End of the Square?
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The Square of Opposition: General Structure

\[ \begin{align*}
  & x & \leftrightarrow & ct(x) \\
  & \downarrow & \uparrow & \downarrow & \uparrow \\
  & sb(x) & \leftrightarrow & cd(x) \\
 \end{align*} \]

\[ \begin{align*}
  & sp(y) & \leftrightarrow & cd(y) \\
  & \uparrow & \downarrow & \uparrow & \downarrow \\
  & y & \leftrightarrow & sct(y) \\
 \end{align*} \]

\textbf{ct}(x): “contrary of } x \text{”
\textbf{cd}(x): “contradictory of } x \text{”
\textbf{sb}(x): “subaltern of } x \text{”

\textbf{sct}(y): “subcontrary of } y \text{”
\textbf{cd}(y): “contradictory of } y \text{”
\textbf{sp}(y): “superaltern of } y \text{”

symmetrical relations

non-symmetrical relations
The Hexagon of Oppositions: General Structure

<table>
<thead>
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<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$ct_1(x)$</td>
<td>“1$^{st}$ contrary of $x$”</td>
</tr>
<tr>
<td>$ct_2(x)$</td>
<td>“2$^{nd}$ contrary of $x$”</td>
</tr>
<tr>
<td>$sb_1(x)$</td>
<td>“1$^{st}$ subaltern of $x$”</td>
</tr>
<tr>
<td>$sb_2(x)$</td>
<td>“2$^{nd}$ subaltern of $x$”</td>
</tr>
<tr>
<td>$cd(x)$</td>
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<tr>
<td>$sct_1(y)$</td>
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</tr>
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<td>“2$^{nd}$ subcontrary of $y$”</td>
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<td>“1$^{st}$ superaltern of $y$”</td>
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<tr>
<td>$cd(y)$</td>
<td>“contradictory of $y$”</td>
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Oppositions
without the square
End of the Square? Costa-Leite’s line segment

“Consider a question: is there a way to represent oppositions without two-dimensional objects such as squares or objects of higher dimensions? The answer is yes.” (Costa-Leite, “Oppositions in a line segment”: 2)

Let $\mathbb{Z}^*$ be a set of non-null integers, $\mathbb{Z}_+$ a set of positive integers, $\mathbb{Z}_-$ a set of negative integers, and $\mathbb{Z}' = \{-r, -q, q, r\} \subseteq \mathbb{Z}$

Let $\mathcal{C}$ be a set of a categorical statements $\{A,E,I,O\}$

$i$ a function on $\mathcal{C}$ s.t. $i: \mathcal{C} \mapsto \mathbb{Z}'$

$j \in \mathbb{Z}'_+$ iff $j \in \{A,E\}$ (universal sentences)

$j \in \mathbb{Z}'_-$ iff $j \in \{I,O\}$ (particular sentences)

Then for every $\alpha, \beta \in \mathcal{C}$:

$i(\alpha)$ and $i(\beta)$ are contraries iff $i(\alpha), i(\beta) \in \mathbb{Z}'_+$

$i(\alpha)$ and $i(\beta)$ are contradictories iff $i(\alpha) + i(\beta) = 0$

$i(\alpha)$ and $i(\beta)$ are subcontraries iff $i(\alpha), i(\beta) \in \mathbb{Z}'_-$

$i(\beta)$ is the subaltern of $i(\alpha)$ iff $i(\alpha) \neq i(\beta)$ and $i(\beta) \in \mathbb{Z}'_-$
Segment Line of Oppositions: Categorical statements (Costa-Leite)

+2 = ct(+1)  
−1 = cd(+1)  
−2 = sb(+1)  

−1 = sct(−2)  
+2 = cd(−2)  
+1 = sp(−2)
End of the Square? Costa-Leite’s line segment

**Problem:** the above definitions fail with the hexagon of oppositions.

\[ \mathbb{Z}'' = \{-s, -r, -q, q, r, s\} \subseteq \mathbb{Z} \]

\[ \mathcal{C}' = \{A, U, E, O, Y, I\} \]

\[ U = A \text{ or } E, \ Y = I \text{ and } O \]

\[ i(U) = i(A) + i(E) \]
\[ i(Y) = i(I) + i(O) \]

Let \( i(A) = +1, \ i(U) = +3, \ i(E) = +2, \ i(O) = -1, \ i(Y) = -3, \ i(I) = -2, \)

\[ Y = \text{ct}(A) \]
now \( i(Y) + i(A) = -3 + 1 = -2, \) therefore \( i(\alpha) + i(\beta) \not\in \mathbb{Z}^*_+ \)

\[ U = \text{sct}(I) \]
now \( i(U) + i(I) = +3 - 2 = +1, \) therefore \( i(\alpha) + i(\beta) \not\in \mathbb{Z}^*_+ \)

\[ U = \text{sb}(A) \]
now \( i(U) = +3, \) therefore \( i(U) \not\in \mathbb{Z}^*_+ \)
End of the Square? Costa-Leite’s line segment

New definitions:

For every $\alpha, \beta, \gamma \in C$:

- $i(\alpha)$ and $i(\beta)$ are *contraries* iff $i(\alpha) + i(\beta) + i(\gamma) = 0$ and $i(\gamma) \in \mathbb{Z}^*_-$
- $i(\alpha)$ and $i(\beta)$ are *contradictories* iff $i(\alpha) + i(\beta) = 0$
- $i(\alpha)$ and $i(\beta)$ are *subcontraries* iff $i(\alpha) + i(\beta) + i(\gamma) = 0$ and $i(\gamma) \in \mathbb{Z}^*_+$
- $i(\beta)$ is the *subaltern* of $i(\alpha)$ iff $i(\alpha) \neq i(\beta)$ and $i(\beta) \in \mathbb{Z}^*_-$
  or $i(\alpha) \neq i(\beta)$ and (a) $i(\beta) > i(\alpha) \in \mathbb{Z}^*_+$
  and (b) $i(\beta) > i(\alpha) \in \mathbb{Z}^*_-$
Segment Line of Oppositions: Categorical statements (Costa-Leite)

\[
\begin{array}{cccccccc}
Y & I & O & A & E & U \\
-3 & -2 & -1 & 0 & +1 & +2 & +3 \\
+2 = ct(+1) & -1 = sct(-2) \\
-1 = cd(+1) & +2 = cd(-2) \\
-2 = sb(+1) & +1 = sp(-2) \\
\end{array}
\]
End of the Square? Costa-Leite’s line segment

Problem:
The new definitions seem to be *ad hoc* (hold for \( \mathbb{Z}'' \) only).
What of the extensions \( \mathbb{Z}'\cdots' \), for any set \( C'\cdots' \) of \( 2^n \) elements?

“There are, notwithstanding, some problems which remain open: the question to determine whether the same procedure can also be applied to solids and higher dimensions, as well as to *more than four oppositions*, are very complicated and still have to investigated in detail.” (Costa-Leite, *ibid.*: 9)

For any family \( C'\cdots' \), there is a maximal number of \( 2^n \) elements

Solution:
An alternative formal semantics based on oppositions
Cf. Sommers & Englebretnen’s “Term-Functor Logic” (TFL)

- 3 kinds of opposition: C-oppositions, Q-oppositions, P-oppositions
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Oppositions
with another square
A formal semantics of oppositions

\[ L_{op} = \langle \mathcal{L}, Q, A, S, \cap, \cup, Op, \text{op} \rangle \]

\[ \mathcal{L} = \{ x, y, \ldots \} \]
\[ Q: \text{question-forming function on } x, \text{ s.t. } Q(x) = \langle q_1(x), \ldots, q_n(x) \rangle \]

\[ A: \text{answer-forming function on } x, \text{ s.t.} \]
\[ A(x) = \langle a_1(x), \ldots, a_n(x) \rangle \]
\[ a(x) \mapsto \{ 1, 0 \} \text{ (1: yes-answer, 0: no-answer) } \]

\[ S: \text{set of bitstrings, i.e. ordered values of } x \text{ s.t. } \text{Card}(S) = 2^n \text{ (with } n \text{ ordered bits)} \]

\[ \text{Op}(x, y) \text{ reads } \text{“} x \text{ and } y \text{ are opposed to each other”} \]
\[ \text{Op}(x, y) = \text{Op}(x, \text{op}(x)) \]
A formal semantics of oppositions

\[ L_{op} = \langle Q, A, S, \cap, \cup, Op, op \rangle \]

\[ \mathcal{L} = \{x, y, \ldots\} \]

\( Q \): question-forming function on \( x \), s.t.
\[ Q(x) = \langle q_1(x), \ldots, q_n(x) \rangle \]

\( A \): answer-forming function on \( x \), s.t.
\[ A(x) = \langle a_1(x), \ldots, a_n(x) \rangle \]
\[ a(x) \mapsto \{1,0\} \] (1: yes-answer, 0: no-answer)

\( S \): set of bitstrings, i.e. ordered values of \( x \) s.t.
\[ \text{Card}(S) = 2^n \] (with \( n \) ordered bits)

\( \text{op}(x) \) reads as “opposite to \( x \)”

is a multifunction s.t.
\[ \text{op}(x) : S \mapsto \varnothing(S) \]

Multifunction: to any value of \( S \) corresponds zero, one, or several elements of \( S \)

: function taking its values in the set of the subparts of \( S \), \( \varnothing(S) \)
A Boolean calculus of oppositions (with binary P-oppositions)

For every $a_i(x)$ and $a_i(y)$ and every opposite-forming operator $\text{op}(x)$ on $x$:

c\text{t}(x) = y \iff a_i(x) = 1 \implies a_i(y) = 0$

c\text{d}(x) = y \iff a_i(x) = 1 \iff a_i(y) = 0$

s\text{c}(x) = y \iff a_i(x) = 0 \implies a_i(y) = 1$

s\text{b}(x) = y \iff a_i(x) = 1 \implies a_i(y) = 1$

s\text{p}(x) = y \iff a_i(x) = 0 \implies a_i(y) = 0$

Examples:

c\text{t}(1000) = 0001

c\text{d}(1000) = 0111

s\text{c}(1110) = 0111

s\text{b}(1000) = 1110

s\text{p}(1110) = 1000
Questions about categorical statements $\Theta = S \times P$

$Q(\Theta) = \langle q_1(\Theta), q_2(\Theta), q_3(\Theta) \rangle$

$q_1(\Theta) = SaP$
$q_2(\Theta) = \overline{SaP} \cap \overline{SeP}$
$q_3(\Theta) = SeP$

Answers to questions about categorical statements $\Theta = S \times P$

$A(\Theta) = \langle a_1(\Theta), a_2(\Theta), a_3(\Theta) \rangle$

$A(SaP) = 100$  
$A(SaP \text{ or } SeP) = 100 \cup 001 = 101$  
$A(SeP) = 001$

$A(\text{SoP}) = 011$  
$A(\text{SiP and SoP}) = 110 \cap 011 = 010$  
$A(\text{SiP}) = 110$
The Hexagon of Opposition: Categorical Statements (Aristotle)

\[ x = \text{SaP or SeP} \]
\[ y = \text{SiP} \]
\[ \text{SiP and SoP} \]

\[
\begin{align*}
\text{ct}_1(100) &= 001 \\
\text{ct}_2(100) &= 110 \land 011 = 010 \\
\text{sb}_1(100) &= 110 \\
\text{sb}_2(110) &= 010 \\
\text{cd}(100) &= 011
\end{align*}
\]

\[
\begin{align*}
\text{sct}_1(110) &= 011 \\
\text{sct}_2(110) &= 100 \lor 001 = 10 \\
\text{sp}_1(110) &= 100 \\
\text{sp}_2(110) &= 110 \land 011 = 010 \\
\text{cd}(110) &= 001
\end{align*}
\]
Questions about modal sentences $\Pi = \Box \varphi$

$Q(\Pi) = \langle q_1(\Pi), q_2(\Pi), q_3(\Pi) \rangle$

$q_1(\Pi) = \Box \varphi$
$q_2(\Pi) = \Box \varphi \cap \Box \neg \varphi$
$q_3(\Pi) = \Box \neg \varphi$

Answers to questions about S5 modal statements $\Pi = \Box \varphi$

$A(\Pi) = \langle a_1(\Pi), a_2(\Pi), a_3(\Pi) \rangle$

$A(\Box \varphi) = 100$ \hspace{2cm} $A(\neg \Box \varphi) = 011$
$A(\Box \varphi \lor \Box \neg \varphi) = 100 \lor 001 = 101$ \hspace{2cm} $A(\neg \Box \neg \varphi \land \Box \neg \neg \varphi) = 110 \land 011 = 010$
$A(\Box \neg \varphi) = 001$ \hspace{2cm} $A(\neg \Box \neg \varphi) = 110$
The Hexagon of Opposition: Modal sentences (Blanché)

\[ x = \Box \varphi \]
\[ y = \neg \Box \neg \varphi \]

\[ \text{ct}_1(100) = 001 \]
\[ \text{ct}_2(100) = 110 \land 011 = 010 \]
\[ \text{sb}_1(100) = 110 \]
\[ \text{sb}_2(110) = 100 \lor 001 = 101 \]
\[ \text{cd}(100) = 011 \]

\[ \text{sct}_1(110) = 011 \]
\[ \text{sct}_2(110) = 100 \lor 001 = 101 \]
\[ \text{sp}_1(110) = 100 \]
\[ \text{sp}_2(110) = 110 \land 011 = 010 \]
\[ \text{cd}(110) = 001 \]
Questions about bivalent binary propositions $\Phi = p \cdot q$

$Q(\Phi) = \langle q_1(\Phi), q_2(\Phi), q_3(\Phi), q_4(\Phi) \rangle$

$q_1(\Phi) = p \cap q$
$q_2(\Phi) = \overline{p} \cap q$
$q_3(\Phi) = p \cap \overline{q}$
$q_4(\Phi) = \overline{p} \cap \overline{q}$

Answers to questions about bivalent binary propositions $\Phi = p \cdot q$

$A(\Phi) = \langle a_1(\Phi), a_2(\Phi), a_3(\Phi), a_4(\Phi) \rangle$

$A(p \land q) = 1000$
$A((p \land q) \lor (\neg p \land \neg q)) = 1000 \lor 0001$
$= 1001$
$A(\neg p \land \neg q) = 0001$

$A(\neg (p \land q)) = 0111$
$A((p \land q) \land (\neg p \land \neg q)) = 1000 \land 0001$
$= 0110$
$A(\neg (\neg p \land \neg q)) = 1110$
The Hexagon of Opposition: Binary sentences (Piaget)

\[ x = p \land q \]
\[ y = \neg (\neg p \land \neg q) \]

\[ (p \land q) \lor (\neg p \land \neg q) \]

\[ ct_1(1000) = 0001 \]
\[ ct_2(1000) = 1110 \cap 0111 = 1001 \]
\[ sb_1(1000) = 1110 \]
\[ sb_2(1110) = 0110 \]
\[ cd(1000) = 0111 \]

\[ sct_1(1110) = 0111 \]
\[ sct_2(1110) = 1000 \cup 0001 \]
\[ sp_1(1110) = 1000 \]
\[ sp_2(1110) = 1110 \cap 0111 = 0110 \]
\[ cd(1110) = 0001 \]
Questions about singular terms $\Omega = S$ is/is not P/not-P

$Q(\Omega) = \langle q_1(\Omega), q_2(\Omega), q_3(\Omega), q_4(\Omega) \rangle$

$q_1(\Omega) = S$ is absolutely $P$  
$q_2(\Omega) = S$ is absolutely $P \cap S$ is absolutely $\overline{P}$  
$q_3(\Omega) = S$ is absolutely not $\overline{P}$

Answers to questions about singular terms $\Omega = S$ is/is not P/not-P

$A(\Omega) = \langle a_1(\Omega), a_2(\Omega), a_3(\Omega) \rangle$

$A(S$ is $P) = 100$  
$A(S$ is $P$ or not-$P$) = $100 \cup 001 = 101$  
$A(S$ is not-$P) = 001$

$A(S$ is not $P) = 011$

$A(S$ is $P$ or not-$P$) = $100 \cup 011 = 101$

$A(S$ is not $P$ and not not-$P$) = $110 \cap 011 = 010$

$A(S$ is not-$P$) = 001  
$A(S$ is not not-$P$) = 110
The Hexagon of Opposition: Term logic (Aristotle, Englebretsen)

\[ x = S \text{ is } P \quad \text{and} \quad S \text{ is not-P} \]

\[ y = S \text{ is not not-P} \quad \text{and} \quad S \text{ is not P} \]

\[ \begin{align*}
ct_1(100) &= 0001 \\
ct_2(100) &= 110 \land 011 = 101 \\
sb_1(100) &= 110 \\
sb_2(110) &= 010 \\
cd(100) &= 011 \\
sct_1(110) &= 011 \\
sct_2(110) &= 100 \lor 001 \\
sp_1(110) &= 100 \\
sp_2(110) &= 110 \land 011 = 010 \\
cd(110) &= 001
\end{align*} \]
**Graphs**: how to determine the values(s) of the multifunction op?

\[ y = f(x) \]
\[ z = g(y) = g(f(x)) \]
\[ x = h(z) = h(g(f(x))) \]
**Graphs**: how to determine the values(s) of the multifunction op?

\[ A(y) = 0111 = \text{sb}(0001) \]
\[ A(z) = 1110 = \text{sct}(0111) = \text{sct}(\text{sb}(0001)) \]
\[ A(x) = 0001 = \text{cd}(1110) = \text{cd}(\text{sct}(\text{sb}(0001))) \]
**Definitions.** For every $x$:

$\text{op}(x) \neq x$  
$\text{cd}(\text{cd}(x)) = x$  

(contradictoriness)

$\text{op}(x) = \text{op}_i(\text{op}_j(x))$ iff $\text{op}^{-1}(x) = \text{op}_j(\text{op}_i(x))$  
$\text{op}_i(\text{op}_j^{-1}(x)) = \text{op}_j(\text{op}_i(x))$  
$\text{sp}(y) = x$ iff $x = \text{sb}(y)$  

(converse)

$\text{cd}(x) = \text{sb}(\text{ct}(x)) = \text{ct}(\text{sp}(x))$  

(contradictoriness)

$\text{ct}(x) = \text{cd}(\text{sb}(x)) = \text{sp}(\text{cd}(x))$  

(contrariety)

$\text{sct}(x) = \text{cd}(\text{sp}(x)) = \text{sb}(\text{cd}(x))$  

(subcontrariety)

$\text{sb}(x) = \text{cd}(\text{ct}(x))$  

(subalternation)
For every bitstring $A(x)$ of length $\lambda(x) = \mu(x) + \nu(x) = n$
Let $\nu(x)$ and $\mu(x)$ be the number of yes- and no-answers in any $A(x)$. Then:

**Proposition 1**
The number of *contraries* of $x$ is
$\text{Card}(\text{ct}(x)) = 2^{\mu(x)} - 1$.

Examples: let $A(x) = 1001$
$\mu(x) = 2$
Hence $\text{Card}(\text{ct}(x)) = 2^2 - 1 = 3$

let $A(y) = 111111$
$\mu(x) = 0$
Hence $\text{Card}(\text{ct}(x)) = 2^0 - 1 = 0$

**Proof**: See Schang, F.: “Logic in Opposition”. 
For every bitstring $A(x)$ of length $\lambda(x) = \mu(x) + \nu(x) = n$
Let $\nu(x)$ and $\mu(x)$ be the number of yes- and no-answers in any $A(x)$. Then:

**Proposition 2**
The number of *subalterns* of $x$ is
$\text{Card}(\text{sb}(x)) = 2^{\mu(x)} - 1$.

Example: let $A(x) = 1001$
$\mu(x) = 2$
Hence $\text{Card}(\text{sb}(x)) = 2^2 - 1 = 3$

*Proof*: $\text{sb}(x) = \text{cd} (\text{ct}(x))$
For every $x$, $\text{Card} (\text{cd}(x)) = \text{Card}(x) = 1$
Hence $\text{Card}(\text{sb}(x)) = \text{Card}(\text{cd}(\text{ct}(x))) = \text{Card}(\text{ct}(x)) = 2^{\mu(x)} - 1$
For every bitstring $A(x)$ of length $\lambda(x) = \mu(x) + \nu(x) = n$
Let $\nu(x)$ and $\mu(x)$ be the number of yes- and no-answers in any $A(x)$. Then:

**Proposition 3**
The number of *superalterns* of $x$ is
Card$(\text{sp}(x)) = 2^{\nu(x)} - 1$.

Example: let $A(x) = 1001$
$\nu(x) = 2$
Hence $\text{Card}(\text{sp}(x)) = 2^2 - 1 = 3$

**Proof:** $\text{sp}(x) = \text{ct}(\text{cd}(x))$
For every $x$, Card$(\text{cd}(x)) = n - \mu(x) = \nu(x)$
Hence Card$(\text{sp}(x)) = \text{Card(ct(cd(x)))} = 2^{\nu(x)} - 1$. \[\square\]
For every bitstring $A(x)$ of length $\lambda(x) = \mu(x) + \nu(x) = n$

Let $\nu(x)$ and $\mu(x)$ be the number of yes- and no-answers in any $A(x)$. Then:

**Proposition 4**
The number of *subcontraries* of $x$ is
Card($\text{sct}(x)$) $= 2^{\nu(x)} - 1$.

Example: let $A(x) = 1001$
$\nu(x) = 2$
Hence Card($\text{sct}(x)$) $= 2^2 - 1 = 3$

*Proof*: $\text{sct}(x) = \text{cd}(\text{sp}(x))$
For every $x$, $\nu(\text{cd}(x)) = \nu(x)$.
Hence Card($\text{sct}(x)$) $= $ Card($\text{cd}(\text{sp}(x))$) $= $ Card($\text{sp}(x)$) $= 2^{\mu(x)} - 1$. $\square$
For every bitstring $A(x)$ of length $\lambda(x) = \mu(x) + \nu(x) = n$
Let $\nu(x)$ and $\mu(x)$ be the number of yes- and no-answers in any $A(x)$. Then:

**Proposition 5**
The number of *indeterminates* of $x$ is
Card(id($x$)) = $(2^n - 1) - \text{Card}(d(x))$.

Examples: let $A(x) = 1001$
Card(d($x$)) = Card(ct($x$) + cd($x$) + sct($x$) + sb($x$) + sp($x$)) = 11
Hence Card(id($x$)) = $2^4 - 1 - 11 = 4$

**Proof**: Determinates are the disjoint union of ct($x$), cd($x$), sct($x$), sb($x$), sp($x$).
For every $x$, Card(ct($x$) $\cap$ sp($x$)) = Card(sct($x$) $\cap$ sb($x$)) = 1.
Hence Card(d($x$)) = Card(ct($x$) + cd($x$) + sct($x$) + sb($x$) + sp($x$)) − 2. □
Vector theory
How to determine the values(s) of op?

\[ \overrightarrow{uv} + \overrightarrow{vw} = \overrightarrow{uw} \]
\[ \overrightarrow{vw} + \overrightarrow{wu} = \overrightarrow{vu} \]
\[ \overrightarrow{wu} + \overrightarrow{uv} = \overrightarrow{wv} \]
An arithmetization of oppositions: bitstrings as base-2 integers

- base-2 integers are turned into base-10 integers with a function $\sigma: S \mapsto \mathbb{N}$
- bitstrings are turned into integers, s.t.:
  $$\sum(x) = \langle \sigma_1(x) + \ldots + \sigma_n(x) \rangle, \text{ with } \sigma_k(x) = 2^{n-k} \times a_k(x)$$
  Example: $\sum(1101) = 8 + 4 + 0 + 1 = 13$
- opposite-forming operators are turned into arithmetic operators $\pm \sigma$, s.t.:
  $$\pm(\sum(x)) = \sum(y)$$

For every $x, y$:

- $x$ and $y$ are *contradictories* iff $\sigma(x) \neq 0 \Leftrightarrow \sigma(y) = 0$
- $x$ and $y$ are *contraries* iff $\sigma(x) \neq 0 \Rightarrow \sigma(y) = 0$
- $x$ and $y$ are *subcontraries* iff $\sigma(x) = 0 \Rightarrow \sigma(y) \neq 0$
- $x$ is *subaltern* of $y$ iff $\sigma(x) \neq 0 \Rightarrow \sigma(y) \neq 0$

Example: $A(x) = 0111, A(y) = 0001$, $\sigma(y) \neq 0 \Rightarrow \sigma(x) \neq 0$, therefore $\text{Op}(x,y) = \text{SB}(x,y)$
How to determine the value(s) of \( op \)?

\[
+6(1) = 7 \\
+13(1) = +7(+6(1)) = 14 \\
\pm0(1) = +7(+6-13(1))) = 1
\]
The Hexagon of Oppositions: General Structure

1 with the square
2 without the square
3 another square

\[ x = 8 \]
\[ y = 14 \]

1 = \((-7)\)8 = 8 \(-\) 7
6 = \((-2)\)8 = 8 \(-\) 2
14 = \((+6)\)8 = 8 + 6
9 = \((+1)\)8 = 8 + 1
7 = \((-1)\)8 = 8 \(-\) 1

7 = \((-7)\)14 = 14 \(-\) 7
9 = \((-5)\)14 = 14 \(-\) 5
8 = \((-6)\)14 = 14 \(-\) 6
6 = \((-8)\)14 = 14 \(-\) 8
1 = \((-13)\)14 = 14 \(-\) 13
PROBLEMS:
- Costa Leite’s segments hold for limited diagrams only
- the vectorial behavior of oppositions holds with 2D diagrams only
  it is lost with, e.g., hypercubes ($n = 3$), tetraicosahedrons ($n = 4$), etc.

SOLUTION:
- a general diagram for oppositions of any structural complexity
- replacing vertices with areas in a diagram of $n$-chotomies
Diagrams with areas (rather than vertices) of $n$-bitstrings ($n = \text{length}$)

$S$ is a square if $L = l$

$S$ is a rectangle if $L \neq l$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

$$
\begin{array}{c|c}
1 & 0 \\
\hline
n = 1 \\
\end{array}
$$

$L = 2^{(1+1)/2} = 2^{2/2} = 2^1 = 2$

$l = 2^{(1-1)/2} = 2^{0/2} = 2^0 = 1$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

<table>
<thead>
<tr>
<th>11</th>
<th>01</th>
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<tbody>
<tr>
<td>10</td>
<td>00</td>
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</table>

$n = 2$

$L = 2^{2/2} = 2^1 = 2$

$l = 2^{2/2} = 2^{2/2} = 2^1 = 2$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

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</table>

$n = 3$

$L = 2^{(3+1)/2} = 2^{4/2} = 2^2 = 4$

$l = 2^{(3-1)/2} = 2^{2/2} = 2^1 = 2$
Diagrams with areas (rather than vertices) of \(n\)-bitstring \((n = \text{length})\)

<table>
<thead>
<tr>
<th>1111</th>
<th>1101</th>
<th>0111</th>
<th>0101</th>
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<tbody>
<tr>
<td>1110</td>
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<td>1011</td>
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<td>1010</td>
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</tbody>
</table>

\(n = 4\)

\[ L = 2^{4/2} = 2^2 = 4 \]

\[ l = 2^{4/2} = 2^2 = 4 \]
Diagrams with areas (rather than vertices) of \( n \)-bitstring \( (n = \text{length}) \)

\[
\begin{array}{cccccccc}
1111 & 1110 & 1011 & 1010 & 0111 & 0110 & 0011 & 0010 \\
1101 & 1100 & 1010 & 1010 & 0110 & 0101 & 0010 & 0001 \\
1011 & 1010 & 1001 & 1001 & 0101 & 0100 & 0001 & 0000 \\
1001 & 1000 & 1000 & 1000 & 0100 & 0100 & 0000 & 0000 \\
\end{array}
\]

\( n = 5 \)

\[
L = 2^{(5+1)/2} = 2^{6/2} = 2^3 = 8 \\
l = 2^{(5-1)/2} = 2^{4/2} = 2^2 = 4
\]
Diagrams with areas (rather than vertices) of \( n \)-bitstring (\( n = \text{length} \))

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<thead>
<tr>
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</table>

\( n = 6 \)

\[
L = 2^{6/2} = 2^3 = 8
\]

\[
l = 2^{6/2} = 2^3 = 8
\]
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

$n = 6$

$A(x) = 101001$

$d(x) = \{\text{ct}(x), \text{cd}(x), \text{sct}(x), \text{sp}(x), \text{sb}(x)\}$

Card(\text{ct}(x)) = 2^3 - 1 = 7

Card(\text{cd}(x)) = 1

Card(\text{sct}(x)) = 2^3 - 1 = 7

Card(\text{sb}(x)) = 2^3 - 1 = 7

Card(\text{sp}(x)) = 2^3 - 1 = 7

Card(\text{id}(x)) = 29 - 2 = 27

Card(\text{d}(x)) = 64 - 1 - 27 = 36
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

Card($ct(x)$) = $2^3 - 1 = 7$

Card($cd(x)$) = 1

Card($sct(x)$) = $2^3 - 1 = 7$

Card($sb(x)$) = $2^3 - 1 = 7$

Card($sp(x)$) = $2^3 - 1 = 7$

Card($d(x)$) = $29 - 2 = 27$

Card($id(x)$) = $64 - 1 - 27 = 36$

$n = 6$

$A(x) = 101001$

$ct(x) = \{000000, 010000, 000100, 000010, 010100, 010010, 000110\}$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n =$ length)

$n = 6$

$A(x) = 101001$
$cd(x) = \{010110\}$

Card(ct($x$)) = $2^3 - 1 = 7$
Card(cd($x$)) = 1
Card(sct($x$)) = $2^3 - 1 = 7$
Card(sb($x$)) = $2^3 - 1 = 7$
Card(sp($x$)) = $2^3 - 1 = 7$
Card(d($x$)) = $29 - 2 = 27$
Card(id($x$)) = $64 - 1 - 27 = 36$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

$n = 6$

$A(x) = 101001$

$\text{sp}(x) = \{000000, 100000, 001000, 000001, 101000, 100001, 001001\}$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

$A(x) = 101001$

$sb(x) = \{111111, 101111, 111011, 111101, 101011, 101101, 111001\}$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

Card$(ct(x)) = 2^3 - 1 = 7$

Card$(cd(x)) = 1$

Card$(sct(x)) = 2^3 - 1 = 7$

Card$(sb(x)) = 2^3 - 1 = 7$

Card$(sp(x)) = 2^3 - 1 = 7$

Card$(d(x)) = 29 - 2 = 27$

Card$(id(x)) = 64 - 1 - 27 = 36$

$n = 6$

$A(x) = 101001$

$sct(x) = \{111111, 011111, 110111, 111110, 010111, 011110, 110110\}$
Diagrams with areas (rather than vertices) of \( n \)-bitstring \( (n = \text{length}) \)

\[
\begin{array}{cccccccc}
\text{1 with the square} & \text{2 without the square} & \text{3 another square} \\
\end{array}
\]

\[n = 6\]

\[111111 = \text{sb}(x)\]

Card\((\text{ct}(x)) = 2^3 - 1 = 7\]

Card\((\text{cd}(x)) = 1\]

Card\((\text{sct}(x) \cap \text{sb}(x)) = 1\]

Card\((\text{sp}(x)) = 2^3 - 1 = 7\]

Card\((\text{d}(x)) = 29 - 2 = 27\]

Card\((\text{id}(x)) = 64 - 1 - 27 = 36\]
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

$n = 6$

$111111 = \text{sct}(x)$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n =$ length)

$\text{Card}(\text{ct}(x) \cap \text{sp}(x)) = 1$

$\text{Card}(d(x)) = 29 - 2 = 27$

$\text{Card}(\text{id}(x)) = 64 - 1 - 27 = 36$

$n = 6$

$000000 = \text{sp}(x)$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

$\text{Card}(\text{ct}(x) \cap \text{sp}(x)) = 1$

$\text{Card}(d(x)) = 29 - 2 = 27$

$\text{Card}(\text{id}(x)) = 64 - 1 - 27 = 36$

$n = 6$

$000000 = \text{ct}(x)$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

$n = 6$

Subalterns are contradictories of contraries.

$sb(x) = cd(ct(x))$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

\begin{align*}
\text{Card}(\text{ct}(x)) &= 2^3 - 1 = 7 \\
\text{Card}(\text{cd}(x)) &= 1 \\
\text{Card}(\text{sct}(x)) &= 2^3 - 1 = 7 \\
\text{Card}(\text{sb}(x)) &= 2^3 - 1 = 7 \\
\text{Card}(\text{sp}(x)) &= 2^3 - 1 = 7 \\
\text{Card}(\text{d}(x)) &= 2^9 - 2 = 27 \\
\text{Card}(\text{id}(x)) &= 2^{64} - 1 - 2^9 = 36
\end{align*}

$n = 6$

Subalterns are contradictories of contraries.

$sb(x) = cd(\text{ct}(x))$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

\[
\begin{align*}
\text{Card}(\text{ct}(x)) &= 2^3 - 1 = 7 \\
\text{Card}(\text{cd}(x)) &= 1 \\
\text{Card}(\text{sc}(x)) &= 2^3 - 1 = 7 \\
\text{Card}(\text{sp}(x)) &= 2^3 - 1 = 7 \\
\text{Card}(\text{d}(x)) &= 29 - 2 = 27 \\
\text{Card}(\text{id}(x)) &= 64 - 1 - 27 = 36
\end{align*}
\]

\[n = 6\]

Subalterns are **contradictories** of contraries.
\[\text{sb}(x) = \text{cd}(\text{ct}(x))\]
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

$n = 6$

Superalterns are contradictories of subcontraries.

$\text{sp}(x) = \text{cd}(\text{sct}(x))$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

Superalterns are *contradictories* of *subcontraries.*

$\text{sp}(x) = \text{cd}(\text{sct}(x))$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

\[
\begin{align*}
\text{Card}(ct(x)) &= 2^3 - 1 = 7 \\
\text{Card}(cd(x)) &= 1 \\
\text{Card}(sct(x)) &= 2^3 - 1 = 7 \\
\text{Card}(sb(x)) &= 2^3 - 1 = 7 \\
\text{Card}(sp(x)) &= 2^3 - 1 = 7 \\
\text{Card}(d(x)) &= 29 - 2 = 27 \\
\text{Card}(\text{id}(x)) &= 64 - 1 - 27 = 36
\end{align*}
\]

$n = 6$

**Indeterminates** with respect to $x$

id$(x)$
Diagrams with areas (rather than vertices) of $n$-bitstring ($n = \text{length}$)

\[ n = 6 \]

**Indeterminates** are contradictories of **determinates** $d(x)$
\[ d(x) = \{ cd(x), ct(x), sct(x), sb(x), sp(x) \} \]
Diagrams with areas (rather than vertices) of \( n \)-bitstring \((n = \text{length})\)

\[
\begin{align*}
\text{Card}(ct(x)) &= 2^3 - 1 = 7 \\
\text{Card}(cd(x)) &= 1 \\
\text{Card}(sct(x)) &= 2^3 - 1 = 7 \\
\text{Card}(sb(x)) &= 2^3 - 1 = 7 \\
\text{Card}(sp(x)) &= 2^3 - 1 = 7 \\
\text{Card}(d(x)) &= 29 - 2 = 27 \\
\text{Card}(id(x)) &= 64 - 1 - 27 = 36
\end{align*}
\]

\( n = 6 \)

**Indeterminates** are **contradictories** of determinates \( d(x) \)

\( id(x) = cd(d(x)) \)
1 with the square

2 without the square

3 another square

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1 with the square

2 without the square

3 another square

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End of the square?
Standard diagrams are diagrams with vertices + oriented graphs (sb/sp)
Open questions: how many diagrams/kinds of oppositions can there be?

Towards another square
A functional calculus of opposites helps to:
- determine complete structures of oppositions with $2^n$ elements ($n$-bitstrings)
- deal with logical oppositions as opposite-forming multifunctions
- device new diagrams of oppositions with areas + colored diagrams

Extended works
Towards a 3-dimensional theory of meaning though 3 kinds of oppositions:
- C-opposition: individual objects $x$ are sets of sets of properties
- Q-opposition: quantified properties over time, space, individuals
- P-opposition: answers to ordered questions (cf. Schang 2017)

Towards a generalized theory of logical values: **Partition Semantics**.
References


Muito obrigado.