Logic and Logical Philosophy
Volume 26 (2017), 197–206
DOI: 10.12775/LLP.2016.025

Yaroslav Petrukhin

NATURAL DEDUCTION FOR THREE-VALUED REGULAR LOGICS

Abstract. In this paper, I consider a family of three-valued regular logics: the well-known strong and weak S. C. Kleene's logics and two intermediate logics, where one was discovered by M. Fitting and the other one by E. Komendantskaya. All these systems were originally presented in the semantical way and based on the theory of recursion. However, the proof theory of them still is not fully developed. Thus, natural deduction systems are built only for strong Kleene's logic both with one (A. Urquhart, G. Priest, A. Tamminga) and two designated values (G. Priest, B. Kooi, A. Tamminga). The purpose of this paper is to provide natural deduction systems for weak and intermediate regular logics both with one and two designated values.

Keywords: natural deduction; regular logic; Kleene's logic; three-valued logic

1. Introduction

Regular logics were first mentioned in the works of S. C. Kleene [4, 5] where he defined two three-valued regular logics (a strong and a weak one). A regular logic is understood as a logic which propositional connectives are regular. What is its regularity, and how it can be useful, are explained in [5] as follows:

> We conclude that, in order for the propositional connectives to be partial recursive operations (or at least to produce partial recursive predicates when applied to partial recursive predicates), we must choose tables for them which are *regular*, in the following sense: A given column (row) contains $\mathfrak{t}$ in the $\mathfrak{u}$ row (column), only if the column (row) consists entirely of $\mathfrak{t}$'s; and likewise for $\mathfrak{f}$. [5, p. 334]

Received May 27, 2016. *Revised* August 23, 2016. *Published online* August 29, 2016

The values $\mathfrak{t}$ and $\mathfrak{f}$ are understood in a usual way, i.e., as "truth" and "falsehood", value $\mathfrak{u}$ is understood as "not defined". For $\mathfrak{t}$, $\mathfrak{u}$, and $\mathfrak{f}$ we will use 1, ½, and 0, respectively.

Let At and Form be, respectively, the set of all propositional variables and the set of all formulas of the propositional language in language that is built with propositional variables and the propositional connectives: $\neg$, $\vee$, and $\wedge$. All logics we will build in the set Form. Let us denote a truth-table f for a connective c by f_c.

In the three-valued case, there are only four regular conjunctions and disjunctions (see [6] for details).[1] Depending on the choice of the number of designated values each collection of connectives yields two different regular logics.

First, we have two logics $\mathbf{K_3}$ and $\mathbf{K_3^2}$, respectively, for the matrixes $\langle\{1, ½, 0\}, f_\neg, f_\vee, f_\wedge, \{1\}\rangle$ and $\langle\{1, ½, 0\}, f_\neg, f_\vee, f_\wedge, \{1, ½\}\rangle$, where

	$f_\neg$
1	0
½	½
0	1

$f_\vee$	1	½	0
1	1	1	1
½	1	½	½
0	1	½	0

$f_\wedge$	1	½	0
1	1	½	0
½	½	½	0
0	0	0	0

Second, we have two logics $\mathbf{K_3^w}$ and $\mathbf{K_3^{w2}}$, respectively, for the matrixes $\langle\{1, ½, 0\}, f_\neg, f_\vee, f_\wedge, \{1\}\rangle$ and $\langle\{1, ½, 0\}, f_\neg, f_\vee, f_\wedge, \{1, ½\}\rangle$, where $f_\neg$ is the same as for $\mathbf{K_3}$ and

$f_\vee$	1	½	0
1	1	½	1
½	½	½	½
0	1	½	0

$f_\wedge$	1	½	0
1	1	½	0
½	½	½	½
0	0	½	0

The logics $\mathbf{K_3}$ and $\mathbf{K_3^w}$ are called *strong Kleene's logic* and *weak Kleene's logic*, respectively. They were introduced by Kleene in [4] in 1938 (see also [5]). However, $\mathbf{K_3}$ appeared in [8] as a fragment of Łukasiewicz's logic $\mathbf{Ł_3}$ (1920), and $\mathbf{K_3^w}$ appeared in [1] as a fragment of Bochvar's logic $\mathbf{B_3}$ (1938) independently by [4]. Note that $\mathbf{K_3^2}$ is also known as **LP** (*Logic of Paradox*) and was carefully studied by Priest [11].

A natural deduction system for $\mathbf{K_3}$ was first created by Urquhart [14]; later Priest [11] and Tamminga [12] independently obtained the same result. A natural deduction system for $\mathbf{K_3^2}$ was created by Priest [11] and later it was independently provided by Kooi and Tamminga [7].

[1] Four-valued regular logics are described in [13].

Thirdly, we have two logics $\mathbf{K_3^{\rightarrow}}$ and $\mathbf{K_3^{\rightarrow 2}}$, respectively, for the matrixes $\langle\{1, ½, 0\}, f_\neg, f_\vee, f_\wedge, \{1\}\rangle$ and $\langle\{1, ½, 0\}, f_\neg, f_\vee, f_\wedge, \{1, ½\}\rangle$, where $f_\neg$ is the same as for $\mathbf{K_3}$ and

$f_\vee$	1	½	0
1	1	1	1
½	½	½	½
0	1	½	0

$f_\wedge$	1	½	0
1	1	½	0
½	½	½	½
0	0	0	0

Fourthly, we have two logics $\mathbf{K_3^{\leftarrow}}$ and $\mathbf{K_3^{\leftarrow 2}}$, respectively, for the matrixes $\langle\{1, ½, 0\}, f_\neg, f_\vee, f_\wedge, \{1\}\rangle$ and $\langle\{1, ½, 0\}, f_\neg, f_\vee, f_\wedge, \{1, ½\}\rangle$, where $f_\neg$ is the same as for $\mathbf{K_3}$ and

$f_\vee$	1	½	0
1	1	½	1
½	1	½	½
0	1	½	0

$f_\wedge$	1	½	0
1	1	½	0
½	½	½	0
0	0	½	0

$\mathbf{K_3^{\rightarrow}}$ and $\mathbf{K_3^{\leftarrow}}$ are called *intermediate logics*. The logic $\mathbf{K_3^{\rightarrow}}$ was first discovered by Fitting [2]. The logic $\mathbf{K_3^{\leftarrow}}$ was discovered by Komendantskaya [6].

2. Rules for natural deduction systems

We will use the following rules of inference:

$$(\mathrm{EFQ})\frac{A \quad \neg A}{B} \qquad (\mathrm{EM})\frac{}{A \vee \neg A} \qquad (\neg\neg\mathrm{I})\frac{A}{\neg\neg A} \qquad (\neg\neg\mathrm{E})\frac{\neg\neg A}{A}$$

$$(\vee\mathrm{I}_1)\frac{A}{A \vee B} \qquad (\vee\mathrm{I}_2)\frac{B}{A \vee B} \qquad (\vee\mathrm{I}_3)\frac{\neg A \wedge B}{A \vee B} \qquad (\vee\mathrm{I}_4)\frac{A \wedge \neg B}{A \vee B} \qquad (\vee\mathrm{I}_5)\frac{A \wedge B}{A \vee B}$$

$$(\wedge\mathrm{I}_1)\frac{A \quad B}{A \wedge B} \qquad (\wedge\mathrm{I}_2)\frac{A \quad \neg A}{A \wedge B} \qquad (\wedge\mathrm{I}_3)\frac{B \quad \neg B}{A \wedge B}$$

$$(\wedge\mathrm{E}_1)\frac{A \wedge B}{A} \qquad (\wedge\mathrm{E}_2)\frac{A \wedge B}{B} \qquad (\wedge\mathrm{E}_3)\frac{A \wedge B}{\neg A \vee B} \qquad (\wedge\mathrm{E}_4)\frac{A \wedge B}{A \vee \neg B}$$

$$(\neg\vee\mathrm{I}_1)\frac{\neg A \wedge \neg B}{\neg(A \vee B)} \qquad (\neg\vee\mathrm{I}_2)\frac{A \wedge \neg A}{\neg(A \vee B)} \qquad (\neg\vee\mathrm{I}_3)\frac{B \wedge \neg B}{\neg(A \vee B)}$$

$$(\neg\vee\mathrm{E}_1)\frac{\neg(A \vee B)}{\neg A \wedge \neg B} \qquad (\neg\vee\mathrm{E}_2)\frac{\neg(A \vee B)}{\neg A \vee B} \qquad (\neg\vee\mathrm{E}_3)\frac{\neg(A \vee B)}{A \vee \neg B}$$

$$(\neg\wedge\mathrm{I}_1)\frac{\neg A \vee \neg B}{\neg(A \wedge B)} \qquad (\neg\wedge\mathrm{I}_2)\frac{\neg A}{\neg(A \wedge B)} \qquad (\neg\wedge\mathrm{I}_3)\frac{\neg B}{\neg(A \wedge B)}$$

$$(\neg\wedge\mathrm{I}_4)\,\dfrac{\neg A\wedge B}{\neg(A\wedge B)}\qquad(\neg\wedge\mathrm{I}_5)\,\dfrac{A\wedge\neg B}{\neg(A\wedge B)}\qquad(\neg\wedge\mathrm{E})\,\dfrac{\neg(A\wedge B)}{\neg A\vee\neg B}$$

Moreover, we consider the following four versions of the proof construction rule ($\vee$E):

$$(\vee\mathrm{E}_1)\,\dfrac{A\vee B\quad\begin{matrix}[A]\\ C\end{matrix}\quad\begin{matrix}[B]\\ C\end{matrix}}{C}$$

$$(\vee\mathrm{E}_2)\,\dfrac{A\vee B\quad\begin{matrix}[A]\\ C\end{matrix}\quad\begin{matrix}[\neg A\wedge B]\\ C\end{matrix}}{C}\qquad(\vee\mathrm{E}_3)\,\dfrac{A\vee B\quad\begin{matrix}[B]\\ C\end{matrix}\quad\begin{matrix}[A\wedge\neg B]\\ C\end{matrix}}{C}$$

$$(\vee\mathrm{E}_4)\,\dfrac{A\vee B\quad\begin{matrix}[A\wedge B]\\ C\end{matrix}\quad\begin{matrix}[A\wedge\neg B]\\ C\end{matrix}\quad\begin{matrix}[\neg A\wedge B]\\ C\end{matrix}}{C}$$

where $[X]$ means that the assumption X is discharged

Note that $\mathcal{R}_{\mathrm{cl}} := \{(\mathrm{EFQ}), (\mathrm{EM}), (\neg\neg\mathrm{I}), (\neg\neg\mathrm{E}), (\vee\mathrm{I}_1), (\vee\mathrm{I}_2), (\vee\mathrm{E}_1), (\wedge\mathrm{I}_1), (\wedge\mathrm{E}_1), (\wedge\mathrm{E}_2), (\neg\vee\mathrm{I}_1), (\neg\vee\mathrm{E}_1), (\neg\wedge\mathrm{I}_1), (\neg\wedge\mathrm{E})\}$ is a set of rules of a natural deduction system for classical logic. For natural deduction systems for $\mathbf{K_3}$ and $\mathbf{K_3^2}$ are suitable sets $\mathcal{R}_{\mathrm{cl}}\setminus\{(\mathrm{EM})\}$ and $\mathcal{R}_{\mathrm{cl}}\setminus\{(\mathrm{EFQ})\}$, respectively. Moreover, $\mathcal{R}_{\mathrm{cl}}\setminus\{(\mathrm{EFQ}), (\mathrm{EM})\}$ is a set of inference rules for **FDE** (it was proven by Priest in [11]; for more detailed proof see [10]).

The notion of a deduction of A from Γ in all natural deduction systems described in this paper is defined as a tree labeled with formulas.

As an example, consider the following deduction of $(A\wedge B)\vee(A\wedge C)$ from $A\wedge(B\vee C)$ in the mentioned systems.

$$\dfrac{\dfrac{A\wedge(B\vee C)}{B\vee C}\,(\wedge\mathrm{E}_2)\quad\dfrac{\dfrac{\dfrac{A\wedge(B\vee C)}{A}\,(\wedge\mathrm{E}_1)\quad\overset{(1)}{[B]}}{A\wedge B}\,(\wedge\mathrm{I}_1)}{(A\wedge B)\vee(A\wedge C)}\,(\vee\mathrm{I}_1)\quad\dfrac{\dfrac{\dfrac{A\wedge(B\vee C)}{A}\,(\wedge\mathrm{E}_1)\quad\overset{(2)}{[C]}}{A\wedge C}\,(\wedge\mathrm{I}_1)}{(A\wedge B)\vee(A\wedge C)}\,(\vee\mathrm{I}_2)}{(A\wedge B)\vee(A\wedge C)}\,(\vee\mathrm{E}_1)^{(1),(2)}$$

3. Natural deduction system for $\mathbf{K_3^{\rightarrow}}$

A set of rules for $\mathbf{K_3^{\rightarrow}}$ is as follows: (EFQ), ($\neg\neg$I), ($\neg\neg$E), ($\vee\mathrm{I}_1$), ($\vee\mathrm{I}_3$), ($\vee\mathrm{E}_2$), ($\wedge\mathrm{I}_1$), ($\wedge\mathrm{E}_1$), ($\wedge\mathrm{E}_2$), ($\neg\vee\mathrm{I}_1$), ($\neg\vee\mathrm{E}_1$), ($\neg\wedge\mathrm{I}_2$), ($\neg\wedge\mathrm{I}_5$), ($\neg\wedge$E).

Soundness follows by a simple routine check.

Theorem 3.1 (Soundness). *For any $\Gamma\subseteq\mathrm{Form}$ and any $A\in\mathrm{Form}$:*

$$\textit{if } \Gamma\vdash_{\mathbf{K_3^{\rightarrow}}} A \textit{ then } \Gamma\models_{\mathbf{K_3^{\rightarrow}}} A.$$

For completeness proof Henkin's method is used. At that the notational conventions of [7, 12] are adopted.

DEFINITION 3.1. A set of formulas Γ is a *nontrivial prime theory* iff the following conditions are met:

(Γ1) $\Gamma \neq \text{Form}$ (*non-triviality*)
(Γ2) $\Gamma \vdash_{\mathbf{K}_3^{\rightarrow}} A$ iff $A \in \Gamma$ (*closure of* $\vdash_{\mathbf{K}_3^{\rightarrow}}$)
(Γ3) if $A \vee B \in \Gamma$ then either $A \in \Gamma$ or both $\neg A \in \Gamma$ and $B \in \Gamma$ (*primeness*)

DEFINITION 3.2. For all $\Gamma \subseteq \text{Form}$ and $A \in \text{Form}$, $e(A, \Gamma)$ is a canonic valuation iff the following conditions hold:

$$e(A,\Gamma) = \begin{cases} 1 \iff A \in \Gamma \text{ and } \neg A \notin \Gamma \\ ½ \iff A \notin \Gamma \text{ and } \neg A \notin \Gamma \\ 0 \iff A \notin \Gamma \text{ and } \neg A \in \Gamma \\ \emptyset \iff A \in \Gamma \text{ and } \neg A \in \Gamma \end{cases}$$

Note. For logics with two designated values conditions for ½ and $\emptyset$ are defined in a different way:

$$e(A,\Gamma) = \begin{cases} ½ \iff A \in \Gamma \text{ and } \neg A \in \Gamma; \\ \emptyset \iff A \notin \Gamma \text{ and } \neg A \notin \Gamma \end{cases}$$

LEMMA 3.1. *For all* $\Gamma \subseteq \text{Form}$ *and* $A, B \in \text{Form}$:

1. $e(A, \Gamma) \neq \emptyset$,
2. $f_{\neg}(e(A, \Gamma)) = e(\neg A, \Gamma)$,
3. $f_{\vee}(e(A, \Gamma), e(B, \Gamma)) = e(A \vee B, \Gamma)$,
4. $f_{\wedge}(e(A, \Gamma), e(B, \Gamma)) = e(A \wedge B, \Gamma)$.

PROOF. *Ad* 1. Suppose $e(A, \Gamma) = \emptyset$. Then $A \in \Gamma$, $\neg A \in \Gamma$. By the rule (EFQ), $B \in \Gamma$, that is $\Gamma = \text{Form}$. This contradicts to (Γ1). Therefore, $e(A, \Gamma) \neq \emptyset$.

Ad 2. If $e(A, \Gamma) = 1$ then $A \in \Gamma$, $\neg A \notin \Gamma$. By the rule ($\neg\neg$I), $\neg\neg A \in \Gamma$. Hence, $e(\neg A, \Gamma) = 0 = f_{\neg}(1) = f_{\neg}(e(A, \Gamma))$.

If $e(A, \Gamma) = ½$ then $A \notin \Gamma$, $\neg A \notin \Gamma$. Suppose $\neg\neg A \in \Gamma$. By the rule ($\neg\neg$E), $A \in \Gamma$. Contradiction. Consequently, $\neg\neg A \notin \Gamma$. Hence $e(\neg A, \Gamma) = ½ = f_{\neg}(½) = f_{\neg}(e(A, \Gamma))$.

If $e(A, \Gamma) = 0$ then $A \notin \Gamma$, $\neg A \in \Gamma$. Suppose $\neg\neg A \in \Gamma$. By the rule ($\neg\neg$E), $A \in \Gamma$. Contradiction. Consequently, $\neg\neg A \notin \Gamma$. Hence $e(\neg A, \Gamma) = 1 = f_{\neg}(0) = f_{\neg}(e(A, \Gamma))$.

Ad 3. If $e(A,\Gamma)=1$ and $e(B,\Gamma)=1$, then $A\in\Gamma$, $\neg A\notin\Gamma$, $B\in\Gamma$, $\neg B\notin\Gamma$. By the rule ($\vee I_1$), $A\vee B\in\Gamma$. Let $\neg(A\vee B)\in\Gamma$. By the rule ($\neg\vee E_1$), $\neg A\wedge\neg B\in\Gamma$. Applying the rules ($\wedge E_1$) and ($\wedge E_2$), get $\neg A\in\Gamma$ and $\neg B\in\Gamma$. Contradiction. Hence $\neg(A\vee B)\notin\Gamma$. Consequently, $e(A\vee B,\Gamma)=1=f_\vee(1,1)=f_\vee(e(A,\Gamma),e(B,\Gamma))$.

If $e(A,\Gamma)=½$ and $e(B,\Gamma)=1$, then $A\notin\Gamma$, $\neg A\notin\Gamma$, $B\in\Gamma$, $\neg B\notin\Gamma$. Let $A\vee B\in\Gamma$. By ($\Gamma3$), either $A\in\Gamma$ or both $\neg A\in\Gamma$ and $B\in\Gamma$. Since $A\notin\Gamma$, so $\neg A\in\Gamma$ and $B\in\Gamma$. But then $\neg A\in\Gamma$. Contradiction. $A\vee B\notin\Gamma$. Let $\neg(A\vee B)\in\Gamma$. Applying the rules ($\neg\vee E_1$), ($\wedge E_1$), and ($\wedge E_2$), get $\neg A\in\Gamma$, $\neg B\in\Gamma$. Contradiction. Hence $\neg(A\vee B)\notin\Gamma$. Consequently, $e(A\vee B,\Gamma)=½=f_\vee(½,1)=f_\vee(e(A,\Gamma),e(B,\Gamma))$.

If $e(A,\Gamma)=0$ and $e(B,\Gamma)=1$, then $A\notin\Gamma$, $\neg A\in\Gamma$, $B\in\Gamma$, $\neg B\notin\Gamma$. Applying the rules ($\wedge I_1$) and ($\vee I_3$), get $A\vee B\in\Gamma$. Let $\neg(A\vee B)\in\Gamma$. Applying the rules ($\neg\vee E_1$) and ($\wedge E_2$), get $\neg B\in\Gamma$. Contradiction. Hence $\neg(A\vee B)\notin\Gamma$. Consequently, $e(A\vee B,\Gamma)=1=f_\vee(0,1)=f_\vee(e(A,\Gamma),e(B,\Gamma))$.

The other cases are proved similarly.

Ad 4. If $e(A,\Gamma)=1$ and $e(B,\Gamma)=1$, then $A\in\Gamma$, $\neg A\notin\Gamma$, $B\in\Gamma$, $\neg B\notin\Gamma$. By the rule ($\wedge I_1$), $A\wedge B\in\Gamma$. Let $\neg(A\wedge B)\in\Gamma$. By the rule ($\neg\wedge E$), $\neg A\vee\neg B\in\Gamma$. By ($\Gamma3$), either $\neg A\in\Gamma$ or both $\neg\neg A\in\Gamma$ and $\neg B\in\Gamma$. Since $\neg A\notin\Gamma$, so $\neg\neg A\in\Gamma$ and $\neg B\in\Gamma$. But then $\neg B\in\Gamma$. Contradiction. Hence $\neg(A\wedge B)\notin\Gamma$. Therefore, $e(A\wedge B,\Gamma)=1=f_\wedge(1,1)=f_\wedge(e(A,\Gamma),e(B,\Gamma))$.

If $e(A,\Gamma)=1$ and $e(B,\Gamma)=0$, then $A\in\Gamma$, $\neg A\notin\Gamma$, $B\notin\Gamma$, $\neg B\in\Gamma$. Let $A\wedge B\in\Gamma$. By the rule ($\wedge E_2$), $B\in\Gamma$. Contradiction. $A\wedge B\notin\Gamma$. Applying the rules ($\wedge I_1$) and ($\neg\wedge I_5$), get $\neg(A\wedge B)\in\Gamma$. Therefore, $e(A\wedge B,\Gamma)=0=f_\wedge(1,0)=f_\wedge(e(A,\Gamma),e(B,\Gamma))$.

If $e(A,\Gamma)=0$ and $e(B,\Gamma)=½$, then $A\notin\Gamma$, $\neg A\in\Gamma$, $B\notin\Gamma$, $\neg B\notin\Gamma$. Let $A\wedge B\in\Gamma$. By the rule ($\wedge E_1$), $A\in\Gamma$. Contradiction. $A\wedge B\notin\Gamma$. By the rule ($\neg\wedge I_2$), $\neg(A\wedge B)\in\Gamma$. Therefore, $e(A\wedge B,\Gamma)=0=f_\wedge(0,½)=f_\wedge(e(A,\Gamma),e(B,\Gamma))$.

The other cases are proved similarly. □

By a structural induction on formulas, using Lemma 3.1 we obtain:

LEMMA 3.2. *Let Γ be any nontrivial prime theory and v_Γ be an arbitrary valuation such that $v_\Gamma(p)=e(p,\Gamma)$, for any $p\in\mathrm{At}$. Then for any $A\in Form$ we have $v_\Gamma(A)=e(A,\Gamma)$.*

Lemma 3.3 (Lindenbaum). *For all* $\Gamma \subseteq$ Form, $A \in$ Form, *if* $\Gamma \nvdash_{\mathbf{K}_3^{\to}} A$, *then there is* $\Gamma^* \subseteq$ Form *such that* (1) $\Gamma \subseteq \Gamma^*$, (2) $\Gamma^* \nvdash_{\mathbf{K}_3^{\to}} A$, *and* (3) Γ^* *is a nontrivial prime theory.*

Proof. Let $B_1, B_2, \ldots$ be an enumeration of all formulas. Now define a sequence of sets of formulas $\Gamma_1, \Gamma_2, \ldots$. Let $\Gamma_1 = \Gamma$ and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \bigcup \{B_{n+1}\} & \text{if } \Gamma_n \bigcup \{B_{n+1}\} \nvdash_{\mathbf{K}_3^{\to}} A, \\ \Gamma_{n+1} = \Gamma_n & \text{otherwise.} \end{cases}$$

We put $\Gamma^* := \bigcup_{n=1}^{\infty} \Gamma_n$. Then:

(1) Follows from the definition of Γ^*.

(2) By a straightforward induction on i. Since $\Gamma_1 = \Gamma$, so $\Gamma_1 \nvdash_{\mathbf{K}_3^{\to}} A$. By the inductive assumption, $\Gamma_i \nvdash_{\mathbf{K}_3^{\to}} A$. If $\Gamma_{i+1} = \Gamma_i$, then $\Gamma_{i+1} \nvdash_{\mathbf{K}_3^{\to}} A$. If $\Gamma_{i+1} \neq \Gamma_i$, then $\Gamma_{i+1} = \Gamma_i \cup \{B_{i+1}\}$. Assume that $\Gamma_i \cup \{B_{i+1}\} \vdash_{\mathbf{K}_3^{\to}} A$. But then $\Gamma_{i+1} = \Gamma_i$, by the definition of the sequence of $\Gamma_1, \Gamma_2, \ldots$. Contradiction. Hence, $\Gamma_i \cup \{B_{i+1}\} \nvdash_{\mathbf{K}_3^{\to}} A$. Thus, if $\Gamma_{i+1} \neq \Gamma_i$, then $\Gamma_{i+1} \nvdash_{\mathbf{K}_3^{\to}} A$. Clearly, that if it holds for Γ_i that $\Gamma_i \nvdash_{\mathbf{K}_3^{\to}} A$, then $\Gamma^* \nvdash_{\mathbf{K}_3^{\to}} A$.

(3) We show the condition (Γ1)–(Γ3).

(Γ1) Since $\Gamma^* \nvdash_{\mathbf{K}_3^{\to}} A$, so it is obvious that $\Gamma^* \neq$ Form.

(Γ2) "$\Rightarrow$" Assume that $\Gamma^* \vdash_{\mathbf{K}_3^{\to}} B$. Then there is i such that $B = B_i$ and for some Γ_i we have $\Gamma_i \vdash_{\mathbf{K}_3^{\to}} B_i$. Assume that $B_i \notin \Gamma_i$. Hence $\Gamma_{i-1} \cup \{B_i\} \vdash_{\mathbf{K}_3^{\to}} A$. But then $\Gamma^* \vdash_{\mathbf{K}_3^{\to}} A$, because $\Gamma_{i-1} \subseteq \Gamma^*$ and $\Gamma^* \vdash_{\mathbf{K}_3^{\to}} B$. Nonetheless, it was proved in (2) that $\Gamma^* \nvdash_{\mathbf{K}_3^{\to}} A$. In this case $B_i \in \Gamma_i$. Therefore, if $\Gamma^* \vdash_{\mathbf{K}_3^{\to}} B$ then $B \in \Gamma^*$.

"$\Leftarrow$" Suppose $B \in \Gamma$, $\Gamma^* \nvdash_{\mathbf{K}_3^{\to}} B$. Then there is i such that $B = B_i$ and for some Γ_{i-1} we have $\Gamma_{i-1} \cup \{B_i\} \vdash_{\mathbf{K}_3^{\to}} A$. Since $\Gamma_{i-1} \subseteq \Gamma^*$, so $\Gamma^* \cup \{B_i\} \vdash_{\mathbf{K}_3^{\to}} A$. From the latter and the fact that $\Gamma^* \nvdash_{\mathbf{K}_3^{\to}} A$, obtain that $B_i \notin \Gamma^*$, that is $B \notin \Gamma^*$. Contradiction. Therefore, $\Gamma^* \vdash_{\mathbf{K}_3^{\to}} B$. Thus, if $B \in \Gamma^*$ then $\Gamma^* \vdash_{\mathbf{K}_3^{\to}} B$.

(Γ3) To show ($\star$): if $B \vee C \in \Gamma^*$ then either $B \in \Gamma^*$ or both $\neg B \in \Gamma^*$ and $C \in \Gamma^*$, we first prove the following statements:

(a) If $B \vee C \in \Gamma^*$ then either $B \in \Gamma^*$ or $\neg B \wedge C \in \Gamma^*$.
(b) If $B \in \Gamma^*$ or $\neg B \wedge C \in \Gamma^*$, then either $B \in \Gamma^*$ or both $\neg B \in \Gamma^*$ and $C \in \Gamma^*$.

Suppose $B \vee C \in \Gamma^*$, but $B \notin \Gamma^*$ and $\neg B \wedge C \notin \Gamma^*$. Since $B \vee C \in \Gamma^*$, so $\Gamma^* \vdash_{\mathbf{K}_3^{\to}} B \vee C$; cf. ($\Gamma$2)). On the other hand, for some i and j we have:

$B = B_i$ and $\neg B \wedge C = B_j$; $\Gamma_{i-1} \cup \{B_i\} \vdash_{\mathbf{K_3^{\rightarrow}}} A$, and $\Gamma_{j-1} \cup \{B_j\} \vdash_{\mathbf{K_3^{\rightarrow}}} A$. Moreover, $\Gamma_{i-1} \subseteq \Gamma^*$ and $\Gamma_{j-1} \subseteq \Gamma^*$. Then $\Gamma^* \cup \{B_i\} \vdash_{\mathbf{K_3^{\rightarrow}}} A$ and $\Gamma^* \cup \{B_j\} \vdash_{\mathbf{K_3^{\rightarrow}}} A$. From the latter and the fact that $\Gamma^* \vdash_{\mathbf{K_3^{\rightarrow}}} B \vee C$, by the rule ($\vee$E$_2$), we obtain that $\Gamma^* \vdash_{\mathbf{K_3^{\rightarrow}}} A$, but according to (2), $\Gamma^* \nvdash_{\mathbf{K_3^{\rightarrow}}} A$. Contradiction. So the statement (a) is proved.

Using the rules ($\wedge$E$_1$) and ($\wedge$E$_2$), it is simple to prove the statement (b). Moreover, using the transitivity and statements (a) and (b), we obtain ($\star$). $\square$

Theorem 3.2 (Completeness). *For all* $\Gamma \subseteq \mathrm{Form}$ *and* $A \in \mathrm{Form}$:

$$\text{if } \Gamma \models_{\mathbf{K_3^{\rightarrow}}} A \text{ then } \Gamma \vdash_{\mathbf{K_3^{\rightarrow}}} A.$$

Proof. Will be provided by contraposition. Let $\Gamma \nvdash_{\mathbf{K_3^{\rightarrow}}} A$. Then, by Lemma 3.3, there is $\Gamma^* \subseteq \mathrm{Form}$ such that $\Gamma \subseteq \Gamma^*$, $\Gamma^* \nvdash_{\mathbf{K_3^{\rightarrow}}} A$, and Γ^* is a nontrivial prime theory. By Lemma 3.2, there is a valuation v_{Γ^*} such that: $v_{\Gamma^*}(B) = 1$, for any $B \in \Gamma$, and $v_{\Gamma^*}(A) \neq 1$. But then $\Gamma \not\models_{\mathbf{K_3^{\rightarrow}}} A$. $\square$

In the light of theorems 3.1 and 3.2 we obtain:

Theorem 3.3 (Adequacy). *For all* $\Gamma \subseteq \mathrm{Form}$ *and* $A \in \mathrm{Form}$:

$$\Gamma \vdash_{\mathbf{K_3^{\rightarrow}}} A \text{ iff } \Gamma \models_{\mathbf{K_3^{\rightarrow}}} A.$$

4. Natural deduction systems for $\mathbf{K_3^{\rightarrow}}$-related logics

For a natural deduction system of the logic $\mathbf{K_3^{\rightarrow 2}}$ we have the following set of rules: (EM), ($\neg\neg$I), ($\neg\neg$E), ($\vee$I$_1$), ($\vee$I$_2$), ($\vee$E$_1$), ($\wedge$I$_1$), ($\wedge$I$_2$), ($\wedge$E$_1$), ($\wedge$E$_3$), ($\neg\vee$I$_1$), ($\neg\vee$I$_2$), ($\neg\vee$E$_1$), ($\neg\vee$E$_3$), ($\neg\wedge$I$_1$), ($\neg\wedge$E).

For a natural deduction system of the logic $\mathbf{K_3^{\leftarrow}}$ we have the following set of rules: (EFQ), ($\neg\neg$I), ($\neg\neg$E), ($\vee$I$_2$), ($\vee$I$_4$), ($\vee$E$_3$), ($\wedge$I$_1$), ($\wedge$E$_1$), ($\wedge$E$_2$), ($\neg\vee$I$_1$), ($\neg\vee$E$_1$), ($\neg\wedge$I$_3$), ($\neg\wedge$I$_4$), ($\neg\wedge$E).

For a natural deduction system of the logic $\mathbf{K_3^{\leftarrow 2}}$ we have the following set of rules: (EM), ($\neg\neg$I), ($\neg\neg$E), ($\vee$I$_1$), ($\vee$I$_2$), ($\vee$E$_1$), ($\wedge$I$_1$), ($\wedge$I$_3$), ($\wedge$E$_2$), ($\wedge$E$_4$), ($\neg\vee$I$_1$), ($\neg\vee$I$_3$), ($\neg\vee$E$_1$), ($\neg\vee$E$_2$), ($\neg\wedge$I$_1$), ($\neg\wedge$E).

For a natural deduction system of the logic $\mathbf{K_3^{w}}$ we have the following set of rules: (EFQ), ($\neg\neg$I), ($\neg\neg$E), ($\vee$I$_3$), ($\vee$I$_4$), ($\vee$I$_5$), ($\vee$E$_4$), ($\wedge$I$_1$), ($\wedge$E$_1$), ($\wedge$E$_2$), ($\neg\vee$I$_1$), ($\neg\vee$E$_1$), ($\neg\wedge$I$_1$), ($\neg\wedge$E).

For a natural deduction system of the logic $\mathbf{K_3^{w2}}$ we have the following set of rules: (EM), ($\neg\neg$I), ($\neg\neg$E), ($\vee \mathrm{I}_1$), ($\vee \mathrm{I}_2$), ($\vee \mathrm{I}_5$), ($\vee \mathrm{E}_1$), ($\wedge \mathrm{I}_1$), ($\wedge \mathrm{I}_2$), ($\wedge \mathrm{I}_3$), ($\wedge \mathrm{E}_3$), ($\wedge \mathrm{E}_4$), ($\neg\vee \mathrm{I}_1$), ($\neg\vee \mathrm{I}_2$), ($\neg\vee \mathrm{I}_3$), ($\neg\vee \mathrm{E}_1$), ($\neg\wedge \mathrm{I}_1$), ($\neg\wedge$E).

Similarly as Theorem 3.3 we obtain:

Theorem 4.1 (Adequacy). *Let* $\boldsymbol{L}$ *be one of the following logics*: $\mathbf{K_3^{\rightarrow 2}}$, $\mathbf{K_3^{\leftarrow}}$, $\mathbf{K_3^{\leftarrow 2}}$, $\mathbf{K_3^{w}}$, $\mathbf{K_3^{w2}}$. *Then for all* $\Gamma \subseteq \mathrm{Form}$ *and* $A \in \mathrm{Form}$:

$$\Gamma \vdash_{\boldsymbol{L}} A \text{ iff } \Gamma \models_{\boldsymbol{L}} A.$$

Concluding remarks

Thus, I have constructed natural deduction systems for weak and intermediate regular logics. Consequently, all three-valued regular logics (both with one and two designated values) are presented in the form of natural deduction systems. However, development of proof-search algorithms suitable for these systems is left untouched and, hopefully, will stimulate further research. One more subject for future investigations could be study of proof theory of four-valued regular logics.

Acknowledgments. I would like to thank an anonymous referee for helpful comments on the earlier version of this paper.

References

[1] Bochvar, D. A., "On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus", *History and Philosophy of Logic* 2 (1981): 87–112. English translation of Bochvar's paper of 1938. DOI: 10.1080/01445348108837023
[2] Fitting, M., "Kleene's three valued logics and their children", *Fundamenta Informaticae* 20 (1992): 113–131.
[3] Karpenko, A. S., *The Development of Many-Valued Logic* (in Russian), LKI, 2010.
[4] Kleene, S. C., "On a notation for ordinal numbers", *The Journal of Symbolic Logic* 3 (1938): 150–155. DOI: 10.2307/2267778
[5] Kleene, S. C., *Introduction to Metamathematics*, Sixth Reprint, Wolters-Noordhoff Publishing and North-Holland Publishing Company, 1971.
[6] Komendantskaya, E. Y., "Functional expressibility of regular Kleene's logics" (in Russian), *Logical Investigations* 15 (2009): 116–128.

[7] Kooi, B., and A. Tamminga, "Completeness via correspondence for extensions of the logic of paradox", *The Review of Symbolic Logic* 5 (2012): 720–730. DOI: 10.1017/S1755020312000196

[8] Łukasiewicz, J., "On three-valued logic", pages 87–88 in *Selected Works*, L. Borkowski (ed.), Amsterdam, North-Holland Publishing Company, 1997 (English translation of Łukasiewicz's paper of 1920).

[9] Mendelson, E., *Introduction to Mathematical Logic*, Fourth Edition, Chapman & Hall, 1997.

[10] Petrukhin, Y. I., "Correspondence analysis for first degree entailment", *Logical Investigations* 22, 1 (2016): 108–124.

[11] Priest, G., "Paraconsistent logic", in *Handbook of Philosophical Logic*, Second Edition, Vol. 6, M. Gabbay and F. Guenthner (eds.), Dordrecht, Kluwer, 2002. DOI: 10.1007/978-94-017-0460-1_4

[12] Tamminga, A., "Correspondence analysis for strong three-valued logic", *Logical Investigations* 20 (2014): 255–268.

[13] Tomova, N. E., "About four-valued regular logics" (in Russian), *Logical Investigations* 15 (2009): 223–228.

[14] Urquhart, A., "Basic many-valued logic", in *Handbook of Philosophical Logic*, Second Edition, Vol. 2, M. Gabbay and F. Guenthner (eds.), Dordrecht, Kluwer, 2001. DOI: 10.1007/978-94-017-0452-6_4

Yaroslav Petrukhin
Department of Philosophy
Moscow State University
Moscow, Russia
yaroslav.petrukhin@mail.ru