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Functorial Semantics for the Advancement of the Science of Cognition

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Abstract

Cognition involves physical stimulation, neural coding, mental conception, and conscious perception. Beyond the neural coding of physical stimuli, it is not clear how exactly these component processes constitute cognition. Within mathematical sciences, category theory provides tools such as category, functor, and adjointness, which are indispensable in the explication of the mathematical calculations involved in acquiring mathematical knowledge. More specifically, functorial semantics, in showing that theories and models can be construed as categories and functors, respectively, and in establishing the adjointness between abstraction (of theories) and interpretation (to obtain models), mathematically accounts for knowing-within-mathematics. Here we show that mathematical knowing recapitulates – in an elementary form – ordinary cognition. The process of going from particulars (physical stimuli) to their concrete models (conscious percepts) via abstract theories (mental concepts) and measured properties (neural coding) is common to both mathematical knowing and ordinary cognition. Our investigation of the similarity between knowing-within-mathematics and knowing-in-general leads us to make a case for the development of the basic science of cognition in terms of the functorial semantics of mathematical knowing.

1. Introduction

Our conscious experiences are a means of knowing. As an illustration consider the following scenario: I know that there is a whiteboard in the classroom. How did I know? I saw the whiteboard. Conscious perception of an object, such as this whiteboard, is mediated by the neural coding of physical stimuli, i.e. sensation. However, there is more to perception than all that is given in the sensation (Albright 2015). In the present example, seeing the whiteboard requires, in addition to the sensation, a mental concept: *whiteboard* (Miller 1999). Thus, cognition can be viewed as a process of knowing that begins with an object and results in knowledge of the object (understood as the ability to distinguish and identify the object), involving (at least) physical stimuli, neural coding, mental concepts, and conscious perception. Beyond the neural coding of physical stimuli, in trying to relate sensation to conception and perception, we encounter the difficult mind-matter problem. It has been recognized that this is a theoretical difficulty, and a clear articulation of the difficulty can facilitate a solution (McGinn 1989, Nagel 1993).

What exactly is the mind-matter problem? Mind is useful in making sense of and maneuvering through the material world. In thinking about things and in making things we think of, we go between the material world of things and the mental realm of thoughts with ease. In the absence of concepts needed for an explicit articulation of the relation between thoughts and things, the ease with which we go between mind and matter becomes the difficult mind-matter problem, with the very comprehensibility of reality remaining beyond the reach of reason (Einstein 1936, Wigner 1960). The mind-matter problem, at its core, is a problem of developing the conceptual repertoire needed to relate two (seemingly) separate universes of discourse or categories: (i) material world, and (ii) its reflections in the mental realm. It is in this context of relating the two categories of objective reality and its subjective reflections that we begin to discern the significance of category theory for providing a scientific account of the effectiveness of mind in the material world (in the words of F. William Lawvere, quoted from Picado 2007, p. 25):

> Everyday human activities, such as building a house on a hill by a stream, laying a network of telephone conduits, navigating the solar system, require plans that can work. Planning any such undertaking requires the development of thinking about space. Each development involves many steps of thought and many related geometrical constructions on spaces. Because of the necessary multistep nature of thinking about space, uniquely mathematical measures must be taken to make it reliable. Only explicit principles of thinking (logic) and explicit principles of space (geometry) can guarantee reliability. The great advance made by the [category] theory invented 60 years ago by Eilenberg and MacLane (1945) permitted making the principles of logic and geometry explicit; this was accomplished by discovering the common form of logic and geometry so that the principles of the relation between the two are also explicit. They thereby solved a problem opened 2300 years earlier by Aristotle with his initial inroads into making explicit the Categories of Concepts. In the 21^{st} century, their solution is applicable not only to plane geometry and to medieval syllogisms, but also to

> infinite-dimensional spaces of transformations, to "spaces" of data, and to other conceptual tools that are applied thousands of times a day. The form of the principles of both logic and geometry was discovered by categorists to rest on "naturality" of the transformations between spaces and the transformations within thought.

In explicating the nature of theories and models of different categories of mathematical objects, Lawvere (1966) put forward the category of categories as a foundation for mathematics. Of immediate significance, for our present purposes, is functorial semantics: the realization that a theory of a category of mathematical objects is an abstract essence (e.g. a type of cohesion, a kind of variation) shared by all objects of the category, that the theory is also a mathematical category, and models are functors interpreting the category embodying the essences (Lawvere 1963, Lawvere and Rosebrugh 2003, pp. 154f, 235f).[1]

To better appreciate the reach of functorial semantics, let us note that experiments do not mechanically lead to theories. Thus, it is extremely difficult to formulate an explicit mathematical account of scientific theorization (Lawvere 2001). This is essentially the problem of developing a scientific account of science. With individual cognition as "scientific theorizing writ small" (Fodor 2006, p. 93), the prospects of developing an explicit science of cognition are apparently bleak: "if God were to tell us how it [mind] works, none of us would understand Him" (Fodor 2006, p. 94). Fortunately, with abstract theories as mathematical categories, the relation between particular objects and general theories is purely mathematical, and hence amenable to scientific investigation. In other words, as an illustration, although we do not have a mathematical account of the temporal emergence of group theory from rotations, we have, in functorial semantics, "a precise mathematical model for a very general scientific process of concept formation" (Lawvere 2013a).

Functorial semantics can be said to constitute an elementary form of ordinary cognition in spelling out the processes involved in mathematical knowing – going from given particulars to measurements of the given particulars, to the conceptualization of the particulars based on their measured properties, to interpretations of the thus formed theories to obtain models, all together resulting in knowledge (Lawvere 1994). This realization is the rationale behind the present in-depth study of the profound resemblance between knowing-within-mathematics and knowing-in-general (Lawvere 2013b).

Mental concepts, which connect physical stimuli to conscious experiences, have been particularly challenging to account for scientifically (Fodor 1998). Cognitive scientists have long recognized the inadequacy of

[1] The Appendices to this paper provide mathematical definitions and informal elaborations of various category theoretic notions used in the main text.

the standard model of a concept as a collection of properties: "Just listing properties does not completely specify the knowledge represented in a concept; people also know about the relations between the properties" (Smith and Medin 1981, p. 83). Mathematicians have also recognized that concepts (e.g. *graph*) cannot be adequately characterized in terms of the properties alone (Lawvere 2003, Lawvere and Schanuel 2009, p. 380). This realization led to refining the Fregean "concept is a set [of properties]" into the present "concept is a [mathematical] category". Concept, when modeled as a category, includes not only properties (as objects) but also their relations (as morphisms; Lawvere 2004; Lawvere and Schanuel 2009, pp. 369f). The basic idea underlying the definition of a mathematical category is (Lawvere 1991a, p. 1):

> A category of objects of thought is not specified until one has specified the category of maps which transform these objects into one another and by means of which they can be compared and distinguished.

In addition to functorial semantics, sketch theory also provides a mathematical account of the relation between mathematical objects and their descriptions in terms of sketches of structures and prototypes of sketches (Bastiani and Ehresmann 1972). Grothendieck's classifying toposes and descent theory also bear on the core concerns of functorial semantics (Barr and Wells 2005, pp. 144ff, Clementino and Picado 2008, p. 22). Since our immediate aim here is to stress the relevance of functorial semantics to cognition, as originally pointed out by Lawvere (1994), in a manner readily accessible to cognitive neuroscientists, we have limited our discussion to elaborating functorial semantics.

Thus, we postpone a detailed comparison of functorial semantics with various models, in contemporary cognitive science, of concept formation such as the prototype theory of concepts (Rosch and Mervis 1975, see also Fodor 1998). More importantly, there is a substantial body of category theoretic studies of cognition in terms of memory evolutive systems (Ehresmann and Vanbremeersch 2007), where the binding problem (Roskies 1999) is modeled as a colimit (Brown and Porter 2003). In our subsequent work, we plan to discuss memory evolutive systems in order to further establish the significance of mathematical knowing for more implicitly developing the science of cognition.

In the following, we provide an accessible category theoretic account of the mathematics of theorizing and modeling involved in acquiring mathematical knowledge. We then elaborate the parallels between ordinary cognition and mathematical knowing, as seen from the perspective of functorial semantics. In closing, we suggest extending the functorial semantics of knowing-within-mathematics to account for knowing-in-general.

2. Mathematical Theories and Models

How do we theorize? There are innumerable facts about any given category of objects. Amongst these facts (expressed as statements), oftentimes there is a small number of statements, referred to as a theory, from which most other true statements can be derived (Lawvere and Rosebrugh 2003, pp. ix-x). However, the statements constituting a theory are but a subjective (in the sense of involving arbitrary choices) presentation of an objective essence (somewhat analogous to the verbal presentation of a concept in a language) in which all objects of the category partake and in terms of which every object of the category can be reconstructed or modeled (Lawvere 2004, p. 8). For example, since every set can be reconstructed in terms of the basic shape of the single-element set $\mathbf{1} = \{\bullet\}$, and since any two functions between sets can be distinguished using $\mathbf{1}$-shaped figures, a single-element set can be considered as the essence of sets (Lawvere 1972, p. 135, Lawvere and Schanuel 2009, p. 214, Reyes *et al.* 2004, p. 30). Theorizing, or the extraction of essences, can be formalized as follows.

Given a category $\mathbf{A}$, functors $P : \mathbf{A} \to \mathbf{B}$ can be thought of as properties of the category $\mathbf{A}$, with values in a category $\mathbf{B}$ (cf. Appendix A1, A2). If we know all these properties, i.e. the functor category $\mathbf{B}^{\mathbf{A}}$ whose objects and morphisms are properties and their relations, then we know all there is to know about $\mathbf{A}$ (Lawvere 1994, p. 49). We can take the totality of properties or the functor category $\mathbf{B}^{\mathbf{A}}$ as the theory of $\mathbf{A}$.

However, oftentimes we notice that certain properties follow from certain other properties. So, we can define theory $\text{Th}(\mathbf{A})$ of $\mathbf{A}$ to be a subcategory of the functor category $\mathbf{B}^{\mathbf{A}}$ of all $\mathbf{B}$-valued properties of $\mathbf{A}$, such that all or most of the properties of $\mathbf{A}$ can be accounted for in terms of the theory $\text{Th}(\mathbf{A})$ of $\mathbf{A}$ (Lawvere 1994, p. 45). In fact, within the functor category $\mathbf{B}^{\mathbf{A}}$, we have a subcategory with representable functors $R : \mathbf{A} \to \mathbf{B}$ as objects, which can be taken as the theory $\text{Th}(\mathbf{A})$ of the category $\mathbf{A}$, since any functor on $\mathbf{A}$ (any property of $\mathbf{A}$) is a colimit (or "sum") of representable functors (Lawvere and Rosebrugh 2003, p. 250).

A simple illustration of the above mathematics of forming mathematical theories is as follows. Given some rectangular figures, we measure their properties such as length and width. Comparison of the measured properties leads to the formation of the concept *square*, where length and width are the same. This basic idea of a concept as a relation between properties can be made more precise and, in the process, shown to underlie much of mathematical practice.

There are various categories of mathematical objects such as sets, functions, dynamical systems, groups, and graphs (see Fig. 1). Corresponding to different categories of mathematical objects, mathematical concepts or theories (e.g. group theory) express the abstract essence of the corre-

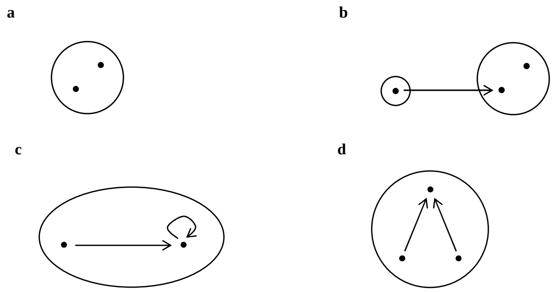

Figure 1: Objects of four mathematical categories: (a) a two-element set, (b) a function from a single-element set to a two-element set, (c) a dynamical system with two states, one of which is an equilibrium state, and (d) a graph consisting of two arrows with a common target dot.

sponding objects (groups). This mathematical scenario, with particular objects and their general concepts, resembles the more elaborate scenario of cognition, wherein we have things and concepts of things expressing the essence(s) of things. Importantly, the mathematical procedure of abstracting the essence of a given category of mathematical objects thereby is well understood (Lawvere 2004), and this mathematical understanding can help clarify more commonplace cognition (Lawvere 1994).

Now we explain the mathematics of going from particulars to generals in terms of graphs (Fig. 1d, Lawvere and Schanuel 2009, pp. 141f, 150f). Let a category **G** of graphs be the given particulars. A first step in extracting the essence of the given particulars is the measurement of properties, such as dots and arrows, of the graphs. A functor

$$D : \mathbf{G} \to \mathbf{S}$$

assigning to each graph X (in the category **G**) its set $\mathrm{X_D}$ of dots in the category **S** of sets is one such measurement (Fig. 2, Lawvere 2013a, Lawvere and Rosebrugh 2003, pp. 236f). This process of measuring dots of graphs is representable, with a graph D consisting of exactly one dot representing the dots functor D (Lawvere and Rosebrugh 2003, pp. 248f; the calculation of objects representing functors is discussed in App. A3). Another measurement is a set-valued functor

$$A : \mathbf{G} \to \mathbf{S}$$

assigning to each graph X (in **G**) its set $\mathrm{X_A}$ of arrows (in **S**). The arrows functor A is represented by a graph A consisting of exactly one arrow (with its source and target dots distinct).

A theory is derived from comparisons of measured properties. With the measured properties as representable functors D and A, represent-

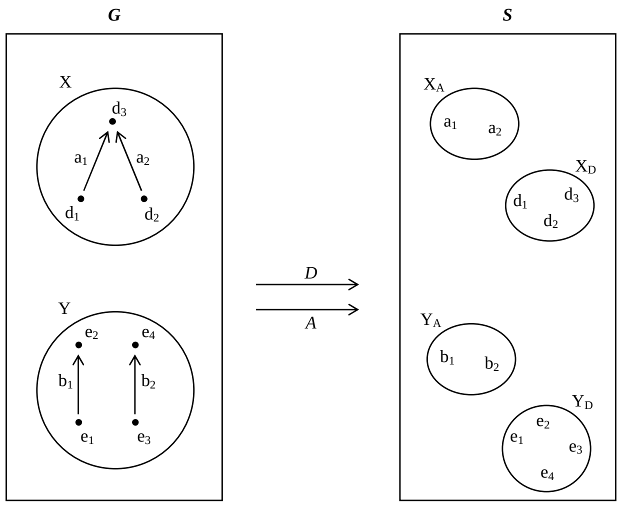

Figure 2: Measurements of properties of objects of the category **G** of graphs, with values in the category **S** of sets, are construed as functors $D, A : \mathbf{G} \to \mathbf{S}$. The dots functor $D : \mathbf{G} \to \mathbf{S}$ assigns to each graph in **G** its set of dots in **S**, i.e. $D(\mathrm{X}) = \mathrm{X_D}$ and $D(\mathrm{Y}) = \mathrm{Y_D}$. The arrows functor $A : \mathbf{G} \to \mathbf{S}$ assigns to each graph in **G** its set of arrows in **S**, i.e. $A(\mathrm{X}) = \mathrm{X_A}$ and $A(\mathrm{Y}) = \mathrm{Y_A}$.

ing graphs D and A along with the graph morphisms between D and A constitute a theory of graphs. There are two morphisms

$$s : \mathrm{D} \to \mathrm{A}$$
$$t : \mathrm{D} \to \mathrm{A}$$

from the graph D to the graph A (Lawvere and Schanuel 2009, p. 150). Thus a two-object (graphs D and A) and two-morphism (graph morphisms s and t) subcategory **T** of the category **G** of graphs is a theory of graphs (Fig. 3). The theory **T** of graphs can be thought of as the essence of graphs.

The theory of graphs, i.e. the category **T**, can be interpreted into a background category, such as the category **S** of sets to obtain models. A model of the theory **T** is a contravariant functor (see App. A2):

$$M : \mathbf{T} \to \mathbf{S}$$

interpreting the two parallel morphisms $s : \mathrm{D} \to \mathrm{A}$ and $t : \mathrm{D} \to \mathrm{A}$ in the category **T** as a parallel pair of functions in the category **S** of sets:

$$\mathit{source} : \text{arrows} \to \text{dots}\,,$$
$$\mathit{target} : \text{arrows} \to \text{dots}\,.$$

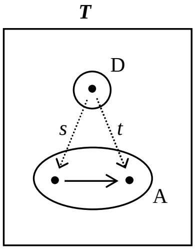

Figure 3: A theory of graphs is a category **T** consisting of two graphs (D, A) and two graph morphisms (s, t : D $\rightarrow$ A). The morphisms s, t are inclusions of the dot in graph D as source, target dot of the arrow in A.

More explicitly, a model of a graph X in **G** can be constructed based on the objects and morphisms in the theory **T** as follows. The set of all morphisms from the dot graph D (in the theory **T**) to a graph X is the set of all dots in the graph X. Similarly, the set of all morphisms from arrow A (in **T**) to X is the set of all arrows in X. Any morphism from A to X can be pre-composed with either one of the two morphisms (s, t : D $\rightarrow$ A) from D to A. Thus we obtain a pair of sets (arrows, dots) equipped with a parallel pair of functions (*source, target*), specifying for each arrow (in the domain set arrows) its source, target dot (in the codomain set dots), as a model of a graph (Fig. 4).

Theories and models of (categories of) functions, dynamical systems, groups, and reflexive graphs (treated as particulars) can also be obtained along the above lines (Fig. 5, Lawvere and Rosebrugh 2003, pp. 154f; Lawvere and Schanuel 2009, pp. 135ff). The theory of functions consists of two objects $U : \mathbf{0} \rightarrow \mathbf{1}$ and $\mathbf{1} : \mathbf{1} \rightarrow \mathbf{1}$ and one morphism $U \rightarrow \mathit{1}$ between the two objects, where $\mathbf{0} = \{\}$ and $\mathbf{1} = \{\bullet\}$ (Lawvere and Rosebrugh 2003, pp. 114ff). The two objects U and $\mathit{1}$ of the theory of functions represent the measurement of codomain and domain, respectively, of functions. A model of the theory of functions (in the category of sets) is a function $f : \mathrm{X}_1 \rightarrow \mathrm{X}_\mathrm{U}$ from a domain set X_1 to a codomain set X_U.

The theory of dynamical systems consists of one object $(\mathrm{N}, n : \mathrm{N} \rightarrow \mathrm{N})$ and one morphism $(n : \mathrm{N} \rightarrow \mathrm{N})$, where $N = \{0, 1, 2, ...\}$ and $n(\mathrm{n}) = \mathrm{n} + 1$. The object $(\mathrm{N}, n : \mathrm{N} \rightarrow \mathrm{N})$ of the theory of dynamical systems represents the measurement of states of dynamical systems (Lawvere and Schanuel 2009, pp. 177ff). A model of the theory of dynamical systems is a set of states equipped with an endomap. The theory of groups has one object $(\mathrm{Z}, z : \mathrm{Z} \rightarrow \mathrm{Z})$ and one invertible endomorphism $(z : \mathrm{Z} \rightarrow \mathrm{Z})$, where $Z = \{..., -1, 0, 1, ...\}$ and $z(\mathrm{z}) = \mathrm{z} + 1$ satisfying $z \circ z' = z' \circ z = 1_\mathrm{Z}$ with $z'(\mathrm{z}) = \mathrm{z} - 1$ and $\circ$ denotes the composition of morphisms (Lawvere and Rosebrugh 2003, p. 24). Models of the group theory (in the category of sets) are permutations (Lawvere 1994, p. 55). Both the theories of dynam-

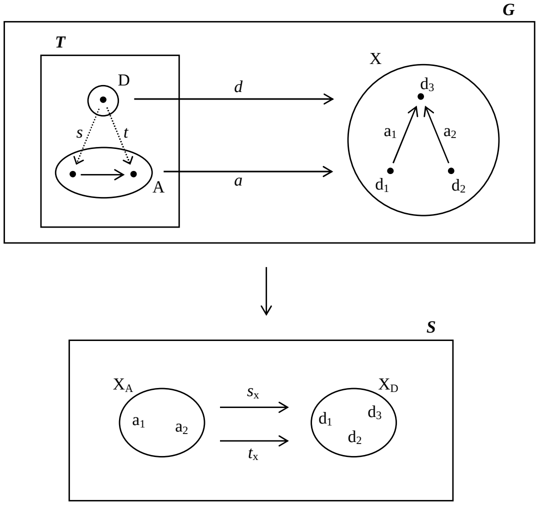

Figure 4: A graph morphism $a : \mathrm{A} \to \mathrm{X}$ from an object A (in the theory **T** of the category **G** of graphs) to a graph X is an A-shaped figure in X, and a morphism $d : \mathrm{D} \to \mathrm{X}$ is a D-shaped figure in X. The set of all morphisms from A to X is the set $\mathrm{X_A}$ of all arrows in X, and the set of all morphisms from D to X is the set $\mathrm{X_D}$ of all dots in X. Any morphism from A to X can be pre-composed with a morphism from D to A to obtain a morphism from D to X. Pre-composing an arrow $a : \mathrm{A} \to \mathrm{X}$ with source $s : \mathrm{D} \to \mathrm{A}$ gives a morphism from D to X, which is the source dot of the arrow $a : \mathrm{A} \to \mathrm{X}$; pre-composing with target $t : \mathrm{D} \to \mathrm{A}$ gives the target dot. Thus we obtain a parallel pair of functions $s_x, t_x : \mathrm{X_A} \to \mathrm{X_D}$, specifying for each arrow (in the set $\mathrm{X_A}$ of arrows) its source, target dot (in the set $\mathrm{X_D}$ of dots), as a model of the graph X in the category **S** of sets.

ical systems and of groups consist of one object and one endomorphism on the only object. The only endomorphism in the theory of groups is required to be invertible, while that in the theory of dynamical systems need not satisfy any additional conditions besides being an endomorphism.

The theory of reflexive graphs is a category consisting of two graphs (point P and arrow A) and three graph morphisms ($s, t : \mathrm{P} \to \mathrm{A}, r : \mathrm{A} \to \mathrm{P}$), where morphism r is the common retraction of both source s and target t, i.e. $r \circ s = r \circ t = 1_\mathrm{P}$ (Lawvere 1994, pp. 46f). The objects P and A of the theory of reflexive graphs represent the measurement of points and arrows, respectively, of reflexive graphs. A model of the theory of reflexive graphs (in the category of sets) is a pair of sets equipped with three functions (f, g, i) such that the function i is the common section of the functions f and g, i.e. $f \circ i = g \circ i = 1$. Broadly, theories can be construed as subcategories of the category of models, and models are

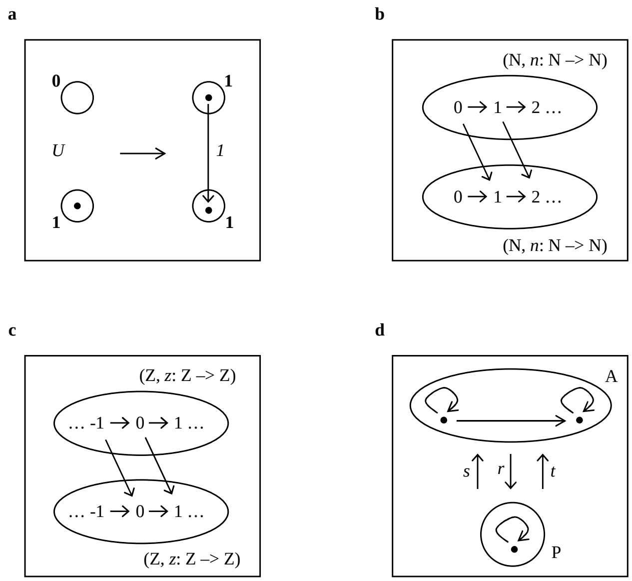

Figure 5: (a) The theory of functions consists of two functions $U : \mathbf{0} \to \mathbf{1}$ and $\mathit{1} : \mathbf{1} \to \mathbf{1}$ along with the morphism $U \to \mathit{1}$. (b) The theory of dynamical systems has one object $(\mathrm{N}, n : \mathrm{N} \to \mathrm{N})$ and one endomorphism $(n : \mathrm{N} \to \mathrm{N})$, where $N = \{0, 1, 2, ...\}$ and $n(\mathrm{n}) = \mathrm{n} + 1$. (c) The theory of groups has one object $(\mathrm{Z}, z : \mathrm{Z} \to \mathrm{Z})$ and one invertible endomorphism $(z : \mathrm{Z} \to \mathrm{Z})$, where $Z = \{..., -1, 0, 1, ...\}$ and $z(\mathrm{z}) = \mathrm{z} + 1$. (d) The theory of reflexive graphs consists of two graphs (P, A) and three graph morphisms $(s, t : \mathrm{P} \to \mathrm{A}, r : \mathrm{A} \to \mathrm{P})$, where r is the common retraction of both s and t.

contravariant functors from the theory to a background category (Lawvere 2004, p. 19). Within this mathematical framework, a theory of a thing is a part of the thing (e.g. graph theory is a subcategory of the category of graphs). This is the quintessence of the mathematical method of trying to find the part of a thing that determines everything about the thing (Lawvere 1972, p. iv, 9f).

3. Functorial Semantics for Cognition

Substantial progress has been made in characterizing the neural coding of physical stimuli and in identifying the neural correlates of conscious experiences (Albright *et al.* 2000). However, we still do not have a comprehensive scientific account of cognition (Fodor 2006). It is not clear how to go from the neural measures of physical properties of given objects to their representations in mind and consciousness, not to mention their expression in language. Certainly, we are in need of a deep insight into cognition. Mathematics can enable us to define this insight as the science of cognition. Recognizing that both cognition and mathematics are about knowing, we look at how we acquire mathematical knowledge and use its characteristics to illuminate and develop the field of cognition. For example, as we have just seen, with graphs as given particulars, we measure properties (e.g. set of dots, set of arrows) of the given graphs (Figs. 1d, 2). Based on comparisons of these measured properties, a theory of graphs is formed (Fig. 3). Interpreting the theory into a background (e.g. sets) yields models of the given particulars (Fig. 4).

Correspondences between knowing-within-mathematics and knowing-in-general can be understood as follows. Looking at given stimuli results in neural sensations; a mental concept of the given stimuli results from comparisons of sensations; interpreting the thus formed mental concept (of the given stimuli) into consciousness (background) results in perception (of the stimuli). In drawing parallels between

stimuli – sensations – concepts – percepts

and

particulars – properties – theories – models

of cognitive neuroscience and mathematical sciences, respectively, we see the profound resemblance between mathematical knowing and ordinary cognition. The process of abstracting theories (mental concepts) from particulars (physical stimuli) via measurement of properties (neural coding) and interpreting theories to obtain models (conscious percepts) is common to both mathematical knowing and ordinary cognition.

In the light of this correspondence, functorial semantics constitutes a unifying framework within which all four categories – physics, brain, mind, and consciousness – can be conceptualized as a coherent whole without having to reduce one into any other. More explicitly, the physical world, brain, mind, and consciousness are conceptualized as four mathematical categories corresponding to physical stimuli, neural codes, mental concepts, and conscious percepts, respectively, and their relationships (measurement, abstraction, and interpretation) as functors between these categories.

Although the physical world, brain, mind, and consciousness are all formalized as categories, the brain (by virtue of being a category of measurements of the category of physical world) is a higher-order category with functors on the world as objects. Along these lines, mind (by virtue of being a category with comparisons of measured properties as objects) and consciousness (by virtue of being a category of interpretations of the objects and morphisms of the mind) are endowed with increasingly higher-order structure.

By elucidating the common elemental form of cognition, functorial semantics can provide an overarching mathematical framework within which the relation between not only things and thoughts, but also between mental concepts and conscious percepts can be formalized. By comparing mental concepts to (mathematical) theories and conscious percepts to (mathematical) models, we can put forward adjointness, a specific form of interdependence, as the relation between concepts and percepts (see App. A4). Note that within this framework, unlike the supposed opposition syntax versus semantics, the opposition is between abstraction (of general concepts from particulars) and interpretation (of abstract theories to obtain concrete models; Lawvere 2004, pp. 8, 16; see also Lawvere 2002, 2006, 2010). Moreover, likening words (needed to communicate concepts) to presentations of theories (needed to calculate), the mathematical relationship between theory, semantics (of models), and syntax (of presentations) can be used to clarify the totality of cognition: stimuli – sensations – concepts – percepts – words.

One immediate theoretical implication of the above idealization of cognition is with regard to the nature of the objective logic of cognition, which is the logic intrinsic to the universe of discourse of cognition (as idealized). The logic of cognition, when modeled as a four-stage process that begins with physical stimuli and results in the conscious perception of stimuli via neural coding and mental concepts, is richer than the Boolean logic of sets (cf. Fig. 6).

Firstly, let us note that the objective logic of a category is embodied in its truth value object (Lawvere and Schanuel 2009, pp. 342ff, 352ff), which can be calculated in terms of the theoretical essences of the category. The truth value object of the category of sets is a two-element set $\Omega_{\mathrm{S}} = \{\mathrm{false}, \mathrm{true}\}$, which corresponds to the two parts ($\mathbf{0} = \{\}, \mathbf{1} = \{\bullet\}$) of the essence $\mathbf{1} = \{\bullet\}$ of the category of sets (Reyes *et al.* 2004, pp. 93ff). The logical operations of negation, conjunction, and disjunction can be characterized in terms of the truth value object. For example, negation is an endomap *not*: $\Omega_{\mathrm{S}} \to \Omega_{\mathrm{S}}$, with $\Omega_S(\mathrm{false}) = \mathrm{true}$, $\Omega_S(\mathrm{true}) = \mathrm{false}$.

Now, as a first approximation, one may identify the cognition of an object with the end-result of a percept modeled as a set of features, for instance: moon = {round, white}. With percepts modeled as sets, the logic of percepts is the Boolean logic of sets. Modeling perception as a measurement of physical stimuli is somewhat more realistic. When percepts are modeled as functions $f : \mathrm{A} \to \mathrm{B}$, their truth-value object has an intermediate truth value between "false" and "true" (Lawvere and Rosebrugh 2003, p. 117).

Nevertheless, various extra-sensory influences on perception have highlighted the inadequacies of the measurement-device conception of percep-

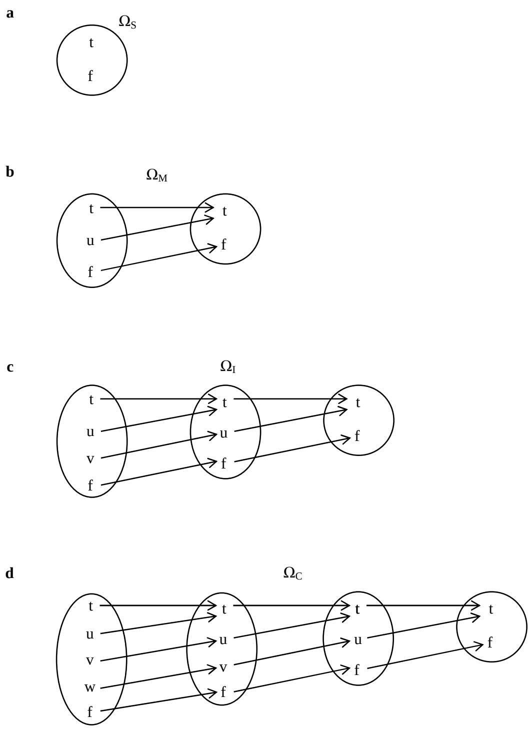

Figure 6: (a) The truth-value object of the category of sets is a two-element set $\Omega_S = \{\text{false, true}\}$. (b) The truth-value object of the category of functions is a function $\Omega_\mathbf{M}$ whose internal diagram is shown. There is a global truth value "u" in between false ("f") and true ("t"). (c) The truth-value object Ω_I of the category of two sequential functions admits two intermediate truth values "u" and "v". (d) The truth-value object Ω_C of the category of three sequential functions has five degrees of truth.

tion (Mausfeld 2002), which has led to the contemporary understanding of perception as involving a process of interpretation in addition to measurement (Albright 2015). Thus, two sequential functions $A - f \to B - g \to C$ objectifying interpretation after measurement constitute a closer approximation, and the corresponding category admits four global truth values (Linton 2004). Our model involving measurement (of properties) followed by abstraction (of concepts) followed by interpretation (giving rise to percepts) is a further refinement. The resulting category of three sequential functions $A - f \to B - g \to C - h \to D$ has a truth-value object with five degrees of truth globally.

In addition to the degrees of truth, the logic of cognition, even at the level of a single process of measurement, admits varieties of negation (*not, non*; Lawvere 1986, 1991b) and contradiction (i.e., X *and non*-X $\neq 0$; Lawvere 1994, p. 48; Lawvere and Rosebrugh 2003, p. 201). Let us note that the processes of sensation, abstraction, and interpretation are much more structured than mere functions. The above simplification is intended to indicate the possibility of calculating the objective logic of cognition. We are investigating the differences between Boolean logic and the logic of cognition (realistically objectified) in greater detail, so as to evaluate real-world situations along the lines of the recent fascinating applications of quantum logic to cognitive science (Roy 2016).

4. Discussion

Mathematics is a time-tested method of knowing. As such, the mathematics of acquiring mathematical knowledge, i.e. functorial semantics, can be used to gain insights into the more elaborate ordinary cognition. Sherrington's recognition of neurons (weighted summation of synaptic inputs, i.e. dot product) as an elementary form of the integrative action of the brain paved the way for awe-inspiring progress in neuroscience (Albright *et al.* 2000, p. S3). In a similar vein, recognizing functorial semantics as an elementary form of cognition can help us put in place the needed foundations for the advancement of the science of knowing (Lawvere 1994).

However, understanding examples such as cohomology, in terms of which functorial semantics is motivated, requires decades of mathematical training. As a result, the relevance of functorial semantics to cognitive science remains to be appreciated. One of the main objectives of our paper is to present functorial semantics in a manner accessible to working cognitive neuroscientists. Herein, we have introduced functorial semantics by using the category of graphs, which is relatively easy to understand. Using elementary categories to introduce functorial semantics has allowed us to focus on the mathematics of knowing: forming mathematical concepts (based on measured properties) of the given particulars, and interpreting the thus formed theory (into a background) to obtain models. Our approach has been inspired by Lawvere and Schanuel's (2009) introduction of category theory via elementary categories, which not only made category theory accessible but also brought out the categorical concepts implicit in elementary mathematics.

The main conclusion to be drawn from the present exercise is the significance of knowing-within-mathematics for the study of knowing-in-general, especially with regard to two foundational questions:

1. What is the relation between the material world of things and the mental realm of thoughts?
2. What is the relation between mental concepts and conscious percepts?

Our conceptualization of things follows from measurements of the things under study. For instance, measurement of dots and arrows of graphs led us to a two-object and two-morphism theory of graphs (Fig. 3). If, instead of arrows, we measured loops of graphs, then we would have obtained a two-object and one-morphism category as graph theory. The way we see things is a consequence of our conceptualization. A model of a graph based on a two-object and two-morphism theory of graphs can be different from one based on a two-object and one-morphism graph theory. For example, from the perspective of two-object and one-morphism graph theory, a graph with two dots is indistinguishable from a graph with one arrow (with its source and target dots distinct), since both have zero loops and two dots.

In light of the kinship between mathematical knowing and ordinary cognition that we have highlighted, the idea that scientific theorizing is separate from ordinary cognition (Pinker 2005), which has been put forth to rationalize the shortcomings of contemporary cognitive science (Fodor 2001), is untenable. Functorial semantics, with its well-defined transformations between particulars and generals, can be elaborated from knowing-within-mathematics (comprised of particulars, measured properties, abstract theories, and concrete models) to encompass cognition (comprised of physical stimuli, neural sensations, mental concepts, and conscious percepts).

This research program involves the investigation of the mathematical structure of the empirical spaces of physical stimuli, neural codes, mental concepts, and conscious percepts from the perspective of functorial semantics: (i) construing physical stimuli (e.g. sound, light), neural codes, mental concepts, and conscious percepts as categories based on their structure (cf. Balduzzi and Tononi 2009, Clark 1993, Gärdenfors 2004, Stanley 1999, Wandell 1995), (ii) construing mediating processes (sensation: physical stimuli $\rightarrow$ neural codes; abstraction: neural codes $\rightarrow$ mental concepts; interpretation: mental concepts $\rightarrow$ conscious percepts) as functors, (iii) examining sensations (e.g. population coding, opponent coding) to see if they can be construed as representable functors, (iv) examining mental concepts to see if they can be construed as colimits of sensations, and (v) examining the empirical relations between abstraction and interpretation to see if they are characterized by adjointness. Extending functorial semantics beyond mathematical knowing to account for ordinary cognition, although monumental, is proposed as a pathway for the advancement of the science of cognition.

Appendices

A1. Category

Consider four neurons A, B, C, D, and three synapses $f : \mathrm{A} \to \mathrm{B}$, $g : \mathrm{B} \to \mathrm{C}$, $h : \mathrm{C} \to \mathrm{D}$ between these neurons. We can compose two synapses if the origin of the second synapse is the same as the destination of the first, and obtain a synaptic path from the origin of the first synapse to the destination of the second. The origin of $g : \mathrm{B} \to \mathrm{C}$ is same as the destination of $f : \mathrm{A} \to \mathrm{B}$, so we can compose them to obtain a path from A to C: $g \circ f = \mathrm{A}{-}f \to \mathrm{B}{-}g \to \mathrm{C} = \mathrm{A} \to \mathrm{C}$, where the composition $g \circ f$ is read as "g after f".

Next, for each neuron, there is an identity synaptic pathway with the neuron as both origin and destination. For example, we have the identity path $1_\mathrm{A} : \mathrm{A} \to \mathrm{A}$ on neuron A. The composition of paths between neurons satisfies two identity laws: pre-composing any path $f : \mathrm{A} \to \mathrm{B}$ with the identity path on its origin A gives the same path: $f \circ 1_\mathrm{A} = \mathrm{A} - 1_\mathrm{A} \to \mathrm{A} - f \to \mathrm{B} = \mathrm{A} - f \to \mathrm{B}$, and post-composing with the identity path on its destination B gives the same path: $1_\mathrm{B} \circ f = \mathrm{A} - f \to \mathrm{B} - 1_\mathrm{B} \to \mathrm{B} = \mathrm{A} - f \to \mathrm{B}$.

Furthermore, composition satisfies the associative law, i.e. given a triple of composable paths: $\mathrm{A} - f \to \mathrm{B} - g \to \mathrm{C} - h \to \mathrm{D}$, we can first compose the paths f and g and then compose the composite path $g \circ f$ with path h to obtain the composite $h \circ (g \circ f)$. Alternatively, we can first compose the paths g and h and then compose the composite path $h \circ g$ with path f to obtain the composite $(h \circ g) \circ f$. Either way, we get the same composite path from A to D, i.e. $h \circ (g \circ f) = (h \circ g) \circ f = h \circ g \circ f : \mathrm{A} \to \mathrm{D}$. Thus, with the composition of paths satisfying both the identity laws and the associative law, the neurons along with their synapses form a mathematical category (Lawvere and Schanuel 2009, p. 21).

Now consider graphs and graph morphisms. A graph morphism $f : \mathrm{X} \to \mathrm{Y}$ is a way of placing the arrows and dots of the domain graph X on the arrows and dots of the codomain graph Y *without tearing apart* the domain graph (Lawvere and Schanuel 2009, pp. 141f, 210). For example, placing the two arrows in graph X (of Fig. 2) on either one of the two arrows in graph Y is a morphism, but placing the two arrows in X on two different arrows in Y is not a morphism. When a graph X is modeled as a pair of functions $s_x, t_x : \mathrm{X_A} \to \mathrm{X_D}$ assigning to each arrow in the set $\mathrm{X_A}$ of arrows of graph X its source, target dot in the set $\mathrm{X_D}$ of dots (Fig. 4), a graph morphism $f : \mathrm{X} \to \mathrm{Y}$ is a pair of functions $f_\mathrm{A} : \mathrm{X_A} \to \mathrm{Y_A}, f_\mathrm{D} : \mathrm{X_D} \to \mathrm{Y_D}$ satisfying $f_\mathrm{D} \circ s_x = s_y \circ f_\mathrm{A}$ and $f_\mathrm{D} \circ t_x = t_y \circ f_\mathrm{A}$, which ensures preservation of source and target relations.

With graph morphisms thus defined, we find that the composite of graph morphisms is a graph morphism. Given another morphism $g : \mathrm{Y} \to \mathrm{Z}$, i.e. a pair of functions $g_\mathrm{A} : \mathrm{Y_A} \to \mathrm{Z_A}$ and $g_\mathrm{D} : \mathrm{Y_D} \to \mathrm{Z_D}$

satisfying: $g_D \circ s_y = s_z \circ g_A$ and $g_D \circ t_y = t_z \circ g_A$, we find that $g \circ f : X \to Z$ is a morphism, i.e. the composite functions $g_A \circ f_A : X_A \to Z_A$ and $g_D \circ f_D : X_D \to Z_D$ satisfy: $g_D \circ f_D \circ s_x = s_z \circ g_A \circ f_A$ and $g_D \circ f_D \circ t_x = t_z \circ g_A \circ f_A$. Furthermore, composition of graph morphisms satisfies both the identity laws and the associative law. Thus, graphs and graph morphisms form a category.

A2. Functor

Consider the perception of light of different wavelengths as different colors. As a first approximation, we can model perception as a function (from a domain set of wavelengths to a codomain set of colors) assigning a perceived color to each wavelength. However, note that in seeing colors, sometimes large wavelength differences are mapped to small perceptual differences (within a color category of, say, red), while small wavelength differences are mapped to large differences in perception (across the color categories of, say, red versus yellow). So, we need a mathematical construct which not only maps elements (wavelengths) to elements (colors) but also maps relations between wavelengths (physical differences) to relations between colors (perceptual differences). The mathematical construct of a functor, mapping objects to objects and relations between objects (morphisms) to relations between objects (morphisms), does just this (Lawvere and Schanuel 2009, p. 369). The notion of a functor between categories can be thought of as a generalization of the more familiar notion of a function between sets.

To see the mathematical construct of a functor more clearly, consider the process of translating text from one language into another. Based on, say, looking at the Telugu-language meaning of English words in an English → Telugu dictionary, we can think of translation as an assignment of a Telugu-language word to a word in English language. Modeling translation as a function

$$translation : \text{words}_{\text{English}} \to \text{words}_{\text{Telugu}}$$

(assigning an element in the codomain set $\text{words}_{\text{Telugu}}$ to each element in the domain set $\text{words}_{\text{English}}$) is a first approximation. In addition to translating words, we also translate sentences to sentences. When translating a sentence (e.g. "Sita married Rama") in one language to a sentence in another language we make sure that subjects (Sita) go to subjects and objects (Rama) to objects. Thus we can summarize the process of translation as:

$$translation(subject(\text{sentence})) = subject(translation(\text{sentence}))$$
$$translation(object(\text{sentence})) = object(translation(\text{sentence}))$$

These equations state that the subject (object) of a translated sentence is same as the translation of the subject (object) of the sentence.

Oftentimes we combine two sentences into a sentence (apple is fruit *and* fruit is edible → apple is edible), and translation is respectful of such combinations. If we translate two sentences S and R (in a language L) into S' and R' (in another language L'), then the combined sentence S + R is translated as S' + R', i.e.

$$translation\,(\mathrm{S}+\mathrm{R}) = translation\,(\mathrm{S}) + translation\,(\mathrm{R})$$

Additionally, if we imagine an identity sentence associated with every word, i.e. for each word such as "fruit" we imagine an identity sentence "fruit is fruit", then the translation of an identity sentence is the same as the identity sentence of the translated word. These requirements of preserving the subject, object, composition, and identity structure in the process of translation constitute the definition of a functor.

Formally, a covariant functor $F : \mathbf{A} \to \mathbf{B}$ assigns to each object A and to each morphism $f : \mathrm{A} \to \mathrm{A}'$ in the domain category $\mathbf{A}$ an object $F(\mathrm{A})$ and a morphism $F(f) : F(\mathrm{A}) \to F(\mathrm{A}')$, respectively in the codomain category $\mathbf{B}$, such that identities and composites are preserved, i.e. $F(1_{\mathrm{A}}) = 1_{F(\mathrm{A})}$ and $F(g \circ f) = F(g) \circ F(f)$. Measurements of graph properties, i.e. the dots functor $D : \mathbf{G} \to \mathbf{S}$ and the arrows functor $A : \mathbf{G} \to \mathbf{S}$ are both covariant functors (Fig. 2). The dots functor $D : \mathbf{G} \to \mathbf{S}$ assigns to each graph X in the category $\mathbf{G}$ of graphs its set $\mathrm{X_D}$ of dots in the category $\mathbf{S}$ of sets: $D(\mathrm{X}) = \mathrm{X_D}$, and to each morphism $f : \mathrm{X} \to \mathrm{Y}$ the function $f_{\mathrm{D}} : \mathrm{X_D} \to \mathrm{Y_D}$ between the corresponding sets of dots. Since the domain (codomain) of the set function f_{D} to which a graph morphism f is assigned is the value of the dots functor D at the domain (codomain) of the graph morphism, the dots functor D is a covariant functor.

A contravariant functor $F : \mathbf{A} \to \mathbf{B}$ assigns to each morphism $f : \mathrm{A} \to \mathrm{A}'$ in $\mathbf{A}$ a morphism $F(f) : F(\mathrm{A}') \to F(\mathrm{A})$ in $\mathbf{B}$ (in the opposite direction), and satisfies $F(g \circ f) = F(f) \circ F(g)$. A model of the graph theory $\mathbf{T}$ is a contravariant functor $M : \mathbf{T} \to \mathbf{S}$ interpreting the category $\mathbf{T}$ into a background category $\mathbf{S}$ of sets. The category $\mathbf{T}$ consists of two objects (dot D, arrow A), and two morphisms ($s : \mathrm{D} \to \mathrm{A}$, $t : \mathrm{D} \to \mathrm{A}$), see Fig. 3. The functor M maps the two objects D, A to two sets: $M(\mathrm{D})$ = dots, $M(\mathrm{A})$ = arrows, and the two morphisms $s : \mathrm{D} \to \mathrm{A}$, $t : \mathrm{D} \to \mathrm{A}$ to the two functions *source* : arrows → dots, *target* : arrows → dots (Fig. 4). The domain (codomain) set of the function *source* : $M(\mathrm{A}) \to M(\mathrm{D})$, to which the graph morphism $s : \mathrm{D} \to \mathrm{A}$ is mapped, is the value of the functor M at the codomain (domain) object of the morphism $s : \mathrm{D} \to \mathrm{A}$. The same is the case with the function *target* : $M(\mathrm{A}) \to M(\mathrm{D})$ to which the morphism $t : \mathrm{D} \to \mathrm{A}$ is mapped. Hence, model $M : \mathbf{T} \to \mathbf{S}$ is a contravariant functor.

A3. Representable Functor

The basic idea underlying the mathematical construct of a representable functor is that in certain cases it is possible to objectify processes of measurement. The measurement of properties of elements of a set is often formalized as a function $f : \mathrm{A} \to \mathrm{B}$. For example, with A = {banana, apple} and B = {green, yellow, red}, *color* : A → B is a B-valued property of A, where *color*(banana) = yellow and *color*(apple) = red. In a similar way, measurements of properties of objects of a category **A** are formalized as functors $F : \mathbf{A} \to \mathbf{B}$. If there is an object in the domain category of a functor representing the functor, then the functor is representable.

We will now show that the dots functor $D : \mathbf{G} \to \mathbf{S}$, which assigns to each graph X in the domain category **G** of graphs its set of dots $\mathrm{X_D}$ in the codomain category **S** of sets, is a representable functor. The dots functor $D : \mathbf{G} \to \mathbf{S}$ is representable if there is a graph D in the domain category **G** and an element d in the set $D(\mathrm{D})$ of dots of the graph D, such that, for any graph X, the function from the set $\mathrm{X^D}$ (of all graph maps from D to X) to the set $D(\mathrm{X}) = \mathrm{X_D}$ (of all dots of the graph X), which assigns to each graph map $x : \mathrm{D} \to \mathrm{X}$ in the set $\mathrm{X^D}$ the element $D(x)(\mathrm{d})$ in the set $\mathrm{X_D}$, is an isomorphism of sets (Lawvere and Rosebrugh 2003, pp. 248f).

Taking a single-dot graph as the graph D, with the only dot of the graph D as the element d of the set $D(\mathrm{D})$ of dots of the graph D, we find that the function $f : \mathrm{X^D} \to \mathrm{X_D}$ assigning to each map $x : \mathrm{D} \to \mathrm{X}$ the element $D(x)(\mathrm{d}) = \mathrm{x}$ is an isomorphism. Since there is only one dot in the graph D, a graph map $x : \mathrm{D} \to \mathrm{X}$ is a listing of a dot x in a graph X. Also, for each dot x in a graph X, there is a graph map $x : \mathrm{D} \to \mathrm{X}$ with the dot x as its value at the only dot in the domain graph D. Thus, graph maps from a single-dot graph D to the graph X are in one-to-one correspondence with the dots of the graph X. Formally, functions $f : \mathrm{X^D} \to \mathrm{X_D}$ (with $f(x : \mathrm{D} \to \mathrm{X}) = D(x)(\mathrm{d}) = \mathrm{x}$) and $g : \mathrm{X_D} \to \mathrm{X^D}$ (with $\mathrm{g(x)} = x : \mathrm{D} \to \mathrm{X}$) satisfy both: $f \circ g = 1_{\mathrm{X_D}}$ and $g \circ f = 1_{\mathrm{X^D}}$ (since $f \circ g(\mathrm{x}) = f(x : \mathrm{D} \to \mathrm{X}) = \mathrm{x}$ and $g \circ f(x : \mathrm{D} \to \mathrm{X}) = g(\mathrm{x}) = x : \mathrm{D} \to \mathrm{X}$). Thus, the dots functor, assigning to each graph its set of dots, is represented by the single-dot graph.

It can also be shown that the arrows functor $A : \mathbf{G} \to \mathbf{S}$, assigning to each graph X in **G** its set $\mathrm{X_A}$ of arrows in the category **S** of sets, is represented by a graph A consisting of one arrow (Figs. 2, 3). In the category of functions, the two functions $U : \mathbf{0} \to \mathbf{1}$ and $\mathit{1} : \mathbf{1} \to \mathbf{1}$, where $\mathbf{0} = \{\}$ and $\mathbf{1} = \{\bullet\}$, represent the measurement of codomain and domain, respectively, of functions (Fig. 5a). In the category of dynamical systems, the dynamical system (N, n: N → N), where $N = \{0, 1, 2, ...\}$ and n(n) = n + 1, represents the measurement of the states of a dynamical system (Fig. 5b; Lawvere and Schanuel 2009, pp. 177ff).

A4. Adjointness

The notion of adjointness can be introduced in terms of an opposite pair of sorting and exemplification processes between particulars and generals (Lawvere and Schanuel 2009, pp. 81ff). Consider two sets A = {cat, dog, rose, lily}, B = {animal, flower}. A function f : A → B, with f(cat) = animal, f(dog) = animal, f(rose) = flower, and f(lily) = flower, is a sorting of A. In the opposite direction, a function g : B → A, with g(animal) = cat and g(flower) = rose is an exemplification of B. The two functions f : A → B and g : B → A satisfy $f \circ g = 1_{\mathrm{B}}$ and $g \circ f = e$ (where 1_{B} is the identity function on B and e is an idempotent endomap on A), which formalizes the interdependence of sorting and exemplification: an example of a general (animal) is one of the particulars (cat) mapped to the general.

However, generals (animal, flower) and particulars (cat, dog, rose, lily) are not discrete structureless elements but they are related to one another in specific ways. So, the process of sorting is also a mapping of relations (between particulars) to relations (between generals), while exemplification is a mapping of relations in the opposite direction. Since a functor is a mapping of relations to relations, a more accurate model of the opposite processes of sorting and exemplification is an opposite pair of functors between particulars and generals. This opposite pair of functors is also constrained by the requirement that a particular sorted into a general is an example of the general. In formalizing these constraints we find that the relation between particulars and generals is an instance of adjointness (Lawvere 2004, p. 8).

Adjoint functors can be discerned in all branches of mathematics (MacLane 1998, pp. 107f). A simple example of adjointness involves the transformation of a function of two variables into a function of one variable, whose values are functions (MacLane 1965, p. 57). Consider a set of numbers A = {4, 9}, a set of operations $B = \{\surd, {}^\wedge\}$, where $\surd$ and $^\wedge$ denote square root and squaring operations, respectively, and a set of values $C = \{2, 3, 16, 81\}$. When operations in B are applied to the numbers in A we get the values in C, which can be written as a function of two variables i.e., f : A × B → C, with $f(4, \surd) = 2$, $f(9, \surd) = 3$, $f(4, {}^\wedge) = 16$, and $f(9, {}^\wedge) = 81$. Corresponding to the two-variable function f : A × B → C, there is a one-variable function g : B → C^{A}, whose values are functions; C^{A} is the set of all functions from A to C. The function g : B → C^{A} maps the square root operation $\surd$ (in B) to the function mapping the numbers 4 and 9 (in A) to the numbers 2 and 3 (in C), respectively, while the squaring operation $^\wedge$ (in B) is mapped to the function mapping the numbers 4 and 9 (in A) to the numbers 16 ans 81 (in C), respectively. In the opposite direction, a one-variable function g : B → C^{A}, whose values are functions, determines a two-variable function f : A × B → C.

This one-to-one correspondence between functions $f : \mathrm{A} \times \mathrm{B} \to \mathrm{C}$ with a product $(\mathrm{A} \times \mathrm{B})$ as domain and functions $g : \mathrm{B} \to \mathrm{C}^{\mathrm{A}}$ with exponential $(\mathrm{C}^{\mathrm{A}})$ as codomain constitutes the adjointness between multiplication and exponentiation functors (Lawvere and Rosebrugh 2003, p. 109).

Now we present a simplified version of the theory-model adjointness (Lawvere 1972, pp. 139ff, Lawvere 2006) as an illustration of the calculations involved in formalizing the interdependence of concepts and percepts. Consider a theory construed as a category $\mathbf{T}$. Models of the theory $\mathbf{T}$ (in a background category $\mathbf{S}$ of sets) are contravariant functors $\mathbf{T}^{\mathrm{op}} \to \mathbf{S}$. The totality of these models constitutes a functor category $\mathbf{S}^{T^{\mathrm{op}}}$, with functors $\mathbf{T}^{\mathrm{op}} \to \mathbf{S}$ as objects. The theory-model interdependence can be characterized in terms of an opposite pair of functors: abstraction $A : \mathbf{S}^{\mathbf{T}^{\mathrm{op}}} \to \mathbf{T}$ and interpretation $I : \mathbf{T} \to \mathbf{S}^{\mathbf{T}^{\mathrm{op}}}$ (Lawvere and Rosebrugh 2003, pp. 249f). Let us take a category with one object T along with its identity morphism $1_T : T \to T$ as a theory $\mathbf{T}$. The opposite category $\mathbf{T}^{\mathrm{op}}$ has the same objects as $\mathbf{T}$, but with arrows reversed (Lawvere and Rosebrugh 2003, p. 5). Since reversing the identity morphism results in the same morphism, we have $\mathbf{T}^{\mathrm{op}} = \mathbf{T}$. So, functors $X : \mathbf{T} \to \mathbf{S}$ (evaluated at the only object T and its identity morphism 1_{T} of the category $\mathbf{T}$, i.e. $X(\mathrm{T}) = \mathrm{X}$ and $X(1_{\mathrm{T}}) = 1_{\mathrm{X}}$) correspond to sets in $\mathbf{S}$.

Thus, with theory $\mathbf{T}$ as a category with one morphism 1_{T}, the category of functors $\mathbf{S}^{\mathbf{T}^{\mathrm{op}}}$ corresponds to the category $\mathbf{S}$ of sets. With $\mathbf{S}^{\mathbf{T}^{\mathrm{op}}} = \mathbf{S}$, interpretation $I : \mathbf{T} \to \mathbf{S}$ is right adjoint to abstraction $A : \mathbf{S} \to \mathbf{T}$. The functor $A : \mathbf{S} \to \mathbf{T}$ maps all sets (in $\mathbf{S}$) to the only object T (in $\mathbf{T}$) and every function to the only morphism 1_{T}. In the opposite direction, there is a functor $I : \mathbf{T} \to \mathbf{S}$, whose value at the object T is the single-element set $\mathbf{1}$, i.e. $I(\mathrm{T}) = \mathbf{1}$ and $I(1_{\mathrm{T}}) = 1_{\mathbf{1}}$. To show that the functor $I : \mathbf{T} \to \mathbf{S}$ is right adjoint to $A : \mathbf{S} \to \mathbf{T}$, we have to show that given an object T (in $\mathbf{T}$), every figure in T whose shape is a value of the functor A i.e. every figure $t : A(\mathrm{S}) \to T$ *is in* the figure $\mathrm{e}_{\mathrm{T}} : A(I(\mathrm{T})) \to T$, and that there is only one function $s : \mathrm{S} \to I(\mathrm{T})$ satisfying $t = \mathrm{e}_{\mathrm{T}} \circ A(s)$ (Lawvere and Schanuel 2009, pp. 374f). Since $A(\mathrm{S}) = \mathrm{T}$, and since there is only one morphism $1_{\mathrm{T}} : \mathrm{T} {\to} \mathrm{T}$ (in $\mathbf{T}$), $t = 1_{\mathrm{T}}$, whose inclusion in $\mathrm{e}_{\mathrm{T}} : A(I(\mathrm{T})) \to \mathrm{T}$ we have to characterize.

Since $A(I(\mathrm{T})) = \mathrm{T}$, it follows that $\mathrm{e}_{\mathrm{T}} = 1_{\mathrm{T}}$. Now we have to show that there is only one function $s : \mathrm{S} \to I(\mathrm{T})$ satisfying $1_{\mathrm{T}} = 1_{\mathrm{T}} \circ A(s)$. Note that $I(\mathrm{T}) = \mathbf{1}$, and that there is only one function $s : \mathrm{S} \to \mathbf{1}$ from any set S to the terminal set $\mathbf{1}$, and the only function s is mapped to 1_{T}, i.e. $A(s) = 1_{\mathrm{T}}$. So, we have to show that $1_{\mathrm{T}} = 1_{\mathrm{T}} \circ 1_{\mathrm{T}}$, which is given by the identity laws defining a category. Thus, the interpretation functor $I : \mathbf{T} \to \mathbf{S}^{\mathbf{T}^{\mathrm{op}}}$ is right adjoint to the abstraction functor $A : \mathbf{S}^{\mathbf{T}^{\mathrm{op}}} \to \mathbf{T}$.

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