Abstract. The aim of this paper is to sketch a topological epistemology that can be characterized as a knowledge first epistemology. For this purpose, the standard topological semantics for knowledge in terms of the interior kernel operator $K$ of a topological space is extended to a topological semantics of belief operators $B$ in a new way. It is shown that a topological structure has a kind of “derivation” (its “assembly” or “lattice of nuclei”) that defines a profusion of belief operators $B$. These operators are compatible with the knowledge operator $K$ in the sense that the all the pairs $(K, B)$ satisfy the rules and axioms of a (weak) Stalnaker logic of knowledge and belief. The family of belief operators $B$ compatible with $K$ is partially ordered such that different belief operators can be compared according to their strength or reliability. Thereby, for a given topological knowledge operator, a kind of intuitionist logic of belief operators $B$ compatible with $K$ is defined.

In sum, the topological knowledge first epistemology presented in this paper amounts to a pluralist knowledge first epistemology that conceives the relation between knowledge and belief not as a 1-1-relation but as a pluralist 1-n-relation, i.e., one knowledge operator $K$ gives rise to a numerous family of compatible belief operators $B$.

Keywords: Topological Epistemology; Knowledge first epistemology; Stalnaker’s logic of knowledge and belief; Plurality of belief operators; Nuclei, Intuitionist logic of belief operators.

1. Introduction. Topological epistemology can be conceived in a natural way as a knowledge first epistemology: According to the topological approach, knowledge is formally modeled by a topological interior kernel operator $K$ operating on a set of possible worlds $X$ endowed with a topological structure $(X, OX)$. Conceiving the subsets $A$ of $X$ as propositions, a proposition $A$ is defined to be known at a world $w \in X$ iff $w \in K(A) \subseteq X$. More precisely the knowledge operator $K$ is defined as the interior kernel operator of the topological structure $OX$ of the topological space $(X, OX)$. 
The classical Kuratowski axioms of topology entail that the interior kernel operator K has several properties that are intuitively quite appealing for a knowledge operator. If we define that a proposition $A \subseteq X$ describes a fact of the world $w \in X$ iff $w \in A$ one immediately obtains from the Kuratowski axioms of topology that knowledge is factive, i.e., that only facts can be known: since $K(A) \subseteq A$. Similarly, topological knowledge satisfies the famous (or notorious) “KK-principle” asserting that knowing a proposition A entails that one knows that one knows A. Formally this is expressed by $K(A) \subseteq K(K(A))$. Other intuitively plausible results for a topological model of knowledge can be obtained analogously. In sum, topological epistemology seems to be a promising starting point for a formal version of a knowledge first epistemology. Topological epistemology clearly reflects the basic idea knowledge first epistemology that knowledge is not derivative from its relation to other epistemic concepts such as justified true belief or rationality (cf. Carter, Gordon, Jarvis (2017), Williamson (2000)).

A genuine knowledge first epistemology requires more, however, than just taking the concept of knowledge as primitive. It is guided by the maxim that other epistemological concepts such as belief should be derivable from the concept of knowledge that is taken as primitive. Looking at the standard formalism of topology, it is not so clear, how this is to be achieved for the concept of belief. In contrast to the case of knowledge, it is not obvious, how to conceptualize belief topologically.

By a well-known theorem of Kuratowski there are exactly six different topological operators that can be defined directly as concatenations from the basic topological operators $\text{int}$ (interior kernel) and $\text{cl}$ (closure), namely, the combinations:

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1 In this paper the interior kernel operator of topology is always interpreted as knowledge K. Thus, if in a context the modal aspects prevail, the term K will be used, if topological aspects are to be emphasized, particularly, if other topological terms occur that have no direct modal interpretation, we will use the term int. This notational ambiguity should not cause any confusion.
It is not directly obvious whether any of the combinations listed in (1.1) can be meaningfully interpreted as belief. For instance, the closure operator cl is certainly not a plausible candidate since the inclusion $A \subseteq cl(A)$ had to be interpreted as the assertion that, if $w$ is an $A$-world, i.e., $w \in A$, then it would be believed that $w$ is an $A$-world. This is certainly not true for a realist concept of belief: There are many facts that are not believed to be facts.

A closer look, however, reveals that there is an operator in (1.1) that scores quite well as a plausible candidate for the office of a belief operator, namely, the operator intclint. More precisely, the pair of topological operators (int, inclint) satisfies all axioms of Stalnaker’s logic KB of knowledge and belief (except the axiom (NI) of negative introspection) (cf. Stalnaker (2006), Baltag, Bezhanishvili, Özgün, Smets (2014, 2019)).

Looking beyond (1.1), however, one may ask, however, whether the pair $(K, B) = (\text{int}, \text{intclint})$ is the only pair of topological operators that satisfies Stalnaker’s axioms. This question has hardly been treated in the literature. As will be shown in this paper, this question has to be No. Indeed, for a topological knowledge operator $K$, there are usually many different belief operators $B$ compatible with $K$ in the sense that the pairs $(K, B)$ satisfy the axioms of Stalnaker’s logic (except (NI)). More precisely, the family of these belief operators $B$ is partially ordered. Some operators $B$ are riskier than others, i.e., they differ more from knowledge $K$ than others. The minimal element of this family is clearly the knowledge operator

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2 It is well-known that int and cl are interdefinable: Denoting by $C$ the set-theoretical complement, one has $\text{int} = C \text{cl} C$ and $\text{cl} = C \text{int} C$.

3 Even more, the partial order of belief operators defines a complete Heyting algebra on the set of these operators. This yields a kind of logic of belief operators $B$ related to a given knowledge operator $K$. Different belief operators may be compared with respect to their strength and reliability. The strongest belief operator $B$ related to $K$ is trivially $K$ itself that does not differ from $K$ at all. The weakest belief operator $B$ related to $K$ and still a consistent belief operator, is Stalnaker’s operator $B_S$. This is a highly non-trivial mathematical result that can be proved as a corollary of Isbell’s famous density theorem. Between $K$ and $B_S$ a partially ordered variety of belief operator is located the structure of which depends on the underlying topological space $X$ of possible worlds.
itself that may be interpreted as the belief operator that differs not at all from knowledge. A deep mathematical result (Isbell’s Density Theorem) shows that the family of belief operators related to \( K \) also has a unique maximal element, namely, Stalnaker’s belief operator \( B_S \) topologically defined as intclint.\(^4\)

As said, the plurality of belief operators \( B \) compatible with a given knowledge operator \( K \) has been virtually unobserved so far. There are different reasons for this blind spot. One reason is that for the familiar Kripke relational semantics of epistemic and doxastic logic this plurality is hardly visible. A second reason for the neglect of this plurality may have been the fact that for Stalnaker’s original logic KB (for which the axiom (NI) of negative introspection is assumed to hold) the belief operator \( B \) turns out to be uniquely defined by the knowledge operator \( K \) (Stalnaker (2006), Baltag et al. (2014, 2019)).

This determination of the concept belief by the concept of knowledge is considered by Stalnaker, Baltag et al., and other authors as a virtue of KB, since thereby the bimodal logic KB turns out to be actually a unimodal logic of the modal operator \( K \), since \( B \) can be uniquely defined by \( K \). One may doubt, however, whether conceptual economy is the only criterion for a “good” epistemological logic. Other criteria for the “goodness” of an epistemological logic are whether it is realistic and flexible. In any case, a pluralist account of knowledge and belief only emerges if the axiom (NI) is given up. Only then, a truly bimodal logic is required for dealing with knowledge and belief.

This shows that Stalnaker’s logic KB that takes into account only \( B_S = \text{intclint} \) as belief operator, does not yield a comprehensive logic of knowledge and belief. The operator \( B_S \) is only one among many topological belief operators \( B \) that accompany \( K \) and jointly satisfy the rules and axioms of Stalnaker’s logic KB of knowledge and belief. Depending on the

\(^4\) Isbell’s theorem is generally considered as one of the most important theorems of (pointfree) topology (cf. Johnstone (1982), Borceux (1994), Picado and Pultr (2012)).
underlying topological structure (X, OX) often there are infinitely many belief operators B compatible with a given knowledge operator K. This mathematical fact strongly supports the epistemological thesis that for a given knowledge operator K many belief operators B should be recognized as compatible with K.

Thus, the widely discussed issue whether belief is basic and knowledge is to be defined in terms of belief, or, the other way round, whether a knowledge first approach is preferable that defines belief in terms knowledge, is too simple. Rather, a more flexible approach should be pursued: The operator K defines a topological structure (X, OX) which gives rise to variety of belief operators B. Perhaps the simplest choice for a compatible B is Stalnaker’s operator BS. This is indeed the only plausible choice from (1.1), but, as will be proved in detail, (1.1) is far being from a complete list of possibilities. Another possibility that will be discussed in detail in the following is the construction of belief operators with the help of dense subsets F of topological spaces. This does not exhaust the profusion of possible belief operators. There are still others that differ from the knowledge operator K in more substantial way. As an example, we discuss in this paper the “perfect” belief operator BPF that topological essence may be traced back to the founding fathers of set theory and topology Cantor and Hausdorff.

In sum, the topological approach presented in this paper is a “knowledge first” epistemology in a flexible sense: belief operators are defined in terms of concepts that rely of the concept of knowledge in some variable form. Thus, the maxim “knowledge first” in epistemology need not mean that other epistemological concepts such as belief are to be definable uniquely in terms of knowledge.

The outline of this paper is as follows. In the next section we recall the rules and laws of Stalnaker’s logic KB of knowledge and belief. In particular we define the concept of a weak Stalnaker system that will be the most appropriate logical system for the topological epistemology discussed in this paper. Section 3 contains the basics of the topological semantics of
the knowledge operator of KB as developed by Baltag et alii in a series of recent papers (cf. Baltag et al. (2019)). In section 4 it is shown that (dense) nuclei can be used to define a profusion of belief operators B that are compatible with a given knowledge operator K in the sense that the pairs of operators (K, B) satisfy the rules and axioms of a weak Stalnaker system. More precisely, the family of belief operators B compatible with K can be shown to be a complete Heyting algebra. In section 5 a new “perfect” belief operator $B_{PF}$ is defined that takes into account some fundamental features of topological spaces already discussed by Cantor and Hausdorff. In section 6 the “Cantor dust” is used to show that the perfect belief operator $B_{PF}$ actually differs from Stalnaker’s operator $B_S$. Some concluding remarks are offered in section 7.

2. Stalnaker’s Logic of Knowledge and Belief. To set the stage, let us recall the basics of the vocabulary and syntax of the bimodal logic KB of knowledge and belief put forward in Stalnaker (2006). We start with a standard unimodal language $L_K$ with a countable set PROP of propositional letters, Boolean operators $\neg$, $\land$, and a modal operator K to be interpreted as a knowledge operator. The formulas of $L_K$ are defined as usual by the grammar

$$\varphi ::= p \mid \neg p \mid \varphi \land \psi \mid K\varphi , \quad p \in \text{PROP}.$$  

The abbreviations for the Boolean connectives $\lor$, $\rightarrow$, and $\leftrightarrow$ are standard. Then, analogously to $L_K$, a bimodal epistemological language $L_{KB}$ for operators K and B is defined by adding $B\varphi$ as another type of well-formed formulas. For a more detailed presentation of topological semantics, the reader may consult the recent papers of Baltag et alii mentioned above. The language of KB is an extension of classical propositional language by two modal operators K and B that have to fulfil the following axioms and rules:

(2.1) Definition (Stalnaker’s axioms and inference rules for knowledge and belief).
All tautologies of classical propositional logic.

(K) $K(\phi \rightarrow \psi) \rightarrow (K\phi \rightarrow K\psi)$  
(Knowledge is additive).

(T) $K\phi \rightarrow \phi$  
(Knowledge implies truth).

(KK) $K\phi \rightarrow KK\phi$  
(Positive introspection for K).

(CB) $B\phi \rightarrow \neg B\neg \phi$  
(Consistency of belief).

(PI) $B\phi \rightarrow KB\phi$  
(Positive introspection of B).

(NI) $\neg B\phi \rightarrow K\neg B\phi$  
(Negative introspection of B).

(KB) $K\phi \rightarrow B\phi$  
(Knowledge implies belief).

(FB) $B\phi \rightarrow BK\phi$  
(Full belief).

Inference Rules:

(MP) From $\phi$ and $\phi \rightarrow \psi$, infer $\psi$.  
(Modus Ponens).

(NEC) From $\phi$, infer $K\phi$.  
(Necessitation). ♦

For the topological approach to knowledge and belief, the axiom (NI) plays a special role. It is easily shown that (NI) holds only for topological models of a special kind, namely, models that are based on extremally disconnected spaces. For the systems of knowledge and belief considered in this paper we will only require that they are weak Stalnaker systems in the following sense:

**Definition.** A bimodal system based on the bimodal language $L_{KB}$ is a weak Stalnaker system iff it satisfies all of Stalnaker's axioms and rules given in (2.1) except possibly the axiom (NI) of negative introspection. ♦

There are various reasons for abandoning (NI): First, (NI) is intuitively not a very plausible requirement for belief. Second, from a topological perspective, the axiom (NI) is very restrictive. Only a restricted class of topological models satisfies (NI), namely, extremally disconnected.
disconnected spaces. Most spaces that “occur in nature”, do not belong to this class. For instance, the familiar Euclidean spaces and their relatives are far from being extremally disconnected. Finally, the axiom (NI) leads to a 1-1-relation between knowledge $K$ and belief $B$. This is an implausible and too simplistic understanding of the complex relation between knowledge and belief. Rather, there are good reasons to assume that a given a topological knowledge operator $K$ is compatible with many belief operators $B$ in the sense that the pairs of $(K, B)$ satisfy all axioms of a weak Stalnaker system (2.2).

3. A Topological semantics of knowledge operators. In recent years Baltag, Bezhanishvili, Özgün, and Smets in various recent publications proposed a topological semantics for KB (cf. Baltag et al. (2014, 2019)). This semantics will be also be the basis of the one used in this paper. A novelty of this paper is the extension of this semantics to a semantics for belief operators $B$ of KB. This requires the introduction of new topological concepts, since, in contrast to Stalnaker (2006) and Baltag et al. (2014, 2019), for the weak version of KB to be presented in this paper, the belief operator $B$ need not be uniquely definable in terms of the knowledge operator $K$.

The main ingredient for a comprehensive semantics of belief operators $B$ is the concept of a nucleus, introduced in the 1980s in modern point-free topology (cf. Johnstone (1982, 2000), Borceux (1994), Picado and Pultr (2012)). This topological semantics will be used throughout the rest of this paper. First of all, recall the definition of a topological space:

(3.1) **Definition.** Let $X$ be a set with power set $PX$. A topological space is an ordered pair $(X, OX)$ with $OX \subseteq PX$ that satisfies the following conditions:

(i) $\emptyset, X \in OX$.

(ii) $OX$ is closed under finite set-theoretical intersections $\cap$ and arbitrary unions $\cup$. ♦
The elements of OX are called the open sets of the topological space (X, OX). The set-theoretical complements CA of open sets A are called a closed sets. The set of closed subsets of (X, OX) is denoted by CX. The interior kernel operator int and the closure operator cl of (X, OX) are defined as usual: The interior kernel int(A) of a set A ∈ PX is the largest open set that is contained in A; the closure cl(A) of A is the smallest closed set containing A. For details, see Willard (2004), Steen and Seebach Jr. (1982), or any other textbook on set-theoretical topology. The operators int and cl are well-known to satisfy the Kuratowski axioms:

(3.2) Proposition (Kuratowski Axioms). Let (X, OX) be a topological space, A, B ∈ PX. The interior kernel operator int and the closure operator of (X, OX) satisfy the following (in)equalities

(i) \( \text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B) \). \( \text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B) \).
(ii) \( \text{int}(\text{int}(A)) = \text{int}(A) \). \( \text{cl}(\text{cl}(A)) = \text{cl}(A) \).
(iii) \( \text{int}(A) \subseteq A \). \( A \subseteq \text{cl}(A) \).
(iv) \( \text{int}(X) = X \). \( \emptyset = \text{cl}(\emptyset) \).

In the following these axioms and some of their elementary consequences are used without explicit mention. Moreover, we will use freely the fact that these operators are inter-definable: \( \text{int}(A) = C\text{cl}(CA) \) and \( \text{cl}(A) = C\text{int}(CA) \).

Often it is expedient to conceive int and cl as operating on PX. This is possible in two (slightly different, but equivalent) ways: One may conceive int as an operator \( PX \longrightarrow OX \) or as an operator \( PX \longrightarrow \text{int} \longrightarrow OX \longrightarrow i \longrightarrow PX \) using implicitly the canonical inclusion \( OX \longrightarrow i \longrightarrow PX \). An analogous assertion holds for the closure operator cl. Relying on these interpretations of int and cl the concatenations of int and cl make perfect sense. In the following,
concatenations such as intcl, intclint will play an important role and both interpretations of int and cl will be used.

For the definition of consistent belief operators B the concept of dense subsets of a topological space will play an important role:

(3.5) Definition. Let int and cl be the topological operators of the space (X, OX), Y, Z ∈ PX.

(i) Y is a dense subset of X iff cl(Y) = X.

(ii) Z is a nowhere dense in X iff int(cl(Z)) = Ø.

(iii) A point x ∈ X is an isolated point of (X, OX) iff {x} ∈ OX. A subset A is dense-in-itself iff A has no isolated points. ♦

(3.6) Examples of dense and nowhere subsets of topological spaces (X, OX).

(i) For the trivial coarse topology (X, {Ø, X}) every non-empty subset A ∈ PX is dense and only Ø is nowhere dense. In contrast, for the discrete topology (X, PX) only X is dense, and only Ø is nowhere dense.

(ii) Let (R, OR) be the real line endowed with the familiar Euclidean topology. Let F ⊆ R be a finite set. Then F is nowhere dense and the complement CF of F is a dense open subset of (R, OR). Analogously, the infinite set of integers Z is a nowhere dense subset of (R, OR).

(iii) A more sophisticated example of a nowhere dense set is given by the Cantor dust D of the real line (R, OR) defined as follows: From the unit interval [0,1] of R remove the middle open interval (1/3, 2/3) obtaining the union of the closed interval [0, 1/3] and [2/3, 1]. This set is denoted by D₁. From D₁ remove the open middle intervals (1/9, 2/9) and (7/9, 8/9) obtaining a set D₂ that consists of the four closed intervals [0, 1/9], [2/9, 1/3], [2/3, 7/9], and [8/9, 1]. And so on. Then the Cantor dust is defined as the infinite intersection D := ∩Dᵢ. The Cantor dust is nowhere dense and perfect (= closed and without isolated points) (cf. Steen and Seebach Jr.
(1978, p. 57-58)). Hence the complement $\mathcal{C}D$ is a dense open subset of $(\mathbb{R}, \mathcal{O}_\mathbb{R})$. In section 6 $\mathcal{C}D$ will be used to show that the “perfect” belief operator $B_{PF}$ differs from Stalnaker’s belief operator $B_S$.

After these preparations, we can now begin to define a topological semantics for the modal languages $L_K$, $L_{KB}$ and $L_B$:

(3.7) **Definition.** Given a topological space $(X, O_X)$, a topo(logical) model for $L_K$ is given by $(X, O_X, \mu)$ where $\text{PROP} \rightarrow \mu \rightarrow PX$ is a valuation function from the set $\text{PROP}$ of propositional letters $p \in \text{PROP}$ into $PX$. The interior semantics for the Boolean connectives $\land$ and $\neg$ is defined as usual. If a formula $\varphi$ of $L$ is interpreted as $\mu(\varphi) = A \in PX$, then the formula $K\varphi$ of $L_K$ is interpreted as $\mu(K\varphi) = \text{int}(A)$. ♦

Usually, it is not necessary to explicitly mention the interpretation $\mu$ of a model $(X, O_X, \mu)$. Hence, in order to simplify denotation we often write $A$, $K(A)$ or $\text{int}(A)$, instead of $\mu(\varphi)$, $K\mu(\varphi)$, for $A = \mu(\varphi)$, $\mu(K\varphi)$ etc. How to interpret formulas that contain a belief operator $B$ will be explained in detail in the next section.

**4. A topological semantics of belief operators.** In this section we extend the topological semantics of knowledge to formulas that contain the belief operator $B$. For this purpose, we introduce the concept of (topological) nuclei (cf. Johnstone (1982), Borceux (1994), Picado and Pultr (2012)). Nuclei are the essential ingredient for defining belief operators $B$ compatible
with a topological knowledge operator $K$.\footnote{Committing a harmless abuse of language the concepts of belief operator and (dense) nucleus may even be identified.} The concept of a topological nucleus is basic for the rest of this paper.\footnote{This paper does not aim to give a full-fledged introduction into the theory of nuclei. Instead, we only intend to provide the basic definitions and facts so that the reader can understand that this theory has interesting applications regarding the modal theory of belief and knowledge. For a fuller account, the reader may consult the excellent references Johnstone (1982, II, 2.4, Lemma), Borceux 1994 (Theorem 1.5.7).}

(4.1) Definition. Let $(X, OX)$ be a space, and let $A, D \in OX$. An operator $OX \rightarrow OX$ is called a nucleus of $(X, OX)$ if it satisfies the following properties:

(i) $A \subseteq B(A)$. \hspace{1cm} (Inflation)

(ii) $B(B(A)) \subseteq B(A)$. \hspace{1cm} (Idempotence)

(iii) $B(A \cap D) = B(A) \cap B(D)$. \hspace{1cm} (Distributivity)

The set of nuclei of a topological space $(X, OX)$ is denoted by $\text{NUC}(OX)$.$\footnote{Simmons calls $\text{NUC}(OX)$ the assembly of $OX$ (cf. Simmons (1978, p. 242), (1982, p. 312)).}$

(4.2) Definition. The set of nuclei $\text{NUC}(OX)$ is partially ordered by the relation $\leq$ defined by

$$B \leq B' := B(A) \subseteq B'(A) \text{ for all } A \in OX. \hspace{1cm} \ddagger$$

As is easily proved, this partial order renders $(\text{NUC}(OX), \leq)$ a complete lattice. Even more, $(\text{NUC}(OX), \leq)$ can be shown to be a complete Heyting algebra (Johnstone (1982 (II, 2.4, Lemma), Borceux 1994 (Theorem 1.5.7))).

(4.3) Definition. A nucleus $B \in \text{NUC}(OX)$ is called a dense nucleus iff $B(\emptyset) = \emptyset$. The subset of dense nuclei of $\text{NUC}(OX)$ is denoted by $\text{NUC}(OX)_d$.\footnote{Using the fact that $\text{NUC}(OX)$ is a complete Heyting algebra it is not difficult to show that $\text{NUC}(OX)_d$ also has the structure of a complete Heyting algebra.}
Dense nuclei will play a central role for the definition of consistent belief operators compatible with a knowledge operator $K$. The following proposition shows that usually there are many different dense nuclei for a given knowledge operator:

\[(4.4)\] Proposition. Let $F \subseteq X$ be a dense subset of a topological space $(X, O_X)$, $D \in O_X$.

(i) Define $B_F(D) := \text{int}(C_F \cup D)$. Then $B_F$ is a dense nucleus.

(ii) The Stalnaker nucleus $B_S(D) := \text{intcl}(D)$ is a dense nucleus.

**Proof.** An elementary calculation using the Kuratowski axioms (3.2). ♦

Almost all topological spaces $(X, O_X)$ have many dense subsets.\(^9\) For the purposes of the present paper the essential point of (4.4) is that it in general ensures the existence of many dense nuclei. As will be shown in a moment, this suffices to prove that there are many belief operators $B$ compatible with one and the same knowledge operator $K$. Thereby the usual restriction of attention to Stalnaker’s operator $B_S$ can be overcome.

Even for familiar spaces like Euclidean spaces the precise structure of $\text{NUC}(O_X)$ is extremally complicated and not completely known. For the purposes of the present paper this need not concern us. Rather, we can be content to take notice of the following fact that is generally considered as the most important single fact of nuclei and related concepts (Johnstone (1982), Picado and Pultr (2012)). It will be shown that it also plays a central role for the topological epistemology of knowledge and belief. Informally stated, Isbell’s theorem can be used to show that consistent belief operators cannot deviate arbitrarily from knowledge. More precisely, that there is always a largest consistent belief operator, namely, $B_S$:

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\[^9\] The only exception are topological spaces $(X, O_X)$ for which $O_X$ is a Boolean algebra. These spaces may be used for defining topological models for the peculiar epistemic logic $S5$. In this case, the only belief operator compatible with $K$ is $K$ itself.
(4.5) Theorem (Isbell’s Density Theorem, cf. Johnstone (1982), Picado and Pultr (2012)). For all topological spaces \((X, O_X)\) the Stalnaker nucleus \(B_S\) is the largest dense nucleus of \((X, O_X)\), i.e., if \(B \in \text{NUC}(O_X)_d\) then \(B \leq B_S\). ♦

The proof of (4.5) goes well beyond the horizon of this paper and cannot be given here. The reader is asked to consult the excellent references (Johnstone (1982), Picado and Pultr (2012)).

In this paper nuclei are used to shed new light on the epistemological concepts of knowledge and belief. More precisely, we will show that dense nuclei define dense belief operators in the following canonical way. For a topological space \((X, O_X)\) let \(PX \rightarrow \text{int} \rightarrow OX\) the interior kernel operator and \(OX \rightarrow i \rightarrow PX\) the canonical inclusion. For a nucleus \(B\) \(OX \rightarrow B \rightarrow OX\) the concatenation \(PX \rightarrow \text{int} \rightarrow OX \rightarrow B \rightarrow OX \rightarrow i \rightarrow PX\) is well defined. Clearly, \(B\) uniquely determines the concatenation \(iB\text{int}\). Thus, the following definition makes sense:

(4.6) Definition. Let \(B \in \text{NUC}(O_X)_d\). The concatenation \(PX \rightarrow \text{int} \rightarrow OX \rightarrow B \rightarrow OX \rightarrow i \rightarrow PX\) is called the belief operator defined by the nucleus \(B\) (related to the knowledge operator \(\text{int}\)). ♦

Since the nucleus \(B\) uniquely determines the belief operator \(iB\text{int}\), the belief operator \(iB\text{int}\) may be also denoted by \(B\). Thus, committing a harmless abuse of language we may say that a (dense) nucleus \(B \in \text{NUC}(O_X)_d\) and a consistent belief operator are essentially one and the same object. Hence, for simplifying denotation nuclei and belief operators may be denoted by \(B\).

The definition (4.6) makes it possible to extend the familiar topological semantics of the unimodal language \(L_K\) to a bimodal language \(L_{KB}\) of modal operators \(K\) and \(B\) as follows:

(4.7) Definition. Given a topological space \((X, O_X)\) with interior operator \(\text{int}\) and \(B\) a belief operator in the sense of (4.6), a topo(logical) model for the bimodal logic \(L_{KB}\) is given by \(M = (X, O_X, v)\), where \(\text{PROP} \rightarrow \mu \rightarrow PX\) is a valuation function from the set \(\text{PROP}\) of propositional
letters to PX. The interior semantics for the Boolean connectives $\land$, $\neg$, and formulas $K\varphi$ are interpreted as before for $L_K$, formulas $B\varphi$ are interpreted by $\mu(B\varphi) := B(\mu(\varphi))$. ♦

The following theorem shows that (4.7) is a reasonable and fruitful definition that defines a family of well-behaved belief operators $B_j$ for a knowledge operator $K$ that enjoy all properties that one intuitively expects from “good” belief operators.

**Theorem (4.8).** Let $(X,OX)$ be a topological space with an interior kernel operator $\text{int}$, and $B$ be any dense belief operator $B \in \text{NUC}(OX)_d$. Then for every valuation $\text{PROP} \rightarrow \mu \rightarrow \text{PX}$ the model $(X,OX,B,\mu)$ defines a weak Stalnaker system, i.e., for $(X,OX,B)$ all rules and axioms (2.2) of a weak KB-logic are satisfied.

For every valuation $\text{PROP} \rightarrow \mu \rightarrow \text{PX}$ the family of models $\{(X,OX,B,\mu); B \in \text{NUC}(OX)_d\}$ is an epimorphic image of $\text{NUC}(OX)_d$, endowed with a partial order induced by $\text{NUC}(OX)_d$ and with maximal element $(X,OX,B_S,\mu)$.

**Proof.** This is easily proved by checking the pertinent definitions (3.7), (4.1), (4.2), and applying Isbell’s density theorem (4.5). ♦

The theorem (4.8) asserts that a topological knowledge operator $K$ is always accompanied by a multitude of compatible belief operators $B$ in the sense that all the pairs $(K, B)$ satisfy the rules and axioms of a weak Stalnaker system (2.2). Moreover, (4.8) ensures that no model $(X,OX,B,\mu)$ is riskier, i.e., more error-prone, than $(X,OX,B_S,\mu)$. In other words, in the family of models $(X,OX,B,\mu)$ a Stalnaker model $(X,OX,B_S,\mu)$ is the largest, i.e., most error-prone, but still consistent model. This fact establishes a kind of moderate pluralism of admissible belief operators for a given knowledge operator: Given a common knowledge base defined by a knowledge operator $K$, cognitive agents that all accept $K$ still have a certain degree of freedom.
to choose between different belief operators, without that necessarily one operator can be distinguished as better or more rational than another.

This pluralism of belief operators renders a combined topological logic of knowledge and belief more complex than one may have previously thought, but, on the other hand, this pluralist aspect renders this logic more interesting. Moreover, the pluralism of different coexisting belief operators renders epistemological logic more realist and flexible. After all, it is simply not plausible to assume that different cognitive agents who rely on the same knowledge operator, have to use the very same belief operator as well. Rather, conceiving beliefs as hypotheses or conjectures that go beyond established knowledge, it cannot be expected that all cognitive agents subscribe to the same hypotheses. This does not mean that all beliefs, how extravagant they may be, are equally reasonable. The very minimum of reasonableness that a belief system should satisfy in order to be acknowledged as acceptable is that it is not inconsistent, i.e., that no contradictions are believed. In topological terms, consistency of belief operators is expressed in terms of density: Only dense nuclei define consistent belief operators. Most important, Isbell’s density theorem ensures that for all topological spaces Stalnaker’s operator $B_S$ is the largest dense operator.

Different types of belief operators can be identified, for instance, Stalnaker’s operator $B_S$ and a profusion of operators $B_F$ that are defined with the help of dense subsets $F$ of $(X, OX)$. For most spaces $(X, OX)$, the determination of $\text{NUC}(OX)$ is difficult and for many spaces only partially known, even for familiar spaces such as Euclidean spaces. It is not the aim of this paper to deal with this difficulty in any greater depth, be it sufficient to mention the existence of various types of belief operators, among them the operators $B_F$ and Stalnaker’s operator $B_S$. In the next section we discuss a new operator, namely, the “perfect” belief operator $B_{PF}$ defined with the help of the concept of the perfect kernel already discussed by Cantor and Hausdorff.

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10 It can be shown that for most spaces the operator $B_S$ is not of the form $B_F$ for any dense subset $F$ of $(X, OX)$. 
more than one hundred years ago (cf. Hausdorff (1937), Zarycki (1930), Vaidyanathaswamy (1947)).

5. The perfect belief operator \( \mathbf{B}_{PF} \). In this section for all dense-in-themselves topological spaces \((X, \mathcal{O}_X)\) we will construct a belief operator \( \mathbf{B}_{PF} \) (strictly smaller than \( \mathbf{B}_S \)) such that the pair \((\text{int}, \mathbf{B}_{PF})\) defines weak Stalnaker systems in the sense of (2.2). The belief operator \( \mathbf{B}_{PF} \) may be considered as a general canonical operator since it relies on some fundamental features of the underlying topological spaces \((X, \mathcal{O}_X)\).

First recall that a subset \( A \subseteq X \) is dense-in-itself in \((X, \mathcal{O}_X)\) iff \( A \) has no isolated points (cf. (3.5) (iii))). Since the arbitrary union of dense-in-themselves subsets of \( X \) is dense-in-itself (Kuratowski (1966)) and the empty set \( \emptyset \) is clearly dense-in-itself, for all subsets \( A \subseteq X \) the largest dense-in-itself subset \( \mathbf{P}_F(A) \) of \( A \) is a well-defined concept. Moreover, the closure \( \text{cl}(A) \) of a dense-in-itself set \( A \) is dense-in-itself (Kuratowski (1966)). \( \mathbf{P}_F(A) \) is called the dense-in-itself kernel of \( A \) (cf. Zarycki (1930), Oxtoby (1976)).

(5.1) Proposition. Let \((X, \mathcal{O}_X)\) be a topological space, \( A, D \in \mathcal{C}_X \). The perfect kernel \( \mathbf{P}_F(A) \) of \( A \) has the following properties:

(i) \( \mathbf{P}_F(A) \subseteq A \) and \( \mathbf{P}_F(A) \) is closed.

(ii) If \( A \subseteq D \) then \( \mathbf{P}_F(A) \subseteq \mathbf{P}_F(D) \). (Monotony)

(iii) \( \mathbf{P}_F(\mathbf{P}_F(A)) = \mathbf{P}_F(A) \). (Idempotence)

(iv) \( \mathbf{P}_F(A \cup D) = \mathbf{P}_F(A) \cup \mathbf{P}_F(D) \). (Distributivity with respect to \( \cup \))\textsuperscript{11}

\textsuperscript{11} In Zarycki (1930) it is erroneously claimed that \( \mathbf{P} \) is distributive with respect to \( \cup \) for all subsets \( A, D \) of \( X \), not only for closed ones. This error was observed by Vaidyanathaswatasmy (1947) and Oxtoby (1976). Oxtoby proved a correct, more complex formula for all subsets \( A, D \) that yields (5.1)(iv) for closed \( A \) and \( D \). Fortunately, for our purposes it is sufficient that distributivity ((5.1)(iv)) holds for closed subsets of \( X \).

Simmons (1978, 1982) stated (without explicit proof) that (5.4) (iv) holds, i.e., that the operation \( \mathbf{P} \) is distributive with respect to \( \cup \) for closed sets \( A \) and \( D \). He then went on to show that \( \mathbf{C}_{PC} \) is a nucleus. Actually, Oxtoby’s proved general results on \( \mathbf{P}(A \cup B) \) only for \( T_1 \)-spaces (A topological space \( X \), ...
Proof. The proofs of (i) – (iii) are obvious. A detailed proof of (a much stronger result than) (iv) can be found in Oxtoby (1976).

Now assume that the space \((X, \mathcal{O}_X)\) is dense-in-itself, i.e., \(\text{PF}(X) = X\). Then we can define a “perfect belief operator” as follows:

\((5.2)\) Proposition. Let \((X, \mathcal{O}_X)\) be topological space that is dense-in-itself, \(A \in \mathcal{O}_X\). Define the operator \(\mathcal{O}_X \xrightarrow{\text{B}_\text{PF}} \mathcal{O}_X\) by \(\text{B}_\text{PF}(A) := \text{CPF}(\text{C}A)\). Then \(\text{B}_\text{PF}\) is a dense nucleus. \(\text{B}_\text{PF}\) is called the perfect nucleus of \((X, \mathcal{O}_X)\).

Proof. We have to prove that \(\text{B}_\text{PF}\) satisfies the conditions \((4.1)\) (i) – (iii) that define a nucleus. ad ((4.1) (i): Since the closure of a dense-in-itself subset of \(\text{C}A\) is dense-in-itself and \(\text{C}A\) is closed one clearly has that \(\text{PF}(\text{C}A) \subseteq \text{C}A\). Hence \(A = \text{C}C \subseteq \text{CPF}(\text{C}A)\). By definition of \(\text{PF}\) the proofs for (ii) and (iii) are obvious. A detailed proof of (a much stronger result than (iv) can be found in Oxtoby (1976, section 2). Thus, \(\text{B}_\text{PF}\) is a nucleus. If \(X\) is dense-in-itself one has \(X = \text{PF}(X)\) and \(\text{B}_\text{PF}\) is a dense nucleus since \(\text{B}_\text{PF}(\emptyset) = \text{CPF}(\text{C}\emptyset) = \text{C}(\text{PF}(X)) = \text{C}X = \emptyset\).

As explained before, the nucleus \(\text{B}_\text{PF}\) defines in a canonical way a belief operator also denoted by \(\text{B}_\text{PF}\). By \((4.8)\) we obtain:

\((5.3)\) Theorem. Let \((X, \mathcal{O}_X)\) be a dense-in-itself topological space. Then the pair \((K, \text{B}_\text{PF})\) of the interior operator \(K\) and the perfect belief \(\text{B}_\text{PF}\) satisfies the rules and axioms of a weak Stalnaker system.

\(\text{OX) is T}_1\) iff \(\{x\}\) is closed for all \(x \in X\). A closer inspection of his proof, however, reveals that it works for general topological spaces.
By Isbell’s theorem (4.5) the perfect nucleus $B_{PF}$ is smaller than or equal to $B_S$, i.e., for all $A \in OX$ one has $B_{PF}(A) \subseteq B_S(A)$. In the next section we will show that $B_{PF}$ is indeed strictly smaller than $B_S$. This means that forming beliefs on the basis of the perfect belief operator $B_{PF}$ is less risky to be in error than doing this on the basis of $B_S$. In other words, believing that the world $w$ is an $A$-world by using $B_{PF} (w \in B_{PF}(A))$ is less error-prone than believing that the world $w$ is an $A$-world by using $B_S ((w \in B_S(A))$.

Examples of dense-in-themselves spaces abound. For instance, Euclidean spaces and other Polish spaces are dense-in-themselves. Hence, (5.3) has wide applications.

6. The Cantor dust is visible by $B_{PF}$. In this section we show that $B_{PF}$ is actually different from $B_S$. For this purpose consider the Cantor dust $D$ located in the unit interval $[0, 1]$. As is well known the set $D$ is a perfect set and nowhere dense in $\mathbb{R}$, i.e., $\text{intcl}(D) = \text{int}(D) = \emptyset$. Hence $CD$ is open and one calculates for the operators $B_S$ and $B_{PF}$, respectively:

(6.1) Theorem. Let $(\mathbb{R}, \mathcal{O}_R)$ be the Euclidean line and $D \subseteq \mathbb{R}$ the Cantor dust. Then the pairs of modal operators $(\text{int}, B_S)$ and $(\text{int}, B_{PF})$ define different weak Stalnaker systems as defined in (2.2). More precisely, with respect to the partial order defined in (4.2) one obtains that $B_{PF}$ is strictly smaller than $B_S$, i.e., $B_{PF} < B_S$.\

Proof. By Isbell’s theorem one already has $B_S \leq B_{PF}$. To clinch the proof of (6.1) one has to find a subset $Y$ of $\mathbb{R}$ such that $B_S(Y) \subset B_{PF}(Y)$. We will show that the complement $CD$ of the Cantor dust $D$ satisfies this condition. $D$ is perfect and nowhere dense in $(\mathbb{R}, \mathcal{O}_R)$ (see (3.6)(ii)). Hence $CD$ is open and one calculates for $B_S(CD)$ and $B_{PF}(CD)$:

$$B_S(CD) = \text{intcl}(CD) = \text{intcl}(CD) = \mathbb{R}$$

(6.2)
\[
B_{PF}(CD) = CPF(CC) = CPF(D) = CD.
\]

The Cantor dust \(D\), as a nowhere dense set, may appear to be a “thin” or topologically “small” subset of \(\mathbb{R}\). Or, expressed in other words, the elements \(x \in D\) may be considered as a kind of rare anomalies compared with the “ordinary” elements \(x \in \mathbb{R} - D\). Thus, ignoring the difference between \(\mathbb{R}\) and \(\mathbb{R} - D\) may be considered as a rather small and negligible error. Or, in other words, it may appear only be slightly riskier to use \(B_S\) instead of \(B_{PF}\) as a device for formulating one’s justified beliefs (hypotheses).

Against this assessment one can argue, however, that the quality of “thinness” depends on the method of how “thinness” is measured. With good arguments one may put forward the thesis that the “thinness” of subsets of \(\mathbb{R}\) should be measured in terms of a topologically well-behaved measure of \((\mathbb{R}, O\mathbb{R})\) such as the Lebesgue measure \(\lambda\). Then the problem arises that there exist “fat versions” of the Cantor dust (topologically equivalent with the ordinary Cantor dust in the sense that they are homeomorphic to \(D\)) such that the difference between the measures \(\lambda(\mathbb{R})\) and \(\lambda(\mathbb{R} - D_F)\) can be arbitrarily large.\(^\text{12}\) Thus, the difference between \(B_S\) and \(B_{PF}\) should not be underestimated.

The perfect belief operator \(B_{PF}\) is a general operator in the same sense as is \(B_S\). Both operators do not depend on the specific structure of the topological spaces \((X, O_X)\) on which they are defined, but only on general features of the topological structure. In contrast, the operators \(B_F\) defined by dense subsets \(F\) of topological spaces are not general in that their definition relies on the specific structure of the spaces \((X, O_X)\) on which they are defined.

\(^{12}\) A nice discussion of (fat) Cantor sets can be found in Lando (2012).
7. Concluding Remarks. The knowledge first topological epistemology sketched in this paper may be characterized as a hierarchical knowledge first epistemology: Its basic level is given by the topological structure \((X, OX)\) defined by the knowledge operator \(K\) itself. The next level of topological epistemology is given by the “derivation” of the topological structure \((X, OX)\), i.e., the complete Heyting algebra \(\text{NUC}(OX)\). On this floor the belief operators \(B\) reside that are compatible with \(K\). This lattice \(\text{NUC}(OX)\) of belief operators has a very complex structure even for familiar spaces such as the Euclidean spaces. Its more detailed investigation may be an issue of future research. Several types of belief operators can be distinguished - particularly important operators are Stalnaker’s \(B_S\) operator characterized as the riskiest belief operator in the sense that it deviates most from \(K\), and the “perfect” belief operator \(B_{PF}\) that takes into account some of the most basic features of topological structures that already for Cantor and Hausdorff played an important role (cf. Zarycki (1930), Hausdorff (1957)).

References:


\(^{13}\) Nuclei can be defined for all Heyting algebras \(H\) whatsoever, not only for \(OX\). Thus, the \(\text{NUC}\) construction of nuclei can be iterated, i.e., one can define iterations \(\text{NUC}(OX), (\text{NUC}(\text{NUC}(OX))), \ldots \) etc. The height of this tower is a measure of complexity of \(OX\). The tower of nuclei may, or may not, become constant. In the former case, it terminates in a Boolean algebra \(\text{NUC}^n(OX)\) for some \(n \geq 0\). The simplest case is the not very plausible epistemic logic \(S5\). According to \(S5\), \(K\) has to satisfy the axiom of negative introspection, i.e., \(\text{NOT}(K(A)) \boxrightarrow{\text{K}} \text{NOT}(K(A))\) for all propositions \(A\). For the topological models \((X, OX)\) of \(S5\) one calculates \(OX = \text{NUC}^0(OX) = \ldots = \text{NUC}^0(OX)\) for all \(n\).


