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ANDERSON AND BELNAP'S MINIMAL POSITIVE LOGIC WITH MINIMAL NEGATION

1. Introduction

We understand the locution "minimal negation" as historically defined by Johansson in [3], where negative theorems are instances of positive intuitionistic theorems with $A \to f$ abbreviating $\neg A$ and f an arbitrary propositional falsity constant. In particular, any logic with axioms for customary properties of conjunction and disjunction is endowed with minimal negation if it contains as theorems:

(i) $A \to \neg\neg A$

(ii) $(A \to B) \to (\neg B \to \neg A)$

(iii) $(A \to \neg B) \to (B \to \neg A)$

(iv) $(A \to \neg A) \to \neg A$

(v) $(A \to B) \to ((A \to \neg B) \to \neg A)$

(vi) $(A \to \neg B) \to ((A \to B) \to \neg A)$

all of which are instances of positive implicative theorems of $I^{\to}$ (intuitionistic implicational logic).

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Our question is: can we embed minimal negation in implicative logics weaker than $I^{\rightarrow}$? [4] shows how to define minimal negation in the positive fragment of the logic of relevance R and [5] in contractionless intuitionistic logic. Is it possible to endow weaker positive logics with minimal negation? In this paper our aim is to prove that minimal negation can be embedded in even such a weak system as Anderson and Belnap's minimal positive logic.

Whenever implicative resources to instantiate minimal negation are absent, we can still impose additional constraints on f to obtain minimal negation. Interestingly, this strategy allows the semantical isolation of different principles of negation. Moreover, finegrained varieties of negation weaker than minimal are naturally considered in this setting, which offers a kind of microscopical companion to [2]. Together with the modelizing of positive and minimal negation, these facts illustrate – we think- the conceptual import and motivation of the proofs.

The structure of the paper is as follows. In §2, Anderson and Belnap's minimal positive logic $ML+$ is presented. §§3-5 recall semantical consistency and completeness proofs for $ML+$ with respect to Routley-Meyer type relational semantics. §§6,7,8 define the logic $ML+$, f (ML with the falsity constant f added to the sentential language). In §§9,10,11, we define the logic MLm ($ML+$ with weak double negation and weak contraposition, but without reductio). In §§12,13,14 the logic $MLmr$ (MLm with the reductio axiom). The paper ends in §15 with three notes on the results obtained.

2. Anderson and Belnap's Minimal Positive Logic: The logic $ML+$

Anderson and Belnap's minimal implicative logic is (see 8.11 of [1], labelled $T^{\rightarrow} - W$):

A1. $A \rightarrow A$

A2. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$

A3. $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

with the rule Modus ponens (if $\vdash A$ and $\vdash A \to B$, then $\vdash B$).

To this logic $ML^{\to}$ we add the axioms:

A4. $(A \wedge B) \to A \quad (A \wedge B) \to B$

A5. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

A6. $A \to (A \vee B) \qquad B \to (A \vee B)$

A7. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

A8. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

and the rule Adjunction (if $\vdash A$ and $\vdash B$, then $\vdash A \wedge B$) thus defining the logic $ML+$. We note that $ML+$ is $BR+$ plus C5 of [1] (p.294), also called $(T - W)+$.

The following formulas (useful in the proof of the completeness theorem) are derivable:

T1. $(A \wedge B) \to (B \wedge A)$

T2. $((A \vee B) \wedge (C \wedge D)) \to ((A \wedge C) \vee (B \wedge D))$

T3. $((A \to C) \vee (B \to D)) \to ((A \wedge B) \to (C \vee D))$

T4. $((A \to C) \wedge (B \to D)) \to ((A \wedge B) \to (C \wedge D))$

T5. ((A→C)∧(B→D))→((A∨B)→(C∨D))

3. Semantics for $ML+$

A $ML+$ model is a quadruple $< O, K, R, \vDash >$where K is a set, O a nonempty subset of K and R a ternary relation on K subject to the following definitions and postulates for all $a, b, c, d \in K$ with quantifiers ranging over K:

d1 $a \leq b =_{def} \exists x(x \in O$ and $Rxab)$

d2 $R^2abcd =_{def} \exists x(Rabx$ and $Rxcd)$

P1 $a \leq a$

P2 $a \leq b$ and $Rbcd \Rightarrow Racd$

P3 $R^2abcd \Rightarrow \exists x(Racx$ and $Rbxd)$

P4 $R^2abcd \Rightarrow \exists x(Rbcx$ and $Raxd)$

Finally, $\vDash$ is a binary valuation relation between elements from K and sentences of $ML+$ satisfying the following conditions for all sentence letters p, formulas A, B and $a, b, c \in K$:

(i) $a \vDash p$ and $a \leq b \Rightarrow b \vDash p$

(ii) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

(iii) $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$

(iv) $a \vDash A \to B$ iff for all $b, c \in K$, $Rabc$ and $b \vDash A \Rightarrow c \vDash B$

A is valid in $ML+$ [$\vDash_{ML+} A$] iff $a \vDash A$ for all $a \in O$ in all models.

4. Semantic consistency of $ML+$

We prove:

Lemma 4.1. *$a \leq b$ and $a \vDash A \Rightarrow b \vDash A$*

Proof. Induction on the length of A (Use P2 in the case of the conditional). ■

Lemma 4.2. *$\vDash_{ML+} A \to B$ iff for all $a \in K$ in all models, $a \vDash A \Rightarrow a \vDash B$*

Proof. Use P1, d1 and Lemma 4.1. ■

Now we can prove:

Theorem 4.1. (Semantic consistency of ML+) *If $\vdash_{ML+} A$, then $\vDash_{ML+} A$.*

Proof. Since all axioms of $ML+$ are conditional formulas, we can use Lemma 4.2. to immediately render A1, A4-A8 and the rules. A2 and A3 are proved with, respectively, P3 and P4. ■

5. Completeness of $ML+$

We begin by recalling some definitions. A theory is a set of formulas of $ML+$ closed under adjunction and provable entailment; a theory a is prime if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; finally, a is regular if all theorems of $ML+$ belong to a.

Now we define the $ML+$ canonical model. Let K^T be the set of all theories and R^T be defined on K^T as follows: for all formulas A, B and $a, b, c \in K^T$, $R^T abc$ just in case if $A \to B \in a$ and $A \in b$, then $B \in c$. Further, let K^C be the set of all prime theories, O^C the set of all regular prime theories and R^C the restriction of R^T to K^C. Finally, let $\vDash^C$ be defined as follows: for any wff A and $a \in K^C$, $a \vDash^C A$ iff $A \in a$. Then, the $ML+$ canonical model is the quadruple $< O^C, K^C; R^C, \vDash^C >$. In what follows in this section we sketch a proof of the completeness theorem, beginning with some previous Lemmas.

Lemma 5.1. *Let A be a wff, $a \in K^T$ and $A \notin a$. Then, $A \notin x$ for some $x \in K^C$ such that $a \subseteq x$.*

Proof. Define from a a maximal theory x without A. If x is not prime, then for some wffs $B, C, B \vee C \in x, B \notin x, C \notin x$. Put $[B, x] = \{E : \exists D(D \in x$ and $\vdash (B \wedge D) \to E\}$. Define $[C, x]$ similarly. Well, both $[B, x]$ and $[C, x]$ are theories strictly including x. As x is maximal, $A \in [B, x]$ and $A \in [C, x]$, whence it is easy to prove that $A \in x$, which is impossible. ■

Lemma 5.2. *Let $R^T abc$, $a, b \in K^T$, $c \in K^C$. Then, $R^T xbc$ for some $x \in K^C$ such that $a \subseteq x$.*

Proof. Define from a a maximal theory x such that $R^T xbc$. Suppose x is not prime and define $[A, x]$ and $[B, x]$ as in Lemma 5.1. Since both $[A, x]$ and $[B, x]$ are theories strictly including x, we deduce not$-R^T[A, x]bc$ and not$-R^T[B, x]bc$. But then, it is easily shown that c is not prime, which is impossible. ■

Lemma 5.3. *Let $R^T abc$, $a, b \in K^T$, $c \in K^C$. Then, $R^T axc$ for some $x \in K^C$ such that $b \subseteq x$.*

Proof. Similar to that of Lemma 5.2. ■

Lemma 5.4. *If* $\nvdash_{ML+} A$, *there is some* $x \in O^C$ *such that* $A \notin x$.

Proof. $ML+$ is the minimal regular theory such that $A \notin ML+$. By Lemma 5.1, there is some $x \in K^C$ such that $ML+ \subseteq x$ and $A \notin x$. But $x \in O^C$. ■

Lemma 5.5. *Let* $a, b \in K^T$. *The set* $x = \{B : \exists A(A \rightarrow B \in a$ *and* $A \in b)\}$ *is a theory such that* $R^T abx$.

Proof. It is smoothly proven that x is closed under adjunction and provable entailment. Obviously then, $R^T abx$. ■

Lemma 5.6. $a \leq^C b$ *iff* $a \subseteq b$.

Proof. Suppose $a \leq^C b$. By d1, $R^C xab$ with $x \in O^C$. As $A \rightarrow A \in x$, if $A \in a$, then $A \in b$, i.e., $a \subseteq b$. Suppose now $a \subseteq b$. Granted that a is a theory, clearly $R^T ML+aa$, and, so, $R^T ML+ab$. By Lemma 5.2, there is some $x \in K^C$ such that $ML+ \subseteq x$ and $R^C xab$. As $x \in O^C$, $a \leq^C b$ by d1. ■

Lemma 5.7. *Let* $a, b, c \in K^T$ *and* $d \in K^C$. *Moreover, let* $R^{T2}abcd$. *Then, there is some* $x \in K^C$ *such that* $R^T acx$ *and* $R^T bxd$.

Proof. Suppose $R^{T2}abcd$, that is, $R^T abx$ and $R^T xcd$ for some $x \in K^T$. We have to prove that there is some $x \in K^C$ such that $R^T acx$ and $R^T bxd$. Define [cf. Lemma 5.5] the theory $z = \{B : \exists A(A \rightarrow B \in a$ and $A \in c)\}$ with $R^T acz$. Deduce now $R^T bzd$ using A2. By Lemma 5.3, $R^T bxd$ with $z \subseteq x$ and $x \in K^C$. By $R^T acz$ and definitions, $R^T acx$. ■

Lemma 5.8. *Let* $a, b, c \in K^T$ *and* $d \in K^C$. *Further, assume* $R^{T2}abcd$. *Then, there is some* $x \in K^C$ *such that* $R^T bcx$ *and* $R^T axd$.

Proof. Proceed as in Lemma 5.7, but use A3. ■

Lemma 5.9. *The canonical postulates hold in the* $ML+$ *canonical model.*

Proof. P1 is trivial by Lemma 5.4. P2 holds by Lemma 5.4 and definition of R^C. Use Lemmas 5.7 and 5.8 to prove, respectively, P3 and P4. ■

Lemma 5.10. *The canonical* $\vDash^C$ *is a valuation relation satisfying conditions (i)-(iv) of §3.*

Proof. (i) is trivial by Lemma 5.4 (ii) and (iii) are easy given $ML+$ and properties of the members of K^C. (iv) from left to right is immediate. So, interest focusses on clause (iv) from right to left. Assume $a \nvDash^C A \to B$. We show that there are $x, y \in K^C$ such that $R^C axy$, $x \vDash^C A$ and $x \nvDash^C B$. Define first $b = \{C :\vdash A \to C\}$. Notice b is a theory and $A \in b$ [$\vdash A \to A$]. Define now (cf. Lemma 5.5.) the theory $c = \{C : \exists D(D \to C \in a$ and $D \in b)\}$ such that $R^T abc$. We do have: $B \notin c$ (if otherwise $B \in c$, then $C \to B \in a$, $C \in b$, therefore, $\vdash A \to C$, and, so, $A \to B \in a$, i.e., $a \vDash^C A \to B$, contradicting the hypothesis). By Lemma 5.1, there is some $y \in K^C$ such that $c \subseteq y$ and $B \notin y$. Applying definitions, $R^T aby$. By Lemma 5.3, there is some $x \in K^C$ including b and $R^C axy$ with $A \in x$ [$A \in b$]. ■

Lemma 5.11. *The* $ML+$ *canonical model is indeed a* $ML+$ *model.*

Proof. By Lemma 5.4, O^C is non-empty. Obviously, R^C is a ternary relation on K^C. Then, Lemma 5.11 follows by Lemmas 5.9 and 5.10. ■

Finally, we prove

Theorem 5.1. (Completeness of ML+) *If* $\vDash_{ML+} A$, *then* $\vdash_{ML+} A$.

Proof. By contraposition, if $\nvdash_{ML+} A$, then $\nvDash_{ML+} A$: Lemmas 5.4 and 5.11. ■

6. The logic $ML + f$

In order to define the logic $ML + f$, we add to the sentential language of $ML+$ the propositional falsity constant f together with the definition: $\neg A =_{def} A \to f$. Note that, for example, the following are provable in $ML + f$:

T6. $(A \to B) \to (\neg B \to \neg A)$

T7. $\neg B \to ((A \to B) \to \neg A)$

T8. $\neg(A \vee B) \rightarrow (\neg A \wedge \neg B)$
T9. $(\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$

7. Semantics for $ML + f$

An $ML+f$ model is a quintuple $< O, K, S, R, \vDash >$ where $< O, K, R, \vDash >$ is an $ML+$ model and S is a subset of K such that $S \cap O \neq \emptyset$. The clause:

(v) $a \vDash f$ iff $a \notin S$

is satisfied in all models.

A is valid in $ML + f$ [$\vDash_{ML+f} A$] iff $a \vDash A$ for all $a \in O$ in all models.

We note that f is not valid: let $a \in S \cap O$. Then, $a \nvDash f$. But $a \in O$. So, $\nvDash_{ML+f} f$.

Theorem 7.1. (Semantic consistency of $ML + f$) *If $\vdash_{ML+f} A$, then $\vDash_{ML+f} A$.*

Proof. Since analogues of Lemmas 4.1 and 4.2 are immediate, the theorem follows by Theorem 4.1. ■

8. Completeness of $ML + f$

We define the $ML + f$ canonical model as the quintuple

$$< O^C, K^C, S^C, R^C, \vDash^C >$$

where $< O^C, K^C, R^C, \vDash^C >$ is the $ML+$ canonical model and S^C is interpreted as the set of all consistent theories. A theory a is consistent iff $f \notin a$.

Now we need to prove:

Lemma 8.1. *$S^C \cap O^C$ is not empty.*

Proof. Given that $\nvDash_{ML+f} f$ [see §7], we have, by theorem 7.1., $\nvdash_{ML+f} f$, i.e., $f \notin ML + f$. As $ML + f$ is a theory, Lemma 5.1. applies and there is some $x \in K^C$ such that $ML + f \subseteq x$ and $f \notin x$. Obviously, $x \in O^C$ [since $ML + f \subseteq x$]. As $f \notin x$, $x \in S^C$. ■

Lemma 8.2. *Clause (v) holds in the canonical model.*

Proof. Lemma 8.1 and definition of S^C. ■

Lemma 8.3. *The $ML+f$ canonical model is in fact a $ML+f$ model.*

Proof. Lemmas 5.11, 8.1 and 8.2. ■

Theorem 8.1. (Completeness of $ML+f$) *If $\vDash_{ML+f} A$, then $\vdash_{ML+f} A$.*

Proof. It suffices to note that an analogue of Lemma 5.4 holds for $ML+f$, since the theorem follows by Lemmas 5.4, 8.1 and 8.3. ■

9. $ML+$ with minimal negation: The logic MLm

MLm is defined when we add to $ML+f$ the axiom:

A9. $A \to ((A \to f) \to f)$

We note that, in addition to T6-T9, the following are exemplary theorems of MLm:

T10. $A \to \neg\neg A$

T11. $(A \to \neg B) \to (B \to \neg A)$

T12. $\neg\neg\neg A \to \neg A$

T13. $(\neg A \wedge \neg B) \to \neg(A \vee B)$

Alternatively, MLm can be axiomatized with

A9'. If $\vdash A \to (B \to f)$, then $\vdash B \to (A \to f)$

or

A9". $(A \to (B \to f)) \to (B \to (A \to f))$

instead of A9, among other possibilities. Proof is left to the reader.

10. Semantics for MLm

A MLm model is any $ML+f$ model with the addition of the postulate:

P5. $Rabc$ and $c \in S \Rightarrow \exists x(x \in S$ and $Rbax)$

A is valid in MLm ($\vDash_{MLm} A$) iff $a \vDash A$ for all $a \in O$ in all models.

Theorem 10.1. (Semantic consistency of MLm) *If* $\vdash_{MLm} A$, *then* $\vDash_{MLm} A$.

Proof. Given Theorem 7.1, just the validity of A9 is at issue. Use P5. ■

11. Completeness of MLm

We define the MLm canonical model exactly as the $ML+f$ canonical model, but with this difference: now a theory a is consistent iff the negation of a theorem does not belong to a.

Lemma 11.1. $f \in a$ *iff* a *is inconsistent.*

Proof. Suppose $f \in a$. By A9, $(f \to f) \to f \in a$. Thus, a is inconsistent. Suppose now a is inconsistent. Then, for a theorem A, $A \to f \in a$. By A9, $(A \to f) \to f$ is a theorem. So, $f \in a$. ■

Next, we note that an analogue of Lemma 8.1 for MLm becomes immediate. We prove

Lemma 11.2. *Let* $a, b, c \in K^T$ *with* c *consistent and* $R^T abc$. *Then, there is some* $x \in S^C$ *such that* $c \subseteq x$ *and* $R^T bax$.

Proof. Assume $R^T abc$, with $a, b, c \in K^T$ and c consistent. Define (cf. Lemma 5.5) the theory $y = \{B : \exists A(A \to B \in b \text{ and } A \in a)\}$. We go into proving that y is consistent. Suppose it is not. Then, (cf. Lemma 11.1) $f \in y$. By definition of y, $A \to f \in b$, $A \in a$. By A7, $(A \to f) \to f \in a$. Given $R^T abc$, $f \in c$, which is impossible c being consistent.

So, we have a consistent theory y such that $R^T bay$. Since $f \notin y$, Lemma 5.1 applies and we have some $x \in K^C$ such that $y \subseteq x$ and $f \notin x$. So x is consistent, that is, $x \in S^C$. Given that $y \subseteq x$ and $R^T bay$, we conclude $R^T bax$. ■

Lemma 11.3. *The canonical P5, i.e.,* $R^C abc$ *and* $c \in S^C \Rightarrow \exists x(x \in S^C$ *and* $R^C bax)$, *holds in the* MLm *canonical model.*

Proof. By Lemma 11.2. ■

Lemma 11.4. *Clause (v) holds in the MLm canonical model.*

Proof. Lemma 11.1. ■

Lemma 11.5. *The MLm canonical model is indeed a MLm model.*

Proof. Lemmas 8.3, 11.3 and 11.4. ■

Finally we turn to

Theorem 11.1. (Completeness of MLm) *If $\vDash_{MLm} A$, then $\vdash_{MLm} A$.*

Proof. First we note that an analogue of Lemma 5.4 for MLm is immediate. Then, theorem 11.1 follows by Lemmas 5.4 and 11.5. ■

12. MLm with the reductio axiom: The logic $MLmr$

We add to MLm the axiom:

A10. $(A \to (A \to f)) \to (A \to f)$

Note that, in addition to T6-T13, the following are provable in $MLmr$:

T14. $(A \to B) \to ((A \to \neg B) \to \neg A)$

T15. $(A \to \neg B) \to ((A \to B) \to \neg A)$

T16.$(A \to \neg B) \to \neg(A \wedge B)$

T17. $(A \to B) \to \neg(A \wedge \neg B)$

T18. $\neg(A \wedge \neg A)$

T19. $\neg\neg(A \vee \neg A)$

$MLmr$ can be axiomatized alternatively with

A10'. $(A \to B) \to ((A \to (B \to f)) \to (A \to f))$

that is, T14, or

A10". $(A \to (B \to f)) \to ((A \to B) \to (A \to f))$

which is T15, or

A10"'. $(A \to (B \to f)) \to ((A \wedge B) \to f)$

i.e., T16, or

A10"". If $\vdash A \rightarrow B$, then $\vdash (A \rightarrow \neg B) \rightarrow \neg A$

among other possibilities instead of A10. Proofs are left to the reader.

13. Semantics for $MLmr$

Models for $MLmr$ are defined as those for MLm but with the addition of the postulate

P6. $Rabc$ and $c \in S \Rightarrow \exists x \exists y(Rabx$ and $Rxby$ and $y \in S)$

Note. The postulate

P6'. $Rabc$ and $c \in S \Rightarrow \exists x(x \in S$ and $Rcbx)$

used in [5] for A10 still would work here. We have preferred P6 because it is exactly what is needed in the semantic consistency and completeness proofs of the alternatives offered in §12 (we leave to the reader the proofs of these facts).

Theorem 13.1. (Semantic consistency of $MLmr$**)** *If* $\vdash_{MLmr} A$*, then* $\vDash_{MLmr} A$*.*

Proof. Given Theorem 10.1, we have to prove that A10 is valid. Use P6. ■

14. Completeness of $MLmr$

The $MLmr$ canonical model is defined similarly as the corresponding one for MLm. Hence, an analogue of Lemma 8.1 for $MLmr$ is immediate. Next we prove,

Lemma 14.1. *Let* $a, b, c \in K^T$ *c being consistent and* $R^T abc$*. Then, there is some* $x \in K^C$ *and* $y \in S^C$ *such that* $R^T abx$ *and* $R^T xby$*.*

Proof. Suppose $R^T abc$, $a, b, c \in K^T$ and c a consistent theory. Define (cf. Lemma 5.5) the theory $u = \{B : \exists A(A \rightarrow B \in a$ and $A \in b)\}$ such that $R^T abu$. Define now the theory $w = \{B : \exists A(A \rightarrow B \in u$ and $A \in b)\}$ such that $R^T ubw$. We prove first that w is consistent. Suppose it is not. Then,

$f \in w$ (cf. Lemma 11.1.). By definition of w, $B \to f \in u$ ($B \in b$). By definition of u, $A \to (B \to f) \in a$ ($A \in b$). Then, by T16, $(A \wedge B) \to f \in a$. But, given that $A \wedge B \in b$ ($A, B \in b$) and $R^T abc$, $f \in c$, which is impossible c being consistent. Therefore, we have $u, w \in K^T$ (w being consistent). Lemma 5.1 applies and in consequence there is some y in S^C such that $w \subseteq y$ and $R^T uby$. Now, by Lemma 5.2 there is some x in K^C such that $u \subseteq x$ and $R^T xby$. As $R^T abu$ (and $u \subseteq x$), $R^T abx$ as required. ■

Lemma 14.2. *The canonical P6 holds in the $MLmr$ canonical model.*

Proof. By Lemma 14.1. ■

Lemma 14.3. *The $MLmr$ canonical model is in fact a $MLmr$ model.*

Proof. Lemmas 11.5 and 14.2. ■

Theorem 14.1. (Completeness of $MLmr$) *If $\vDash_{MLmr} A$, then $\vdash_{MLmr} A$.*

Proof. Lemma 5.4 for $MLmr$ and Lemma 14.3. ■

15. Additional remarks

1. Both MLm and $MLmr$ can also be defined with a negation connective instead of the falsity constant f, by means of the general strategy followed in [5].

2. MLm and $MLmr$ are different logics. We refer to the results of [5]. There, the logic $I + -C$ [positive intuitionistic logic without contraction] was defined and next extended with a minimal negation [$Im - C$] and reductio [$Imr - C$]. These extensions are defined from $I + -C$ exactly as MLm and $MLmr$ were defined from $ML+$. Well, in that paper it was proved that $Im - C$ and $Imr - C$ are different logics, the first included in the second. Therefore, MLm and $MLmr$ are different logics, $MLmr$ including MLm.

3. Can a genuine intuitionistic negation, that is, a minimal negation plus

$$\neg A \to (A \to B)$$

be introduced in $ML+$? The answer is affirmative, but we leave the matter for another paper.

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