Space, Time and Number in the Brain: Searching for the Foundations of Mathematical Thought edited by Stanislas Dehaene and Elizabeth Brannon

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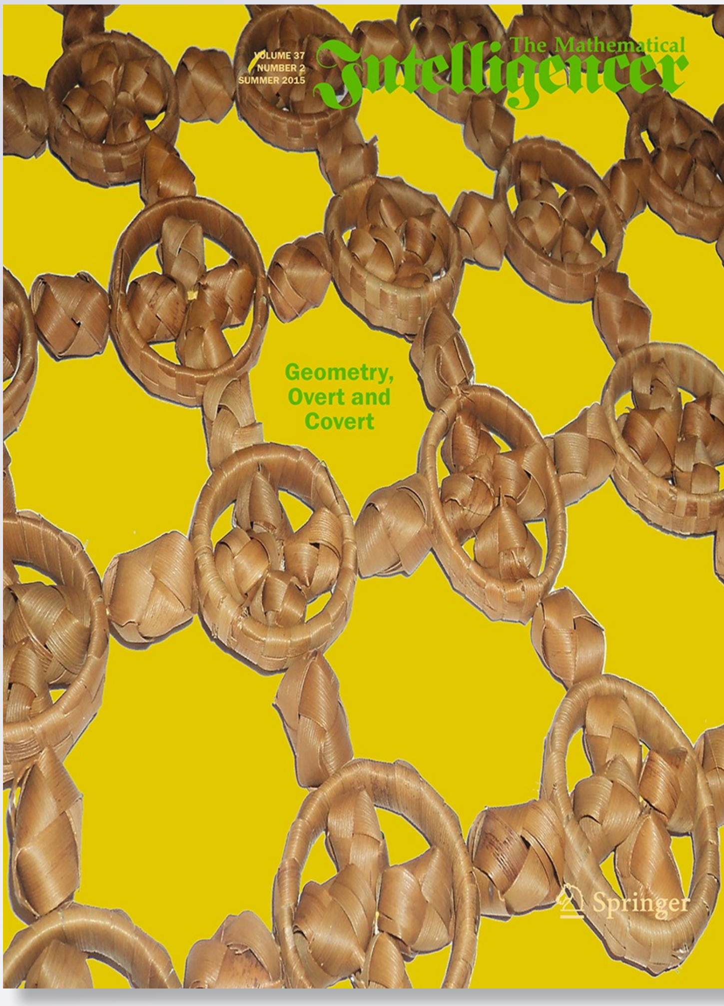

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Space, Time and Number in the Brain: Searching for the Foundations of Mathematical Thought

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Albert Einstein once said, about "the world of our sense experiences," "the fact that it is comprehensible is a miracle" (1936, p. 351). A few decades later, another physicist, Eugene Wigner, wondered about the unreasonable effectiveness of mathematics in the natural sciences, concluding that "the miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve" (1960, p. 14). At least three factors are involved in Einstein's and Wigner's miracles: the physical world, mathematics, and human cognition. One way to relate these factors is to ask how the universe could possibly be structured in such a way that mathematics would be applicable to it, and that we would be able to understand that application. This is roughly Wigner's question. Alternatively, the way of the mathematical naturalist is to argue that we abstract certain properties from the world, perhaps using our bodies and physical tools, thereby articulating basic mathematical concepts, which we continue building into the complex formal structures of mathematics. John Stuart Mill, Penelope Maddy, and Rafael Nuñez teach this strategy of cognitive abstraction in very different manners. But what if the very concepts and basic principles of mathematics were built into our cognitive structure itself? Given such a cognitive a priori mathematical endowment, would the miracles of the link between world and cognition (Einstein) and mathematics and world (Wigner) not vanish, or at least significantly diminish? This is the stance of Stanislas Dehaene's and Elizabeth Brannon's 2011 anthology, following a venerable rationalist tradition including Plato and Immanuel Kant.

Space, Time and Number in the Brain (henceforth: *STNB*) searches for the foundations of mathematical thought across an impressive collection of 21 articles. The book is divided into five sections: (1) Mental Magnitudes and Their Transformations, (2) Neural Codes for Space,

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Time and Number, (3) Shared Mechanisms for Space, Time and Number?, (4) Origins of Proto-Mathematical Intuitions, (5) Representational Change and Education. The breadth of research and results represented in this anthology are staggering. The primary thrust of the empirical topics of the essays within is an exploration of the subconscious, automatic, and prelinguistic neurological and mental elements necessary for mathematical cognition—that is, space, time, and number. Importantly, these essays do *not* address either the data or the theories behind conscious or reflective mathematical cognition, such as basic arithmetical, algebraic, or geometrical inferences, let alone proofs. Despite the empirical progress they describe, the challenge remains of developing and testing predictive and explanatory models of the data. Examples of such models and theories include Elizabeth Spelke's "core systems" of geometry and number (*STNB*, Chapter 18), Vincent Walsh's ATOM "A Theory Of Magnitude," which links space, time, and number through a single underlying magnitude, and dual-process theories segregating automatic, analogical systems from explicit, rule-based systems.

Given the depth, complexity, and importance of many ideas in this anthology, it is most effective to organize a review of *STNB* not by chapter, but with respect to general themes. The three general topics we address are: Data: A Clean Consensus; Cognitive-Epistemic Foundations of Mathematics (ceFOM): Coding, computability, and cognition; and Representation: Language and Approximation. In the first section, we review robust empirical findings, such as the SNARC effect, the ANS, and Weber's law. The second section addresses the relations between mathematics, computation, and cognition, which we call ceFOM. Given that investigations of these topics are almost as old a topic in philosophy as the foundations of mathematics itself (Hacking 2014), and given that many philosophers have appealed in various ways to possible epistemic, psychological, and neurological mechanisms underlying mathematical cognition (e.g., Luitzen E. J. Brouwer, William James, and Ludwig Wittgenstein), we discuss it here even though it is only mentioned marginally in the anthology. The third section turns to another important area of inquiry: the role of linguistic representation in the cognition of mathematics. We conclude with a summary of these broader themes of *STNB*, a brief allusion to the history of mathematics, and a statement of some outstanding problems.

Data: A Clean Consensus

As the book demonstrates, a vast number of experiments confirm the existence of a system of fundamental magnitude representations for space, time, and number that allow human and nonhuman animals to navigate and make perceptual calculations. We highlight two of the most robust, without implying either that they are the only relevant data about such system or that there is a uniform theoretical account of the cognitive mechanisms underlying a fundamental magnitude system.

1. The Spatial Numerical Association of Response Codes (SNARC) Effect

The Spatial Numerical Association of Response Codes Effect is well confirmed and its prominence in the anthology is entirely justified. In his introduction to Section 3 (*STNB*, Chapters 9-12), the book's coeditor and the discoverer of the effect, Stanislas Deheane, describes the SNARC effect thus:

> My research [in the early 1990s] concerned how quickly human participants could decide whether a digit was odd or even. The central issue was whether they could do this digitally, like a computer, by simply looking at the last bit or the last digit, or whether they would be influenced by conceptual variables such as the magnitude of the number. Unexpectedly, the latter effect turned out to be massive. Independently of whether the digit was odd or even, number magnitude biased the subject's responses in such a way that large numbers led to faster key presses on the right-hand side of space, and small numbers to faster key presses on the left-hand side of space. …number and space are intimately related concepts… (*STNB*, p. 119)

This effect indicates that there is a behavioral and cognitive correlation between the estimation of numeric size and spatial representations concerning length. Such correlation is not only systematic, but also analogical-continuous, in the sense that spatial lengths and numerical values are associated in terms of the "number line." Besides the preponderance of this effect in many conditions and cognitive tasks, its implications for acquiring basic knowledge of the mapping between numbers and geometric figures is quite remarkable because it could serve as basis for the analogical reasoning involved in the visualization of geometric axioms. In all of this, however, it is important to keep in mind that SNARC is an effect—a behavioral symptom—indicating and pointing to the workings of underlying cognitive structures and processes about which we continue theorizing.

The evidence for SNARC includes behavioral measures (responses with the left or the right arm); perceptual illusions concerning compression of spatial, temporal, and numeric values that could, in principle, be *modal specific* properties of the visual system; and neurological data concerning brain lesions. The book, notably, also includes a chapter on synesthesia (*STNB*, Chapter 9), which adds to the quite significant number of experiments confirming SNARC. The findings on synesthesia highlight the variety of cognitive processes involved in SNARC and the theoretical challenges they raise.

Unlike most of the chapters in the anthology, the discussion on synesthesia makes explicit reference to conscious awareness—what Ned Block (1995) calls phenomenal consciousness. Although this is not surprising because synesthesia necessitates experiences with phenomenal qualitative character, the theoretical implication of having such experiences remains unclear. Many of the mechanisms responsible for the representation of numbers in humans and other animals seem to be innate, and deeply embedded in the navigational system, as interfaced with space and time (*STNB*, Chapters 4, 17-19). A plausible way of characterizing all these cognitive processes is in terms of either unconscious cognitive processing for motor control or as precursors to experiences of space and number, which cannot play an equally fundamental role in the motor-control system (for discussion about this issue, see Gallistel 1990, Rosenbaum 2002, and Montemayor 2013). So the specific role of having such experiences about space and number remains unclear, as is the interaction between these experiences and the well-confirmed unconscious processes that combine numeric, temporal, and spatial information in navigation (e.g., the computation of the solar ephemeris function in insect navigation).

Although the data confirming the SNARC effect are abundant, the findings corroborating them concern very different cognitive processes. This may be considered a consequence of the robustness of such an effect, but it could also be interpreted as a potential source of confounding factors. In other words, the data require careful theoretical unpacking that remains undone. Outstanding questions vis-à-vis the SNARC effect include: Is SNARC a symptom of the primitive approximate system or is it the result of the interface of this system with conceptual and linguistic capacities? Which of these processes occur at a personal level and which are related to subpersonal processes? What is the interaction between conscious and unconscious processing? Could the unconscious SNARC findings be analog or magnitude-like and the conscious ones linguistically and introspectively mediated? What are the exact neurological bases for SNARC, and how could SNARC be computationally and mechanistically reconstructed?

2. The Approximate Number System (ANS) and Weber's Law

The approximate nature of magnitude-based cognition and its compliance with Weber's law is arguably one of the most robust findings in psychophysics (*STNB*, Chapters 1, 8, 12, 14-17). Weber's law is a ubiquitous feature of the comparison of magnitude representations in animals and humans. Indeed, conformity to Weber's law has been found in adult humans, infants, and animals both in the temporal and number domains (see Montemayor 2013 for review). Weber's law captures the scalar factor of interval timing, and it is expressed by the formula $\Delta I/I = k$. The difference threshold (ΔI)—the minimal change required for discrimination—, divided by the value of the initial stimulus or magnitude, is constant (k). The value of (k)—the constant—has to be found through experiment (see especially Chapters 16 and 17). Any changes in value that occur within the difference threshold are not noticeable by the cognitive organism. Weber's law explicitly captures the approximate nature of magnitude representations and it applies generally and at different levels of processing (similar to the SNARC effect) with respect to these magnitude-based representations. Thus, the interrelation between the number line and the magnitude representation of time seems as strong as the analog between the number line and the magnitude representation of space.

In time cognition, this systematic ratio effect inspired the accumulator model and the scalar expectancy theory—to date, one of the main models for time cognition (consult Church and Gibbon 1982). Because time cognition seems also to be fundamentally related to numerical cognition, this model could explain the mechanisms (presumably innate and part of the cognitive a priori) for approximate number cognition that also comply with Weber's law.

The scalar factor associated with Weber's law is a basic and systematic feature of time estimation, as corroborated by many experiments, in humans and nonhumans. Weber's law also governs other kinds of magnitude-based representations, such as number and ratio. (*STNB*, Chapter 1, Gallistel 1990) An important theoretical issue is whether time representation is in some sense more fundamental than other magnitude-based representations. Time seems to be a primitive magnitude in the sense that it cannot be decomposed into other magnitudes. (But see *STNB*, Chapter 20, for an analysis of time in terms of social dynamics and metaphor, comparing English and Mandarin speakers.) For example, preserving temporal metric relations (temporal order and simultaneity) is fundamental for computing other magnitudes. The case of the computation of the solar ephemeris function in which insects identify their location by mapping readings from their circadian clocks to spatial representations is one example, but computations of rate and speed are other, equally relevant examples (see Montemayor 2013 for a review of these findings).

Because of this intimate relation between time cognition and number, one finds an interesting tension in the interpretations of the remarkably consistent data supporting the Approximate Number System (ANS). First, there is the tension between the approximate analog system and the culturally mediated discrete or symbolic one, mentioned previously. Second, there is the philosophical problem of the cognitive origins of mathematical cognition, and the distinction between geometry and arithmetic. Is spatial cognition more fundamentally related to the number line, or is time more fundamentally related to numbers and arithmetic? In the philosophical tradition, theorists have thought that whereas one need not have spatial representations to represent successive numbers, temporal representations are essential (Kant, Arthur Schopenhauer, and L. E. J. Brouwer held such a view). These are important issues to consider in future analyses on the SNARC effect and ANS data.

Cognitive-Epistemic Foundations of Mathematics (ceFOM): Coding, Computability, and Cognition

This evidence provides new vistas into the area of cognitive-epistemic Foundations of Mathematics (ceFOM). Questions about how we can know numbers, shapes, and rules, and how it is even possible to prove necessary arithmetical, algebraic, and geometric relations; about whether mathematics is an internal cognitive structure (that evolved), a learned one and/or one existing in an independent realm; and about whether we can build a machine or a computer simulation that emulates our knowledge structures and conscious experience, are quite old. That is, questions about the *nature* and *sources* of mathematical knowledge and proof, and the uses of mathematics in the *emulation* of consciousness itself, remain very much with us (e.g., Hacking 2014).

The data presented in *STNB* pertain to discussions regarding cognition and computation, and mathematics and computation. These complex discussions originate in Kurt Gödel's work on incompleteness (or undecidability), Alan Turing's finite-state machines, and Roger Penrose's physicalist and physics-oriented antimechanism. If it had turned out that humans and other animals acquired various mathematical concepts solely from the environment, that there were no cognitive universals (or near universals) regarding mathematics, and/or that diverse mathematical structures are in no way internally correlated or structured, then the analogy between computer and mind would have been much less powerful and would have had much less justification. We would also have had to worry less about the exact relations between proof and algorithm, and between mathematical structures and algorithms, at least vis-à-vis the relevance of those concerns to (and from) cognitive structures and processes. However, the data *STNB* reviews suggest that significant amounts of internal cognitive mathematical content exist. As fully explanatory research programs, naturalism in mathematics and embodiment of consciousness are thus no longer viable options. Perhaps they will remain part of the story. (See Winther 2011 for discussion of how a plurality of research programs on consciousness could potentially be integrated, thereby avoiding the pernicious reification of any one of them.) A central lesson *STNB* teaches is that we must continue wrestling with whether the mind is a computer (Turing), and with how effective the algorithms of that computer are, given the limitations of arithmetic and set theory (Gödel). In all of this, Penrose's work remains powerful and interesting (consult Solomon Feferman's clear and tireless analyses, e.g., Feferman 1995, 2009).

Here is the section on "Human Turing Machine" from Dehaene and Brannon's foreword:

> In humans at least, quantities enter into sophisticated multi-step calculation and decision algorithms which can be likened to computer programs. Do these computations imply specifically human brain mechanisms that grant us the computational power of a Turing machine? Does the human brain contain dedicated mechanisms for the necessary operations of "routing" (selecting one out of many input-output mappings), "chaining" (re-using the output of a process as the input to another), "if-then" branching, or "for" and "while" loops? Can multi-step operations unfold automatically or are they necessarily under conscious control? (*STNB*, p. xi, citations suppressed)

With respect to the cognitive-epistemic Foundations of Mathematics, the editors of *STNB* clearly understand that the empirical results of their volume have ramifications for how psychologists and cognitive scientists, mathematicians, and computer scientists interpret and model the mind/brain, with respect to (i) neuronal activity, (ii) reasoning in general, and (iii) mathematical reasoning, and mathematical proofs in particular.

To investigate further what is at stake in ceFOM, it would be useful to turn to Penrose's framings of some of the questions. Penrose (1994, p. 12, 2000, p. 101) presents the following table "about the relationship between conscious thinking and computation":

A. All thinking is computation; in particular, feelings of conscious awareness are evoked merely by the carrying out of appropriate computations.
B. Awareness is a feature of the brain's physical action; and, whereas any physical action can be simulated computationally, computational simulation cannot by itself evoke awareness.
C. Appropriate physical action of the brain evokes awareness, but this physical action cannot even be properly simulated computationally.
D. Awareness cannot be explained by physical, computational, or any other scientific terms.

As is well known, Penrose is an advocate of C. (We here set aside his distinction between strong vs. weak C.) Together with Stuart Hameroff, Penrose has argued that there are quantum-mechanical processes occurring in cellular microtubules and that these give rise to consciousness. Such processes cannot be simulated computationally, but they are physical and physics-based, a fact that Nancy Cartwright takes Penrose to task for in Penrose (2000)—"What are Roger's [Penrose] reasons for thinking answers to questions about the mind and consciousness are to be found in physics rather than in biology?" she asks (p. 161). To provide perspective on the table, Daniel Dennett would be an advocate of A, David Chalmers of D. You can decide where you find yourself along the spectrum. Interestingly, at least options A and B are consistent with the empirical results reviewed in *STNB*. Option D seems out of the question vis-à-vis *STNB*, but what about option C? Again, open questions remain, and *STNB* advances our empirical knowledge significantly. Although it leaves theory underdetermined, the evidence found in *STNB* strongly suggests that theoreticians must revisit and rethink the computational theory of mind, and the role of Gödel, Turing, and Penrose in that terrain. (For a related framing, see Chalmers 1995, who distinguishes three questions: "What does it take to simulate our physical *action*?" "What does it take to *evoke* conscious awareness?," and "What does it take to *explain* conscious awareness?".)

Representation: Language and Approximation

In all of this, questions regarding the relationship between language and the number-space-time system(s) lurk. These concerns may be related to the issue of magnitude or digital formats, but they need not be. In our discussion of synesthesia, we highlighted the contrast of approaches between the analog approximate system (which is supposed to be evolutionarily more ancient) and the linguistic-dependent symbolic elements (which might even be culturally mediated) that are supposedly related to synesthesia. The contrast between language-based and nonlinguistic approaches is most explicitly addressed in Chapters 18 (Spelke) and 20 (Lera Boroditsky). Although their views may not be incompatible with respect to some issues (e.g., Spelke grants language the role of generalizing the core systems), there seem to be important theoretical disagreements.

On the one hand, Spelke defends the hypothesis that there is a set of cognitive systems for core mathematical knowledge of number, which is phylogenetically ancient and innate. Similar to Noam Chomsky's notion of Universal Grammar (UG), this core knowledge is a universal and necessary condition for the acquisition of numeric knowledge. According to Spelke, one of the core systems (for tracking small numbers) is discrete, whereas the other core system is approximate (the system corroborated by the findings on the SNARC effect and ANS). The role of language with respect to these systems remains unclear. One possibility is Brian Butterworth's claim (Chapter 16) that the discrete system seems to be more heavily involved in calculations of arithmetic, but the ANS also is very accurate at calculations of time and number, too. Another option is that language helps generalize or functionalize the representations of these systems in recursive ways. Spelke argues that there are two other systems for shapes and surfaces that capture information in terms of Euclidean geometry. This seems to be an endorsement of a view that Hans Reichenbach (1958) criticized, which he named "the visual a priori," the Kantian claim that the axioms of Euclidean geometry are basic principles of our visual system (Reichenbach 1958, pp. 32-33). Reichenbach objects that in the visualization of any geometric axioms "the normative function of visualization is not of visual but of logical origin" (Reichenbach 1958, p. 91).

On the other hand, Boroditsky proposes that there are very strong interactions between representations of space and time, but that these representations are entirely dependent on language and culture. Bluntly put, she endorses the Sapir-Whorf hypothesis of linguistic relativity. This claim, when contrasted with Spelke's UG-type account of core knowledge, presents the following problem. Even assuming that language is indispensible for human capacities to learn and know mathematics, there are Chomsky's (1986) related distinctions of competence versus performance, and I-language versus E-language. I-language refers to the set of mental representations that constitute our linguistic competence. I-language ("I" stands for "internal") is an abstract set of computational principles that operate without us being aware of them. Chomsky proposed that given the poverty of linguistic stimuli, the acquisition of language by humans shows that our linguistic competence is innate. This means that our language faculty is the result of our genetic makeup, and not of exposure to stimuli and external guidance. In contrast, E-language is the external or public manifestation of the internal representation of language on which linguistic competence depends. Unlike I-language, E-language is learned, and humans are aware of the express principles and symbols that constitute E-language. Thus, interpreted in terms of E-language, Boroditsky's claim would be in strong disagreement with the findings on ANS, SNARC, and the core knowledge systems, which suggest that these systems are implicit and innate. Although there may be ways of making these claims compatible (for instance by distinguishing conscious from unconscious processing), these details need to be provided by a comprehensive theory. Again, this is part of the work remaining to be performed.

To conclude, *STNB* is an important contribution to the literature on the psychology of space and time perception, the foundations of mathematics, and the relation between analogical-continuous representations of number and language-based representations of mathematical relations. Although *STNB* is an impressive collection of papers by leading scholars in the field of mathematical cognition, important theoretical difficulties remain unsolved. We discussed three of these problems concerning the interpretation of the findings about the SNARC effect and Weber's law, the nature of mathematical cognition in the light of debates concerning mathematics and computation, and the relation between mathematics and language.

Brouwer described the "first act of intuitionism" in terms of: "Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time" (1951, p. 4). This idea, which finds philosophical expression in the work of Kant and Schopenhauer, remains at the center of debates regarding the systems underlying mathematical cognition. Although Brouwer was probably wrong in generalizing this claim, the role of time and space perception in determining mathematical capacities remains a subject of research possibilities going forward.

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