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Logic in Opposition[1]

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Abstract

It is claimed hereby that, against a current view of logic as a theory of consequence, opposition is a basic logical concept that can be used to define consequence itself. This requires some substantial changes in the underlying framework, including: a non-Fregean semantics of questions and answers, instead of the usual truth-conditional semantics; an extension of opposition as a relation between any structured objects; a definition of oppositions in terms of basic negation. Objections to this claim will be reviewed.

Keywords. consequence, existential import, multifunction, negation, opposite, opposition

1. Introduction

The paper wants to do justice to the central contribution of opposition to the way meaning is currently formed and conveyed. For this purpose, let us tell some words about the meaning of "meaning" while turning to the very content of opposition, from Aristotle's works to a general theory between logic, ontology, and algebra.

1.1. Opposition and meaning

Meaning has to do with information, and information is not a ready-made collection of related objects. Moreover, existence does not seem to be so a crucial property of an object once information has more to do with how people interact with each other. Theses precisions may help to bring some light upon the philosophical background of our logic of opposition, where the central concept of "truth" has to be treated very cautiously in an intersubjective sense of accepted information.

That a formal semantics equally applies to different categories of things like individuals, concept or sentences entails that our so-called "logic" of opposition lies between formal ontology and algebra. However, it can be called a theory of meaning safely insofar as it relies upon any questions and answers liable to present something as a relevant information beyond the sole case of sentences. To put it in other words, the following semantics departs from the realist-minded notion of truth by relating meaning to the way in which any piece of information is given about a putative object. The more objects there are in a given local ontology, the more questions are to be asked in order to make order between them. Borrowing from the Goodmanian parlance [6], there are several ways of making worlds and, correspondingly, one and the same object can have a different meaning-in-a-model (a local ontology) according to the number of properties it is provided with.

That it *exists* is a thing; but another thing is that, according to us, existence is neither a necessary nor a sufficient condition to say anything meaningful about it. Therefore, one and the same object can belong to different worlds (or models) once different properties are assigned to it or, better, different perspectives are entertained to make a description of it. To push the line further,

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let us say that the so-called "actual" world is a maximal lexical field, i.e. a proper set of overlapping sets of information; each element of this common world can (and, indeed, does often) belong to different such subsets that are to be compared with "possible worlds", i.e. different perspectives (lexical fields) from which they are entertained as valuable pieces of information.

1.2. Plan of work

The theory of opposition is investigated and revisited in the present talk. This will be made in two main steps, defensive and constructive in turn. (1) Against a widespread opinion, it is argued that such a theory is not just an old-fashioned legacy of Aristotle's traditional logic that would have definitely failed because of the problem of *existential import*. (2) Beyond the current view that logic is a theory of consequence, it is suggested that opposition is a more basic relation encompassing Tarski's consequence as a particular case of opposition. (3) Objections to this counterintuitive view of opposition will be reviewed and lead to a more dialogical view of the logical discipline: the aim of logic is not so much preserving truth than expressing structured differences.

Logic as a theory of difference will be defended as follows.

(1) According to Aristotle's logical legacy [1], there are four kinds of logical opposition between universal and particular propositions: *contrariety*, *contradiction*, *subcontrariety*, and *subalternation*.

After defining these, attention will be paid first upon the so-called problem of existential import; the logical square of opposition is said to be invalidated once the predications are about empty terms, leading to a radical depreciation of the theory of opposition because of its allegedly limited application and dependence upon some preconceptions of traditional logic. Against this definite view, it is shown that existential import does not invalidate the logical square under some alternative formalization of the propositions [4].

(2) Then the concept of opposition is abstracted from its historical context and developed into a set-theoretical approach [14,15,16]. Firstly, opposition is given as a binary *relation* between *structured objects*. Secondly, a correlated theory of opposites depicts oppositions as a relation between a relatum and its opposite. Thirdly, a non-Fregean semantics leads to a Boolean calculus of opposites: *Question-Answer Semantics* (hereafter: **QAS**), where the logical value of any structured proposition is an ordered set of answers to corresponding questions. In the case of logical oppositions, the meaning of structured propositions is afforded by questions about their *disjunctive normal form*. A Boolean algebra of the classical oppositions follows from it and matches with Piaget's INRC Group [12]. Fourthly, the crucial role of *negation* accounts for the oppositional roots of logical consequence, and its oppositional nature is justified by claiming that subalternation proceeds as a double mixed negation.

(3) Finally, a number of objections will be addressed about this revisited theory of opposition. These can be summed up by the following questions:
(a) Can opposition be something else than a relation of incompatibility? (c) Isn't subalternation a restrictedly standard view of logical consequence? (d) Can one set up a proper calculus with the opposite-forming operators?

A way to reply to this set of objections requires an alternative view of logic: not a theory of truth-preserving consequence, but a theory of difference-forming negation. A way to uphold this trend within **QAS** requires the epistemological primacy of *negation* upon truth. The variety of logical negations must be distinguished from the unique operator of *denial* for every opposite relation between structured objects.

2. The historical background of opposition

Two reasons may be advocated at the least to show that the theory of opposition is on a par with Aristotle's logical works. For one thing, the famous "square of opposition" is currently assigned to the philosopher's name, although it has been argued elsewhere (e.g. in [14]) why

Aristotle never mentioned any such figure in his logical writings. On the other hand, each of the well-known relations of opposition finds its roots in Aristotle's texts, too. This is not the whole story, however, in the sense that a properly logical theory of opposition can be displayed without resorting to traditional logic. In this respect, a formal device can be used to set up a Boolean algebra of opposition which doesn't take into account any other information than logical values.
Let us return to the historical background of logical oppositions, however, in order to see more clearly how an algebraic logic of oppositions can be freely abstracted from the Aristotelian theory of quantified propositions while embracing it altogether.

2.1. Definitions

Aristotelian oppositions are characterized by some constraints upon the truth-values of related propositions *a* and *b*.

Proposition 1
a and *b* are *contrary* to each other iff they cannot be *true* together.

Proposition 2
a and *b* are *contradictory* to each other iff they cannot be *true* together and cannot be *false* together.

Proposition 3
a and *b* are *subcontrary* to each other iff they cannot be *false* together.

Proposition 4
b is *subaltern* to *a* iff *b* cannot be false whenever *a* is true.

A number of questions arise from this preliminary presentation, including the three following ones. First: why does one deal with logical oppositions in the form of a square, i.e. why should one stick to four logical relations among its six edges (four straight lines and two diagonals)? Next: does it make sense to talk about subcontrariety and subalternation within a theory of *opposition*? Aristotle depicted the former in terms of "verbal oppositions" (see e.g. [3], p. 416) while ignoring the latter as an opposition altogether, after all. Last, but not least: what of non-classical negations with respect to the theory of oppositions? While Aristotle clearly linked opposition and negation through the so-called laws of *non-contradiction* (a proposition and its negation cannot be true together) and *excluded middle* (a proposition and its negation cannot be false together), contradiction is the only kind of opposition that relates to negation from this classical (bivalent) perspective.
Another focus is in order before answering these questions in the sequel, namely, one of the logical problems that led to the historical fall of the theory of opposition.

2.2. Existential import

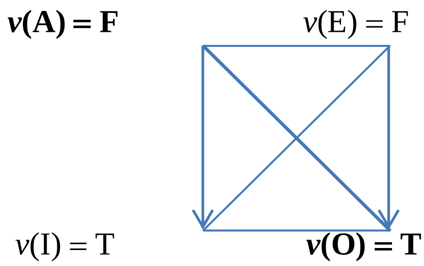
According to the so-called problem of *existential import*, the logical square of opposition is made invalid by a standard, truth-functional semantics once propositions refer to empty names, i.e. dummy individuals that don't exist (like "the present king of France", "griffins", and the like). If so, then its applicability is restricted to non-empty models and thereby weakens the scientific relevance of a logical theory of opposition. Such a semantic difficulty has to do with the way truth-values are assigned to propositions, since a predication like "S is P" assumes for its truth that S be instantiated by at least one individual (*x*, say).

A modern formal translation of Aristotle's traditional logic turns predications into quantified propositions like "(…*x*)(S(*x*) … P(*x*))", where the blanks are to be filled by quantifiers (either universal or existential) and logical connectives (either conditional or conjunction). Except for the ambiguous case of singular propositions, the result is a clear correspondence between traditional and modern formulas.

Proposition 5

Formulas from traditional logic can be translated in modern first-order logic as follows. (A) Universal affirmative: “Every S is P” := $(\forall x)(Sx \supset Px)$; (E) Universal negative: “Every S is not P” := $(\forall x)(Sx \supset \sim Px)$; (I) Particular affirmative: “Some S is P” := $(\exists x)(Sx \wedge Px)$; (O) Particular negative: “Some S is not P” := $(\exists x)(Sx \wedge \sim Px)$.

Let us consider the sentence “Some griffins are nice”. The truth-value of this particular affirmative relies upon whether there is some griffin which happens to be nice. But there cannot be some such creature, for no griffin exists at all. Hence the first conjunct S*x* is made false, and so is the entire conjunctive proposition. Let us write by *v*(I) = F the case that the I-proposition is false. This entails that its contradictory, i.e. the corresponding universal affirmative, is true, according to the definition of contradictories just given above: *v*(E) = T. This sounds intuitively right, since no griffin could be nice once no such creature exists. But the tricky point is about its subaltern, i.e. the related particular negative to the effect that some griffin is not nice. Such a proposition cannot be true whenever no griffin exists, so that *v*(O) = F. The whole set of logical oppositions is thus at odds with their aforementioned definitions, as witnessed by the following invalid square and its troublesome relation (in bold face).



A number of replies have been proposed to settle this problem, namely: restricting the existential import of propositions; discarding their formal modern interpretation; invalidating the square as it stands, otherwise. Our own solution would consist in changing the formalization of particulars, as argued in a recent paper (see [4]); in a nutshell, our point is that the contradictories of universals should not be rendered in the form of existential propositions whose truth-conditions require the existence of their subject-term. Whatever the explanatory value of this formal reply may be, it helps to save the square and enhances its scientific value within the realm of logic.

Once the square of quantified propositions is restored, we can push the line further by abstracting from the category of sentences Aristotle was strictly concerned with. Logical values are the essential information required to define logical oppositions, indeed, and any sort of meaningful object is included in our discussion. But to do so actually requires another formal semantics than the truth-conditional one.

3. A formal theory of opposition

The subsequent formal semantics makes a primary distinction between oppositions and opposites, before defining their features by means of Boolean bitstrings. Just as Tarski suggested an abstract view of consequence as either a relation between sets of formulas or an operator [20,21], the same treatment will be reserved to the logical concept of opposition.

3.1. Opposition as a relation

It is taken to be granted that opposition proceeds as a relation between objects, irrespective of how many and what these are exactly. Although the mainstream theory of opposition usually refers to the binary relation between propositions (as e.g. in [14]), it will be argued in the following that our proposed semantics needn't apply to propositions and equally applies to individuals, concepts, or whatever does make sense by means of a question-answer game.

Proposition 6

An opposition Op is an ordered binary relation between any meaningful objects a and b: Op(a,b), such that it holds iff the 2-tuple of objects a,b satisfies Op.

It is worthwhile to note that Op has been restricted hereby to the arity $n = 2$, although more than two contrary oppositions can be related to each other satisfactorily. It is thus the case that e.g. necessary, impossible and contingent propositions are contrary to each other. Yet this is not the case for most of the other relations like, e.g., contradictories: if a is contradictory with b and b is contradictory with c, then a is not contradictory but, rather, identical to c. For take "white" as an instance of a; then its contradictory b is "not-white", and the contradictory c of the latter is "not-not-white", i.e. "white", while the contrary of "white" is "black". We will return to these peculiarities later on (see section 3.3).

For the time being, let us note not only that any n-tuple of a valid relation of opposition can be reduced to a set of 2-tuples (see [13] for a similar rationale with classical 3-ary connectives); but also, that these relations can be constructed through intermediary operations within a more fine-grained formal semantics.

3.2. Opposite as an operation

Taking the preceding example again, the concepts "white" and "black" stand into a contrary relation. We propose in the following to investigate the logical properties of "blacken", that is, the operation by means of which anything white is turned into something black.

Proposition 7

An opposite O is a mapping O(a) upon a relatum a of an opposition, such that it turns it into the second relatum b of the given opposition: Op(a,O(a)) = Op(a,b).

A logic of colours has already been set up by Jaspers (see [8]) in the same vein, where chromatic oppositions are displayed by a set of Boolean bitstrings that is going to be explained in the following semantics.

3.3. An algebraic semantics for oppositions and opposites

Our formal semantics has a twofold purpose: to afford a formal theory of meaning for any sort of objects from mere individuals to usual sentences; to set up this semantics with the help of Boolean algebra. While it is *a locus classicus* to say that only sentences make properly sense by their truth-conditions, the following leads to a more comprehensive "non-Fregean" semantics that characterizes the sense by means of questions and answers.

3.3.1. Question-Answer Semantics

A special attention is paid to the way in which information is conveyed about an individual, concept, or sentence; indeed, how they are depicted by some of their properties may have a deep influence on their general meaning. This leads to a question-dependent view of meaning, where the value of any given information relies upon the sorts of properties put into focus.

Our Question-Answer Semantics (hereafter: **QAS**) resorts to a non-Fregean theory of sense and reference, assuming that no reference is a truth-value. By doing so, **QAS** is on a par with Roman Suszko's critics of the so-called "Fregean Axiom" in [19].

Proposition 8

The *meaning* of any object a is determined by its sense and its reference, *sense* being a finite ordered set $\mathbf{Q}(a) = \langle \mathbf{q}_1(a),...,\mathbf{q}_n(a)\rangle$ of n *questions* about a (where $n \geq 1$) and *reference* being the set $\mathbf{A}(a) = \langle \mathbf{a}_1(a),...,\mathbf{a}_n(a)\rangle$ of corresponding *answers*.

The standard, truth-conditional semantics can be embedded as a special case of our question-answer framework, by using the words "true" and "false" as the metalinguistic predicates of specific questions among other ones. By contrast, our non-standard semantics results in a calculus of logical values while going beyond the prominent case of "truth-values".

3.3.2. Boolean algebra of oppositions

Given that m sorts of answers can be given to n questions, there are m^n possible values for each a. For instance, asking $n = 3$ questions about a and having $m = 2$ available answers yields a set of $m^n = 2^3 = 8$ logical values including $\mathbf{A}(a) = 111, 110, \ldots$ until 000. The number of such logical values is relative to the formal ontology within which a is presented; that is, it depends upon how many data are needed in order to be able to individuate a, i.e. to make it logically different from any other object in a given set. In a nutshell, n is a sufficient amount of questions iff $\mathbf{A}(a) \neq \mathbf{A}(b)$, assuming that these questions can characterize anything meaningful by a finite set of properties (i.e. the semantic predicates of a question). The perplexing cases of vague predicates and ensuing paradoxes should lead to a Boolean counterpart of infinite-valued matrices; but they won't be considered in the present paper.

It is worthwhile to note that the objects are not provided with a *single* value like "true" or "false" in **QAS**; rather, their reference amounts to an ordered combination of single *sub-values* that stand for each of the answers. We stick to the Boolean values 1 and 0 in the sequel, where $m = 1$ is a yes-answer and $m = 0$ a no-answer, while pointing out that a question-answer game needn't be confined into such binary answers. Let us call by a *bitstring* any such structured string of ordered answers; in the case of a Boolean algebra, each sub-value of a string takes either 1 or 0 and is thereby reminiscent of logical bivalence. At the same time, the m^n possible values of an object go largely beyond two cases whenever $m > 1$ and result in something analogous with a many-valued calculus of Boolean bistrings.

A calculus of logical oppositions is made possible by making use of bitwise operations.

Proposition 9

$\cap$ and $\cup$ are the operations of *meet* and *join* upon values 1 and 0, such that $1 > 0$. Then:

$x \cap y = max(x,y)$;

$x \cup y = min(x,y)$.

The Aristotelian relations can be rendered algebraically by asking questions about the compossibility of truth-values between any two propositions a and b. Assuming that every classical (bivalent) proposition a can be translated by a disjunctive normal form $\mathbf{A}(a) = \langle \mathbf{a}_1(a),\mathbf{a}_2(a),\mathbf{a}_3(a),\mathbf{a}_4(a)\rangle$, to characterize such a propositional opposition between a and b amounts to a questioning about their various compossibilities among $n = 4$ possible cases, namely: whether a and b can be true together; whether a can be true while b is false; whether a can be false while b is true; whether a and b can be false together.

More generally, oppositions go beyond the sole logical category of propositions and are to be defined in common terms of compossible yes- or no-answers for their arbitrary objects a,b, irrespective of the sorts of questions to be asked about them.

Proposition 10

Opposition Op(a,b) is a set Op = {CT,CD,SCT,SB} of relations to be defined:

10.1 by the logical values $\mathbf{A}(a)$ and $\mathbf{A}(b)$ of any two objects a and b such that, for any i^{th} question of the same question-answer game, these stand into a relation of:

contrariety CT(a,b)	iff	$\forall \mathbf{a}_i$: $\mathbf{a}_i(a) = 1 \Rightarrow \mathbf{a}_i(b) = 0$
contradiction CD(a,b)	iff	$\forall \mathbf{a}_i$: $\mathbf{a}_i(a) = 1 \Leftrightarrow \mathbf{a}_i(b) = 0$
subcontrariety SCT(a,b)	iff	$\forall \mathbf{a}_i$: $\mathbf{a}_i(a) = 0 \Rightarrow \mathbf{a}_i(b) = 1$
subalternation SB(a,b)	iff	$\forall \mathbf{a}_i$: $\mathbf{a}_i(a) = 1 \Rightarrow \mathbf{a}_i(b) = 1$

10.2. by the Booleans operations of meet and join, together with the logical values of *tautology* $\top$ (only yes-anwers) and *antilogy* $\bot$ (only no-answers):

contrariety CT(a,b)	iff	$\mathbf{A}(a) \cap \mathbf{A}(b) = \bot$ and $\mathbf{A}(a) \cup \mathbf{A}(b) \neq \top$
contradiction CD(a,b)	iff	$\mathbf{A}(a) \cap \mathbf{A}(b) = \bot$ and $\mathbf{A}(a) \cup \mathbf{A}(b) = \top$
subcontrariety SCT(a,b)	iff	$\mathbf{A}(a) \cap \mathbf{A}(b) \neq \bot$ and $\mathbf{A}(a) \cup \mathbf{A}(b) = \top$
subalternation SB(a,b)	iff	$\mathbf{A}(a) \cap \mathbf{A}(b) = \mathbf{A}(a)$ and $\mathbf{A}(a) \cup \mathbf{A}(b) = \mathbf{A}(b)$

Two notes are in order, in connection with the above definitions of opposition.
On the one hand, a minimal *number* of questions is required to preserve the relations of contrariety and subcontrariety between any objects a, b.

Proof. Let $i < 3$, e.g. $i = 2$ or $i = 1$. Suppose that $\mathbf{A}(a) = 10$. If $\mathbf{A}(b) = 00$, then Op(b,a) = SB(b,a); if $\mathbf{A}(b) = 01$, then Op(a,b) = CD(a,b); if $\mathbf{A}(b) = 11$, then Op(a,b) = SB(a,b). No other relation occurs whenever $i = 2$, and, *a fortiori*, with $i < 2$. ■

The case where $i = 1$ corresponds to the usual truth-functional semantics where each proposition is given a unique value 1 (for True) and 0 (for False), and this is the reason why McCall rightly claimed in [10] that no other operator than a contradiction-forming one can be devised in it.
On the other hand, the above definitions betray a real difference between *subalternation* (in symbols: SB) and the other relations: not only does the former not hold when a and b are interchanged, since SB is not a *symmetrical* relation; but also, the $n = 4$ questions used to characterize opposition are not sufficient to identify SB. Indeed, the latter holds once every given question about a and b can be answered positively or negatively together; but an additional condition must be added to it, to the effect that a given question cannot be answered negatively about b once answered positively about a. By omitting this further constraint, the result is a mere relation of non-contradiction or *independence* (see [2]) with respect to which subalternation is a subcase.

Proposition 11
Op(a,b) is a relation of *independence* IND iff:
IND(a,b) iff $\mathbf{A}(a) \cap \mathbf{A}(b) \neq \bot$ and $\mathbf{A}(a) \cup \mathbf{A}(b) \neq \top$

It may be replied that SB is not a relation of opposition at all, in the light of the preceding difficulty. For example, Demey & Smessaert argued in [5] that the Aristotelian square is a complex gathering of two different sorts of relation from two separate question-answer games, namely: opposition Op(a,b), and implication Imp(a,b). While Imp can be equated with the Tarskian relation of consequence Cn, we argue that subalternation can be embedded into the unique question-answer game defining logical oppositions (see section 4.2). By doing so, consequence is made a particular case of opposition in the sense that its very definition calls for the relation Op. More precisely, subalternation is formed by a kind of *double negation*. In accordance with our structuralist view of meaning as a synchronic set of different objects, let us see how negation takes in our algebraic logic of opposition.

3.3.3. Opposites as negations

As an alternative to the systematic treatment through sequent calculi (e.g. in [11]), Piaget paved the way to a general theory of negation by proposing in [12] a so-called *theory of reversibility* and its corresponding INRC Group of group-theoretical operations. In order to account for his genetic epistemology, Piaget claimed that intelligent reasoning consists in transforming structured elements with the help of a number of basic operations such as switch and permutation. A brief look at the former definitions of oppositions (see Definition 10) shows how reversibility is on a par with our main concern.

To begin with, Piaget's INRC Group is a set of 4 operations N,R,C, together with a trivial one I. Albeit restricted to the special case of binary propositions of classical logic, this whole device can be rendered within **QAS** as follows.

Proposition 12

Let $\mathbf{A}(a) = \langle \mathbf{a}_1(a),...,\mathbf{a}_n(a)\rangle$ be an arbitrary object individuated by n questions, and let § be a switching operation of denial that applies to single values $\mathbf{a}_i(a)$ such that §(1) = 0. Then the INRC Group can be defined by operations of switching and permuting upon every single value of $\mathbf{A}(a)$:

Identity I	(not switching, not permuting)	$\mathrm{I}(a) = \langle(\mathbf{a}_1(a),...,\mathbf{a}_n(a)\rangle$
Inversion N	(switching, not permuting)	$\mathrm{N}(a) = \langle\S(\mathbf{a}_1(a)),...,\S(\mathbf{a}_n(a))\rangle$
Reciprocity	(not switching, permuting)	$\mathrm{R}(a) = \langle(\mathbf{a}_n(a)),...,(\mathbf{a}_1(a))\rangle$
Correlation	(switching, permuting)	$\mathrm{C}(a) = \langle\S(\mathbf{a}_n(a)),...,\S(\mathbf{a}_1(a))\rangle$

Each of these operations can be obtained through a combination of other ones. Thus

Proposition 13

INRC Group includes the following rules of iteration:

Identity	II = NN = RR = CC = I
Commutation	For every X,Y $\in$ {I,N,R,C}, XY = YX
Idempotence	For every X $\in$ {I,N,R,C}, IX = X
Complementarity	NR = C, NC = R, RC = N

While stressing the link between reversibility and the opposite-forming operators O, let us note the difference between the operations of *denial* and *negation*: the former is applied to single values, whereas the latter applies to whole structured values. Denial is a sort of *proto-negation* that helps to form logical negations, just as Humberstone suggested in [7] by proposing to iterate negation such that $\S\S a = {\sim}a$.

Moreover, N exactly matches with a contradiction-forming operator in that it proceeds by reverting any single value and thereby satisfies the definition of contradiction (see Definition 10). Nevertheless, there is no such one-one correspondence between each of the four operations of Piaget's INRC Group and the four opposite-forming operators op = {ct,cd,sct,sb}. Apart from the special case N(a) = cd(a), which opposite is constructed by R and C depends upon which logical value these reversibility operators are applied to. Taking $\mathbf{A}(a)$ = 1000 as an example, R(a) = 0001 = ct(a) and C(a) = 1110 = sb(a); while taking $\mathbf{A}(a)$ = 1100 entails that R(a) = 0011 = cd(a) and C(a) = 1100 = I(a).

More interestingly, negation can be characterized in two ways through our opposite-forming operators and, thus, in terms of opposition. First, more than three non-trivial operators like Piaget's ones can be devised to create opposite terms O(a) from a; it consists in applying the operator of denial to some single values of a but not all of them, the result of which is a distinction between *global* and *local* negations (see [15,16]). Second, such usual non-classical negations as *paracomplete* (intuitionist) and *paraconsistent* negations can be rendered within our logical theory of opposition. Starting from a result by Béziau [3], it has been shown that a logical hexagon of modal oppositions includes three sorts of logical negations, namely: classical negation is the contradiction-forming operator, whereas paracomplete and paraconsistent negations correspond to the contrary- and subcontrary-forming operators, respectively. More generally, a distinction is thus made between extensional and intensional negations.

Proposition 14
For any object a:
The contradiction-forming operator cd is an *extensional* operator of negation such that there is only one b resulting from cd(a) = b.
The contrary- and subcontrary-forming operators ct and sct are *intensional* operators of negation such that there are more than one b resulting from ct(a) = b.

Proof. By Proposition 10.
A logical negation is *paracomplete* iff the Law of Excluded Middle (LEM) fails with it, i.e. there is a logical negation O such that LEM: $a \vee$ O(a) is not tautological. Let $\mathbf{A}(a) \cup \mathbf{A}(O(a)) \neq \top$ the algebraic counterpart of the statement that LEM is not tautological in **QAS**. If $\mathbf{A}(a) \cup \mathbf{A}(O(a)) \neq \top$, then $\mathbf{a}_i(a) = \mathbf{a}_i(O(a)) = 0$ for some single value $\mathbf{a}_i(a)$ of $\mathbf{A}(a)$. By definition of CT, $\mathbf{A}(a) \cup \mathbf{A}(b) \neq \top$ when ct(a) = b. Hence LEM fails if O = ct.
A logical negation is *paraconsistent* iff the Law of Explosion (LE) fails, i.e. there is a logical negation O such that, for any b, $a \wedge$ O(a) does not entail b. Let SB($a \wedge$ O(a),b) the counterpart of LE. The failure of LE is to be proved by a counterexample such that $\mathbf{A}(a \wedge O(a)) \cap \mathbf{A}(b) \neq \mathbf{A}(a \wedge O(a))$, i.e. $\mathbf{A}(a \wedge O(a)) \neq \perp$. By definition of SCT, $\mathbf{A}(a) \cap \mathbf{A}(b) \neq \top$ when sct(a) = b. Hence LE fails if O = sct. ■

To sum up, the debate launched by Slater about the meaning of logical negation in [18] led to the construction of opposite-forming operators, doing justice to the occurrence of non-classical negations within the theory of opposition. Such a rationale had been foreshadowed by Piaget's INRC Group, while noting again that the latter are to be clearly distinguished from the class op of opposite-forming operators (i.e. there is no one-one correspondence between the pairs {R,C} and {ct,sb}, respectively).

4. Objections (and its replies)

A number of objections can be raised against our whole enterprise, from the structuralist-minded view of meaning to the translation of standard logics into **QAS**. Let us see a sample of these, while attempting to give sufficient replies.

4.1. Opposition is nothing but incompatibility

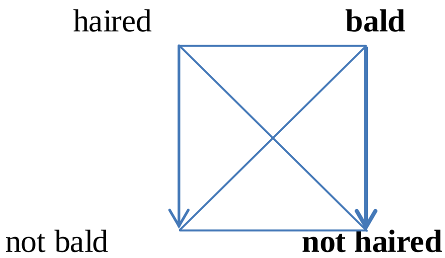
Aristotle claimed himself that subcontrariety is an opposition "only verbally", in contrast to the genuine instances of contrariety and contradiction. This suggests that an Aristotelian opposition between any two sentences a and b is synonymous with *incompatibility*, in the sense that both cannot be true at once. If so, then our logical theory of opposition should be renamed as a theory of non-identity or, better, a theory of *difference* that accounts for the logical connections between different objects within a structured set of objects (possible worlds, or lexical fields).

A look at the Platonic process of "diaeresis" should argue for our case, however. Indeed, the dialectic process of definition can be seen as a diachronic question-answer game where different objects are more and more individuated by increasing the number of questions characterizing them. Moreover, it has been seen that the operator of denial § applies to a single Boolean value by switching it from 1 to 0 (and conversely), just as the contradiction-forming operator cd applies to ordered values.

In a nutshell, our algebraic view of logical values as structured bitstrings helps to explain why opposition produces the meaning of different objects without implying their mutual incompatibility. This also means that contradiction is the primary opposition underlying any other one, including the "verbal" case of subcontrariety and even subalternation.

4.2. Consequence is not subalternation

That a man is bald *entails* that it is not haired, in accordance to the contrary relation between "bald" and "haired". Indeed, "not haired" is the contradictory of "haired" and, given that any contradictory of a contrary is a subaltern, the contradictory of the contrary of "bald", "not haired", stands for its subaltern.



In semi-formal words: ct(haired) = bald, and cd(bald) = not bald; hence cd(ct(haired)) = sb(haired) = not bald. This calculus is another evidence for the fact that Piaget's reversibility operators differ from our opposite-forming operators, by passing, insofar as the latter are not commutative.

Proposition 15
Let ~ the symbol for classical negation, ¬ for paracomplete negation, and – for paraconsistent negation. Then:
15.1 $\sim(a) := \text{cd}(a)$; $\neg(a) := \text{ct}(a)$; $-(a) := \text{sct}(a)$
15.2 subalternation results from the double mixed negation $\sim\neg(\text{a}) := \text{cd}(\text{ct}(\text{a}))$
15.3 the members of op are not *commutative* operators: for any $x,y \in \{\text{ct,cd,sct,sb}\}$, $x(y(a)) \neq y(x(a))$ (where $x \neq y$)

A proof of 15.3 can be given thanks to the intensional behavior of the so-called non-classical negations, where there is a one-many mapping from the input value to the resulting opposite outputs (between brackets in the sequel).

Proof. By induction upon the members of the class op of opposing-forming operators.
Let $\mathbf{A}(a) = 1000$. Then:
$\text{ct}(a) = \{0000,0100,0010,0001,0110,0011,0101\}$
$\text{cd}(\text{ct}(a)) = \{1111,1011,1101,1110,1001,1100,1010\}$
$\text{cd}(a) = 0111$
$\text{ct}(\text{cd}(a)) = \varnothing$
Therefore $\text{cd}(\text{ct}(a)) \neq \text{ct}(\text{cd}(a))$.
(The reader is pleased to go through the entire inductive proof.) ■

The sole exception is the case where the iterated operator is the extensional case of contradiction, reproducing the classical law of double negation in **QAS**: $\text{cd}(\text{cd}(a)) := \sim\sim(a) = a$. It is obviously not so with non-classical negations, especially with the paracomplete operator that famously violates the aforementioned inference rule: $\text{ct}(\text{ct}(a)) \neq a$.

It could be replied to all of this that subalternation is nothing but a very restrictive counterpart of logical consequence. Whatever the case may be about the crucial properties of consequence, it is taken to be granted that our Boolean treatment is on a par with the semantic view of logical consequence as *truth-preservation*. Besides, the former helps to abstract from the notion of truth by claiming that any yes-answer to premises must lead to the same answers in the conclusion. In other words, any object occurs as a consequence whenever it confirms anything accepted about its premises. For this very reason, consequence, entailment, and subalternation are equated with each other from our point of view. Although there might be alternative views of consequence, let us argue that our **QAS** should be able to account for such non-standard versions by changing the central clauses of its question-answer game.

4.3. There is no calculus for opposite-forming functions

It has been noted in the preceding section that most of the opposite-forming operators proceed as one-many mappings, that is, operators with one input value and several output values. Mathematically speaking, this is a sufficient reason to establish that op is not a proper *function*: only one-one or many-one mappings are entitled to be called by this name, whereas one-many mappings do not. This is not a sufficient reason to conclude that no calculus can be devised for a theory of opposition and its constructive operators, however. Following the calculus of iterated negations by Kaneiwa [9], and by analogy with the arithmetic operation of square root, it clearly appears that $\sqrt{4}$ has a definite number of output values, i.e. $\sqrt{4} = \{-2,2\}$. In the same line, a definite number of values $b_1,\ldots,b_n$ can be assigned to any opposite of a such that $\mathrm{op}(a) = \{b_1,\ldots,b_n\}$. This calculus leads to a set op of *multifunctions* (or many-valued functions), instead of usual functions.

Admittedly, the resulting calculus is complicated by a more complex range of possible values. For instance, how many contraries of an increasing width of bitstrings there can be should be an increasing set of outputs … or the null set, in case the input value couldn't be said to have contraries at all. To clarify this complex situation, let us return to the structured values and their set-theoretical properties.

Proposition 16

Let Card be the symbol of cardinality. Then for any value of a, $\mathrm{Card}(\mathrm{cd}(a)) = 1$.

Proof. By Proposition 14, every logical object cannot have but one contradictory. Hence the cardinality of $\mathrm{cd}(a)$ is 1.

Proposition 17

Let m, n and $\mathrm{y}(a)$ be the number of answers, questions, and yes-answers in the logical value of a, respectively. Then for any a, $\mathrm{Card}(\mathrm{ct}(a)) = m^{n\text{-}\mathrm{y}(a)} - 1$.

Proof. By truncating the yes-answers **A**(a).
According to the definition of contrariety in Proposition 10, any yes-answer to a i^{th} question about a entails a corresponding no-answer for its contrary b. That is, $\mathbf{a}_i(b) = \S(1) = 0$ whenever $\mathbf{a}_i(a) = 1$. By truncating every valuation where $\mathbf{a}_i(a) = 1$, there remains a subset of n-$\mathrm{y}(a)$ cases with only no-answers for a, i.e. $\mathbf{a}_{n/i}(a) = 0$. Then $\mathbf{a}_{n/i}(b) = 1$ or 0, which yields a maximal number of possible valuations while excluding the special case with only $\mathbf{a}_{n/i}(a) = 1$ (a and b would be contradictories, otherwise). As there are m^n possible valuations for m sorts of answers and n questions, the non-truncated bitstring of n-$\mathrm{y}(a)$ elements results in a set of $m^{n\text{-}\mathrm{y}(a)}$ possible valuations minus the aforementioned excluded case with only yes-answers. Hence $\mathrm{Card}(\mathrm{ct}(a)) = m^{n\text{-}\mathrm{y}(a)} - 1$. ■

Example: let $\mathbf{A}(a) = 0100$, with $m = 2$, $n = 4$, and $\mathrm{y}(a) = 1$. Hence:
$\mathbf{a}_2(a) = 1$, therefore $\mathbf{a}_2(\mathrm{ct}(a)) = \S(1) = 0$; by truncating the latter case, there remains a set of n-$\mathrm{y}(a)$ = 3 cases where $\mathbf{a}_{n/2}(a) = 1$ or 0. That is:

	$\mathbf{a}_1(a)$	$\mathbf{a}_2(a)$	$\mathbf{a}_3(a)$	$\mathbf{a}_4(a)$	
$\mathbf{A}(a)$	0	1	0	0	
$\mathbf{A}(\mathrm{ct}(a))$	0	0	0	0	(1)

1	0	0	0	(2)
0	0	1	0	(3)
0	0	0	1	(4)
1	0	1	0	(5)
1	0	0	1	(6)
0	0	1	1	(7)
~~1~~	~~0~~	~~1~~	~~1~~	(= cd(*a*))

Card(ct(*a*)) = $m^{n\text{-y}(a)}$ - 1 = $2^{4\text{-}1}$ - 1 = 2^3 - 1 = 8 - 1 = 7, namely:
ct(*a*) = {0000,1000,0010,0001,1010,1001,0011}
Note: **A**(*a*) = 0100 and **A**(*b*) = 0000 stand into a relation of contrariety and subalternation at once, since we have both CT(*a*,*b*) = CT(*b*,*a*) and SB(*b*,*a*). This is allowed by the definitions of CT and SB, however (see Proposition 10), merely excluding the case where *a* and *b* cannot be *false* at once (by CT).

Proposition 18
Let *m*, *n* and y(*a*) be the number of answers, questions, and yes-answers in the logical value of *a*, respectively. Then for any *a*, Card(sct(*a*)) = $m^{\text{y}(a)}$ - 1.

Proof. By truncating the no-answers in **A**(*a*).
According to the definition of subcontrariety in Proposition 10, any no-answer to a i^{th} question about *a* entails a corresponding yes-answer for its subcontrary *b*. That is, $\mathbf{a}_i(b)$ = §(0) = 1 whenever $\mathbf{a}_i(a)$ = 0. By truncating every valuation where $\mathbf{a}_i(a)$ = 0, it remains a subset of y(*a*) cases with only yes-answers for *a*, i.e. $\mathbf{a}_{n/i}(a)$ = 1. Then $\mathbf{a}_{n/i}(b)$ = 1 or 0, which yields a maximal number of possible valuations while excluding the special case with only $\mathbf{a}_{n/i}(a)$ = 0 (*a* and *b* would be contradictories, otherwise). As there are m^n possible valuations for *m* sorts of answers and *n* questions, the non-truncated bitstring of y(*a*) elements results in a set of $m^{\text{y}(a)}$ possible valuations minus the aforementioned excluded case. Hence Card(sct(*a*)) = $m^{\text{y}(a)}$ - 1. ■

Example: let **A**(*a*) = 1011, with *m* = 2, *n* = 4, and y(*a*) = 3. Hence:
$\mathbf{a}_2(a)$ = 0, therefore $\mathbf{a}_2$(sct(*a*)) = §(0) = 1; by truncating the latter case, there remains a set of *n*-y(*a*) = 3 cases where $\mathbf{a}_{n/2}(a)$ = 1 or 0. That is:

	$\mathbf{a}_1(a)$	$\mathbf{a}_2(a)$	$\mathbf{a}_3(a)$	$\mathbf{a}_4(a)$	
A(*a*)	1	0	1	1	
A(sct(*a*))	1	1	1	1	(1)
	0	1	1	1	(2)
	1	1	0	1	(3)
	1	1	1	0	(4)
	0	1	0	1	(5)
	0	1	1	0	(6)
	1	1	0	0	(7)
	~~0~~	~~0~~	~~1~~	~~0~~	(= cd(*a*))

Card(sct(*a*)) = $m^{\text{y}(a)}$ - 1 = 2^3 - 1 = 8 - 1 = 7, namely:
sct(*a*) = {1111,0111,1101,1110,0101,0110,1100}

Note: $\mathbf{A}(a)$ = 1101 and $\mathbf{A}(b)$ = 1111 stand into a relation of subcontrariety and subalternation at once, since we have both SCT(a,b) = SCT(b,a) and SB(a,b). This is allowed by the definitions of SCT and SB (see Proposition 10), merely excluding the case where a and b cannot be *true* at once (by definition of SCT).

The above computations nicely match with the definition Aristotle gave to subcontraries as "contradictories of contraries" (see e.g. [3]). This plural expression should be clearly distinguished from the singular characterization of a subaltern as the "contradictory of a contrary".

Proposition 19

For any objects a,b:

19.1 a and b are *subcontrary* to each other iff their contradictories are contrary to each other, so that:

SCT(a,b) = CT(cd(a),cd(b))

Proof. According to Proposition 10, contradiction proceeds by switching every answer $\mathbf{a}_i(a)$ such that $\mathbf{a}_i(\text{cd}(a)) = \S(\mathbf{a}_i(a))$. According to Proposition 17 and Proposition 18, the non-truncated subsets of contraries and subcontraries are respectively such that $\mathbf{a}_{n/i}(a) = 0$ and $\mathbf{a}_{n/i}(a) = 1$, i.e. $\mathbf{a}_{n/i}(a) = \S(\mathbf{a}_{n/i}(a))$. Now these are contradictory to each other. Therefore, SCT(a,b) = CT(cd(a),cd(b)). ■

19.2 b is a *subaltern* of a iff b is the contradictory of a contrary of a, so that:

Card(sb(a)) = Card(ct(a))

Proof. By Proposition 15.2, sb(a) = cd(ct(a)). There is only one contradictory of any opposite term op(a) of a, by Proposition 16: Card(cd(op(a))) = Card(op(a)), hence Card(sb(a)) = Card(ct(a)).

An alternative proof of the later result can be obtained through the definition of subalternation by Proposition 10: each yes-answer being preserved in the subaltern sb(a), truncate every yes-answer of a while excluding the case where $\mathbf{a}_{n/i}(\text{sb}(a)) = 1$ (a and sb(a) would be identical, otherwise). Thus compute the non-truncated bitstring of no-answers as $m^{n\text{-y}(a)} - 1$.

Example: let $\mathbf{A}(a)$ = 0100, with $m = 2$, $n = 4$, and y(a) = 3. Hence:

$\mathbf{a}_2(a) = 1$, therefore $\mathbf{a}_2(\text{sb}(a)) = 1$; by truncating the latter case, there remains a set of n-y(a) = 3 cases where $\mathbf{a}_{n/2}(a)$ = 1 or 0. That is:

	$\mathbf{a}_1(a)$	$\mathbf{a}_2(a)$	$\mathbf{a}_3(a)$	$\mathbf{a}_4(a)$	
$\mathbf{A}(a)$	0	1	0	0	
$\mathbf{A}(\text{sb}(a))$	1	1	0	0	(1)
	0	1	1	0	(2)
	0	1	0	1	(3)
	1	1	1	0	(4)
	1	1	0	1	(5)
	0	1	1	1	(6)
	1	1	1	1	(7)
	~~0~~	~~1~~	~~0~~	~~0~~	(= a)

Card(sb(a)) = $m^{\text{y}(a)} - 1 = 2^3 - 1 = 8 - 1 = 7$, namely:

sct(a) = {1100,0110,0101,1110,1101,0111,1111}

5. Conclusion

The gist of the present paper relied upon an algebraic analysis of opposition, in the name of a structural view of meaning. Not everything has been said about it, admittedly: although logical consequence is depicted as a by-product of the larger relation of opposition, no counterpart of Tarski's systematic work about consequence is available until now with respect to opposition.

This should lead to a twofold investigation in later works. Firstly, a general theory of iterated oppositions for n iterations, to generalize the above section 4.3 and its multifunctional calculus of opposites: what can be the contrary of the subcontrary of the subaltern of some object a, for instance? Secondly, the construction of an abstract operator of opposition in line with Tarski's operator of consequence (see especially [21]): can there be such an operator to be characterized either in logic, or algebra, or topology?

Whether what has been displayed in the paper belongs to the area of algebra or logic of opposition is questionable. For one thing, our formal theory of opposition crucially relies upon Boolean bitstrings, and this has much more to do with algebra than logic. At the same time, such a distinction between logic and algebra assumes that the former be considered as a pair $\langle L,Cn\rangle$ including a formal language (set of formulas) L and a basic operator of consequence Cn upon elements of L. A next step towards a more comprehensive approach of logic would consist in embedding logical consequence within a broader pair $\langle L,Op\rangle$, accordingly: just as consequence has been investigated in the form of either a relation or an operator [20,21], opposition could be viewed from the perspective of a general relation Op or a general opposite-forming operator op.

Finally, our treatment of meaning through Boolean translations of information amounts to a finitist version of possible-world semantics, i.e. an algebraic semantics where models are finite sets of sets of objects. Meaning as a set of lexical fields is thus treated by a finite set of overlapping question-answer games about definite objects. If so, then whoever aspiring to a general model theory should blame **QAS** for limiting the use of logic to finitely many models. Two replies could be given in turn: if finite question-answer games lead to finitely many-valued sets of objects, then their infinite counterparts might lead to infinitely many-valued objects (by analogy to the infinitely many-valued matrices); eventually, our constructive treatment of meaning as a questioning process is played by bona fide speakers who don't practice with infinite set of data. For who plays with infinity, if not God (if any)?

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