{Elim}_\& \\
@_\iota(\phi \,\&\, \psi) \Vdash @_\iota\, \phi & \text{Gen}_@ \\
@_\iota(\phi \,\&\, \psi) \Vdash @_\iota\, \psi & \text{similarly} \\
@_\iota(\phi \,\&\, \psi) \Vdash @_\iota\, \phi \,\&\, @_\iota\, \psi & \text{Intro}_\&.
\end{array}$$

  Right-to-left:

$$\begin{array}{rl}
\iota, @_\iota\, \phi, @_\iota\, \psi \Vdash \phi \,\&\, \psi & \text{Elim}_@\text{, Intro}_\& \\
@_\iota\, \phi \,\&\, @_\iota\, \psi \Vdash @_\iota(\phi \,\&\, \psi) & \text{Gen}_@\text{, Ref, Red, Elim}_\&.
\end{array}$$

- $\mathrm{VDist}_{@}$ (with &): Left-to-right:

$$
\begin{array}{rl}
@_\iota\,\phi \,\&\, \psi \Vdash @_\iota\,\phi & \text{Elim}_{\&} \\
@_\kappa(@_\iota\,\phi \,\&\, \psi) \Vdash @_\iota\,\phi & \text{Gen}_{@}\text{, Red} \\
@_\iota\,\phi \,\&\, \psi \Vdash \psi & \text{Elim}_{\&} \\
@_\kappa(@_\iota\,\phi \,\&\, \psi) \Vdash @_\kappa\,\psi & \text{Gen}_{@} \\
@_\kappa(@_\iota\,\phi \,\&\, \psi) \Vdash @_\iota\,\phi \,\&\, @_\kappa\,\psi & \text{Intro}_{\&}.
\end{array}
$$

Right-to-left:

$$
\begin{array}{rl}
@_\iota\,\phi, \psi \Vdash @_\iota\,\phi \,\&\, \psi & \text{Intro}_{\&} \\
@_\iota\,\phi \,\&\, @_\kappa\,\psi \Vdash @_\kappa(@_\iota\,\phi \,\&\, \psi) & \text{Gen}_{@}\text{, Red, Elim}_{\&}.
\end{array}
$$

- $\mathrm{VDist}_{@}$ (with +): First, observe that $\sim @_\iota\,\phi \dashv\Vdash @_\kappa \sim @_\iota\,\phi$:

$$
\begin{array}{rl}
\sim @_\iota\,\phi \dashv\Vdash \downarrow i.\, @_{cl} \neg @_i\, @_\iota\,\phi & \text{def. of } \sim \\
\dashv\Vdash @_{cl} \neg @_\iota\,\phi & \text{Red, Vac}_{\downarrow} \\
\dashv\Vdash @_\kappa\, @_{cl} \neg @_\iota\,\phi & \text{Red} \\
\dashv\Vdash @_\kappa \downarrow i.\, @_{cl} \neg @_\iota\,\phi & \text{Vac}_{\downarrow}\text{, Gen}_{@} \\
\dashv\Vdash @_\kappa \downarrow i.\, @_{cl} \neg @_i\, @_\iota\,\phi & \text{Red, Gen}_{\downarrow}\text{, Gen}_{@} \\
\dashv\Vdash @_\kappa \sim @_\iota\,\phi & \text{def. of } \sim.
\end{array}
$$

With this, we can derive $\mathrm{VDist}_{@}$. Left-to-right:

$$
\begin{array}{rl}
@_\iota\,\phi + \psi, \sim @_\iota\,\phi \Vdash \psi & \text{S5} \\
@_\kappa(@_\iota\,\phi + \psi), @_\kappa \sim @_\iota\,\phi \Vdash @_\kappa\,\psi & \text{Gen}_{@} \\
@_\kappa(@_\iota\,\phi + \psi) \Vdash \sim @_\kappa \sim @_\iota\,\phi + @_\kappa\,\psi & \text{S5} \\
@_\kappa(@_\iota\,\phi + \psi) \Vdash @_\iota\,\phi + @_\kappa\,\psi & \sim @_\iota\,\phi \dashv\Vdash @_\kappa \sim @_\iota\,\phi\text{, S5}.
\end{array}
$$

Right-to-left:

$$
\begin{array}{rl}
@_\iota\,\phi \Vdash @_\iota\,\phi + \psi & \text{S5} \\
@_\kappa\, @_\iota\,\phi \Vdash @_\kappa(@_\iota\,\phi + \psi) & \text{Gen}_{@} \\
@_\iota\,\phi \Vdash @_\kappa(@_\iota\,\phi + \psi) & \text{Red} \\
\psi \Vdash @_\iota\,\phi + \psi & \text{S5} \\
@_\kappa\,\psi \Vdash @_\kappa(@_\iota\,\phi + \psi) & \text{Gen}_{@} \\
@_\iota\,\phi + @_\kappa\,\psi \Vdash @_\kappa(@_\iota\,\phi + \psi) & \text{S5}.
\end{array}
$$

- VDist$_@$ (with $\supset$):[3] Left-to-right:

$$\begin{array}{rl} @_\kappa(@_\iota\,\phi \supset \psi), @_\kappa\,@_\iota\,\phi \Vdash @_\kappa\,\psi & \text{Gen}_@ \\ @_\kappa(@_\iota\,\phi \supset \psi) \Vdash @_\iota\,\phi \supset @_\kappa\,\psi & \text{Red, Ded.} \end{array}$$

Right-to-left:

$$\begin{array}{rl} @_\kappa \sim @_\iota\,\phi \Vdash @_\kappa(@_\iota\,\phi \supset \psi) & \text{Gen}_@ \\ \sim @_\iota\,\phi \Vdash @_\kappa(@_\iota\,\phi \supset \psi) & @_\kappa \sim @_\iota\,\phi \dashVdash \sim @_\iota\,\phi \\ @_\kappa\,\psi \Vdash @_\kappa(@_\iota\,\phi \supset \psi) & \text{Gen}_@ \\ @_\iota\,\phi \supset @_\kappa\,\psi \Vdash @_\kappa(@_\iota\,\phi \supset \psi) & \text{S5.} \end{array}$$

- Bool (with $\neg$, $\wedge$, and $\Box$): I just do the $\neg$-case to illustrate.

$$\begin{array}{rl} \neg\,\phi \vdash \sim\phi & \text{Bool (for } \vdash) \\ cl, \neg\,\phi \Vdash \sim\phi & \text{C2U} \\ @_\iota\,cl, @_\iota\,\neg\,\phi \Vdash @_\iota \sim\phi & \text{Gen}_@ \\ @_\iota\,cl, @_\iota \sim\phi \Vdash @_\iota\,\neg\,\phi & \text{similarly} \\ @_\iota\,cl \Vdash @_\iota\,\neg\,\phi \equiv @_\iota \sim\phi & \text{Ded, Intro}_\&. \end{array}$$

- Bool (with $\vee$, $\rightarrow$, and $\Diamond$): I just do the $\Diamond$-case to illustrate. First, by S5 (for $\Vdash$), $\blacklozenge\,\phi \dashVdash \sim\blacksquare\sim\phi$. Second, by Nec, S5, and Rigid, $@_\iota\,cl \Vdash @_\iota\,\neg\,\phi = @_\iota \sim\phi$ (and similarly for & and $\blacksquare$). Thus:

$$\begin{array}{ll} \Vdash @_\iota\,\blacklozenge\,\phi \equiv @_\iota \sim\blacksquare\sim\phi & \text{Gen}_@\text{, Ded} \\ @_\iota\,cl \Vdash @_\iota\,\blacklozenge\,\phi \equiv @_\iota\,\neg\,\Box\,\neg\,\phi & \text{Bool (with} \sim \text{and } \blacksquare\text{), Rep} \\ @_\iota\,cl \Vdash @_\iota\,\blacklozenge\,\phi \equiv @_\iota\,\Diamond\,\phi & \text{S5, C2U, Gen}_@. \end{array}$$

Derivations for Table S1

- RK: standard.
- U2C+:

$$\begin{array}{rl} cl, \phi_1, \ldots, \phi_n \Vdash \psi & \text{premise} \\ cl, \phi_1, \ldots, \phi_n \vdash \psi & \text{U2C} \\ \phi_1, \ldots, \phi_n \vdash \psi & \text{Cl.} \end{array}$$

[3] This instance of VDist$_@$ isn't strictly needed for the proofs in Part A. Also, note that VDist$_@$ does not hold with the converse of $\supset$: we don't have $@_\kappa(\psi \supset @_\iota\,\phi) \dashVdash @_\kappa\,\psi \supset @_\iota\,\phi$. (This shouldn't be surprising since $@_\kappa$ acts like a universal quantifier.)

- $\text{Gen}_{\downarrow}$:

$$\begin{array}{rl} |i|_1, \phi_1, \ldots, \phi_n \vdash \psi & \text{premise} \\ cl, |i|_1, \phi_1, \ldots, \phi_n \Vdash \psi & \text{C2U} \\ \downarrow i.cl, \downarrow i.\phi_1, \ldots, \downarrow i.\phi_n \Vdash \downarrow i.\psi & \text{Gen}_{\downarrow} \text{ (for } \Vdash) \\ cl, \downarrow i.\phi_1, \ldots, \downarrow i.\phi_n \Vdash \downarrow i.\psi & \text{Vac}_{\downarrow} \\ \downarrow i.\phi_1, \ldots, \downarrow i.\phi_n \vdash \downarrow i.\psi & \text{U2C+.} \end{array}$$

- Bool (with $\vee$, $\rightarrow$, $\Diamond$, and $=$): I just do the $\Diamond$-case to illustrate.

$$\begin{array}{rl} i, |i|_1, @_i\, cl, @_i \Diamond \phi \Vdash @_i \blacklozenge \phi & \text{Bool (for } \Vdash\text{), Ded} \\ cl, \Diamond \phi \Vdash \blacklozenge \phi & \text{Gen}_{\downarrow}\text{, Idle}_{\downarrow} \\ \Diamond \phi \vdash \blacklozenge \phi & \text{Elim}_{\&}\text{, Ded, U2C+} \\ \blacklozenge \phi \vdash \Diamond \phi & \text{similarly.} \end{array}$$

- $\text{Dist}_{@}$: I'll just do the $\Box$-case to illustrate. First observe:

$$\begin{array}{rl} \vdash \blacksquare \phi \leftrightarrow \Box \phi & \text{Bool} \\ cl \Vdash \blacksquare \phi \equiv \Box \phi & \text{C2U, Ded, Intro}_{\&} \\ @_\iota\, cl \Vdash @_\iota \blacksquare \phi \equiv @_\iota \Box \phi & \text{Elim}_{\&}\text{, Ded, Gen}_{@}\text{, Intro}_{\&} \\ @_\iota\, cl \vdash @_\iota \blacksquare \phi \leftrightarrow @_\iota \Box \phi & \text{U2C, Bool.} \end{array}$$

Next observe:

$$\begin{array}{rl} |\iota|_1 \Vdash @_\iota \blacksquare \phi \equiv \blacksquare @_\iota \phi & \text{Dist}_{@} \text{ for } \Vdash \\ |\iota|_1 \vdash @_\iota \blacksquare \phi \leftrightarrow \Box @_\iota \phi & \text{U2C, Bool.} \end{array}$$

Combining these together, we get our desired conclusion.

- ClIntro: I just illustrate with $\neg$.

$$\begin{array}{rll} \neg @_\iota \phi & \dashv\vdash \sim @_\iota \phi & \text{Bool} \\ & \dashv\vdash \downarrow i.\, @_{cl} \neg @_i @_\iota \phi & \text{def. of } \sim \\ & \dashv\vdash \downarrow i.\, @_{cl} \neg @_\iota \phi & \text{Red} \\ & \dashv\vdash @_{cl} \neg @_\iota \phi & \text{Vac}_{\downarrow}. \end{array}$$

- Rigid: by Rigid (for $\Vdash$) and Bool (and S5).
- Rep: I just illustrate with $\neg$.

$\blacksquare(\phi \equiv \psi) \Vdash \blacksquare(\neg\,\phi \equiv \neg\,\psi)$	Rep (for $\Vdash$)
$\downarrow i.\, @_{cl}\, \Box\, @_i \downarrow j.\, @_{cl}(@_j\, \phi \leftrightarrow @_j\, \psi) \Vdash\, \downarrow i.\, @_{cl}\, \Box\, @_i \downarrow j.\, @_{cl}(@_j \neg\,\phi \leftrightarrow @_j \neg\,\psi)$	def. of $\blacksquare$ and $\equiv$
$k, \lvert k\rvert_1, @_k \downarrow i.\, @_{cl}\, \Box\, @_i \downarrow j.\, @_{cl}(@_j\, \phi \leftrightarrow @_j\, \psi) \Vdash @_k \downarrow i.\, @_{cl}\, \Box\, @_i \downarrow j.\, @_{cl}(@_j \neg\,\phi \leftrightarrow @_j \neg\,\psi)$	Intro$_@$ and Elim$_@$
$k, \lvert k\rvert_1, @_k\, @_{cl}\, \Box\, @_k\, @_{cl}(@_k\, \phi \leftrightarrow @_k\, \psi) \Vdash @_k\, @_{cl}\, \Box\, @_k\, @_{cl}(@_k \neg\,\phi \leftrightarrow @_k \neg\,\psi)$	DA$_@$
$\lvert k\rvert_1, @_{cl}\, \Box\, @_{cl}(@_k\, \phi \leftrightarrow @_k\, \psi) \Vdash @_{cl}\, \Box\, @_{cl}(@_k \neg\,\phi \leftrightarrow @_k \neg\,\psi)$	Gen$_@$, Ref, Red
$\lvert k\rvert_1, @_{cl}\, \Box\, @_{cl}(@_k\, \phi \leftrightarrow @_k\, \psi) \vdash @_{cl}\, \Box\, @_{cl}(@_k \neg\,\phi \leftrightarrow @_k \neg\,\psi)$	U2C
$\lvert k\rvert_1, \Box(@_k\, \phi \leftrightarrow @_k\, \psi) \vdash \Box(@_k \neg\,\phi \leftrightarrow @_k \neg\,\psi)$	ClIntro.

Derivations in **QH** (Tables S3–S4)

Tables S3–S4 contain the theorems and rules from Table A7 along with auxiliary theorems and rules. We start by deriving Gen$_\forall$ and RK$_\forall$ from Table S3. Then we derive the theorems and rules in Table S4, followed by the remaining ones in Table S3.

QH$_\vdash$ (theorems and derivable rules)	
Theorems	
Dual$_\forall$	$\neg\,\forall p\, \phi \dashv\vdash \exists p\, \neg\,\phi$
VDist$_\exists$	$\exists p(\phi \wedge \psi) \dashv\vdash (\phi \wedge \exists p\, \psi)$ where p does not occur free in ϕ
Derivable Rules	
Gen$_\forall$	if $\vdash \phi$, then $\vdash \forall p\, \phi$
RK$_\forall$	if $\phi_1, \ldots, \phi_n \vdash \psi$, then $\forall p\, \phi_1, \ldots, \forall p\, \phi_n \vdash \forall p\, \psi$

Table S3: Some useful theorems and derivable rules for $\vdash$ in **QH**

QH$_\Vdash$ (theorems and derivable rules)	
Theorems	
K$_\forall$	$\forall p(\phi \supset \psi), \forall p\, \phi \Vdash \forall p\, \psi$
Intro$_\exists$	$\phi[q/p] \Vdash \exists p\, \phi$ where q is free for p in ϕ
Vac$_\exists$	$\exists p\, \phi \Vdash \phi$ where p does not occur free in ϕ
VDist$_\exists$	$\exists p(\phi\,\&\,\psi) \dashv\Vdash \phi\,\&\,\exists p\, \psi$ where p does not occur free in ϕ
VE$_\forall$	$\forall p\, \phi \Vdash \forall q\, \phi[q/p]$ where q is free for p in ϕ and q does not occur free in $\forall p\, \phi$
NecEx	$\mathsf{E}\phi \Vdash \blacksquare\, \mathsf{E}\phi$
	$\sim \mathsf{E}\phi \Vdash \blacksquare \sim \mathsf{E}\phi$

$\text{BF}_{\blacklozenge}$	$\blacklozenge\, \exists p\, \phi \Vdash \exists p \blacklozenge \phi$
$\text{BF}^{+}_{@}$	$\lvert\iota\rvert_1, @_\iota\, \exists p\, \phi \Vdash \exists p\, @_\iota\, \phi$
$\text{CBF}_{@}$	$@_\iota\, \forall p\, \phi \Vdash \forall p\, @_\iota\, \phi$
$\text{BF}_{\downarrow}$	$\downarrow i.\, \exists p\, \phi \Vdash \exists p \downarrow i.\phi$
Derivable Rules	
$\text{RK}_{\forall}$	if $\phi_1, \ldots, \phi_n \Vdash \psi$, then $\forall p\, \phi_1, \ldots, \forall p\, \phi_n \Vdash \forall p\, \psi$

Table S4: Some useful theorems and derivable rules for $\Vdash$ in **QH**

- $\text{Gen}_{\forall}$ (for $\vdash$):

$\vdash \phi$	
$\Vdash @_{cl}\, \phi$	C2U+
$\Vdash \forall p\, @_{cl}\, \phi$	$\text{Gen}_{\forall}$ (for $\Vdash$)
$\Vdash @_{cl}\, \forall p\, \phi$	$\text{BF}_{@}$
$\vdash @_{cl}\, \forall p\, \phi$	U2C
$\vdash \forall p\, \phi$	Cl, $\text{Elim}_{@}$.

- $\text{RK}_{\forall}$ (for $\vdash$): by $\text{Gen}_{\forall}$ (for $\vdash$) and $\text{K}_{\forall}$.
- $\text{CBF}_{@}$:

$\forall p\, \phi \Vdash \phi$	$\text{Elim}_{\forall}$
$@_i\, @_\iota\, \forall p\, \phi \Vdash @_i\, @_\iota\, \phi$	$\text{Gen}_{@}$
$@_i\, @_\iota\, \forall p\, \phi \vdash @_i\, @_\iota\, \phi$	U2C
$\forall p\, @_i\, @_\iota\, \forall p\, \phi \vdash \forall p\, @_i\, @_\iota\, \phi$	$\text{RK}_{\forall}$ (for $\vdash$)
$@_i\, @_\iota\, \forall p\, \phi \vdash \forall p\, @_i\, @_\iota\, \phi$	$\text{Vac}_{\forall}$, U2C
$@_i\, @_\iota\, \forall p\, \phi \vdash @_i\, \forall p\, @_\iota\, \phi$	$\text{BF}_{@}$
$@_i\, @_\iota\, \forall p\, \phi \Vdash @_i\, \forall p\, @_\iota\, \phi$	U2C+, Red
$@_\iota\, \forall p\, \phi \Vdash \forall p\, @_\iota\, \phi$	$\text{Gen}_{\downarrow}$, $\text{Idle}_{\downarrow}$.

- $\text{K}_{\forall}$ (for $\Vdash$):

$@_i(\phi \supset \psi), @_i\, \phi \Vdash @_i\, \psi$	Ded, $\text{Gen}_{@}$
$@_i(\phi \supset \psi), @_i\, \phi \vdash @_i\, \psi$	U2C
$\forall p\, @_i(\phi \supset \psi), \forall p\, @_i\, \phi \vdash \forall p\, @_i\, \psi$	$\text{RK}_{\forall}$ (for $\vdash$)
$@_i\, \forall p(\phi \supset \psi), @_i\, \forall p\, \phi \vdash @_i\, \forall p\, \psi$	$\text{CBF}_{@}$, $\text{BF}_{@}$, U2C
$@_i\, \forall p(\phi \supset \psi), @_i\, \forall p\, \phi \Vdash @_i\, \forall p\, \psi$	C2U+, Red
$\forall p(\phi \supset \psi), \forall p\, \phi \Vdash \forall p\, \psi$	$\text{Gen}_{\downarrow}$, $\text{Idle}_{\downarrow}$.

- $\mathrm{RK}_\forall$ (for $\Vdash$): by Ded, $\mathrm{Gen}_\forall$, and $\mathrm{K}_\forall$ (for $\Vdash$).
- RE: we extend the proof to quantifiers.

$$
\begin{array}{lll}
\phi \dashVdash \psi & & \\
\quad \Vdash \phi \equiv \psi & & \text{Ded, Intro}_\& \\
\quad \Vdash \forall p\, \phi \equiv \forall p\, \psi & & \mathrm{RK}_\forall \\
\forall p\, \phi \dashVdash \forall p\, \psi & & \text{Ded, Elim}_\&.
\end{array}
$$

  For $\exists$, use $\mathrm{RK}_\forall$ and $\mathrm{Dual}_\forall$.
- $\mathrm{Intro}_\exists$ and $\mathrm{Vac}_\exists$: use $\mathrm{Dual}_\forall$.
- $\mathrm{VDist}_\exists$: standard.
- $\mathrm{VE}_\forall$:

$$
\begin{array}{rll}
\forall p\, \phi \Vdash \phi[q/p] & & \mathrm{Elim}_\forall \\
\forall q\, \forall p\, \phi \Vdash \forall q\, \phi[q/p] & & \mathrm{RK}_\forall \\
\forall p\, \phi \Vdash \forall q\, \phi[q/p] & & \mathrm{Vac}_\forall.
\end{array}
$$

- NecEx: it suffices to establish the first form, since the second follows using Nec and S5.

$$
\begin{array}{rll}
\blacksquare(p \equiv \phi) \Vdash \exists p\, \blacksquare(p \equiv \phi) & & \mathrm{Intro}_\exists \\
\blacksquare\,\blacksquare(p \equiv \phi) \Vdash \blacksquare\, \exists p\, \blacksquare(p \equiv \phi) & & \text{Nec, S5} \\
\blacksquare(p \equiv \phi) \Vdash \blacksquare\, \exists p\, \blacksquare(p \equiv \phi) & & \text{S5} \\
\exists p\, \blacksquare(p \equiv \phi) \Vdash \exists p\, \blacksquare\, \exists p\, \blacksquare(p \equiv \phi) & & \mathrm{RK}_\forall, \mathrm{Dual}_\forall \\
\exists p\, \blacksquare(p \equiv \phi) \Vdash \blacksquare\, \exists p\, \blacksquare(p \equiv \phi) & & \mathrm{Vac}_\exists.
\end{array}
$$

- $\mathrm{BF}_\blacklozenge$: standard.
- $\mathrm{BF}^+_@$:

$$
\begin{array}{lll}
|\iota|_1\,, @_\iota\, \exists p\, \phi \Vdash \sim @_\iota \sim \exists p\, \phi & & \mathrm{Dist}_@ \\
|\iota|_1\,, @_\iota\, \exists p\, \phi \Vdash \sim @_\iota\, \forall p \sim \phi & & \mathrm{Dual}_\forall \\
|\iota|_1\,, @_\iota\, \exists p\, \phi \Vdash \sim \forall p\, @_\iota \sim \phi & & \mathrm{BF}_@ \\
|\iota|_1\,, @_\iota\, \exists p\, \phi \Vdash \exists p \sim @_\iota \sim \phi & & \mathrm{Dual}_\forall \\
|\iota|_1\,, @_\iota\, \exists p\, \phi \Vdash \exists p\, @_\iota\, \phi & & \mathrm{Dist}_@.
\end{array}
$$

- $\mathrm{BF}_\downarrow$: similar to $\mathrm{BF}^+_@$.
- $\mathrm{Dual}_\forall$: First, observe $\vdash \forall p(\sim\neg\phi \supset \phi)$ by Bool and $\mathrm{Gen}_\forall$ (for $\vdash$). Since $\forall p(\sim\neg\phi \supset \phi) \Vdash \forall p \sim\neg\phi \supset \forall p\, \phi$ by $\mathrm{K}_\forall$, it follows by U2C that $\forall p \sim\neg\phi \vdash \forall p\, \phi$. Similarly, $\forall p\, \phi \vdash \forall p \sim\neg\phi$. Thus:

$$
\sim \forall p \sim \neg \phi \dashVdash \exists p \sim\sim\neg\phi \qquad \mathrm{Dual}_\forall
$$

$$
\begin{array}{rll}
\sim \forall p \sim \neg \phi \dashv\vdash \exists p \neg \phi & & \text{RE, U2C} \\
\neg \forall p \sim \neg \phi \dashv\vdash \exists p \neg \phi & & \text{Bool} \\
\neg \forall p\, \phi \dashv\vdash \exists p \neg \phi & & \forall p\, \phi \dashv\vdash \forall p \sim \neg \phi.
\end{array}
$$

- VDist$_\exists$: standard.