The Contact Relation Algebra of the
Euclidean Plane has Infinitely Many Elements

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Abstract. Let REL(O*E) be the relation algebra of binary relations defined on the Boolean algebra O*E of regular open regions of the Euclidean plane E. The aim of this paper is to prove that the canonical contact relation C of O*E generates a subalgebra REL(O*E, C) of REL(O*E) that has infinitely many elements. More precisely, REL(O*,C) contains an infinite family $\{SPP^n, n \geq 1\}$ of relations generated by the relation SPP (**S**eparable **P**roper **P**art). This relation can be used to define point-free concept of connectedness that for the regular open regions of E coincides with the standard topological notion of connectedness, i.e., a region of the plane E is connected in the sense of topology if and only if it has no separable proper part. Moreover, it is shown that the contact relation algebra REL(O*E, C) and the relation algebra REL(O*E, NTPP) generated by the non-tangential proper parthood relation NTPP, coincide. This entails that the allegedly purely topological notion of connectedness can be defined in mereological terms.

Key words: Relation algebras, Relational calculus, Boolean contact algebras, mereo(topo)logy, Connectedness, Euclidean plane.

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1. Introduction. A mereological system may be defined as a relational system (M, P) with M being a class of mereological individuals and P a binary relation $P \subseteq M \times M$ such that for a, b $\in$ M the relation a P b is to be interpreted as the fact that the mereological individual a is

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part of the mereological individual b. The relation P is called the parthood relation. (M, P) is called a classical mereological system if it has the structure of a complete Boolean algebra from which the bottom element 0 has been deleted. Typical examples of mereological systems are the Boolean algebras of power sets PX and the Boolean algebras O*Y of regular open subsets of topological spaces Y. [1]

The parthood relation P is in no way the only relation that plays a role in mereology. For instance, the relations O of overlapping and D of disjointness have been amply discussed in the literature (cf. Lewis (1991)). The relations P, O, and D are not, of course, independent from each other. Rather, one may be defined in terms of the other (cf. Lewis (1991)). To render precise this claim, let us introduce some terminology and recall some elementary facts.

Denoting the power set of the Cartesian product $X \times X$ of a set X by REL(X) binary relations such as P, D, and O on some appropriate X may be conceived of as elements of REL(X). The set REL(X) of binary relations is a Boolean algebra with respect to the set-theoretical operations intersection ($\cap$), union ($\cup$), and complement (**C**). Given R, S, T $\in$ REL(X) binary relations such as R $\cap$ S, R $\cup$ S, **C**T may be said to be generated by R, S, and T using the Boolean structure of REL(X). Due to the fact that REL(X) is the power set of the Cartesian product $X \times X$, REL(X) has some further structure that is not present in ordinary Boolean algebras. This structure will play a crucial role in this paper. It is encapsulated in the operators of conversion ° and relational multiplication • as will be explained now.

(1.1) Definition. Let R, S $\in$ REL(X). (a) The converse R° of R is the binary relation defined by R° := {(b, a); (a, b) $\in$ R}; (b) The relational product R • S of R and S is defined as R • S := {(a, c); $\exists$b $\in$ X (a, b)$\in$ R AND (b, c) $\in$ S}.♦

Clearly, the operator ° of conversion is an involution, i.e., R°° = R. The relations between the operations of conversion, relational multiplication, and the Boolean operations defined are complicated and described by the so called Schröder equations (see (1.2) below). The essential structure of REL(X) is then succinctly captured by the concept of a relation algebra (cf. Bennett and Düntsch (2007, 112-113)):

(1. 2) Definition. A relation algebra (A, 0, 1, $\leq$, °, •, id) is a structure which satisfies for all a, b, c $\in$ A the following requirements:

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[1] As is common usage in mathematics relational systems such as (PX, $\subseteq$), (B, $\leq$), ... are denoted by PX, B, ... , if there is no danger of confusion.

(1)   (A, 0, 1, ≤) is a Boolean algebra.

(2)   (A °, •, id) is an involuted monoid. That is to say, (A, °, id) is a semigroup with identity id, the multiplication ° is an involution with $a°° = a$, $(a • b)° = (b° • a°)$.

(3)   For all a, b, c ∈ A the following equations ("Schröder equations") are equivalent

$$(a • b) \cap c = 0 , (a° • c) \cap b = 0 , (c • b°) \cap a = 0.♦$$

Clearly, REL(X) satisfies the conditions (1.2)(1) and (1.2)(2). The equivalence of the Schröder equations for binary relations is well known (see for instance Düntsch, Schmidt, Winter (2001, 383, henceforth DLW) or Schmidt (2001, 158). Hence REL(X) is a relation algebra in the sense of (1.2).[2] The bottom element 0 of REL(X) is the empty binary relation Ø on X, and the top element 1 is the relation X × X ∈ REL(X). The identity of the semigroup (REL(X), °) is the identity relation id := {(x, x), x ∈ X} ∈ REL(X). The relation algebra REL(X) is called the (full) relation algebra over X.

The main aim of this paper is to investigate certain relation subalgebras of REL(M), M being a classical Boolean mereological system, R a binary relation on M and REL(M, R) the relation algebra generated by R by the Boolean operations ∩, ∪, **C**, the conversion ° and relational multiplication •. The mereologically most important examples of such relation algebras REL(M, R) are REL(M, P) and REL(M, C), P being the parthood relation of M, and C a "contact relation" - to be defined precisely in the next section. For the moment let us concentrate on REL(M, P) and consider some of its elements:

(1.4) Examples of Mereological Relations. Let (M, P) be a classical Boolean mereological system. Then the relations of overlapping O and identity id can be defined in terms of P and therefore are elements of REL(M, P). On the other hand, the parthood relation P is also an element of the relation algebra REL(M, O):

$$O = (P° • P) , \quad P = NOT(O • NOT\ O) , \quad id = P \cap P°.♦$$

Actually, much more is true, namely, that the relation algebras REL(M, P) and REL(M, O) coincide, since O can be defined in terms of P and, vice versa, P can be defined in terms of O. The elementary examples of (1.4) suggest a more general problem, namely the task of

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[2] The Boolean operators of REL(X) will sometimes be written in a set-theoretical guise as ∩, ∪, and **C**, sometimes as AND, OR and NOT, sometimes a ∩ b will be written simply as ab to keep some complicated formulas as legible as possible.

calculating the relation algebra REL(M, P) for a classical Boolean mereological system (M, P). For an important special case the reader finds a complete solution of this problem In Düntsch, Wang, and McCloskey (1999):

<u>(1.5) Theorem</u>. Let (M, P) be a classical mereological system without atoms.[3] Then the relation algebra REL(M, P) generated by the parthood relation P is an atomic Boolean algebra of cardinality 128 generated by seven atoms.♦

Classical Boolean systems M are not the only mereological systems on the market. Many systems, in particular those dealing with the spatial mereology come along with some further structure that may be relevant for defining interesting relation subalgebras of REL(M) (see Gerla (1995, section 3) for some historical remarks). Among the best-known examples of these structures are so-called mereotopological structures arising from topological spaces.

The following definition recollects the necessary conceptual apparatus, in particular, the definition of a topological space in terms of a Kuratowski closure operator (cf. Kuratowski and Mostowski (1976):

<u>(1.6) Definition</u>. A topology on a set X is defined as a relational structure (X, cl) with cl an operator $PX \overset{cl}{\longrightarrow} PX$ satisfying the Kuratowski axioms (A, B $\in$ PX):

(1) $cl(\emptyset) = \emptyset$.   (2) $A \subseteq cl(A)$.

(3) $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$.   (4) $cl(cl(A)) \subseteq cl(A)$.♦

A subset $A \subseteq X$ with $cl(A) = A$ is called closed. The set theoretical complement **C**A of a closed set is called an open set. The set of closed sets is denoted by CX, and the set of open sets by OX. Dually to the closure operator cl an interior kernel operator int is defined as $int(A) := \mathbf{C}cl(\mathbf{C}A)$. A subset $A \subseteq X$ is called regular open iff $A = int(cl(A))$. The set of regular open sets of X is denoted by O*X. A subset $A \subseteq X$ is called regular closed iff $A = cl(int(A))$. The set of regular closed sets of a topological space X is denoted by C*X. For V The boundary of $V \in PX$ is defined as $bd(V) := cl(V) \cap cl(\mathbf{C}V)$.♦

As is well-known, for a topological space X the order structure OX is a complete Heyting algebra. Dually, the order structure CX of closed sets has the structure of a complete co-

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[3] That M has no atoms is essential for the validity of this theorem.

Heyting algebra (or Brouwer algebra). Finally, O*X is a complete Boolean algebra.[4] Mereological systems such as O*X that arise from topological structures come along with a canonical contact relation:

(1.7) Definition. Let O*X be the complete Boolean algebra of regular open sets of the topological space X. Then a canonical binary contact relation $C \in REL(O*X)$ is defined as

$$a\ C\ b := cl(a) \cap cl(b) \neq 0$$

Th relational structure (O*X, C) is called the canonical Boolean contact algebra of the topological space (X, cl). REL(O*X, C) is called the contact relation algebra of X (or of (O*X, C).♦

Now we can precisely characterize the relational systems that will occupy center stage in this paper, namely, relation algebras REL(O*X, C) that arise from topological spaces X. In order to avoid technicalties we restrict our attention to well-behaved familiar spaces such as the Euclidean plane E and its relatives. Nevertheless it is not hard to see that the results of this paper can be readily generalized to a much wider class of topological spaces.

The starting point for the investigation of Boolean relational algebra REL(O*X, C) $\subseteq$ REL(O*X) is the following well-known proposition:

(1.9) Proposition. Let X be a topological space. Then the parthood relation P of the Boolean mereological system O*X can be defined in relational terms as

$$P = NOT(C \cdot NOTC).♦$$

Proposition (1.9) evidences that the contact relation C is more basic than P, since P is definable in terms of C but C is not definable in terms of P. This entails that cardinality of the relation algebra REL(O*X, C) is at least as great as that of REL(O*X, P) (= 128). The main aim of this paper to show that in many cases it is actually <u>much</u> greater:

(1.10) Theorem. Let (E, cl) be the Euclidean plane endowed with the canonical closure operator cl. Then the canonical contact relation C generates a relation algebra REL(O*E, C) $\subseteq$ REL(O*E) that has infinitely many elements.♦

Before we go on the following remark may be in order: One can easily prove that (1.10) holds for a much wider class of topological spaces than Euclidean spaces. The restriction to

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[4] Since O*X and C*X are canonically isomorphic, it suffices to deal only with O*X in the following.

the Euclidean plane E is motivated solely by the intention to keep things as simple as possible. Since the 1990s many authors studied the relation algebra REL(O*X, C) generated by a contact relation C (cf. (DSW, 385), Bennett and Düntsch (2007, 123ff)). It became more and more obvious REL(O*X, C) was much larger than one had originally thought, and that it had an extremally complicated structure with respect to relational multiplication •.

Arguably the most comprehensive study of REL(O*X, C) for Boolean mereological systems (O*X, ⊆) is (DSW). The authors identified not less than 25 jointly exhaustive and pairwise disjoint generators of REL(O*X, C). Mormann (2001) proved that this "necessary relation algebra" was only a part of REL(O*X, C). He defined a so-called "hole relation" H, which did not appear in the list of relations presented in (DSW), and which split several of the 25 generators of the algebra mentioned there. Hence, these relations turned out to be not atoms of REL(O*X, C). In (Li, Li, and Ying (2005)) the authors showed that the relation H even generated infinitely many distinct non-trivial relations.

The main aim of this paper is to prove the existence of an infinite family $\{SPP^n, n \geq 1\}$ of relations in REL(O*E, C), O*E being the Boolean mereological system of regular open subsets of the Euclidean plane E. The relation SPP is a subrelation of the proper parthood relation PP, namely, "separable proper parthood" (to be defined precisely in the following). Structurally, SPP resembles the relation H in certain respects, but that the relation products SPP^n are all different and non-trivial is intuitively more appealing than the analogous one for the hole relation H.

The outline of this paper is as follows: In section 2 we will introduce some binary relations that are generated by the canonical contact relation C and that will be used in the later sections to prove the main theorem of this paper, namely, the existence of an infinite family of relations generated by the contact relation C. Based on some intuitive examples taken from the Euclidean plane E we will discuss the geometrical meaning of these relations. In section 3 a new family of relations SPP^n is introduced that are generated by the contact relation C for Boolean contact algebras.[5] As an application of the relation SPP (**s**eparable **p**roper **p**art), we show in section 4 that the topological concept of connectedness can be defined relationally in terms of the contact relation C. Our result can be succinctly formulated as follows: A regular open region w of Euclidean space is connected in the sense of topology if and only if it has no separable proper part, i.e., iff there is no region a such that a SPP w obtains. In section 5 it is pointed out that the relation algebra REL(O*E, C) generated by the contact relation has still another, more mereological generator, to wit, the

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[5] For $n \geq 1$, the relation SPP^n is defined as the n-times relation product SPP • SPP • ... • SPP.

relation NTPP of non-tangential parthood. This entails, in particular, that the ordinary parthood relation P is generated by NTPP. This fact may be taken as evidence that the expressive power of mereology is greater than was previously thought. It suffices to express many allegedly topological notions such as connectedness.

2. Binary Relations on Boolean Contact Algebras. The aim of this section is to prepare the ground for a proof that the relation algebra REL(O*E, C) has infinitely many elements. For this purpose we recall the definitions of some mereotopological relations that will play a role for the definition of the already mentioned infinite family {SPP^n, n ≥ 1}. Let us begin with a precise definition of the basic notion of a Boolean contact algebra (cf. Bennett and Düntsch 2007, 123ff).

(2.1) Definition. Let B be a non-trivial Boolean algebra (0 ≠ 1). A Boolean contact algebra is a pair (B, C) such that C is a binary relation defined on B – {0} satisfying the following axioms:

(BCA0)   a C b implies a, b ≠ 0.
(BCA1)   a ≠ 0 implies a C a.
(BCA2)   a C b ⇔ b C a.
(BCA3)   a C b and b ≤ c implies a C c.
(BCA4)   a C (b + c) implies a C b or a C c.
(BCA5)   {x; a C x} ⊆ {x; b C x} implies a ≤ b.
(BCA6)   If a (NOTC) b there is d such that a (NOTC)d and NOT d (NOTC)b.
(BCA7)   a ≠ 0, 1 implies a C a* (a* Boolean complement of a).

It is well known that for the Euclidean plane E (O*E, C) defines a Boolean contact algebra in the sense of (2.1).

Now let us have a closer look on REL(O*E, C). Up to now, the most comprehensive list of binary mereotopological relations, which are definable in relational terms from the contact relation C, may be found in (DLW). In order to pave the ground for the introduction of the family of relations {SPP^n, n ≥ 1} in the next section, some of the relations discussed in (DSW) are collected in the following list:

(2.2) Mereotopological relations generated by the contact relation C . The standard contact relation C defined by a C b := cl(a) ∩ cl(b) ≠ Ø generates (among others) the following elements of REL(O*E, C):

(1)	DC	:=	NOT C	(Disconnectedness)
(2)	#	:=	NOT (P ∪ P°)	(Incomparability)
(3)	P	:=	NOT (C* • C)	(Parthood)
(4)	PP	:=	(P ∩ NOT id)	(Proper Parthood)
(5)	O	:=	(P° • P)	(Overlapping)
(6)	PO	:=	O ∩ (NOT (P ∪ P°))	(Partial Overlapping)
(7)	EC	:=	(C ∩ NOT O)	(External Contact)
(8)	TPP	:=	(PP ∩ (EC • EC))	(Tangential Proper Parthood)
(9)	NTPP	:=	(PP ∩ NOT TPP)	(Non-tangential proper Parthood)
(10)	ECD	:=	NOT(PP • PP° ∪ PP° • PP)	(Boolean Complement)

All these relations have a straightforward geometrical meaning that may be read off by drawing some figures in the Euclidean plane.

Perhaps the most interesting ones are the relations TPP(8) and NTPP (9). Intuitively, a NTPP b may be interpreted as the fact that a is located in the interior of b, i.e. has no contact with the surface or the boundary of b.[6]

The distinction between tangential and non-tangential parthood is essential for geometrical and topological reasoning which distinguishes it from a merely set-theoretical reasoning. This idea is backed by the observation that for trivial (i.e. discrete) topological spaces (X, cl) with cl = id the distinction between TPP and NTPP collapses. Moreover, for the trivial Boolean contact algebra (PX, id) the contact relation C boils down to the overlapping relation O, and therefore TPP = NTPP = PP.

The specific features of the topological structure on X may have a bearing on REL(O*X, C) in many ways. For instance, if O*X is a non-atomistic Boolean algebra this entails that TPP • TPP = TPP, and therefore TPP^n = TPP, for n ≥ 1. Non-atomicity, however, does not suffice to ensure that $NTPP^n$ = NTPP. Rather, to ensure this one has to assume that X is a normal topological space[7].

As is well-known the Euclidean space (E, cl) is normal non-atomistic Hausdorff space. Hence the relational products TPP^n, $NTPP^m$, and PP^k do not contribute anything to the cardinality of

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[6] In order to improve the legibility and to emphasize this geometrical interpretation of NTPP as an interior parthood relation sometimes we will write a << b instead of a NTPP b, even if this runs counter to the terminology used for other mereotopological relations. A further notational simplification is achieved if we replace ECD by the more common expression a* for the Boolean complement of a. For instance, instead of a ECD • NTPP • ECD b we will write a* << b*.

[7] Restricting our attention to Hausdorff spaces recall that a Hausdorff topological space (X, cl) is normal if and only if for disjoint closed subsets A and B of X there exist disjoint open sets U(A) and U(B) containing A and B respectively.

REL(O*E, C). Nevertheless, one can prove that even in this case C generates infinitely many distinct relations.

The list (2.1) does not mean to be exhaustive. Actually there are many other relations definable in relational terms from C. In (DSW) the authors define not less than 25 "basic" relations $R_1, ..., R_{25} \in$ REL(E, C). "Basic" in this context is to convey the meaning that the R_i are mutually disjoint jointly exhaust the top element E × E of REL(O*E, C) in that $\cup_{i=1,...,25} R_i =$ E × E. Recalling that 7 atomic elements suffice to generate REL(O*E, P) one may ask whether the R_i suffice to generate REL(O*E, C). The answer is no. In Mormann (2001) it was shown, however, that there are relations in REL(O*E, C) that are not treated in (DSW). Among them, the so called "hole relation" H (cf. Mormann 2001) may be considered as the most important one:

(2.3) Definition. Let (B, C) be a Boolean contact algebra, and a, b $\in$ B. Then the relation a H b ("a is a hole of b") is defined as

$$a \; H \; b := (a \; EC \; b \cap a \; NOT(EC \cdot NOT(O)) \; b).\blacklozenge$$

Informally a is a hole of b if and only if a and b are in exterior contact and every y, which is in exterior contact with a, has a non-trivially overlap with b. Looking at some intuitive examples the reader may convince himself that this definition captures at least some important aspects of the intuitive concept "hole".

The following geographical examples will hopefully strengthen this impression: Consider the politicl map of Africa. Then the territory of Lesotho can be considered as a hole of the territory of South Africa, but Swaziland is not a hole since it is not completely surrounded by South Africa but has a common border with Mozambique. On the other hand, the territory of Swaziland would be a hole with respect to the union of the territories of South Africa and Mozambique. Somewhat differently, the union of Switzerland and Liechtenstein can be considered as hole with respect to the territory of the European Union, while Switzerland and Liechtenstein taken separately, cannot, since both have a common boundary with each other and therefore are not completely included by the territory of the European Union.[8]

Beside its usefulness for the describing complex real-world geographical situations the most interesting feature of H is to show that the jointly exhaustive and pairwise disjoint set of mereotopological relations given in (DLW) does not yield the ultimate partition of the top

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[8] A political map of the "Holy Roman Empire of the German Nation" in the Late Middle Ages would have provided, of course, a much richer collection of very complex cartographical hole structures.

element 1 of REL(E). Rather, with the help of H one can show that many of the 25 „basic" relations defined in (DLW) split further. Indeed, Li, Li, and Ying pointed out that relational products of H define infinitely many distinct non-trivial relations and that REL(O*E, C) is not an integral algebra, i.e., the relation id properly splits into subrelations (cf. (Li, Li, Ying (2005), Proposition 3.3)).[9]

As will turn out in the following, the most interesting component of H is (EC • NOT(O)). This relational product will also play a crucial role in the definition of SPP. In the rest of this section some of its most important properties are recalled.

2.4) Lemma. Let (B, C) be a Boolean contact algebra. Then the following equivalence holds:

$$a \ (EC \bullet NOT(O)) \ b \ \Leftrightarrow \ a \ TPP \ (a + b)$$

Proof. First let us prove the direction $\Leftarrow$. Assume that a TPP $(a \cup b)$. As is easily seen this entails that $(a \cup b)^* = a^* \wedge b^* \neq 0$. One has to find a region y with a EC y NOT(O) b, i.e., y has to satisfy the following properties:

(2.5) $a \wedge y = 0$,   a C y, $b \wedge y = 0$

By the assumption that a TPP $(a+b)$ and the very definition of TPP there is a y such that y EC a and y EC $(a \vee b)$. Hence $a \wedge y = 0$ and a C y. The third clause of (2.5) follows from the fact that y EC $(a \vee b)$ entails that $y \wedge (a \vee b) = 0$ and a fortiori $y \wedge b = 0$. This proves the first half of the lemma.

The proof of the direction $\Rightarrow$ is carried out by reductio. We show that a (EC • NOT(O)) b AND a NTPP $(a \vee b)$ leads to a contradiction. From the first component of of this expression we get that there is a y such that $a \wedge y = 0$, a C y, $b \wedge y = 0$. The first two clauses yield a TPP y^*. On the other hand, one has a P y^* and b P y^*, therefore $(a \vee b)$ P y^*. From a NTPP $(a \vee b)$ we get a NTPP y^*. This is a contradiction.♦

(2.6) Lemma. Let (O*X, C) be a Boolean contact algebra. Recall that for a $\in$ O*X the boundary bd(a) of a is defined as bd(a) := cl(a) $\cap$ cl(**C**a). Then the following equivalences hold:

(1)   x NTPP $(x \vee y) \Leftrightarrow cl(x) \subseteq x \vee y$
(2)   x TPP $(x \vee y) \Leftrightarrow bd(x) \cap bd(x \vee y) \neq 0$
(3)   x NTPP $(x \vee y) \Leftrightarrow bd(x) \cap bd(x \vee y) = 0$

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[9] Mormann (2001) has shown that H is a generator of REL(O*E, C), i.e., the relation algebras REL(O*E, C) and REL(O*E, H) coincide.

Proof. The assertion (1) is proved in (DSW, Theorem 6.3 (6.6); (2) is proved in (DSW, Theorem 6.3 (6.5)), and (3) is just the contraposition of (2).♦

3. Separable Proper Parts. In this section we show that there exists an infinite family of different, non-trivial relations in the Boolean contact relation algebra REL(O*E, C) of the Euclidean space E. More precisely we will show the existence of a non-ending strictly decreasing series of relation products $\{SPP^n, n \geq 1\}$ that have a non-trivial intersection:

$$\cap_{n \geq 1} SPP^n \subset \ldots \subset SPP^3 \subset SPP^2 \subset SPP^1 = SPP$$

The existence of the SPP does not depend on the specific topological features of the Euclidean plane E. Rather, the existence of SPP can be proved for a much wider class of spaces.

(3.1) Proposition. Let (B, C) be a Boolean contact algebra. Define the relation SPP by

$$bSPPa := (b\ PP\ a)\ AND\ (z)(z << a \Rightarrow (bz << b)$$

By a general result of Tarski on the definability properties of definitions of this kind, this definition entails that SPP can be expressed in the relational calculus as an element of the relation algebra REL(B, C) (cf. Tarski and Givant 1987). Nevertheless it may be illuminating to give an explicit definition of SPP in relational terms, which reveals a certain structural similarity of SPP and the hole relation H:

(3.2) Proposition.   SPP = (PP ∩ NOT(NTPP° • ECD • NOT(EC • NOT(O))• ECD).

Proof. The proof consists in showing that the term $(z)(z << a \Rightarrow (bz << b)$ can be defined in terms of the relational calculus. This is done in several steps as follows:

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\begin{aligned}
(z)(z << a \Rightarrow bz << b) &\Leftrightarrow && NOT\ \exists\ z\ NOT\ (z << a\ \Rightarrow (bz << b) \\
&\Leftrightarrow && NOT\ \exists\ z\ NOT\ (NOT(z << a)\ OR\ (bz << b)) \\
&\Leftrightarrow && NOT\ \exists\ z\ (z << a\ AND\ NOT\ (bz << b) \\
&\Leftrightarrow && NOT\ \exists\ z\ (z << a\ AND\ bz\ TPP\ b) \\
&\Leftrightarrow && NOT\ \exists\ z\ (z << a\ AND\ b^*\ TPP\ (b^* \vee z^*)
\end{aligned}
$$

By (2.2)(8) one has $b^*TPP(b^* \vee z^*) \Leftrightarrow z^*NOT(EC \cdot NOT(O))b^*$. By definition (2.2)(10) xECDy iff x = y*. Hence we finally obtain

$$(z)(z << a \Rightarrow bz << b) \Leftrightarrow a(NOT(NTPP^* \cdot ECD \cdot NOT(EC \cdot NOT(O)) \cdot ECD))b$$

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and we are done.♦

The relation SPP does not appear among the relations studied in (DSW) nor, as far as I know, in any other paper dealing with matters of relation algebras. Informally, for topological contact algebras (O*X, C) a SPP b holds if and only if one of the following situations obtains:

(1)   a DC b - a.
(2) $a \vee (b - a) = int(cl(a \cup (b - a)))$

An elementary example of (2) is provided by two open disks a and (b – a) that touch each other only in a common point of their boundaries. The following example shows that already for the real line **R** the relations SPP^n define an infinite family of different non-trivial elements of the contact relation algebra REL(**R**, C):

(3.3) Lemma. Let **R** be the real line endowed with the standard metrical topology. Consider the following disjoint unions A_i of open intervals:

$A_0 := (0,1)$
$A_1 := (0, 1) \cup (2, 3)$
$A_2 := (0, 1) \cup (2, 3) \cup (4, 5)$
$A_3 := (0, 1) \cup (2, 3) \cup (4, 5) \cup (6, 7)$
....

The sets A_i and the union $A := \cup (2n, 2n+1)$, n = 0, 1, 2, ... of the disjoint regular open intervals A_i are elements of O*R and the following relations obtain between them:

(3.4) A_0 SPP A_1 SPP A_2 SPP A_3 SPP ... SPP A_n SPP ... SPP A

More generally, the following relations hold:

(3.5) $(A_m SPP^n A_{n+m}$ AND NOT $(A_m SPP^{n+k} A_{n+m+k})$,   for k, n ≥ 1, m ≥ 0.♦

In particular, one has $(A_0 SPP^n A_n)$, but NOT$(A_0 SPP^{n-k} A_n)$, for k ≥ 1. Hence all SPP^n are non-trivial and distinct from each other. Moreover, A_m SPP^n A holds for all n ≥ 1, m ≥ 0. Hence, the intersection of the relations SPP^n is non-empty. In other words, the relations SPP^n are an infinite descending chain, and the relational contact algebra REL(O*E, C) turns out to be non-atomistic.

Clearly, the ever-decreasing chain of SPP^n may not only be set up for O*E but also for the mereotopological systems (O*X, C) of many other topological spaces X whose algebras OX of open sets have no atoms.

4. Connectedness. One of the basic concepts in topology is the concept of connectedness. A topological space X is connected if and only if there are no two disjoint open subsets Y and Z of X such that Y ∪ Z. More generally, an open subset A of X is connected if and only if there are no disjoint open sets B and C such that B ∪ C. This definition strongly depends on the point-set structure of A, B, and C, or so it seems. Hence it is not directly obvious whether it has an analogue for general Boolean mereological systems the elements of which may be not point sets. This transpires if one attempts to apply directly the topological concept of connectedness to Boolean mereological systems such as O*X conceived of as abstract lattice-theoretical structures without taking into account the point-set structure of their elements.

A direct translation of the topological concept of connectedness amounts to the following:

(4.1) Definition. Let O*E be the Boolean mereological system of the regular open regions of the Euclidean plane E. The region a is said to be *connected* if and only if there are no disjoint regions b and c such that a = b ∨ c.♦

This definition is designed after the standard characterization of connectedness of open subsets of topological spaces. Nevertheless the following proposition reveals that *connectedness* is a quite useless concept as:

(4.2) Proposition. No non-trivial region of O*E is *connected*.

Proof. Let a be a non-empty regular open region of E and α any point of a. Since a is open there is a regular open neighborhood U(α) of α that is completely contained in a. As is easily seen U(α) may be chosen to be a proper part of a. Then

$$U(\alpha) \wedge (a \wedge U(\alpha)^*) = \varnothing \qquad \text{and} \qquad U(\alpha) \vee (a \wedge U(\alpha)^*) = a.$$

Hence, the region a is not *connected*.♦

The aim of this section is to show that the following point-free definition of connectedness scores better:

(4.3) Definition. Let (B, C) be a topological contact algebra. The mereological individual a is said to be SPP-connected if and only if for all b NOT(b SPP a). ♦

In the rest of this section we will show that the relation SPP of separable proper parthood does provide a meaningful generalization of the topological concept of connection for general contact Boolean algebras (B, C) in the sense of the following theorem:

(4.4) Theorem. Let (O*E, C) be the Boolean contact algebra of regular open regions of the Euclidean plane E. Then a Euclidean region $a \in O*E$ is connected in the traditional topological sense if and only if it is SPP-connected in the sense of (4.3).♦

The proof of (4.4) is carried out in a series of lemmas.

(4.5) Lemma. Let X be a topological space. A regular open region $a \in O*X$ is connected iff there are no disjoint regular open regions b and c such that $b \cup c = a$.

Proof. Assume $a \in O*X$ to be non-connected. We show that a has a separation by regular open sets. By our assumption there are disjoint open sets b and c such that $b \cup c = a$. Denote the set of open sets of X by OX and recall that OX is a Heyting algebra. Hence every $a \in OX$ has a Heyting complement $a^{\#}$ defined as $a^{\#} := \mathbf{C}int\mathbf{C}(a)$. Due to some well-known laws for Heyting complements one eventually obtains

$$a = b \cup c \subseteq b^{\#\#} \cup c^{\#\#} \subseteq (b \cup c)^{\#\#} = a^{\#\#} = a$$

Hence $b^{\#\#} \cup c^{\#\#} = a$. Since $(b \cap c) = 0$ one concludes that $0 = 0^{\#\#} = (b \cap c)^{\#\#} = (b^{\#\#} \cap c^{\#\#})$. Hence, $b^{\#\#}$ and $c^{\#\#}$ define a regular open separation of a. The other direction is trivial.♦

(4.6) Definition. Let (B, C) a Boolean contact algebra with non-tangential proper parthood relation <<. Then the region a is called strongly SPP-connected iff

NOT ∃b (b < a AND (z) (z << a ⇒ (zb) << b) AND (z (a - b) << (a - b)).♦

Clearly, if a region is strongly SPP-connected, then it is SPP-connected, i.e. it has no separable proper part. Later it will be shown that SPP-connectedness and strong SPP-connectedness are equivalent. Abbreviating temporarily the standard topological connection by "T-connection" we first prove the following crucial lemma:

(4.7) Lemma. Let X be a regular topological space. Then a region a ∈ O*X is <u>T- connected</u> if and only if it is <u>strongly SPP-connected</u>.

<u>Proof.</u> The direction ⇒ is proved by reductio. Assume that a is T-connected but not strongly SPP-connected. Then by the definition of strong SPP-connectedness there is a region b < a such that

(4.8) $(z) (z << a \Rightarrow (bz << b)$ AND $((a - b)z << (a - b))$

Obviously b ∩ (a - b) = O. Due to the assumption that a is T-connected (b, a - b) cannot be separation of a. Hence (b ∪ (a - b)) must be a proper subset of a. On the other hand a = b ∨ (a - b)) = int(cl((b ∪ (a - b)). Hence there must be a point α ∈ a with a regular open neighborhood U(α) << a and U(α) ⊆ cl(b ∪ (a - b)) = cl(b) ∪ cl(a - b) but α ∉ (b ∪ (a - b)). Hence α ∈ bd(b) ∪ bd(a - b).

Assume α ∈ bd(b). Since the space X is assumed to be regular, U(α) contains a <u>closed</u> neighborhood cl(V(α)) of α. Then one has cl(b ∩ cl(V(α))) = cl(b) ∩ cl(V(α)). Since cl(b) ∩ cl(V(α)) ⊆ cl(b) ∩ U(α) one gets α ∈ bd(b) ∩ bd(b ∩ U(α)). This is a contradiction to the first clause of the implication (4.8) since it yields (U(α) ∩ b) TPP b.

In the same way one argues if one assumes α ∈ bd(a - b). Hence there cannot exist an element α ∈ (b ∨ (a - b)) - (b ∪ (a - b)). In other words (b ∨ (a - b)) = (b ∪ (a - b)). That means, (b, (a - b)) is a separation of a. This is a contradiction to our assumption that a is T-connected. Thereby we have proved that T-connectedness implies strong SPP-connectedness.

The direction ⇒ that strong SPP-connectedness entails T-connectedness is also proved by reductio. Assume that a is strongly SPP-connected but not T-connected. Hence there is a separation (b, c) of a, i.e., b and c are disjoint and (b ∪ c) = a. One may assume that b and c are regular open since a = (b ∪ c) = (b ∪ c)## = b## ∪ c## and 0 = (b ∩ c) ## = (b## ∩ c##). In particular one has c = (a - b). Since a is assumed to be strongly SPP-connected one may assume that there is a region z with (z << a) and NOT (zb << b), that is to say that (zb TPP b) or (zc TPP c) holds. This means that there is an α ∈ bd(zb) ∩ bd(b). This entails α ∈ c, since (b, c) is a separation. Since c is open there is an open neighborhood U(α) completely contained in c. This implies that b and c have a non-empty intersection, since α is a boundary point of b.[10] This is a contradiction. An analogous contradiction is obtained if one assumes zc TPP c. Hence, strong SPP-connectedness implies T-connectedness.♦

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[10] Recall that α is a boundary point of b iff every open neighborhood U(α) of α has a non-empty intersection with b and **C**b.

This result can be formulated succinctly as the assertion that for Boolean contact algebras (O*X, C) the notions of T-connectedness and strong SPP-connectedness are equivalent. Now we are going to prove that also strong SPP-connectedness and SPP-connectedness are equivalent:

(4.9) Lemma. Assume (b < a) AND (z << a) AND (a - b)z TPP (a - b) for some z. Then there is a z' such that (z' << a) AND (bz' TPP b).

Proof. From (a - b)z TPP (a - b) one obtains $bd((a - b)z) \cap bd(a - b) \neq \emptyset$. The assumption z << a entails that $bd((a - b)z) \subseteq cl(z) \subseteq a$. Hence $bd((a - b)z) \cap bd(a - b) \cap a \neq \emptyset$. Hence a fortiori $bd(a - b) \cap a \neq \emptyset$. Calculating

$$bd(a - b) \cap a = \quad bd(a \cap b^*) \cap a = bd(a) \cap cl(b^*) \cup (cl(a) \cap bd(b^*)) \cap a$$
$$= \quad (bd(a) \cap cl(b^*) \cap a) \cup (cl(a) \cap bd(b^*) \cap a)$$
$$= \quad \emptyset \cup (bd(b^*) \cap a) = bd(b) \cap a.$$

We finally obtain $bd(b) \cap a \neq \emptyset$. Assume $\alpha \in bd(b) \cap a$ and $U(\alpha) << a$. Defining $z' := b \cap U(\alpha)$ we have z' << a and z' = bz' TPP b.♦

(4.10) Proposition. The notions of strong SPP-connectedness, SPP-connectedness, and T-connectedness are equivalent.

Proof. Assume a to be SPP-connected but not strongly SPP-connected. Hence a has a part (a -b) such that (z) (z << a ⇒ ((a - b)z << (a - b)). Applying (4.8) yields z' such that (z' << a) AND (bz' TPP b). But then (z ∨ z') is an interior part of a, i.e., (z ∨ z') << a that does not satisfy (z ∨ z')b << b AND (z ∨ z') << (a - b). That is to say, a is not SPP-connected. This is a contradiction. ♦

Proposition (4.10) shows that the concept of SPP-connectedness is an adequate generalization of the notion of the tradional topological concept of connectedness for general Boolean contact algebras. Thus, the relation calculus enables us to formulate for Boolean contact algebras a concept of connectedness that generalizes the standard topological definition of this concept for regular topological spaces.
Let us conclude this section with the observation that the relation SPP offers a solution to a basic problem of mereology, namely, to provide a working and intuitively plausible definition of the notion of a whole:

(4.11) Definition. Let (M, C) a classical Boolean mereological system endowed with a contact relation C. Then a mereological individual b ∈ M is said to be a whole if and only if it has no separable proper part, i.e., there is no a ∈ M such a SPP b.♦

As has been explained in detail, for mereological systems arising from topology this definition yields plausible results. Nevertheless one may doubt that the solution of the "whole problem" offered by (4.11) can be considered as a mereological solution. One may object that (4.11) heavily depends on the "unmereological" concept of the contact relation C which "obviously" is not a mereological, but a topological (or mereotopological) concept. The aim of the next section is to refute this objection by showing that the role of C in (4.11) can be taken over by another, fully mereological concept, namely, the non-tangential parthood relation NTPP.

5. NTPP as a Generator of the Contact Relation Algebra. Usually mereology has been characterized as the theory of parts and wholes. Recently, some authors contended that this familiar definition is misleading since it "suggests that mereology has something to say about both parts and whole, which is not true." (Varzi 2007, 947, also Casati and Varzi 1999). According to Varzi, "the notion of a whole goes beyond the conceptional resources of mereology and calls for topological concepts and principles of various sorts." (ibd.) As evidence for this claim, Varzi and others contend that one has to rely on genuinely non-mereological topological concepts such as contact or connectedness in order to formulate a satisfying notion of a whole. This assertion is in need of qualification, to say the least.

First it should be mentioned that a reasonable concept of connectedness and other allegedly unmereological concept such as boundary, interior parthood and others can be defined in the framework of Heyting algebras (cf. Mormann 2012). That is to say, if one uses Heyting mereological systems instead of Boolean ones these allegedly unmereological concepts are definable in a purely mereological framework whose only primitive concept is the notion of parthood. Secondly, one may argue that concepts such as contact and connectedness are not so far away from the conceptual sphere of mereological concepts as Varzi and others want to make us believe. This can be seen when we have a closer look on our paradigmatic example of a topological Boolean contact algebra, namely, (O*E, C).

If X is a topological space it is easily proved by checking the relevant definitions that the following equivalence holds for all $a, b \in O^*X$:

(5.1) $\qquad\qquad cl(a) \subseteq b \quad \Leftrightarrow \quad a \text{ NTPP } b$

Now recall that a topological space X is regular if and only if every open neighborhood $U(\alpha)$ of a point $\alpha \in X$ contains an open neighborhood $V(\alpha)$ such that $cl(V(\alpha)) \subseteq U(\alpha)$ (cf. Willard 2004, Theorem 14.3 (b), 92). This is equivalent with the assertion that every open subset $a \subseteq X$ is the union of open subsets b_i of a such that $cl(b_i) \subseteq a$. This entails that for regular topological spaces X the parthood relation P can be defined in terms of NTPP:

(5.2)   a P b := (z) (z NTPP a $\Rightarrow$ z NTTP b)

This definition can be expressed in the relational calculus as

(5.3)   P = NOT(NTPP° • NOT NTPP)

Therefore P, and all relations derived from P such as P°, PP, and PP° are elements of REL(O*E, NTPP). Now recall that ECD := {(x, x*); x ∈ X}} can be expressed as:

$$\text{ECD} \Leftrightarrow \text{NOT(PP • PP° ∪ PP° • PP)}$$

(cf. (2.2)(10) and (DSW, 4.2). Hence, ECD ∈ REL(OX*, P) ⊆ REL(O*X, NTPP). From (DSW, Lemma (5.3), 2) one obtains that the contact relation C itself can be expressed in terms of NTPP and ECD as

(5.4)   C = NOT (NTPP • ECD)

Since E is well-known to be a regular topological space we eventually arrive at the following theorem:

(5.5) Theorem. Let E be the Euclidean plane. The non-tangential parthood relation NTPP is a generator of the contact relation algebra REL(O*E, C). Since according to (2.2)(9) C is a generator of REL(O'E, NTPP), the relation algebras REL(O*E, NTPP) and REL(O*E, C) coincide.♦

In other words, the allegedly "unmereological" contact relation C can be expressed in terms of the non-tangential parthood relation, since by (5.4) NTPP turns out to be a generator of the relation algebra REL(O*E, C). This entails that the relation SPP, which was crucial in a point-free definition of a good notion of connectedness, can also be defined in terms of NTPP.[11]

Now, intuitively there are good reasons to consider NTPP as a genuinely mereological relation that successfully captures the idea of "real" parthood. Compared with it, the ordinary parthood relation P is a rather weak concept. Even mereological purists, who want to stick as closely as possible to a narrow concept of parthood, should have no qualms to

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[11] In the terminology of Coppola, Gerla and Miranda (2010), taking the contact relation C as basic, amounts to subscribe to a connection-based approach, while taking a relation such as NTPP as basic may be characterized as an inclusion-based approach. Both accounts may be traced back to Whitehead's work on point-free geometry the first decades of the last century (cf. Whitehead (1929) and Whitehead (1920). Theorem (5.5) can be read as the contention that these two approaches can be reformulated in such a way that they become equivalent. It is interesting to note that Coppola, Gerla, and Miranda (2010) present a different reformulation according to which the inclusion relation can be defined from the connection relation but not vice versa.

accept NTPP as a genuine mereological relation. Hence it is misleading to assert that mereology is conceptually too weak to deal successfully with "topological" concepts such as ""connectedness", whole", "boundary", "interior", etc. which allegedly go beyond its conceptual resources (cf. Varzi (2007, 947)).

The relationship between mereology and topology turns out to be more complex as that it could be described adequately by saying that mereology is a kind of poor man's topology.

References:

Aiello, M., Pratt-Hartmann, I., and van Benthem, J., (eds.) Handbook of Spatial Logics, Springer.

Bennett, B., Düntsch, I., 2007, Axioms, Algebras and Topology, in M. Aiello, I. Pratt-Hartmann, J. van Benthem (eds.), Handbook of Spatial Logics, Springer, 99 – 159.

Biancino, L, Gerla, G., 1991, Connection Structures, Notre Dame Journal of Formal Logic, 32, 242 – 247.

Casati, R., Varzi, A., 1999, Parts and Places, The Structures of Spatial Representation, MIT Press.

Cohen, A.G., Varzi, A., 2003, Mereotopological Connection, Journal of Philosophical Logic 32, 357 – 390.

Coppola, C., Gerla, G, Miranda, A., 2010, Point-Free Foundations of Geometry and Multi-valued Logic, Notre Dame Journal of Formal Logic 51(3), 383 -405.

Davey, B. A, Priestley, H. A., 1990, Introduction to Lattices and Order, Cambridge, Cambridge University Press.

Düntsch, I., Wang, H., McCloskey, S., 1999, Relation Algebras in Qualitative Spatial Reasoning, Fundamenta Informaticae 39, 229 – 248.

Düntsch, I., Wang, H., McCloskey, S., 2001, A relation-algebraic approach to the region

Düntsch, I., Schmidt, G., Winter, M., 2001, A Necessary Relation Algebra for Mereotopology, Studia Mathematica 69, 381 – 409. (DLW)

Gerla, G. 1995, Pointless Geometries, in F. Buekenhout (ed.), Handbook of Incidence Geometry. Buildings and Foundations, Elsevier, Amsterdam, 1015 – 1031.

Kuratowski, K., Mostowski, A., 1976, Set Theory. With an Introduction to Descriptive Set Theory, North-Holland, Amsterdam, New York, Oxford.

Lewis, D., 1991, Parts and Classes, Oxford, Oxford University Press.

Li, Y., Li, S., Ying, M., 2005, Relational Reasoning in the Region Connection Calculus, arXiv:cs/0505041v1.

Mormann, T., 2001, Holes in the Region Connection Calculus, Unpublished Manuscript, presented at RelMICS 6, Osterwijk (Netherlands), October 2001.

Mormann, T., 2012, Heyting Mereology as a Framework for Spatial Reasoning, Axiomathes, Online First, DOI 10.1007/s10516-011-9180-x.

Schmidt, G., 2011, Relational Mathematics, Encyclopedia of Mathematics and Its Application 132, Cambridge, Cambridge University Press.

Tarski, A., 1941, On the calculus of relations, Journal of Symbolic Logic 6, 73 – 89.

Tarski, A., Givant, S., 1987, A formalization of set theory without variables, Colloquium Publications 41, American Mathematical Society, Providence.

Varzi, A.C., 2007, Spatial Reasoning and Ontology: Parts, Wholes, and Locations, in M. Aiello, I. Pratt-Hartmann, and J. van Benthem (eds.) Handbook of Spatial Logics, Springer, 945 – 1038.

Whitehead, A.N., 1920, The Concept of Nature, Cambridge, Cambridge University Press.

Whitehead, A.N., 1929, Process and Reality, MacMillan, New York.

Willard, S., 2004, General Topology, Mineola, New York, Dover.