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From Maximal Intersubjectivity to Objectivity: An Argument from the Development of Arithmetical Cognition

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Abstract

One main challenge of non-platonist philosophy of mathematics is to account for the apparent objectivity of mathematical knowledge. Cole and Feferman have proposed accounts that aim to explain objectivity through the intersubjectivity of mathematical knowledge. In this paper, focusing on arithmetic, I will argue that these accounts as such cannot explain the apparent objectivity of mathematical knowledge. However, with support from recent progress in the empirical study of the development of arithmetical cognition, a stronger argument can be provided. I will show that since the development of arithmetic is (partly) determined by biologically evolved proto-arithmetical abilities, arithmetical knowledge can be understood as maximally intersubjective. This maximal intersubjectivity, I argue, can lead to the experience of objectivity, thus providing a solution to the problem of reconciling non-platonist philosophy of mathematics with the (apparent) objectivity of mathematical knowledge.

Keywords Epistemology of arithmetic · Philosophy of mathematics · Maximal intersubjectivity · Objectivity

1 Introduction

Platonism in the traditional sense is widely seen as problematic in modern philosophy of mathematics (see, e.g., Shapiro (2005) for an overview). Although the reasons for this are not uniform, one common source can be traced to Benacerraf's (1973) epistemological problem, which asked how we as physical subjects can gain knowledge of abstract—i.e., non-temporal, non-spatial and causally inactive—mathematical objects. The rejection of platonism in mathematics comes, however, with important challenges. Mathematical knowledge in the platonist tradition is associated with a particular set of characteristics, most importantly objectivity, apriority, necessity and universalness (Shapiro 1997; Linnebo 2018a). Platonism in its many formulations has an explanation for those characteristics. Whether the platonist ontology is specified in terms of ideal abstract objects (Plato 1992), *ante rem* structures (Shapiro 1997), or thin or trivial objects (Rayo 2015; Linnebo 2018b), assuming the existence of mind-independent timeless subject matter for mathematics can explain the traditional characteristics associated with mathematics. Without that assumption, all four characteristics listed above pose *bona fide* philosophical challenges, ones that have not proven to be easy to tackle.

Yet we must tackle those problems if we want to keep associating mathematical knowledge with the traditional characteristics, unless we are to ignore Benacerraf's problem. While I don't intend to suggest that Benacerraf provided a conclusive refutation of platonism, I believe that the current platonist literature fails to offer satisfactory solution to the epistemological problem. At the same time, a great deal of progress has been made in developing non-platonist accounts of mathematics. However, that progress has not come only from within philosophy. Indeed, I will argue in this paper, it is from the domain of empirical research that we have recently received the most important support for non-platonist epistemology.

In this paper, I focus on explaining the *apparent* objectivity of arithmetical knowledge. As we will see, we should carefully consider what is understood by mathematical objectivity, but based on practices and applications, there is at least a strong *impression* of mathematical knowledge being objective (Pantsar 2021a). While this may explain

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the appeal of platonism, here I argue that also non-platonist epistemology of arithmetic can feasibly explain the apparent objectivity. I show that based on empirical evidence, arithmetic is a cultural development based (partly) on biologically evolved *proto-arithmetical* abilities for observing quantities. Evidence strongly suggests that arithmetic, if developed by cultures, develops in convergent ways at least when it comes to basic operations (addition, multiplication, etc.) on finite numbers. Furthermore, in learning and developing arithmetic, we employ our proto-arithmetical abilities, which ensure that we acquire similar number concepts and grasp arithmetical operations in a similar way. This explains why arithmetic is strongly intersubjective. In fact, I will conclude, it is intersubjective in a strong enough sense to be experienced as objectivity.

2 Intersubjectivity and Objectivity

The problem at hand can be spelled out as a simple dilemma. As a human endeavour, mathematics is generally thought to be strongly intersubjective. Indeed, it is often seen as objective. Mathematicians may disagree on some matters—for example, whether the axiom of choice should be included in set theory or whether the continuum hypothesis should be accepted—but the field of agreement is overwhelmingly greater. But why is there such strong intersubjectivity and, indeed, a common experience of objectivity (Pantsar 2021a)? Plato's (1992) answer would be that this is due to mathematics being about an objective, abstract domain. However, such platonist views have become to be seen as increasingly problematic, and some philosophers have argued that mathematics is entirely a human creation (see, e.g., Field 1980; Wittgenstein 1976). But if mathematics is a human creation, do we not lose objectivity in any strong sense of the word?

To keep things manageable, in this paper I will focus only on the arithmetic of positive integers, generally considered to be one of the most basic—if not *the* most basic—areas of mathematics. Arithmetic is not a field without disagreement (concerning, e.g., whether it should be axiomatized in first or second-order logic) but there is no disagreement over basic arithmetical operations. Indeed, such disagreement is seen as so unlikely that divergent statements have played key roles in famous works of fiction as exceptional beliefs. In Dostoevsky's (1864) novel *Notes from Underground*, a character implies that believing that $2 + 2 = 5$ is a mark of human freedom from the oppression of logic and reason. In Orwell's (1961) novel *Nineteen Eighty-Four*, turning the setting around, the statement $2 + 2 = 5$ is turned into a vehicle of oppression when the protagonist is faced with a political system's power to distort even arithmetic.

Why do the examples of Dostoevsky and Orwell resonate with readers? I believe the reason is the shared view of the inherent impossibility of $2 + 2 = 5$. The novelists could count on this being a striking example since few would actually entertain the thought that arithmetical equations could be changed either by personal conviction or political decree. However, this wide intersubjectivity of arithmetical beliefs does not by itself imply that $2 + 2 = 4$ is an *objective* truth. It could well be a convention by either implicit or explicit human agreement, but one that is so ingrained that we are unable to conceive of disagreeing with it.

This kind of position was suggested by Cole (2009), who argued that "*mathematical entities are pure constitutive social constructs constituted by mathematical activities*" (p. 599, emphases original). As the results of such activities, Cole argued, "all socially acceptable mathematical axioms are objectively true" (p. 604). Predictably, this social constructivist view was criticized for not being able to account for the objectivity and necessity of mathematics. Dieterle (2010), for example, pointed out that the kind of "epistemic objectivity" that Cole presents in support of social constructivism is too weak and cannot account for the kind of objectivity that is associated with mathematics. In his subsequent work, Cole (2013, 2015) has presented a stronger account, based on Searle's (1997, 2010) theory of construction of social reality. According to this theory, humans can impose functions onto reality. A piece of paper, for example, becomes currency through such imposition. The important point is that impositions like money are *institutional* in that they depend on there being constitutive rules for their existence. Laws, rules of games, and other such impositions are recognized collectively by populations, and thus get their intersubjective character. Cole (2013, 2015) argues that mathematical domains are such institutional entities.

But if mathematical objects are social constructs, how can we account for the apparent objectivity of truths like $2 + 2 = 4$, which seems to be stronger than that of rules of chess? In his revised view, Cole argues that there are various institutional entities whose "function is to facilitate our abilities to represent, analyze, reason about, discover truths concerning, etc. facets of reality that are not the entities in question." (Cole 2013 pp. 13–14). In Cole's account, numbers and other mathematical objects are institutional entities that play this kind of *representational function* (RF) (p. 14):

> [T]he primary reason why we introduce facets of reality to serve RFs is to allow us to represent the world using intentional states that structure it into entities with features, for, as a result of the cognitive constitution of human beings at this evolutionary stage, we find it much easier to engage in the aforementioned types of activities using such states. (Cole 2013, p. 14)

Therefore we introduce entities with RFs, such as natural numbers, because it is cognitively advantageous for us to handle a representation of reality in terms of numbers. This representation in terms of abstract objects like numbers emerges both from basic arithmetical principles and "spatio-temporal entities that we find around us" (Cole 2013, p. 34). In this institutional account of socially constructed objects, arithmetical facts are "fairly robustly objective" because "truths, such as $7+5=12$, could *not* be any different from how they are and the natural numbers still serve their RFs" (Cole 2013, p. 34, emphasis original). An important part of Cole's (2015) account is that there are different levels of objectivity, and they are strongly connected to the question of different levels of *constraint* on social constructs. In the case of numbers, their representative function provides strong constraints, since they can only *be* numbers if they serve their representative function.

Rytilä (2021) has criticized Cole's account for not being able to explain the objectivity and applicability of mathematics. Both of these are *bona fide* problems, but in the rest of this paper I focus on the question of objectivity. I want to start the analysis from the basic assumption mentioned by Cole in the above quotation, i.e., "as a result of the cognitive constitution of human beings at this evolutionary stage, we find it much easier to engage in the aforementioned types of activities". Granting that entities with representational functions help us in cognitive activities, we must ask *why* this is the case with numbers. What are the features of the world that make the RFs of natural numbers beneficial for us, and how do they account for the apparent objectivity of arithmetical truths?

To set the topic up more clearly in a wider context, let us move on to another account of mathematical objectivity, proposed by Feferman (2009). He argues that "mathematics emerges from humanly constructed, intersubjectively established, basic structural conceptions" and presents a list of ten theses of how this happens (Feferman 2009, p. 1). For the present purposes, two are particularly important:

> The basic objects of mathematical thought exist only as mental conceptions, though the source of these conceptions lies in everyday experience in manifold ways, in the processes of counting, ordering, matching, combining, separating, and locating in space and time. […] The basic conceptions of mathematics are of certain kinds of relatively simple ideal-world pictures which are not of objects in isolation but of structures, i.e. coherently conceived groups of objects interconnected by a few simple relations and operations. They are communicated and understood prior to any axiomatics, indeed prior to any systematic logical development. (Feferman 2009, p. 3)

Accounts consistent with the first point, rooting mathematics in everyday experience, have been developed by Kitcher (1983), Mac Lane (1985) and Lakoff and Núñez (2000). Those accounts are also consistent with the other point (Feferman's third thesis, to be exact), according to which basic conceptions of mathematics are ideal-world pictures emerging from the everyday experiences. Unfortunately there is no space here to go into the details, but it is easy to see how Feferman can argue for strong intersubjectivity of mathematics based on this account. If we share our everyday experiences and form similar ideal-world pictures, it is understandable that we end up sharing mathematical notions. This can add strength also to Cole's account, since it suggests why some rules and concepts are institutionalized as social constructs, while others are not. But, again, the question is *why* we share everyday experiences and form similar ideal-world pictures.

Indeed, that is the main question I tackle in the rest of this paper, and it pertains to both Cole's and Feferman's account: why *are* some rules and concepts institutionalized? If this is because we as humans engage in similar experiences, why is that the case? Why do we observe the world in similar ways and end up having similar ideal representations of it? As I see it, the strength of accounts like those of Cole and Feferman depends on how well we can answer these questions, i.e., how well we can justify strong intersubjectivity of mathematics, and whether this can explain the (apparent) objectivity of mathematics.

3 What is Objectivity?

Before we continue, the meaning of terms like "intersubjective" and "objective" needs to be made more specific. The meaning of intersubjectivity is in the present context a simpler question, although at least seven different definitions of the term seem to be in use in social psychology (Gillespie and Cornish 2010). In philosophy, intersubjectivity is often associated with the phenomenology of Husserl (1913), but here I proceed with a general definition according to which intersubjectivity refers to two or more persons sharing a subjective cognitive state or experience. How this can be established is a traditional problem in philosophy, but for the task at hand we can safely assume that intersubjective experiences exist and concepts can be shared in an intersubjective manner. We will later see how this intersubjectivity can be explained in the context of arithmetical cognition, but for now it is enough to assume that intersubjectivity can generally take place.

However, the relation between intersubjectivity and objectivity is a more problematic issue. Feferman (2009), for example, presents his view as supporting "intersubjective objectivity" and writes (p. 2) about "objective subjectivity".

What could this mean? At first glance, the term seems to be self-contradictory: however we define objectivity and subjectivity in relation to each other, surely there cannot be such a thing as objective subjectivity? While Feferman's account ultimately avoids this problem, I believe that the terminological problem remains. Indeed, I contend that at best we can *explain* the *apparent* objectivity of mathematics based on intersubjectivity. If our epistemology of mathematics is based on subjective experiences, I will argue, we cannot save objectivity in the platonist sense. However, this should not worry us, because platonist objectivity should not be our target phenomenon in philosophy of mathematics. What we should aim to do is to explain how the apparent objectivity be explained in strong enough sense to account for mathematics as a human phenomenon, and then see whether a platonist explanation is still needed. I will argue that at least in the case of arithmetic it is not.

Thus my initial focus is on explaining the apparent objectivity of mathematics, rather than robust, platonist objectivity. In order to achieve that, however, we need to have a clearer idea of what is meant by objectivity, apparent or robust. Here I will follow the influential criteria presented by Wright (1992) and used by Shapiro (2007) in his treatment of the objectivity of mathematical discourse. The three criteria I will focus on are *epistemic constraint*, *cognitive command*, and *wider cosmological role*.[1] The criterion of epistemic constraint states that non-objective matters are always knowable, meaning that in non-objective discourse it is always the case that: $P \leftrightarrow (P\ may\ be\ known)$ (Wright 1992, p. 75). In objective discourse, however, there can be true propositions whose truth cannot be known (Shapiro 2007, p. 339). The criterion of cognitive command means that in objective discourse we can a priori rule out the possibility of what Shapiro (2007 p. 356) calls "blameless" disagreements: i.e., disagreements that cannot be explained by a reason such as divergent information or different conditions (Wright 1992, p. 92). The criterion of wider cosmological role states that for a discourse to be objective, it must feature also in explanations that are not exclusive to the domain of the discourse (Wright 1992, p. 198).

What do these criteria entail in the case of mathematical discourse? For one thing, they appear to establish objectivity in stronger sense than suggested by Feferman and Cole. For example, Feferman and Cole both discuss chess in terms of Searle's (1997) idea of objectivity of social constructs (Feferman 2009; Cole 2015). Yet discourse on chess does not fulfil Wright's criteria for objectivity. In particular, the criterion of wider cosmological role is not fulfilled since discourse on chess is essentially exclusive to the domain of chess. Of course chess can be discussed in a variety of ways regarding, e.g., its place in society or the arts, and analogies using chess may be used in other domains. But ultimately, aside from analogies, discourse on chess as a game does not feature in non-chess explanations. The fact mentioned by Feferman (2009, p. 4) as being objective, namely that it is not possible to force a checkmate with a king and two knights against a king, is a good example of this. It follows from the rules of chess and is thus in this weak sense objective. However, like any other fact about chess endgames, it is a fact that only features in the discourse of chess. If the rules of chess were changed by common agreement, the change would only directly pertain to chess itself.

With mathematics, this is different due to the wide existence of mathematical applications in science, as well as in everyday life. If the rules of arithmetic were changed so that basic operations would get different results, our financial system, for example, would be immediately thrown off the rails. If rules of calculus, complex analysis, probability theory, and other areas of mathematics were substantially changed, all science as we know it would be altered.[2] Mathematical discourse clearly has a wider cosmological role. In mathematics, we can also rule out blameless disagreements, at least to a significant part. While there may be disagreement over, for example, whether the axiom of choice should be included in set theory, there are no such potentially blameless disagreements over arithmetical operations.[3]

Finally, also the criterion of epistemic constraint appears to make mathematical discourse objective. There are several reasons to believe that there can be unknowable mathematical truths. First, there are mathematical problems that are generally considered to be too complex computationally to be solved for sufficiently large inputs (Arora and Barak 2007; Pantsar 2021b). Second, Gödel (1931) proved that no consistent formal system strong enough to express arithmetic can be complete, i.e., prove all true sentences in the system. If we accept that proof from axioms is our way of knowing things in mathematics, this alone implies that there are unknowable truths in our mathematical systems.[4]

[1] In Shapiro's analysis of Wright's work, he also includes the criteria of *response dependency* and *judgment dependency*. In the present context, however, I don't believe they add anything substantial to the criterion of cognitive command.

[2] This discussion is often carried out in terms of whether mathematics is *indispensable* to scientific explanations (e.g., Field 1980; Colyvan 2001). However, I believe the question of wider cosmological role is relevant even if we didn't consider mathematics to be indispensable for science, as long as the mathematical explanations are useful and fruitful (for more, see Pantsar 2018a, b).

[3] If Dostoevsky's character would genuinely insist that $2+2=5$, we would not hesitate to say that he is wrong with blame.

[4] It is important to note that it does not follow that any particular mathematical truth would be unknowable, since the unprovable sentences depend on the particular axiomatizations and encodings. See (Pantsar 2009) for more.

Third, mathematical languages are not limited to particular lengths of expression. Both formal and informal mathematical languages can be complex enough to make the prospect of knowing every truth expressible in them unfeasible.

Thus mathematical discourse appears to fulfil Wright's criteria for objectivity, as argued by Shapiro (2007). However, I am not ready to agree with Shapiro that consequently mathematical discourse indeed *is* objective. What we have established is the weaker position that mathematical discourse is *apparently* objective, which does not necessarily imply robust objectivity. Appearances can be deceiving. Perhaps mathematics is not objective after all, but it consists of extremely widely accepted and deeply ingrained conventions, as suggested by, e.g., Warren (2020). If that were the case, could we distinguish between a convention and an objective truth? Let me present one example to show why this could be difficult. Is the statement $4 * 4 = 16$ more objective than the statement $(-4) * (-4) = 16$? The first impression would seem to be that both are equally objective. However, as pointed out by Stewart (2006) and others, the algebraic fact that multiplying two negative numbers gives a positive number is ultimately a convention. There are good mathematical reasons for accepting it, but what could be an extra-mathematical interpretation of "negative times negative"? But if that objective-appearing fact is in fact a convention, should we be worried that much—perhaps all—of mathematics consists of deeply ingrained conventions like that? I think that this is a genuine worry for accounts of mathematical objectivity and something more is needed, a more "bottom-up" approach that can establish why mathematics appears to be objective to us, as well as explain what this reason for the apparent objectivity amounts to in the question of real objectivity.

4 The Origins of Intersubjective Arithmetic

Recall Cole's (2013) view that numbers are "institutional entities" that serve representational functions that allow us to engage in representing the world in an easier manner, based on our cognitive constitution. Now the question is, what exactly is the connection between numbers as institutional entities and our cognitive constitution? Indeed, in order to evaluate the general feasibility of Cole's account, the first question to ask is *whether* such a connection exists. If not, then numbers as institutional entities could be arbitrary conventions, in which case the problems of objectivity and applicability, as mentioned in (Rytilä 2021), could be devastating for Cole's account. After all, why would arbitrary conventions have any success in representing the world in a way that can be used in scientific applications?

The way I think such questions should be approached is through first explaining how and why mathematical cognition is intersubjective. As established in Sect. 2, the strength of Cole's (2013, 2015) and Feferman's (2009) accounts depends on how we can justify strong intersubjectivity of mathematical knowledge. In order to evaluate that justification, we must have a proper understanding of what mathematical knowledge is like. Unlike traditionally in philosophy of mathematics, I don't see how such understanding can be formed without empirical study of the development of mathematical cognition. Fortunately, when it comes to the development of arithmetical cognition, a great deal of progress has been made in the cognitive sciences. Recently, this progress has also lead to an improved philosophical understanding of the character of arithmetical knowledge.

I will first present what I deem to be the most plausible hypothesis of how arithmetical knowledge has developed. Then I will assess the consequences of accepting this hypothesis for the epistemology of arithmetic. Due to considerations of space, there will not be an opportunity for detailed comparison of the hypothesis to others presented in the literature. For details regarding that and other aspects of the discussion, I prompt the reader to (Pantsar 2014, 2015, 2018b, 2019, 2020, 2021c).

In a nutshell, I believe that arithmetical cognition in individual ontogeny is the result of an *enculturated* development based on evolutionarily ancient *proto-arithmetical* abilities. In cultural history and phylogeny, the development of arithmetical knowledge is made possible by *cumulative cultural evolution* based on the biologically evolved proto-arithmetical capacities. Let me start explaining this by focusing first on the ontogenetic aspect.

In the field of research called *numerical cognition*, it is generally believed that human are endowed with numerical abilities already in infancy, and they are shared with many non-human animals (for overviews, see, e.g., Dehaene and Brannon 2011; Cohen Kadosh and Dowker 2015). Some argue that there are innate number concepts (Gallistel 2017), others support innate "number modules" (Butterworth 1999), and yet others postulate an innate "number sense" (Dehaene 2011). As explained elsewhere (Pantsar 2014, 2019), I believe this terminology to be highly misleading. The infant and non-human animal abilities themselves are not in doubt, as they are based on solid empirical evidence. The problem is in calling such abilities *numerical*. Even more worryingly, some authors have written about infant and (non-human) animal *arithmetic* (Wynn 1992; Rugani et al. 2009).

It is important to note that the abilities in question have very different characteristics from arithmetical abilities. The infant and non-human abilities are approximate and/or limited, and do not involve exact number concepts, whereas arithmetical abilities are general and concern exact number concepts. I have previously labelled the evolutionary ancient abilities that infants and non-human animals possess *proto-arithmetical* and to concern *numerosities*, while reserving the term *numbers* for properly arithmetical abilities (Pantsar 2014). However, while I think it is crucial to distinguish between the two kinds of abilities, as the term already suggests, I believe that proto-arithmetical abilities form a (partial) cognitive basis for arithmetical abilities.

It is commonplace now to accept the existence of two proto-arithmetical abilities, both associated with a particular "core cognitive" system. Core cognition refers to evolutionarily ancient, innate independent cognitive systems (Carey 2009).[5] The first ability is *subitizing* and it is associated with the *object tracking system* (OTS) (Trick and Pylyshyn 1994). Subitizing refers to the ability to determine the amount of objects in our field of vision without counting and it has been shown to be present both in neonate humans and many non-human animals (Starkey and Cooper 1980; Dehaene 2011). The subitizing ability is exact but it stops working when the amount of objects is higher than three or four, which is thought to be the limit of the OTS (Carey 2009; Knops 2020). The second ability is *estimating* and it is associated with the approximate number system (ANS). Unlike subitizing, the ANS-based estimating ability is not limited to small numerosities, but it is increasingly inaccurate as the numerosity of the observed objects becomes larger (Dehaene 2011; Knops 2020).

The subitizing and estimating abilities have been established to be present also in members of non-arithmetical cultures, like the Amazonian cultures of Pirahã and Munduruku (Gordon 2004; Pica et al. 2004). Therefore, proto-arithmetical abilities are shared intersubjectively so widely that they can be feasibly considered to be universal for neurotypical humans. If arithmetical abilities were similarly universal, we would have a solid explanation for the objectivity and applicability of arithmetic. If some ability is shared by all neurotypical humans, surely knowledge associated with this ability would be objective in a strong sense. Similarly, if all neurotypical humans cognized in a similar way about quantitative information, it should be expected that knowledge associated with these cognitive abilities would find its way to scientific quantifying applications.[6]

The problem, however, is that proto-arithmetical abilities are *not* arithmetical abilities. Thus, any objectivity associated with proto-arithmetical abilities does not imply the objectivity of arithmetical knowledge. Indeed, there are clear differences in intersubjectivity between proto-arithmetical and arithmetical abilities. As mentioned above, there are completely non-arithmetical cultures that possess neither number concepts nor numeral words (e.g., the Pirahã and the Munduruku). People in such cultures share the same proto-arithmetical abilities as people in arithmetical cultures, yet simple arithmetic like $2 + 2 = 4$ is not included in their abilities. Therefore, even if proto-arithmetical abilities can be feasibly considered to be universally intersubjective (in neurotypical cases), the intersubjectivity of arithmetical abilities is—at best—limited to members of arithmetical cultures.[7]

However, the topic at hand is not whether arithmetical abilities are universal, it is whether arithmetical *knowledge* can feasibly be seen as objective. Clearly it is possible for knowledge to be objective even though it is not universally possessed by humans. Thus the key question becomes whether arithmetical knowledge based on proto-arithmetical abilities, can be considered objective. But since arithmetical knowledge is clearly culturally developed (more on how this happens later), in order to answer this question, we need to be sensitive to cultural factors. In relation to ontogeny, we therefore need to answer how cultural factors enable the learning of culturally developed knowledge and skills, while applying also the core cognitive proto-arithmetical systems.

A highly promising theoretical framework for such answers is provided by the *enculturation* account, as proposed by Menary (2015) and others (Fabry 2020; Jones 2020). Enculturation refers to the transformative process in which interactions with the surrounding culture shape the acquisition and development of cognitive abilities and practices (Menary 2015; Fabry 2018). The central idea behind enculturation is that the neural plasticity of the brain enables both structural and functional variations, which make it possible to acquire new cognitive capacities in a specific cultural context (Dehaene 2009; Anderson 2015). For example, when learning to read, the plasticity of the brain allows acquiring a new cognitive capacity by recycling or reusing neural resources developed originally for other purposes, e.g., visual processing and language comprehension (Dehaene 2009; Menary 2014; Fabry 2018). Recently, philosophical accounts have been developed according to which acquiring and developing arithmetical cognition also happens through a process of enculturation (Menary 2015; Pantsar 2019; Fabry 2020; Jones 2020). These accounts are

[5] It should be noted that the term "innate" is itself problematic and can refer to several different views (Griffiths 2001). Here "innate" is used in a sense that an innate capacity is possessed independently of cultural contribution.

[6] See (Pantsar 2021c) for a detailed argument to this effect.

[7] This is where my account differs from that presented by Ferreirós, who writes that: "I dare to suggest the idea that mathematics, in its most elementary strata, may be the best expression of that which we humans have in common, merely in virtue of being human." (Ferreirós 2016, p. 188). I do not think that mathematics expresses something that all humans have in common. Even in its most elementary strata, mathematics—unlike proto-mathematical abilities—expresses also cultural developments, which do not apply universally to humans.

based on cultural learning transforming neural resources associated with proto-arithmetical capacities by using them for culturally specific arithmetical learning.[8] This does not mean, however, that proto-arithmetical abilities are lost in the process of enculturation. Instead, arithmetical abilities develop as parallel capacities to proto-arithmetical abilities, thus giving people in arithmetical cultures two systems for treating quantities (Dehaene 2011).

There is a great deal of empirical evidence to support this account. From experiments on monkeys and humans, it is known that (partly) the same brain regions activate in proto-arithmetical processing of quantities as with numerical and arithmetical processing (Nieder 2006; Cantlon and Brannon 2007; Piazza et al. 2007). Researchers disagree over which proto-arithmetical ability, and thus which core cognitive system, plays a more important role in this development. Some argue for OTS as the primary system (Carey 2009; Beck 2017), while others see ANS as the key system (Halberda and Feigenson 2008; Dehaene 2011). Yet others argue for hybrid accounts in which both core cognitive systems play an integral role (Spelke 2011; vanMarle et al. 2018; Pantsar 2021c). The details of the accounts differ, but there is increasing consensus over the general position that number concepts and arithmetical abilities are acquired and developed based on proto-arithmetical abilities. But given that this development is only possible in culturally specific contexts, in which cultural practices like counting and linguistic tools like numeral words are in place, the account also needs to include cultural influences. Thus the enculturation framework should be applied in explaining the ontogeny of arithmetical abilities.[9]

However, the required enculturation process can only take place if the cultural setting already includes the needed cultural practices, linguistic tools, and other factors. This prompts the question how such cultural developments can initially take place. To answer this, we need to move the focus from ontogeny to phylogeny and cultural history. The phylogeny of proto-arithmetical abilities, given that it takes place in the frame of biological evolution, must be explained through processes of natural selection that (mainly) drive evolution. Here I don't want to speculate on the character of such processes, but it is plausible that proto-arithmetical abilities are evolutionarily advantageous. From keeping track of geographical features (e.g., nest being in the fourth hole) to keeping track of offspring and avoiding predators (e.g., three wolves posing a different threat than two wolves), there are various ways in which keeping track of numerosity—as opposed to other magnitudes—can be helpful for survival. While the origin and development of proto-arithmetical abilities is an important topic, here I will say nothing more about it. Instead, I will focus on the question how humans have culturally developed arithmetic based on proto-arithmetical abilities, starting from acquiring number concepts.

There is a fundamental difficulty in explaining how humans possess number concepts. Nativist accounts over number concepts are not supported by any empirical data (for details, see (Pantsar 2021c)), but if not innate, humans must acquire number concepts in ontogeny. The enculturation framework has great potential in explaining this, but such cultural learning is possible only when the culture already possesses number concepts. However, as pointed out, among others, by Pelland (2020), there must have been a stage when humans moved from only possessing proto-arithmetical abilities to having number concepts. How can this be explained in the enculturation framework?

Given the lack of numeral words in non-numerical cultures like the Pirahã and the Munduruku, one is tempted to hypothesize that the introduction of numeral words was crucial in the development of number concepts. However, this poses another dilemma: how could numeral words develop if there were no number concepts for them to refer to? I believe that we should look for answers to this dilemma in the enculturated development based on proto-arithmetical abilities. In modern arithmetical cultures, an important part of acquiring number concepts is learning an ordered counting list of numeral words (Carey 2009; Beck 2017). But counting is itself an advanced process that requires explaining. Thus the challenge starts from the early origins of the history of number concepts.

It is not possible to accurately trace the origins of number concepts in numerical cultures, since there is no historical record of spoken numeral words. However, both from pre-historical and historical material record, as well as anthropological data on other cultures, some light can be shed on the matter. Based on this evidence, it is likely that counting procedures with numeral words and later symbols have emerged from a combination of application of proto-arithmetical abilities and material engagement with objects. Proto-arithmetical abilities can explain why humans discriminate collections of items in terms of quantity in the first place (see, e.g., Zahidi (2021)). Material engagement can explain how the practice of one-to-one matching and the notion of one-to-one correspondence can emerge.[10]

[8] According to Menary's (2015) account, the neural circuits associated with proto-arithmetical abilities are recycled for culturally developed, arithmetical purposes. Fabry (2020) and Jones (2020) argue for a similar view, but based on a different, more general, type of reuse of neural resources (Anderson 2015).

[9] For a detailed argument, see Pantsar (2021c, 2019).

[10] Zahidi (2021) argues that placing items in one-to-one correspondence could be the result of applying proto-arithmetical abilities. However, it has been reported that the Pirahã cannot do one-to-one matching for collections of items larger than three (Gordon 2004; Everett and Madora 2012). This suggests that one-to-one matching is an ability that also needs cultural input, e.g., the application of marking tools (see (Pantsar 2021a) for more).

Practices of tallying and finger counting, for example, can emerge from the introduction of the practice of marking, say, one observed animal with one notch. At this point, no number concepts are needed: simply by comparing amounts of notches, it is possible to establish which group of animals is more numerous. Such practices, whether due to tallying or finger counting, can lead to developing properties of counting sequences, such as their linear order (Bender and Beller 2012; Overmann 2018). In this kind of development, words for body parts can gradually evolve new meanings as numeral words and tallying marks can evolve new uses as numeral symbols (Ifrah 1998).

Anthropological studies strongly suggest that this kind of transfer of meaning of words and symbols takes place. Many cases have been established in which words originally not used to signify quantity have acquired numerical purposes. The Hup language word for two, for example, means "eyes" and the word for three is also the word for a three-chambered rubber plant seed (Epps 2006). This is evidence that number concepts and numeral words have thus co-evolved culturally from practices of material engagement, such as tallying and body-part counting (Wiese 2007; dos Santos 2021). With numeral words in place, it is possible to associate number concepts with further material practices. By grouping objects such as pebbles, it is possible to derive norms about the "plus one" operation and addition in general (Overmann 2018). By applying these practices to financial transactions, for example, they can then be associated with new forms of material engagement and acquire a wider status within cultures. There is strong evidence that this is what happened in Mesopotamia, where clay tokens were used for accounting since the Neolithic era (8300–4500 BC) (Schmandt-Besserat 1992; Overmann 2018). Groupings of clay tokens of units of different sizes started to be associated with different numerical values; and with the introduction of cuneiform writing systems these relations between different units and their groupings could be transferred into numerical symbols (Nissen et al. 1994; Overmann 2018).

Thus we arrive at an answer to the question of origins above. We don't need a nativist account of number concepts, because number concepts and numeral words developed concurrently through practices of material engagement. This is compatible with the theoretical framework of *cumulative cultural evolution*, according to which knowledge and skills are developed in small increments and transmitted across generations (Tomasello 1999; Boyd and Richerson 2005; Heyes 2018). Of course not all cultures need to develop knowledge and skills independently. Cultures that have regular interactions with other cultures can acquire new cultural practices, which they can then develop further. This can account for the long line of cultural development of number concepts and arithmetic, in which numeral words and symbols, physical artifacts, and other factors have played crucial roles (Everett 2017; Pantsar 2019).

How does all this relate to the intersubjectivity of arithmetic? Could it not be the case that cultures start developing arithmetic in divergent ways? Indeed, to some degree, this is evident when comparing different cultures that developed arithmetic independently. The Mayans, for example, were expert calculators but, as far as we know, did not engage in proofs of arithmetical theorems, which became central in the Mesopotamian/Greek line of development (Merzbach and Boyer 2011). However, in no known cases where cultures have developed arithmetic is there divergence in basic operations on finite numbers. $2 + 2 = 4$ is equally much an arithmetical truth for us as it was for the Mayans and the ancient Chinese.

How can we account for this apparently universal convergence of basic arithmetical operations toward the same results in independently developed arithmetical cultures? I propose that the above considerations on ontogeny, phylogeny and cultural history of number concepts and arithmetic suggest the answer. Since our proto-arithmetical abilities determine how we engage with quantities, we cannot develop "deviant" systems of arithmetic that clash with them. Cultures do not necessarily develop number concepts, but when they do, it happens through the application of proto-arithmetical abilities. Therefore the linear, discrete number concepts that we have for arithmetic are, at least for the part concerning basic finite operations, equivalent with the number concepts that the Mayans possessed. In ontogeny, then, members of an arithmetical culture are taught the same number concepts and arithmetical rules, in a process in which they apply their biologically evolved proto-arithmetical abilities. That is the reason why number concepts and arithmetic are intersubjective, both within and across cultures.

5 From Intersubjectivity to Objectivity

In the previous section, I have presented an argument why arithmetical knowledge and skills are intersubjective. However, since arithmetical abilities are not universal, this intersubjectivity is not as strong as that of the biologically evolved proto-arithmetical abilities. As a consequence, it could be argued that arithmetic is not objective, or at least not as objective as proto-arithmetical abilities. But this kind of reasoning would be misleading. After all, we do not generally judge the objectivity of a discourse based on how many people or cultures engage in it. Instead, the objectivity is decided by the status of the subject matter. In this case, the subject matter is arithmetical knowledge which, as was established at the end of the previous section, is (partly) determined by proto-arithmetical abilities. Hence, while there are differences in the ways different cultures develop arithmetic, these differences do not concern divergent paths when it comes to results of basic arithmetical operations. Arithmetic, if developed, develops along similar lines, due to our universally shared proto-arithmetical abilities. I have argued that this kind of intersubjectivity is so strong that arithmetical knowledge should be called *maximally intersubjective* (Pantsar 2014).

I call the intersubjectivity "maximal" for a good reason. While individuals may share experiences and concepts intersubjectively within cultures, with proto-arithmetical abilities this intersubjectivity spans cultures. Evidence from different cultures implies that humans share proto-mathematical abilities as widely as any other cognitive or physical abilities. Proto-arithmetical abilities have evolved as evolutionary adaptations, which implies that—in addition to being useful—they are constantly exercised. Thus humans share a wealth of experience exercising proto-arithmetical abilities, which are not dependent on languages or culturally developed practices. If it is true that arithmetical knowledge and skills are fundamentally determined by proto-arithmetical abilities, it is to be expected that many would accept such maximally intersubjective knowledge as being *objective*. Whether maximal intersubjectivity should indeed count as objectivity is something I cannot take a stand on here. However, if something is maximally intersubjective, we may not be able to *distinguish* it from being objective.[11]

Now we also have an answer to the problem that Cole (2013, 2015) faced. Since he did not provide an explanation why mathematical objects like numbers can perform a representative function, there remained the possibility that mathematical knowledge is merely a matter of institutional conventions, like the rules of chess and other games. However, now we can explain why this is not the case. While we could change the rules of chess by a common agreement, we could *not* change the laws or arithmetic so that it would be the case that $2 + 2 = 5$. The reason for this is that such deviant arithmetic would clash with our proto-arithmetical abilities. Dostoevsky's character and Orwell's oppressive regime could not convince us that $2 + 2 = 5$. This is not because $2 + 2 = 4$ is such a firmly entrenched convention. It is because developing an arithmetical theory in which $2 + 2 = 5$ goes against our cognitive architecture, as it has evolved through millions of years. In this way, the maximal intersubjectivity of arithmetical knowledge can explain why it is experienced as being objective.[12]

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[11] It should be added here that nothing in the present account demands assuming the existence of mind-independent mathematical *objects*. See (Pantsar 2021a) for an argument on how mathematical objectivity does not demand mathematical objects and (Pantsar 2021d) for more on the existence of mathematical objects.

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