Florentin Smarandache

(author and editor)

Collected Papers

(on Neutrosophics Theory and Applications)

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Introductory Note

This twelfth volume of Collected Papers includes 86 papers comprising 976 pages on Neutrosophic Theories and their Applications, published between 2013-2021 in the international book series about neutrosophic sets and systems by the author alone or in collaboration with the following 112 co-authors (alphabetically ordered) from 21 countries: Abdel Nasser H. Zaied, Muhammad Akram, Bobin Albert, S. A. Alblowi, S. Anitha, Guennoun Asmae, Assia Bakali, Ayman M. Manie, Abdul Sami Awan, Azeddine Elhassouny, Erick González-Caballero, D. Dafik, Mithun Datta, Arindam Dey, Mamouni Dhar, Christopher Dyer, Nur Ain Ebas, Mohamed Eisa, Ahmed K. Essa, Faruk Karaaslan, João Alcione Sganderla Figueiredo, Jorge Fernando Goyes García, N. Ramila Gandhi, Sudipta Gayen, Gustavo Alvarez Gómez, Sharon Dinarza Álvarez Gómez, Haitham A. El-Ghareeb, Hamiden Abd El-Wahed Khalifa, Masooma Raza Hashmi, Ibrahim M. Hezam, German Acurio Hidalgo, Le Hoang Son, R. Jahir Hussain, S. Satham Hussain, Ali Hussein Mahmood Al-Obaidi, Hays Hatem Imran, Nabeela Ishfaq, Saeid Jafari, R. Jansi, V. Jeyanthi, M. Jeyaraman, Sripati Jha, Jun Ye, W.B. Vasantha Kandasamy, Abdullah Kargın, J. Kavikumar, Kawther Fawzi Hamza Alhasan, Huda E. Khalid, Neha Andalleb Khalid, Mohsin Khalid, Madad Khan, D. Koley, Valeri Kroumov, Manoranjan Kumar Singh, Pavan Kumar, Prem Kumar Singh, Ranjan Kumar, Malayalan Lathamaheswari, A.N. Mangayarkkarasi, Carlos Rosero Martínez, Marvelio Alfaro Matos, Mai Mohamed, Nivetha Martin, Mohamed Abdel-Basset, Mohamed Talea, K. Mohana, Muhammad Irfan Ahamad, Rana Muhammad Zulqarnain, Muhammad Riaz, Muhammad Saeed, Muhammad Saqlain, Muhammad Shabir, Muhammad Zeeshan, Anjan Mukherjee, Mumtaz Ali, Deivanayagampillai Nagarajan, Iqra Nawaz, Munazza Naz, Roan Thi Ngan, Necati Olgun, Rodolfo González Ortega, P. Pandiammal, I. Pradeepa, R. Princy, Marcos David Oviedo Rodríguez, Jesús Estupiñán Ricardo, A. Rohini, Sabu Sebastian, Abhijit Saha, Mehmet Șahin, Said Broumi, Saima Anis, A.A. Salama, Ganeshsree Selvachandran, Seyed Ahmad Edalatpanah, Sajana Shaik, Soufiane Idbrahim, S. Sowndrarajan, Mohamed Talea, Ruipu Tan, Chalapathi Tekuri, Selçuk Topal, S. P. Tiwari, Vakkas Uluçay, Maikel Leyva Vázquez, Chinnadurai Veerappan, M. Venkatachalam, Luige Vlădăreanu, Ştefan Vlăduţescu, Young Bae Jun, Wadei F. Al-Omeri, Xiao Long Xin.

Keywords

Neutrosophy; Neutrosophic Logic; Neutrosophic Sets; Neutrosophic Crisp Set; Neutrosophic Topology; Neutrosophic Crisp Topology; Interval-Valued Neutrosophic Set; Interval-Valued Neutrosophic Subring; Interval-Valued Neutrosophic Normal Subring; Interval-Valued Neutrosophic Hypersoft Set; Neutrosophic Multiple Regression; Neutrosophic Regression; Neutrosophic Correlation; Neutrosophic Implication; Single Valued Neutrosophic Numbers; Neutrosophic Uninorm; Neutrosophic Implicatory; Neutrosophic Components; Neutrosophic Offset Components; Neutrosophic Distance; Similarity Measure; Bipolar Neutrosophic Sets; Neutrosophic Soft Rough Set; Single-Valued Neutrosophic Triplet Numbers; Single-Valued Neutrosophic Score Function; Single-Valued Neutrosophic Accuracy Function; Single-Valued Neutrosophic Certainty Function; Entropy Measure; Medical Diagnosis; Autoimmune Disease; Fuzzy Cognitive Maps; Neutrosophic Hypergraphs; Optimal Decision Making; Neutrosophic Cubic Translation; Neutrosophic Cubic Multiplication; Neutrosophic Cubic BF Ideal; Neutrosophic Cubic BF Subalgebra; Neutrosophic Cubic Magnified Translation; Quadratic Residues; Quadratic Nonresidues; Neutrosophic Quadratic Residues; Neutrosophic Quadratic Nonresidues; Neutrosophic Quadratic Residue Graph; Neutrosophic Quadratic Nonresidue Graph; Fuzzy Neutrosophic Soft Mapping; Coincidence Point; Fixed Point; Centroid Points; Neutrosophic Metric Space; Banach Contraction; Edelstein Contraction; Trapezoidal Fuzzy Neutrosophic Numbers; TriVariate Truth-Value; MultiVariate Truth-Value; UniVariate Truth-Value; Automata Theory; Box Function; Sociogram; Neutrosophic Sociogram; Neutrosociology; Group Analysis; Sociometry Analysis; Communication; Information; Extensics.

Florentin Smarandache is an emeritus prof. dr. of mathematics at the University of New Mexico, United States. He got his MSc in Mathematics and Computer Science from the University of Craiova, Romania, PhD in Mathematics from the State University of Kishinev, and Postdoctoral in Applied Mathematics from Okayama University of Sciences, Japan, and The Guangdong University of Technology, Guangzhou, China. He is the founder of neutrosophy (generalization of dialectics), neutrosophic set, logic, probability and statistics since 1995 and has published hundreds of papers and books on neutrosophic physics, superluminal and instantaneous physics, unmatter, quantum paradoxes, absolute theory of relativity, redshift and blueshift due to the medium gradient and refraction index besides the Doppler effect, paradoxism, outerart, neutrosophy as a new branch of philosophy, Law of Included Multiple-Middle, multispace and multistructure, hypersoft set, IndetermSoft Set and IndetermHyperSoft Set, SuperHyperGraph, SuperHyperTopology, SuperHyperAlgebra, Neutrosophic SuperHyperAlgebra, degree of dependence and independence between neutrosophic components, refined neutrosophic set, neutrosophic over-under-off-set, plithogenic set / logic / probability / statistics, neutrosophic triplet and duplet structures, quadruple neutrosophic structures, extension of algebraic structures to NeutroAlgebra and AntiAlgebra, NeutroGeometry & AntiGeometry, Dezert-Smarandache Theory and so on to many peer-reviewed international journals and many books and he presented papers and plenary lectures to many international conferences around the world.

In addition, he published many books of poetry, dramas, children' stories, translations, essays, a novel, folklore collections, traveling memories, and art albums
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Neutrosophic Measure and Neutrosophic Integral

Florentin Smarandache

Florentin Smarandache (2013). Neutrosophic Measure and Neutrosophic Integral. *Neutrosophic Sets and Systems* 1, 3-7

Abstract. Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. We now introduce for the first time the notions of neutrosophic measure and neutrosophic integral. Neutrosophic Science means development and applications of neutrosophic logic/set/measure/integral/ probability etc. and their applications in any field. It is possible to define the neutrosophic measure and consequently the neutrosophic integral and neutrosophic probability in many ways, because there are various types of indeterminacies, depending on the problem we need to solve. Indeterminacy is different from randomness. Indeterminacy can be caused by physical space materials and type of construction, by items involved in the space, or by other factors. Neutrosophic measure is a generalization of the classical measure for the case when the space contains some indeterminacy. Neutrosophic Integral is defined on neutrosophic measure. Simple examples of neutrosophic integrals are given.

Keywords: neutrosophy, neutrosophic measure, neutrosophic integral, indeterminacy, randomness, probability.

1 Introduction to Neutrosophic Measure

1.1 Introduction

Let <A> be an item. <A> can be a notion, an attribute, an idea, a proposition, a theorem, a theory, etc.

And let <antiA> be the opposite of <A>; while <neutA> be neither <A> nor <antiA> but the neutral (or indeterminacy, unknown) related to <A>.

For example, if <A> = victory, then <antiA> = defeat, while <neutA> = tie game.

If <A> is the degree of truth value of a proposition, then <antiA> is the degree of falsehood of the proposition, while <neutA> is the degree of indeterminacy (i.e. neither true nor false) of the proposition.

Also, if <A> = voting for a candidate, <antiA> = voting against that candidate, while <neutA> = not voting at all, or casting a blank vote, or casting a black vote. In the case when <antiA> does not exist, we consider its measure be null {m(antiA)=0}. And similarly when <neutA> does not exist, its measure is null { m(neutA) = 0}.

1.2 Definition of Neutrosophic Measure

We introduce for the first time the scientific notion of neutrosophic measure.

Let X be a neutrosophic space, and Σ a σ-neutrosophic algebra over X. A *neutrosophic measure* ν is defined by for neutrosophic set $A \in \Sigma$ by

$\nu : X \to R^3$,

$$\nu(A) = (m(A), m(neutA), m(antiA)), \quad (1)$$

with *antiA* = the opposite of A, and *neutA* = the neutral (indeterminacy) neither A nor anti A (as defined above);

for any $A \subseteq X$ and $A \in \Sigma$,

m(A) means *measure of the determinate part of A;*

m(neutA) means *measure of indeterminate part of A;*

and *m(antiA)* means *measure of the determinate part of antiA*;

where ν is a function that satisfies the following two properties:

a) Null empty set: $\nu(\Phi) = (0,0,0)$.

b) Countable additivity (or σ-additivity): For all countable collections $\{A_n\}_{n\in L}$ of disjoint neutrosophic sets in Σ, one has:

$$\nu\left(\bigcup_{n\in L} A_n\right) = \left(\sum_{n\in L} m(A_n), \sum_{n\in L} m(neutA_n), \sum_{n\in L} m(antiA_n) - (n-1)m(X)\right)$$

where X is the whole neutrosophic space,

and (2)

$$\sum_{n\in L} m(antiA_n) - (n-1)m(X) = m(X) - \sum_{n\in L} m(A_n) = m(\bigcap_{n\in L} antiA_n).$$

1.3 Neutrosophic Measure Space

A neutrosophic measure space is a triplet (X, Σ, ν).

1.4 Normalized Neutrosophic Measure

A neutrosophic measure is called normalized if

$\nu(X) = (m(X), m(neutX), m(antiX)) = (x_1, x_2, x_3)$,

with $x_1 + x_2 + x_3 = 1$,

$$\text{and } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \quad (3)$$

Where, of course, X is the whole neutrosophic measure space.

1.5 Finite Neutrosophic Measure Space

Let $A \subset X$. We say that $\nu(A) = (a_1, a_2, a_3)$ is finite if all a_1, a_2, and a_3 are finite real numbers.

A *neutrosophic measure space* (X, Σ, ν) is called *finite* if $\nu(X) = (a, b, c)$ such that all *a*, *b*, and *c* are finite (rather than infinite).

1.6 σ-Finite Neutrosophic Measure

A neutrosophic measure is called σ-finite if *X* can be decomposed into a countable union of neutrosophically measurable sets of fine neutrosophic measure.

Analogously, a set *A* in *X* is said to have a σ-finite neutrosophic measure if it is a countable union of sets with finite neutrosophic measure.

1.7 Neutrosophic Axiom of Non-Negativity

We say that the neutrosophic measure ν *satisfies the axiom of non-negativity*, if:

$$\forall A \in \Sigma ,$$

$$\nu(A) = (a_1, a_2, a_3) \geq 0 \text{ if } a_1 \geq 0, a_2 \geq 0, \text{ and } a_3 \geq 0 . \quad (4)$$

While a neutrosophic measure ν, that satisfies only the null empty set and countable additivity axioms (hence not the non-negativity axiom), takes on at most one of the $\pm\infty$ values.

1.8 Measurable Neutrosophic Set and Measurable Neutrosophic Space

The members of Σ are called measurable neutrosophic sets, while (X, Σ) is called a measurable neutrosophic space.

1.9 Neutrosophic Measurable Function

A function $f : (X, \Sigma_X) \to (Y, \Sigma_Y)$, mapping two measurable neutrosophic spaces, is called *neutrosophic measurable function* if $\forall B \in \Sigma_Y, f^{-1}(B) \in \Sigma_X$ (the inverse image of a neutrosophic Y-measurable set is a neutrosophic X-measurable set).

1.10 Neutrosophic Probability Measure

As a particular case of neutrosophic measure ν is th neutrosophic probability measure, i.e. a neutrosophic measure that measures probable/possible propositions

$$^{-}0 \leq \nu(X) \leq 3^{+}, \quad (5)$$

where X is the whole neutrosophic probability sample space.

We use nonstandard numbers, such 1^+ for example, to denominate the absolute measure (measure in all possible worlds), and standard numbers such as *1* to denominate the relative measure (measure in at least one world). Etc.

We denote the neutrosophic probability measure by $\mathcal{NP}$ for a closer connection with the classical probability $\mathcal{P}$.

1.11 Neutrosophic Category Theory

The neutrosophic measurable functions and their neutrosophic measurable spaces form a neutrosophic category, where the functions are arrows and the spaces objects.

We introduce the neutrosophic category theory, which means the study of the neutrosophic structures and of the neutrosophic mappings that preserve these structures.

The classical category theory was introduced about 1940 by Eilenberg and Mac Lane.

A neutrosophic category is formed by a class of neutrosophic objects $X, Y, Z, \ldots$ and a class of neutrosophic morphisms (arrows) $\nu, \xi, \omega, \ldots$ such that:

a) If $Hom(X, Y)$ represent the neutrosophic morphisms from X to Y, then $Hom(X, Y)$ and $Hom(X', Y')$ are disjoint, except when $X = X'$ and $Y = Y'$;

b) The composition of the neutrosophic morphisms verify the axioms of

i) Associativity: $(\nu \circ \xi) \circ \omega = \nu \circ (\xi \circ \omega)$

ii) Identity unit: for each neutrosophic object X there exists a neutrosophic morphism denoted id_X, called *neutrosophic identity* of X such that $id_X \circ \nu = \nu$ and $\xi \circ id_X = \xi$

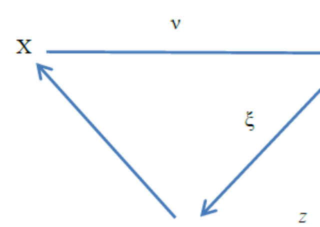

Fig. 2

1.12 Properties of Neutrosophic Measure

a) *Monotonicity.*

If A_1 and A_2 are neutrosophically measurable, with $A_1 \subseteq A_2$, where

$\nu(A_1) = (m(A_1), m(neutA_1), m(antiA_1))$,

and $\nu(A_2) = (m(A_2), m(neutA_2), m(antiA_2))$,

then

$$m(A_1) \leq m(A_2), m(neutA_1) \leq m(neutA_2), m(antiA_1) \geq m(antiA_2) \quad (6)$$

Let $\nu(X) = (x_1, x_2, x_3)$ and $\nu(Y) = (y_1, y_2, y_3)$. We say that $\nu(X) \leq \nu(Y)$, if $x_1 \leq y_1$, $x_2 \leq y_2$, and $x_3 \geq y_3$.

b) *Additivity.*

If $A_1 \cap A_2 = \Phi$, then $\nu(A_1 \cup A_2) = \nu(A_1) + \nu(A_2)$, (7)

where we define

$$(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, a_3 + b_3 - m(X)) \tag{8}$$

where X is the whole neutrosophic space, and

$$a_3 + b_3 - m(X) = m(X) - m(A) - m(B) = m(X) - a_1 - a_2$$
$$= m(antiA \cap antiB). \tag{9}$$

1.13 Neutrosophic Measure Continuous from Below or Above

A *neutrosophic measure* ν *is continuous from below* if, for $A_1, A_2, ...$ neutrosophically measurable sets with $A_n \subseteq A_{n+1}$ for all n, the union of the sets A_n is neutrosophically measurable, and

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n) \tag{10}$$

And a *neutrosophic measure* ν *is continuous from above* if for $A_1, A_2, ...$ neutrosophically measurable sets, with $A_n \supseteq A_{n+1}$ for all n, and at least one A_n has finite neutrosophic measure, the intersection of the sets A_n and neutrosophically measurable, and

$$\nu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n). \tag{11}$$

1.14 Generalizations

Neutrosophic measure is a generalization of the *fuzzy measure*, because when $m(neutA) = 0$ and *m(antiA)* is ignored, we get

$$\nu(A) = (m(A), 0, 0) \equiv m(A) \tag{12}$$

and the two fuzzy measure axioms are verified:

a) If $A = \Phi$, then $\nu(A) = (0,0,0) \equiv 0$

b) If $A \subseteq B$, then $\nu(A) \leq \nu(B)$.

The neutrosophic measure is practically a triple classical measure: a classical measure of the determinate part of a neutrosophic object, a classical part of the indeterminate part of the neutrosophic object, and another classical measure of the determinate part of the opposite neutrosophic object. Of course, if the indeterminate part does not exist (its measure is zero) and the measure of the opposite object is ignored, the neutrosophic measure is reduced to the classical measure.

1.15 Examples

Let's see some examples of neutrosophic objects and neutrosophic measures.

a) If a book of *100* sheets (covers included) has *3* missing sheets, then

$$\nu(book) = (97, 3, 0) \tag{13}$$

where ν is the neutrosophic measure of the book number of pages.

b) If a surface of *5 × 5* square meters has cracks of *0.1 × 0.2* square meters, then $\nu(surface) = (24.98, 0.02, 0)$, (14), where ν is the neutrosophic measure of the surface.

c) If a die has two erased faces then

$$\nu(die) = (4, 2, 0), \tag{14}$$

where ν is the neutrosophic measure of the die's number of correct faces.

d) An approximate number N can be interpreted as a neutrosophic measure $N = \underline{d} + \underline{i}$, where $\underline{d}$ is its determinate part, and $\underline{i}$ its indeterminate part. Its anti part is considered *0*.

For example if we don't know exactly a quantity q, but only that it is between let's say $q \in [0.8, 0.9]$, then $q = 0.8 + i$, where *0.8* is the determinate part of q, and its indeterminate part $i \in [0, 0.1]$.

We get a negative neutrosophic measure if we approximate a quantity measured in an inverse direction on the *x*-axis to an equivalent positive quantity.

For example, if $r \in [-6, -4]$, then $r = -6 + i$, where -6 is the determinate part of r, and $i \in [0, 2]$ is its indeterminate part. Its anti part is also 0.

e) Let's measure the truth-value of the proposition *G* = "through a point exterior to a line one can draw only one parallel to the given line".

The proposition is incomplete, since it does not specify the type of geometrical space it belongs to. In an *Euclidean geometric space* the proposition *G* is <u>true</u>; in a *Riemannian geometric space* the proposition *G* is <u>false</u> (since there is no parallel passing through an exterior point to a given line); in a *Smarandache geometric space* (constructed from mixed spaces, for example from a part of Euclidean subspace together with another part of Riemannian space) the proposition *G* is <u>indeterminate</u> (true and false in the same time).

$$\nu(G) = (1, 1, 1). \tag{15}$$

f) In general, not well determined objects, notions, ideas, etc. can become subject to the neutrosophic theory.

2 Introduction to Neutrosophic Integral

2.1 Definition of Neutrosophic Integral

Using the neutrosophic measure, we can define a neutrosophic integral.

The neutrosophic integral of a function *f* is written as:

$$\int_X f d\nu \tag{16}$$

where *X* is the a neutrosophic measure space, and the integral is taken with respect to the neutrosophic measure ν.

Indeterminacy related to integration can occur in multiple ways: with respect to value of the function to be integrated, or with respect to the lower or upper limit of integration, or with respect to the space and its measure.

2.2 First Example of Neutrosophic Integral: Indeterminacy Related to Function's Values

Let $fN: [a, b] \rightarrow R$ (17)

where the neutrosophic function is defined as:

$fN(x) = g(x)+i(x)$ (18)

with $g(x)$ the determinate part of $fN(x)$, and $i(x)$ the indeterminate part of $fN(x)$, where for all x in $[a, b]$ one has: $i(x) \in [0, h(x)], h(x) \geq 0$. (19)

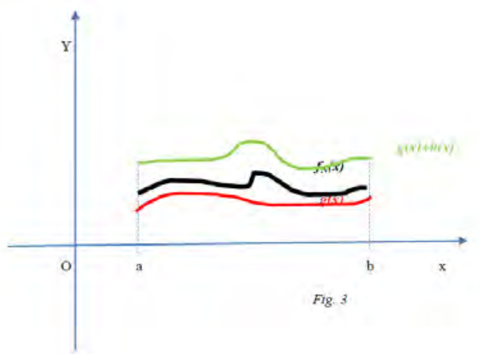

Fig. 3

Therefore the values of the function fN(x) are approximate, i.e. $f_N(x) \in [g(x), g(x)+h(x)]$. (20)

Similarly, the neutrosophic integral is an approximation:

$$\int_a^b f_N(x)d\nu = \int_a^b g(x)dx + \int_a^b i(x)dx \qquad (21)$$

1.10 Second Example of Neutrosophic Integral: Indeterminacy Related to the Lower Limit

Suppose we need to integrate the function

$f: X \rightarrow R$ 22)

on the interval $[a, b]$ from X, but we are unsure about the lower limit a. Let's suppose that the lower limit "a" has a

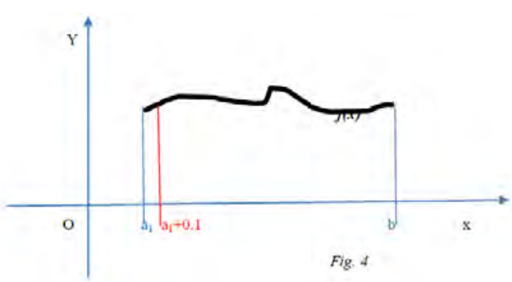

Fig. 4

determinant part "a_1" and an indeterminate part ε, i.e.

$$a = a_1+\varepsilon \qquad (23)$$

where

$$\varepsilon \in [0, 0.1]. \qquad (24)$$

Therefore

$$\int_a^b {}_X\, fd\nu = \int_{a_1}^b f(x)dx - \mathrm{i}_1 \qquad (25)$$

where the indeterminacy i_1 belongs to the interval:

$$i_1 \in [0, \int_{a_1}^{a_1+0.1} f(x)dx]. \qquad (26)$$

Or, in a different way:

$$\int_a^b {}_X\, fd\nu = \int_{a_1+0.1}^b f(x)dx + \mathrm{i}_2 \qquad (27)$$

where similarly the indeterminacy i_2 belongs to the interval:

$$i_2 \in [0, \int_{a_1}^{a_1+0.1} f(x)dx]. \qquad (28)$$

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Soft Neutrosophic Group

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Abstract.In this paper we extend the neutrosophic group and subgroup to soft neutrosophic group and soft neutrosophic subgroup respectively. Properties and theorems related to them are proved and many examples are given.

Keywords: Neutrosophic group,neutrosophic subgroup,soft set,soft subset,soft group,soft subgroup,soft neutrosophic group, soft ,neutrosophic subgroup.

1 Introduction

The concept of neutrosophic set was first introduced by Smarandache [13,16] which is a generalization of the classical sets, fuzzy set [18], intuitionistic fuzzy set [4] and interval valued fuzzy set [7]. Soft Set theory was initiated by Molodstov as a new mathematical tool which is free from the problems of parameterization inadequacy. In his paper [11], he presented the fundamental results of new theory and successfully applied it into several directions such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, theory of probability. Later on many researchers followed him and worked on soft set theory as well as applications of soft sets in decision making problems and artificial intelligence. Now, this idea has a wide range of research in many fields, such as databases [5, 6], medical diagnosis problem [7], decision making problem [8], topology [9], algebra and so on.Maji gave the concept of neutrosophic soft set in [8] and later on Broumi and Smarandache defined intuitionistic neutrosophic soft set. We have worked with neutrosophic soft set and its applications in group theory.

2 Preliminaries

2.1 Nuetrosophic Groups

Definition 1 $[14]$ *Let* $(G,*)$ *be any group and let* $\langle G \cup I\rangle = \{a + bI : a,b \in G\}$ *. Then neutrosophic group is generated by* I *and* G *under* $*$ *denoted by* $N(G) = \{\langle G \cup I\rangle, *\}$ *.* I *is called the neutrosophic element with the property* $I^2 = I$ *. For an integer* n *,* $n + I$ *and* nI *are neutrosophic elements and* $0.I = 0$ *.*

I^{-1} , the inverse of I is not defined and hence does not exist.

Theorem 1 $[14]$ *Let* $N(G)$ *be a neutrosophic group. Then*

1) $N(G)$ in general is not a group;
2) $N(G)$ always contains a group.

Definition 2 *A pseudo neutrosophic group is defined as a neutrosophic group, which does not contain a proper subset which is a group.*

Definition 3 *Let* $N(G)$ *be a neutrosophic group. Then,*

1) A proper subset $N(H)$ of $N(G)$ is said to be a neutrosophic subgroup of $N(G)$ if $N(H)$ is a neutrosophic group, that is, $N(H)$ contains a proper subset which is a group.
2) $N(H)$ is said to be a pseudo neutrosophic subgroup if it does not contain a proper subset which is a group.

Example 1 $(N(Z),+)$, $(N(Q),+)$ $(N(R),+)$ *and* $(N(C),+)$ *are neutrosophic groups of integer, rational, real and complex numbers, respectively.*

Example 2 *Let* $Z_7 = \{o,1,2,...,6\}$ *be a group under addition modulo* 7 *.* $N(G) = \{\langle Z_7 \cup I\rangle, '+' \mathrm{mod}\, ulo 7\}$ *is a neutrosophic group which is in fact a group. For* $N(G) = \{a + bI : a,b \in Z_7\}$ *is a group under `* $+$ *' modulo* 7 *.*

Definition 4 *Let* $N(G)$ *be a finite neutrosophic group. Let* P *be a proper subset of* $N(G)$ *which under the* *operations of* $N(G)$ *is a neutrosophic group. If* $o(P)/o(N(G))$ *then we call* P *to be a Lagrange neutrosophic subgroup.*

Definition 5 $N(G)$ *is called weakly Lagrange neutrosophic group if* $N(G)$ *has at least one Lagrange neutrosophic subgroup.*

Definition 6 $N(G)$ *is called Lagrange free neutrosophic group if* $N(G)$ *has no Lagrange neutrosophic subgroup.*

Definition7 *Let* $N(G)$ *be a finite neutrosophic group. Suppose* L *is a pseudo neutrosophic subgroup of* $N(G)$ *and if* $o(L)/o(N(G))$ *then we call* L *to be a pseudo Lagrange neutrosophic subgroup.*

Definition 8 *If* $N(G)$ *has at least one pseudo Lagrange neutrosophic subgroup then we call* $N(G)$ *to be a weakly pseudo Lagrange neutrosophic group.*

Definition 9 *If* $N(G)$ *has no pseudo Lagrange neutrosophic subgroup then we call* $N(G)$ *to be pseudo Lagrange free neutrosophic group.*

Definition 10 *Let* $N(G)$ *be a neutrosophic group. We say a neutrosophic subgroup* H *of* $N(G)$ *is normal if we can find* x *and* y *in* $N(G)$ *such that* $H = xHy$ *for all* $x, y \in N(G)$ *(Note* $x = y$ *or* $y = x^{-1}$ *can also occur).*

Definition 11 *A neutrosophic group* $N(G)$ *which has no nontrivial neutrosophic normal subgroup is called a simple neutrosophic group.*

Definition 12 *Let* $N(G)$ *be a neutrosophic group. A proper pseudo neutrosophic subgroup* P *of* $N(G)$ *is said to be normal if we have* $P = xPy$ *for all* $x, y \in N(G)$. *A neutrosophic group is said to be pseudo simple neutrosophic group if* $N(G)$ *has no nontrivial pseudo normal subgroups.*

2.2 Soft Sets

Throughout this subsection U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U, and $A \subset E$. Molodtsov [12] defined the soft set in the following manner:

Definition13 $[11]$ *A pair* (F, A) *is called a soft set over* U *where* F *is a mapping given by* F : $A \rightarrow P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ may be considered as the set of e -elements of the soft set (F, A), or as the set of e-approximate elements of the soft set.

Example 3 *Suppose that* U *is the set of shops.* E *is the set of parameters and each parameter is a word or sentence. Let*

$$E = \left\{ \begin{array}{l} \text{high rent,normal rent,} \\ \text{in good condition,in bad condition} \end{array} \right\}.$$

Let us consider a soft set (F, A) *which describes the attractiveness of shops that Mr.* Z *is taking on rent. Suppose that there are five houses in the universe* $U = \{h_1, h_2, h_3, h_4, h_5\}$ *under consideration, and that* $A = \{e_1, e_2, e_3\}$ *be the set of parameters where*

e_1 stands for the parameter 'high rent,

e_2 stands for the parameter 'normal rent,

e_3 stands for the parameter 'in good condition.

Suppose that

$F(e_1) = \{h_1, h_4\}$,

$F(e_2) = \{h_2, h_5\}$,

$F(e_3) = \{h_3, h_4, h_5\}$.

The soft set (F, A) is an approximated family $\{F(e_i), i = 1,2,3\}$ of subsets of the set U which gives us a collection of approximate description of an object. Thus, we have the soft set (F, A) as a collection of approximations as below:

$(F, A) = \{$ high rent $= \{h_1, h_4\}$, normal rent $= \{h_2, h_5\}$, in good condition $= \{h_3, h_4, h_5\}\}$.

Definition 14 $[3]$. *For two soft sets* (F, A) *and*

$\left(H,B\right)$ over U, $\left(F,A\right)$ is called a soft subset of $\left(H,B\right)$ if

1) $A \subseteq B$ and
2) $F(e) \subseteq H(e)$, for all $e \in A$.

This relationship is denoted by $\left(F,A\right) \subset \left(H,B\right)$.

Similarly $\left(F,A\right)$ is called a soft superset of $\left(H,B\right)$ if $\left(H,B\right)$ is a soft subset of $\left(F,A\right)$ which is denoted by $\left(F,A\right) \supset \left(H,B\right)$.

Definition 15 $[3]$. *Two soft sets $\left(F,A\right)$ and $\left(H,B\right)$ over U are called soft equal if $\left(F,A\right)$ is a soft subset of $\left(H,B\right)$ and $\left(H,B\right)$ is a soft subset of $\left(F,A\right)$.*

Definition 16 *Let $[3]$ $\left(F,A\right)$ and $\left(G,B\right)$ be two soft sets over a common universe U such that $A \cap B \neq \phi$. Then their restricted intersection is denoted by $(F,A) \cap_R (G,B) = (H,C)$ where $\left(H,C\right)$ is defined as $H(c) = F(c) \cap G(c)$ for all $c \in C = A \cap B$.*

Definition 17$[3]$ *The extended intersection of two soft sets $\left(F,A\right)$ and $\left(G,B\right)$ over a common universe U is the soft set $\left(H,C\right)$, where $C = A \cup B$, and for all $e \in C$, $H\left(e\right)$ is defined as*

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \cap G(e) & \text{if } e \in A \cap B. \end{cases}$$

We write $(F,A) \cap_\varepsilon (G,B) = (H,C)$.

Definition 18 $[3]$ *The restricted union of two soft sets $\left(F,A\right)$ and $\left(G,B\right)$ over a common universe U is the soft set $\left(H,C\right)$, where $C = A \cup B$, and for all $e \in C$, $H\left(e\right)$ is defined as the soft set $\left(H,C\right) = \left(F,A\right) \cup_R \left(G,B\right)$ where $C = A \cap B$ and $H(c) = F(c) \cup G(c)$ for all $c \in C$.*

Definition 19$[3]$ *The extended union of two soft sets $\left(F,A\right)$ and $\left(G,B\right)$ over a common universe U is the soft set $\left(H,C\right)$, where $C = A \cup B$, and for all $e \in C$, $H\left(e\right)$ is defined as*

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \cup G(e) & \text{if } e \in A \cap B. \end{cases}$$

We write $(F,A) \cup_\varepsilon (G,B) = (H,C)$.

2.3 Soft Groups

Definition 20$[2]$ *Let $\left(F,A\right)$ be a soft set over G. Then $\left(F,A\right)$ is said to be a soft group over G if and only if $F\left(x\right) \prec G$ forall $x \in A$.*

Example 4 *Suppose that*
$G = A = S_3 = \{e,(12),(13),(23),(123),(132)\}$
. Then $\left(F,A\right)$ is a soft group over S_3 where

$$F\left(e\right) = \left\{e\right\},$$
$$F\left(12\right) = \left\{e,\left(12\right)\right\},$$
$$F\left(13\right) = \left\{e,\left(13\right)\right\},$$
$$F\left(23\right) = \left\{e,\left(23\right)\right\},$$
$$F\left(123\right) = F\left(132\right) = \left\{e,\left(123\right),\left(132\right)\right\}.$$

Definition 21$[2]$ *Let $\left(F,A\right)$ be a soft group over G. Then*

1) $\left(F,A\right)$ is said to be an identity soft group over G if $F\left(x\right) = \left\{e\right\}$ for all $x \in A$, where e is the identity element of G and
2) $\left(F,A\right)$ is said to be an absolute soft group if $F\left(x\right) = G$ for all $x \in A$.

Definition 22 *The restricted product $\left(H,C\right)$ of two soft groups $\left(F,A\right)$ and $\left(K,B\right)$ over G is denoted by the soft set $\left(H,C\right) = \left(F,A\right)_{\circ}^{\wedge}\left(K,B\right)$ where $C = A \cap B$ and H is a set valued function from C*

to $P(G)$ *and is defined as* $H(c) = F(c)K(c)$ *for all* $c \in C$ *. The soft set* (H,C) *is called the restricted soft product of* (F,A) *and* (K,B) *over* G.

3 Soft Neutrosophic Group

Definition 23 *Let* $N(G)$ *be a neutrosophic group and* (F,A) *be soft set over* $N(G)$ *.Then* (F,A) *is called soft neutrosophic group over* $N(G)$ *if and only if* $F(x) \prec N(G)$ *, for all* $x \in A$.

Example 5 *Let*

$$N(Z_4) = \left\{ \begin{array}{c} 0,1,2,3,I,2I,3I,1+I,1+2I,1+3I, \\ 2+I,2+2I,2+3I,3+I,3+2I,3+3I \end{array} \right\}$$

be a neutrosophic group under addition modulo 4*. Let* $A = \{e_1, e_2, e_3, e_4\}$ *be the set of parameters, then* (F,A) *is soft neutrosophic group over* $N(Z_4)$ *where*

$$F(e_1) = \{0,1,2,3\}, F(e_2) = \{0,I,2I,3I\},$$
$$F(e_3) = \{0,2,2I,2+2I\},$$
$$F(e_4) = \{0,I,2I,3I,2,2+2I,2+I,2+3I\}.$$

Theorem 2 *Let* (F,A) *and* (H,A) *be two soft neutrosophic groups over* $N(G)$ *. Then their intersection* $(F,A) \cap (H,A)$ *is again a soft neutrosophic group over* $N(G)$ *.*

Proof The proof is straightforward.

Theorem 3 *Let* (F,A) *and* (H,B) *be two soft neutrosophic groups over N(G). If* $A \cap B = \phi$ *, then* $(F,A) \cup (H,B)$ *is a soft neutrosophic group over* $N(G)$ *.*

Theorem 4 *Let* (F,A) *and* (H,A) *be two soft neutrosophic groups over* $N(G)$ *. If* $F(e) \subseteq H(e)$ *for all* $e \in A$ *, then* (F,A) *is a soft neutrosophic subgroup of* (H,A) *.*

Theorem 5 *The extended union of two soft neutrosophic groups* (F,A) *and* (K,B) *over* $N(G)$ *is not a soft neutrosophic group over* $N(G)$ *.*

Proof Let (F,A) and (K,B) be two soft neutrosophic groups over $N(G)$. Let $C = A \cup B$, then for all $e \in C$, $(F,A) \cup_{\varepsilon} (K,B) = (H,C)$ where

$$H(e) \quad \begin{array}{ll} = F(e) & \text{If } e \in A - B, \\ = K(e) & \text{If } e \in B - A, \\ = F(e) \cup K(e) & \text{If } e \in A \cap B. \end{array}$$

As union of two subgroups may not be again a subgroup. Clearly if $e \in C = A \cap B$, then $H(e)$ may not be a subgroup of $N(G)$. Hence the extended union (H,C) is not a soft neutrosophic group over $N(G)$.

Example 6 *Let* (F,A) *and* (K,B) *be two soft neutrosophic groups over* $N(Z_2)$ *under addition modulo* 2 *, where*

$$F(e_1) = \{0,1\}, F(e_2) = \{0,I\}$$

And

$$K(e_2) = \{0,1\}, K(e_3) = \{0,1+I\}.$$

Then clearly their extended union is not a soft neutrosophic group as $H(e_2) = F(e_2) \cup K(e_2) = \{0,1,I\}$ *is not a subgroup of* $N(Z_2)$ *.*

Theorem 6 *The extended intersection of two soft neutrosophic groups over* $N(G)$ *is soft neutrosophic group over* $N(G)$ *.*

Theorem 7 *The restricted union of two soft neutrosophic groups* (F,A) *and* (K,B) *over* $N(G)$ *is not a soft neutrosophic group over* $N(G)$.

Theorem 8 *The restricted intersection of two soft neutrosophic groups over* $N(G)$ *is soft neutrosophic group over* $N(G)$ *.*

Theorem 9 *The restricted product of two soft neutrosoph-*

ict groups $\left(F,A\right)$ *and* $\left(K,B\right)$ *over* $N\left(G\right)$ *is a soft neutrosophic group over* $N\left(G\right)$.

Theorem 10 *The* AND *operation of two soft neutrosophic groups over* $N\left(G\right)$ *is soft neutrosophic group over* $N\left(G\right)$.

Theorem 11 *The* OR *operation of two soft neutrosophic groups over* $N\left(G\right)$ *may not be a soft neutrosophic group.*

Definition 24 *A soft neutrosophic group which does not contain a proper soft group is called soft pseudo neutrosophic group.*

Example 7 *Let* $N(Z_2)=\left\langle Z_2\cup I\right\rangle=\left\{0,1,I,1+I\right\}$ *be a neutrosophic group under addition modulo* 2. *Let* $A=\left\{e_1,e_2,e_3\right\}$ *be the set of parameters, then* $\left(F,A\right)$ *is a soft pseudo neutrosophic group over* $N\left(G\right)$ *where*

$$F\left(e_1\right)=\left\{0,1\right\},$$
$$F\left(e_2\right)=\left\{0,I\right\},$$
$$F\left(e_3\right)=\left\{0,1+I\right\}.$$

Theorem 12 *The extended union of two soft pseudo neutrosophic groups* $\left(F,A\right)$ *and* $\left(K,B\right)$ *over* $N\left(G\right)$ *is not a soft pseudo neutrosophic group over* $N\left(G\right)$.

Example 8 *Let* $N(Z_2)=\left\langle Z_2\cup I\right\rangle=\left\{0,1,I,1+I\right\}$ *be a neutrosophic group under addition modulo* 2. *Let* $\left(F,A\right)$ *and* $\left(K,B\right)$ *be two soft pseudo neutrosophic groups over* $N\left(G\right)$, *where*

$$F(e_1)=\left\{0,1\right\},F(e_2)=\left\{0,I\right\},$$
$$F(e_3)=\left\{0,1+I\right\}.$$

And

$$K\left(e_1\right)=\left\{0,1+I\right\},K\left(e_2\right)=\left\{0,1\right\}.$$

Clearly their restricted union is not a soft pseudo neutrosophic group as union of two subgroups is not a subgroup.

Theorem 13 *The extended intersection of two soft pseudo neutrosophic groups* $\left(F,A\right)$ *and* $\left(K,B\right)$ *over* $N\left(G\right)$ *is again a soft pseudo neutrosophic group over* $N\left(G\right)$.

Theorem 14 *The restricted union of two soft pseudo neutrosophic groups* $\left(F,A\right)$ *and* $\left(K,B\right)$ *over* $N\left(G\right)$ *is not a soft pseudo neutrosophic group over* $N\left(G\right)$.

Theorem 15 *The restricted intersection of two soft pseudo neutrosophic groups* $\left(F,A\right)$ *and* $\left(K,B\right)$ *over* $N\left(G\right)$ *is again a soft pseudo neutrosophic group over* $N\left(G\right)$.

Theorem 16 *The restricted product of two soft pseudo neutrosophic groups* $\left(F,A\right)$ *and* $\left(K,B\right)$ *over* $N\left(G\right)$ *is a soft pseudo neutrosophic group over* $N\left(G\right)$.

Theorem 17 *The* AND *operation of two soft pseudo neutrosophic groups over* $N\left(G\right)$ *soft pseudo neutrosophic soft group over* $N\left(G\right)$.

Theorem 18 *The* OR *operation of two soft pseudo neutrosophic groups over* $N\left(G\right)$ *may not be a soft pseudo neutrosophic group.*

Theorem19 *Every soft pseudo neutrosophic group is a soft neutrosophic group.*

Proof The proof is straight forward.

Remark 1 *The converse of above theorem does not hold.*

Example 9 *Let* $N\left(Z_4\right)$ *be a neutrosophic group and* $\left(F,A\right)$ *be a soft neutrosophic group over* $N\left(Z_4\right)$. *Then*

$$F\left(e_1\right)=\left\{0,1,2,3\right\},F\left(e_2\right)=\left\{0,I,2I,3I\right\},$$
$$F\left(e_3\right)=\left\{0,2,2I,2+2I\right\}.$$

But $\left(F,A\right)$ *is not a soft pseudo neutrosophic group as* $\left(H,B\right)$ *is clearly a proper soft subgroup of* $\left(F,A\right)$. *where*

$$H\left(e_1\right)=\left\{0,2\right\},H\left(e_2\right)=\left\{0,2\right\}.$$

Theorem 20 $\left(F,A\right)$ *over* $N\left(G\right)$ *is a soft pseudo* *neutrosophic group if* $N\left(G\right)$ *is a pseudo neutrosophic group.*

Proof Suppose that $N\left(G\right)$ be a pseudo neutrosophic group, then it does not contain a proper group and for all $e \in A,$ the soft neutrosophic group $\left(F,A\right)$ over $N\left(G\right)$ is such that $F\left(e\right) \prec N\left(G\right).$ Since each $F\left(e\right)$ is a pseudo neutrosophic subgroup which does not contain a proper group which make $\left(F,A\right)$ is soft pseudo neutrosophic group.

Example 10 *Let* $N(Z_2) = \left\langle Z_2 \cup I \right\rangle = \left\{0,1,I,1+I\right\}$ *be a pseudo neutrosophic group under addition modulo* $2.$ *Then clearly* $\left(F,A\right)$ *a soft pseudo neutrosophic soft group over* $N\left(Z_2\right)$ *, where*

$$F\left(e_1\right) = \left\{0,1\right\}, F\left(e_2\right) = \left\{0,I\right\},$$
$$F\left(e_3\right) = \left\{0,1+I\right\}.$$

Definition 25 *Let* $\left(F,A\right)$ *and* $\left(H,B\right)$ *be two soft neutrosophic groups over* $N\left(G\right)$ *. Then* $\left(H,B\right)$ *is a soft neutrosophic subgroup of* $\left(F,A\right)$ *, denoted as* $\left(H,B\right) \prec \left(F,A\right)$ *, if*

1) $B \subset A$ and
2) $H\left(e\right) \prec F\left(e\right)$, for all $e \in A$.

Example 11 *Let* $N(Z_4) = \left\langle Z_4 \cup I \right\rangle$ *be a soft neutrosophic group under addition modulo* 4 *, that is*

$$N\left(Z_4\right) = \begin{Bmatrix} 0,1,2,3,I,2I,3I,1+I,1+2I,1+3I, \\ 2+I,2+2I,2+3I,3+I,3+2I,3+3I \end{Bmatrix}.$$

Let $\left(F,A\right)$ *be a soft neutrosophic group over* $N\left(Z_4\right)$ *, then*

$$F\left(e_1\right) = \left\{0,1,2,3\right\}, F\left(e_2\right) = \left\{0,I,2I,3I\right\},$$
$$F\left(e_3\right) = \left\{0,2,2I,2+2I\right\},$$
$$F\left(e_4\right) = \left\{0,I,2I,3I,2,2+2I,2+I,2+3I\right\}.$$

$\left(H,B\right)$ is a soft neutrosophic subgroup of $\left(F,A\right)$, where

$$H\left(e_1\right) = \left\{0,2\right\}, H\left(e_2\right) = \left\{0,2I\right\},$$
$$H\left(e_4\right) = \left\{0,I,2I,3I\right\}.$$

Theorem 21 *A soft group over* G *is always a soft neutrosophic subgroup of a soft neutrosophic group over* $N\left(G\right)$ *if* $A \subset B.$

Proof Let $\left(F,A\right)$ be a soft neutrosophic group over $N\left(G\right)$ and $\left(H,B\right)$ be a soft group over $G.$ As $G \subset N\left(G\right)$ and for all $b \in B, H\left(b\right) \prec G \subset N\left(G\right)$. This implies $H\left(e\right) \prec F\left(e\right)$, for all $e \in A$ as $B \subset A.$ Hence $\left(H,B\right) \prec \left(F,A\right).$

Example 12 *Let* $\left(F,A\right)$ *be a soft neutrosophic group over* $N\left(Z_4\right)$ *, then*

$$F\left(e_1\right) = \left\{0,1,2,3\right\}, F\left(e_2\right) = \left\{0,I,2I,3I\right\},$$
$$F\left(e_3\right) = \left\{0,2,2I,2+2I\right\}.$$

Let $B = \left\{e_1,e_3\right\}$ *such that* $\left(H,B\right) \prec \left(F,A\right)$ *, where*

$$H\left(e_1\right) = \left\{0,2\right\}, H\left(_3\right) = \left\{0,2\right\}.$$

Clearly $B \subset A$ *and* $H\left(e\right) \prec F\left(e\right)$ *for all* $e \in B.$

Theorem 22 *A soft neutrosophic group over* $N\left(G\right)$ *always contains a soft group over G.*

Proof The proof is followed from above Theorem.

Definition 26 *Let* $\left(F,A\right)$ *and* $\left(H,B\right)$ *be two soft pseudo neutrosophic groups over* $N\left(G\right)$ *. Then* $\left(H,B\right)$ *is called soft pseudo neutrosophic subgroup of* $\left(F,A\right)$ *, denoted as* $\left(H,B\right) \prec \left(F,A\right)$ *, if*

1) $B \subset A$
2) $H\left(e\right) \prec F\left(e\right)$, for all $e \in A$.

Example 13 *Let* $\left(F,A\right)$ *be a soft pseudo neutrosophic group over* $N\left(Z_4\right)$ *, where*

$$F(e_1) = \{0, I, 2I, 3I\}, F(e_2) = \{0, 2I\}.$$

Hence $(H,B) \prec (F,A)$ where

$$H(e_1) = \{0, 2I\}.$$

Theorem 23 *Every soft neutrosophic group* (F,A) *over* $N(G)$ *has soft neutrosophic subgroup as well as soft pseudo neutrosophic subgroup.*

Proof Straightforward.

Definition 27 *Let* (F,A) *be a soft neutrosophic group over* $N(G)$*, then* (F,A) *is called the identity soft neutrosophic group over* $N(G)$ *if* $F(x) = \{e\}$*, for all* $x \in A$*, where* e *is the identity element of* G*.*

Definition 28 *Let* (H,B) *be a soft neutrosophic group over* $N(G)$*, then* (H,B) *is called Full-soft neutrosophic group over* $N(G)$ *if* $F(x) = N(G)$*, for all* $x \in A$*.*

Example 14 *Let*

$$N(R) = \left\{ \begin{matrix} a + bI \ : \ a, b \in R \text{ and} \\ I \text{ is indeterminacy} \end{matrix} \right\}$$

is a neutrosophic real group where R *is set of real numbers and* $I^2 = I$*, therefore* $I^n = I$*, for* n *a positive integer. Then* (F,A) *is a Full-soft neutrosophic real group where*

$$F(e) = N(R), \text{ for all } e \in A$$

Theorem 24 *Every Full-soft neutrosophic group contain absolute soft group.*

Theorem 25 *Every absolute soft group over* G *is a soft neutrosophic subgroup of Full-soft neutrosophic group over* $N(G)$*.*

Theorem 26 *Let* $N(G)$ *be a neutrosophic group. If order of* $N(G)$ *is prime number, then the soft neutrosophic group* (F,A) *over* $N(G)$ *is either identity soft neutrosophic group or Full-soft neutrosophic group.*

Proof Straightforward.

Definition 29 *Let* (F,A) *be a soft neutrosophic group over* $N(G)$*. If for all* $e \in A$*, each* $F(e)$ *is Lagrange neutrosophic subgroup of* $N(G)$*, then* (F,A) *is called soft Lagrange neutrosophic group over* $N(G)$*.*

Example 15 *Let* $N(Z_3 / \{0\}) = \{1, 2, I, 2I\}$ *is a neutrosophic group under multiplication modulo* 3*. Now* $\{1,2\}$*,* $\{1, I\}$ *are subgroups of* $N(Z_3 / \{0\})$ *which divides order of* $N(Z_3 / \{0\})$*. Then the soft neutrosophic group*

$$(F,A) = \{F(e_1) = \{1,2\}, F(e_2) = \{1, I\}\}$$

is an example of soft Lagrange neutrosophic group.

Theorem 27 *If* $N(G)$ *is Lagrange neutrosophic group, then* (F,A) *over* $N(G)$ *is soft Lagrange neutrosophic group but the converse is not true in general.*

Theorem 28 *Every soft Lagrange neutrosophic group is a soft neutrosophic group.*

Proof Straightforward.

Remark 2 *The converse of the above theorem does not hold.*

Example 16 *Let* $N(G) = \{1, 2, 3, 4, I, 2I, 3I, 4I\}$ *be a neutrosophic group under multiplication modulo* 5 *and* (F,A) *be a soft neutrosophic group over* $N(G)$*, where*

$$F(e_1) = \{1, 4, I, 2I, 3I, 4I\}, F(e_2) = \{1, 2, 3, 4\},$$
$$F(e_3) = \{1, I, 2I, 3I, 4I\}.$$

But clearly it is not soft Lagrange neutrosophic group as $F(e_1)$ *which is a subgroup of* $N(G)$ *does not divide order of* $N(G)$*.*

Theorem 29 *If* $N(G)$ *is a neutrosophic group, then the soft Lagrange neutrosophic group is a soft neutrosophic group.*

Proof Suppose that $N(G)$ be a neutrosophic group and (F,A) be a soft Lagrange neutrosophic group over $N(G)$. Then by above theorem (F,A) is also soft neutrosophic group.

Example 17 *Let* $N(Z_4)$ *be a neutrosophic group and* (F, A) *is a soft Lagrange neutrosophic group over* $N(Z_4)$ *under addition modulo* 4 *, where*

$F(e_1) = \{0,1,2,3\}, F(e_2) = \{0, I, 2I, 3I\},$
$F(e_3) = \{0,2,2I,2+2I\}.$

But (F, A) *has a proper soft group* (H, B) *, where*
$H(e_1) = \{0,2\}, H(e_3) = \{0,2\}.$

Hence (F, A) is soft neutrosophic group.

Theorem 30 *Let* (F, A) *and* (K, B) *be two soft Lagrange neutrosophic groups over* $N(G)$ *. Then*

1) Their extended union $(F, A) \cup_{\varepsilon} (K, B)$ over $N(G)$ is not soft Lagrange neutrosophic group over $N(G)$.
2) Their extended intersection $(F, A) \cap_{\varepsilon} (K, B)$ over $N(G)$ is not soft Lagrange neutrosophic group over $N(G)$.
3) Their restricted union $(F, A) \cup_R (K, B)$ over $N(G)$ is not soft Lagrange neutrosophic group over $N(G)$.
4) Their restricted intersection $(F, A) \cap_R (K, B)$ over $N(G)$ is not soft Lagrange neutrosophic group over $N(G)$.
5) Their restricted product $(F, A)_{\circ}^{\wedge} (K, B)$ over $N(G)$ is not soft Lagrange neutrosophic group over $N(G)$.

Theorem 31 *Let* (F, A) *and* (H, B) *be two soft Lagrange neutrosophic groups over* $N(G)$ *.Then*

1) Their AND operation $(F, A) \wedge (K, B)$ is not soft Lagrange neutrosophic group over $N(G)$.
2) Their OR operation $(F, A) \vee (K, B)$ is not a soft Lagrange neutrosophic group over $N(G)$.

Definition 30 *Let* (F, A) *be a soft neutrosophic group over* $N(G)$ *. Then* (F, A) *is called soft weakly Lagrange neutrosophic group if atleast one* $F(e)$ *is a Lagrange neutrosophic subgroup of* $N(G)$ *, for some* $e \in A$.

Example 18 *Let* $N(G) = \{1,2,3,4, I, 2I, 3I, 4I\}$ *be a neutrosophic group under multiplication modulo* 5 *, then* (F, A) *is a soft weakly Lagrange neutrosophic group over* $N(G)$ *, where*

$F(e_1) = \{1,4, I, 2I, 3I, 4I\}, F(e_2) = \{1,2,3,4\},$
$F(e_3) = \{1, I, 2I, 3I, 4I\}.$

As $F(e_1)$ *and* $F(e_3)$ *which are subgroups of* $N(G)$ *do not divide order of* $N(G)$ *.*

Theorem 32 *Every soft weakly Lagrange neutrosophic group* (F, A) *is soft neutrosophic group.*

Remark 3 *The converse of the above theorem does not hold in general.*

Example 19 *Let* $N(Z_4)$ *be a neutrosophic group under addition modulo* 4 *and* $A = \{e_1, e_2\}$ *be the set of parameters, then* (F, A) *is a soft neutrosophic group over* $N(Z_4)$ *, where*

$F(e_1) = \{0, I, 2I, 3I\}, F(e_2) = \{0, 2I\}.$

But not soft weakly Lagrange neutrosophic group over $N(Z_4)$.

Definition 31 *Let* (F, A) *be a soft neutrosophic group over* $N(G)$ *. Then* (F, A) *is called soft Lagrange free neutrosophic group if* $F(e)$ *is not Lagrangeneutrosophic subgroup of* $N(G)$ *, for all* $e \in A$.

Example 20 *Let* $N(G) = \{1,2,3,4, I, 2I, 3I, 4I\}$ *be a neutrosophic group under multiplication modulo* 5

and then (F,A) *be a soft Lagrange free neutrosophic group over* $N(G)$ *, where*

$$F(e_1)=\{1,4,I,2I,3I,4I\}, F(e_2)=\{1,I,2I,3I,4I\}.$$

As $F(e_1)$ *and* $F(e_2)$ *which are subgroups of* $N(G)$ *do not divide order of* $N(G)$.

Theorem 33 *Every soft Lagrange free neutrosophic group* (F,A) *over* $N(G)$ *is a soft neutrosophic group but the converse is not true.*

Definition 32 *Let* (F,A) *be a soft neutrosophic group over* $N(G)$ *. If for all* $e\in A$ *, each* $F(e)$ *is a pseudo Lagrange neutrosophic subgroup of* $N(G)$ *, then* (F,A) *is called soft pseudo Lagrange neutrosophic group over* $N(G)$.

Example 21 *Let* $N(Z_4)$ *be a neutrosophic group under addition modulo* 4 *and* $A=\{e_1,e_2\}$ *be the set of parameters, then* (F,A) *is a soft pseudo Lagrange neutrosophic group over* $N(Z_4)$ *where*

$$F(e_1)=\{0,I,2I,3I\}, F(e_2)=\{0,2I\}.$$

Theorem 34 *Every soft pseudo Lagrange neutrosophic group is a soft neutrosophic group but the converse may not be true.*

Proof Straightforward.

Theorem 35 *Let* (F,A) *and* (K,B) *be two soft pseudo Lagrange neutrosophic groups over* $N(G)$ *. Then*

1) Their extended union $(F,A)\cup_{\varepsilon}(K,B)$ over $N(G)$ is not a soft pseudo Lagrange neutrosophic group over $N(G)$.
2) Their extended intersection $(F,A)\cap_{\varepsilon}(K,B)$ over $N(G)$ is not pseudo Lagrange neutrosophic soft group over $N(G)$.
3) Their restricted union $(F,A)\cup_{R}(K,B)$ over $N(G)$ is not pseudo Lagrange neutrosophic soft group over $N(G)$.
4) Their restricted intersection $(F,A)\cap_{R}(K,B)$ over $N(G)$ is also not soft pseudo Lagrange neutrosophic group over $N(G)$.
5) Their restricted product $(F,A)\overset{\wedge}{\underset{\circ}{}}(K,B)$ over $N(G)$ is not soft pseudo Lagrange neutrosophic group over $N(G)$.

Theorem 36 *Let* (F,A) *and* (H,B) *be two soft pseudo Lagrange neutrosophic groups over* $N(G)$ *. Then*

1) Their AND operation $(F,A)\wedge(K,B)$ is not soft pseudo Lagrange neutrosophic group over $N(G)$.
2) Their OR operation $(F,A)\vee(K,B)$ is not a soft pseudo Lagrange neutrosophic soft group over $N(G)$.

Definition 33 *Let* (F,A) *be a soft neutrosophic group over* $N(G)$ *. Then* (F,A) *is called soft weakly pseudo Lagrange neutrosophic group if atleast one* $F(e)$ *is a pseudo Lagrange neutrosophic subgroup of* $N(G)$ *, for some* $e\in A$.

Example 22 *Let* $N(G)=\{1,2,3,4,I,2I,3I,4I\}$ *be a neutrosophic group under multiplication modulo* 5 *Then* (F,A) *is a soft weakly pseudo Lagrange neutrosophic group over* $N(G)$ *, where*

$$F(e_1)=\{1,I,2I,3I,4I\}, F(e_2)=\{1,I\}.$$

As $F(e_1)$ *which is a subgroup of* $N(G)$ *does not divide order of* $N(G)$.

Theorem 37 *Every soft weakly pseudo Lagrange neutrosophic group* (F,A) *is soft neutrosophic group.*

Remark 4 *The converse of the above theorem is not true in general.*

Example 23 *Let $N(Z_4)$ be a neutrosophic group under addition modulo 4 and $A=\{e_1,e_2\}$ be the set of parameters, then (F,A) is a soft neutrosophic group over $N(Z_4)$,where*

$$F(e_1)=\{),I,2I,3I\},F(e_2)=\{0,2I\}.$$

But it is not soft weakly pseudo Lagrange neutrosophic group.

Theorem 38 *Let (F,A) and (K,B) be two soft weakly pseudo Lagrange neutrosophic groups over $N(G)$. Then*

1) Their extended union $(F,A)\cup_{\varepsilon}(K,B)$ over $N(G)$ is not soft weakly pseudo Lagrange neutrosophic group over $N(G)$.
2) Their extended intersection $(F,A)\cap_{\varepsilon}(K,B)$ over $N(G)$ is not soft weakly pseudo Lagrange neutrosophic group over $N(G)$.
3) Their restricted union $(F,A)\cup_{R}(K,B)$ over $N(G)$ is not soft weakly pseudo Lagrange neutrosophic group over $N(G)$.
4) Their restricted intersection $(F,A)\cap_{R}(K,B)$ over $N(G)$ is not soft weakly pseudo Lagrange neutrosophic group over $N(G)$.
5) Their restricted product $(F,A)\overset{\wedge}{\circ}(K,B)$ over $N(G)$ is not soft weakly pseudo Lagrange neutrosophic group over $N(G)$.

Definition 34 *Let (F,A) be a soft neutrosophic group over $N(G)$. Then (F,A) is called soft pseudo Lagrange free neutrosophic group if $F(e)$ is not pseudo Lagrange neutrosophic subgroup of $N(G)$, for all $e\in A$.*

Example 24 *Let $N(G)=\{1,2,3,4,I,2I,3I,4I\}$ be a neutrosophic group under multiplication modulo 5 Then (F,A) is a soft pseudo Lagrange free neutrosophic group over $N(G)$, where*

$$F(e_1)=\{1,I,2I,3I,4I\},F(e_2)=\{1,I,2I,3I,4I\}.$$

As $F(e_1)$ and $F(e_2)$ which are subgroups of $N(G)$ do not divide order of $N(G)$.

Theorem 39 *Every soft pseudo Lagrange free neutrosophic group (F,A) over $N(G)$ is a soft neutrosophic group but the converse is not true.*

Theorem 40 *Let (F,A) and (K,B) be two soft pseudo Lagrange free neutrosophic groups over $N(G)$. Then*

1) Their extended union $(F,A)\cup_{\varepsilon}(K,B)$ over $N(G)$ is not soft pseudo Lagrange free neutrosophic group over $N(G)$.
2) Their extended intersection $(F,A)\cap_{\varepsilon}(K,B)$ over $N(G)$ is not soft pseudo Lagrange free neutrosophic group over $N(G)$.
3) Their restricted union $(F,A)\cup_{R}(K,B)$ over $N(G)$ is not pseudo Lagrange free neutrosophic soft group over $N(G)$.
4) Their restricted intersection $(F,A)\cap_{R}(K,B)$ over $N(G)$ is not soft pseudo Lagrange free neutrosophic group over $N(G)$.
5) Their restricted product $(F,A)\overset{\wedge}{\circ}(K,B)$ over $N(G)$ is not soft pseudo Lagrange free neutrosophic group over $N(G)$.

Definition 35 *A soft neutrosophic group (F,A) over $N(G)$ is called soft normal neutrosophic group over*

$N(G)$ *if* $F(e)$ *is a normal neutrosophic subgroup of* $N(G)$*, for all* $e \in A$.

Example 25 *Let* $N(G) = \{e, a, b, c, I, aI, bI, cI\}$ *be a neutrosophic group under multiplicationwhere* $a^2 = b^2 = c^2 = e, bc = cb = a, ac = ca = b, ab = ba = c$. *Then* (F, A) *is a soft normal neutrosophic group over* $N(G)$ *where*

$$F(e_1) = \{e, a, I, aI\},$$
$$F(e_2) = \{e, b, I, bI\},$$
$$F(e_3) = \{e, c, I, cI\}.$$

Theorem 42 *Every soft normal neutrosophic group* (F, A) *over* $N(G)$ *is a soft neutrosophic group but the converse is not true.*

Theorem 42 *Let* (F, A) *and* (H, B) *be two soft normal neutrosophic groups over* $N(G)$ *. Then*

1) Their extended union $(F, A) \cup_\varepsilon (K, B)$ over $N(G)$ is not soft normal neutrosophic group over $N(G)$.
2) Their extended intersection $(F, A) \cap_\varepsilon (K, B)$ over $N(G)$ is soft normal neutrosophic group over $N(G)$.
3) Their restricted union $(F, A) \cup_R (K, B)$ over $N(G)$ is not soft normal neutrosophic group over $N(G)$.
4) Their restricted intersection $(F, A) \cap_R (K, B)$ over $N(G)$ is soft normal neutrosophic group over $N(G)$.
5) Their restricted product $(F, A)_\circ^\wedge (K, B)$ over $N(G)$ is not soft normal neutrosophic soft group over $N(G)$.

Theorem 43 *Let* (F, A) *and* (H, B) *be two soft normal neutrosophic groups over* $N(G)$ *. Then*

1) Their AND operation $(F, A) \wedge (K, B)$ is soft normal neutrosophic group over $N(G)$.
2) Their OR operation $(F, A) \vee (K, B)$ is not soft normal neutrosophic group over $N(G)$.

Definition 36 *Let* (F, A) *be a soft neutrosophic group over* $N(G)$ *. Then* (F, A) *is called soft pseudo normal neutrosophic group if* $F(e)$ *is a pseudo normal neutrosophic subgroup of* $N(G)$ *, for all* $e \in A$.

Example 26 *Let* $N(Z_2) = \langle Z_2 \cup I \rangle = \{0, 1, I, 1 + I\}$ *be a neutrosophic group under addition modulo* 2 *and let* $A = \{e_1, e_2\}$ *be the set of parameters, then* (F, A) *is soft pseudo normal neutrosophic group over* $N(G)$*, where*

$$F(e_1) = \{0, I\}, F(e_2) = \{0, 1 + I\}.$$

As $F(e_1)$ *and* $F(e_2)$ *are pseudo normal subgroup of* $N(G)$.

Theorem 44 *Every soft pseudo normal neutrosophic group* (F, A) *over* $N(G)$ *is a soft neutrosophic group but the converse is not true.*

Theorem 45 *Let* (F, A) *and* (K, B) *be two soft pseudo normal neutrosophic groups over* $N(G)$ *. Then*

1) Their extended union $(F, A) \cup_\varepsilon (K, B)$ over $N(G)$ is not soft pseudo normal neutrosophic group over $N(G)$.
2) Their extended intersection $(F, A) \cap_\varepsilon (K, B)$ over $N(G)$ is soft pseudo normal neutrosophic group over $N(G)$.
3) Their restricted union $(F, A) \cup_R (K, B)$ over $N(G)$ is not soft pseudo normal neutrosophic

group over $N(G)$.

4) Their restricted intersection $(F,A)\cap_R (K,B)$ over $N(G)$ is soft pseudo normal neutrosophic group over $N(G)$.

5) Their restricted product $(F,A)\overset{\wedge}{\circ}(K,B)$ over $N(G)$ is not soft pseudo normal neutrosophic group over $N(G)$.

Theorem 46 Let (F,A) *and* (K,B) *be two soft pseudo normal neutrosophic groups over* $N(G)$. *Then*

1) Their AND operation $(F,A)\wedge(K,B)$ is soft pseudo normal neutrosophic group over $N(G)$.

2) Their OR operation $(F,A)\vee(K,B)$ is not soft pseudo normal neutrosophic group over $N(G)$.

Definition 37 *Let* $N(G)$ *be a neutrosophic group. Then* (F,A) *is called soft conjugate neutrosophic group over* $N(G)$ *if and only if* $F(e)$ *is conjugate neutrosophic subgroup of* $N(G)$, *for all* $e\in A$.

Example 27 *Let*

$$N(G)=\begin{Bmatrix}0,1,2,3,4,5,I,2I,3I,4I,5I,\\1+I,2+I,3+I,...,5+5I\end{Bmatrix}$$

be a neutrosophic group under addition modulo 6 *and let* $P=\{0,3,3I,3+3I\}$ *and* $K=\{0,2,4,2+2I,4+4I,2I,4I\}$ *are conjugate neutrosophic subgroups of* $N(G)$. *Then* (F,A) *is soft conjugate neutrosophic group over* $N(G)$, *where*

$$F(e_1)=\{0,3,3I,3+3I\},$$
$$F(e_2)=\{0,2,4,2+2I,4+4I,2I,4I\}.$$

Theorem 47 *Let* (F,A) *and* (K,B) *be two soft conjugate neutrosophic groups over* $N(G)$. *Then*

1) Their extended union $(F,A)\cup_\varepsilon (K,B)$ over $N(G)$ is not soft conjugate neutrosophic group over $N(G)$.

2) Their extended intersection $(F,A)\cap_\varepsilon (K,B)$ over $N(G)$ is again soft conjugate neutrosophic group over $N(G)$.

3) Their restricted union $(F,A)\cup_R (K,B)$ over $N(G)$ is not soft conjugate neutrosophic group over $N(G)$.

4) Their restricted intersection $(F,A)\cap_R (K,B)$ over $N(G)$ is soft conjugate neutrosophic group over $N(G)$.

5) Their restricted product $(F,A)\overset{\wedge}{\circ}(K,B)$ over $N(G)$ is not soft conjugate neutrosophic group over $N(G)$.

Theorem 48 *Let* (F,A) *and* (K,B) *be two soft conjugate neutrosophic groups over* $N(G)$. *Then*

1) Their AND operation $(F,A)\wedge(K,B)$ is again soft conjugate neutrosophic group over $N(G)$.

2) Their OR operation $(F,A)\vee(K,B)$ is not soft conjugate neutrosophic group over $N(G)$.

Conclusion

In this paper we extend the neutrosophic group and sub-group, pseudo neutrosophic group and subgroup to soft neutrosophic group and soft neutrosophic subgroup and respectively soft pseudo neutrosophic group and soft pseu-do neutrosophic subgroup. The normal neutrosophic sub-group is extended to soft normal neutrosophic subgroup. We showed all these by giving various examples in order to illustrate the soft part of the neutrosophic notions used.

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Filters via Neutrosophic Crisp Sets

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Abstract. In this paper we introduce the notion of filter on the neutrosophic crisp set, then we consider a generalization of the filter's studies. Afterwards, we present the important neutrosophic crisp filters. We also study several relations between different neutrosophic crisp filters and neutrosophic topologies. Possible applications to database systems are touched upon.

Keywords: Filters; Neutrosophic Sets; Neutrosophic crisp filters; Neutrosophic Topology; Neutrosophic Crisp Ultra Filters; Neutrosophic Crisp Sets.

1 Introduction

The fundamental concept of neutrosophic set, introduced by Smarandache in [6, 7, 8] and studied by Salama in [1, 2, 3, 4, 5, 9, 10], provides a groundwork to mathematically act towards the neutrosophic phenomena which exists pervasively in our real world and expand to building new branches of neutrosophic mathematics. Neutrosophy has laid the foundation for a whole family of new mathematical theories, generalizing both their crisp and fuzzy counterparts, such as the neutrosophic crisp set theory.

2 Preliminaries

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [6, 7, 8] and Salama et al. [1, 2, 3, 4, 5, 9, 10]. Smarandache introduced the neutrosophic components T, I, and F which represent the membership, indeterminacy, and non-membership values respectively, where $]^-0, 1^+[$ is the non- standard unit interval.

3 Neutrosophic Crisp Filters

3.1 Definition 1

First we recall that a neutrosophic crisp set A is an object of the form A = <A$_1$, A$_2$, A$_3$>, where A$_1$, A$_2$, A$_3$ are subsets of X, and

$$A_1 \cap A_2 = \phi, A_1 \cap A_3 = \phi, A_2 \cap A_3 = \phi.$$

Let Ψ be a neutrosophic crisp set in the set X. We call Ψ a neutrosophic crisp filter on X if it satisfies the following conditions:

(N_1) Every neutrosophic crisp set in X, containing a member of Ψ, belongs to Ψ.

(N_2) Every finite intersection of members of Ψ belongs to Ψ.

(N_3) ϕ_N is not in Ψ.

In this case, the pair (X, Ψ) is neutrosophically filtered by Ψ.

It follows from (N_2) and (N_3) that every finite intersection of members of Ψ is not ϕ_N (not empty). We obtain the following results.

3.2 Proposition 1

The conditions (N_2) and *(N$_1$)* are equivalent to the following two conditions:

(N_{2a}) The intersection of two members of Ψ belongs to Ψ.

(N_{1a}) X_N belongs to Ψ.

3.3 Proposition 1.2

Let Ψ be a non-empty neutrosophic subsets in X satisfying (N_1).

Then,

(1) $X_N \in \Psi$ iff $\Psi \neq \phi_N$;

(2) $\phi_N \notin \Psi$ iff $\Psi \neq$ all neutrosophic crisp subsets of X.

From the above Propositions (1) and (2), we can characterize the concept of neutrosophic crisp filter.

3.4 Theorem 1.1

Let Ψ be a neutrosophic crisp subsets in a set X. Then Ψ is neutrosophic crisp filter on X, if and only if it satisfies the following conditions:

(i) Every neutrosophic crisp set in X, containing a member of Ψ, belongs to Ψ.

(ii) If $A, B \in \Psi$, then $A \cap B \in \Psi$.

(iii) $\Psi^X \neq \Psi \neq \phi_N$.

Proof: It's clear.

3.5 Theorem 1.2

Let $X \neq \phi$. Then the set $\{X_N\}$ is a neutrosophic crisp filter on X. Moreover if A is a non-empty neutrosophic crisp set in X, then $\{B \in \Psi^X : A \subseteq B\}$ is a neutrosophic crisp filter on X.

Proof: Let $N = \{B \in \Psi^X : A \subseteq B\}$. Since $X_N \in \Psi$ and $\phi_N \notin \Psi$, $\phi_N \neq \Psi \neq \Psi^X$. Suppose $U, V \in \Psi$, then $A \subseteq U, A \subseteq V$. Thus $A_1 \subseteq U_1 \cap V_1$, $A_2 \subseteq U_2 \cap V_2$ or $A_2 \subseteq U_2 \cup V_2$, and $A_3 \subseteq U_3 \cup V_3$ for all $x \in X$. So $A \subseteq U \cap V$ and hence $U \cap V \in N$.

4 Comparison of Neutrosophic Crisp Filters

4.1 Definition 2

Let Ψ_1 and Ψ_2 be two neutrosophic crisp filters on a set X. We say that Ψ_2 is finer than Ψ_1, or Ψ_1 is coarser than Ψ_2, if $\Psi_1 \subset \Psi_2$.

If also $\Psi_1 \neq \Psi_2$, then we say that Ψ_2 is strictly finer than Ψ_1, or Ψ_1 is strictly coarser than Ψ_2.

We say that two neutrosophic crisp filters are comparable if one is finer than the other. The set of all neutrosophic crisp filters on X is ordered by the relation: Ψ_1 coarser than Ψ_2, this relation inducing the inclusion relation in Ψ^X.

4.2 Proposition 2

Let $(\Psi_j)_{j \in J}$ be any non-empty family of neutrosophic crisp filters on X. Then $\Psi = \cap_{j \in J} \Psi_j$ is a neutrosophic crisp filter on X. In fact, Ψ is the greatest lower bound of the neutrosophic crisp set $(\Psi_j)_{j \in J}$ in the ordered set of all neutrosophic crisp filters on X.

4.3 Remark 2

The neutrosophic crisp filter induced by the single neutrosophic set X_N is the smallest element of the ordered set of all neutrosophic crisp filters on X.

4.4 Theorem 2

Let A be a neutrosophic set in X. Then there exists a neutrosophic filter $\Psi(A)$ on X containing A if for any given finite subset $\{S_1, S_2, ..., S_n\}$ of A, the intersection $\cap_{i=1} S_i \neq \phi_N$. In fact $\Psi(A)$ is the coarsest neutrosophic crisp filter containing A.

Proof $(\Rightarrow)$ Suppose there exists a neutrosophic filter $\Psi(A)$ on X containing A. Let B be the set of all finite intersections of members of A. Then by axiom (N_2), $B \subset \Psi(A)$. By axiom (N_3), $\phi_N \notin \Psi(A)$. Thus for each member B of B, we get that the necessary condition holds

$(\Leftarrow)$ Suppose the necessary condition holds.

Let $\Psi(A) = \{A \in \Psi^X : A \text{ contains a member of B}\}$, where B is the family of all finite intersections of members of A. Then we can easily check that $\Psi(A)$ satisfies the conditions in Definition 1. We say that the neutrosophic crisp filter $\Psi(A)$ defined above is generated by A, and A is called a *sub-base* of $\Psi(A)$.

4.5 Corollary 2.1

Let Ψ be a neutrosophic crisp filter in a set X, and A a neutrosophic set. Then there is a neutrosophic crisp filter $\Psi^/$ which is finer than Ψ and such that $A \in \Psi^/$ if and A is a neutrosophic set. Then there is a neutrosophic crisp filter $\Psi^/$ which is finer than Ψ and such that $A \in \Psi^/$ iff $A \cap U \neq \phi_N$ for each $U \in \Psi$.

4.6 Corollary 2.2

A set φ_N of a neutrosophic crisp filter on a non-empty set X, has a least upper bound in the set of all neutrosophic crisp filters on X if for all finite sequence $(\Psi_j)_{j \in J}, 0 \leq j \leq n$ of elements of φ_N and all $A_j \in \Psi_j (1 \leq j \leq n)$, $\cap_{j=1} A_j \neq \phi_N$.

4.7 Corollary 2.3

The ordered set of all neutrosophic crisp filters on a non-empty set X is inductive.

If Λ is a sub-base of a neutrosophic filter N on X, then Ψ is not in general the set of neutrosophic sets in X containing an element of Λ; for Λ to have this property it is necessary and sufficient that every finite intersection of members of Λ should contain an element of Λ. Hence, we have the following results.

4.8 Theorem 3

Let β be a set of neutrosophic crisp sets on a set X. Then the set of neutrosophic crisp sets in X containing an element of β is a neutrosophic crisp filter on X if β possesses the following two conditions:

(β_1) The intersection of two members of β contain a member of β.

(β_2) $\beta \neq \phi_N$ and $\phi_N \notin \beta$.

4.9 Definition 3

Let Λ and β be two neutrosophic sets on X satisfying conditions (β_1) and (β_2). We call them bases of neutrosophic crisp filters they generate. We consider two neutrosophic bases equivalent, if they generate the same neutrosophic crisp filter.

4.10 Remark 3

Let Λ be a sub-base of neutrosophic filter Ψ. Then the set β of finite intersections of members of Λ is a base of a neutrosophic filter Ψ.

4.11 Proposition 3.1

A subset β of a neutrosophic crisp filter Ψ on X is a base of Ψ if every member of Ψ contains a member of β.

Proof $(\Rightarrow)$ Suppose β is a base of N. Then clearly, every member of Ψ contains an element of β. $(\Leftarrow)$ Suppose the necessary condition holds. Then the set of neutrosophic sets in X containing a member of β coincides with Ψ by reason of $(\Psi_j)_{j\in J}$.

4.12 Proposition 3.2

On a set X, a neutrosophic crisp filter $\Psi^{/}$ with base $\beta^{/}$ is finer than a neutrosophic crisp filter Ψ with base β if every member of β contains a member of $\beta^{/}$.

Proof: This is an immediate consequence of Definitions 2 and 3.

4.13 Proposition 3.3

Two neutrosophic crisp filters bases β and $\beta^{/}$ on a set X are equivalent if every member of β contains a member of $\beta^{/}$ and every member of $\beta^{/}$ and every member of $\beta^{/}$ contains a member of β.

5 Neutrosophic Crisp Ultrafilters

5.1 Definition 4

A neutrosophic ultrafilter on a set X is a neutrosophic crisp filter Ψ such that there is no neutrosophic crisp filter on X which is strictly finer than Ψ (in other words, a maximal element in the ordered set of all neutrosophic crisp filters on X).

Since the ordered set of all neutrosophic crisp filters on X is inductive, Zorn's lemma shows that:

5.2 Theorem 4

Let Ψ be any neutrosophic ultrafilter on a set X; then there is a neutrosophic ultrafilter other than Ψ.

5.3 Proposition 4

Let Ψ be a neutrosophic ultrafilter on a set X. If A and B are two neutrosophic subsets such that $A \cup B \in \Psi$, then $A \in \Psi$ or $B \in \Psi$.

Proof: Suppose not. Then there are neutrosophic sets A and B in X such that $A \notin \Psi, B \notin \Psi$ and $A \cup B \in \Psi$ Let $\Lambda = \{M \in \Psi^X : A \cup M \in \Psi\}$. It is straightforward to check that Λ is a neutrosophic crisp filter on X, and Λ is strictly finer than Ψ, since $B \in \Lambda$.This contradiction proves the hypothesis that Ψ is a neutrosophic crisp ultrafilter.

5.4 Corollary 4

Let Ψ be a neutrosophic crisp ultrafilter on a set X and let $(\Psi_j)_{1\le j\le n}$ be a finite sequence of neutrosophic crisp sets in X. If $\bigcup_{j=1} \Psi_j \in \Psi$, then at least one of the Ψ_j belongs to Ψ.

5.5 Definition 5

Let A be a neutrosophic crisp set in a set X. If U is any neutrosophic crisp set in X, then the neutrosophic crisp set $A \cap U$ is called trace of U on A, and it is denoted by U_A. For all neutrosophic crisp sets U and V in X, we have $(U \cap V)_A = U_A \cap V_A$.

5.6 Definition 6

Let A be a neutrosophic crisp set in a set X. Then the set Λ_A of traces $A \in \Psi^X$ of members of Λ is called the trace of Λ on A.

5.7 Proposition 5

Let Ψ be a neutrosophic crisp filter on a set X and $A \in \Psi^X$. Then the trace Ψ_A of Ψ on A is a neutrosophic crisp filter if each member of Ψ intersects with A.

Proof: The result in Definition 6 shows that Ψ_A satisfies (N_2). If $M \cap A \subset P \subset A$, then $P = (M \cup P) \cap A$. Thus Ψ_A satisfies (N_1). Hence Ψ_A is a neutrosophic crisp filter if it satisfies (N_3), i.e. if each member of Ψ intersects with A.

5.8 Definition 7

Let Ψ be a neutrosophic crisp filter on a set X and $A \in \Psi^{X}$. If the trace is Ψ_A of Ψ on A, then Ψ_A is said to be induced by Ψ and A.

5.9 Proposition 6

Let Ψ be a neutrosophic crisp filter on a set X inducing a neutrosophic filter N_A on $A \in \Psi^{X}$. Then the trace β_A on A of a base β of Ψ is a base of Ψ_A.

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Communication vs. Information, an Axiomatic Neutrosophic Solution

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Florentin Smarandache, Ştefan Vlăduțescu (2013). Communication vs. Information, an Axiomatic Neutrosophic Solution. *Neutrosophic Sets and Systems* 1, 38-45

Abstract. Study represents an application of the neutrosophic method, for solving the contradiction between communication and information. In addition, it recourse to an appropriate method of approaching the contradictions: Extensics, as the method and the science of solving the contradictions.

The research core is the reality that the scientific research of communication-information relationship has reached a dead end. The bivalent relationship communication-information, information-communication has come to be contradictory, and the two concepts to block each other.

After the critical examination of conflicting positions expressed by many experts in the field, the extensic and inclusive hypothesis is issued that information is a form of communication. The object of communication is the sending of a message. The message may consist of thoughts, ideas, opinions, feelings, beliefs, facts, information, intelligence or other significational elements. When the message content is primarily informational, communication will become information or intelligence.

The arguments of supporting the hypothesis are:

a) linguistic (the most important being that there is "communication of information" but not "information of communication"; also, it is clarified and reinforced the over situated referent, that of the communication as a process),

b) systemic-procedural (in the communication system is developing an information system; the informing actant is a type of communicator, the information process is a communication process),

c) practical (the delimitation eliminates the efforts of disparate and inconsistent understanding of the two concepts),

d) epistemological arguments (the possibility of inter-subjective thinking of reality is created), linguistic arguments,

e) logical and realistic arguments (it is noted the situation that allows to think coherently in a system of concepts - derivative series or integrative groups)

f) and arguments from historical experience (the concept of communication has temporal priority, it appears 13 times in Julius Caesar's writings).

In an axiomatic conclusion, the main arguments are summarized in four axioms: three are based on the pertinent observations of specialists, and the fourth is a relevant application of Florentin Smarandache's neutrosophic theory.

Keywords: neutrosophy, communication, information, message, extensics

1. Clarification on the used methodological tool

With the Extensics as a science of solving the conflicting issues, "extensical procedures" will be used to solve the contradiction. In this respect, considering that the matter-elements are defined, their properties will be explored ("The key to solve contradictory problems, Wen Cai argues, the founder of Extensics (Cai, 1999, p. 1540), is the study of properties about matter-elements"). According to „The basic method of Extensics is called extension methodology" (...), and "the application of the extension methodology in every field is the extension engineering methods" (Weihai Li & Chunyan Yang, 2008, p. 34).

With neutrosophic, linguistic, systemic, and hermeneutical methods, grafted on "extension methodology" a) are "open up the things", b) is marked "divergent nature of matter-element", c) "extensibility of matter-element" takes place and c) "extension communication" allows a new inclusion perspective to open, a sequential ranging of things to emphasize at a higher level and the contradictory elements to be solved. "Extension" is, as postulated by Wen Cai (Cai, 1999, p. 1538) "opening up carried out".

2. The subject of communication: the message. The subject of informing: the information. The information thesis as species of message

In order to finish our basic thesis that of the information as a form of communication, new arguments may be revealed which corroborate with those previously mentioned. As phenomena, processes, the communication and information occur in a unique communication system. In communication, information has acquired a specialized profile. In the information field, the intelligence, in his turn, strengthened a specific, detectable, identifiable and discriminative profile. It is therefore acceptable under the pressure of practical argument that one may speak of a general communication system which in relation to the message sent and configured in the communication process could be imagined as information system or intelligence system. Under the influence of the systemic assumption that a (unitary) communicator transmits or customize transactionally with another (receiving) communicator a message, one may understand the communicational system as the interactional unit of the factors that exerts and fulfill the function of communicating a message.

In his books "Messages: building interpersonal communication skills" (attained in 1993 its fourth edition and in 2010 its twelfth) and "Human Communication" (2000), Joseph De Vito (the renowned specialist who has proposed the name "Communicology" for the sciences of communication - 1978), develops a concept of a simple and productive message. The message is, as content, what is communicated. As a systemic factor, it is emerging as what is communicated. To remember in this context is that the German Otto Kade insisted that what it is communicated to receive the title of "release". According to Joseph De Vito, through communication meanings are transmitted. "The communicated message" is only a part of the meanings (De Vito, 1993, p. 116). Among the shared meanings feelings and perceptions are found (De Vito J., 1993, p. 298). Likewise, information can be communicated (De Vito, 1990, p. 42), (De Vito, 2000, p. 347) (also, Fârte, 2004; Ciupercă, 2009; Cojocaru, Bragaru & Ciuchi, 2012; Cobley & Schulz, 2013).

In a "message theory" called "Angelitics", Rafael Capurro argues that the message and information are concepts that designate similar but not identical phenomena. In Greek "Angelia" meant message; from here, "Angelitics" or theory of the message (Angelitics is different from Angeologia dealing, in the field of religion and theology, with the study of angels). R. Capurro set four criteria for assessing the relationship between message and information. The similarity of the two extends over three of them. The message, as well as the information, is characterized as follows: „is supposed to bring something new and/or relevant to the receiver; can be coded and transmitted through different media or messengers; is an utterance that gives rise to the receiver's selection through a release mechanism of interpretation". "The difference between these two is the next: „a message is sender-dependent, i.e. it is based on a heteronomic or assymetric structure. This is not the case of information: we receive a message but we ask for information" (http://www.capurro.de/angeletics_zkm.html) (see also, Capurro, 2011; Holgate, 2011). To request information is to send a message of requesting information. Therefore, the message is similar to the information in this respect too. In our opinion, the difference between them is from genus to species: information is a species of message. The message depends on the transmitter and the information, as well. Information is still a specification of the message, is an informative message. C. Shannon asserts that the message is the defining subject of the communication. He is the stake of the communication because „the fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point" (Shannon, 1948, p. 31).

The communication process is in fact the "communication" of a complex and multilayered message. 'Thoughts, interests, talents, experiences"(Duck & McMahan, 2011, p. 222), "information, ideas, beliefs, feelings "(Wood, 2009, p. 19 and p. 260) can be found in a message. G. A. Miller, T. M. Newcomb and Brent R. Ruben consider that the subject of communication is information: "Communication - Miller shows – means that information is passed from one place to another" (Miller, 1951, p. 6). In his turn, T. M. Newcomb asserts: „very communication act is viewed as a transmission of information" (Newcomb, 1966, p. 66) and Brent R. Ruben argues: „Human communication is the process through which individuals in relationships, groups, organizations and societies create, transmit and use information to relate to the environment and one another" (Ruben, 1992, p. 18).

Professor Nicolae Drăgulănescu, member of the American Society of Information Science and Technology, is the most important of Romanian specialists in the Science of information. According to him, "communicating information" is the third of the four processes that form the "informational cycle", along with generating the information, processing/storing the information and the use of information. The process of communication, Nicolae Drăgulanescu argues, is one of the processes whose object is the information (http://ndragulanescu.ro/publicatii/CP54.pdf, p. 8) (also, Drăgulănescu, 2002; Drăgulănescu, 2005). The same line is followed by Gabriel Zamfir too; he sees the information as "what is communicated in one or other of the available languages" (Zamfir, 1998, p. 7), as well as teacher Sultana Craia: communication is a "process of transmitting a piece of information, a message" (Craia, 2008, p. 53). In general, it is accepted that information means transmitting or receiving information. However, when speaking of transmitting information, the process is considered not to be information but communication. Therefore, it is created the appearance that the information is the product and communication would only be the transmitting process. Teodoru Ştefan, Ion Ivan şi Cristian Popa assert: "Communication is the process of transmitting information, so the ratio of the two categories is from the basic product to its transmission" (Popa, Teodoru & Ivan I., 2008, p. 22). The professors Vasile Tran and Irina Stănciugelu see communication as an "exchange of information with symbolic content" (Tran & Stănciugelu, 2003, p. 109). The communication is an over-ranged concept and an ontological category more extended than informing or information. On the other hand, information is generated even in the global communication process. From this point of view, information (whose subject-message is information) is a regional, sectorial communication. Information is that communication whose message consists of new, relevant, pertinent and useful significances, i.e. of information. This position is shared by Doru Enache too (Enache, 2010, p. 26).

The position set by Norbert Wiener, consolidated by L. Brillouin and endorsed by many others makes from the information the only content of the message. N. Wiener argues that the message "contains information" (Wiener N., 1965, p. 16), L. Brillouin talks about "information contained in the message" (Brillouin, 2004, p. 94 and p. 28).

Through communication "information, concepts, emotions, beliefs are conveyed" and communication "means (and subsumes) information" (Rotaru, 2007, p.10). Well-known teachers Marius Petrescu and Neculae Năbârjoiu consider that the distinction between communication and information must be achieved depending on the message. A communication with an informational message becomes information. As a form of communication, information is characterized by an informative message and a "message is informative as long as it contains something unknown yet" (Petrescu & Năbârjoiu, 2006, p. 25). One of the possible significant elements that could form the message content is thus the information as well. Other components could be thoughts, ideas, beliefs, knowledge, feelings, emotions, experiences, news facts. Communication is "communicating" a message regardless of its significant content.

3. The information thesis as a form of communication

The question of the relationship between communication and information as fields of existence is the fingerprint axis of communication and information ontology. The ontological format allows two formulas: the existence in the act and the virtual existence. The ontological component of the concepts integrates a presence or a potency and an existential fact or at a potential of existence (Zins, 2007; Allo, 2007; Stan, 2009; Burgin, 2010; Case, 2013).

In addition to the categorial-ontological element, in the nuclear ratio of communication-information concepts it shows comparative specificities and regarding attributes and characteristics, on three components, epistemological, methodological and hermeneutical.

In a science which would have firmly taken a strong subject, a methodology and a specific set of concepts, this ontological founding decision would be taken in an axiom. It is known that, in principle, axioms solve within the limits of that type of argument called evidence (clear and distinct situation), the relations between the systemic, structural, basic concepts. Specifically, in Extensics, scientists with an advanced vision, substantiated by professor Wen Cai, axioms govern the relationship between two matter-elements with divergent profiles. For the communication and information issues that have occurred relatively recently (about three quarters of a century) in subjects of study or areas of scientific concern not a scientific authority to settle the issue was found. The weaknesses of these sciences of soft type are visible even today when after non accredited proposals of science ("comunicology" - communicology Joseph De Vito, "communicatics," - "comunicatique" of Metayer G., informatology - Klaus Otten and Anthony Debons, 1970) it was resorted to the remaining in the ambiguity of validating the subject "The sciences of communication and information" or "The sciences of information and communication", enjoying the support of some courses, books, studies and dictionaries (Toma, 1999; Tudor, 2001; Strechie, 2009; Ţenescu, 2009).

This generic vision of unity and cohesion wrongs both the communication and information (Vlăduţescu, 2004; Vlăduţescu, 2006). In practice, the apparent unjust overall, integrative, altogether treatment has not an entirely and covering confirmation. In almost all humanist universities of the world the faculties and the communication courses are prevailing, including those of Romania and China. Professor Nicolae Drăgulănescu ascertained in what Romania is concerned, that in 20 colleagues communication (with various denominations) is taught and in only two the informing-information is taught.

The main perspectives from which the contradictory relationship of communication-information was approached are the ontological, the epistemological and the systemic. In most cases, opinions were incidental. When it was about the dedicated studies, the most common comparative approach was not programmatically made on one or more criteria and neither directly and applied.

In his study "Communication and Information" (19 March 9, pp. 3-31), J. R. Schement starts from the observation that "in the rhetoric of the Information Age, the communication and information converge in synonymous meanings." On the other hand, he retains that there are specialists who declare in favor of stating a firming distinction of their meanings. To clarify exactly the relationship between the two phenomena, i.e. concepts, he examines the definitions of information and communication that have marked the evolution of the "information studies" and the "communication studies". For informing (information) three fundamental themes result: information-as-thing (M. K. Buckland), information-as-process (N. J. Belkin - 1978, R. M. Hayes, Machlup & Mansfield, Elstner - 2010 etc.), Information-as-product-of - manipulation (C. J. Fox, R. M. Hayes). It is also noted that these three subjects involve the assessing of their issuers, a "connection to the phenomenon of communication". In parallel, from examining the definitions of communication it is revealed that the specialists "implicitly or explicitly introduce the notion of information in defining communication". There are also three the central themes of defining communication: communication-as-transmission (C. Shannon, W. Weaver, E. Emery, C. Cherry, B. Berelson, G. Steiner), communication-as-sharing-process (R. S. Gover, W. Schramm), communication-as-interaction (G. Gerbner, L. Thayer). Comparing the six thematic nodes, Schement emphasizes that the link between information and communication is "highly complex" and dynamic "information and communication is ever present and connected" (Schement, 1993, p. 17). In addition, in order that “information exist, the potential for communication must be present”. The result at the ontological level of these findings is that the existence of information is (strictly) conditioned by the presence of communication. That is for the information to occur communication must be present. Communication will precede and always condition the existence of information. And more detailed: communication is part of the information ontology. Ontologically, information occurs in communication also as potency of communication (Vlăduțescu, 2002). J. R. Schement is focused on finding a way to census a coherent image leading to a theory of communication and information ("Toward a Theory of Communication and Information" - Schement, 1993, p. 6). He avoids to conclusively asserting the temporal and linguistic priority, the ontological precedence and the amplitude of communication in relation to information. The study concludes that

1. "Information and communication are social structures" ("two words are used as interchangeable, even as synonyms" – it is argued) (Schement, 1993, p. 17),

2. "The study of information and communication share concepts in common" (in both of them communication, information, "symbol, cognition, content, structure, process, interaction, technology and system are to be found" - Schement, 1993, p. 18),

3. "Information and communication form dual aspects of a broader phenomenon" (Schement J.R., 1993, p. 18).

In other words, we understand that: a) linguistically ("words", "terms", "notions", "concepts", "idea of") communication and information are synonyms; b) as area of study the two resort the same conceptual arsenal. Situation produced by these two elements of the conclusion allows, in our opinion, a hierarchy between communication and information. If it is true that ontologically and temporally the communication precedes information, if this latter phenomenon is an extension smaller than the first, if eventual sciences having communication as object, respectively information, benefit from the one and the same conceptual vocabulary, then the information can be a form of communication. Despite this line followed coherently by the linguistic, categorical-ontological, conceptual and definitional epistemological arguments brought in the reasoning, the third part of the conclusion postulates the existence of a unique phenomenon which would include communication and information (3. "Information and communication form two aspects of the same phenomenon "- Schement JR, 1993, p. 18). This phenomenon is not named. The conclusive line followed by the arguments and the previous conclusive elements enabled us to articulate information as one of the forms of communication. Confirmatively, the fact that J. R. Schement does not name a phenomenon situated over communication and information, gives us the possibility of attracting the argument in order to strengthen our thesis that information is a form of communication. That is because a category of phenomena encompassing communication and information cannot be found. J. R. Schement tends towards a leveling perspective and of convergence in the communication and information ontology. Instead, M. Norton supports an emphasized differentiation between communication and information. He belongs to those who see communication as one of the processes and one of the methods "for making information available". The two phenomena "are intricately connected and have some aspects that seem similar, but they are not the same" (Norton, 2000, p. 48 and p. 39). Harmut B. Mokros and Brent R. Ruben (1991) lay the foundation of a systemic vision and leveling understanding of the communication-information relationship. Taking into account the context of reporting as a core element of the internal structure of communication and information systems, they mark the information as a criterion for the radiography of relationship. The systemic-theoretical non-linear method of research founded in 1983 by B. R. Ruben is applied to the subject represented by the phenomena of communication and information. Research lays in the "Information Age" and creates an informational reporting image. The main merit of the investigation comes from the relevance given to the non-subordination between communication and information in terms of a unipolar communication that relates to a leveling information. Interesting is the approach of information in three constituent aspects: "informatione" (potential information - that which exists in a particular context, but never received a significance in the system), "information" (active information in the system) and "information" (information created socially and culturally in the system). The leveling information is related to a unified communication (Hofkirchner, 2010; Floridi, 2011; Fuchs, 2013; Hofkirchner, 2013). On each level of information there is communication. Information and communication is co-present: communication is inherent to information. Information has inherent properties of communication. Research brings a systemic-contextual elucidation to the relationship between communication and information and only subsidiarily a firm ontological positioning. In any case: in information communication never misses.

In the most important studies of the professor Stan Petrescu: "Information, the fourth weapon" (1999) and "About intelligence. Espionage-Counterespionage" (2007), information is understood as "a type of communication" (Petrescu, 1999, p. 143) and situated in the broader context of "knowledge on the internal and international information environment " (Petrescu, 2007, p. 32).

4. Axiomatic conclusion: four axioms of communication-information ontology

4.1. The message axiom.

We call the ontological segregation axiom on the subject or the Tom D. Wilson - Solomon Marcus' axiom, the thesis that not any communication is information, but any information is communication. Whenever the message contains information, the communicational process will acquire an informational profile. Moreover, the communicational system becomes informational system. Derivatively, the communicator becomes the "informer" and the communicational relationship turns into informational relationship. The interactional basis of society, even in the Information Age, is the communicational interaction. Most social interactions are non-informational. In this respect, T. D. Wilson has noted: „We frequently receive communications of facts, data, news, or whatever which leave us more confused than ever. Under formal definition, these communications contain no information" (Wilson, 1987, p. 410). Academician Solomon Marcus takes into account the undeniable existence of a communication "without a transfer of information" (Marcus, 2011a, p. 220; Marcus, 2011b). For communications that do not contain information we do not have a separate and specific term. Communications containing information or just information are called informing.

Communication involves a kind of information, but as Jean Baudrillard stated (Apud Dâncu, 1999, p. 39), "it is not necessarily based on information". More specifically, any communication contains cognition that can be knowledge, data or information. Therefore, in communication, information may be missing, may be adjacent, incidental or collateral. Communication can be informational in nature or its destination. That communication which by its nature and organization is communication of information is called informing.

The main process ran in Information System is informing. The function of such a system is to inform. The actants can be informants, producers-consumers of information, transmitters of information, etc. The information action takes identity by the cover enabled onto-categorial by the verb "to inform". In his turn, Petros A. Gelepithis considers the two concepts, communication and information to be crucial for "the study of information system" (Gelepithis, 1999, p. 69).

Confirming the information axiom as post reductionist message, as reduced object of communication, Soren Brier substantiates: „communication system actually does not exchange information" (Brier, 1999, p. 96). Sometimes, within the communication system information is no longer exchanged.

However, communication remains; communication system preserves its validity, which indicates and, subsequently, proves that there can be communication that does not involve information (Bates, 2006; Dejica, 2006; Chapman & Ramage, 2013).

On the other hand, then

a) when in the Information System functional principles such as "need to know"/"need to share" are introduced,

b) when running processes for collecting, analyzing and disseminating information,

c) when the beneficiaries are deciders, "decision maker", "ministry", "government", "policymakers" and

d) when the caginess item occurs, this Information System will become Intelligence System (see Gill, Marrin & Phytian, 2009, p. 16, p. 17, p. 112, p. 217), (Sims & Gerber, 2005, p. 46, p. 234; Gill P.& Phytian, 2006, p. 9, p. 236, p. 88; Johnson, 2010, p. 5, p. 6, p. 61, p. 392, p. 279; Maior, 2009; Maior, 2010). Peter Gill shows that "Secrecy is the Key to Understanding the essence of intelligence" (Gill, 2009, p. 18), and Professor George Cristian Maior emphasizes: "in intelligence, collecting and processing information from secret sources remain essential" (Major, 2010, p. 11).

Sherman Kent, W. Laqueur, M. M. Lowenthal, G.-C. Maior etc. start from a complex and multilayered concept of intelligence, understood as meaning knowledge, activity, organization, product, process and information. Subsequently, the question of ontology, epistemology, hermeneutics and methodology of intelligence occurs. Like Peter Gill, G.-C. Maior does pioneering work to separate the ontological approach of intelligence from the epistemological one and to analyze the "epistemological foundation of intelligence" (Maior, 2010, p. 33 and p. 43).

The intelligence must be also considered in terms of ontological axiom of the object. In this regard, noticeable is that one of its meanings, perhaps the critical one, places it in some way in the information area. In our opinion, the information that has critical significance for accredited operators of the state, economic, financial and political power, and holds or acquires confidential, secret feature is or becomes intelligence. Information from intelligence systems can be by itself intelligence or end up being intelligence after some specialized processing. "Intelligence is not just information that merely exists" (Marinică & Ivan, 2010, p. 108), Mariana Marinică and Ion Ivan assert, it is acquired after a "conscious act of creation, collection, analysis, interpretation and modeling information" (Marinică & Ivan, 2010, p. 105).

4.2. Linguistic axiom.

A second axiom of communication-information ontological segregation can be drawn in relation to the linguistic argument of the acceptable grammatical context. Richard Varey considers that understanding "the difference between communication and information is the central factor" and finds in the linguistic context the criterion to validate the difference: „we speak of giving information **to** while communicate **with** other" (Varey, 1997, p. 220). The transmission of information takes place "to" or to someone, and communication takes place "with". Along with this variant of grammatical context it might also emerge the situation of acceptability of some statements in relation to the object of the communication process, respectively the object of the information process.

The statement "to communicate a message, information" is acceptable. Instead, the statement "to inform communication" is not. The phrase "communication of messages-information" is valid, but the phrase "informing of communication", is not. Therefore, language bears knowledge and "lead us" (Martin Heidegger states) to note that, linguistically, communication is more ontological extensive and that information ontology is subsumed to it (Henno, 2013; Gîfu & Cristea, 2013; Gorun & Gorun, 2011).

The ontical and ontological nature of language allows it to express the existence and to achieve a functional-grammatical specification. Language allows only grammatical existences. As message, the information can be "communicated" or "communicable". There is also the case in which a piece of information cannot be "communicated" or "communicable". Related, communication cannot be "informed". The semantic field of communication is therefore larger, richer and more versatile (Ștefan Buzărnescu, 2006). Communication allows the "incommunicable".

4.3. Teleological axiom.

In addition to the axiom of segregating communication, of informing in relation to the object (message), it may be stated as an axiom a Magoroh Maruyama's contribution to the demythologization of information. In the article "Information and Communication in Poly Epistemological System" in "The Myths of Information", he states: „The transmission of information is not the purpose of communication. In Danish culture, for example, the purpose of communication is frequently to perpetuate the familiar, rather than to introduce new information" (Maruyama, 1980, p. 29).

The ontological axiom of segregation in relation to the purpose determines information as that type of communication with low emergence in which the purpose of the interaction is transmitting information.

4.4. The neutrosophic communication axiom.

Understanding the frame set by the three axioms, we find that some communicational elements are heterogeneous and neutral in relation to the criterion of informativity. In a speech some elements can be suppressed without the message suffering informational alterations. This means that some message-discursive meanings are redundant; others are not essential in relation to the orexis-the practical course or of practical touch in the order of reasoning. Redundancies and non-nuclear significational components can be elided and informational and the message remains informationally unchanged. This proves the existence of cores with neutral, neutrosophic meanings. (In the epistemological foundations of the concept of neutrosophy we refer to Florentin Smarandache's work, *A Unifying Field in Logics, Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics*, 1998) (Smarandache, 1998; Smarandache, 1999; Smarandache, 2002; Smarandache, 2005; Smarandache, 2010a; Smarandache, 2010b; Smarandache & Păroiu, 2012).

On the operation of this phenomenon are based the procedures of textual contraction, of grouping, of serial registration, of associating, summarizing, synthesizing, integrating.

We propose to understand by neutrosophic communication that type of communication in which the message consists of and it is based on neutrosophic significational elements: non-informational, redundant, elidable, contradictory, incomplete, vague, imprecise, contemplative, non-practical, of relational cultivation. Informational communication is that type of communication whose purpose is sharing an informational message. The issuer's fundamental approach is, in informational communication, to inform. To inform is to transmit information or, specifically, in the professor's Ilie Rad words: "to inform, that is just send information" (Moldovan, 2011, p. 70) (also, Rad, 2005; Rad, 2008). In general, any communication contains some or certain neutrosophic elements, suppressible, redundant, elidable, non-nuclear elements. But when neutrosophic elements are prevailing communication is no longer informational, but neutrosophic. Therefore, the neutrosophic axiom allows us to distinguish two types of communication: neutrosophic communication and informational communication. In most of the time our communication is neutrosophic. The neutrosophic communication is the rule. The informational communication is the exception. In the ocean of the neutrosophic communication, diamantine islands of informational communication are distinguished.

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Several Similarity Measures of Neutrosophic Sets

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Abstract. Smarandache (1995) defined the notion of neutrosophic sets, which is a generalization of Zadeh's fuzzy set and Atanassov's intuitionistic fuzzy set. In this paper, we first develop some similarity measures of neutrosophic sets. We will present a method to calculate the distance between neutrosophic sets (NS) on the basis of the Hausdorff distance. Then we will use this distance to generate a new similarity measure to calculate the degree of similarity between NS. Finally we will prove some properties of the proposed similarity measures.

Keywords- Neutrosophic Set, Matching Function, Hausdorff Distance, Similarity Measure.

1 Introduction

Smarandache introduced a concept of neutrosophic set which has been a mathematical tool for handling problems involving imprecise, indeterminacy, and inconsistent data [1, 2].The concept of similarity is fundamentally important in almost every scientific field. Many methods have been proposed for measuring the degree of similarity between fuzzy sets (Chen, [11]; Chen et al., [12]; Hyung, Song, & Lee, [14]; Pappis& Karacapilidis, [10]; Wang, [13]...). But these methods are unsuitable for dealing with the similarity measures of neutrosophic set (NS). Few researchers have dealt with similarity measures for neutrosophic set and single valued neutrosophic set ([3, 4,17,18]), (i.e. the crisp neutrosophic sets, where the components T, I, F are all crisp numbers). Recently, Jun [3] discussed similarity measures on interval neutrosophic set (which an instance of NS) based on Hamming distance and Euclidean distance and showed how these measures may be used in decision making problems. Furthermore, A.A.Salama [4] defined the correlation coefficient, on the domain of neutrosophic sets, which is another kind of similarity measurement. In this paper we first extend the Hausdorff distance to neutrosophic set which plays an important role in practical application, especially in many visual tasks, computer assisted surgery and so on. After that a new series of similarity measures has been proposed for neutrosophic set using different approaches.

Similarity measures have extensive application in several areas such as pattern recognition, image processing, region extraction, psychology [5], handwriting recognition [6], decision making [7], coding theory etc.

This paper is organized as follows: Section2 briefly reviews the definition of Hausdorff distance and the neutrosophic set. Section 3 presents the new extended Hausdorff distance between neutrosophic sets. Section 4 provides the new series of similarity measure between neutrosophic sets, some of its properties are discussed. In section 5 a comparative study was done. Finally the section 6 outlines some conclusions.

2 Preliminaries

In this section we briefly review some definitions and examples which will be used in the rest of the paper.

2.1Definition: Hausdorff Distance

The Hausdorff distance (Nadler, 1978) is the maximum distance of a set to the nearest point in the other set. More formal description is given by the following

Given two finite sets $A = \{a_1, ..., a_p\}$ and $B = \{b_1, ..., b_q\}$, the Hausdorff distance H (A, B) is defined as:

$$H(A, B) = \max\{h(A, B), h(B, A)\} \quad (1)$$

where

$$H(A, B) = \max_{a \in A} \min_{b \in B} d(a, b) \quad (2)$$

a and b are elements of sets A and B respectively; d (a, b) is any metric between these elements.

The two distances h (A, B) and h (B, A) are called directed Hausdorff distances.

The function h (A, B) (the directed Hausdorff distance from A to B) ranks each element of A based on its distance to the nearest element of B, and then the largest ranked such element (the most mismatched element of A) specifies the value of the distance. Intuitively, if h(A, B) = c, then each element of A must be within distance c of some element of B, and there also is some element of A that is exactly distance c from the nearest element of B (the most mismatched element). In general h (A, B) and h (B, A) can attain very different values (the directed distances are not symmetric).

Let us consider the real space R, for any two intervals A= $[a_1,a_2]$ and B= $[b_1,b_2]$, the Hausdorff distance H(A,B) is given by

$$H(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\} \quad (3)$$

2.2 Definition (see [2]). Let U be an universe of discourse then the neutrosophic set A is an object having the form $A = \{< x: T_{A(x)}, I_{A(x)}, F_{A(x)} >, x \in U\}$, where the functions T, I, F : $U \to]^-0,1^+[$ define respectively the degree of membership (or Truth) , the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element x ∈ U to the set A with the condition.

$$^-0 \le T_{A(x)} + I_{A(x)} + F_{A(x)} \le 3^+. \quad (4)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. So instead of $]^-0,1^+[$ we need to take the interval [0,1] for technical applications, because $]^-0,1^+[$will be difficult to apply in the real applications such as in scientific and engineering problems.

2.3 Definition (see [18]): Let X be a space of points (objects) with generic elements in X denoted by x (Wang et al., 2010). An SVNS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$ for each point x in X, $T_A(x), I_A(x), F_A(x) \in [0, 1]$.

When X is continuous, an SVNS A can be written as

$$A=\int_X \frac{<T_A(x), I_A(x), F_A(x)>}{x}, x \in X. \quad (5)$$

When X is discrete, an SVNS A can be written as

$$A=\sum_1^n \frac{<T_A(x_i), I_A(x_i), F_A(x_i)>}{x_i}, x_i \in X. \quad (6)$$

2.4 Definition (see [2,18]). A neutrosophic set or single valued neutrosophic set (SVNS) A is contained in another neutrosophic set B i.e. A ⊆ B if $\forall x \in U$, $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, $F_A(x) \ge F_B(x)$.

2.5 Definition (see [2]). The complement of a neutrosophic set A is denoted by A^c and is defined as $T_{A^c(x)} = F_{A(x)}$, $I_{A^c(x)} = I_{A(x)}$, and $F_{A^c(x)} = T_{A(x)}$ for every x in X.

A complete study of the operations and application of neutrosophic set can be found in [1] [2] [18].

In this paper we are concerned with neutrosophic sets whose T_A, I_A and F_A values are single points in [0, 1] instead of subintervals/subsets in [0, 1].

3 Extended Hausdorff Distance Between Two Neutrosophic Sets

Based on the Hausdorff metric, Eulalia Szmidt and Janusz Kacprzyk defined a new distance between intuitionistic fuzzy sets and/or interval-valued fuzzy sets in[8], taking into account three parameter representation (membership, non-membership values, and the hesitation margins) of A-IFSs which fulfill the properties of the Hausdorff distances. Their definition is defined by:

$$H_3(A,B) = \frac{1}{n}\sum_{i=1}^{n} max\{|\mu_A(x) - \mu_B(x)|, |\nu_A(x) - \nu_B(x)|, |\pi_A(x) - \pi_B(x)|\} \quad (7)$$

where $A = \{< x, \mu_A(x), \nu_A(x), \pi_A(x) >\}$ and $B = \{< x, \mu_B(x), \nu_B(x), \pi_B(x)>\}$.

The terms and symbols used in [8] are changed so that they are consistent with those in this section.

In this paper we are interested in extending the Hausdorff distance formulation in constructing a new distance for neutrosophic set due to its simplicity in the calculation.

Let $X=\{x_1,x_2, \ldots, x_n\}$ be a discrete finite set. Consider a neutrosophic set A in X, where $T_{A(xi)}$, $I_{A(xi)}$, $F_{A(xi)} \in [0, 1]$, for every $x_i \in X$, represent its membership, indeterminacy, and non-membership values respectively denoted by $A = \{< x, T_{A(xi)}, I_{A(xi)}, F_{A(xi)} >\}$.

Then we propose a new distance between A ∈ NS and B ∈ NS defined by

$$d_H(A, B) = \frac{1}{n}\sum_{i=1}^{n} \max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\} \quad (8)$$

Where $d_H(A,B)$ = H (A, B) denote the extended Hausdorff distance between two neutrosophic sets A and B.

Let A, B and C be three neutrosophic sets. For all $x_i \in X$ we have:

$$d_H(A, B) = H(A, B) = \max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_{B(x_i)}|, |F_A(x_i) - F_B(x_i)|\} \quad (9)$$

The same between A and C are written as:

For all $x_i \in X$

H (A, C)

=

$\max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$ (10)

and between B and C is written as:

For all $x_i \in X$

H (B , C) = $\max\{|T_B(x_i) - T_C(x_i)|, |I_B(x_i) - I_C(x_i)|, |F_B(x_i) - F_C(x_i)|\}$ (11)

3.1 Proposition:

The above defined distance $d_H(A,B)$ between NS A and B satisfies the following properties (D1-D4):

(D1) $d_H(A,B) \geq 0$. (12)

(D2) $d_H(A,B) = 0$ if and only if A = B; for all A, B $\in$ NS. (13)

(D3) $d_H(A,B) = d_H(B,A)$. (14)

(D4) If A⊆B⊆C, C is an NS in X, then

$d_H(A,C) \geq d_H(A,B)$ (15)

And

$d_H(A,C) \geq d_H(B,C)$ (16)

Remark: Let A, B $\in$ NS, A⊆ B if and only if , for all x_i in X

$T_A(x_i) \leq T_B(x_i), I_A(x_i) \geq I_B(x_i), F_A(x_i) \geq F_B(x_i)$ (17)

It is easy to see that the defined measure $d_H(A,B)$ satisfies the above properties (D1)-(D3). Therefore, we only prove (D4).

Proof of (D4) for the extended Hausdorff distance between two neutrosophic sets. Since

A⊆ B ⊆ C implies , for all x_i in X

$T_A(x_i) \leq T_B(x_i) \leq T_C(x_i), I_A(x_i) \geq I_B(x_i) \geq I_C(x_i), F_A(x_i) \geq F_B(x_i) \geq F_C(x_i)$

We prove that $d_H(A,B) \leq d_H(A,C)$ (18)

α - If $|T_A(x_i) - T_C(x_i)| \geq |I_A(x_i) - I_C(x_i)| \geq |F_A(x_i) - F_C(x_i)|$ (19)

Then

H (A, C) = $|T_A(x_i) - T_C(x_i)|$ but we have

(i) For all x_i in X, $|I_A(x_i) - I_B(x_i)| \leq |I_A(x_i) - I_C(x_i)|$ (ii) (20)

$\leq |T_A(x_i) - T_C(x_i)|$

And , $\forall x_i \in X$

$|F_A(x_i) - F_B(x_i)| \leq |F_A(x_i) - F_C(x_i)|$ (21)

$\leq |T_A(x_i) - T_C(x_i)|$

(iii) $\forall x_i \in X$ 55, 55

$|I_B(x_i) - I_C(x_i)| \leq |I_A(x_i) - I_C(x_i)|$ (22)

$\leq |T_A(x_i) - T_C(x_i)|$

And ,for all x_i in X

$|F_B(x_i) - F_C(x_i)| \leq |F_A(x_i) - F_C(x_i)|$

$\leq |T_A(x_i) - T_C(x_i)|$ (23)

On the other hand we have, $\forall x_i \in X$

(iv) $|T_A(x_i) - T_B(x_i)| \leq |T_A(x_i) - T_C(x_i)|$ (24)

and $|T_B(x_i) - T_C(x_i)| \leq |T_A(x_i) - T_C(x_i)|$

Combining (i), (ii), and (iii) we obtain

Therefore, for all x_i in X

$\frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\}$

$\leq \frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$

And

$\frac{1}{n}\sum_1^n \max\{|T_B(x_i) - T_C(x_i)|, |I_B(x_i) - I_C(x_i)|, |F_B(x_i) - F_C(x_i)|\}$

$\leq \frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$

That is

$d_H(A,B) \leq d_H(A,C)$ and $d_H(B,C) \leq d_H(A,C)$. (25)

β - If $|T_A(x_i) - T_C(x_i)| \leq |F_A(x_i) - F_C(x_i)| \leq |I_A(x_i) - I_C(x_i)|$ (26)

Then

H (A, C) = $|I_A(x_i) - I_C(x_i)|$ but we have $\forall x_i \in X$ (27)

(a) $|T_A(x_i) - T_B(x_i)| \leq |T_A(x_i) - T_C(x_i)|$ (28)

$\leq |I_A(x_i) - I_C(x_i)|$

And $|F_A(x_i) - F_B(x_i)| \leq |F_A(x_i) - F_C(x_i)|$
$\leq |I_A(x_i) - I_C(x_i)|$

(b) $|T_B(x_i) - T_C(x_i)| \leq |T_A(x_i) - T_C(x_i)|$ (29)
$\leq |I_A(x_i) - I_C(x_i)|$

And $|F_B(x_i) - F_C(x_i)| \leq |F_A(x_i) - F_C(x_i)|$ (30)
$\leq |I_A(x_i) - I_C(x_i)|$

On the other hand we have $\forall\ x_i \in X$:

(c) $|I_A(x_i) - I_B(x_i)| \leq |I_A(x_i) - I_C(x_i)|$ and (31)

$|I_B(x_i) - I_C(x_i)| \leq |I_A(x_i) - I_C(x_i)|$

Combining (a) and (c) we obtain:
Therefore, $\forall\ x_i \in X$

$\frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\}$
$\leq \frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$
56

And

$\frac{1}{n}\sum_1^n \max\{|T_B(x_i) - T_C(x_i)|, |I_B(x_i) - I_C(x_i)|, |F_B(x_i) - F_C(x_i)|\}$

$\leq \frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$

That is

$d_H(A,B) \leq d_H(A,C)$ and $d_H(B,C) \leq d_H(A,C)$
(32)

γ - If
$|T_A(x_i) - T_C(x_i)| \leq |I_A(x_i) - I_C(x_i)| \leq |F_A(x_i) - F_C(x_i)|$
(33)

Then

H (A, C) = $|F_A(x_i) - F_C(x_i)|$ but we have for all x_i in X
(34)

(a) $|T_A(x_i) - T_B(x_i)| \leq |T_A(x_i) - T_C(x_i)|$
(35)
$\leq |F_A(x_i) - F_C(x_i)|$

and $|I_A(x_i) - I_B(x_i)| \leq |I_A(x_i) - I_C(x_i)|$ (36)

$\leq |F_A(x_i) - F_C(x_i)|$

(b) $\forall\ x_i \in X$ $|T_B(x_i) - T_C(x_i)| \leq |T_A(x_i) - T_C(x_i)|$
(37)

$\leq |F_A(x_i) - F_C(x_i)|$

and $\forall\, x_i \in X$ $|I_B(x_i) - I_C(x_i)| \leq |I_A(x_i) - I_C(x_i)|$
(38)

$\leq |F_A(x_i) - F_C(x_i)|$

On the other hand we have for all x_i in X

(c) $\forall\ x_i \in X$ $|F_A(x_i) - F_B(x_i)| \leq |F_A(x_i) - F_C(x_i)|$
(39)
and
$|F_B(x_i) - F_C(x_i)| \leq |F_A(x_i) - F_C(x_i)|$
(40)

Combining (a), (b), and (c) we obtain

Therefore, for all x_i in X

$\frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_B(x_i)|, |I_A(x_i) - I_B(x_i)|, |F_A(x_i) - F_B(x_i)|\}$

$\leq \frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$
.

And

$\frac{1}{n}\sum_1^n \max\{|T_B(x_i) - T_C(x_i)|, |I_B(x_i) - I_C(x_i)|, |F_B(x_i) - F_C(x_i)|\}$

$\leq \frac{1}{n}\sum_1^n \max\{|T_A(x_i) - T_C(x_i)|, |I_A(x_i) - I_C(x_i)|, |F_A(x_i) - F_C(x_i)|\}$

That is

$d_H(A,B) \leq d_H(A,C)$ and $d_H(B,C) \leq d_H(A,C)$.
(41)

From α, β , and γ, we can obtain the property (D4).

3.2 Weighted Extended Hausdorff Distance Between Two Neutrosophic Sets.

In many situations the weight of the element $x_i \in X$ should be taken into account. Usually the elements have different importance. We need to consider the weight of the element so that we have the following weighted distance between NS. Assume that the weight of $x_i \in X$ is w_i where $X=\{x_1, x_2,.., x_n\}$, $w_i \in [0,1]$, $i=\{1,2,3,.., n\}$ and $\sum_1^n w_i = 1$. Then the weighted extended Hausdorff distance between NS A and B is defined as:

$$d_{HW}(A,B) = \sum_1^n w_i\, d_H(A(x_i), B(x_i)) \quad (42)$$

It is easy to check that $d_{HW}(A,B)$ satisfies the four properties D1-D4 defined above.

4 Some new similarity measures for neutrosophic sets

The distance measure between two NS is used in finding the similarity between neutrosophic sets. We found in the literature different similarity measures, and we extend them to neutrosophic sets (NS), several of them defined below: Liu [9] also gave an axiom definition for the similarity measure of fuzzy sets, which also can be expressed for neutrosophic sets (NS) as follow:

4.1.Definition: Axioms of a Similarity Measure

A mapping S:NS(X)×NS(X)→[0,1], NS(X) denotes the set of all NS in X={x_1,x_2,…,x_n}, S(A, B) is said to be the degree of similarity between A∈ NS and B ∈ NS, if S(A,B) satisfies the properties of conditions (P1-P4):

(P1) S (A, B) = S (B, A). (43)

(P2) S(A,B) = (1,0,0) = $\underline{1}$ .If A = B for all A,B ∈ NS. (44)

(P3) $S_T(A,B) \geq 0$, $S_I(A,B) \geq 0$, $S_F(A,B) \geq 0$. (45)

(P4) If A⊆B⊆C for all A, B, C ∈ NS, then S (A, B) ≥S (A, C) and S (B, C) ≥ S (A, C). (46)

Numerical Example:

Let A ≤ B ≤ C. with $T_A \leq T_B \leq T_C$ and $I_A \geq I_B \geq I_C$ and $F_A \geq F_B \geq F_C$ for each x_i∈ NS.

For example:

A= { x_1 (0.2, 0.5, 0.6); x_2 (0.2, 0.4, 0.4) }

B= { x_1 (0.2, 0.4, 0.4); x_2 (0.4, 0.2, 0.3) }

C= { x_1 (0.3, 0.3, 0.4); x_2 (0.5, 0.0, 0.3) }

In the following we define a new similarity measure of neutrosophic set and discuss its properties.

4.2 Similarity Measures Based on the Set – Theoretic Approach.

In this section we extend the similarity measure for intuitionistic and fuzzy set defined by Hung and Yung [16] to neutrosophic set which is based on set-theoretic approach as follow.

4.2.Definition: Let A,B be two neutrosophic sets in X={x_1,x_2,.., x_n}, if A = {< x, $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i)$ >} and B= {< x, $T_B(x_i)$, $I_B(x_i)$, $F_B(x_i)$ >} are neutrosophic values of X in A and B respectively, then the similarity measure between the neutrosophic sets A and B can be evaluated by the function

For all x_i in X

$$S_T(A,B) = \left(\sum_1^N \left[\frac{\min(T_A(x_i),T_B(x_i))}{Max(T_A(x_i),T_B(x_i))}\right]\right)/n \quad (47)$$

$$S_I(A,B) = 1 - \left(\sum_1^N \left[\frac{\min(I_A(x_i),I_B(x_i))}{Max(I_A(x_i),I_B(x_i))}\right]\right)/n \quad (48)$$

$$S_F(A,B) = 1 - \left(\sum_1^N \left[\frac{\min(F_A(x_i),F_B(x_i))}{Max(F_A(x_i),F_B(x_i))}\right]\right)/n \quad (49)$$

and $S(A,B) = (S_T(A,B), S_I(A,B), S_F(A,B))$ (50)

where

$S_T(A,B)$ denote the degree of similarity (where we take only the T's).

$S_I(A,B)$ denote the degree of indeterminate similarity (where we take only the I's).

$S_F(A,B)$ denote degree of nonsimilarity (where we take only the F's).

Min denotes the minimum between each element of A and B.

Max denotes the minimum between each element of A and B.

Proof of (P4) for the (1).

Since A⊆B⊆C implies, for all x_i in X

$T_A(x_i) \leq T_B(x_i) \leq T_C(x_i)$, $I_A(x_i) \geq I_B(x_i) \geq I_C(x_i)$, $F_A(x_i) \geq F_B(x_i) \geq F_C(x_i)$

Then, for all x_i in X

$$\frac{\min(T_A(x_i),T_B(x_i))}{Max(T_A(x_i),T_B(x_i))} = \frac{T_A(x_i)}{T_B(x_i)} \quad (51)$$

$$\frac{\min(T_A(x_i),T_C(x_i))}{Max(T_A(x_i),T_C(x_i))} = \frac{T_A(x_i)}{T_C(x_i)} \quad (52)$$

$$\frac{\min(T_B(x_i),T_C(x_i))}{Max(T_B(x_i),T_C(x_i))} = \frac{T_B(x_i)}{T_C(x_i)} \quad (53)$$

Therefore, for all x_i in X

$$\frac{T_A(x_i)}{T_C(x_i)} = \frac{T_B(x_i)}{T_C(x_i)} + \frac{T_A(x_i)-T_B(x_i)}{T_C(x_i)} \leq \frac{T_B(x_i)}{T_C(x_i)} \quad (54)$$

57

(since $T_A(x_i) \leq T_B(x_i)$)

Furthermore, for all x_i in X

$$\frac{\min(T_A(x_i),T_B(x_i))}{Max(T_A(x_i),T_B(x_i))} \geq \frac{\min(T_A(x_i),T_C(x_i))}{Max(T_A(x_i),T_C(x_i))} \quad (55)$$

Or

$$\frac{T_A(x_i)}{T_B(x_i)} \geq \frac{T_A(x_i)}{T_C(x_i)} \text{ or } T_B(x_i) \leq T_C(x_i) \quad (56)$$

(since $T_C(x_i) \geq T_B(x_i)$)

Inequality (53) implies that, for all x_i in X

$$\frac{T_A(x_i)}{T_C(x_i)} \leq \frac{T_A(x_i)}{T_B(x_i)} \quad (57)$$

From the inequalities (54) and (57), the property (P4) for $S_T(A,B) \geq S_T(A,C)$ is proven.

In a similar way we can prove that $S_I(A,B)$ and $S_F(A,B)$.

We will to prove that $S_I(A,C) \geq S_I(A,B)$. For all x_i∈ X we have:

$$S_I(A,C) = 1 - \frac{\min(I_A(x_i),I_C(x_i))}{Max(I_A(x_i),I_C(x_i))} = 1 - \frac{I_C(x_i)}{I_A(x_i)} \geq 1 - \frac{I_B(x_i)}{I_A(x_i)} \quad (58)$$

Since $I_C(x_i) \leq I_B(x_i)$

Similarly we prove $S_F(A,C) \geq S_F(A,B)$ for all x_i in X

(59)

$$S_F(A,C) = 1 - \frac{min(F_A(x_i),F_C(x_i))}{Max(F_A(x_i),F_C(x_i))} = 1 - \frac{F_C(x_i)}{F_A(x_i)} \geq 1 - \frac{F_B(x_i)}{F_A(x_i)}$$

(60)

Since $F_C(x_i) \leq F_B(x_i)$

Then S(A, C)≤S(A, B) where

S(A,C)=($S_T(A,C)$, $S_I(A,C)$, $S_F(A,C)$) and S (A, B) = ($S_T(A,B)$, $S_I(A,B)$, $S_F(A,B)$).

(61)

In a similar way we can prove that S (B, C) ≥ S (A, C). If A⊆B⊆C therefore S (A, B) satisfies (P4) of definition 4.1.

By applying (50), the degree of similarity between the neutrosophic sets (A, B), (A, C) and (B, C) are:

S(A, B) =($S_T(A,B)$, $S_I(A,B)$, $S_F(A,B)$)= (0.75, 0.35, 0.30)

S (A, C) = ($S_T(A,C)$, $S_I(A,C)$, $S_F(A,C)$)= (0.53, 0.7, 0.30)

S (B, C) =($S_T(B,C)$, $S_I(B,C)$, $S_F(B,C)$)= (0.73, 0.63, 0)

Then (49) satisfies property P4: S(A, C) ≤ S(A, B) and S(A, C) ≤ S(B, C).

Usually, the weight of the element $x_i \in X$ should be taken into account, then we present the following weighted similarity between NS. Assume that the weight of $x_i \in X=\{1,2,\dots,n\}$ is w_i (i=1,2,…, n) when $w_i \in [0,1], \sum_1^n w_i = 1$.

Denote $S_w^T(A,B) = (\sum_1^N w_i \left[\frac{min(T_A(x_i),T_B(x_i))}{Max(T_A(x_i),T_B(x_i))}\right])/n$

(62)

$$S_w^I(A,B) = 1 - (\sum_1^N w_i \left[\frac{min(I_A(x_i),I_B(x_i))}{Max(I_A(x_i),I_B(x_i))}\right])/n$$

(63)

$$S_w^F(A,B) = 1 - (\sum_1^N w_i \left[\frac{min(F_A(x_i),F_B(x_i))}{Max(F_A(x_i),F_B(x_i))}\right])/n$$

(64)

and $S_w(A,B) = (\ S_w^T((A,B),\ S_w^I((A,B),\ S_w^F((A,B))$

(65)

It is easy to check that $S_w(A,B)$ satisfies the four properties P1-P4 defined above.

4.3 Similarity Measure Based on the Type1 Geometric Distance Model

In the following, we express the definition of similarity measure between fuzzy sets based on the model of geometric distance proposed by Pappis and Karacapilidis in [10] to similarity of neutrosophic set.

4.3.Definition: Let A,B be two neutrosophic sets in X={x_1, x_2,..., x_n}, if A = {< x, $T_A(x_i)$, $I_A(x_i)$, $F_A(x_i)$ >} and B= {< x, $T_B(x_i)$, $I_B(x_i)$, $F_B(x_i)$ >} are neutrosophic values of X in A and B respectively, then the similarity measure between the neutrosophic sets A and B can be evaluated by the function

For all x_i in X

$$L_T(A,B) = 1 - \frac{\sum_1^n |T_A(x_i) - T_B(x_i)|}{\sum_1^n (T_A(x_i) + T_B(x_i))}$$

(66)

$$L_I(A,B) = \frac{\sum_1^n |I_A(x_i) - I_B(x_i)|}{\sum_1^n (I_A(x_i) + I_B(x_i))}$$

(67)

$$L_F(A,B) = \frac{\sum_1^n |F_A(x_i) - F_B(x_i)|}{\sum_1^n (F_A(x_i) + F_B(x_i))}$$

(68)

and

$$L(A,B) = (L_T(A,B),\ L_I(A,B),\ L_F(A,B))$$

(69)

We will prove this similarity measure satisfies the properties 1-4 as above. The property (P1) for the similarity measure (69) is obtained directly from the definition 4.1.

Proof: obviously, (68) satisfies P1-P3-P4 of definition 4.1. In the following L (A, B) will be proved to satisfy (P2) and (P4).

Proof of (P2) for the (69)

For all x_i in X

First of all, $L_T(A,B) = 1 \leftrightarrow \frac{\sum_1^n |T_A(x_i) - T_B(x_i)|}{\sum_1^n (T_A(x_i) + T_B(x_i))} = 0$

(70)

$$\leftrightarrow |T_A(x_i) - T_B(x_i)| = 0$$

$$\leftrightarrow T_A(x_i) = T_B(x_i)$$

$$L_I(A,B) = 0 \leftrightarrow \frac{\sum_1^n |I_A(x_i) - I_B(x_i)|}{\sum_1^n (I_A(x_i) + I_B(x_i))} = 0$$

(71)

$$\leftrightarrow |I_A(x_i) - I_B(x_i)| = 0 \ \leftrightarrow I_A(x_i) = I_B(x_i)$$

$$L_F(A,B) = 0 \leftrightarrow \frac{\sum_1^n |F_A(x_i) - F_B(x_i)|}{\sum_1^n (F_A(x_i) + F_B(x_i))} = 0$$

(72)

$$\leftrightarrow |F_A(x_i) - F_B(x_i)| = 0 \leftrightarrow F_A(x_i) = F_B(x_i)$$

Then $L(A,B) = (L_T(A,B),\ L_I(A,B),\ L_F(A,B))$ = (1, 0, 0) if A=B for all A, B ∈ NS.

(73)

Proof of P3 for the (69) is obvious.

By applying (69) the degree of similarity between the neutrosophic sets (A, B), (A, C) and (B, C) are:

L (A, B) $=(L_T(A,B), L_I(A,B), L_F(A,B))=$ (0.8, 0.2, 0.17).
L (A, C) $= (L_T(A,C), L_I(A,C), L_F(A,C))=$ (0.67, 0.5, 0.17).
L (B, C) $= (L_T(B,C), L_I(B,C), L_F(B,C))=$ (0.85, 0.33, 0).

The result indicates that the degree of similarity between neutrosophic sets A and B $\in [0, 1]$. Then (69) satisfies property P4: L(A, C) $\leq$ L(A, B) and L(A, C) $\leq$ L(B, C).

4.4 Similarity Measure Based on the Type 2 Geometric Distance model

In this section we extend the similarity measure proposed by Yang and Hang [16] to neutrosophic set as follow:

4.4.Definition: Let A, B be two neutrosophic set in X=$\{x_1,x_2,.., x_n\}$, if A = $\{< x, T_A(x_i), I_A(x_i), F_A(x_i) >\}$ and B= $\{<x, T_B(x_i), I_B(x_i), F_B(x_i) >\}$ are neutrosophic values of X in A and B respectively, then the similarity measure between the neutrosophic set A and B can be evaluated by the function:

For all x_i in X

$$M_T\ (A, B) = \frac{1}{n}\sum_1^n\left(1 - \frac{|T_A(x_i)-T_B(x_i)|}{2}\right). \tag{74}$$

$$M_I\ (A, B) = \frac{1}{n}\sum_1^n\left(\frac{|I_A(x_i)-I_B(x_i)|}{2}\right). \tag{75}$$

$$M_F\ (A, B) = \frac{1}{n}\sum_1^n\left(\frac{|F_A(x_i)-F_B(x_i)|}{2}\right).$$

And

$$M_{T,I,F} = (M_T(A,B), M_I(A,B), M_F(A,B)) \quad \text{for all } i=\{x_1,x_2,.., x_n\} \tag{76}$$

The proofs of the properties P1-P2-P3 in definition 4.1 (Axioms of a Similarity Measure) of the similarity measure in definition 4.4 are obvious.

Proof of (P4) for the (76).
Since for all x_i in X

$$T_A(x_i) \leq T_B(x_i) \leq T_C(x_i), I_A(x_i) \geq I_B(x_i) \geq I_C(x_i), F_A(x_i) \geq F_B(x_i) \geq F_C(x_i)$$

Then for all x_i in X

$$1 - \frac{|T_C(x_i)-T_A(x_i)|}{2} = 1 - \frac{(T_C(x_i)-T_A(x_i))}{2} \tag{77}$$

$$= 1 - \left(\frac{(T_C(x_i)-T_B(x_i))}{2} + \frac{(T_B(x_i)-T_A(x_i))}{2}\right)$$

$$\leq 1 - \left(\frac{(T_C(x_i)-T_B(x_i))}{2}\right)$$

$$= 1 - \frac{|T_C(x_i)-T_B(x_i)|}{2}$$

Then M_T(A, C) $\leq$ M_T(B, C). (78)

Similarly, M_T(A, C) $\leq$ M_T(A, B) can be proved easily.

For M_I(A, C) $\geq$ M_I(B, C) and M_F(A, C) $\geq$ M_F(B, C) the proof is easy.

Then by the definition 4.4, (P4) for definition 4.1, is satisfied as well.

By applying (76), the degree of similarity between the neutrosophic sets (A, B), (A, C) and (B, C) are:

M(A, B)=(M_T (A,B), M_I (A,B), M_F (A,B))=(0.95 , 0.075 , 0.075)
M(A, C)= (M_T (A,C), M_I (A,C), M_F (A,C))=(0.9, 0.15 , 0.075)
M(B, C)= (M_T (B,C), M_I (B,C), M_F (B,C))=(0.9, 0.075 , 0)

Then (76) satisfies property P4:
M (A, C) $\leq$ M (A, B) and M (A, C) $\leq$ M (B, C). (79)

Another way of calculating similarity (degree) of neutrosophic sets is based on their distance. There are more approaches on how the relation between the two notions in form of a function can be expressed. Two of them are presented below (in section 4.5 and 4.6).

4.5 Similarity Measure Based on the Type3 Geometric Distance Model.

In the following we extended the similarity measure proposed by Koczy in [15] to neutrosophic set (NS).

4.5.Definition: Let A, B be two neutrosophic sets in X=$\{x_1,x_2,.., x_n\}$, if A = $\{< x, T_A(x_i), I_A(x_i), F_A(x_i) >\}$ and B= $\{< x, T_B(x_i), I_B(x_i), F_B(x_i) >\}$ are neutrosophic values of x in A and B respectively, then the similarity measure between the neutrosophic sets A and B can be evaluated by the function

$\mathbf{H_T(A,B) = \frac{1}{1+d_\infty^T(AB)}}$ denotes the degree of similarity. (80)

$\mathbf{H_I(A,B) = 1 - \frac{1}{1+d_\infty^I(AB)}}$ denotes the degree of indeterminate similarity. (81)

$d_\infty^T(A,B) = \max\{|T_A(x_i) - T_B(x_i)|\}$.
(83)

$d_\infty^I(A,B) = \max\{|I_A(x_i) - I_B(x_i)|\}$.
(84)

$d_\infty^F(A,B) = \max\{|F_A(x_i) - F_B(x_i)|\}$.
(85)

and H (A, B) = ($H_T(A, B), H_I(A, B), H_F(A, B)$).
(86)

By applying the (86) in numerical example we obtain:

$d_\infty(A,B)$= (0.2, 0.2, 0.2), then H (A, B) = (0.83, 0.17, 0.17).

$d_\infty(A,C)$= (0.3, 0.4, 0.1), then H (A, C) = (0.76, 0.29, 0.17).

$d_\infty(B,C)$= (0.1, 0.2, 0), then H (B, C) = (0.90, 0.17, 0).

It can be verified that H (A, B) also has the properties (P1)-(P4).

4.6 Similarity Measure Based on Extended Hausdorff Distance

It is well known that similarity measures can be generated from distance measures. Therefore, we may use the proposed distance measure based on extended Hausdorff distance to define similarity measures. Based on the relationship of similarity measures and distance measures, we can define a new similarity measure between NS A and B as follows:

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$$N(A,B) = 1 - d_H(A,B) \qquad (87)$$

Where $d_H(A,B)$ represent the extended Hausdorff distance between neutrosophic sets (NS) A and B.

According to the above distance properties (D1-D4).It is easy to check that the similarity measure (87) satisfies the four properties of axiom similarity defined in 4.1

By applying the (87) in numerical example we obtain:

$N(A,B)$ =0.8

$N(A,C)$ =0.7

$N(B,C)$ =0.85 61

Then (5) satisfies property P4:

N(A, C) ≤ N(A, B) and N(A, C) ≤ N(B, C)

Remark: It is clear that the larger the value of N(A, B), the more the similarity between NS A and B.

Next we define similarity measure between NS A and B using a matching function.

4.7 Similarity Measure of two Neutrosophic Sets Based on Matching Function.

Chen [11] and Chen et al. [12] introduced a matching function to calculate the degree of similarity between fuzzy sets. In the following, we extend the matching function to deal with the similarity measure of NS.

4.7 Definition Let F and E be two neutrosophic sets over U. Then the similarity between them, denoted by K (F, G) or $K_{F,G}$ has been defined based on the matching function as:

For all x_i in X

$$K(F,G) = K_{F,G} = \frac{\sum_{i=1}^{n}(T_F(x_i)\cdot T_G(x_i) + I_F(x_i)\cdot I_G(x_i) + F_F(x_i)\cdot F_G(x_i))}{\max\left(\sum_{i=1}^{n}((T_F(x_i))^2 + (I_F(x_i))^2 + (F_F(x_i))^2), \sum_{i=1}^{n}((T_G(x_i))^2 + (I_G(x_i))^2 + (F_G(x_i))^2)\right)} \qquad (88)$$

Considering the weight $w_j \in [0, 1]$ of each element $x_i \in X$, we get the weighting similarity measure between NS as:

For all x_i in X

$$K_w(F,G) = \frac{\sum_{i=1}^{n} w_i(T_F(x_i)\cdot T_G(x_i) + I_F(x_i)\cdot I_G(x_i) + F_F(x_i)\cdot F_G(x_i))}{\max\left(\sum_{i=1}^{n} w_i((T_F(x_i))^2 + (I_F(x_i))^2 + (F_F(x_i))^2), \sum_{i=1}^{n} w_i((T_G(x_i))^2 + (I_G(x_i))^2 + (F_G(x_i))^2)\right)} \qquad \textbf{(89)}$$

If each element $x_i \in X$ has the same importance, then (89) is reduced to (88). The larger the value of $K(F,G)$ the more the similarity between F and G. Here $K(F,G)$ has all the properties described as listed in the definition 4.1.

By applying the (88) in numerical example we obtain:

$K(A,B)$ = 0.75, $K(A,C)$ = 0.66, and $K(B,C)$ = 0.92

Then (87) satisfies property P4: K(A, C) ≤ K(A, B) and K(A, C) ≤ K(B, C)

2 Comparision of various similarity measures

In this section, we make a comparison among similarity measures proposed in the paper. Table 1 show the comparison of various similarity measures between two neutrosophic sets respectively.

	A, B	A, C	B, C
(50)	(0.75, 0.35, 0.3)	(0.53, 0.7, 0.3)	(0.73, 0.63, 0)
(69) 62	(0.8, 0.2,0.17)	(0.67, 0.5, 0.17)	(0.85, 0.33, 0)
(76)	(0.95, 0.075, 0.075)	(0.9, 0.15, 0.075)	(0.9, 0.075, 0)
(86)	(0.83, 0.17, 0.17)	(0.76, 0.29, 0.17)	(0.9, 0.17, 0)
(87)	0.8	0.7	0.85
(88)	0.75	0.66	0.92

Table 1: Example results obtained from the similarity measures between neutrosophic sets A , B and C.

Each similarity measure expression has its own measuring. They all evaluate the similarities in neutrosophic sets, and they can meet all or most of the properties of similarity measure.

(87)	0.8	0.7	0.85
(88)	0.75	0.66	0.92

Table 1: Example results obtained from the similarity measures between neutrosophic sets A , B and C.

Each similarity measure expression has its own measuring. They all evaluate the similarities in neutrosophic sets, and they can meet all or most of the properties of similarity measure.

In definition 4.1, that is P1-P4. It seems from the table above that from the results of similarity measures between neutrosophic sets can be classified in two type of similarity measures: the first type which we called "crisp similarity measure" is illustrated by similarity measures (N and K) and the second type called "neutrosophic similarity measures" illustrated by similarity measures (S, L, M and H). The computation of measure **H , N** and **S** are much simpler than that of **L, M** and **K.**

Conclusions

In this paper we have presented a new distance called "extended Hausdorff distance for neutrosophic sets" or "neutrosophic Hausdorff distance". Then, we defined a new series of similarity measures to calculate the similarity between neutrosophic sets. It's hoped that our findings will help enhancing this study on neutrosophic set for researchers.

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On Neutrosophic Implications

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Abstract: In this paper, we firstly review the neutrosophic set, and then construct two new concepts called neutrosophic implication of type 1 and of type 2 for neutrosophic sets. Furthermore, some of their basic properties and some results associated with the two neutrosophic implications are proven.

Keywords: *Neutrosophic Implication, Neutrosophic Set, N-norm, N-conorm.*

1 Introduction

Neutrosophic set (NS) was introduced by Florentin Smarandache in 1995 [1], as a generalization of the fuzzy set proposed by Zadeh [2], interval-valued fuzzy set [3], intuitionistic fuzzy set [4], interval-valued intuitionistic fuzzy set [5], and so on. This concept represents uncertain, imprecise, incomplete and inconsistent information existing in the real world. A NS is a set where each element of the universe has a degree of truth, indeterminacy and falsity respectively and with lies in] 0^-, 1^+ [, the non-standard unit interval.

NS has been studied and applied in different fields including decision making problems [6, 7, 8], Databases [10], Medical diagnosis problem [11], topology [12], control theory [13], image processing [14, 15, 16] and so on.

In this paper, motivated by fuzzy implication [17] and intutionistic fuzzy implication [18], we will introduce the definitions of two new concepts called neutrosophic implication for neutrosophic set.

This paper is organized as follow: In section 2 some basic definitions of neutrosophic sets are presented. In section 3, we propose some sets operations on neutrosophic sets. Then, two kind of neutrosophic implication are proposed. Finally, we conclude the paper.

2 Preliminaries

This section gives a brief overview of concepts of neutrosophic sets, single valued neutrosophic sets, neutrosophic norm and neutrosophic conorm which will be utilized in the rest of the paper.

Definition 1 **(Neutrosophic set) [1]**

Let X be a universe of discourse then, the neutrosophic set A is an object having the form:

A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$ >,x ∈ X}, where the functions T, I, F : X→ $]^-0, 1^+[$ define respectively the degree of membership (or Truth), the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element x ∈ X to the set A with the condition.

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+. \qquad (1)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. So instead of $]^-0, 1^+[$, we need to take the interval [0, 1] for technical applications, because $]^-0, 1^+[$will be difficult to apply in the real applications such as in scientific and engineering problems.

Definition 2 **(Single-valued Neutrosophic sets) [20]**

Let X be an universe of discourse with generic elements in X denoted by x. An SVNS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$, for each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x)$, ∈ [0, 1].

When X is continuous, an SVNS A can be written as

$$A= \int_X \frac{<T_A(x), I_A(x), F_A(x)>}{x}, x \in X. \qquad (2)$$

When X is discrete, an SVNS A can be written as

$$A= \sum_1^n \frac{<T_A(x_i), I_A(x_i), F_A(x_i)>}{x_i}, x_i \in X \qquad (3)$$

Definition 3 (Neutrosophic norm, n-norm) [19]

Mapping N_n: $(]-0,1+[\times]-0,1+[\times]-0,1+[)^2 \rightarrow]-0,1+[\times]-0,1+[\times]-0,1+[$

N_n (x(T_1, I_1, F_1), y(T_2, I_2, F_2)) = ($N_n T$(x,y), $N_n I$(x,y), $N_n F$(x,y), where

$N_n T(.,.)$,$N_n I(.,.)$,$N_n F(.,.)$

are the truth/membership, indeterminacy, and respectively falsehood/ nonmembership components.

N_n have to satisfy, for any x, y, z in the neutrosophic logic/set M of the universe of discourse X, the following axioms

a) Boundary Conditions: N_n (x, 0) = 0, N_n (x, 1) = x.

b) Commutativity: N_n (x, y) = N_n (y, x).

c) Monotonicity: If x ≤y, then N_n (x, z) ≤ N_n (y, z).

d) Associativity: N_n (N_n (x, y), z) = N_n (x, N_n (y, z)).

N_n represents the intersection operator in neutrosophic set theory.

Let J ∈ {T, I, F} be a component.

Most known N-norms, as in fuzzy logic and set the T-norms, are:

• The Algebraic Product N-norm: $N_{n-\text{algebraic}}J(x, y) = x \cdot y$

• The Bounded N-Norm: $N_{n-\text{bounded}}J(x, y) = \max\{0, x + y - 1\}$

• The Default (min) N-norm: $N_{n-\min}$ (x, y) = min{x, y}.

A general example of N-norm would be this.

Let x(T_1, I_1, F_1) and y (T_2, I_2, F_2) be in the neutrosophic set M. Then:

N_n (x, y) = ($T_1 \wedge T_2$, $I_1 \vee I_2$, $F_1 \vee F_2$) (4)

where the "∧" operator is a N-norm (verifying the above N-norms axioms); while the "∨" operator, is a N-conorm.

For example, ∧ can be the Algebraic Product T-norm/N-norm, so $T_1 \wedge T_2 = T_1 \cdot T_2$ and ∨ can be the Algebraic Product T-conorm/N-conorm, so $T_1 \vee T_2 = T_1 + T_2 - T_1 \cdot T_2$

Or ∧ can be any T-norm/N-norm, and ∨ any T-conorm/N-conorm from the above.

Definition 4 **(Neutrosophic conorm, N-conorm) [19]**

Mapping N_c: ( ]-0,1+[×]-0,1+[×]-0,1+[ )2→]-0,1+[×]-0,1+[×]-0,1+[

N_c (x(T_1, I_1, F_1), y(T_2, I_2, F_2)) = (N_cT(x,y), N_cI(x,y), N_cF(x,y)),

where N_cT(.,.),N_cI(.,.),N_cF(.,.) are the truth/membership, indeterminacy, and respectively falsehood/non membership components.

N_c have to satisfy, for any x, y, z in the neutrosophic logic/set M of universe of discourse X, the following axioms:

a) Boundary Conditions: N_c (x, 1) = 1, N_c (x, 0) = x.

b) Commutativity: N_c (x, y) = N_c (y, x).

c) Monotonicity: if x ≤y, then N_c (x, z) ≤ N_c (y, z).

d) Associativity: N_c (N_c (x, y), z) = N_c (x, N_c (y, z))

N_c represents respectively the union operator in neutrosophic set theory.

Let J ∈ {T, I, F} be a component. Most known N-conorms, as in fuzzy logic and set the T-conorms, are:

• The Algebraic Product N-conorm: $N_{c-\text{algebraic}}$ J(x, y) = x + y −x · y

• The Bounded N-conorm: $N_{c-\text{bounded}}$ J(x, y) = min{1, x + y}

• The Default (max) N-conorm: $N_{c-\max}$ J(x, y) = max{x, y}.

A general example of N-conorm would be this.

Let x(T_1, I_1, F_1) and y(T_2, I_2, F_2) be in the neutrosophic set/logic M. Then:

N_c (x, y) = (T1∨T2, I1∧I2, F1∧F2) (5)

where the "∧" operator is a N-norm (verifying the above N-conorms axioms); while the "∨" operator, is a N-norm.

For example, ∧ can be the Algebraic Product T-norm/N-norm, so T1∧T2= T1·T2 and ∨ can be the Algebraic Product T-conorm/N-conorm, so T1∨T2= T1+T2-T1·T2.

Or ∧ can be any T-norm/N-norm, and ∨ any T-conorm/N-conorm from the above.

In 2013, A. Salama [21] introduced beside the intersection and union operations between two neutrosophic set A and B, another operations defined as follows:

Definition 5

Let A, B two neutrosophic sets

A $\cap_1$ B = min (T_A , T_B) ,max (I_A , I_B) , max(F_A , F_B)

A $\cup_1$ B = (max (T_A , T_B) , max (I_A , I_B) ,min(F_A , F_B))

A $\cap_2$ B={ min (T_A , T_B), min (I_A , I_B), max (F_A , F_B)}

A $\cup_2$ B = (max (T_A , T_B) , min (I_A , I_B) ,min(F_A , F_B))

A^C= (F_A , I_A , T_A).

Remark

For the sake of simplicity we have denoted:

$\cap_2$ = min min max, $\cup_2$ = max min min

$\cap_1$ = min max max, $\cup_1$ = max max min.

Where $\cap_1$, $\cup_2$ represent the intersection set and the union set proposed by Florentin Smarandache and $\cap_2$, $\cup_1$ represent the intersection set and the union set proposed by A.Salama.

3 Neutrosophic Implications

In this subsection, we introduce the set operations on neutrosophic set, which we will work with. Then, two neutrosophic implication are constructed on the basis of single valued neutrosophic set .The two neutrosophic implications are denoted by $_{NS1}$ and $_{NS2}$. Also, important properties of $_{NS1}$ and $_{NS2}$ are demonstrated and proved.

Definition 6 **(Set Operations on Neutrosophic sets)**

Let A and B two neutrosophic sets , we propose the following operations on NSs as follows:

$A\ @\ B = (\frac{T_A + T_B}{2}, \frac{I_A + I_B}{2}, \frac{F_A + F_B}{2})$ where $< T_A, I_A, F_A > \in A$,$< T_B, I_B, F_B > \in B$

$A\ \$\ B = (\overline{T_A\, T_B}, \overline{I_A\, I_B}, \overline{F_A\, F_B})$,where $< T_A, I_A, F_A > \in A$,$< T_B, I_B, F_B > \in B$

$A\ \#\ B = (\frac{2\,T_A T_B}{T_A + T_B}, \frac{2\,I_A I_B}{I_A + I_B}, \frac{2\,F_A F_B}{F_A + F_B})$, where $< T_A, I_A, F_A > \in A$,$< T_B, I_B, F_B > \in B$

$A \oplus$ B=(T_A +T_B-T_A T_B , I_A I_B, F_A F_B) ,where $< T_A, I_A, F_A > \in A$,$< T_B, I_B, F_B > \in B$

$A \otimes$ B= (T_B T_A , I_A+I_B-I_A I_B, F_A +F_B-F_A F_B), where $< T_A, I_A, F_A > \in A$,$< T_B, I_B, F_B > \in B$

Obviously, for every two A and B, (A @ B), (A $ B), (A# B) , $A\oplus$ B and $A\otimes$ B are also NSs.
Based on definition of standard implication denoted by "A → B", which is equivalent to "non A or B". We extended it for neutrosophic set as follows:

Definition 7
Let A(x) ={<x, $T_A(x)$, $I_A(x)$, $F_A(x)$> | **x** ∈ **X**} and B(x) ={<x, $T_B(x)$, $I_B(x)$, $F_B(x)$> | x ∈ X} , A, B ∈ NS(X). So, depending on how we handle the indeterminacy, we can defined two types of neutrosophic implication, then $_{NS1}$ is the neutrosophic type1 defined as

A $_{NS1}$ B ={< x, $F_A(x) \vee T_B(x)$, $I_A(x) \wedge I_B(x)$, $T_A(x) \wedge F_B(x)$ > | x ∈ X} (6)

And
$_{NS2}$ is the neutrosophic type 2 defined as

A $_{NS2}$ B =={< x, $F_A(x) \vee T_B(x)$, $I_A(x) \vee I_B(x)$, $T_A(x) \wedge F_B(x)$ > | x ∈ X} (7)

by ∨ and ∧ we denote a neutrosophic norm (N-norm) and neutrosophic conorm (N-conorm).

Note: The neutrosophic implications are not unique, as this depends on the type of functions used in N-norm and N-conorm.
Throughout this paper, we used the function (dual) min/ max.

Theorem 1
For A, B and C ∈ NS(X),

i. A $\cup_1$ B $_{NS1}$ C =(A $_{NS1}$ C) $\cap_1$ (B $_{NS1}$ C)
ii. A $_{NS1}$ B $\cap_1$ C =(A $_{NS1}$ B) $\cap_1$ (A $_{NS1}$ C)
iii. A $\cap_1$ B $_{NS1}$ C = (A $_{NS1}$ C) $\cup_1$ (B $_{NS1}$ C)
iv. A $_{NS1}$ B $\cup_1$ C =(A $_{NS1}$ B)$\cup_1$ (A $_{NS1}$ C)

Proof
(i) From definition in (5) ,we have
A $\cup_1$ B $_{NS1}$ C ={<x ,Max(min(F_A , F_B), T_C) , Min(max (I_A, I_B), I_C) , Min(max (T_A , T_B), F_C) >| x ∈ X} (8)
and
(A $_{NS1}$ C) $\cap_1$ (B $_{NS1}$ C)= {<x, Min(max(F_A , T_C), max(F_B, T_C)) , Max (min (I_A , I_C), min (I_B , I_C)), Max(min (T_A , F_C), min (T_B , F_C)) >| x ∈ X} (9)
Comparing the result of (8) and (9), we get
Max(min(F_A , F_B), T_C)= Min(max(F_A , T_C), max(F_B , T_C))
Min(max (I_A , I_B), I_C)= Max (min (I_A , I_C), min (I_B , I_C))
Min(max (T_A , T_B), F_C)= Max(min (T_A , F_C), min (T_B , F_C))

Hence, A $\cup_1$ B $_{NS1}$ C = (A $_{NS1}$ C) $\cap_1$ (B $_{NS1}$ C)

(ii) From definition in (5), we have
A $_{NS1}$ B $\cap_1$ C={Max(F_A, min(T_B , T_C)) , Min(I_A ,max (I_B, I_C)) , Min(T_A, max (F_B , F_C) >| x ∈ X} (10)
and (A $_{NS1}$ B) $\cap_1$ (A $_{NS1}$ C) = {<x, Min (max (F_A , T_B), max(F_A , T_C)) , Max (min (I_A , I_B), min (I_A , I_C)), Max(min (T_A , F_B), min (T_A , F_C) >| x ∈ X}
(11)
Comparing the result of (10) and (11), we get
Max(F_A, min(T_B , T_C))= Min(max(F_A , T_B), max(F_A , T_C))
Min(I_A ,max (I_B , I_C))= Max (min (I_A , I_B), min (I_A , I_C))
Min(T_A, max (F_B , F_C)= Max(min (T_A , F_B), min (T_A , F_C))
Hence, A $\cap_1$ B $_{NS1}$ C = (A $_{NS1}$ C) $\cup_1$ (B $_{NS1}$ C)

(iii) From definition in (5), we have
A $\cap_1$ B $_{NS1}$ C ={< x , Max(max(F_A , F_B), T_C) , Min(min(I_A , I_B), I_C) , Min(min (T_A , T_B), F_C) >| x ∈ X} (12)
and
(A $_{NS1}$ C)$\cup_1$ (B $_{NS1}$ C) = {<x, Max(max(F_A , T_C), max(F_B , T_C)) , Max (min (I_A , I_C), min (I_B , I_C)), Min(min (T_A , F_C), min (T_B , F_C)) >| x ∈ X}
(13)
Comparing the result of (12) and (13), we get
Max(max(F_A , F_B), T_C)= Max(max(F_A , T_C), max(F_B , T_C))
Min(min(I_A , I_B), I_C)= Max (min (I_A , I_C), min (I_B, I_C))
Min(min (T_A , T_B), F_C)= Min(min (T_A , F_C), min (T_B , F_C)),
Hence, A $\cap_1$ B $_{NS1}$ C = (A $_{NS1}$ C) $\cup_1$ (B $_{NS1}$ C)
(iv) From definition in (5), we have
A $_{NS1}$ B $\cup_1$ C ={<x, Max (F_A ,Max (T_B , T_C)), Min (I_A , Max (I_B , I_C)), Min(T_A, Min(F_B , F_C)) >| x ∈ X} (14)
and
(A $_{NS1}$ B) $\cup_1$ (A $_{NS1}$ C) = {<x, Max(max (F_A, T_B), max(F_A , T_C)) , Max (min (I_A , I_B), min (I_A , I_C)), Min(min (T_A , F_B), min (T_A , F_C)) >| x ∈ X} (15)
Comparing the result of (14) and (15), we get
Max (F_A , Max (T_B , T_C)) = Max(max(F_A , T_B), max(F_A , T_C))
Min (I_A , Max (I_B , I_C)) = Max (min (I_A , I_B), min (I_A , I_C))
Min (T_A, Min(F_B , F_C)) = Min(min (T_A , F_B), min (T_A , F_C))
hence, A $_{NS1}$ B $\cup_1$ C = (A $_{NS1}$ B)$\cup_1$ (A $_{NS1}$ C)

In the following theorem, we use the operators: $\cap_2$ = min min max , $\cup_2$ = max min min.

Theorem 2 For A, B and C ∈ NS(X),

i. A $\cup_2$ B $_{NS1}$ C =(A $_{NS1}$ C) $\cap_2$ (B $_{NS1}$ C)

ii. A $_{NS1}$ B $\cap_2$ C =(A $_{NS1}$ B) $\cap_2$ (A $_{NS1}$ C)
iii. A $\cap_2$ B $_{NS1}$ C = (A $_{NS1}$ C) $\cup_2$ (B $_{NS1}$ C)
iv. A $_{NS1}$ B $\cup_2$ C =(A $_{NS1}$ B) $\cup_2$ (A $_{NS1}$ C)

Proof
The proof is straightforward.
In view of A $_{NS2}$ B ={< x, $F_A \vee T_B$, $I_A \vee I_B$, $T_A \wedge F_B$ >| x ∈ X} , we have the following theorem:

Theorem 3
For A, B and C ∈ NS(X),
i. A $\cup_1$ B $_{NS2}$ C =(A $_{NS2}$ C) $\cap_1$ (B $_{NS2}$ C)
ii. A $_{NS2}$ B $\cap_1$ C =(A $_{NS2}$ B) $\cap_1$ (A $_{NS2}$ C)
iii. A ∩ B $_{NS2}$ C = (A $_{NS2}$ C) $\cup_1$ (B $_{NS2}$ C)
iv. A $_{NS2}$ B $\cup_1$ C =(A $_{NS2}$ B)$\cup_1$ (A $_{NS2}$ C)

Proof
(i) From definition in (5), we have
A $\cup_1$ B $_{NS2}$ C ={<x, Max(min(F_A , F_B), T_C), Max(max (I_A, I_B), I_C) , Min(max (T_A , T_B), F_C) >| x ∈ X} (16)
and
(A $_{NS2}$ C)$\cap_1$ (B $_{NS2}$ C)= {<x, Min(max(F_A , T_C), max(F_B , T_C)) , Max (max (I_A , I_C), max (I_B , I_C)), Max(min (T_A , F_C), min (T_B , F_C)) >| x ∈ X} (17)
Comparing the result of (16) and (17), we get
Max(min(F_A , F_B), T_C)= Min(max(F_A , T_C), max(F_B , T_C))
Max(max (I_A , I_B), I_C)= Max (max(I_A , I_C), max (I_B , I_C))
Min(max (T_A , T_B), F_C)= Max(min (T_A , F_C), min (T_B , F_C))
hence, A $\cup_1$ B $_{NS2}$ C = (A $_{NS2}$ C) $\cap_1$ (B $_{NS2}$ C)
(ii) From definition in (5) ,we have
A $_{NS2}$ B $\cap_1$ C={<x ,Max(F_A, min(T_B , T_C)) , Max(I_A , max (I_B , I_C)) , Min(T_A, max (F_B , F_C) >| x ∈ X} (18)
and
(A $_{NS2}$ B) ∩ (A $_{NS2}$ C) = {<x,Min(max(F_A , T_B), max(F_A , T_C)) , Max (max (I_A , I_B),max (I_A , I_C)), Max(min (T_A , F_B), min (T_A , F_C)) >| x ∈ X} (19)
Comparing the result of (18) and (19), we get
Max(F_A, min(T_B , T_C))= Min(max(F_A , T_B), max(F_A , T_C))
Max(I_A ,max (I_B , I_C))= Max (max(I_A , I_B), max (I_A , I_C))
Min(T_A, max (F_B , F_C)= Max(min (T_A , F_B), min (T_A , F_C))
Hence , A $_{NS2}$ B$\cap_1$ C =(A $_{NS2}$ B) $\cap_1$ (A $_{NS2}$ C)
(iii) From definition in (5), we have
A $\cap_1$ B $_{NS2}$ C ={<x, Max(max(F_A , F_B), T_C) , Max(max (I_A, I_B), I_C) , Min(min (T_A , T_B), F_C) >| x ∈ X} (20)
and
(A $_{NS2}$ C) $\cup_1$ (B $_{NS2}$ C) = {Max(max(F_A , T_C), max(F_B, T_C)) , Max (max (I_A , I_C), max (I_B , I_C)), Min(min (T_A , F_C), min (T_B , F_C)) } (21)
Comparing the result of (20) and (21), we get
Max(max(F_A , F_B), T_C)= Max(max(F_A , T_C), max(F_B , T_C))
Max(max(I_A , I_B), I_C)= Max (max (I_A , I_C), max (I_B , I_C))
Min(min (T_A , T_B), F_C)= Min(min (T_A , F_C), min (T_B , F_C)),
hence, A $\cap_1$ B $_{NS2}$ C = (A $_{NS2}$ C) $\cup_1$ (B $_{NS2}$ C)
(iv) From definition in (5) ,we have
A $_{NS2}$ B $\cup_1$ C ={<x, Max (F_A , Max (T_B , T_C)), Max (I_A , Max (I_B , I_C)) , Min (T_A, Min(F_B , F_C))> | x ∈ X} (22)
and
(A $_{NS2}$ B) $\cup_1$ (A $_{NS2}$ C)= Max(max(F_A , T_B), max(F_A , T_C)) , Max (max (I_A , I_B), max (I_A , I_C)), Min(min (T_A , F_B), min (T_A , F_C)) (23).
Comparing the result of (22) and (23), we get
Max (F_A , Max (T_B , T_C)) = Max(max(F_A , T_B), max(F_A , T_C))
Max (I_A , Max (I_B , I_C)) = Max (max (I_A , I_B), max (I_A , I_C))
Min (T_A, Min(F_B , F_C)) = Min(min (T_A , F_B), min (T_A , F_C))
hence , A $_{NS2}$ B $\cup_1$ C =(A $_{NS2}$ B)$\cup_1$ (A $_{NS2}$ C)
Using the two operators $\cap_2$ = min min max ,
$\cup_2$ = max min min, we have

Theorem 4
For A, B and C ∈ NS(X),
i. A $\cup_2$ B $_{NS2}$ C =(A $_{NS2}$ C) $\cap_2$ (B $_{NS2}$ C)
ii. A $_{NS2}$ B $\cap_2$ C =(A $_{NS2}$ B) $\cap_2$ (A $_{NS2}$ C)
iii. A $\cap_2$ B $_{NS2}$ C = (A $_{NS2}$ C) $\cup_2$ (B $_{NS2}$ C)
iv. A $_{NS2}$ B $\cup_2$ C =(A $_{NS2}$ B)$\cup_2$ (A $_{NS2}$ C)

Proof
The proof is straightforward.

Theorem 5
For A, B ∈ NS(X),

i. A $_{NS2}$ B^C = A^C $\cup_1$ B^C
ii. (A $_{NS2}$ B^C)^c =(A^C $\cup_1$ B^C)^c = A $\cap_1$ B
iii. (A $_{NS1}$ B^C)^c = A $\cap_2$ B
iv. A^C $_{NS1}$ B = A $\cup_2$ B
v. A $_{NS1}$ B^C = (A $\cap_2$ B)^c

Proof
(i) From definition in (5) ,we have

A $_{NS2}$ B^C ={<x, max (F_A , F_B) ,min (I_A , I_B) , min (T_A , T_B) | x ∈ X} (24)

and
A^C $\cup_1$ B^C ={ max (F_A , F_B) ,min (I_A , I_B) , min (T_A, T_B)} (25)

From (24) and (25), we get A $_{NS2}$ $B^C = A^C \cup_1 B^C$

(ii) From definition in (5), we have

$A^C \cup_1 B^C$ ={<x, max (F_A , F_B) ,min (I_A , I_B) , min (T_A, T_B) > | x ∈ X} (26)

and

$(A^C \cup_1 B^C)^c$= {<x, min (T_A , T_B),min (I_A , I_B) ,max (F_A , F_B) > | x ∈ X}

(27)

From (26) and (27), we get (A $_{NS2}$ $B^C)^c$ =$(A^C \cup_1 B^C)^c$= A $\cap_1$ B

(iii) From definition in (5) ,we have

(A $_{NS1}$ $B^C)^c$ ={ <x, min (T_A , T_B), min (I_A , I_B), max (F_A, F_B) > | x ∈ X} (28)

and

A $\cap_2$ B={ min (T_A , T_B), min (I_A , I_B), max (F_A , F_B)} (29)

From (28) and (29), we get (A $_{NS1}$ $B^C)^c$ = A $\cap_2$ B

(iv)

A^C $_{NS1}$ B = $A \cup_2 B$ ={ <x, max (T_A , T_B) , min (I_A , I_B), min (F_A , F_B) > | x ∈ X}

(v)

A $_{NS1}$ B^C ={<x, max (F_A , F_B),min (I_A , I_B) , max (T_A , T_B) > | x ∈ X} (30)

and

$(A \cap_2 B)^c$ ={<x, max (F_A , F_B),min (I_A , I_B) , max (T_A, T_B) > | x ∈ X} (31)

From (30) and (31), we get A $_{NS1}$ B^C = $(A \cap_2 B)^c$

Theorem 6

For A, B ∈ NS(X),

i. $(A\ \ B)^c$ $_{NS1}$ (A @ B) = $(A@B)^c$ $_{NS1}$ (A B) =$(A⊕B)$

ii. $(A⊗B)^c$ $_{NS1}$ $(A$ @ $B)$ = $(A@B)^c$ $_{NS1}$ $(A ⊗ B)$ = $(A$@ $B)$

iii. $(A⊗B)^c$ $_{NS1}$ (A # B) = $(A\# B)^c$ $_{NS1}$ (A ⊗ B) = $(A\#\ B)$

iv. $(A⊕B)^c$ $_{NS1}$ (A \$ B) = $(A\$ B)^c$ $_{NS1}$ (A ⊕ B) = $(A$ ⊕$B)$

v. $(A⊗B)^c$ $_{NS1}$ (A \$ B) = $(A\$ B)^c$ $_{NS1}$ (A ⊗ B) = $(A$ \$ $B)$

vi. $(A⊗B)^c$ $_{NS1}$ (A ⊕ B) = $(A⊕ B)^c$ $_{NS1}$ (A ⊗ B) = $(A$ ⊕ $B)$

Proof

Let us recall following simple fact for any two real numbers a and b.

Max(a, b) +Min(a, b) = a +b.

Max(a, b) x Min(a, b) = a x b.

(i) From definition in (6), we have

$(A⊕$ B$)^c$ $_{NS1}$ (A @ B) = {<x,Max(T_A +T_B-T_A T_B, $\frac{T_A+T_B}{2}$) ,Min(I_A I_B, $\frac{I_A+I_B}{2}$) ,Min(F_A F_B, $\frac{F_A+F_B}{2}$) > | x ∈ X} = (T_A +T_B- T_A T_B, I_A I_B, F_A F_B)

= $A⊕B$ (32)

and

$(A@B)^c$ $_{NS1}$ (A ⊕B) = ($\frac{F_A+F_B}{2}$, $\frac{I_A+I_B}{2}$, $\frac{T_A+T_B}{2}$) $_{NS1}$ (T_A +T_B-T_A T_B , I_A I_B, F_A F_B)

= {<x, Max($\frac{T_A+T_B}{2}$, T_A +T_B-T_A T_B) ,Min($\frac{I_A+I_B}{2}$,I_A I_B) ,Min($\frac{F_A+F_B}{2}$, F_A F_B) > | x ∈ X} (33)

=(T_A +T_B-T_A T_B, I_A I_B, F_A F_B)

= $A⊕B$

From (32) and (33), we get the result (i)

(ii) From definition in (6), we have

$(A⊗B)^c = (T_B\, T_A\, , I_A + I_B - I_A\, I_B, F_A\ + F_B - F_A\, F_B)^c$= $(F_A\ + F_B - F_A\, F_B, I_A + I_B - I_A\, I_B, T_B\, T_A)$

$(A⊗B)^c$ $_{NS1}$ (A @ B) =

= $(F_A\ + F_B - F_A\, F_B, I_A + I_B - I_A\, I_B, T_B\, T_A)$ $_{NS1}$ ($\frac{T_A+T_B}{2}$, $\frac{I_A+I_B}{2}$, $\frac{F_A+F_B}{2}$)

={<x, Max (T_B T_A, $\frac{T_A+T_B}{2}$) ,Min($I_B + I_B - I_A\, I_B$, $\frac{I_A+I_B}{2}$) ,Min($F_A\ + F_B - F_A\, F_B$, $\frac{F_A+F_B}{2}$) > | x ∈ X}

=($\frac{T_A+T_B}{2}$, $\frac{I_A+I_B}{2}$, $\frac{F_A+F_B}{2}$) = (A @ B) (34)

and

$(A@B)^c$ $_{NS1}$ (A ⊗ B) =

=($\frac{F_A+F_B}{2}$, $\frac{I_A+I_B}{2}$, $\frac{T_A+T_B}{2}$) $_{NS1}$(T_A T_B, I_A +I_B-I_A I_B, F_A +F_B-F_A F_B)

={< x, Max($\frac{T_A+T_B}{2}$, T_A T_B), Min ($\frac{I_A+I_B}{2}$, I_A +I_B-I_A I_B), Min ($\frac{F_A+F_B}{2}$, F_A +F_B-F_A F_B) | x ∈ X}

=($\frac{T_A+T_B}{2}$, $\frac{I_A+I_B}{2}$, $\frac{F_A+F_B}{2}$) = (A @ B) (35)

From (34) and (35), we get the result (ii)

(iii) From definition in (6) ,we have

$(A⊗B)^c$ $_{NS1}$ (A # B) =$(F_A\ + F_B - F_A\, F_B, I_A + I_B - I_A\, I_B, T_B\, T_A)$ $_{NS1}$ ($\frac{2\,T_A T_B}{T_A+T_B}$, $\frac{2\,I_A I_B}{I_A+I_B}$, $\frac{2\,F_A F_B}{F_A+F_B}$)

= {<x , Max (T_B T_A , $\frac{2\,T_A T_B}{T_A+T_B}$) ,Min($I_A + I_B - I_A\, I_B$, $\frac{2\,I_A I_B}{I_A+I_B}$), Min($F_A\ + F_B - F_A\, F_B$, $\frac{2\,F_A F_B}{F_A+F_B}$) > | x ∈ X}

=($\frac{2\,T_A T_B}{T_A+T_B}$, $\frac{2\,I_A I_B}{I_A+I_B}$, $\frac{2\,F_A F_B}{F_A+F_B}$)

=$(A\#\ B)$ (36)

and

$(A\# B)^c$ $_{NS1}$ (A ⊗ B)=($\frac{2\,F_A F_B}{F_A+F_B}$, $\frac{2\,I_A I_B}{I_A+I_B}$, $\frac{2\,T_A T_B}{T_A+T_B}$) $_{NS1}$ (T_B T_A , I_A+I_B-I_A I_B, F_A +F_B-F_A F_B)

={<x, Max ($\frac{2\,T_A T_B}{T_A+T_B}$, T_B T_A) ,Min($\frac{2\,I_A I_B}{I_A+I_B}$, I_A+I_B-I_A I_B), Min($\frac{2\,F_A F_B}{F_A+F_B}$, F_A +F_B-F_A F_B) > | x ∈ X}

$= \left(\frac{2T_A T_B}{T_A + T_B}, \frac{2I_A I_B}{I_A + I_B}, \frac{2F_A F_B}{F_A + F_B} \right) = (A\#\ B)$ (37)

From (36) and (37), we get the result (iii).

(iv) From definition in (6), we have

$(A \oplus B)^c \ _{NS1}$ $(A\ \$\ B) = (F_A F_B, \ , I_A I_B, T_A + T_B - T_A T_B)$ $_{NS1}(\overline{T_A T_B}, \ \overline{I_A I_B}, \ \overline{F_A F_B})$

$= \{<x, \text{Max}(T_A + T_B - T_A T_B, \ \overline{T_A T_B}), \text{Min}(I_A I_B, \overline{I_A I_B}), \text{Min}(F_A F_B, \ \overline{F_A F_B}) > | \ x \in X\}$

$= (\overline{T_A T_B}, \ \overline{I_A I_B}, \ \overline{F_A F_B})$

$= (A\ \$\ B)$ (38)

and

$(A\$ B)^c \ _{NS1}$ $(A \oplus B) = (\overline{F_A F_B}, \ \overline{I_A I_B}, \ \overline{T_A T_B})_{NS1}(T_A + T_B - T_A T_B, I_A I_B, F_A F_B)$

$= \{< x, \text{Max}(\overline{T_A T_B}, T_A + T_B - T_A T_B), \text{Min}(\overline{I_A I_B}, I_A I_B), \text{Min}(\overline{F_A F_B}, F_A F_B) > | \ x \in X\}$

$= (\overline{T_A T_B}, \ \overline{I_A I_B}, \ \overline{F_A F_B})$

$= (A\ \$\ B)$ (39)

From (38) and (39), we get the result (iv).

(v) From definition in (6), we have

$(A \otimes B)^c \ _{NS1}$ $(A\ \$\ B) = (F_A + F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A)_{NS1} (\overline{T_A T_B}, \ \overline{I_A I_B}, \ \overline{F_A F_B})$

$= \{<x, \text{Max}(T_B T_A, \ \overline{T_A T_B}), \text{Min}(I_A + I_B - I_A I_B, \overline{I_A I_B}), \text{Min}(F_A + F_B - F_A F_B, \ \overline{F_A F_B}) > | \ x \in X\}$

$= (\overline{T_A T_B}, \ \overline{I_A I_B}, \ \overline{F_A F_B})$

$= (A\ \$\ B)$ (40)

and

$(A\$ B)^c \ _{NS1}$ $(A \otimes B) = (\overline{F_A F_B}, \ \overline{I_A I_B}, \ \overline{T_A T_B})_{NS1}(T_B T_A, I_A + I_B - I_A I_B, F_A + F_B - F_A F_B)$

$= \{<x, \text{Max}(\overline{T_A T_B}, T_B T_A), \text{Min}(\overline{I_A I_B}, I_A + I_B - I_A I_B), \text{Min}(\overline{F_A F_B}, F_A + F_B - F_A F_B) > | \ x \in X\}$

$= (\overline{T_A T_B}, \ \overline{I_A I_B}, \ \overline{F_A F_B})$

$= (A\ \$\ B)$ (41)

From (40) and (41), we get the result (v).

(vi) From definition in (6), we have

$(A \otimes B)^c \ _{NS1}$ $(A \oplus B) = (A \oplus B)^c \ _{NS1}$ $(A \otimes B) = (A \oplus B)$

$(A \otimes B)^c \ _{NS1}$ $(A \oplus B) = (F_A + F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A)_{NS1}(T_A + T_B - T_A T_B, I_A I_B, F_A F_B)$

$= \{<x, \text{Max}(T_B T_A, T_A + T_B - T_A T_B), \text{Min}(I_A + I_B - I_A I_B, I_A I_B), \text{Min}(F_A + F_B - F_A F_B, F_A F_B) > | \ x \in X\}$

$= (T_A + T_B - T_A T_B, I_A I_B, F_A F_B)$

$= (A \oplus B)$ (42)

and

$(A \oplus B)^c \ _{NS1}$ $(A \otimes B) = (F_A F_B, I_A I_B, T_A + T_B - T_A T_B)_{NS1}(T_B T_A, I_A + I_B - I_A I_B, F_A + F_B - F_A F_B)$

$= \{<x, \text{Max}(T_A + T_B - T_A T_B, T_B T_A), \text{Min}(I_B I_A, I_A + I_B - I_A I_B), \text{Min}(F_A F_B, F_A + F_B - F_A F_B) > | \ x \in X\}$

$= (T_A + T_B - T_A T_B, I_A I_B, F_A F_B)$

$= (A \oplus B)$ (43)

From (42) and (43), we get the result (vi).

The following theorem is not valid.

Theorem 7

For A, B $\in$ NS(X),

i. $(A \quad B)_{NS1} \ (A @ B)^c \ {}^c = (A@B)_{NS1} \ (A \quad B)^c \ {}^c = (A@B)$

ii. $(A \otimes B)_{NS1} \ (A @ B)^c \ {}^c = (A@B)_{NS1} \ (A \otimes B)^c \ {}^c = (A \otimes B)$

iii. $(A \oplus B)_{NS1} \ (A\#\ B)^c \ {}^c = (A\# B)_{NS1} \ (A \oplus B)^c \ {}^c = (A\#\ B)$

iv. $(A \otimes B)_{NS1} \ (A\#\ B)^c \ {}^c = (A\# B)_{NS1} \ (A \otimes B)^c \ {}^c = (A \otimes B)$

v. $(A \oplus B)_{NS1} \ (A\ \$\ B)^c \ {}^c = (A\$ B)_{NS1} \ (A \oplus B)^c \ {}^c = (A\ \$\ B)$

vi. $(A \otimes B)_{NS1} \ (A\ \$\ B)^c \ {}^c = (A\$ B)_{NS1} \ (A \otimes B)^c \ {}^c = (A \otimes B)$

Proof

The proof is straightforward.

Theorem 8

For A, B $\in$ NS(X),

i. $(A \quad B)_{NS2} \ (A @ B)^c \ {}^c = (A@B)_{NS2} \ (A \quad B)^c \ {}^c = (A@B)$

ii. $(A \otimes B)_{NS2} \ (A @ B)^c \ {}^c = (A@B)_{NS2} \ (A \otimes B)^c \ {}^c = (A \otimes B)$

iii. $(A \oplus B)_{NS2} \ (A\#\ B)^c \ {}^c = (A\# B)_{NS2} \ (A \oplus B)^c \ {}^c = (A\#\ B)$

iv. $(A \otimes B)_{NS2} \ (A\#\ B)^c \ {}^c = (A\# B)_{NS2} \ (A \otimes B)^c \ {}^c = (A \otimes B)$

v. $(A \oplus B)_{NS2} \ (A\ \$\ B)^c \ {}^c = (A\$ B)_{NS2} \ (A \oplus B)^c \ {}^c = (A\ \$\ B)$

vi. $(A \otimes B)_{NS2} \ (A\ \$\ B)^c \ {}^c = (A\$ B)_{NS2} \ (A \otimes B)^c \ {}^c = (A \otimes B)$

Proof

(i) From definition in (6), we have

$(A \quad B)_{NS2}\ (A @ B)^{C\ c} = (T_A + T_B - T_A T_B\ , I_A I_B\ , F_A F_B)_{NS2}\ (\frac{F_A+F_B}{2}, \frac{I_A+I_B}{2}, \frac{T_A+T_B}{2})$

$=\{<x\ , \begin{matrix} \text{Max}\ \ F_A F_B, \frac{F_A+F_B}{2}\ , \text{Max}\ \ I_A I_B\ , \frac{I_A+I_B}{2}\ , \\ \text{Min}\ \ T_A + T_B - T_A T_B\ , \frac{T_A+T_B}{2} \end{matrix}{}^{c} >|$

$x \in X\}$

$= \frac{F_A+F_B}{2}, \frac{I_A+I_B}{2}, \frac{T_A+T_B}{2}{}^{c}$

$=(\frac{T_A+T_B}{2}, \frac{I_A+I_B}{2}, \frac{F_A+F_B}{2})$

$=(A@B)$ (44)

and

$(A@B)_{NS2}\ (A \oplus B)^{C\ c}$

$= (\frac{T_A+T_B}{2}, \frac{I_A+I_B}{2}, \frac{F_A+F_B}{2})_{NS2}\ (F_A F_B, I_A I_B, T_A + T_B - T_A T_B)^{c}$

$=$

$\text{Max}\ \frac{F_A+F_B}{2}, F_A F_B\ , Max\ \frac{I_A+I_B}{2}, I_A I_B\ , Min(\frac{T_A+T_B}{2}, T_A + T_B - T_A T_B)^{c}$

$= \frac{F_A+F_B}{2}, \frac{I_A+I_B}{2}, \frac{T_A+T_B}{2}{}^{c}$

$=(\frac{T_A+T_B}{2}, \frac{I_A+I_B}{2}, \frac{F_A+F_B}{2})$

$=(A@B)$ (45)

From (44) and (45), we get the result (i).

(ii) From definition in (6) ,we have

$(A \otimes B)_{NS2}\ (A @ B)^{C\ c} =\ \text{Max}\ \ F_A + F_B - F_A F_B, \frac{F_A+F_B}{2}\ , \text{Max}\ \ I_A + I_B - I_A I_B\ , \frac{I_A+I_B}{2}\ , \text{Min}\ \ T_B T_A\ , \frac{T_A+T_B}{2}{}^{c}$

$= F_A + F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A{}^{c}$

$=(T_B T_A\ , I_A + I_B - I_A I_B\ , F_A + F_B - F_A F_B)$

$= (A \otimes B)$ (46)

and

$(A@B)_{NS2}\ (A \otimes B)^{C\ c} = \{<x, (\frac{T_A+T_B}{2}, \frac{I_A+I_B}{2}, \frac{F_A+F_B}{2})$
$(F_A + F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A) > |\ x \in X\}$

$=$

$\begin{matrix} \text{Max}\ \ \frac{F_A+F_B}{2}, F_A + F_B - F_A F_B\ , \\ \text{Max}\ \ \frac{I_A+I_B}{2}, I_A + I_B - I_A I_B\ , \text{Min}\ \ \frac{T_A+T_B}{2}, T_B T_A \end{matrix}{}^{c}$

$= F_A + F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A{}^{c}$

$=(T_B T_A, I_A + I_B - I_A I_B\ , F_A + F_B - F_A F_B)$

$=(A \otimes B)$ (47)

From (46) and (47), we get the result (ii).

(iii)From definition in (6), we have

$(A \oplus B)_{NS2}\ (A \#\ B)^{C\ c} =$

$=$

$\text{Max}\ \ F_A F_B, \frac{2 F_A F_B}{F_A+F_B}\ , \text{Max}\ \ I_A I_B\ , \frac{2 I_A I_B}{I_A+I_B}\ , \text{Min}\ \ T_A + T_B - T_A T_B\ , \frac{2 T_A T_B}{T_A+T_B}{}^{c}$

$= \frac{2 F_A F_B}{F_A+F_B}, \frac{2 I_A I_B}{I_A+I_B}, \frac{2 T_A T_B}{T_A+T_B}{}^{c}$

$= \frac{2 T_A T_B}{T_A+T_B}, \frac{2 I_A I_B}{I_A+I_B}, \frac{2 F_A F_B}{F_A+F_B}$

$= (A \#\ B)$ (48)

and

$(A \#\ B)_{NS2}\ (A \oplus B)^{C\ c} = \frac{2 T_A T_B}{T_A+T_B}, \frac{2 I_A I_B}{I_A+I_B}, \frac{2 F_A F_B}{F_A+F_B}{}_{NS2}\ (F_A F_B,\ , I_A I_B, T_A + T_B - T_A T_B)$

$=$

$\begin{matrix} \text{Max}\ \ \frac{2 F_A F_B}{F_A+F_B}, F_A F_B\ , \text{Max}\ \frac{2 I_A I_B}{I_A+I_B}, I_A I_B\ , \\ \text{Min}\ \ \frac{2 T_A T_B}{T_A+T_B}, T_A + T_B - T_A T_B \end{matrix}{}^{c}$

$= \frac{2 F_A F_B}{F_A+F_B}, \frac{2 I_A I_B}{I_A+I_B}, \frac{2 T_A T_B}{T_A+T_B}{}^{c}$

$= \frac{2 T_A T_B}{T_A+T_B}, \frac{2 I_A I_B}{I_A+I_B}, \frac{2 F_A F_B}{F_A+F_B}$

$= (A \#\ B)$ (49)

From (48) and (49), we get the result (iii).

(iv) From definition in (6), we have

$(A \otimes B)_{NS2}\ (A \#\ B)^{C\ c} =$

$=$

$\begin{matrix} \text{Max}\ \ F_A + F_B - F_A F_B, \frac{2 F_A F_B}{F_A+F_B}\ , \\ \text{Max}\ \ I_A + I_B - I_A I_B\ , \frac{2 I_A I_B}{I_A+I_B}\ , \text{Min}\ \ T_B T_A\ , \frac{2 T_A T_B}{T_A+T_B} \end{matrix}{}^{c}$

$= F_A + F_B - F_A F_B,\ \ I_A + I_B - I_A I_B,\ T_B T_A{}^{c}$

$= T_B T_A,\ \ I_A + I_B - I_A I_B,\ F_A + F_B - F_A F_B$

$=(A \otimes B)$ (50)

and

$(A \#\ B)_{NS2}\ (A \otimes B)^{C\ c} =$

$= \text{Max}\ \ \frac{2 F_A F_B}{F_A+F_B},\ F_A + F_B - F_A F_B\ , \text{Max}\ \frac{2 I_A I_B}{I_A+I_B}, I_A + I_B - I_A I_B\ ,\ , \text{Min}\ \frac{2 T_A T_B}{T_A+T_B}, T_B T_A{}^{c}$

$= F_A + F_B - F_A F_B,\ \ I_A + I_B - I_A I_B,\ T_B T_A{}^{c}$

$= T_B T_A,\ \ I_A + I_B - I_A I_B,\ F_A + F_B - F_A F_B$

$=(A \otimes B)$ (51)

From (50) and (51), we get the result (iv).

(v) From definition in (6), we have

$(A \oplus B)_{NS2}\ (A\ \$\ B)^{C\ c} =$

$= \begin{matrix} \text{Max}\ \ F_A F_B,\ \ \overline{F_A F_B}\ , \text{Max}\ I_A I_B\ ,\ \ \overline{I_A\ I_B}\ , \\ \text{Min}\ T_A + T_B - T_A T_B,\ \ \overline{T_A T_B} \end{matrix}{}^{c}$

$= \overline{F_A F_B},\ \overline{I_A\ I_B},\ \overline{T_A T_B}{}^{c}$

$=(\ \overline{T_A T_B}\ ,\ \ \overline{I_A\ I_B}\ ,\ \ \overline{F_A F_B})$

$=(A\ \$\ B)$ (52)

and

$(A\$ B)$ $_{NS2}$ $(A \oplus B)^{C}$ ^{c} =

= Max $\overline{F_A F_B}$, $F_A F_B$, Max $\overline{I_A\ I_B}$, $I_A I_B$, ^{c}

Min $\overline{T_A T_B}$, $T_A + T_B - T_A T_B$

= $\overline{F_A F_B}$, $\overline{I_A\ I_B}$, $\overline{T_A T_B}$ ^{c}

=($\overline{T_A T_B}$, $\overline{I_A\ I_B}$, $\overline{F_A F_B}$)

=$(A \$ B)$ (53)

From (52) and (53), we get the result (v).

(vi) From definition in (2), we have

$(A\otimes B)$ $_{NS2}$ $(A \$ B)^{C}$ ^{c}

=

Max $F_A + F_B - F_A F_B$, $\overline{F_A F_B}$, ^{c}

Max $I_A + I_B - I_A I_B$, $\overline{I_A\ I_B}$, Min $T_B T_A$, $\overline{T_A T_B}$

= $F_A + F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A$ ^{c}

= $T_B T_A, I_A + I_B - I_A I_B, F_A + F_B - F_A F_B$

=$(A \otimes B)$ (54)

and

$(A\$ B)$ $_{NS2}$ $(A \otimes B)^{C}$ ^{c} =

= Max $\overline{F_A F_B}$, $F_A + F_B - F_A F_B$, ^{c}

Max $\overline{I_A\ I_B}$, $I_A + I_B - I_A I_B$, Min $\overline{T_A T_B}$, $T_B T_A$

= $F_A + F_B - F_A F_B, I_A + I_B - I_A I_B, T_B T_A$ ^{c}

= $T_B T_A, I_A + I_B - I_A I_B, F_A + F_B - F_A F_B$

=$(A \otimes B)$ (55)

From (54) and (55), we get the result (v).

The following are not valid.

$< T_A$, $F_A>$	$< T_B$, $F_B>$	A $_{NS1}$ B	A $_{NS1}$ B	$V(A\rightarrow B)$
< 0 ,1>	**< 0 ,1>**	**< 1 ,0>**	**< 1 ,0>**	**< 1 ,0>**
< 0 ,1>	**< 1 ,0>**	**< 1 ,0>**	**< 1 ,0>**	**< 1 ,0>**
< 1 ,0>	**< 0 ,1>**	**< 0 ,1>**	**< 0 ,1>**	**< 0 ,1>**
< 1 ,0>	**< 1 ,0>**	**< 1 ,0>**	**< 1 ,0>**	**< 1 ,0>**

Theorem 9

1- $(A \quad B)^{c}$ $_{NS2}$ $(A @ B)$ = $(A@B)^{c}$ $_{NS2}$ $(A \quad B)$ =$(A\oplus B)$

2- $(A\otimes B)^{c}$ $_{NS2}$ $(A @ B)$ =

$(A@B)^{c}$ $_{NS2}$ $(A \otimes B)$ = $(A@ B)$

3- $(A\oplus B)$ $_{NS2}$ $(A \# B)^{C}$ ^{c} =

$(A\#B)$ $_{NS2}$ $(A \oplus B)^{C}$ ^{c} = $(A\# B)$

4- $(A\otimes B)^{c}$ $_{NS2}$ $(A \# B)$ =

$(A\# B)^{c}$ $_{NS2}$ $(A \otimes B)$ = $(A\# B)$

5- $(A\oplus B)^{c}$ $_{NS2}$ $(A \$ B)$ = $(A\$ B)^{c}$ $_{NS2}$ $(A \oplus B)$ = $(A \oplus B)$

6- $(A\otimes B)^{c}$ $_{NS2}$ $(A \$ B)$ = $(A\$ B)^{c}$ $_{NS2}$ $(A \otimes B)$ = $(A \$ B)$

8- $(A\otimes B)^{c}$ $_{NS2}$ $(A \$ B)$ = $(A\$ B)^{c}$ $_{NS2}$ $(A \otimes B)$ = $(A \$ B)$

9- $(A\otimes B)^{c}$ $_{NS2}$ $(A \oplus B)$ = $(A\oplus B)^{c}$ $_{NS2}$ $(A \otimes B)$ = $(A \oplus B)$

Example

We prove only the (i)

1- $(A \quad B)^{c}$ $_{NS2}$ $(A @ B)$ =

$F_A F_B, I_A I_B, T_A + T_B - T_A T_B$ $_{NS2}$ ($\frac{T_A + T_B}{2}$, $\frac{I_A + I_B}{2}$, $\frac{F_A + F_B}{2}$)

={<x, max $(T_A + T_B - T_A T_B, \frac{T_A + T_B}{2})$,max$(I_A I_B, \frac{I_A + I_B}{2})$,min $(F_A F_B, \frac{F_A + F_B}{2})$ > | **x** ∈ $\boldsymbol{X}$}

={<x, $T_A + T_B - T_A T_B$, $\frac{I_A + I_B}{2}$, $\frac{F_A + F_B}{2}$ > | **x** ∈ $\boldsymbol{X}$} ≠ $(A\oplus B)$

The same thing, for $(A@B)^{c}$ $_{NS2}$ $(A \quad B)$

Then,

$(A \quad B)^{c}$ $_{NS2}$ $(A @ B)$ = $(A@B)^{c}$ $_{NS2}$ $(A \quad B)$ ≠ $(A\oplus B)$.

Remark

We remark that if the indeterminacy values are restricted to 0, and the membership /non-membership are restricted to 0 and 1. The results of the two neutrosophic implications $_{NS1}$ and $_{NS2}$ collapse to the fuzzy /intuitionistic fuzzy implications defined ($V(A\rightarrow B)$) in [17]

Table

Comparison of three kind of implications

From the table, we conclude that fuzzy /intuitionistic fuzzy implications are special case of neutrosophic implication.

Conclusion

In this paper, the neutrosophic implication is studied. The basic knowledge of the neutrosophic set is firstly reviewed, a two kind of neutrosophic implications are constructed, and its properties.

These implications may be the subject of further research, both in terms of their properties or comparison with other neutrosophic implication, and possible applications.

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Neutrosophic Crisp Sets & Neutrosophic Crisp Topological Spaces

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Abstract. In this paper, we generalize the crisp topological spaces to the notion of neutrosophic crisp topological space, and we construct the basic concepts of the neutrosophic crisp topology. In addition to these, we introduce the definitions of neutrosophic crisp continuous function and neutrosophic crisp compact spaces. Finally, some characterizations concerning neutrosophic crisp compact spaces are presented and one obtains several properties. Possible application to GIS topology rules are touched upon.

Keywords: *Neutrosophic Crisp Set; Neutrosophic Topology; Neutrosophic Crisp Topology.*

1 Introduction

Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their crisp and fuzzy counterparts, the most used one being the neutrosophic set theory [6, 7, 8]. After the introduction of the neutrosophic set concepts in [1, 2, 3, 4, 5, 9, 10, 11, 12] and after haven given the fundamental definitions of neutrosophic set operations, we generalize the crisp topological space to the notion of neutrosophic crisp set. Finally, we introduce the definitions of neutrosophic crisp continuous function and neutrosophic crisp compact space, and we obtain several properties and some characterizations concerning the neutrosophic crisp compact space.

2 Terminology

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [6, 7, 8, 12], and Salama et al. [1, 2, 3, 4, 5, 9, 10, 11, 12]. Smarandache introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where $]^-0, 1^+[$ is non-standard unit interval.

Hanafy and Salama et al. [10, 12] considered some possible definitions for basic concepts of the neutrosophic crisp set and its operations. We now improve some results by the following.

3 Neutrosophic Crisp Sets

3.1 Definition

Let X be a non-empty fixed set. A neutrosophic crisp set (NCS for short) A is an object having the form $A = \langle A_1, A_2, A_3 \rangle$ where A_1, A_2 and A_3 are subsets of X satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$.

3.1 Remark

A neutrosophic crisp set $A = \langle A_1, A_2, A_3 \rangle$ can be identified as an ordered triple $\langle A_1, A_2, A_3 \rangle$, where A_1, A_2, A_3 are subsets on X, and one can define several relations and operations between NCSs.

Since our purpose is to construct the tools for developing neutrosophic crisp sets, we must introduce the types of NCSs ϕ_N, X_N in X as follows:

1) ϕ_N may be defined in many ways as a NCS, as follows:

i) $\phi_N = \langle \phi, \phi, X \rangle$, or

ii) $\phi_N = \langle \phi, X, X \rangle$, or

iii) $\phi_N = \langle \phi, X, \phi \rangle$, or

iv) $\phi_N = \langle \phi, \phi, \phi \rangle$

2) X_N may also be defined in many ways as a NCS:

i) $X_N = \langle X, \phi, \phi \rangle$,

ii) $X_N = \langle X, X, \phi \rangle$,

iii) $X_N = \langle X, X, \phi \rangle$,

Every crisp set A formed by three disjoint subsets of a non-empty set is obviously a NCS having the form $A = \langle A_1, A_2, A_3 \rangle$.

3.2 Definition

Let $A = \langle A_1, A_2, A_3 \rangle$ a NCS on , then the complement of the set A , (A^c for short may be defined in three different ways:

C_1 $A^c = \langle A_1{}^c, A_2{}^c, A_3{}^c \rangle$,

C_2 $A^c = \langle A_3, A_2, A_1 \rangle$

C_3 $A^c = \langle A_3, A_2{}^c, A_1 \rangle$

One can define several relations and operations between NCSs as follows:

3.3 Definition

Let X be a non-empty set, and the NCSs A and B in the form $A = \langle A_1, A_2, A_3 \rangle$, $B = \langle B_1, B_2, B_3 \rangle$, then we may consider two possible definitions for subsets $A \subseteq B$

$A \subseteq B$ may be defined in two ways:

1) $A \subseteq B \Leftrightarrow A_1 \subseteq B_1, A_2 \subseteq B_2$ and $A_3 \supseteq B_3$
or

2) $A \subseteq B \Leftrightarrow A_1 \subseteq B_1, A_2 \supseteq B_2$ and $A_3 \supseteq B_3$

3.1 Proposition

For any neutrosophic crisp set A the following hold:

i) $\phi_N \subseteq A$, $\phi_N \subseteq \phi_N$.

ii) $A \subseteq X_N$, $X_N \subseteq X_N$.

3.4 Definition

Let X is a non-empty set, and the NCSs A and B in the form $A = \langle A_1, A_2, A_3 \rangle$, $B = \langle B_1, B_2, B_3 \rangle$. Then:

1) $A \bigcap B$ may be defined in two ways:

i) $A \cap B = \langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3 \rangle$ or

ii) $A \cap B = \langle A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3 \rangle$

2) $A \bigcup B$ may also be defined in two ways:

i) $A \cup B = \langle A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3 \rangle$ or

ii) $A \cup B = \langle A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3 \rangle$

3) $[\ ]A = \langle A_1, A_2, A_1{}^c \rangle$.

4) $<> A = \langle A_3{}^c, A_2, A_3 \rangle$.

3.2 Proposition

For all two neutrosophic crisp sets A and B on X, then the followings are true:

1) $(A \cap B)^c = A^c \cup B^c$.

2) $(A \cup B)^c = A^c \cap B^c$.

We can easily generalize the operations of intersection and union in definition 3.2 to arbitrary family of neutrosophic crisp subsets as follows:

3.3 Proposition

Let $\{A_j : j \in J\}$ be arbitrary family of neutrosophic crisp subsets in X, then

1) $\cap A_j$ may be defined as the following types :

i) $\cap A_j = \langle \cap Aj_1, \cap A_{j2}, \cup A_{j3} \rangle$,or

ii) $\cap A_j = \langle \cap Aj_1, \cup A_{j2}, \cup A_{j3} \rangle$.

2) $\cup A_j$ may be defined as the following types :

i) $\cup A_j = \langle \cup Aj_1, \cap A_{j2}, \cap A_{j3} \rangle$ or

ii) $\cup A_j = \langle \cup Aj_1, \cup A_{j2}, \cap A_{j3} \rangle$.

3.5 Definition

The product of two neutrosophic crisp sets A and B is a neutrosophic crisp set $A \times B$ given by

$A \times B = \langle A_1 \times B_1, A_2 \times B_2, A_3 \times B_3 \rangle$.

4 Neutrosophic Crisp Topological Spaces

Here we extend the concepts of topological space and intuitionistic topological space to the case of neutrosophic crisp sets.

4.1 Definition

A neutrosophic crisp topology (NCT for short) on a non-empty set is a family of neutrosophic crisp subsets in satisfying the following axioms

i) $\phi_N, X_N \in \Gamma$.

ii) $A_1 \cap A_2 \in \Gamma$ for any A_1 and $A_2 \in \Gamma$.

iii) $\cup A_j \in \Gamma \quad \forall \ \{A_j : j \in J\} \subseteq \Gamma$.

In this case the pair (X,Γ) is called a neutrosophic crisp topological space ($NCTS$ for short) in X. The elements in Γ are called neutrosophic crisp open sets (NCOSs for short) in X. A neutrosophic crisp set F is closed if and only if its complement F^C is an open neutrosophic crisp set.

4.1 Remark

Neutrosophic crisp topological spaces are very natural generalizations of topological spaces and intuitionistic topological spaces, and they allow more general functions to be members of topology.

$$TS \longrightarrow ITS \longrightarrow NCTS$$

4.1 Example

Let $X=\{a,b,c,d\}$, ϕ_N, X_N be any types of the universal and empty subsets, and A, B two neutrosophic crisp subsets on X defined by $A=\langle\{a\},\{b,d\},\{c\}\rangle$, $B=\langle\{a\},\{b\},\{c\}\rangle$, then the family $\Gamma=\{\phi_N, X_N, A, B\}$ is a neutrosophic crisp topology on X.

4.2 Example

Let $(X,\tau_\circ)$ be a topological space such that $\tau_\circ$ is not indiscrete. Suppose $\{G_i : i\in J\}$ be a family and $\tau_\circ=\{X,\phi\}\cup\{G_i : i\in J\}$. Then we can construct the following topologies as follows

i) Two intuitionistic topologies

a) $\tau_1=\{\phi_I, X_I\}\cup\{\langle G_i,\phi\rangle, i\in J\}$.

b) $\tau_2=\{\phi_I, X_I\}\cup\left\{\left\langle \phi, G_i^c\right\rangle, i\in J\right\}$

ii) Four neutrosophic crisp topologies

a) $\Gamma_1=\{\phi_N, X_N\}\cup\left\{\left\langle\phi,\phi,G_i^c\right\rangle, i\in J\right\}$

b) $\Gamma_2=\{\phi_N, X_N\}\cup\{\langle G_i,\phi,\phi\rangle, i\in J\}$

c) $\Gamma_3=\{\phi_N, X_N\}\cup\left\{\left\langle G_i,\phi,G_i^c\right\rangle, i\in J\right\}$,

d) $\Gamma_4=\{\phi_N, X_N\}\cup\left\{\left\langle G_i^c\ ,\phi,\phi\right\rangle, i\in J\right\}$

4.2 Definition

Let $(X,\Gamma_1),(X,\Gamma_2)$ be two neutrosophic crisp topological spaces on X. Then Γ_1 is said be contained in Γ_2 (in symbols $\Gamma_1\subseteq\Gamma_2$) if $G\in\Gamma_2$ for each $G\in\Gamma_1$. In this case, we also say that Γ_1 is coarser than Γ_2.

4.1 Proposition

Let $\{\Gamma_j : j\in J\}$ be a family of NCTs on X. Then $\cap\Gamma_j$ is a neutrosophic crisp topology on X. Furthermore, $\cap\Gamma_j$ is the coarsest NCT on X containing all topologies.

Proof

Obvious. Now, we define the neutrosophic crisp closure and neutrosophic crisp interior operations on neutrosophic crisp topological spaces:

4.3 Definition

Let (X,Γ) be NCTS and $A=\langle A_1,A_2,A_3\rangle$ be a NCS in X. Then the *neutrosophic crisp closure* of A ($NCCl(A)$ for short) and *neutrosophic interior crisp* ($NCInt\,(A)$ for short) of A are defined by

$$NCCl(A)=\cap\{K : K \text{ is an NCS in X and } A\subseteq K\}$$
$$NCInt\,(A)=\cup\{G : G \text{ is an NCOS in X and } G\subseteq A\},$$

where NCS is a neutrosophic crisp set, and NCOS is a neutrosophic crisp open set.

It can be also shown that $NCCl\ (A)$ is a NCCS (neutrosophic crisp closed set) and $NCInt\,(A)$ is a CNOS in X

a) A is in X if and only if $NCCl(A)\supseteq A$.

b) A is a NCCS in X if and only if $NCInt(A)=A$.

4.2 Proposition

For any neutrosophic crisp set A in (X,Γ) we have

(a) $NCCl(A^c)=(NCInt(A))^c$,

(b) $NCInt(A^c)=(NCCl(A))^c$.

Proof

a) Let $A=\langle A_1,A_2,A_3\rangle$ and suppose that the family of neutrosophic crisp subsets contained in A are indexed by the family if NCSs contained in A are indexed by the family $A=\{< A_{j_1}, A_{j_2}, A_{j3} >: i\in J\}$. Then we see that we have two types of $NCInt(A)=\{<\cup A_{j1},\cup A_{j_2},\cap A_{j3}>\}$ or $NCInt(A)=\{<\cup A_{j1},\cap A_{j_2},\cap A_{j3}>\}$ hence $(NCInt(A))^c=\{<\cap A_{j1},\cap A_{j_2},\cup A_{j3}>\}$ or $(NCInt(A))^c=\{<\cap A_{j1},\cup A_{j_2},\cup A_{j3}>\}$.

Hence $NCCl(A^c)=(NCInt(A))^c$, which is analogous to (a).

4.3 Proposition

Let (X,Γ) be a NCTS and A,B be two neutrosophic crisp sets in X. Then the following properties hold:

(a) $NCInt(A) \subseteq A$,
(b) $A \subseteq NCCl(A)$,
(c) $A \subseteq B \Rightarrow NCInt(A) \subseteq NCInt(B)$,
(d) $A \subseteq B \Rightarrow NCCl(A) \subseteq NCCl(B)$,
(e) $NCInt(A \cap B) = NCInt(A) \cap NCInt(B)$,
(f) $NCCl(A \cup B) = NCCl(A) \cup NCCl(B)$,
(g) $NCInt(X_N) = X_N$,
(h) $NCCl(\phi_N) = \phi_N$

Proof. (a), (b) and (e) are obvious; (c) follows from (a) and from definitions.

5 Neutrosophic Crisp Continuity

Here come the basic definitions first

5.1 Definition

(a) If $B = \langle B_1, B_2, B_3 \rangle$ is a NCS in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is a NCS in X defined by $f^{-1}(B) = \langle f^{-1}(B_1), f^{-1}(B_2), f^{-1}(B_3) \rangle$.

(b) If $A = \langle A_1, A_2, A_3 \rangle$ is a NCS in X, then the image of A under f, denoted by $f(A)$, is the a NCS in Y defined by $f(A) = \langle f(A_1), f(A_2), f(A_3)^c \rangle$.

Here we introduce the properties of images and preimages some of which we shall frequently use in the following sections .

5.1 Corollary

Let A, $\{A_i : i \in J\}$, be NCSs in X, and B, $\{B_j : j \in K\}$ NCS in Y, and $f : X \to Y$ a function. Then

(a) $A_1 \subseteq A_2 \Leftrightarrow f(A_1) \subseteq f(A_2)$,
$B_1 \subseteq B_2 \Leftrightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,

(b) $A \subseteq f^{-1}(f(A))$ and if f is injective, then $A = f^{-1}(f(A))$.

(c) $f^{-1}(f(B)) \subseteq B$ and if f is surjective, then $f^{-1}(f(B)) = B$, .

(d) $f^{-1}(\cup B_i) = \cup f^{-1}(B_i)$, $f^{-1}(\cap B_i) = \cap f^{-1}(B_i)$,

(e) $f(\cup A_i) = \cup f(A_i)$; $f(\cap A_i) \subseteq \cap f(A_i)$; and if f is injective, then $f(\cap A_i) = \cap f(A_i)$;

(f) $f^{-1}(Y_N) = X_N$, $f^{-1}(\phi_N) = \phi_N$.

(g) $f(\phi_N) = \phi_N$, $f(X_N) = Y_N$, if f is subjective.

Proof

Obvious.

5.2 Definition

Let (X,Γ_1) and (Y,Γ_2) be two NCTSs, and let $f : X \to Y$ be a function. Then f is said to be continuous iff the preimage of each NCS in Γ_2 is a NCS in Γ_1.

5.3 Definition

Let (X,Γ_1) and (Y,Γ_2) be two NCTSs and let $f : X \to Y$ be a function. Then f is said to be open iff the image of each NCS in Γ_1 is a NCS in Γ_2.

5.1 Example

Let (X,Γ_o) and (Y,ψ_o) be two NCTSs

(a) If $f : X \to Y$ is continuous in the usual sense, then in this case, f is continuous in the sense of Definition 5.1 too. Here we consider the NCTs on X and Y, respectively, as follows :
$\Gamma_1 = \{\langle G, \phi, G^c \rangle : G \in \Gamma_o\}$ and
$\Gamma_2 = \{\langle H, \phi, H^c \rangle : H \in \Psi_o\}$,
In this case we have, for each $\langle H, \phi, H^c \rangle \in \Gamma_2$, $H \in \Psi_o$,

$$f^{-1}\langle H, \phi, H^c \rangle = \langle f^{-1}(H), f^{-1}(\phi), f^{-1}(H^c) \rangle$$
$$= \langle f^{-1}H, f(\phi), (f(H))^c \rangle \in \Gamma_1 .$$

(b) If $f : X \to Y$ is open in the usual sense, then in this case, f is open in the sense of Definition 3.2. Now we obtain some characterizations of continuity:

5.1 Proposition

Let $f : (X,\Gamma_1) \to (Y,\Gamma_2)$.

f is continuous if the preimage of each CNCS (crisp neutrosophic closed set) in Γ_2 is a CNCS in Γ_2.

5.2 Proposition

The following are equivalent to each other:

(a) $f : (X,\Gamma_1) \to (Y,\Gamma_2)$ is continuous.
(b) $f^{-1}(CNInt(B) \subseteq CNInt(f^{-1}(B))$ for each CNS B in Y.
(c) $CNCl(f^{-1}(B)) \subseteq f^{-1}(CNCl(B))$ for each CNC B in Y.

5.2 Example

Let (Y, Γ_2) be a NCTS and $f: X \to Y$ be a function. In this case $\Gamma_1 = \{f^{-1}(H) : H \in \Gamma_2\}$ is a NCT on X. Indeed, it is the coarsest NCT on X which makes the function $f: X \to Y$ continuous. One may call it the initial neutrosophic crisp topology with respect to f.

6 Neutrosophic Crisp Compact Space (NCCS)

First we present the basic concepts:

6.1 Definition

Let (X, Γ) be an NCTS.

(a) If a family $\{\langle G_{i1}, G_{i2}, G_{i3}\rangle : i \in J\}$ of NCOSs in X satisfies the condition $\cup\{\langle X, G_{i1}, G_{i2}, G_{i3}\rangle : i \in J\} = X_N$, then it is called an neutrosophic open cover of X.

(b) A finite subfamily of an open cover $\{\langle G_{i1}, G_{i2}, G_{i3}\rangle : i \in J\}$ on X, which is also a neutrosophic open cover of X , is called a neutrosophic finite subcover $\{\langle G_{i1}, G_{i2}, G_{i3}\rangle : i \in J\}$.

(c) A family $\{\langle K_{i1}, K_{i2}, K_{i3}\rangle : i \in J\}$ of NCCSs in X satisfies the finite intersection property (FIP for short) iff every finite subfamily $\{\langle K_{i1}, K_{i2}, K_{i3}\rangle : i = 1,2,\ldots,n\}$ of the family satisfies the condition $\cap\{\langle K_{i_1}, K_{i_2}, K_{i_3}\rangle : i \in J\} \neq \phi_N$.

6.2 Definition

A NCTS (X, Γ) is called neutrosophic crisp compact iff each crisp neutrosophic open cover of X has a finite subcover.

6.1 Example

a) Let $X = N$ and let's consider the NCSs (neutrosophic crisp sets) given below:

$A_1 = \langle\{2,3,4,\ldots\}, \phi, \phi\rangle$, $A_2 = \langle\{3,4,\ldots\}, \phi, \{1\}\rangle$,
$A_3 = \langle\{4,5,6,\ldots\}, \phi, \{1,2\}\rangle$, …

$$A_n = \langle\{n+1, n+2, n+3, \ldots\}, \phi, \{1,2,3,\ldots n-1\}\rangle .$$

Then $\Gamma = \{\phi_N, X_N\} \cup \{A_{n:} = 3,4,5,\ldots\}$ is a NCT on X and (X, Γ) is **a** neutrosophic crisp compact.

b) Let $X = (0,1)$ and let's take the NCSs

$$A_n = \left\langle X, \left(\frac{1}{n}, \frac{n-1}{n}\right), \phi, \left(0, \frac{1}{n}\right)\right\rangle, n = 3,4,5,\ldots \text{in X}.$$

In this case $\Gamma = \{\phi_N, X_N\} \cup \{A_{n:} = 3,4,5,\ldots\}$ is an NCT on X, which is not a neutrosophic crisp compact.

6.1 Corollary

A NCTS (X, Γ) is a neutrosophic crisp compact iff every family $\{\langle X, G_{i1}, G_{i2}, G_{i3}\rangle : i \in J\}$ of NCCSs in X having the FIP has nonempty intersection.

6.2 Corollary

Let (X, Γ_1), (Y, Γ_2) be NCTSs and $f: X \to Y$ be a continuous surjection. If (X, Γ_1) is a neutrosophic crisp compact, then so is (Y, Γ_2)

6.3 Definition

(a) If a family $\{\langle G_{i1}, G_{i2}, G_{i3}\rangle : i \in J\}$ of NCCSs in X satisfies the condition $A \subseteq \cup\{\langle G_{i1}, G_{i2}, G_{i3}\rangle : i \in J\}$, then it is called a neutrosophic crisp open cover of A.

(b) Let's consider a finite subfamily of a neutrosophic crisp open subcover of $\{\langle G_{i1}, G_{i2}, G_{i3}\rangle : i \in J\}$.

A neutrosophic crisp set $A = \langle A_1, A_2, A_3\rangle$ in a NCTS (X, Γ) is called neutrosophic crisp compact iff every neutrosophic crisp open cover of A has a finite neutrosophic crisp open subcover.

6.3 Corollary

Let (X, Γ_1), (Y, Γ_2) be NCTSs and $f: X \to Y$ be a continuous surjection. If A is a neutrosophic crisp compact in (X, Γ_1), then so is $f(A)$ in (Y, Γ_2).

7 Conclusion

In this paper we introduce both the neutrosophic crisp topology and the neutrosophic crisp compact space, and we present properties related to them.

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Neutrosophic Lattices

Vasantha Kandasamy, Florentin Smarandache

Vasantha Kandasamy, Florentin Smarandache (2013). Neutrosophic Lattices. Neutrosophic Sets and Systems 2, 42-47

Abstract**.** In this paper authors for the first time define a new notion called neutrosophic lattices. We define few properties related with them. Three types of neutrosophic lattices are defined and the special properties about these new class of lattices are discussed and developed. This paper is organised into three sections. First section introduces the concept of partially ordered neutrosophic set and neutrosophic lattices. Section two introduces different types of neutrosophic lattices and the final section studies neutrosophic Boolean algebras. Conclusions and results are provided in section three.

Keywords: Neutrosophic set, neutrosophic lattices and neutrosophic partially ordered set.

1 Introduction to partially ordered neutrosophic set

Here we define the notion of a partial order on a neutrosophic set and the greatest element and the least element of it. Let N(P) denote a neutrosophic set which must contain I, 0, 1 and 1 + I; that is 0, 1, I and $1+ I \in$ N(P). We call 0 to be the least element so $0 < 1$ and $0 < I$ is assumed for the working. Further by this N(P) becomes a partially ordered set. We define 0 of N(P) to be the least element and $I \cup 1 = 1 + I$ to be the greatest element of N(P).

Suppose $N(P) = \{0, 1, I, 1 + I, a_1, a_2, a_3, a_1I, a_2I, a_3I\}$ then N(P) with $0 < a_i$, $0 < a_iI$, $1 \le i \le 3$. Further $1 > a_i$; $I > a_iI$; $1 \le i \le 3$ $a_i \not< a_j$ if $i \ne j$ for $1 \le i, j \le 3$ and $Ia_i \not< Ia_j$; $i \ne j$ for $1 \le i, j \le 3$.

We will define the notion of Neutrosophic lattice.

DEFINITION 1.1: *Let N(P) be a partially ordered set with 0, 1, I, $1+I = 1 \cup I \in N(P)$.*

Define min and max on N(P) that is max {x, y} and min {x, y} $\in$ N(P). 0 is the least element and $1 \cup I = 1 + I$ is the greatest element of N(P). {N(P), min, max} is defined as the neutrosophic lattice.

We will illustrate this by some examples.

Example 1.1: Let $N(P) = \{0, 1, I, I \cup 1 = 1 + I, a, aI\}$ be a partially ordered set; N(P) is a neutrosophic lattice.

We know in case of usual lattices [1-4]. Hasse defined the notion of representing finite lattices by diagrams known as Hasse diagrams [1-4]. We in case of Neutrosophic lattices represent them by the diagram which will be known as the neutrosophic Hasse diagram.
The neutrosophic lattice given in example 1.1 will have the following Hasse neutrosophic diagram.

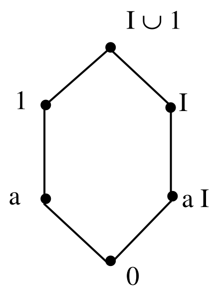

Figure 1.1

Example 1.2: Let $N(P) = \{0, 1, I, 1 \cup I, a_1, a_2, a_1I, a_2I\}$ be a neutrosophic lattice associated with the following Hasse neutrosophic diagram.

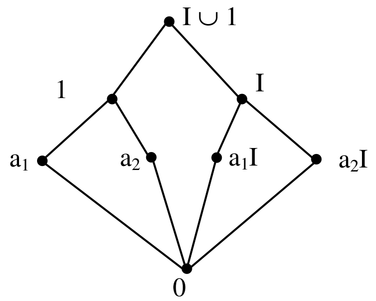

Figure 1.2

Example 1.3: Let $N(P) = \{0\ 1, I, 1 \cup I\}$ be a neutrosophic lattice given by the following neutrosophic Hasse diagram.

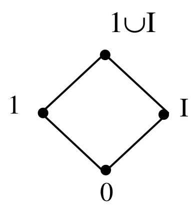

Figure 1.3

It is pertinent to observe that if N(P) is a neutrosophic lattice then 0, 1, I, $1 \cup I \in$ N(P) and so that N(P) given in example 1.3 is the smallest neutrosophic lattice.

Example 1.4: Let N(P) = $\{0, 1, I, 1 \cup I = 1 + I, a_1, a_2, a_1I, a_2I, a_1 < a_2\}$ be the neutrosophic lattice. The Hasse diagram of the neutrosophic lattice N(P) is as follows:

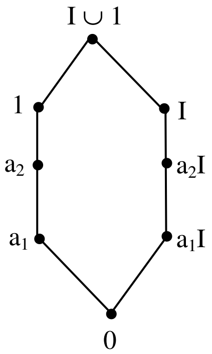

Figure 1.4

We can have neutrosophic lattices which are different.

Example 1.5: Let N(P) = $\{0, 1, I, a_1, a_2, a_3, a_4, a_1I, a_2I, a_3I, a_4I, 1 + I = I \cup 1\}$ be the neutrosophic lattice of finite order. (a_i is not comparable with a_j if $i \neq j$, $1 \leq i, j \leq 4$).

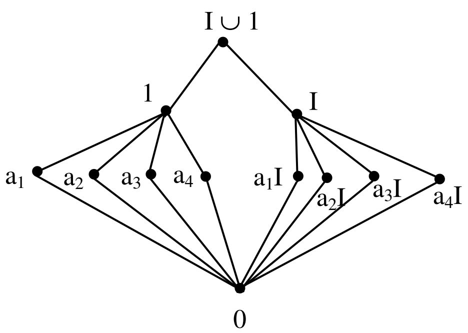

Figure 1.5

We see N(P) is a neutrosophic lattice with the above neutrosophic Hasse diagram.

In the following section we proceed onto discuss various types of neutrosophic lattices.

2. Types of Neutrosophic Lattices

The concept of modular lattice, distributive lattice, super modular lattice and chain lattices can be had from [1-4]. We just give examples of them and derive a few properties associated with them. In the first place we say a neutrosophic lattice to be a pure neutrosophic lattice if it has only neutrosophic coordinates or equivalently all the co ordinates (vertices) are neutrosophic barring 0.

In the example 1.5 we see the pure neutrosophic part of the neutrosophic lattice figure 2.1;

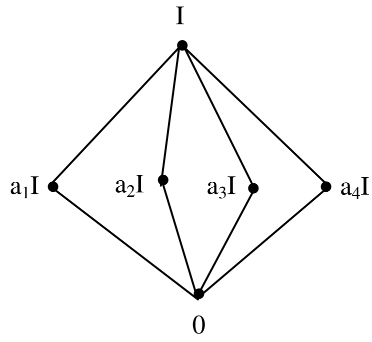

Figure 2.1

whose Hasse diagram is given is the pure neutrosophic sublattice lattice from figure 1.5. Likewise we can have the Hasse diagram of the usual lattice from example 1.5.

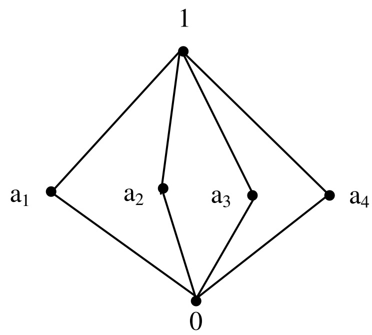

Figure 2.2

We see the diagrams are identical as diagrams one is pure neutrosophic where as the other is a usual lattice. As we have no method to compare a neutrosophic number and a non neutrosophic number, we get two sublattices identical in diagram of a neutrosophic lattice. For the modular identity, distributive identity and the super modular identity and their related properties refer [1-4].

The neutrosophic lattice given in example 1.5 has a sublattice which is a modular pure neutrosophic lattice and sublattice which is a usual modular lattice.

The neutrosophic lattice given in example 1.3 is a distributive lattice with four elements. However the neutrosophic lattice given in example 1.5 is not distributive as it contains sublattices whose homomorphic image is isomorphic to the neutrosophic modular lattice $N(M_4)$; where $N(M_4)$ is a lattice of the form

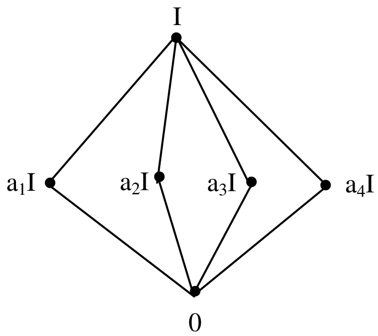

Figure 2.3

Likewise by $N(M_n)$ we have a pure neutrosophic lattice of the form given below in figure 2.4.

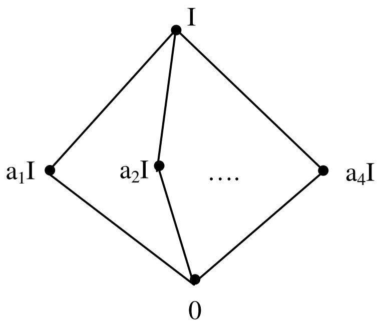

Figure 2.4

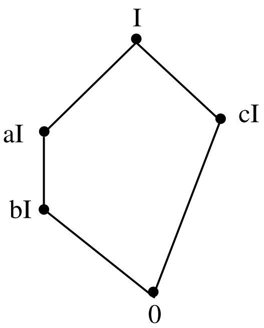

Figure 2.5

The neutrosophic pentagon lattice is given in figure 2.5 which is neither distributive nor modular.

The lattice $N(M_4)$ is not neutrosophic super modular we see the neutrosophic lattice in example 1.5 is not modular for it has sublattices whose homomorphic image is isomorphic to the pentagon lattice.

So we define a neutrosophic lattice N(L) to be a quasi modular lattice if it has atleast one sublattice (usual) which is modular and one sublattice which is a pure neutrosophic modular lattice.

Thus we need to modify the set S and the neutrosophic set N(S) of S. For if $S = \{a_1, \ldots, a_n\}$ we define $N(S) = \{a_1I, a_2I, \ldots, a_nI\}$ and take with $S \cup N(S)$ and the elements 0, 1, I, and $1 \cup I = 1 + I$. Thus to work in this way is not interesting and in general does not yield modular neutrosophic lattices.

We define the strong neutrosophic set of a set S as follows

Let $A = \{a_1, a_2, \ldots, a_n\}$, the strong neutropshic set of A;

$SN(A) = \{a_i, a_jI, a_i \cup a_jI = a_i + a_jI; 0, 1, I, 1 + I, 1 \leq i, j \leq n\}$.

S(L) the strong neutrosophic lattice is defined as follows:

$S(L) = \{0, 1, I, 1 + I, a_i, a_jI,$

$I \cup a_i = I + a_i \;\; a_iI \cup 1 = a_iI + 1, a_i + a_jI = a_i \cup a_jI \;\; 0 \lneq a_i \lneq 1; 0 < a_jI < I, 1 \leq i, j \leq n\}$.

S(L) with max, min is defined as the strong neutrosophic lattice.

We will illustrate this situation by some examples.

Example 2.1: Let S(L) = {0, 1, I, 1 + I, a, aI, a + aI, 1 + aI, I + a}

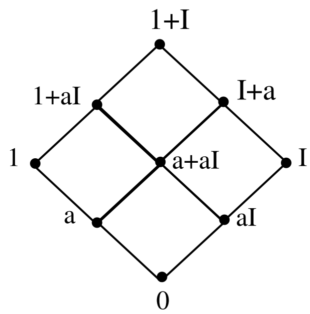

Figure 2.6

be a strong neutrosophic lattice.

We have several sublattices both strong neutrosophic sublattice as well as usual lattice.

For

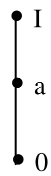

Figure 2.7

is the usual lattice.

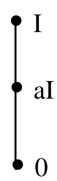

Figure 2.8

is the pure neutrosophic lattice.

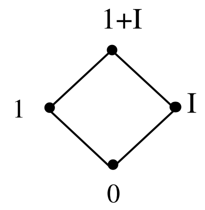

Figure 2.9

is the strong neutrosophic lattice.

These lattices have the edges to be real. Only vertices are indeterminates or neutrosophic numbers. However we can have lattices where all its vertices are real but some of the lines (or edges) are indeterminates.

Example 2.2: For consider

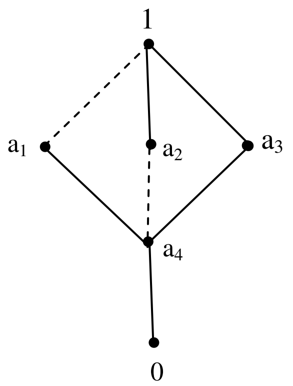

Figure 2.10

Such type of lattices will be known as edge neutrosophic lattices.

In case of edge neutrosophic lattices, we can have edge neutrosophic distributive lattices, edge neutrosophic modular lattices and edge neutrosophic super modular lattices and so on.

We will only illustrate these by some examples.

Example 2.3: Consider the following Hasse diagram.

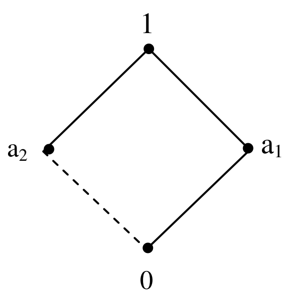

Figure 2.11

This is a edge neutrosophic lattice as the edge connecting 0 to a_2 is an indeterminate.

Example 2.4: Let us consider the following Hasse diagram of a lattice L.

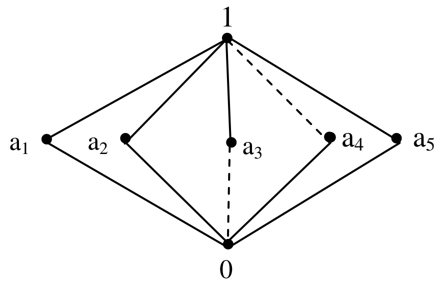

Figure 2.12

L is a edge neutrosophic modular lattice.
The edges connecting 0 to a_3 and 1 to a_4 are neutrosophic edges and the rest of the edges are reals. However all the vertices are real and it is a partially ordered set. We take some of the edges to be an indeterminate.

Example 2.5: Let L be the edge neutrosophic lattice whose Hassee diagram is as follows:

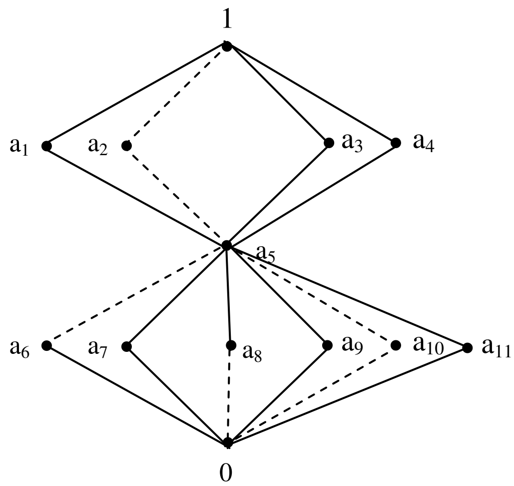

Figure 2.13

Clearly L is not a distributive edge neutrosophic lattice. However L has modular edge neutrosophic sublattices as well as modular lattices which are not neutrosophic.

Inview of this we have the following theorem.

THEOREM 2.1: *Let L be a edge neutrosophic lattice. Then L in general have sublattices which are not edge neutrosophic.*

Proof follows from the simple fact that every vertex is a sublattice and all vertices of the edge neutrosophic lattice which are not neutrosophic; but real is an instance of a not an edge neutropshic lattice.

We can have pure neutrosophic lattice which have the edges as well the vertices to be neutrosophic.

The following lattices with the Hasse diagram are pure neutrosophic lattices.

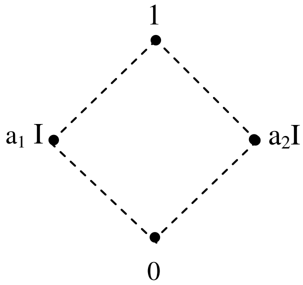

Figure 2.14

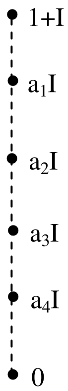

Figure 2.15

These two pure neutrosophic lattices cannot have edge neutrosophic sublattice or vertex neutrosophic sublattice.

3. Neutrosophic Boolean Algebras

Let us consider the power set of a neutrosophic set S = {a + bI | a = 0 or b = 0 can occur with 0 as the least element and 1 + I as the largest element}. P(S) = {Collection of all subsets of the set S} {P(S), ∪, ∩, ϕ, S} is a lattice defined as the neutrosophic Boolean algebra of order $2^{|P(S)|}$.

We will give examples of them.

Example 3.1: Let S = {0, 1, 1 +I, I}. P(S) = {ϕ, {0}, {1}, {1+I}, {I}, {0,1}, {0, I}, {0, 1+I}, {1, I}, {1, 1+I}, {I, 1+I}, {0, 1, I}, {0, 1, 1+I}, {0, I, 1+I}, {1, I, 1+I}, S} be the collection of all subsets of S including the empty set ϕ and the set S. |P(S)| = 16. P(S) is a neutrosophic Boolean algebra under '∪' and '∩' as the operations on P(S) and the containment relation of subsets as the partial order relation on P(S).

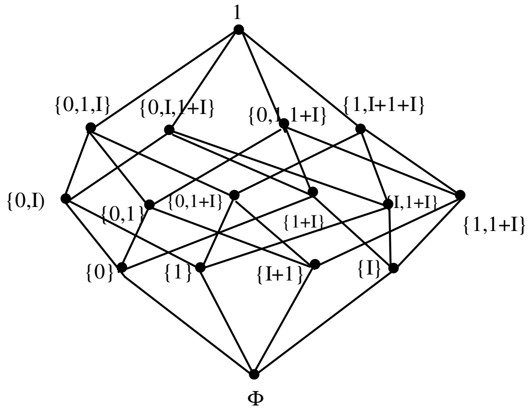

Figure 3.1

Example 3.2: Let S = {0, 1, I, 1+I, a, aI, a+I, aI+1, aI+a} be the neutrosophic set; 0 < a < 1. P(S) be the power set of S. |P(S)| = 2^9. P(S) is a neutrosophic Boolean algebra of order 2^9.

Example 3.3: Let S = {0, 1, I, 1+I, a_1, a_2, a_1I, a_2I, a_1+I, a_2+I, 1+a_1I, 1+a_2I, 1+a_1I+a_2, a_1+a_2, 1 + a_1I + a_2I, …} be the neutrosophic set with $a_1 \not< a_2$ or $a_2 \not< a_1$, 0 < a_1 < 1, 0 < a_2 < 1. P(S) is a neutrosophic Boolean algebra.

Now these neutrosophic Boolean algebras cannot be edge neutrosophic lattices. We make it possible to define edge neutrosophic lattice. Let L be a lattice given by the following Hasse-diagram.

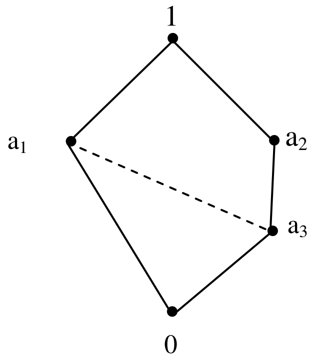

Figure 3.2

a_1 and a_3 are not comparable but we can have a neutrosophic edge given by the above diagram.

So we see the lattice has become a edge neutrosophic lattice.

Let L be a lattice given by the following diagram.

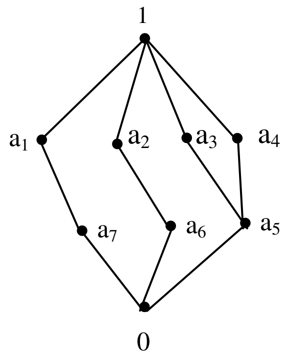

Figure 3.3

Clearly a_1 and a_6 are not comparable, a_2 and a_5 are not comparable a_4 and a_7 are not comparable.

We can have the following Hasse diagram which has neutrosophic edges.

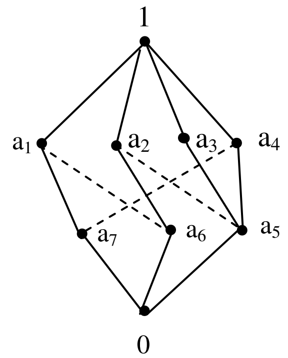

Figure 3.4

Clearly L is a edge neutrosophic lattice where we have some neutrosophic edges which are not comparable in the original lattice.

So we can on usual lattices L remake it into a edge neutrosophic lattice this is done if one doubts that a pair of elements $\{a_1, a_2\}$ of L with $a_1 \neq a_2$, min $\{a_1, a_2\} \neq a_1$ or a_2 or max $\{a_1, a_2\} \neq a_1$ or a_2.

If some experts needs to connect a_1 with a_2 by edge then the resultant lattice becomes a edge neutrosophic lattice.

Conclusion: Here for the first time we introduce the concept of neutrosophic lattices. Certainly these lattices will find applications in all places where lattices find their applications together with some indeterminancy. When one doubts a connection between two vertices one can have a neutrosophic edge.

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Soft Neutrosophic Bigroup and Soft Neutrosophic N-Group

Mumtaz Ali, Florentin Smarandache, Muhammad Shabir, Munazza Naz

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Abstract. Soft neutrosophic group and soft neutrosophic subgroup are generalized to soft neutrosophic bigroup and soft neutrosophic N-group respectively in this paper. Different kinds of soft neutrosophic bigroup and soft neutrosophic N-group are given. The structural properties and theorems have been discussed with a lot of examples to disclose many aspects of this beautiful man made structure.

Keywords: Neutrosophic bigroup, Neutrosophic N-group, soft set, soft group, soft subgroup, soft neutrosophic bigroup, soft neutrosophic subbigroup, soft neutrosophic N-group, soft neutrosophic sub N-group.

1 Introduction

Neutrosophy is a new branch of philosophy which is in fact the birth stage of neutrosophic logic first found by Florentin Smarandache in 1995. Each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F so that this neutrosophic logic is called an extension of fuzzy logic. In fact neutrosophic set is the generalization of classical sets, conventional fuzzy set $[1]$, intuitionistic fuzzy set $[2]$ and interval valued fuzzy set $[3]$. This mathematical tool is handling problems like imprecise, indeterminacy and inconsistent data etc. By utilizing neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache dig out neutrosophic algebraic structures in $[11]$. Some of them are neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, and neutrosophic bigroupoids and so on.

Molodtsov in $[11]$ laid down the stone foundation of a richer structure called soft set theory which is free from the parameterization inadequacy, syndrome of fuzzy se theory, rough set theory, probability theory and so on. In many areas it has been successfully applied such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, and probability. Recently soft set theory has attained much attention since its appearance and the work based on several operations of soft sets introduced in $[2,9,10]$. Some more exciting properties and algebra may be found in $[1]$. Feng et al. introduced the soft semirings $[5]$. By means of level soft sets an adjustable approach to fuzzy soft sets based decision making can be seen in $[6]$. Some other new concept combined with fuzzy sets and rough sets was presented in $[7,8]$. AygÄunoglu et al. introduced the Fuzzy soft groups $[4]$. This paper is a mixture of neutrosophic bigroup,neutrosophic N -group and soft set theory which is infact a generalization of soft neutrosophic group. This combination gave birth to a new and fantastic approach called "Soft Neutrosophic Bigroup and Soft Neutrosophic N -group".

2.1 Neutrosophic Bigroup and N-Group

Definition 1 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a non empty subset with two binary operations on $B_N(G)$ satisfying the following conditions:

1) $B_N(G)=\{B(G_1)\cup B(G_2)\}$ where $B(G_1)$ and $B(G_2)$ are proper subsets of $B_N(G)$.
2) $(B(G_1),*_1)$ is a neutrosophic group.
3) $(B(G_2),*_2)$ is a group .

Then we define $(B_N(G),*_1,*_2)$ to be a neutrosophic bigroup. If both $B(G_1)$ and $B(G_2)$ are neutrosophic groups. We say $B_N(G)$ is a strong neutrosophic bigroup. If both the groups are not neutrosophic group, we say $B_N(G)$ is just a bigroup.

Example 1 Let $B_N(G)=\{B(G_1)\cup B(G_2)\}$ where $B(G_1)=\{g/g^9=1\}$ be a cyclic group of order 9 and $B(G_2)=\{1,2,I,2I\}$ neutrosophic group under multiplication modulo 3. We call $B_N(G)$ a neutrosophic bigroup.

Example 2 Let $B_N(G)=\{B(G_1)\cup B(G_2)\}$

Where $B(G_1)=\{1,2,3,4,I,2I,3I,4I\}$ a neutrosoph ic group under multiplication modulo 5.

$B(G_2)=\{0,1,2,I,2I,1+I,2+I,1+2I,2+2I\}$

is a neutrosophic group under multiplication modulo 3. Clearly $B_N(G)$ is a strong neutrosophic bi group.

Definition 2 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup. A proper subset $P=\{P_1\cup P_2,*_1,*_2\}$ is a neutrosophic subbi group of $B_N(G)$ if the following conditions are satisfied $P=\{P_1\cup P_2,*_1,*_2\}$ is a neutroso phic bigroup under the operations $*_1,*_2$ i.e. $(P_1,*_1)$ is a neutrosophic subgroup of $(B_1,*_1)$ and $(P_2,*_2)$ is a subgroup of $(B_2,*_2)$.

$P_1=P\cap B_1$ and $P_2=P\cap B_2$ are subgroups of B_1 and B_2 respectively. If both of P_1 and P_2 are not neutrosophic then we call $P=P_1\cup P_2$ to be just a bigroup.

Definition 3 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup. If both $B(G_1)$ and $B(G_2)$ are commutative groups, then we call $B_N(G)$ to be a commutative bigroup.

Definition 4 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup. If both $B(G_1)$ and $B(G_2)$ are cyclic, we call $B_N(G)$ a cyclic bigroup.

Definition 5 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup. $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic bigroup. $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ is said to be a neutrosophic normal subbigroup of $B_N(G)$ if $P(G)$ is a neutrosophic subbigroup and both $P(G_1)$ and $P(G_2)$ are normal subgroups of $B(G_1)$ and $B(G_2)$ respectively.

Definition 6 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup of finite order. Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic subbigroup of $B_N(G)$. If $o(P(G))/o(B_N(G))$ then we call $P(G)$ a Lagrange neutrosophic subbigroup, if every neutrosophic subbigroup P is such that $o(P)/o(B_N(G))$ then we call $B_N(G)$ to be a Lagrange neutrosophic bigroup.

Definition 7 If $B_N(G)$ has atleast one Lagrange neutrosophic subbigroup then we call $B_N(G)$ to be a weak Lagrange neutrosophic bigroup.

Definition 8 If $B_N(G)$ has no Lagrange neutrosophic subbigroup then $B_N(G)$ is called Lagrange free neutrosophic bigroup.

Definition 9 Let $B_N(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup. Suppose $P=\{P(G_1)\cup P(G_2),*_1,*_2\}$ and

$K = \{K(G_1) \cup K(G_2), *_1, *_2\}$ be any two neutrosophic subbigroups. we say P and K are conjugate if each $P(G_i)$ is conjugate with $K(G_i), i = 1,2$, then we say P and K are neutrosophic conjugate subbigroups of $B_N(G)$.

Definition 10 A set $(\langle G \cup I \rangle, +, \circ)$ with two binary operations ` $+$ ' and ` $\circ$ ' is called a strong neutrosophic bigroup if

1) $\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle$,
2) $(\langle G_1 \cup I \rangle, +)$ is a neutrosophic group and
3) $(\langle G_2 \cup I \rangle, \circ)$ is a neutrosophic group.

Example 3 Let $\{\langle G \cup I \rangle, *_1, *_2\}$ be a strong neutrosophic bigroup where $\langle G \cup I \rangle = \langle Z \cup I \rangle \cup \{0,1,2,3,4,I,2I,3I,4I\}$. $\langle Z \cup I \rangle$ under ` $+$ ' is a neutrosophic group and $\{0,1,2,3,4,I,2I,3I,4I\}$ under multiplication modulo 5 is a neutrosophic group.

Definition 11 A subset $H \neq \phi$ of a strong neutrosophic bigroup $(\langle G \cup I \rangle, *, \circ)$ is called a strong neutrosophic subbigroup if H itself is a strong neutrosophic bigroup under ` $*$ ' and ` $\circ$ ' operations defined on $\langle G \cup I \rangle$.

Definition 12 Let $(\langle G \cup I \rangle, *, \circ)$ be a strong neutrosophic bigroup of finite order. Let $H \neq \phi$ be a strong neutrosophic subbigroup of $(\langle G \cup I \rangle, *, \circ)$. If $o(H) / o(\langle G \cup I \rangle)$ then we call H, a Lagrange strong neutrosophic subbigroup of $\langle G \cup I \rangle$. If every strong neutrosophic subbigroup of $\langle G \cup I \rangle$ is a Lagrange strong neutrosophic subbigroup then we call $\langle G \cup I \rangle$ a Lagrange strong neutrosophic bigroup.

Definition 13 If the strong neutrosophic bigroup has at least one Lagrange strong neutrosophic subbigroup then we call $\langle G \cup I \rangle$ a weakly Lagrange strong neutrosophic bigroup.

Definition 14 If $\langle G \cup I \rangle$ has no Lagrange strong neutrosophic subbigroup then we call $\langle G \cup I \rangle$ a Lagrange free strong neutrosophic bigroup.

Definition 15 Let $(\langle G \cup I \rangle, +, \circ)$ be a strong neutrosophic bigroup with $\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle$. Let $(H, +, \circ)$ be a neutrosophic subbigroup where $H = H_1 \cup H_2$. We say H is a neutrosophic normal subbigroup of G if both H_1 and H_2 are neutrosophic normal subgroups of $\langle G_1 \cup I \rangle$ and $\langle G_2 \cup I \rangle$ respectively.

Definition 16 Let $G = \langle G_1 \cup G_2, *, \otimes \rangle$, be a neutrosophic bigroup. We say two neutrosophic strong subbigroups $H = H_1 \cup H_2$ and $K = K_1 \cup K_2$ are conjugate neutrosophic subbigroups of $\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle$ if H_1 is conjugate to K_1 and H_2 is conjugate to K_2 as neutrosophic subgroups of $\langle G_1 \cup I \rangle$ and $\langle G_1 \cup I \rangle$ respectively.

Definition 17 Let $(\langle G \cup I \rangle, *_1, ..., *_N)$ be a nonempty set with N -binary operations defined on it. We say $\langle G \cup I \rangle$ is a strong neutrosophic N -group if the following conditions are true.

1) $\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle \cup ... \cup \langle G_N \cup I \rangle$ where $\langle G_i \cup I \rangle$ are proper subsets of $\langle G \cup I \rangle$.
2) $(\langle G_i \cup I \rangle, *_i)$ is a neutrosophic group, $i = 1,2,...,N$.
3) If in the above definition we have
   a. $\langle G \cup I \rangle = G_1 \cup \langle G_2 \cup I \rangle \cup ... \cup \langle G_k \cup I \rangle \cup \langle G_{k+1} \cup I \rangle \cup ... \cup G_N$
   b. $(G_i, *_i)$ is a group for some i or
4) $(\langle G_j \cup I \rangle, *_j)$ is a neutrosophic group for some j. Then we call $\langle G \cup I \rangle$ to be a neutrosophic N -group.

Example 4 Let $\langle G \cup I \rangle = (\langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle \cup \langle G_3 \cup I \rangle \cup \langle G_4 \cup I \rangle, *_1, *_2, *_3, *_4)$ be a neutrosophic 4 -group where

$\langle G_1 \cup I \rangle = \{1,2,3,4,I,2I,3I,4I\}$
neutrosophic group under multiplication modulo 5.

$\langle G_2 \cup I \rangle = \{0,1,2,I,2I,1+I,2+I,1+2I,2+2I\}$
a neutrosophic group under multiplication modulo 3,

$\langle G_3 \cup I \rangle = \langle Z \cup I \rangle$, a neutrosophic group under addition and $\langle G_4 \cup I \rangle = \{(a,b) : a,b \in \{1, I, 4, 4I\}\}$, component-wise multiplication modulo $5\}$.

Hence $\langle G \cup I \rangle$ is a strong neutrosophic 4-group.

Example 5 Let

$\left(\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle \cup G_3 \cup G_4, *_1, *_2, *_3, *_4\right)$

be a neutrosophic 4-group, where

$\langle G_1 \cup I \rangle = \{1,2,3,4,I,2I,3I,4I\}$ a neutrosophic group under multiplication modulo 5.

$\langle G_2 \cup I \rangle = \{0,1,I,1+I\}$,a neutrosophic group under multiplication modulo 2. $G_3 = S_3$ and $G_4 = A_5$, the alternating group. $\langle G \cup I \rangle$ is a neutrosophic 4-group.

Definition 18 Let

$\left(\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle \cup \ldots \cup \langle G_N \cup I \rangle, *_1, \ldots, *_N\right)$

be a neutrosophic N-group. A proper subset $(P, *_1, \ldots, *_N)$ is said to be a neutrosophic sub N-group of $\langle G \cup I \rangle$ if $P = (P_1 \cup \ldots \cup P_N)$ and each $(P_i, *_i)$ is a neutrosophic subgroup (subgroup) of $(G_i, *_i), 1 \le i \le N$.

It is important to note $(P, *_i)$ for no i is a neutrosophic group.

Thus we see a strong neutrosophic N-group can have 3 types of subgroups viz.

1) Strong neutrosophic sub N-groups.
2) Neutrosophic sub N-groups.
3) Sub N-groups.

Also a neutrosophic N-group can have two types of sub N-groups.

1) Neutrosophic sub N-groups.
2) Sub N-groups.

Definition 19 If $\langle G \cup I \rangle$ is a neutrosophic N-group and if $\langle G \cup I \rangle$ has a proper subset T such that T is a neutrosophic sub N-group and not a strong neutrosophic sub N-group and $o(T) / o(\langle G \cup I \rangle)$ then we call T a Lagrange sub N-group. If every sub N-group of $\langle G \cup I \rangle$ is a Lagrange sub N-group then we call $\langle G \cup I \rangle$ a Lagrange N-group.

Definition 20 If $\langle G \cup I \rangle$ has atleast one Lagrange sub N-group then we call $\langle G \cup I \rangle$ a weakly Lagrange neutrosophic N-group.

Definition 21 If $\langle G \cup I \rangle$ has no Lagrange sub N-group then we call $\langle G \cup I \rangle$ to be a Lagrange free N-group.

Definition 22 Let

$\left(\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle \cup \ldots \cup \langle G_N \cup I \rangle, *_1, \ldots, *_N\right)$

be a neutrosophic N-group. Suppose

$H = \{H_1 \cup H_2 \cup \ldots \cup H_N, *_1, \ldots, *_N\}$ and

$K = \{K_1 \cup K_2 \cup \ldots \cup K_N, *_1, \ldots, *_N\}$ are two sub N-groups of $\langle G \cup I \rangle$, we say K is a conjugate to H or H is conjugate to K if each H_i is conjugate to K_i $(i = 1,2,\ldots,N)$ as subgroups of G_i .

2.2 Soft Sets

Throughout this subsection U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U , and $A \subset E$. Molodtsov defined the soft set in the following manner:

Definition 23 A pair (F, A) is called a soft set over U where F is a mapping given by $F : A \to P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U . For $x \in A$, $F(x)$ may be considered as the set of x-elements of the soft set (F, A) , or as the set of e-approximate elements of the soft set.

Example 6 Suppose that U is the set of shops. E is the set of parameters and each parameter is a word or sentence. Let

$$E = \begin{Bmatrix} \text{high rent,normal rent,} \\ \text{in good condition,in bad condition} \end{Bmatrix}.$$

Let us consider a soft set (F, A) which describes the attractiveness of shops that Mr. Z is taking on rent. Suppose that there are five houses in the universe $U = \{s_1, s_2, s_3, s_4, s_5\}$ under consideration, and that $A = \{x_1, x_2, x_3\}$ be the set of parameters where

x_1 stands for the parameter 'high rent,

x_2 stands for the parameter 'normal rent,

x_3 stands for the parameter 'in good condition.

Suppose that

$$F(x_1) = \{s_1, s_4\},$$
$$F(x_2) = \{s_2, s_5\},$$
$$F(x_3) = \{s_3, s_4, s_5\}.$$

The soft set (F, A) is an approximated family $\{F(e_i), i = 1,2,3\}$ of subsets of the set U which gives us a collection of approximate description of an object. Then (F, A) is a soft set as a collection of approximations over U, where

$$F(x_1) = high\ rent = \{s_1, s_2\},$$
$$F(x_2) = normal\ rent = \{s_2, s_5\},$$
$$F(x_3) = in\ good\ condition = \{s_3, s_4, s_5\}.$$

Definition 24 For two soft sets (F, A) and (H, B) over U, (F, A) is called a soft subset of (H, B) if

1. $A \subseteq B$ and
2. $F(x) \subseteq H(x)$, for all $x \in A$.

This relationship is denoted by $(F, A) \subset (H, B)$. Similarly (F, A) is called a soft superset of (H, B) if (H, B) is a soft subset of (F, A) which is denoted by $(F, A) \supset (H, B)$.

Definition 25 Two soft sets (F, A) and (H, B) over U are called soft equal if (F, A) is a soft subset of (H, B) and (H, B) is a soft subset of (F, A).

Definition 26 Let (F, A) and (K, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$. Then their restricted intersection is denoted by $(F, A) \cap_R (K, B) = (H, C)$ where (H, C) is defined as $H(c) = F(c) \cap \mathrm{K}(c)$ for all $c \in C = A \cap B$.

Definition 27 The extended intersection of two soft sets (F, A) and (K, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as

$$H(c) = \begin{cases} F(c) & \text{if c} \in A - B, \\ G(c) & \text{if c} \in B - A, \\ F(c) \cap G(c) & \text{if c} \in A \cap B. \end{cases}$$

We write $(F, A) \cap_\varepsilon (K, B) = (H, C)$.

Definition 28 The restricted union of two soft sets (F, A) and (K, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as $H(c) = F(c) \cup G(c)$ for all $c \in C$. We write it as $(F, A) \cup_R (K, B) = (H, C)$.

Definition 29 The extended union of two soft sets (F, A) and (K, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as

$$H(c) = \begin{cases} F(c) & \text{if c} \in A - B, \\ G(c) & \text{if c} \in B - A, \\ F(c) \cup G(c) & \text{if c} \in A \cap B. \end{cases}$$

We write $(F, A) \cup_\varepsilon (K, B) = (H, C)$.

2.3 Soft Groups

Definition 30 Let (F, A) be a soft set over G. Then (F, A) is said to be a soft group over G if and only if $F(x) \prec G$ for all $x \in A$.

Example 7 Suppose that

$$G = A = S_3 = \{e, (12), (13), (23), (123), (132)\}.$$

Then (F, A) is a soft group over S_3 where

$$F(e) = \{e\},$$
$$F(12) = \{e, (12)\},$$
$$F(13) = \{e, (13)\},$$
$$F(23) = \{e, (23)\},$$
$$F(123) = F(132) = \{e, (123), (132)\}.$$

Definition 31 Let (F, A) be a soft group over G. Then

1. (F, A) is said to be an identity soft group over G if $F(x) = \{e\}$ for all $x \in A$, where e is the identity element of G and
2. (F, A) is said to be an absolute soft group if $F(x) = G$ for all $x \in A$.

3 Soft Neutrosophic Bigroup

Definition 32 Let

$$B_N(G) = \{B(G_1) \cup B(G_2), *_1, *_2\}$$

be a neutrosophic bigroup and let (F, A) be a soft set over $B_N(G)$. Then (F, A) is said to be soft neutrosophic bigroup over $B_N(G)$ if and only if $F(x)$ is a subbigroup of $B_N(G)$ for all $x \in A$.

Example 8 Let

$$B_N(G) = \{B(G_1) \cup B(G_2), *_1, *_2\}$$

be a neutrosophic bigroup, where

$$B(G_1) = \{0,1,2,3,4,I,2I,3I,4I\}$$

is a neutrosophic group under multiplication modulo 5. $B(G_2) = \{g : g^{12} = 1\}$ is a cyclic group of order 12. Let $P(G) = \{P(G_1) \cup P(G_2), *_1, *_2\}$ be a neutrosophic subbigroup where $P(G_1) = \{1,4,I,4I\}$ and $P(G_2) = \{1, g^2, g^4, g^6, g^8, g^{10}\}$.

Also $Q(G) = \{Q(G_1) \cup Q(G_2), *_1, *_2\}$ be another neutrosophic subbigroup where $Q(G_1) = \{1, I\}$ and $Q(G_2) = \{1, g^3, g^6, g^9\}$.

Then (F, A) is a soft neutrosophic bigroup over $B_N(G)$, where

$$F(e_1) = \{1,4,I,4I,1,g^2,g^4,g^6,g^8,g^{10}\}$$

$$F(e_2) = \{1,I,1,g^3,g^6,g^9\}.$$

Theorem 1 Let (F, A) and (H, A) be two soft neutrosophic bigroup over $B_N(G)$. Then their intersection $(F, A) \cap (H, A)$ is again a soft neutrosophic bigroup over $B_N(G)$.

Proof Straight forward.

Theorem 2 Let (F, A) and (H, B) be two soft neutrosophic bigroups over $B_N(G)$ such that $A \cap B = \phi$, then their union is soft neutrosophic bigroup over $B_N(G)$.

Proof Straight forward.

Proposition 1 The extended union of two soft neutrosophic bigroups (F, A) and (K, D) over $B_N(G)$ is not a soft neutrosophic bigroup over $B_N(G)$.

To prove it, see the following example.

Example 9 Let $B_N(G) = \{B(G_1) \cup B(G_2), *_1, *_2\}$, where $B(G_1) = \{1,2,3,4I,2I,3I,4I\}$ and $B(G_2) = S_3$.

Let $P(G) = \{P(G_1) \cup P(G_2), *_1, *_2\}$ be a neutrosophic subbigroup where $P(G_1) = \{1,4,I,4I\}$ and $P(G_2) = \{e, (12)\}$.

Also $Q(G) = \{Q(G_1) \cup Q(G_2), *_1, *_2\}$ be another neutrosophic subbigroup where $Q(G_1) = \{1, I\}$ and $Q(G_2) = \{e, (123), (132)\}$.

Then (F, A) is a soft neutrosophic bigroup over $B_N(G)$, where

$$F(x_1) = \{1,4,I,4I,e,(12)\}$$

$$F(x_2) = \{1,I,e,(123),(132)\}.$$

Again let $R(G) = \{R(G_1) \cup R(G_2), *_1, *_2\}$ be another neutrosophic subbigroup where $R(G_1) = \{1,4,I,4I\}$ and $R(G_2) = \{e, (13)\}$.

Also $T(G) = \{T(G_1) \cup T(G_2), *_1, *_2\}$ be a neutrosophic subbigroup where $T(G_1) = \{1, I\}$ and $T(G_2) = \{e, (23)\}$.

Then (K, D) is a soft neutrosophic bigroup over $B_N(G)$, where

$$K(x_2) = \{1,4,I,4I,e,(13)\},$$

$$K(x_3) = \{1,I,e,(23)\}.$$

The extended union $(F, A) \cup_\varepsilon (K, D) = (H, C)$ such that $C = A \cup D$ and for $x_2 \in C$, we have

$$H(x_2) = F(x_2) \cup K(x_2) = \{1,4,I,4I,e,(13)(123),(132)\}$$

is not a subbigroup of $B_N(G)$.

Proposition 2 The extended intersection of two soft neutrosophic bigroups (F, A) and (K, D) over $B_N(G)$ is again a soft neutrosophic bigroup over $B_N(G)$.

Proposition 3 The restricted union of two soft neutrosophic bigroups (F, A) and (K, D) over $B_N(G)$ is not a soft neutrosophic bigroup over $B_N(G)$.

Proposition 4 The restricted intersection of two soft neutrosophic bigroups (F, A) and (K, D) over $B_N(G)$ is a soft neutrosophic bigroup over $B_N(G)$.

Proposition 5 The AND operation of two soft neutrosophic bigroups over $B_N(G)$ is again soft neutrosophic bigroup over $B_N(G)$.

Proposition 6 The OR operation of two soft neutrosophic bigroups over $B_N(G)$ may not be a soft nuetrosophic bigroup.

Definition 33 Let (F, A) be a soft neutrosophic bigroup over $B_N(G)$. Then

1) (F, A) is called identity soft neutrosophic bigroup if $F(x) = \{e_1, e_2\}$ for all $x \in A$, where e_1 and e_2 are the identities of $B(G_1)$ and $B(G_2)$ respectively.
2) (F, A) is called Full-soft neutrosophic bigroup if $F(x) = B_N(G)$ for all $x \in A$.

Theorem 3 Let $B_N(G)$ be a neutrosophic bigroup of prime order P, then (F, A) over $B_N(G)$ is either identity soft neutrosophic bigroup or Full-soft neutrosophic bigroup.

Definition 34 Let (F, A) and (H, K) be two soft neutrosophic bigroups over $B_N(G)$. Then (H, K) is soft neutrosophi subbigroup of (F, A) written as $(H, K) \prec (F, A)$, if

1) $K \subset A$,
2) $K(x) \prec F(x)$ for all $x \in A$.

Example 10 Let

$B(G) = \{B(G_1) \cup B(G_2), *_1, *_2\}$ where

$$B(G_1) = \begin{Bmatrix} 0,1,2,3,4,I,2I,3I,4I,1+I,2+I,3+I,4+I, \\ 1+2I,2+2I,3+2I,4+2I,1+3I,2+3I, \\ 3+3I,4+3I,1+4I,2+4I,3+4I,4+4I \end{Bmatrix}$$

be a neutrosophic group under multiplication modulo 5 and $B(G_2) = \{g : g^{16} = 1\}$ a cyclic group of order 16. Let $P(G) = \{P(G_1) \cup P(G_2), *_1, *_2\}$ be a neutrosophic subbigroup where

$P(G_1) = \{0,1,2,3,4,I,2I,3I,4I\}$

and

be another neutrosophic subbigroup where

$P(G_2) = \{g^2, g^4, g^6, g^8, g^{10}, g^{12}, g^{14}, 1\}$.

Also $Q(G) = \{Q(G_1) \cup Q(G_2), *_1, *_2\}$

$Q(G_1) = \{0,1,4,I,4I\}$

and

$Q(G_2) = \{g^4, g^8, g^{12}, 1\}$.

Again let $R(G) = \{R(G_1) \cup R(G_2), *_1, *_2\}$ be a neutrosophic subbigroup where

$R(G_1) = \{0,1,I\}$ and $R(G_2) = \{1, g^8\}$.

Let (F, A) be a soft neutrsophic bigroup over $B_N(G)$ where

$F(x_1) = \{0,1,2,3,4,I,2I,3I,4I,g^2,g^4,g^6,g^8,g^{10},g^{12},g^{14},1\}$,

$F(x_2) = \{0,1,4,I,4I,g^4,g^8,g^{12},1\}$,

$F(x_3) = \{0,1,I,g^8,1\}$.

Let (H, K) be another soft neutrosophic bigroup over $B_N(G)$, where

$H(x_1) = \{0,1,2,3,4,g^4,g^8,g^{12},1\}$,

$H(x_2) = \{0,1,I,g^8,1\}$.

Clearly $(H, K) \prec (F, A)$.

Definition 35 Let $B_N(G)$ be a neutrosophic bigroup. Then (F, A) over $B_N(G)$ is called commutative soft neutrosophic bigroup if and only if $F(x)$ is a commutative subbigroup of $B_N(G)$ for all $x \in A$.

Example 11 Let $B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup where $B(G_1)=\{g:g^{10}=1\}$ be a cyclic group of order 10 and $B(G_2)=\{1,2,3,4,I,2I,3I,4I\}$ be a neutrosophic group under mltiplication modulo 5.
Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a commutative neutrosophic subbigroup where $P(G_1)=\{1,g^5\}$ and $P(G_2)=\{1,4,I,4I\}$. Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another commutative neutrosophic subbigroup where $Q(G_1)=\{1,g^2,g^4,g^6,g^8\}$ and $Q(G_2)=\{1,I\}$.
Then (F,A) is commutative soft neutrosophic bigroup over $B_N(G)$, where

$$F(x_1)=\{1,g^5,1,4,I,4I\},$$
$$F(x_2)=\{1,g^2,g^4,g^6,g^8,1,I\}.$$

Theorem 4 Every commutative *soft neutrosophic bigroup* (F,A) over $B_N(G)$ is a soft *neutrosophic bigroup* but the converse is not true.

Theorem 5 If $B_N(G)$ is commutative neutrosophic bigroup. Then (F,A) over $B_N(G)$ is commutative soft neutrosophic bigroup but the converse is not true.

Theorem 6 If $B_N(G)$ is cyclic neutrosophic bigroup. Then (F,A) over $B_N(G)$ is commutative soft neutrosophic bigroup.

Proposition 7 Let (F,A) and (K,D) be two commutative soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_{\varepsilon}(K,D)$ over $B_N(G)$ is not commutative soft neutrosophic bigroup over $B_N(G)$.
2) Their extended intersection $(F,A)\cap_{\varepsilon}(K,D)$ over $B_N(G)$ is commutative soft neutrosophic bigroup over $B_N(G)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $B_N(G)$ is not commutative soft neutrosophic bigroup over $B_N(G)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $B_N(G)$ is commutative soft neutrosophic bigroup over $B_N(G)$.

Proposition 8 Let (F,A) and (K,D) be two commutative soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is commutative soft neutrosophic bigroup over $B_N(G)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not commutative soft neutrosophic bigroup over $B_N(G)$.

Definition 36 Let $B_N(G)$ be a neutrosophic bigroup. Then (F,A) over $B_N(G)$ is called cyclic soft neutrosophic bigroup if and only if $F(x)$ is a cyclic subbigroup of $B_N(G)$ for all $x\in A.$

Example 12 Let $B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup where $B(G_1)=\{g:g^{10}=1\}$ be a cyclic group of order 10 and
$B(G_2)=\{0,1,2,I,2I,1+I,2+I,1+2I,2+2I\}$
be a neutrosophic group under multiplication modulo 3.
Le $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a cyclic neutrosophic subbigroup where $P(G_1)=\{1,g^5\}$ and $\{1,1+I\}$.
Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another cyclic neutrosophic subbigroup where $Q(G_1)=\{1,g^2,g^4,g^6,g^8\}$ and $Q(G_2)=\{1,2+2I\}$.
Then (F,A) is cyclic soft neutrosophic bigroup over $B_N(G)$, where

$$F(x_1)=\{1,g^5,1,1+I\},$$
$$F(x_2)=\{1,g^2,g^4,g^6,g^8,1,2+2I\}.$$

Theorem 7 If $B_N(G)$ is a cyclic neutrosophic soft bigroup, then (F,A) over $B_N(G)$ is also cyclic soft neutrosophic bigroup.

Theorem 8 Every cyclic soft neutrosophic bigroup (F,A) over $B_N(G)$ is a soft neutrosophic bigroup but the converse is not true.

Proposition 9 Let (F,A) and (K,D) be two cyclic soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_{\varepsilon}(K,D)$ over $B_N(G)$ is not cyclic soft neutrosophic bigroup over $B_N(G)$.
2) Their extended intersection $(F,A)\bigcap_{\varepsilon}(K,D)$ over $B_N(G)$ is cyclic soft neutrosophic bigroup over $B_N(G)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $B_N(G)$ is not cyclic soft neutrosophic bigroup over $B_N(G)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $B_N(G)$ is cyclic soft neutrosophic bigroup over $B_N(G)$.

Proposition 10 Let (F,A) and (K,D) be two cyclic soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is cyclic soft neutrosophic bigroup over $B_N(G)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not cyclic soft neutrosophic bigroup over $B_N(G)$.

Definition 37 Let $B_N(G)$ be a neutrosophic bigroup. Then (F,A) over $B_N(G)$ is called normal soft neutrosophic bigroup if and only if $F(x)$ is normal subbigroup of $B_N(G)$ for all $x\in A$.

Example 13 Let $B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup, where
$$B(G_1)=\begin{Bmatrix} e,y,x,x^2,xy,x^2y,I, \\ Iy,Ix,Ix^2,Ixy,Ix^2y \end{Bmatrix}$$
is a neutrosophic group under multiplaction and $B(G_2)=\{g:g^6=1\}$ is a cyclic group of order 6.

Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a normal neutrosophic subbigroup where $P(G_1)=\{e,y\}$ and
$$P(G_2)=\{1,g^2,g^4\}.$$
Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another normal neutrosophic subbigroup where
$Q(G_1)=\{e,x,x^2\}$ and $Q(G_2)=\{1,g^3\}$.

Then (F,A) is a normal soft neutrosophic bigroup over $B_N(G)$ where
$$F(x_1)=\{e,y,1,g^2,g^4\},$$
$$F(x_2)=\{e,x,x^2,1,g^3\}.$$

Theorem 9 Every normal soft neutrosophic bigroup (F,A) over $B_N(G)$ is a soft neutrosophic bigroup but the converse is not true.

Theorem 10 If $B_N(G)$ is a normal neutrosophic bigroup. Then (F,A) over $B_N(G)$ is also normal soft neutrosophic bigroup.

Theorem 11 If $B_N(G)$ is a commutative neutrosophic bigroup. Then (F,A) over $B_N(G)$ is normal soft neutrosophic bigroup.

Theorem 12 If $B_N(G)$ is a cyclic neutrosophic bigroup. Then (F,A) over $B_N(G)$ is normal soft neutrosophic bigroup.

Proposition 11 Let (F,A) and (K,D) be two normal soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_{\varepsilon}(K,D)$ over $B_N(G)$ is not normal soft neutrosophic bigroup over

$B_N(G)$.

2) Their extended intersection $(F,A)\cap_{\varepsilon}(K,D)$ over $B_N(G)$ is normal soft neutrosophic bigroup over $B_N(G)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $B_N(G)$ is not normal soft neutrosophic bigroup over $B_N(G)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $B_N(G)$ is normal soft neutrosophic bigroup over $B_N(G)$.

Proposition 12 Let (F,A) and (K,D) be two normal soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is normal soft neutrosophic bigroup over $B_N(G)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not normal soft neutrosophic bigroup over $B_N(G)$.

Definition 38 Let (F,A) be a soft neutrosophic bigroup over $B_N(G)$. If for all $x\in A$ each $F(x)$ is a Lagrange subbigroup of $B_N(G)$, then (F,A) is called Lagrange soft neutosophic bigroup over $B_N(G)$.

Example 14 Let $B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a neutrosophic bigroup, where

$$B(G_1)=\begin{Bmatrix} e,y,x,x^2,xy,x^2y,I, \\ Iy,Ix,Ix^2,Ixy,Ix^2y \end{Bmatrix}$$

is a neutrosophic symmetric group of and $B(G_2)=\{0,1,I,1+I\}$ be a neutrosophic group under addition modulo 2. Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic subbigroup where $P(G_1)=\{e,y\}$ and $P(G_2)=\{0,1\}$.

Also $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another neutrosophic subbigroup where $Q(G_1)=\{e,Iy\}$ and $Q(G_2)=\{0,1+I\}$.

Then (F,A) is Lagrange soft neutrosophic bigroup over $B_N(G)$, where

$F(x_2)=\{e,y,0,1\}$,

$F(x_2)=\{e,yI,0,1+I\}$.

Theorem 13 If $B_N(G)$ is a Lagrange neutrosophic bigroup, then (F,A) over $B_N(G)$ is Lagrange soft neutrosophic bigroup.

Theorem 14 Every Lagrange soft *neutrosophic bigroup* (F,A) over $B_N(G)$ is a soft *neutrosophic bigroup* but the converse is not true.

Proposition 13 Let (F,A) and (K,D) be two Lagrange soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_{\varepsilon}(K,D)$ over $B_N(G)$ is not Lagrange soft neutrosophic bigroup over $B_N(G)$.
2) Their extended intersection $(F,A)\cap_{\varepsilon}(K,D)$ over $B_N(G)$ is not Lagrange soft neutrosophic bigroup over $B_N(G)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $B_N(G)$ is not Lagrange soft neutrosophic bigroup over $B_N(G)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $B_N(G)$ is not Lagrange soft neutrosophic bigroup over $B_N(G)$.

Proposition 14 Let (F,A) and (K,D) be two Lagrange soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not Lagrange soft neutrosophic bigroup over $B_N(G)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not Lagrange soft neutrosophic bigroup over $B_N(G)$.

Definition 39 Let (F, A) be a soft neutrosophic bigroup over $B_N(G)$. Then (F, A) is called weakly Lagrange soft neutosophic bigroup over $B_N(G)$ if at-least one $F(x)$ is a Lagrange subbigroup of $B_N(G)$, for some $x \in A$.

Example 15 Let $B(G) = \{B(G_1) \cup B(G_2), *_1, *_2\}$ be a neutrosophic bigroup, where

$$B(G_1) = \begin{cases} 0,1,2,3,4,I,2I,3I,4I,1+I,2+I,3+I,4+I, \\ 1+2I,2+2I,3+2I,4+2I,1+3I,2+3I, \\ 3+3I,4+3I,1+4I,2+4I,3+4I,4+4I \end{cases}$$

is a neutrosophic group under multiplication modulo 5 and $B(G_2) = \{g : g^{10} = 1\}$ is a cyclic group of order 10. Let $P(G) = \{P(G_1) \cup P(G_2), *_1, *_2\}$ be a neutrosophic subbigroup where $P(G_1) = \{0,1,4,I,4I\}$ and $P(G_2) = \{g^2, g^4, g^6, g^8, 1\}$. Also $Q(G) = \{Q(G_1) \cup Q(G_2), *_1, *_2\}$ be another neutrosophic subbigroup where $Q(G_1) = \{0,1,4,I,4I\}$ and $Q(G_2) = \{g^5, 1\}$. Then (F, A) is a weakly Lagrange soft neutrosophic bigroup over $B_N(G)$, where

$$F(x_1) = \{0,1,4,I,4I,g^2,g^4,g^6,g^8,1\},$$
$$F(x_2) = \{0,1,4,I,4I,g^5,1\}.$$

Theorem 15 Every weakly Lagrange *soft neutrosophic bigroup* (F, A) over $B_N(G)$ is a soft *neutrosophic bigroup* but the converse is not true.

Proposition 15 Let (F, A) and (K, D) be two weakly Lagrange soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F, A) \cup_\varepsilon (K, D)$ over $B_N(G)$ is not weakly Lagrange soft neutrosophic bigroup over $B_N(G)$.
2) Their extended intersection $(F, A) \cap_\varepsilon (K, D)$ over $B_N(G)$ is not weakly Lagrange soft neutrosophic bigroup over $B_N(G)$.
3) Their restricted union $(F, A) \cup_R (K, D)$ over $B_N(G)$ is not weakly Lagrange soft neutrosophic bigroup over $B_N(G)$.
4) Their restricted intersection $(F, A) \cap_R (K, D)$ over $B_N(G)$ is not weakly Lagrange soft neutrosophic bigroup over $B_N(G)$.

Proposition 16 Let (F, A) and (K, D) be two weakly Lagrange soft neutrosophic bigroups over $B_N(G)$. Then

1) Their *AND* operation $(F, A) \wedge (K, D)$ is not weakly Lagrange soft neutrosophic bigroup over $B_N(G)$.
2) Their *OR* operation $(F, A) \vee (K, D)$ is not weakly Lagrange soft neutrosophic bigroup over $B_N(G)$.

Definition 40 Let (F, A) be a soft neutrosophic bigroup over $B_N(G)$. Then (F, A) is called Lagrange free soft neutrosophic bigroup if each $F(x)$ is not Lagrange subbigroup of $B_N(G)$, for all $x \in A$.

Example 16 Let $B(G) = \{B(G_1) \cup B(G_2), *_1, *_2\}$ be a neutrosophic bigroup, where $B(G_1) = \{0,1,I,1+I\}$ is a neutrosophic group under addition modulo 2 of order 4 and $B(G_2) = \{g : g^{12} = 1\}$ is a cyclic group of order 12. Let $P(G) = \{P(G_1) \cup P(G_2), *_1, *_2\}$ be a neutrosophic subbigroup where $P(G_1) = \{0, I\}$ and $P(G_2) = \{g^4, g^8, 1\}$. Also $Q(G) = \{Q(G_1) \cup Q(G_2), *_1, *_2\}$ be another neutrosophic subbigroup where $Q(G_1) = \{0, 1+I\}$ and $Q(G_2) = \{1, g^3, g^6, g^9\}$. Then (F, A) is Lagrange free soft neutrosophic bigroup over $B_N(G)$, where

$$F(x_1) = \{0, I, 1, g^4, g^8\},$$
$$F(x_2) = \{0, 1+I, 1, g^3, g^6, g^9\}.$$

Theorem 16 If $B_N(G)$ is Lagrange free neutrosophic bigroup, and then (F,A) over $B_N(G)$ is Lagrange free soft neutrosophic bigroup.

Theorem 17 Every Lagrange free *soft neutrosophic bigroup* (F,A) over $B_N(G)$ is a soft *neutrosophic bigroup* but the converse is not true.

Proposition 17 Let (F,A) and (K,D) be two Lagrange free soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_{\varepsilon}(K,D)$ over $B_N(G)$ is not Lagrange free soft neutrosophic bigroup over $B_N(G)$.
2) Their extended intersection $(F,A)\cap_{\varepsilon}(K,D)$ over $B_N(G)$ is not Lagrange free soft neutrosophic bigroup over $B_N(G)$.
3) Their restricted union $(F,A)\cup_{R}(K,D)$ over $B_N(G)$ is not Lagrange free soft neutrosophic bigroup over $B_N(G)$.
4) Their restricted intersection $(F,A)\cap_{R}(K,D)$ over $B_N(G)$ is not Lagrange free soft neutrosophic bigroup over $B_N(G)$.

Proposition 18 Let (F,A) and (K,D) be two Lagrange free soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not Lagrange free soft neutrosophic bigroup over $B_N(G)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not Lagrange free soft neutrosophic bigroup over $B_N(G)$.

Definition 41 Let $B_N(G)$ be a neutrosophic bigroup. Then (F,A) is called conjugate soft neutrosophic bigroup over $B_N(G)$ if and only if $F(x)$ is neutrosophic conjugate subbigroup of $B_N(G)$ for all $x\in A$.

Example 17 Let $B(G)=\{B(G_1)\cup B(G_2),*_1,*_2\}$ be a soft neutrosophic bigroup, where
$$B(G_1)=\{e,y,x,x^2,xy,x^2y\}$$
is Klien 4 -group and
$$B(G_2)=\begin{Bmatrix}0,1,2,3,4,5,I,2I,3I,4I,5I,\\1+I,2+I,3+I,...,5+5I\end{Bmatrix}$$
be a neutrosophic group under addition modulo 6.
Let $P(G)=\{P(G_1)\cup P(G_2),*_1,*_2\}$ be a neutrosophic subbigroup of $B_N(G)$, where $P(G_1)=\{e,y\}$ and $P(G_2)=\{0,3,3I,3+3I\}$. Again let $Q(G)=\{Q(G_1)\cup Q(G_2),*_1,*_2\}$ be another neutrosophic subbigroup of $B_N(G)$, where $Q(G_1)=\{e,x,x^2\}$ and $Q(G_2)=\{0,2,4,2+2I,4+4I,2I,4I\}$. Then (F,A) is conjugate soft neutrosophic bigroup over $B_N(G)$, where
$$F(x_1)=\{e,y,0,3,3I,3+3I\},$$
$$F(x_2)=\{e,x,x^2,0,2,4,2+2I,4+4I,2I,4I\}.$$

Theorem 18 If $B_N(G)$ is conjugate neutrosophic bigroup, then (F,A) over $B_N(G)$ is conjugate soft neutrosophic bigroup.

Theorem 19 Every conjugate soft neutrosophic bigroup (F,A) over $B_N(G)$ is a soft neutrosophic bigroup but the converse is not true.

Proposition 19 Let (F,A) and (K,D) be two conjugate soft neutrosophic bigroups over $B_N(G)$. Then

1) Their extended union $(F,A)\cup_{\varepsilon}(K,D)$ over $B_N(G)$ is not conjugate soft neutrosophic bigroup over $B_N(G)$.
2) Their extended intersection $(F,A)\cap_{\varepsilon}(K,D)$ over $B_N(G)$ is conjugate soft neutrosophic bigroup over $B_N(G)$.

3) Their restricted union $(F,A)\cup_R(K,D)$ over $B_N(G)$ is not conjugate soft neutrosophic bigroup over $B_N(G)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $B_N(G)$ is conjgate soft neutrosophic bigroup over $B_N(G)$.

Proposition 20 Let (F,A) and (K,D) be two conjugate soft neutrosophic bigroups over $B_N(G)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is conjugate soft neutrosophic bigroup over $B_N(G)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not conjugate soft neutrosophic bigroup over $B_N(G)$.

3.3 Soft Strong Neutrosophic Bigroup

Definition 42 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup. Then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is called soft strong neutrosophic bigroup if and only if $F(x)$ is a strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$ for all $x\in A$.

Example 18 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup, where $\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle$ with $\langle G_1\cup I\rangle=\langle Z\cup I\rangle$, the neutrosophic group under addition and $\langle G_2\cup I\rangle=\{0,1,2,3,4,I,2I,3I,4I\}$ a neutrosophic group under multiplication modulo 5. Let $H=H_1\cup H_2$ be a strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$, where $H_1=\{\langle 2Z\cup I\rangle,+\}$ is a neutrosophic subgroup and $H_2=\{0,1,4,I,4I\}$ is a neutrosophic subgroup. Again let $K=K_1\cup K_2$ be another strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$, where $K_1=\{\langle 3Z\cup I\rangle,+\}$ is a neutrosophic subgroup and $K_2=\{0,1,I,2I,3I,4I\}$ is a neutrosophic subgroup. Then clearly (F,A) is a soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$, where

$$F(x_1)=\{0,\pm2,\pm4,\ldots,1,4,I,4I\},$$
$$F(x_2)=\{0,\pm3,\pm6,\ldots,1,I,2I,3I,4I\}.$$

Theorem 20 Every soft strong neutrosophic bigroup (F,A) is a soft neutrosophic bigroup but the converse is not true.

Theorem 21 If $(\langle G\cup I\rangle,*_1,*_2)$ is a strong neutrosophic bigroup, then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is soft strong neutrosophic bigroup.

Proposition 21 Let (F,A) and (K,D) be two soft strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Proposition 22 Let (F,A) and (K,D) be two soft strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Definition 43 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup. Then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is called Lagrange soft strong neutrosophic bigroup if and only if $F(x)$ is Lagrange subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$ for all $x\in A.$

Example 19 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup of order 15, where $\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle$ with $\langle G_1\cup I\rangle=\{0,1,2,1+I,I,2I,2+I,2+2I,1+2I\}$, the neutrosophic group under mltiplication modulo 3 and $\langle G_2\cup I\rangle=\langle A_3\cup I\rangle=\{e,x,x^2,I,xI,x^2I\}$. Let $H=H_1\cup H_2$ be a strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$, where $H_1=\{1,2+2I\}$ is a neutrosophic subgroup and $H_2=\{e,x,x^2\}$ is a neutrosophic subgroup. Again let $K=K_1\cup K_2$ be another strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$, where $K_1=\{1,1+I\}$ is a neutrosophic subgroup and $K_2=\{I,xI,x^2I\}$ is a neutrosophic subgroup. Then clearly (F,A) is Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$, where

$$F(x_1)=\{1,2+2I,e,x,x^2\},$$
$$F(x_2)=\{1,1+I,I,xI,x^2I\}.$$

Theorem 22 Every Lagrange soft strong neutrosophic bigroup (F,A) is a soft neutrosophic bigroup but the converse is not true.

Theorem 23 Every Lagrange soft strong neutrosophic bigroup (F,A) is a soft strong neutrosophic bigroup but the converse is not true.

Theorem 24 If $(\langle G\cup I\rangle,*_1,*_2)$ is a Lagrange strong neutrosophic bigroup, then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is a Lagrange soft strong neutrosophic soft bigroup.

Proposition 23 Let (F,A) and (K,D) be two Lagrange soft strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Proposition 24 Let (F,A) and (K,D) be two Lagrange soft strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not Lagrange soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Definition 44 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup. Then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is called weakly Lagrange soft strong neutrosophic bigroup if atleast one $F(x)$ is a Lagrange subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$ for some $x\in A.$

Example 20 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup of order 15, where $\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle$ with $\langle G_1\cup I\rangle=\{0,1,2,1+I,I,2I,2+I,2+2I,1+2I\}$, the neutrosophic under mltiplication modulo 3 and

$\langle G_2 \cup I \rangle = \{e, x, x^2, I, xI, x^2I\}$. Let $H = H_1 \cup H_2$ be a strong neutrosophic subbigroup of $(\langle G \cup I \rangle, *_1, *_2)$, where $H_1 = \{1, 2, I, 2I\}$ is a neutrosophic subgroup and $H_2 = \{e, x, x^2\}$ is a neutrosophic subgroup. Again let $K = K_1 \cup K_2$ be another strong neutrosophic subbigroup of $(\langle G \cup I \rangle, *_1, *_2)$, where $K_1 = \{1, 1+I\}$ is a neutrosophic subgroup and $K_2 = \{e, I, xI, x^2I\}$ is a neutrosophic subgroup.

Then clearly (F, A) is weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$, where

$$F(x_1) = \{1, 2, I, 2I, e, x, x^2\},$$

$$F(x_2) = \{1, 1+I, e, I, xI, x^2I\}.$$

Theorem 25 Every weakly Lagrange soft strong neutrosophic bigroup (F, A) is a soft neutrosophic bigroup but the converse is not true.

Theorem 26 Every weakly Lagrange soft strong neutrosophic bigroup (F, A) is a soft strong neutrosophic bigroup but the converse is not true.

Proposition 25 Let (F, A) and (K, D) be two weakly Lagrange soft strong neutrosophic bigroups over $(\langle G \cup I \rangle, *_1, *_2)$. Then

1) Their extended union $(F, A) \cup_\varepsilon (K, D)$ over $(\langle G \cup I \rangle, *_1, *_2)$ is not weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$.
.
2) Their extended intersection $(F, A) \cap_\varepsilon (K, D)$ over $(\langle G \cup I \rangle, *_1, *_2)$ is not weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$.
3) Their restricted union $(F, A) \cup_R (K, D)$ over $(\langle G \cup I \rangle, *_1, *_2)$ is not weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$. .
4) Their restricted intersection $(F, A) \cap_R (K, D)$ over $(\langle G \cup I \rangle, *_1, *_2)$ is not weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$.

Proposition 26 Let (F, A) and (K, D) be two weakly Lagrange soft strong neutrosophic bigroups over $(\langle G \cup I \rangle, *_1, *_2)$. Then

1) Their AND operation $(F, A) \wedge (K, D)$ is not weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$. .
2) Their OR operation $(F, A) \vee (K, D)$ is not weakly Lagrange soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$.

Definition 45 Let $(\langle G \cup I \rangle, *_1, *_2)$ be a strong neutrosophic bigroup. Then (F, A) over $(\langle G \cup I \rangle, *_1, *_2)$ is called Lagrange free soft strong neutrosophic bigroup if and only if $F(x)$ is not Lagrange subbigroup of $(\langle G \cup I \rangle, *_1, *_2)$ for all $x \in A$.

Example 21 Let $(\langle G \cup I \rangle, *_1, *_2)$ be a strong neutrosophic bigroup of order 15, where $\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle$ with $\langle G_1 \cup I \rangle = \{0, 1, 2, 3, 4, I, 2I, 3I, 4I\}$, the neutrosophic under mltiplication modulo 5 and $\langle G_2 \cup I \rangle = \{e, x, x^2, I, xI, x^2I\}$, a neutrosophic symmetric group. Let $H = H_1 \cup H_2$ be a strong neutrosophic subbigroup of $(\langle G \cup I \rangle, *_1, *_2)$, where $H_1 = \{1, 4, I, 4I\}$ is a neutrosophic subgroup and $H_2 = \{e, x, x^2\}$ is a neutrosophic subgroup. Again let $K = K_1 \cup K_2$ be another strong neutrosophic subbigroup of $(\langle G \cup I \rangle, *_1, *_2)$, where $K_1 = \{1, I, 2I, 3I, 4I\}$ is a neutrosophic subgroup and $K_2 = \{e, x, x^2\}$ is a neutrosophic subgroup.

Then clearly (F, A) is Lagrange free soft strong neutrosophic bigroup over $(\langle G \cup I \rangle, *_1, *_2)$, where

$$F(x_1)=\{1,4,I,4I,e,x,x^2\},$$
$$F(x_2)=\{1,I,2I,3I,4I,e,x,x^2\}.$$

Theorem 27 Every Lagrange free soft strong neutrosophic bigroup (F,A) is a soft neutrosophic bigroup but the converse is not true.

Theorem 28 Every Lagrange free soft strong neutrosophic bigroup (F,A) is a soft strong neutrosophic bigroup but the converse is not true.

Theorem 29 If $(\langle G\cup I\rangle,*_1,*_2)$ is a Lagrange free strong neutrosophic bigroup, then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is also Lagrange free soft strong neutrosophic bigroup.

Proposition 27 Let (F,A) and (K,D) be weakly Lagrange free soft strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange free soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange free soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange free soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not Lagrange free soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Proposition 28 Let (F,A) and (K,D) be two Lagrange free soft strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not Lagrange free soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not Lagrange free soft strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Definition 46 Let $(\langle G\cup I\rangle,*_1,*_2)$ be a strong neutrosophic bigroup. Then (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is called soft normal strong neutrosophic bigroup if and only if $F(x)$ is normal strong neutrosophic subbigroup of $(\langle G\cup I\rangle,*_1,*_2)$ for all $x\in A$.

Theorem 30 Every soft normal *strong neutrosophic bigroup* (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is a soft *neutrosophic bigroup* but the converse is not true.

Theorem 31 Every soft normal *strong neutrosophic bigroup* (F,A) over $(\langle G\cup I\rangle,*_1,*_2)$ is a soft strong *neutrosophic bigroup* but the converse is not true.

Proposition 29 Let (F,A) and (K,D) be two soft normal strong neutrosophic bigroups over $(\langle G\cup I\rangle,*_1,*_2)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not soft normal strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is soft normal strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is not soft normal strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $(\langle G\cup I\rangle,*_1,*_2)$ is soft normal strong neutrosophic bigroup over $(\langle G\cup I\rangle,*_1,*_2)$.

Proposition 30 Let (F,A) and (K,D) be two soft normal strong neutrosophic bigroups over $\left(\langle G\cup I\rangle,*_1,*_2\right)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is soft normal strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft normal strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.

Definition 47 Let $\left(\langle G\cup I\rangle,*_1,*_2\right)$ be a strong neutrosophic bigroup. Then (F,A) over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is called soft conjugate strong neutrosophic bigroup if and only if $F(x)$ is conjugate neutrosophic subbigroup of $\left(\langle G\cup I\rangle,*_1,*_2\right)$ for all $x\in A.$

Theorem 32 Every soft conjugate *strong neutrosophic bigroup* (F,A) over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is a soft *neutrosophic bigroup* but the converse is not true.

Theorem 33 Every soft conjugate *strong neutrosophic bigroup* (F,A) over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is a soft strong neutrosophic bigroup *but the converse is not true.*

Proposition 31 Let (F,A) and (K,D) be two soft conjugate strong neutrosophic bigroups over $\left(\langle G\cup I\rangle,*_1,*_2\right)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is not soft conjugate strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is soft conjugate strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is not soft conjugate strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ over $\left(\langle G\cup I\rangle,*_1,*_2\right)$ is soft conjugate strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.

Proposition 32 Let (F,A) and (K,D) be two soft conjugate strong neutrosophic bigroups over $\left(\langle G\cup I\rangle,*_1,*_2\right)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is soft conjugate strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft conjgate strong neutrosophic bigroup over $\left(\langle G\cup I\rangle,*_1,*_2\right)$.

4.1 Soft Neutrosophic N-Group

Definition 48 Let $\left(\langle G\cup I\rangle,*_1,\dots,*_N\right)$ be a neutrosophic N-group. Then (F,A) over $\left(\langle G\cup I\rangle,*_1,\dots,*_2\right)$ is called soft neutrosophic N-group if and only if $F(x)$ is a sub N-group of $\left(\langle G\cup I\rangle,*_1,\dots,*_2\right)$ for all $x\in A.$

Example 22 Let $\left(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle\cup\langle G_3\cup I\rangle,*_1,*_2,*_3\right)$ be a neutrosophic 3-group, where $\langle G_1\cup I\rangle=\langle Q\cup I\rangle$ a neutrosophic group under multiplication.
$\langle G_2\cup I\rangle=\{0,1,2,3,4,I,2I,3I,4I\}$ neutrosophic group under multiplication modulo 5 and
$\langle G_3\cup I\rangle=\{0,1,2,1+I,2+I,I,2I,1+2I,2+2I\}$ a neutrosophic group under multiplication modulo 3. Let

$$P=\left\{\left\{\left\langle\frac{1}{2^n},2^n,\frac{1}{(2I)^n},(2I)^n,I,1\right\rangle\right\},(1,4,I,4I),(1,2,I,2I)\right\},$$

$T=\{Q\setminus\{0\},\{1,2,3,4\},\{1,2\}\}$ and
$X=\{Q\setminus\{0\},\{1,2,I,2I\},\{1,4,I,4I\}\}$ are sub 3-groups.

Then (F,A) is clearly soft neutrosophic 3-group over $\left(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle\cup\langle G_3\cup I\rangle,*_1,*_2,*_3\right)$, where

$$F(x_1)=\left\{\left\{\left\langle \frac{1}{2^n},2^n,\frac{1}{(2I)^n},(2I)^n,I,1\right\rangle\right\},(1,4,I,4I),(1,2,I,2I)\right\},$$

$$F(x_2)=\{Q\setminus\{0\},\{1,2,3,4\},\{1,2\}\},$$

$$F(x_3)=\{Q\setminus\{0\},\{1,2,I,2I\},\{1,4,I,4I\}\}.$$

Theorem 34 Let (F,A) and (H,A) be two soft neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then their intersection $(F,A)\cap(H,A)$ is again a soft neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Proof The proof is straight forward.

Theorem 35 Let (F,A) and (H,B) be two soft neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$ such that $A\cap B-=\phi,$ then their union is soft neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Proof The proof can be established easily.

Proposition 33 Let (F,A) and (K,D) be two soft neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ is not soft neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ is soft neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ is not soft neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ is soft neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Proposition 34 Let (F,A) and (K,D) be two soft neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is soft neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Definition 49 Let (F,A) be a soft neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) (F,A) is called identity soft neutrosophic N -group if $F(x)=\{e_1,...,e_N\}$ for all $x\in A,$ where $e_1,...,e_N$ are the identities of $\langle G_1\cup I\rangle,...,\langle G_N\cup I\rangle$ respectively.
2) (F,A) is called Full soft neutrosophic N -group if $F(x)=(\langle G\cup I\rangle,*_1,...,*_N)$ for all $x\in A.$

Definition 50 Let (F,A) and (K,D) be two soft neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then (K,D) is soft neutrosophic sub N -group of (F,A) written as $(K,D)\prec(F,A)$, if

1) $D\subset A,$
2) $K(x)\prec F(x)$ for all $x\in A.$

Example 23 Let (F,A) be as in example 22. Let (K,D) be another soft neutrosophic soft N -group over $(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup\langle G_2\cup I\rangle\cup\langle G_3\cup I\rangle,*_1,*_2,*_3)$, where

$$K(x_1)=\left\{\left\{\left\langle \frac{1}{2^n},2^n\right\rangle\right\},\{1,4,I,4I\},\{1,2,I,2I\}\right\},$$

$$K(x_2)=\{Q\setminus\{0\},\{1,4\},\{1,2\}\}.$$

Clearly $(K,D)\prec(F,A)$.

Thus a soft neutrosophic N -group can have two types of soft neutrosophic sub N -groups, which are following

Definition 51 A soft neutrosophic sub N -group (K,D) of a soft neutrosophic N -group (F,A) is called soft strong neutrosophic sub N -group if

1) $D\subset A,$
2) $K(x)$ is neutrosophic sub N -group of $F(x)$ for

all $x \in A$.

Definition 52 A soft neutrosophic sub N -group (K,D) of a soft neutrosophic N -group (F,A) is called soft sub N -group if

1) $D \subset A$,
2) $K(x)$ is only sub N -group of $F(x)$ for all $x \in A$.

Definition 53 Let $(\langle G \cup I \rangle, *_1, ..., *_N)$ be a neutrosophic N -group. Then (F,A) over $(\langle G \cup I \rangle, *_1, ..., *_N)$ is called soft Lagrange neutrosophic N -group if and only if $F(x)$ is Lagrange sub N -group of $(\langle G \cup I \rangle, *_1, ..., *_N)$ for all $x \in A$.

Example 24 Let $(\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup G_2 \cup G_3, *_1, *_2, *_3)$ be neutrosophic N -group, where $\langle G_1 \cup I \rangle = \{\langle Z_6 \cup I \rangle\}$ is a group under addition modulo 6 , $G_2 = A_4$ and $G_3 = \langle g : g^{12} = 1 \rangle$, a cyclic group of order 12, $o(\langle G \cup I \rangle) = 60$.

Take $P = (\langle P_1 \cup I \rangle \cup P_2 \cup P_3, *_1, *_2, *_3)$, a neutrosophic sub 3 -group where

$\langle T_1 \cup I \rangle = \{0, 3, 3I, 3+3I\}$,

$$P_2 = \left\{ \begin{pmatrix} 1234 \\ 1234 \end{pmatrix}, \begin{pmatrix} 1234 \\ 2143 \end{pmatrix}, \begin{pmatrix} 1234 \\ 4321 \end{pmatrix}, \begin{pmatrix} 1234 \\ 3412 \end{pmatrix} \right\},$$

$P_3 = \{1, g^6\}$. Since P is a Lagrange neutrosophic sub 3 -group where order of $P = 10$.

Let us Take $T = (\langle T_1 \cup I \rangle \cup T_2 \cup T_3, *_1, *_2, *_3)$, where $\langle T_1 \cup I \rangle = \{0, 3, 3I, 3+3I\}, T_2 = P_2$ and $T_3 = \{g^3, g^6, g^9, 1\}$ is another Lagrange sub 3 -group where $o(T) = 12$.

Let (F,A) is soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup G_2 \cup G_3, *_1, *_2, *_3)$, where

$$F(x_1) = \left\{ 0, 3, 3I, 3+3I, 1, g^6, \begin{pmatrix} 1234 \\ 1234 \end{pmatrix}, \begin{pmatrix} 1234 \\ 2143 \end{pmatrix}, \begin{pmatrix} 1234 \\ 4321 \end{pmatrix}, \begin{pmatrix} 1234 \\ 3412 \end{pmatrix} \right\},$$

$$F(x_2) = \left\{ 0, 3, 3I, 3+3I, 1, g^3, g^6, g^9, \begin{pmatrix} 1234 \\ 1234 \end{pmatrix}, \begin{pmatrix} 1234 \\ 2143 \end{pmatrix}, \begin{pmatrix} 1234 \\ 4321 \end{pmatrix}, \begin{pmatrix} 1234 \\ 3412 \end{pmatrix} \right\}.$$

Theorem 36 Every soft Lagrange neutrosophic N -group (F,A) over $(\langle G \cup I \rangle, *_1, ..., *_N)$ is a soft neutrosophic N -group but the converse is not true.

Theorem 37 If $(\langle G \cup I \rangle, *_1, ..., *_N)$ is a Lagrange neutrosophic N -group, then (F,A) over $(\langle G \cup I \rangle, *_1, ..., *_N)$ is also soft Lagrange neutrosophic N -group.

Proposition 35 Let (F,A) and (K,D) be two soft Lagrange neutrosophic N -groups over $(\langle G \cup I \rangle, *_1, ..., *_N)$. Then

1) Their extended union $(F,A) \cup_{\varepsilon} (K,D)$ is not soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
2) Their extended intersection $(F,A) \cap_{\varepsilon} (K,D)$ is not soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
3) Their restricted union $(F,A) \cup_R (K,D)$ is not soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
4) Their restricted intersection $(F,A) \cap_R (K,D)$ is not soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

Proposition 36 Let (F,A) and (K,D) be two soft Lagrange neutrosophic N -groups over $(\langle G \cup I \rangle, *_1, ..., *_N)$. Then

1) Their AND operation $(F,A) \wedge (K,D)$ is not soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.
2) Their OR operation $(F,A) \vee (K,D)$ is not soft Lagrange neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

Definition 54 Let $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ be a neutrosophic N -group. Then (F,A) over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is called soft weakly Lagrange neutrosophic N -group if atleast one $F(x)$ is Lagrange sub N -group of $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ for some $x\in A.$

Examp 25 Let $\left(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup G_2\cup G_3,*_1,*_2,*_3\right)$ be neutrosophic N -group, where $\langle G_1\cup I\rangle=\{\langle Z_6\cup I\rangle\}$ is a group under addition modulo 6 , $G_2=A_4$ and $G_3=\langle g:g^{12}=1\rangle,$ a cyclic group of order $12,$ $o\left(\langle G\cup I\rangle\right)=60.$

Take $P=\left(\langle P_1\cup I\rangle\cup P_2\cup P_3,*_1,*_2,*_3\right)$, a neutrosophic sub 3 -group where

$\langle T_1\cup I\rangle=\{0,3,3I,3+3I\},$

$$P_2=\left\{\begin{pmatrix}1234\\1234\end{pmatrix},\begin{pmatrix}1234\\2143\end{pmatrix},\begin{pmatrix}1234\\4321\end{pmatrix},\begin{pmatrix}1234\\3412\end{pmatrix}\right\},$$

$P_3=\{1,g^6\}.$ Since P is a Lagrange neutrosophic sub 3 -group where order of $P=10.$

Let us Take $T=\left(\langle T_1\cup I\rangle\cup T_2\cup T_3,*_1,*_2,*_3\right),$ where $\langle T_1\cup I\rangle=\{0,3,3I,3+3I\},T_2=P_2$ and $T_3=\{g^4,g^8,1\}$ is another Lagrange sub 3 -group.

Then (F,A) is soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup G_2\cup G_3,*_1,*_2,*_3\right)$, where

$$F(x_1)=\left\{0,3,3I,3+3I,1,g^6,\begin{pmatrix}1234\\1234\end{pmatrix},\begin{pmatrix}1234\\2143\end{pmatrix},\begin{pmatrix}1234\\4321\end{pmatrix},\begin{pmatrix}1234\\3412\end{pmatrix}\right\},$$

$$F(x_2)=\left\{0,3,3I,3+3I,1,g^4,g^8,\begin{pmatrix}1234\\1234\end{pmatrix},\begin{pmatrix}1234\\2143\end{pmatrix},\begin{pmatrix}1234\\4321\end{pmatrix},\begin{pmatrix}1234\\3412\end{pmatrix}\right\}.$$

Theorem 38 Every soft weakly Lagrange neutrosophic N -group (F,A) over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is a soft neutrosophic N -group but the converse is not tue.

Theorem 39 If $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is a weakly Lagrange neutrosophi N -group, then (F,A) over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is also soft weakly Lagrange neutrosophic N -group.

Proposition 37 Let (F,A) and (K,D) be two soft weakly Lagrange neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$. Then

1. Their extended union $(F,A)\cup_\varepsilon(K,D)$ is not soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
2. Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ is not soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
3. Their restricted union $(F,A)\cup_R(K,D)$ is not soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
4. Their restricted intersection $(F,A)\cap_R(K,D)$ is not soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.

Proposition 38 Let (F,A) and (K,D) be two soft weakly Lagrange neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft weakly Lagrange neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.

Definition 55 Let $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ be a neutrosophic N -group. Then (F,A) over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is called soft Lagrange free neutrosophic N -group if $F(x)$ is not Lagrange sub N -group of $(\langle G\cup I\rangle,*_1,...,*_N)$ for all $x\in A.$

Example 26 Let $(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup G_2\cup G_3,*_1,*_2,*_3)$ be neutrosophic 3-group, where $\langle G_1\cup I\rangle=\{\langle Z_6\cup I\rangle\}$ is a group under addition modulo 6 , $G_2=A_4$ and $G_3=\langle g:g^{12}=1\rangle,$ a cyclic group of order $12,$ $o(\langle G\cup I\rangle)=60.$

Take $P=(\langle P_1\cup I\rangle\cup P_2\cup P_3,*_1,*_2,*_3)$, a neutrosophic sub 3-group where

$P_1=\{0,2,4\},$

$$P_2=\left\{\begin{pmatrix}1234\\1234\end{pmatrix},\begin{pmatrix}1234\\2143\end{pmatrix},\begin{pmatrix}1234\\4321\end{pmatrix},\begin{pmatrix}1234\\3412\end{pmatrix}\right\},$$

$P_3=\{1,g^6\}.$ Since P is a Lagrange neutrosophic sub 3-group where order of $P=10.$

Let us Take $T=(\langle T_1\cup I\rangle\cup T_2\cup T_3,*_1,*_2,*_3),$ where $\langle T_1\cup I\rangle=\{0,3,3I,3+3I\},T_2=P_2$ and $T_3=\{g^4,g^8,1\}$ is another Lagrange sub 3-group. Then (F,A) is soft Lagrange free neutrosophic 3-group over $(\langle G\cup I\rangle=\langle G_1\cup I\rangle\cup G_2\cup G_3,*_1,*_2,*_3),$ where

$$F(x_1)=\left\{0,2,4,1,g^6,\begin{pmatrix}1234\\1234\end{pmatrix},\begin{pmatrix}1234\\2143\end{pmatrix},\begin{pmatrix}1234\\4321\end{pmatrix},\begin{pmatrix}1234\\3412\end{pmatrix}\right\},$$

$$F(x_2)=\left\{0,3,3I,3+3I,1,g^4,g^8,\begin{pmatrix}1234\\1234\end{pmatrix},\begin{pmatrix}1234\\2143\end{pmatrix},\begin{pmatrix}1234\\4321\end{pmatrix},\begin{pmatrix}1234\\3412\end{pmatrix}\right\}$$

Theorem 40 Every soft Lagrange free neutrosophic N -group (F,A) over $(\langle G\cup I\rangle,*_1,...,*_N)$ is a soft neutrosophic N -group but the converse is not true.

Theorem 41 If $(\langle G\cup I\rangle,*_1,...,*_N)$ is a Lagrange free neutrosophic N -group, then (F,A) over $(\langle G\cup I\rangle,*_1,...,*_N)$ is also soft Lagrange free neutrosophic N -group.

Proposition 39 Let (F,A) and (K,D) be two soft Lagrange free neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ is not soft Lagrange free neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ is not soft Lagrange free neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ is not soft Lagrange free neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ is not soft Lagrange free neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Proposition 40 Let (F,A) and (K,D) be two soft Lagrange free neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not soft Lagrange free neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft Lagrange free neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Definition 56 Let $(\langle G\cup I\rangle,*_1,...,*_N)$ be a neutrosophic N -group. Then (F,A) over $(\langle G\cup I\rangle,*_1,...,*_N)$ is called soft normal neutrosophic N -group if $F(x)$ is normal sub N -group of $(\langle G\cup I\rangle,*_1,...,*_N)$ for all $x\in A.$

Example 27 Let $(\langle G_1\cup I\rangle=\langle G_1\cup I\rangle\cup G_2\cup\langle G_3\cup I\rangle,*_1,*_2,*_3)$ be a soft neutrosophic N -group, where

$\langle G_1\cup I\rangle=\{e,y,x,x^2,xy,x^2y,I,yI,xI,x^2I,xyI,x^2yI\}$

is a neutrosophic group under multiplaction, $G_2 = \{g : g^6 = 1\}$, a cyclic group of order 6 and $\langle G_3 \cup I\rangle = \langle Q_8 \cup I\rangle = \{\pm1, \pm i, \pm j, \pm k, \pm I, \pm iI, \pm jI, \pm kI\}$ is a group under multiplication. Let $P = (\langle P_1 \cup I\rangle \cup P_2 \cup \langle P_3 \cup I\rangle, *_1, *_2, *_3)$, a normal sub 3-group where $P_1 = \{e, y, I, yI\}$, $P_2 = \{1, g^2, g^4\}$ and $P_3 = \{1, -1\}$. Also $T = (\langle T_1 \cup I\rangle \cup T_2 \cup \langle T_3 \cup I\rangle, *_1, *_2, *_3)$ be another normal sub 3-group where $\langle T_1 \cup I\rangle = \{e, I, xI, x^2I\}, T_2 = \{1, g^3\}$ and $\langle T_3 \cup I\rangle = \{\pm1, \pm i\}$. Then (F, A) is a soft normal neutrosophic N-group over $(\langle G_1 \cup I\rangle = \langle G_1 \cup I\rangle \cup G_2 \cup \langle G_3 \cup I\rangle, *_1, *_2, *_3)$, where

$$F(x_1) = \{e, y, I, yI, 1, g^2, g^4, \pm1\},$$
$$F(x_2) = \{e, I, xI, x^2I, 1, g^3, \pm1, \pm i\}.$$

Theorem 42 Every soft normal neutrosophic N-group (F, A) over $(\langle G \cup I\rangle, *_1, ..., *_N)$ is a soft neutrosophic N-group but the converse is not true.

Proposition 41 Let (F, A) and (K, D) be two soft normal neutrosophic N-groups over $(\langle G \cup I\rangle, *_1, ..., *_N)$. Then

1) Their extended union $(F, A) \cup_\varepsilon (K, D)$ is not soft normal neutrosophic soft N-group over $(\langle G \cup I\rangle, *_1, ..., *_N)$.
2) Their extended intersection $(F, A) \cap_\varepsilon (K, D)$ is soft normal neutrosophic N-group over $(\langle G \cup I\rangle, *_1, ..., *_N)$.
3) Their restricted union $(F, A) \cup_R (K, D)$ is not soft normal neutrosophic N-group over $(\langle G \cup I\rangle, *_1, ..., *_N)$.
4) Their restricted intersection $(F, A) \cap_R (K, D)$ is soft normal neutrosophic N-group over $(\langle G \cup I\rangle, *_1, ..., *_N)$.

Proposition 42 Let (F, A) and (K, D) be two soft normal neutrosophic N-groups over $(\langle G \cup I\rangle, *_1, ..., *_N)$. Then

1) Their *AND* operation $(F, A) \wedge (K, D)$ is soft normal neutrosophic N-group over $(\langle G \cup I\rangle, *_1, ..., *_N)$.
2) Their *OR* operation $(F, A) \vee (K, D)$ is not soft normal neutrosophic N-group over $(\langle G \cup I\rangle, *_1, ..., *_N)$.

Definition 56 Let $(\langle G \cup I\rangle, *_1, ..., *_N)$ be a neutrosophic N-group. Then (F, A) over $(\langle G \cup I\rangle, *_1, ..., *_N)$ is called soft conjugate neutrosophic N-group if $F(x)$ is conjugate sub N-group of $(\langle G \cup I\rangle, *_1, ..., *_N)$ for all $x \in A$.

Theorem 43 Every soft conjugate neutrosophic N-group (F, A) over $(\langle G \cup I\rangle, *_1, ..., *_N)$ is a soft neutrosophic N-group but the converse is not true.

Proposition 43 Let (F, A) and (K, D) be two soft conjugate neutrosophic N-groups over $(\langle G \cup I\rangle, *_1, ..., *_N)$. Then

1) Their extended union $(F, A) \cup_\varepsilon (K, D)$ is not soft conjugate neutrosophic N-group over $(\langle G \cup I\rangle, *_1, ..., *_N)$.
2) Their extended intersection $(F, A) \cap_\varepsilon (K, D)$ is soft conjugate neutrosophic N-group over $(\langle G \cup I\rangle, *_1, ..., *_N)$.
3) Their restricted union $(F, A) \cup_R (K, D)$ is not soft conjugate neutrosophic N-group over $(\langle G \cup I\rangle, *_1, ..., *_N)$.
4) Their restricted intersection $(F, A) \cap_R (K, D)$ is soft conjugate neutrosophic N-group over $(\langle G \cup I\rangle, *_1, ..., *_N)$.

Proposition 44 Let (F, A) and (K, D) be two soft conjugate neutrosophic N-groups over

$(\langle G \cup I \rangle, *_1, ..., *_N)$. Then

1) Their AND operation $(F, A) \wedge (K, D)$ is soft conjugate neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

2) Their OR operation $(F, A) \vee (K, D)$ is not soft conjugate neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

4.2 Soft Strong Neutrosophic N-Group

Definition 57 Let $(\langle G \cup I \rangle, *_1, ..., *_N)$ be a neutrosophic N -group. Then (F, A) over $(\langle G \cup I \rangle, *_1, ..., *_N)$ is called soft strong neutrosophic N -group if and only if $F(x)$ is a strong neutrosophic sub N -group for all $x \in A$.

Example 28 Let $(\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle \cup \langle G_3 \cup I \rangle, *_1, *_2, *_3)$ be a neutrosophic 3 -group, where $\langle G_1 \cup I \rangle = \langle Z_2 \cup I \rangle = \{0, 1, I, 1+I\}$, a neutrosophic group under multiplication modulo 2. $\langle G_2 \cup I \rangle = \{O, 1, 2, 3, 4, I, 2I, 3I, 4I\}$, neutrosophic group under multiplication modulo 5 and $\langle G_3 \cup I \rangle = \{0, 1, 2, I, 2I\}$, a neutrosophic group under multiplication modulo 3. Let

$$P = \left\{ \left\{ \left\langle \frac{1}{2^n}, 2^n, \frac{1}{(2I)^n}, (2I)^n, I, 1 \right\rangle \right\}, \{1, 4, I, 4I\}, \{1, 2, I, 2I\} \right\},$$

and $X = \{Q \setminus \{0\}, \{1, 2, I, 2I\}, \{1, I\}\}$ are neutrosophic sub 3 -groups.

Then (F, A) is clearly soft strong neutrosophic 3 -group over

$$(\langle G \cup I \rangle = \langle G_1 \cup I \rangle \cup \langle G_2 \cup I \rangle \cup \langle G_3 \cup I \rangle, *_1, *_2, *_3),$$

where

$$F(x_1) = \left\{ \left\{ \left\langle \frac{1}{2^n}, 2^n, \frac{1}{(2I)^n}, (2I)^n, I, 1 \right\rangle \right\}, \{1, 4, I, 4I\}, \{1, I\} \right\},$$

$$F(x_2) = \{Q \setminus \{0\}, \{1, 2, I, 2I\}, \{1, I\}\}.$$

Theorem 44 Every soft strong neutrosophic soft N -group (F, A) is a soft neutrosophic N -group but the converse is not true.

Theorem 89 (F, A) over $(\langle G \cup I \rangle, *_1, ..., *_N)$ is soft strong neutrosophic N -group if $(\langle G \cup I \rangle, *_1, ..., *_N)$ is a strong neutrosophic N -group.

Proposition 45 Let (F, A) and (K, D) be two soft strong neutrosophic N -groups over $(\langle G \cup I \rangle, *_1, ..., *_N)$. Then

1) Their extended union $(F, A) \cup_{\varepsilon} (K, D)$ is not soft strong neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

2) Their extended intersection $(F, A) \cap_{\varepsilon} (K, D)$ is not soft strong neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

3) Their restricted union $(F, A) \cup_R (K, D)$ is not soft strong neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

4) Their restricted intersection $(F, A) \cap_R (K, D)$ is not soft strong neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

Proposition 46 Let (F, A) and (K, D) be two soft strong neutrosophic N -groups over $(\langle G \cup I \rangle, *_1, ..., *_N)$. Then

1) Their AND operation $(F, A) \wedge (K, D)$ is not soft strong neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

2) Their OR operation $(F, A) \vee (K, D)$ is not soft strong neutrosophic N -group over $(\langle G \cup I \rangle, *_1, ..., *_N)$.

Definition 58

Let (F, A) and (H, K) be two soft strong neutrosophic N -groups over $(\langle G \cup I \rangle, *_1, ..., *_N)$. Then (H, K) is called soft strong neutrosophic sub $x \in A$ -group of (F, A) written as $(H, K) \prec (F, A)$, if

1) $K \subset A$,
2) $K(x)$ is soft neutrosophic soft sub N-group of $F(x)$ for all $x \in A$.

Theorem 45 If $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ is a strong neutrosophic N-group. Then every soft neutrosophic sub N-group of (F, A) is soft strong neutosophic sub N-group.

Definition 59 Let $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ be a strong neutrosophic N-group. Then (F, A) over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ is called soft Lagrange strong neutrosophic N-group if $F(x)$ is a Lagrange neutrosophic sub N-group of $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ for all $x \in A$.

Theorem 46 Every soft Lagrange strong neutrosophic N-group (F, A) over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ is a soft neutrosophic soft N-group but the converse is not true.

Theorem 47 Every soft Lagrange strong neutrosophic N-group (F, A) over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ is a soft srtong neutrosophic N-group but the converse is not tue.

Theorem 48 If $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ is a Lagrange strong neutrosophic N-group, then (F, A) over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ is also soft Lagrange strong neutrosophic N-group.

Proposition 47 Let (F, A) and (K, D) be two soft Lagrange strong neutrosophic N-groups over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$. Then

1) Their extended union $(F, A) \cup_\varepsilon (K, D)$ is not soft Lagrange strong neutrosophic N-group over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$.
2) Their extended intersection $(F, A) \cap_\varepsilon (K, D)$ is not soft Lagrange strong neutrosophic N-group over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$.
3) Their restricted union $(F, A) \cup_R (K, D)$ is not soft Lagrange strong neutrosophic N-group over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$.
4) Their restricted intersection $(F, A) \cap_R (K, D)$ is not soft Lagrange strong neutrosophic N-group over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$.

Proposition 48 Let (F, A) and (K, D) be two soft Lagrange strong neutrosophic N-groups over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$. Then

1) Their AND operation $(F, A) \wedge (K, D)$ is not soft Lagrange strong neutrosophic N-group over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$.
2) Their OR operation $(F, A) \vee (K, D)$ is not soft Lagrange strong neutrosophic N-group over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$.

Definition 60 Let $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ be a strong neutrosophic N-group. Then (F, A) over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ is called soft weakly Lagrange strong neutrosophic soft N-group if atleast one $F(x)$ is a Lagrange neutrosophic sub N-group of $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ for some $x \in A$.

Theorem 49 Every soft weakly Lagrange strong neutrosophic N-group (F, A) over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ is a soft neutrosophic soft N-group but the converse is not true.

Theorem 50 Every soft weakly Lagrange strong neutrosophic N-group (F, A) over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$ is a soft strong neutrosophic N-group but the converse is not true.

Proposition 49 Let (F, A) and (K, D) be two soft weakly Lagrange strong neutrosophic N-groups over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$. Then

1) Their extended union $(F, A) \cup_\varepsilon (K, D)$ is not soft weakly Lagrange strong neutrosophic N-group over $(\langle G \cup I \rangle, *_1, \ldots, *_N)$.
2) Their extended intersection $(F, A) \cap_\varepsilon (K, D)$ is not soft weakly Lagrange strong neutrosophic N-group over

$(\langle G\cup I\rangle,*_1,...,*_N)$.

3) Their restricted union $(F,A)\cup_R(K,D)$ is not soft weakly Lagrange strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ is not soft weakly Lagrange strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Proposition 50 Let (F,A) and (K,D) be two soft weakly Lagrange strong neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not soft weakly Lagrange strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft weakly Lagrange strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Definition 61 Let $(\langle G\cup I\rangle,*_1,...,*_N)$ be a strong neutrosophic N -group. Then (F,A) over $(\langle G\cup I\rangle,*_1,...,*_N)$ is called soft Lagrange free strong neutrosophic N -group if $F(x)$ is not Lagrange neutrosophic sub N -group of $(\langle G\cup I\rangle,*_1,...,*_N)$ for all N .

Theorem 51 Every soft Lagrange free strong neutrosophic N -group (F,A) over $(\langle G\cup I\rangle,*_1,...,*_N)$ is a soft neutrosophic N -group but the converse is not true.

Theorem 52 Every soft Lagrange free strong neutrosophic N -group (F,A) over $(\langle G\cup I\rangle,*_1,...,*_N)$ is a soft strong neutrosophic N -group but the converse is not true.

Theorem 53 If $(\langle G\cup I\rangle,*_1,...,*_N)$ is a Lagrange free strong neutrosophic N -group, then (F,A) over $(\langle G\cup I\rangle,*_1,...,*_N)$ is also soft Lagrange free strong neutrosophic N -group.

Proposition 51 Let (F,A) and (K,D) be two soft Lagrange free strong neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ is not soft Lagrange free strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ is not soft Lagrange free strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ is not soft Lagrange free strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ is not soft Lagrange free strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Proposition 52 Let (F,A) and (K,D) be two soft Lagrange free strong neutrosophic N -groups over $(\langle G\cup I\rangle,*_1,...,*_N)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is not soft Lagrange free strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft Lagrange free strong neutrosophic N -group over $(\langle G\cup I\rangle,*_1,...,*_N)$.

Definition 62 Let N be a strong neutrosophic N -group. Then (F,A) over $(\langle G\cup I\rangle,*_1,...,*_N)$ is called sofyt normal strong neutrosophic N -group if $F(x)$ is normal neutrosophic sub N -group of $(\langle G\cup I\rangle,*_1,...,*_N)$ for all $x\in A$.

Theorem 54 Every soft normal strong neutrosophic N -group (F,A) over $(\langle G\cup I\rangle,*_1,...,*_N)$ is a soft neutrosophic N -group but the converse is not true.

Theorem 55 Every soft normal strong neutrosophic N -group (F,A) over $(\langle G\cup I\rangle,*_1,...,*_N)$ is a soft strong neutrosophic N -group but the converse is not true.

Proposition 53 Let (F,A) and (K,D) be two soft normal strong neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ is not soft normal strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ is soft normal strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ is not soft normal strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ is soft normal strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.

Proposition 54 Let (F,A) and (K,D) be two soft normal strong neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is soft normal strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft normal strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.

Definition 63 Let $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ be a strong neutrosophic N -group. Then (F,A) over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is called soft conjugate strong neutrosophic N -group if $F(x)$ is conjugate neutrosophic sub N -group of $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ for all $x\in A$.

Theorem 56 Every soft conjugate strong neutrosophic N -group (F,A) over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is a soft neutrosophic N -group but the converse is not true.

Theorem 57 Every soft conjugate strong neutrosophic N -group (F,A) over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$ is a soft strong neutrosophic N -group but the converse is not true.

Proposition 55 Let (F,A) and (K,D) be two soft conjugate strong neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$. Then

1) Their extended union $(F,A)\cup_\varepsilon(K,D)$ is not soft conjugate strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
2) Their extended intersection $(F,A)\cap_\varepsilon(K,D)$ is soft conjugate strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
3) Their restricted union $(F,A)\cup_R(K,D)$ is not soft conjugate strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
4) Their restricted intersection $(F,A)\cap_R(K,D)$ is soft conjugate strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.

Proposition 56 Let (F,A) and (K,D) be two soft conjugate strong neutrosophic N -groups over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$. Then

1) Their AND operation $(F,A)\wedge(K,D)$ is soft conjugate strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.
2) Their OR operation $(F,A)\vee(K,D)$ is not soft conjugate strong neutrosophic N -group over $\left(\langle G\cup I\rangle,*_1,...,*_N\right)$.

Conclusion

This paper is about the generalization of soft neutrosophic groups. We have extended the concept of soft neutrosophic group and soft neutrosophic subgroup to soft neutrosophic bigroup and soft neutrosophic N-group. The notions of soft normal neutrosophic bigroup, soft normal neutrosophic N-group, soft conjugate neutrosophic bigroup and soft conjugate neutrosophic N-group are defined. We have given various examples and important theorems to illustrate the aspect of soft neutrosophic bigroup and soft neutrosophic N-group.

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The Characteristic Function of a Neutrosophic Set

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Abstract. The purpose of this paper is to introduce and study the characteristic function of a neutrosophic set. After given the fundamental definitions of neutrosophic set operations generated by the characteristic function of a neutrosophic set (Ng for short), we obtain several properties, and discussed the relationship between neutrosophic sets generated by Ng and others. Finally, we introduce the neutrosophic topological spaces generated by Ng. Possible application to GIS topology rules are touched upon.

Keywords: Neutrosophic Set; Neutrosophic Topology; Characteristic Function.

1 Introduction

Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts, such as a neutrosophic set theory. After the introduction of the neutrosophic set concepts in [2-13]. In this paper we introduce definitions of neutrosophic sets by characteristic function. After given the fundamental definitions of neutrosophic set operations by Ng, we obtain several properties, and discussed the relationship between neutrosophic sets and others. Added to, we introduce the neutrosophic topological spaces generated by Ng.

2 Terminologies

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [7- 9], Hanafy, Salama et al. [2- 13] and Demirci in [1].

3 Neutrosophic Sets generated by Ng

We shall now consider some possible definitions for basic concepts of the neutrosophic sets generated by Ng and its operations.

3.1 Definition

Let X is a non-empty fixed set. A neutrosophic set (NS for short) A is an object having the form $A=\langle x,\mu_A(x),\sigma_A(x),\nu_A(x)\rangle$ where $\mu_A(x),\sigma_A(x)$ and $\gamma_A(x)$ which represent the degree of member ship function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-member ship (namely $\gamma_A(x)$) respectively of each element $x\in X$ to the set A .and let $g_A : X\times[0,1]\to[0,1]=I$ be reality function, then $Ng_A(\lambda)=Ng_A(\langle x,\lambda_1,\lambda_2,\lambda_3\rangle)$ is said to be the characteristic function of a neutrosophic set on X if

$$Ng_A(\lambda)=\begin{cases}1 & \text{if } \mu_A(x)=\lambda_1,\sigma_{A(x)}=\lambda_2,\nu_A(x)=\lambda_3\\ 0 & \text{otherwise}\end{cases}$$

Where $\lambda=(\langle x,\lambda_1,\lambda_2,\lambda_3\rangle)$. Then the object $G(A)=\langle x,\mu_{G(A)}(x),\sigma_{G(A)}(x),\nu_{G(A)}(x)\rangle$ is a neutrosophic set generated by Ng where

$$\mu_{G(A)}=\sup_{\lambda_1}\{Ng_A(\lambda)\wedge\lambda\}$$

$$\sigma_{G(A)}=\sup_{\lambda_2}\{Ng_A(\lambda)\wedge\lambda\}$$

$$\nu_{G(A)}=\sup_{\lambda_3}\{Ng_A(\lambda)\wedge\lambda\}$$

3.1 Proposition

1) $A\subseteq^{Ng} B\Leftrightarrow G(A)\subseteq G(B)$.

2) $A =^{Ng} B \Leftrightarrow G(A) = G(B)$

3.2 Definition

Let A be neutrosophic set of X. Then the neutrosophic complement of A generated by Ng denoted by A^{Ngc} iff $[G(A)]^c$ may be defined as the following:

(Ng^{c1}) $\langle x, \mu^c{}_A(x), \sigma^c{}_A(x), \nu^c{}_A(x)\rangle$

(Ng^{c2}) $\langle x, \nu_A(x), \sigma\ _A(x), \mu_A(x)\rangle$

(Ng^{c3}) $\langle x, \nu_A(x), \sigma^c{}_A(x), \mu_A(x)\rangle$

3.1 Example. Let $X = \{x\}$, $A = \langle x, 0.5, 0.7, 0.6\rangle$, $Ng_A = 1$, $Ng_A = 0$. Then $G(A) = (\langle x, 0.5, 0.7, 0.6\rangle)$

Since our main purpose is to construct the tools for developing neutrosophic set and neutrosophic topology, we must introduce the $G(0_N)$ and $G(1_N)$ as follows $G(0_N)$ may be defined as:

i) $G(0_N) = \langle x, 0, 0, 1\rangle$

ii) $G(0_N) = \langle x, 0, 1, 1\rangle$

iii) $G(0_N) = \langle x, 0, 1, 0\rangle$

iv) $G(0_N) = \langle x, 0, 0, 0\rangle$

$G(1_N)$ may be defined as:

i) $G(1_N) = \langle x, 1, 0, 0\rangle$

ii) $G(1_N) = \langle x, 1, 0, 1\rangle$

iii) $G(1_N) = \langle x, 1, 1, 0\rangle$

iv) $G(1_N) = \langle x, 1, 1, 1\rangle$

We will define the following operations intersection and union for neutrosophic sets generated by Ng denoted by $\cap^{Ng}$ and $\cup^{Ng}$ respectively.

3.3 Definition. Let two neutrosophic sets $A = \langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle$ and $B = \langle x, \mu_B(x), \sigma_B(x), \nu_B(x)\rangle$ on X, and

$G(A) = \langle x, \mu_{G(A)}(x), \sigma_{G(A)}(x), \nu_{G(A)}(x)\rangle$,

$G(B) = \langle x, \mu_{G(B)}(x), \sigma_{G(B)}(x), \nu_{G(B)}(x)\rangle$. Then $A \cap^{Ng} B$ may be defined as three types:

i) type I: $G(A \cap B) =$

$\langle \mu_{G(A)}(x) \wedge \mu_{G(B)}, \sigma_{G(A)}(x) \wedge \sigma_{G(B)}(x), \nu_{G(A)}(x) \vee \nu_{G(B)}(x)\rangle$

ii) Type II:

$G(A \cap B) =$

$\langle \mu_{G(A)}(x) \wedge \mu_{G(B)}, \sigma_{G(A)}(x) \vee \sigma_{G(B)}(x), \nu_{G(A)}(x) \vee \nu_{G(B)}(x)\rangle$.

ii) Type III:

$G(A \cap B) =$

$\langle \mu_{G(A)}(x) \times \mu_{G(B)}, \sigma_{G(A)}(x) \times \sigma_{G(B)}(x), \nu_{G(A)}(x) \times \nu_{G(B)}(x)\rangle$

$A \cup^{Ng} B$ may be defined as two types:

Type I :

$G(A \cup B) =$

$\langle \mu_{G(A)}(x) \vee \mu_{G(B)}, \sigma_{G(A)}(x) \wedge \sigma_{G(B)}(x), \nu_{G(A)}(x) \wedge \nu_{G(B)}(x)\rangle$ ii) Type II:

$G(A \cup B) =$

$\langle \mu_{G(A)}(x) \vee \mu_{G(B)}, \sigma_{G(A)}(x) \vee \sigma_{G(B)}(x), \nu_{G(A)}(x) \wedge \nu_{G(B)}(x)\rangle$

. **3.4 Definition**

Let a neutrosophic set $A = \langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle$ and $G(A) = \langle x, \mu_{G(A)}(x), \sigma_{G(A)}(x), \nu_{G(A)}(x)\rangle$. Then

(1) $[\ \ ]^{Ng} A = \langle x : \mu_{G(A)}(x), \sigma_{G(A)}(x), 1 - \nu_{G(A)}(x)\rangle$

(2) $<>^{Ng} A =$
$\langle x : 1 - \mu_{G(A)}(x), \sigma_{G(A)}(x), \nu_{G(A)}(x)\rangle$

3.2 Proposition

For all two neutrosophic sets A and B on X generated by Ng, then the following are true

1) $(A \cap B)^{cNg} = A^{cNg} \cup B^{cNg}$.

2) $(A \cup B)^{cNg} = A^{cNg} \cap B^{cNg}$.

We can easily generalize the operations of intersection and union in definition 3.2 to arbitrary family of neutrosophic subsets by generated by Ng as follows:

3.3 Proposition.

Let $\{A_j : j \in J\}$ be arbitrary family of neutrosophic subsets in X generated by Ng, then

a) $\cap^{Ng} A_j$ may be defined as :

1) Type I :
$G(\cap A_j) = \langle \wedge \mu_{G(A_j)}(x), \wedge\sigma_{G(A_j)}(x), \vee\nu_{G(A_j)}(x)\rangle$,

2) Type II:
$G(\cap A_j) = \langle \wedge \mu_{G(A_j)}(x), \vee\sigma_{G(A_j)}(x), \vee\nu_{G(A_j)}(x)\rangle$,

b) $\cup^{Ng} A_j$ may be defined as :

1) $G(\cup A_j) = \langle \vee \mu_{G(A_j)}(x), \wedge\sigma_{G(A_j)}(x), \wedge\nu_{G(A_j)}(x)\rangle$ or

2) $G(\cup A_j) = \left\langle \vee \mu_{G(A_j)}(x), \vee \sigma_{G(A_j)}(x), \wedge \nu_{G(A_j)}(x) \right\rangle$.

3.4 Definition

Let f. X → Y be a mapping .

(i) The image of a neutrosophic set A generated by Ng on X under f is a neutrosophic set B on Y generated by Ng , denoted by f (A) whose reality function g_B : Y x I → I=[0, 1] satisfies the property

$$\mu_{G(B)} = \sup_{\lambda_1} \{Ng_A(\lambda) \wedge \lambda\}$$

$$\sigma_{G(B)} = \sup_{\lambda_2} \{Ng_A(\lambda) \wedge \lambda\}$$

$$\nu_{G(B)} = \sup_{\lambda_3} \{Ng_A(\lambda) \wedge \lambda\}$$

(ii) The preimage of a neutrosophic set B on Y generated by Ng under f is a neutrosophic set A on X generated by Ng, denoted by f^{-1} (B) , whose reality function g_A : X x [0, 1] →[0, 1] satisfies the property G(A) = G (B) o f

3.4 Proposition

Let $\{A_j : j \in J\}$ and $\{B_j : j \in J\}$ be families of neutrosophic sets on X and Y generated by Ng, respectively. Then for a function f: X → Y, the following properties hold:

(i) If $A_j \subseteq^{Ng} A_k$;i , j∈J, then $f(A_j) \subseteq^{Ng} f(A_k)$

(ii) If $B_j \subseteq^{Ng} B_k$, for j , K ∈ J, then

$f^{-1}(B_j) \subseteq^{Ng} f^{-1}(B_K)$

(iii) $f^{-1}(\underset{jEJ}{\cup}^{Ng} B_j) =^{Ng} \underset{jEJ}{\cup}^{Ng} f^{-1}(B_j)$

3.5 Proposition

Let A and B be neutrosophic sets on X and Y generated by Ng, respectively. Then, for a mappings f : X → Y , we have :

(i) $A \subseteq^{Ng} f^{-1}(f(A))$ (if f is injective the equality holds) .

(ii) $f(f^{-1}(B)) \subseteq^{Ng} B$ (if f is surjective the equality holds) .

(iii) $[f^{-1}(B)]^{Ngc} \subseteq^{Ng} f^{-1}(B^{N\,gc})$.

3.5 Definition . Let X be a nonempty set, Ψ a family of neutrosophic sets generated by Ng and let us use the notation

$$G(\Psi) = \{ G(A) : A \in \Psi \}.$$

If (X , G (Ψ)= $N\tau$) is a neutrosophic topological space on X is Salama's sense [3] , then we say that Ψ is a neutrosophic topology on X generated by Ng and the pair (X , Ψ) is said to be a neutrosophic topological space generated by Ng (ngts , for short). The elements in Ψ are called genuine neutrosophic open sets. also , we define the family

$$G(\Psi^c) = \{ 1 - G(A) : A \in \Psi \}.$$

3.6 Definition

Let (X , Ψ) be a ngts . A neutrosophic set C in X generated by Ng is said to be a neutrosophic closed set generated by Ng , if 1- G(C) ∈G (Ψ) = $N\tau$.

3.7 Definition

Let (X , Ψ) be a ngts and A a neutrosophic set on X generated by Ng . Then the neutrosophic interior of A generated by Ng , denoted by, ngintA, is a set characterized by G(intA) = $\underset{G(\Psi)}{\text{int}}$ G(A) , where $\underset{G(\Psi)}{\text{int}}$ denotes the interior operation in neutrosophic topological spaces generated by Ng .Similarly, the neutrosophic closure of A generated by Ng , denoted by ngclA , is a neutrosophic set characterized by G(ngclA)= $\underset{G(\psi)}{cl}$ G(A) , where $\underset{G(\psi)}{cl}$ denotes the closure operation in neutrosophic topological spaces generated by Ng .

The neutrosophic interior gnint(A) and the genuine neutrosophic closure gnclA generated by Ng can be characterized by :

gnintA $=^{Ng} \cup^{Ng}$ { U : U ∈ Ψ and U $\subseteq^{Ng}$ A }

gnclA $=^{Ng} \cap^{Ng}$ { C : C is neutrosophic closed generated by Ng and A $\subseteq^{Ng}$ C }

Since : G (gnint A) = ∪ { G (U) : G (U) ∈ G (Ψ) , G (U) ⊆ G (A) }

G (gncl A) = ∩ { G (C) : G (C) ∈ G (Ψ^c) , G (A) ⊆ G(C) }.

3.6 Proposition . For any neutrosophic set A generated by Ng on a NTS (X , Ψ) , we have

(i) $\mathrm{cl}\, A^{Ngc} =^{Ng} (\mathrm{int}\, A)^{Ngc}$

(ii) $\mathrm{Int}\, A^{Ngc} =^{Ng} (\mathrm{cl}\, A)^{Ngc}$

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A Note on Square Neutrosophic Fuzzy Matrices

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Abstract. In this article, we shall define the addition and multiplication of two neutrosophic fuzzy matrices. Thereafter, some properties of addition and multiplication of these matrices are also put forward.

Keywords: Neutrosophic fuzzy matrice, Neutrosophic Set.

1 Introduction

Neutrosophic sets theory was proposed by Florentin Smarandache [1] in 1999, where each element had three associated defining functions, namely the membership function (T), the non-membership (F) function and the indeterminacy function (I) defined on the universe of discourse X, the three functions are completely independent. The theory has been found extensive application in various field [2,3,4,5,6,7,8,9,10,11] for dealing with indeterminate and inconsistent information in real world.Neutrosophic set is a part of neutrosophy which studied the origin, nature and scope of neutralities, as well as their interactions with ideational spectra. The neutrosophic set generalized the concept of classical fuzzy set [12, 13], interval-valued fuzzy set, intuitionistic fuzzy set [14, 15], and so on.

Also as we know, matrices play an important role in science and technology. However, the classical matrix theory sometimes fails to solve the problems involving uncertainties, occurring in an imprecise environment. In [17] Thomason, introduced the fuzzy matrices to represent fuzzy relation in a system based on fuzzy set theory and discussed about the convergence of powers of fuzzy matrix. In 2004, W. B. V. Kandasamy and F. Smarandache introduced fuzzy relational maps and neutrosophic relational maps.

Our aim ,In this paper is to propose another type of fuzzy neutrosophic matrices ,called "square neutrosophic fuzzy matrices", whose entries is of the form a+Ib (neutrosophic number) , where a,b are the elements of [0,1] and I is an indeterminate such that I^n=I, n being a positive integer. In this study we will focus on square neutrosophic fuzzy matrices.The paper unfolds as follows. The next section briefly introduces some definitions related to neutrosophic set, neutrosophic matrices, Fuzzy integral neutrosophic matrices and fuzzy matrix. Section 3 presents a new type of fuzzy neutrosophic matrices and investigated some properties such as addition and multiplication. Conclusions appear in the last section.

2 Preliminaries

In this section we recall some concept such as , neutrosophic set, neutrosophic matrices and fuzzy neutrosophic matrices proposed by W. B. V. Kandasamy and F. Smarandache in their books [16] , and also the concept of fuzzy matrix .

Definition 2.1 (Neutrosophic Sets).[1]

Let U be an universe of discourse then the neutrosophic set A is an object having the form

A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>,x $\in$ U}, where the functions T, I, F : U→ $]^-0, 1^+[$ define respectively the degree of membership (or Truth) , the degree of indeterminacy, and the degree of non-membership (or

Falsehood) of the element x ∈ U to the set A with the condition.

$$^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}.$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$. So instead of $]^{-}0, 1^{+}[$ we need to take the interval [0, 1] for technical applications, because $]^{-}0, 1^{+}[$will be difficult to apply in the real applications such as in scientific and engineering problems.

Definition 2.2 (Neutrosophic matrix) [16].

Let $\boldsymbol{M}_{m\,x\,n}= \{(\boldsymbol{a}_{ij}) / \boldsymbol{a}_{ij} \in K(I)\}$, where K (I), is a neutrosophic field. We call $\boldsymbol{M}_{m\,x\,n}$ to be the neutrosophic matrix.

Example 1: Let Q(I) = ⟨Q ∪ I ⟩be the neutrosophic field

$$\boldsymbol{M}_{3\,x\,3}=\begin{pmatrix} 0 & 1 & 0 \\ -2 & 4I & 2 \\ 3I & 1 & -I \end{pmatrix}$$

$M_{3\,x\,3}$ denotes the neutrosophic matrix, with entries from rationals and the indeterminacy.

Definition 2.3 (Fuzzy integral neutrosophic matrices)

Let N = [0, 1] ∪ I where I is the indeterminacy. The m ×n matrices $\mathbf{M}_{m\,x\,n}= \{(\boldsymbol{a}_{ij}) / \boldsymbol{a}_{ij} \in [0, 1] \cup I\}$ is called the fuzzy integral neutrosophic matrices. Clearly the class of m ×n matrices is contained in the class of fuzzy integral neutrosophic matrices.

An integral fuzzy neutrosophic row vector is a **1 × n** integral fuzzy neutrosophic matrix, Similarly an integral fuzzy neutrosophic column vector is a m ×1 integral fuzzy neutrosophic matrix.

Example 2 : Let $\boldsymbol{A}_{3\,x\,3}=\begin{pmatrix} 0 & 1 & 0.3 \\ 0.9 & I & 0.2 \\ I & 1 & I \end{pmatrix}$

A is a 3 ×3 integral fuzzy neutrosophic matrix.

Definition 2.5 (Fuzzy neutrosophic matrix) [16]

Let $\mathbf{N_s}$= [0, 1] ∪ nI / n ∈ (0, 1]}; we call the set $\mathbf{N_s}$ to be the fuzzy neutrosophic set. Let $\mathbf{N_s}$ be the fuzzy neutrosophic set. $\mathbf{M}_{m\,x\,n}= \{(\boldsymbol{a}_{ij}) / \boldsymbol{a}_{ij} \in \mathbf{N_s}\}$ we call the matrices with entries from $\mathbf{N_s}$ to be the fuzzy neutrosophic matrices.

Example 3: Let $\mathbf{N_s}$= [0,1] ∪{nI/ n ∈ (0,1]} be the set

$$P = \begin{pmatrix} 0 & 0.2I & I \\ I & 0.01I & 0 \\ 0.31I & 0.53I & 0.1 \end{pmatrix}$$

is a 3 ×3 fuzzy neutrosophic matrix

Definition 2.6 (Fuzzy matrix) [17]

A fuzzy matrix is a matrix which has its elements from the interval [0, 1], called the unit fuzzy interval. A m x n fuzzy matrix for which m = n (i.e the number of rows is equal to the number of columns) and whose elements belong to the unit interval [0, 1] is called a fuzzy square matrix of order n. A fuzzy square matrix of order two is expressed in the following way

$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where the entries a,b,c,d all belongs to the interval [0,1].

3 Some Properties of Square Neutrosophic Fuzzy Matrices

In this section ,we define a new type of fuzzy neutrosophic set and define some operations on this neutrosophic fuzzy matrice.

3.1 .Definition (Neutrosophic Fuzzy Matrices)

Let A be a neutrosophic fuzzy matrices, whose entries is of the form a+Ib (neutrosophic number) , where a,b are the elements of [0,1] and I is an indeterminate such that I^n=I, n being a positive integer.

$$A=\begin{pmatrix} a_1 + I\,b_1 & a_2 + I\,b_2 \\ a_3 + I\,b_3 & a_4 + I\,b_4 \end{pmatrix}$$

3.2.Arithmetic with Square Neutrosophic Fuzzy Matrices

In this section we shall define the addition and multiplication of neutrosophic fuzzy matrices along with some properties associated with such matrices.

3.2.1. Addition Operation of two Neutrosophic Fuzzy Matrices

Let us consider two neutrosophic fuzzy matrices as

A=$\begin{pmatrix} a_1 + I\, b_1 & a_2 + I\, b_2 \\ a_3 + I\, b_3 & a_4 + I\, b_4 \end{pmatrix}$and B=$\begin{pmatrix} c_1 + I\, d_1 & c_2 + I\, d_2 \\ c_3 + I\, d_3 & c_4 + I\, d_4 \end{pmatrix}$

Then we would like to define the addition of these two matrices as

$$A + B = [C_{ij}]$$

Where

$C_{11} = \max(a_1, c_1) + I\max(b_1, d_1)$

$C_{12} = \max(a_2, c_2) + I\max(b_2, d_2)$

$C_{21} = \max(a_3, c_3) + I\max(b_3, d_3)$

$\mathbf{C_{22} = \max(a_4, c_4) + I\max(b_4, d_4)}$

It is noted that the matrices defined by our way is reduced to fuzzy neutrosophic matrix when a =

Properties 1

The following properties can be found to hold in cases of neutrosophic fuzzy matrix multiplication

(i) A+B = B+A

(ii) (A+B)+C = A+(B+C)

3.2.2 Multiplication Operation of Neutrosophic Fuzzy Matrices

Let us consider two neutrosophic fuzzy matrices as $A = [a_{ij} + Ib_{ij}]$ and $B = [c_{ij} + Id_{ij}]$. Then we shall define the multiplication of these two neutrosophic fuzzy matrices as

$AB = [\max\min(a_{ij}, c_{ji}) + I\max\min(b_{ij}, d_{ji})]$. It can be defined in the following way:

If the above mentioned neutrosophic fuzzy matrices are considered then we can define the product of the above matrices as

A B = $[D_{ij}]$, where

$D_{11} = [\max\min\{(a_1, c_1), (a_2, c_3)\} + I\max\min\{(b_1, d_1), (b_2, d_3)\}]$

$D_{12} = [\max\min\{(a_1, c_2), (a_2, c_4)\} + I\max\min\{(b_1, d_2), (b_2, d_4)\}]$

$D_{21} = [\max\min\{(a_3, c_1), (a_4, c_3)\} + I\max\min\{(b_3, d_1), (b_4, d_3)\}]$

$D_{22} = [\max\min\{(a_3, c_2), (a_4, c_4)\} + I\max\min\{(b_3, d_2), (b_4, d_4)\}]$

It is important to mention here that if the multiplication of two neutrosophic fuzzy matrices is defined in the above way then the following properties can be observed to hold:

Properties

(i) $AB \neq BA$

(ii) A(B+C)=AB+AC

2.4.1 Numerical Example

Let us consider three neutrosophic fuzzy matrices as

A=$\begin{pmatrix} 0.1 + I\, 0.3 & 0.4 + I\, 0.1 \\ 0.2 + I\, 0.4 & 0.1 + I\, 0.7 \end{pmatrix}$

B=$\begin{pmatrix} 0.2 + I\, 0.3 & 0.5 + I\, 0.4 \\ 0.3 + I\, 0.8 & 0.9 + I\, 0.1 \end{pmatrix}$

C= $\begin{pmatrix} 0.4 + I\, 0.3 & 0.4 + I\, 0.3 \\ 0.2 + I\, 0.7 & 0.2 + I\, 0.4 \end{pmatrix}$

B+C = $\begin{pmatrix} 0.4 + I\, 0.6 & 0.5 + I\, 0.4 \\ 0.6 + I\, 0.8 & 0.9 + I\, 0.2 \end{pmatrix}$

A (B +C)= $\begin{pmatrix} 0.1 + I\, 0.3 & 0.4 + I\, 0.1 \\ 0.2 + I\, 0.4 & 0.1 + I\, 0.7 \end{pmatrix} \begin{pmatrix} 0.4 + I\, 0.6 & 0.5 + I\, 0.4 \\ 0.6 + I\, 0.8 & 0.9 + I\, 0.2 \end{pmatrix}$

Let us take

A (B +C) = $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where

A_{11} =max{min(0.1, 0.4), min(0.4,0.6)}+I max{min(0.3, 0.6), min(0.1, 0.8)}

= max(0.1, 0.4) + I max (0.3, 0.1)

= 0.4 + I 0.3

A_{12} = max{min(0.1, 0.5), min(0.4,0.9)}+I max{min(0.3, 0.4), min(0.1, 0.2)}

= max(0.1, 0.4) + I max (0.3, 0.1)

= 0.4 + I 0.3

A_{21} = max{min(0.2, 0.4), min(0.1,0.6)}+I max{min(0.4, 0.6), min(0.7, 0.8)}

= max(0.2, 0.1) + I max (0.4, 0.7)

= 0.2 + I 0.7

A_{22} = max{min(0.2, 0.5), min(0.1,0.9)}+I max{min(0.4, 0.4), min(0.7, 0.2)}

= max(0.2, 0.1) + I max (0.4, 0.2)

= 0.2 + I 0.4

Therefore we have

A (B + C)= $\begin{pmatrix} 0.4 + I\,0.3 & 0.4 + I\,0.3 \\ 0.2 + I\,0.7 & 0.2 + I\,0.4 \end{pmatrix}$

Now we shall see what happens to AB+BC

Then let us calculate AB

A B = $\begin{pmatrix} 0.1 + I\,0.3 & 0.4 + I\,0.1 \\ 0.2 + I\,0.4 & 0.1 + I\,0.7 \end{pmatrix} \begin{pmatrix} 0.2 + I\,0.3 & 0.5 + I\,0.4 \\ 0.3 + I\,0.8 & 0.9 + I\,0.1 \end{pmatrix}$

Let is now consider

A B= $\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$, where

C_{11} = max{min(0.1, 0.2), min(0.4,0.3)}+I max{min(0.3, 0.3), min(0.1, 0.8)}

= max(0.1, 0.3) + I max (0.3, 0.1)

= 0.3 + I 0.3

C_{12} = max{min(0.1, 0.5), min(0.4,0.9)}+I max{min(0.3, 0.4), min(0.1, 0.1)}

= max(0.1, 0.4) + I max (0.3, 0.1)

= 0.4 + I 0.3

C_{21} = max{min(0.2, 0.2), min(0.1,0.3)}+I max{min(0.4, 0.3), min(0.7, 0.8)}

= max(0.2, 0.1) + I max (0.3, 0.7)

= 0.2 + I 0.7

C_{22} = max{min(0.2, 0.5), min(0.1,0.9)}+I max{min(0.4, 0.4), min(0.7, 0.1)}

= max(0.2, 0.1) + I max (0.4, 0.1)

= 0.2 + I 0.4

Let us consider A C= $\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$, where

E_{11} = max{min(0.1, 0.4), min(0.4,0.6)}+I max{min(0.3, 0.6), min(0.1, 0.2)}

= max(0.1, 0.4) + I max (0.3, 0.1)

= 0.4 + I 0.3

E_{12} = max{min(0.1, 0.5), min(0.4,0.3)}+I max{min(0.3, 0.3), min(0.1, 0.2)}

= max(0.1, 0.3) + I max (0.3, 0.1)

= 0.3 + I 0.3

E_{21} = max{min(0.2, 0.4), min(0.1,0.6)}+I max{min(0.4, 0.6), min(0.7, 0.2)}

= max(0.2, 0.1) + I max (0.4, 0.2)

= 0.2 + I 0.2

E_{22} = max{min(0.2, 0.5), min(0.1,0.3)}+I max{min(0.4, 0.3), min(0.7, 0.2)}

= max(0.2, 0.1) + I max (0.3, 0.2)

= 0.2 + I 0.3

Thus we have

$C_{11} + E_{11}$= (0.3 + I 0.3) + (0.4 + I 0.3)

= 0.4 + I 0.3

$C_{12} + E_{12}$= (0.4+ I 0.3) + (0.3 + I 0.3)

= 0.4 + I 0.3

$C_{21} + E_{21}$= (0.2+ I 0.7) + (0.2 + I 0.2)

= 0.2+ I 0.7

$C_{22} + E_{22}$= (0.2+ I 0.4) + (0.2 + I 0.3)

=0.2+ I 0.4

Thus, we get, A B + A C= $\begin{pmatrix} 0.4 + I\,0.3 & 0.3 + I\,0.3 \\ 0.2 + I\,0.7 & 0.2 + I\,0.4 \end{pmatrix}$

From the above results, it can be established that

A (B+C) = AB + AC

4. Conclusions

According the newly defined addition and multiplication operation of neutrosophic fuzzy matrices, it can be seen that some of the properties of arithmetic operation of these matrices are analogous to the classical matrices. Further some future works are necessary to deal with some more properties and operations of such kind of matrices.

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Introduction to Develop Some Software Programs for Dealing with Neutrosophic Sets

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A.A. Salama, Haitham A. El-Ghareeb, Ayman M. Manie, Florentin Smarandache (2014). Introduction to Develop Some Software Programs for Dealing with Neutrosophic Sets. *Neutrosophic Sets and Systems* 3, 51-52

Abstract. In this paper, we have developed an Excel package to be utilized for calculating neutrosophic data and analyze them. The use of object oriented programming techniques and concepts as they may apply to the design and development a new framework to implement neutrosophic data operations, the c# programming language, NET Framework and Microsoft Visual Studio are used to implement the neutrosophic classes. We have used Excel as it is a powerful tool that is widely accepted and used for statistical analysis. Figure 1 shows Class Diagram of the implemented package. Figure 2 presents a working example of the package interface calculating the complement. Our implemented Neutrosophic package can calculate Intersection, Union, and Complement of the nuetrosophic set. Figure 3 presents our neutrosphic package capability to draw figures of presented neutrosphic set. Figure 4 presents charting of Union operation calculation, and figure 5 Intersection Operation. nuetrosophic set are characterized by its efficiency as it takes into consideration the three data items: True, Intermediate, and False.

Keywords: Neutrosophic Data; Software Programs.

1 Introduction

The fundamental concepts of neutrosophic set, introduced by Smarandache in [8, 9] and Salama at el. in [1, 2, 3, 4, 5, 6, 7], provides a natural foundation for treating mathematically the neutrosophic phenomena which exist pervasively in our real world and for building new branches of neutrosophic mathematics. In this paper, we have developed an Excel package to be utilized for calculating neutrosophic data and analyze them. We have used Excel as it is a powerful tool that is widely accepted and used for statistical analysis. In this paper, we have developed an Excel package to be utilized for calculating neutrosophic data and analyze them. The use of object oriented programming techniques and concepts as they may apply to the design and development a new framework to implement neutrosophic data operations, the c# programming language, NET Framework and Microsoft Visual Studio are used to implement the neutrosophic classes.

2 Related Works

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [8, 9], and Salama at el. [1, 2, 3, 4, 5, 6, 7]. The c# programming language, NET Framework and Microsoft Visual Studio are used to implement the neutrosophic classes.

3 Proposed frameworks

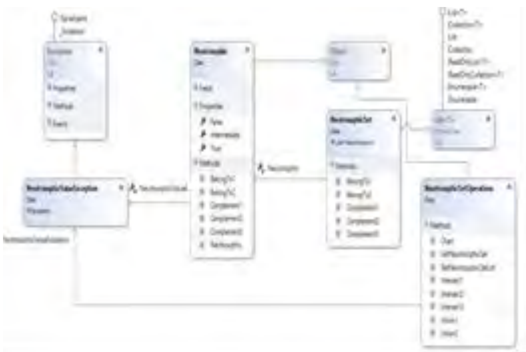

Figure 1: Neutrosophic Package Class Diagram.

We introduce the neutrosophic package class diagram

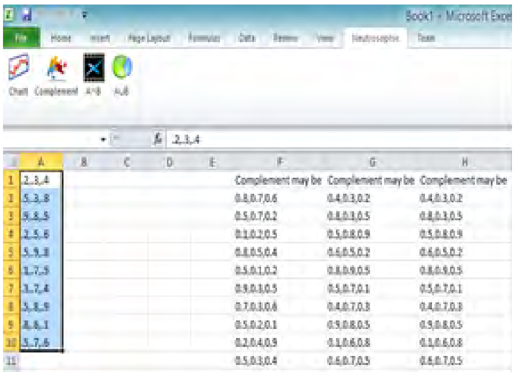

Figure 2: Neutrosophic Package Interface and Calculating Complement.

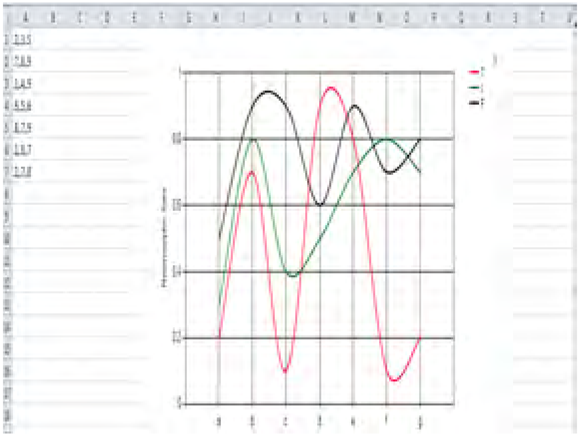

Figure 3: Neutrosophic Chart

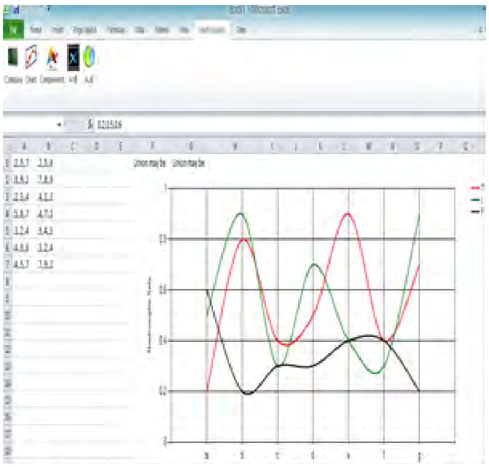

Figure 4: Neutrosophic Packege Union Chart

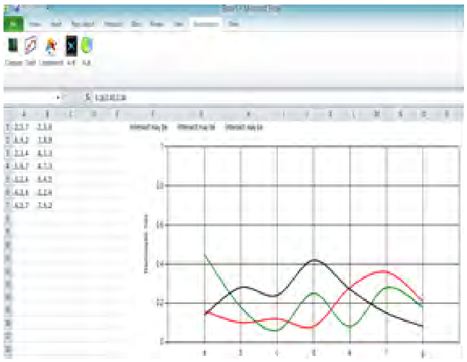

Figure 5: Neutrosophic Packege Intersection Chart

4 Conclusions and Future Work

In future studies we will develop some software programs to deal with the statistical analysis of the neutrosophic data.

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Soft Neutrosophic Ring and Soft Neutrosophic Field

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Mumtaz Ali, Florentin Smarandache, Muhammad Shabir, Munazza Naz (2014). Soft Neutrosophic Ring and Soft Neutrosophic Field. *Neutrosophic Sets and Systems* 3, 53-59

Abstract. In this paper we extend the theory of neutrosophic rings and neutrosophic fields to soft sets and construct soft neutrosophic rings and soft neutrosophic fields. We also extend neutrosophic ideal theory to form soft neutrosophic ideal over a neutrosophic ring and soft neutrosophic ideal of a soft neutrosophic ring . We have given many examples to illustrate the theory of soft neutrosophic rings and soft neutrosophic fields and display many properties of of these. At the end of this paper we gave soft neutrosophic ring homomorphism.

Keywords: Neutrosophic ring, neutrosophic field,neutrosophic ring homomorphism, soft neutrosophic

1 Introduction

Neutrosophy is a new branch of philosophy which studies the origin and features of neutralities in the nature. Florentin Smarandache in 1980 firstly introduced the concept of neutrosophic logic where each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F so that this neutrosophic logic is called an extension of fuzzy logic. In fact neutrosophic set is the generalization of classical sets, conventional fuzzy set, intuitionistic fuzzy set and interval valued fuzzy set. This mathematical tool is used to handle problems like imprecise, indeterminacy and inconsistent data etc. By utilizing neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache dig out neutrosophic algebraic structures. Some of them are neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, and neutrosophic bigroupoids and so on.

Molodtsov in $[8]$ laid down the stone foundation of a richer structure called soft set theory which is free from t**he** parameterization inadequacy, syndrome of fuzzy se theory, rough set theory, probability theory and so on. In many areas it has been successfully applied such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, and probability. Recently soft set theory has attained much attention since its appearance and the work based on several operations of soft sets introduced in $[2,9,10]$. Some more exciting properties and algebra may be found in $[1]$. Feng et al. introduced the soft semirings $[5]$. By means of level soft sets an adjustable approach to fuzzy soft sets based decision making can be seen in $[6]$. Some other new concept combined with fuzzy sets and rough sets was presented in $[7,8]$. AygÄunoglu et al. introduced the Fuzzy soft groups $[4]$.

Firstly, fundamental and basic concepts are given for neutrosophic rings neutrosohic fields and soft rings. In the next section we presents the newly defined notions and results in soft neutrosophic rings and neutrosophic fields. Various types of soft neutrosophic ideals of rings are defined and elaborated with the help of examples. Furthermore, the homomorphisms of soft neutrosophic rings are discussed at the end.

2 **Fundamental Concepts**

Neutrosophic Rings and Neutrosophic Fields

Definition 1. Let R be any ring. The neutrosophic ring $\langle R \cup I \rangle$ is also a ring generated by R and I under the operations of R. I is called the neutrosophic element with the property $I^2 = I$. For an integer n, $n + I$ and nI are neutrosophic elements and $0.I = 0$. I^{-1}, the inverse of I is not defined and hence does not exist.

Definition 2. Let $\langle R \cup I \rangle$ be a neutrosophic ring. A proper subset P of $\langle R \cup I \rangle$ is said to be a neutrosophic subring if P itself is a neutrosophic ring under the operations of $\langle R \cup I \rangle$.

Definition 2. Let $\langle R \cup I \rangle$ be any neutrosophic ring, a non empty subset P of $\langle R \cup I \rangle$ is defined to be a neutrosophic ideal of $\langle R \cup I \rangle$ if the following conditions are satisfied;

1. P is a neutrosophic subring of $\langle R \cup I \rangle$.
2. For every $p \in P$ and $r \in \langle R \cup I \rangle$, rp and $pr \in P$.

Definition 4. Let K be a field . We call the field generated by $K \cup I$ to be the neutrosophic field for it involves the indeterminacy factor in it. We define $I^2 = I$, $I + I = 2I$ i.e., $I + I +, \ldots, + I = nI$, and if $k \in K$ then $k.I = kI, 0I = 0$. We denote the neutrosophic field by $K(I)$ which is generated by $K \cup I$ that is $K(I) = \langle K \cup I \rangle$. $\langle K \cup I \rangle$ denotes the field generated by K and I.

Definition 5. Let $K(I)$ be a neutrosophic field, $P \subset K(I)$ is a neutrosophic subfield of P if P itself is a neutrosophic field.

Soft Sets

Throughout this subsection U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U, and $A \subset E$. Molodtsov [8] defined the soft set in the following manner:

Definition 6. *A pair* (F, A) *is called a soft set over* U *where* F *is a mapping given by* $F : A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $a \in A$, $F(a)$ may be considered as the set of a -elements of the soft set (F, A), or as the set of a -approximate elements of the soft set.

Definition 7. For two soft sets (F, A) and (H, B) over U, (F, A) is called a soft subset of (H, B) if

1) $A \subseteq B$ and
2) $F(a) \subseteq H(a)$, for all $a \in A$.

This relationship is denoted by $(F, A) \subset (H, B)$. Similarly (F, A) is called a soft superset of (H, B) if (H, B) is a soft subset of (F, A) which is denoted by $(F, A) \supset (H, B)$.

Definition 8. Two soft sets (F, A) and (H, B) over U are called soft equal if (F, A) is a soft subset of (H, B) and (H, B) is a soft subset of (F, A).

Definition 9. Let (F, A) and (K, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$. Then their restricted intersection is denoted by $(F, A) \cap_R (\mathrm{K}, B) = (H, C)$ where (H, C) is defined as $H(c) = F(c) \cap \mathrm{K}(c)$ for all $c \in C = A \cap B$.

Definition 10. The extended intersection of two soft sets (F, A) and (K, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H\ c$ is defined as

$$H(\mathrm{c})=\begin{cases} F(\mathrm{c}) & \text{if } \mathrm{c} \in A-B \\ \mathrm{K}(\mathrm{c}) & \text{if } \mathrm{c} \in B-A \\ F(\mathrm{c}) \cap \mathrm{K}(\mathrm{c}) & \text{if } \mathrm{c} \in A \cap B. \end{cases}$$

We write $(F,A) \cap_{\varepsilon} (\mathrm{K},B) = (H,C)$.

Definition 11. The restricted union of two soft sets (F,A) and (K,B) over a common universe U is the soft set (H,C) , where $C = A \cup B$, and for all $c \in C$, $H\ c$ is defined as the soft set $(H,C) = (F,A) \cup_{R} (K,B)$ where $C = A \cap B$ and $H(\mathrm{c}) = F(c) \cup \mathrm{K}(c)$ for all $c \in C$.

Definition 12. The extended union of two soft sets (F,A) and (K,B) over a common universe U is the soft set (H,C) , where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as

$$H(\mathrm{c})=\begin{cases} F(\mathrm{c}) & \text{if } \mathrm{c} \in A-B \\ \mathrm{K}(\mathrm{c}) & \text{if } \mathrm{c} \in B-A \\ F(\mathrm{c}) \cup K(\mathrm{c}) & \text{if } \mathrm{c} \in A \cap B. \end{cases}$$

We write $(F,A) \cup_{\varepsilon} (\mathrm{K},B) = (H,C)$.

Soft Rings

Definition 13. Let R be a ring and let (F,A) be a non-null soft set over R. Then (F,A) is called a soft ring over R if $F(a)$ is a subring of R , for all $a \in A$.

Definition 14. Let (F,A) and (K,B) be soft rings over R. Then (K,B) is called a soft sub ring of (F,A), If it satisfies the following;

1. $B \subset A$
2. $K(a)$ is a sub ring of $F(a)$, for all $a \in A$.

Definition 15. Let (F,A) and (K,B) be soft rings over R. Then (K,B) is called a soft ideal of F,A , If it satisfies the following;

1. $B \subset A$
2. $K(a)$ is an idela of $F(a)$, for all $a \in A$.

3 Soft Neutrosophic Ring

Definition. Let $\langle R \cup I \rangle$ be a neutrosophic ring and (F,A) be a soft set over $\langle R \cup I \rangle$. Then (F,A) is called soft neutrosophic ring if and only if $F(a)$ is a neutrosophic subring of $\langle R \cup I \rangle$ for all $a \in A$.

Example. Let $\langle Z \cup I \rangle$ be a neutrosophic ring of integers and let (F,A) be a soft set over $\langle Z \cup I \rangle$. Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of parameters. Then clearly (F,A) is a soft neutrosophic ring over $\langle Z \cup I \rangle$, where

$$F(a_1) = \langle 2Z \cup I \rangle, F(a_2) = \langle 3Z \cup I \rangle$$

$$F(a_3) = \langle 5Z \cup I \rangle, F(a_4) = \langle 6Z \cup I \rangle.$$

Theorem . Let (F,A) and (H,A) be two soft neutrosophic rings over $\langle R \cup I \rangle$. Then their intersection $(F,A) \cap (H,A)$ is again a soft neutrosophic ring over $\langle R \cup I \rangle$.

Proof. The proof is straightforward.

Theorem. Let (F,A) and (H,B) be two soft neutrosophic rings over $\langle R \cup I \rangle$. If $A \cap B = \phi$, then $(F,A) \cup (H,B)$ is a soft neutrosophic ring over $\langle R \cup I \rangle$.

Proof. This is straightforward.

Remark. The extended union of two soft neutrosophic rings (F,A) and (K,B) over $\langle R \cup I \rangle$ is not a soft neutrosophic ring over $\langle R \cup I \rangle$.

We check this by the help of following Example.

Example. Let $\langle Z \cup I \rangle$ be a neutrosophic ring of integers. Let (F,A) and (K,B) be two soft neutrosophic rings over $\langle Z \cup I \rangle$, where

$$F(a_1) = \langle 2Z \cup I \rangle, F(a_2) = \langle 3Z \cup I \rangle, F(a_3) = \langle 4Z \cup I \rangle,$$

And

$$K(a_1) = \langle 5Z \cup I \rangle, K(a_3) = \langle 7Z \cup I \rangle.$$

Their extended union

$(F,A) \cup_E (K,B) = (H,C)$, where

$$H(a_1) = \langle 2Z \cup I \rangle \cup \langle 5Z \cup I \rangle,$$

$$H(a_2) = \langle 3Z \cup I \rangle,$$

$$H(a_3) = \langle 5Z \cup I \rangle \cup \langle 7Z \cup I \rangle.$$

Thus clearly $H(a_1) = \langle 2Z \cup I \rangle \cup \langle 5Z \cup I \rangle$, $H(a_3) = \langle 5Z \cup I \rangle \cup \langle 7Z \cup I \rangle$ is not a neutrosophic rings.

Remark. The restricted union of two soft neutrosophic rings (F,A) and (K,B) over $\langle R \cup I \rangle$ is not a soft neutrosophic ring over $\langle R \cup I \rangle$.

Theorem. The OR operation of two soft neutrosophic rings over $\langle R \cup I \rangle$ may not be a soft neutrosophic ring over $\langle R \cup I \rangle$.

One can easily check these remarks with the help of Examples.

Theorem. The extended intersection of two soft neutrosophic rings over $\langle R \cup I \rangle$ is soft neutrosophic ring over $\langle R \cup I \rangle$.

Proof. The proof is straightforward.

Theorem. The restricted intersection of two soft neutrosophic rings over $\langle R \cup I \rangle$ is soft neutrosophic ring over $\langle R \cup I \rangle$.

Proof. It is obvious.

Theorem. The AND operation of two soft neutrosophic rings over $\langle R \cup I \rangle$ is soft neutrosophic ring over $\langle R \cup I \rangle$.

Proof. Easy.

Definition. Let (F,A) be a soft set over a neutrosophic ring $\langle R \cup I \rangle$. Then (F,A) is called an absolute soft neutrosophic ring if $F(a) = \langle R \cup I \rangle$ for all $a \in A$.

Definition. Let (F,A) be a soft set over a neutrosophic ring $\langle R \cup I \rangle$. Then (F,A) is called soft neutrosophic ideal over $\langle R \cup I \rangle$ if and only if $F(a)$ is a neutrosophic ideal over $\langle R \cup I \rangle$.

Example. Let $\langle Z_{12} \cup I \rangle$ be a neutrosophic ring. Let $A = \{a_1, a_2\}$ be a set of parameters and (F,A) be a soft set over $\langle Z_{12} \cup I \rangle$. Then clearly (F,A) is a soft neutrosophic ideal over $\langle R \cup I \rangle$, where

$$F(a_1) = \{0,6,2I,4I,6I,8I,10I,6+2I,...,6+10I\},$$

$$F(a_2) = \{0,6,6I,6+6I\}.$$

Theorem. Every soft neutrosophic ideal (F,A) over a neutrosophic ring $\langle R \cup I \rangle$ is trivially a soft neutrosophic ring.

Proof. Let (F,A) be a soft neutrosophic ideal over a neutrosophic ring $\langle R \cup I \rangle$. Then by definition $F(a)$ is a neutrosophic ideal for all $a \in A$. Since we know that every neutrosophic ideal is a neutrosophic subring. It follows that $F(a)$ is a neutrosophic subring of $\langle R \cup I \rangle$. Thus by definition of soft neutrosophic ring, this implies that (F,A) is a soft neutrosophic ring.

Remark. The converse of the above theorem is not true.

To check the converse, we take the following Example.

Example. Let $\langle Z_{10} \cup I \rangle$ be a neutrosophic ring. Let $A = \{a_1, a_2\}$ be a set of parameters and (F, A) be a soft neutrosophic ring over $\langle Z_{10} \cup I \rangle$, where

$$F(a_1) = \{0, 2, 4, 6, 8, 2I, 4I, 6I, 8I\},$$

$$\mathrm{F}(a_2) = \{0, 2I, 4I, 6I, 8I\}.$$

Then obviously (F, A) is not a soft neutrosophic ideal over $\langle Z_{10} \cup I \rangle$.

Proposition. Let (F, A) and (K, B) be two soft neutosophic ideals over a neutrosophic ring $\langle R \cup I \rangle$. Then

1. Their extended union $(F, A) \cup_E (K, B)$ is again a soft neutrosophic ideal over $\langle R \cup I \rangle$.
2. Their extended intersection $(F, A) \cap_E (K, B)$ is again a soft neutrosophic ideal over $\langle R \cup I \rangle$.
3. Their restricted union $(F, A) \cup_R (K, B)$ is again a soft neutrosophic ideal over $\langle R \cup I \rangle$.
4. Their restricted intersection $(F, A) \cap_R (K, B)$ is again a soft neutrosophic ideal over $\langle R \cup I \rangle$.
5. Their OR operation $(F, A) \vee (K, B)$ is again a soft neutrosophic ideal over $\langle R \cup I \rangle$.
6. Their AND operation $(F, A) \vee (K, B)$ is again a soft neutrosophic ideal over $\langle R \cup I \rangle$.

Proof. Supoose (F, A) and (K, B) be two soft neutrosophic ideals over $\langle R \cup I \rangle$. Let $C = A \cup B$. Then for all $c \in C$, The extended union is $(F, A) \cup_E (K, B) = (H, C)$, where

$$H(\mathrm{c}) = \begin{cases} F(c), \text{If c} \in A - B, \\ K(c), \text{If c} \in B - A, \\ F(c) \cup K(c), \text{If c} \in A \cap B. \end{cases}$$

As union of two neutrosophic ideals is again a neutrosophic ideal of $\langle R \cup I \rangle$. Hence the extended union H, C is a soft neutrosophic ideal over $\langle R \cup I \rangle$.

Similarly $(2), (3), (4), (5),$ and (6) can be proved respectively.

Definition. Let (F, A) and (K, B) be two soft neutrosophic rings over $\langle R \cup I \rangle$. Then (K, B) is called soft neutrosophic subring of (F, A), if

1. $B \subseteq A$, and
2. $K(a)$ is a neutrosophic subring of $F(a)$ for all $a \in A$.

Example. Let $\langle C \cup I \rangle$ be the neutrosophic ring of complex numbers. Let $A = \{a_1, a_2, a_3\}$ be a set of parameters. Then (F, A) be a soft neutrosophic ring over $\langle C \cup I \rangle$, where

$$F(a_1) = \langle Z \cup I \rangle, F(a_2) = \langle Q \cup I \rangle,$$

$$F(a_3) = \langle R \cup I \rangle.$$

Where $\langle Z \cup I \rangle, \langle Q \cup I \rangle$ and $\langle R \cup I \rangle$ are neutrosophic rings of integers, rational numbers, and real numbers respectively.

Let $B = \{a_2, a_3\}$ be a set of parmeters . Let (K, B) be the neutrosophic subring of (F, A) over $\langle C \cup I \rangle$, where

$$\mathrm{K}(a_2) = \langle Z \cup I \rangle, \mathrm{K}(a_3) = \langle Q \cup I \rangle.$$

Theorem. Every soft ring (H, B) over a ring R is a soft neutrosophic subring of a soft neutrosophic ring (F, A) over the corresponding neutrosophic ring $\langle R \cup I \rangle$ if $B \subseteq A$.

Proof. Straightforward.

Definition. Let (F, A) and (K, B) be two soft neutrosophic rings over $\langle R \cup I \rangle$. Then (K, B) is called soft neutrosophic ideal of (F, A), if

1. $B \subseteq A$, and

2. $K(a)$ is a neutrosophic ideal of $F(a)$ for all $a \in A$.

Example. Let $\langle Z_{12} \cup I \rangle$ be a neutrosophic ring. Let $A = \{a_1, a_2\}$ be a set of parameters and (F, A) be a soft set over $\langle Z_{12} \cup I \rangle$. Then clearly (F, A) is a soft neutrosophic ring over $\langle Z_{12} \cup I \rangle$,where

$F(a_1) = \{0, 6, 2I, 4I, 6I, 8I, 10I, 6 + 2I, ..., 6 + 10I\}$,

$F(a_2) = \{0, 2, 4, 6, 8, 2\mathrm{I}, 4\mathrm{I}, 6\mathrm{I}, 8\mathrm{I}\}$.

Let $B = \{a_1, a_2\}$ be a set of parameters. Then clearly (H, B) is a soft neutrosophic ideal of (F, A) over $\langle Z_{12} \cup I \rangle$, where

$$H(a_1) = \{0, 6, 6 + 6I\},$$

$$H(a_2) = \{0, 2, 4, 6, 8\}.$$

Proposition. All soft neutrosophic ideals are trivially soft neutrosophic subrings.

Proof. Straightforward.

4 Soft Neutrosophic Field

Defintion. Let $K(I) = \langle K \cup I \rangle$ be a neutrosophic field and let (F, A) be a soft set over $K(I)$. Then (F, A) is said to be soft neutrosophic field if and only if $F(a)$ is a neutrosophic subfield of $K(I)$ for all $a \in A$.

Example. Let $\langle C \cup I \rangle$ be a neutrosophic field of complex numbers. Let $A = \{a_1, a_2\}$ be a set of parameters and let (F, A) be a soft set of $\langle C \cup I \rangle$. Then (F, A) is called soft neutrosophic field over $\langle C \cup I \rangle$, where

$$F(a_1) = \langle R \cup I \rangle, F(a_2) = \langle Q \cup I \rangle.$$

Where $\langle R \cup I \rangle$ and $\langle Q \cup I \rangle$ are the neutosophic fields of real numbers and rational numbers.

Proposition. Every soft neutrosophic field is trivially a soft neutrosophic ring.

Proof. The proof is trivial.

Remark. The converse of above proposition is not true.

To see the converse, lets take a look to the following example.

Example. Let $\langle Z \cup I \rangle$ be a neutrosophic ring of integers. Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of parameters and let (F, A) be a soft set over $\langle Z \cup I \rangle$. Then (F, A) is a soft neutrosophic ring over $\langle Z \cup I \rangle$, where

$$F(a_1) = \langle 2Z \cup I \rangle, F(a_2) = \langle 3Z \cup I \rangle$$

$$F(a_3) = \langle 5Z \cup I \rangle, F(a_4) = \langle 6Z \cup I \rangle.$$

Clearly (F, A) is not a soft neutrosophic field.

Definition. Let (F, A) be a soft neutrosophic field over a neutrosophic field $\langle K \cup I \rangle$. Then (F, A) is called an absolute soft neutrosophic field if $F(a) = \langle K \cup I \rangle$, for all $a \in A$.

5 Soft Neutrosophic Ring Homomorphism

Definition. Let (F, A) and (K, B) be the soft neutrosophic rings over $\langle R \cup I \rangle$ and $\langle R' \cup I \rangle$ respectively. Let $f : \langle R \cup I \rangle \to \langle R' \cup I \rangle$ and $g : A \to B$ be mappings. Let $(f, g) : (\mathrm{F}, \mathrm{A}) \to (\mathrm{K}, \mathrm{B})$ be another mapping. Then (f, g) is called a soft neutrosophic ring homomorphism if the following conditions are hold.

1. f is a neutrosophic ring homomorphism from $\langle R \cup I \rangle$ to $\langle R' \cup I \rangle$.
2. g is onto mapping from A to B, and
3. $f(F(a)) = K(g(a))$ for all $a \in A$.

If f is an isomorphicm and g is a bijective mapping. Then (f, g) is called soft neutrosophic ring isomorphism.

Conclusions

In this paper we extend the neutrosophic ring, neutrosophic field and neutrosophic subring to soft neutrosophic ring, soft neutrosophic field and soft neutrosophic subring respectively. The neutrosophic ideal of a ring is extended to soft neutrosophic ideal. We showed all these by giving various examples in order to illustrate the soft part of the neutrosophic notions used.

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New Neutrosophic Crisp Topological Concepts

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Abstract
In this paper, we introduce the concept of ""neutrosophic crisp neighborhoods system for the neutrosophic crisp point ". Added to, we introduce and study the concept of neutrosophic crisp local function, and construct a new type of neutrosophic crisp topological space via neutrosophic crisp ideals. Possible application to GIS topology rules are touched upon.

Keywords: Neutrosophic Crisp Point, Neutrosophic Crisp Ideal; Neutrosophic Crisp Topology; Neutrosophic Crisp Neighborhoods

1 INTRODUCTION

The idea of "neutrosophic set" was first given by Smarandache [14, 15]. In 2012 neutrosophic operations have been investigated by Salama et al. [4-13]. The fuzzy set was introduced by Zadeh [17]. The intuitionstic fuzzy set was introduced by Atanassov [1, 2, 3]. Salama et al. [11]defined intuitionistic fuzzy ideal for a set and generalized the concept of fuzzy ideal concepts, first initiated by Sarker [16]. Neutrosophy has laid the foundation for a whole family of new mathematical theories, generalizing both their crisp and fuzzy counterparts. Here we shall present the neutrosophic crisp version of these concepts. In this paper, we introduce the concept of "neutrosophic crisp points "and "neutrosophic crisp neigbourhoods systems". Added to we define the new concept of neutrosophic crisp local function, and construct new type of neutrosophic crisp topological space via neutrosophic crisp ideals.

2 TERMINOLOGIES

We recollect some relevant basic preliminaries, and in particular the work of Smarandache in [14, 15], and Salama et al. [4 -13].

2.1 Definition [13]
Let X be a non-empty fixed set. A neutrosophic crisp set (NCS for short) A is an object having the form $A=\langle A_1, A_2, A_3\rangle$ where A_1, A_2 and A_3 are subsets of X satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$.

2.2 Definition [13].
Let X be a nonempty set and $p \in X$ Then the neutrosophic crisp point p_N defined by $p_N = \langle \{p\}, \phi, \{p\}^c \rangle$ is called a neutrosophic crisp point (NCP for short) in X, where NCP is a triple ({only one element in X}, the empty set,{the complement of the same element in X}).

2.3 Definition [13]
Let X be a nonempty set, and $p \in X$ a fixed element in X. Then the neutrosophic crisp set $p_{N_N} = \langle \phi, \{p\}, \{p\}^c \rangle$ is called "vanishing neutrosophic crisp point" (VNCP for short) in X, where VNCP is a triple (the empty set,{only one element in X},{the complement of the same element in X}).

2.4 Definition [13]
Let $p_N = \langle \{p\}, \phi, \{p\}^c \rangle$ be a NCP in X and $A=\langle A_1, A_2, A_3\rangle$ a neutrosophic crisp set in X.

(a) p_N is said to be contained in A ($p_N \in A$ for short) iff $p \in A_1$.

(b) Let p_{N_N} be a VNCP in X, and $A=\langle A_1, A_2, A_3\rangle$ a neutrosophic crisp set in X. Then p_{N_N} is said to be

(b) Let p_{N_N} be a VNCP in X, and $A=\langle A_1, A_2, A_3\rangle$ a neutrosophic crisp set in X. Then p_{N_N} is said to be contained in A ($p_{N_N} \in A$ for short) iff $p \notin A_3$.

2.5 Definition [13].

Let X be non-empty set, and L a non–empty family of NCSs. We call L a neutrosophic crisp ideal (NCL for short) on X if

i. $A \in L$ and $B \subseteq A \Rightarrow B \in L$ [heredity],

ii. $A \in L$ and $B \in \mathrm{L} \Rightarrow A \vee B \in \mathrm{L}$ [Finite additivity].

A neutrosophic crisp ideal L is called a σ - neutrosophic crisp ideal if $\{M_j\}_{j\in N} \leq L$, implies $\underset{j\in J}{\cup} M_{j\in L}$ (countable additivity).

The smallest and largest neutrosophic crisp ideals on a non-empty set X are $\{\phi_N\}$ and the NSs on X. Also, $NC\mathrm{L}_{\mathrm{f}}$, $\mathrm{NCL}_{\mathrm{c}}$ are denoting the neutrosophic crisp ideals (NCL for short) of neutrosophic subsets having finite and countable support of X respectively. Moreover, if A is a nonempty NS in X, then $\{B \in NCS : B \subseteq A\}$ is an NCL on X. This is called the principal NCL of all NCSs, denoted by NCL$\langle A\rangle$.

2.1 Proposition [13]

Let $\{L_j : j \in J\}$ be any non - empty family of neutrosophic crisp ideals on a set X. Then $\bigcap_{j\in J} L_j$ and $\bigcup_{j\in J} L_j$ are neutrosophic crisp ideals on X, where

$$\underset{j\in J}{\cap} L_j = \left\langle \underset{j\in J}{\cap} A_{j1}, \underset{j\in J}{\cap} A_{j2}, \underset{j\in J}{\cup} A_{j3}\right\rangle \text{ or}$$

$$\underset{j\in J}{\cap} L_j = \left\langle \underset{j\in J}{\cap} A_{j1}, \underset{j\in J}{\cup} A_{j2}, \underset{j\in J}{\cup} A_{j3}\right\rangle \text{ and}$$

$$\underset{j\in J}{\cup} L_j = \left\langle \underset{j\in J}{\cup} A_{j1}, \underset{j\in J}{\cup} A_{j2}, \underset{j\in J}{\cap} A_{j3}\right\rangle \text{ or}$$

$$\underset{j\in J}{\cup} L_j = \left\langle \underset{j\in J}{\cup} A_{j1}, \underset{j\in J}{\cap} A_{j2}, \underset{j\in J}{\cap} A_{j3}\right\rangle.$$

2,2 Remark [13]

The neutrosophic crisp ideal defined by the single neutrosophic set ϕ_N is the smallest element of the ordered set of all neutrosophic crisp ideals on X.

2.1 Proposition [13]

A neutrosophic crisp set $A=\langle A_1, A_2, A_3\rangle$ in the neutrosophic crisp ideal L on X is a base of L iff every member of L is contained in A.

3. Neutrosophic Crisp Neigborhoods System

3.1 Definition.

Let $A=\langle A_1, A_2, A_3\rangle$, be a neutrosophic crisp set on a set X, then $p=\langle\{p_1\},\{p_2\},\{p_3\}\rangle$, $p_1 \neq p_2 \neq p_3 \in X$ is called a neutrosophic crisp point

An NCP $p=\langle\{p_1\},\{p_2\},\{p_3\}\rangle$, is said to be belong to a neutrosophic crisp set $A=\langle A_1, A_2, A_3\rangle$, of X, denoted by $p \in A$, if may be defined by two types

i) **Type 1:** $\{p_1\} \subseteq A_1, \{p_2\} \subseteq A_2$ and $\{p_3\} \subseteq A_3$

ii) **Type 2:** $\{p_1\} \subseteq A_1, \{p_2\} \supseteq A_2$ and $\{p_3\} \subseteq A_3$

3.1 Theorem

Let $A=\langle\langle A_1, A_2, A_3\rangle\rangle$, and $B=\langle\langle B_1, B_2, B_3\rangle\rangle$, be neutrosophic crisp subsets of X. Then $A \subseteq B$ iff $p \in A$ implies $p \in B$ for any neutrosophic crisp point p in X.

Proof

Let $A \subseteq B$ and $p \in A$. Then two types

Type 1: $\{p_1\} \subseteq A_1, \{p_2\} \subseteq A_2$ and $\{p_3\} \subseteq A_3$ or

Type 2: $\{p_1\} \subseteq A_1, \{p_2\} \supseteq A_2$ and $\{p_3\} \subseteq A_3$. Thus $p \in B$. Conversely, take any x in X. Let $p_1 \in A_1$ and $p_2 \in A_2$ and $p_3 \in A_3$. Then p is a neutrosophic crisp point in X. and $p \in A$. By the hypothesis $p \in B$. Thus $p_1 \in B_1$, or Type 1: $\{p_1\} \subseteq B_1, \{p_2\} \subseteq B_2$ and $\{p_3\} \subseteq B_3$ or Type 2: $\{p_1\} \subseteq B_1, \{p_2\} \supseteq B_2$ and $\{p_3\} \subseteq B_3$. Hence. $A \subseteq B$.

3.2 Theorem

Let $A=\langle A_1, A_2, A_3\rangle$, be a neutrosophic crisp subset of X. Then $A=\cup\{p : p \in A\}$.

Proof

Since $\cup\{p : p \in A\}$ may be two types

Type 1:
$\langle \cup \{p_1 : p_1 \in A_1\}, \cup\{p_2 : p_2 \in A_2\}, \cap\{p_3 : p_3 \in A_3\}\rangle$ or
Type 2:
$\langle \cup \{p_1 : p_1 \in A_1\}, \cap\{p_2 : p_2 \in A_2\}, \cap\{p_3 : p_3 \in A_3\}\rangle$. Hence
$A = \langle A_1, A_2, A_3 \rangle$

3.1 Proposition

Let $\{A_j : j \in J\}$ is a family of NCSs in X. Then

(a_1) $p = \langle\{p_1\},\{p_2\},\{p_3\}\rangle \in \bigcap_{j\in J} A_j$ iff $p \in A_j$ for each $j \in J$.

(a_2) $p \in \bigcup_{j\in J} A_j$ iff $\exists j \in J$ such that $p \in A_j$.

.

3.2 Proposition

Let $A = \langle A_1, A_2, A_3 \rangle$ and $B = \langle B_1, B_2, B_3 \rangle$ be two neutrosophic crisp sets in X. Then

a) $A \subseteq B$ iff for each p we have

$p \in A \Leftrightarrow p \in B$ and for each p we have

$p \in A \Rightarrow p \in B$.

b) $A = B$ iff for each p we have
$p \in A \Rightarrow p \in B$ and for each p we have $p \in A \Leftrightarrow p \in B$.

3.3 Proposition

Let $A = \langle A_1, A_2, A_3 \rangle$ be a neutrosophic crisp set in X. Then

$A = \cup < \{p_1 : p_1 \in A_1\}, \{p_2 : p_2 \in A_2\}, \{p_3 : p_3 \in A_3\}$.

3.2 Definition

Let $f : X \to Y$ be a function and p be a nutrosophic crisp point in X. Then the image of p under f, denoted by $f(p)$, is defined by
$f(p) = \langle\{q_1\},\{q_2\},\{q_3\}\rangle$, where $q_1 = f(p_1), q_2 = f(p_2)$ .and $q_3 = f(p_3)$.

It is easy to see that $f(p)$ is indeed a NCP in Y, namely $f(p) = q$, where $q = f(p)$, and it is exactly the same meaning of the image of a NCP under the function f.

4 4. Neutrosophic Crisp Local functions

4.1 Definition
Let p be a neutrosophic crisp point of a neutrosophic crisp topological space (X, τ). A neutrosophic crisp neighbourhood (NCNBD for short) of a neutrosophic crisp point p if there is a neutrosophic crisp open set(NCOS for short) B in X such that $p \in B \subseteq A$.

4.1 Theorem
Let (X, τ) be a neutrosophic crisp topological space (NCTS for short) of X. Then the neutrosophic crisp set A of X is NCOS iff A is a NCNBD of p for every neutrosophic crisp set $p \in A$.

Proof
Let A be NCOS of X . Clearly A is a NCBD of any $p \in A$. Conversely, let $p \in A$. Since A is a NCBD of p, there is a NCOS B in X such that $p \in B \subseteq A$. So we have $A = \cup\{p : p \in A\} \subseteq \cup\{B : p \in A\} \subseteq A$ and hence $A = \cup\{B : p \in A\}$. Since each B is NCOS.

4.2 Definition
Let (X, τ) be a neutrosophic crisp topological spaces (NCTS for short) and L be neutrsophic crisp ideal (NCL, for short) on X. Let A be any NCS of X. Then the neutrosophic crisp local function $NCA^*(L,\tau)$ of A is the union of all neutrosophic crisp points(NCP, for short) $P = \langle\{p_1\},\{p_2\},\{p_3\}\rangle$, such that if $U \in N((p))$ and $NA^*(L,\tau) = \cup\{p \in X : A \wedge U \notin L$ for every U nbd of N(P)$\}$, $NCA^*(L,\tau)$ is called a neutrosophic crisp local function of A with respect to τ and L which it will be denoted by $NCA^*(L,\tau)$, or simply $NCA^*(\mathrm{L})$.

4.1 Example
One may easily verify that.
If L=$\{\phi_N\}$, then $NCA^*(L,\tau) = NCcl(A)$, for any neutrosophic crisp set $A \in NCSs$ on X.
If $\mathrm{L} = \{\text{all NCSs on X}\}$ then $NCA^*(L,\tau) = \phi_N$, for any $A \in NCSs$ on X .

4.2 Theorem
Let (X, τ) be a NCTS and L_1, L_2 be two topological neutrosophic crisp ideals on X. Then for any neutrosophic crisp sets A, B of X. then the following statements are verified

i) $A \subseteq B \Rightarrow NCA^*(L,\tau) \subseteq NCB^*(L,\tau)$,
ii) $L_1 \subseteq L_2 \Rightarrow NCA^*(L_2,\tau) \subseteq NCA^*(L_1,\tau)$.
iii) $NCA^* = NCcl(A^*) \subseteq NCcl(A)$.

iv) $NCA^{**} \subseteq NCA^{*}$.

v) $NC(A \cup B)^{*} = NCA^{*} \cup NCB^{*}$.,

vi) $NC(A \cap B)^{*}(L) \subseteq NCA^{*}(L) \cap NCB^{*}(L)$.

vii) $\ell \in L \Rightarrow NC(A \cup \ell)^{*} = NCA^{*}$.

viii) $NCA^{*}(L, \tau)$ is neutrosophic crisp closed set .

Proof

i) Since $A \subseteq B$, let $p = \langle\{p_1\},\{p_2\},\{p_3\}\rangle \in NCA^{*}(L_1)$ then $A \cap U \notin L$ for every $U \in N(p)$. By hypothesis we get $B \cap U \notin L$, then $p = \langle\{p_1\},\{p_2\},\{p_3\}\rangle \in NB^{*}(L_1)$.

ii) Clearly. $L_1 \subseteq L_2$ implies $NCA^{*}(L_2,\tau) \subseteq NCA^{*}(L_1,\tau)$ as there may be other IFSs which belong to L_2 so that for GIFP $p = \langle\{p_1\},\{p_2\},\{p_3\}\rangle \in NCA^{*}(L_1)$ but P may not be contained in $NCA^{*}(L_2)$.

iii) Since $\{\phi_N\} \subseteq L$ for any NCL on X, therefore by (ii) and Example 3.1, $NCA^{*}(L) \subseteq NCA^{*}(\{O_N\}) = NCcl(A)$ for any NCS A on X. Suppose $P_1 = \langle\{p_1\},\{p_2\},\{p_3\}\rangle \in NCcl(A^{*}(L_1))$. So for every $U \in NC(P_1)$, $NC(A^{*}) \cap U \neq \phi_N$, there exists $P_2 = \langle\{q_1\},\{q_2\},\{q_3\}\rangle \in NCA^{*}(L_1) \cap U$ such that for every V NCNBD of $P_2 \in N(P_2)$, $A \cap U \notin L$. Since $U \wedge V \in N(p_2)$ then $A \cap (U \cap V) \notin L$ which leads to $A \wedge U \notin L$, for every $U \in N(P_1)$ therefore $P_1 \in NC(A^{*}(L))$ and so $NCcl(NA^{*}) \subseteq NCA^{*}$ While, the other inclusion follows directly. Hence $NCA^{*} = NCcl(NCA^{*})$ .But the inequality $NCA^{*} \subseteq Ncl(NCA^{*})$.

iv) The inclusion $NCA^{*} \cup NCB^{*} \subseteq NC(A \cup B)^{*}$ follows directly by (i). To show the other implication, let $p \in NC(A \cup B)^{*}$ then for every $U \in NC(p)$, $(A \cup B) \cap U \notin L, i.e, (A \cap U) \cup (B \cap U) \notin L.$ then, we have two cases $A \cap U \notin L$ and $B \cap U \in L$ or the converse, this means that exist $U_1, U_2 \in N(P)$ such that $A \cap U_1 \notin L$, $B \cap U_1 \notin L$, $A \cap U_2 \notin L$ and $B \cap U_2 \notin L$. Then $A \cap (U_1 \cap U_2) \in L$ and $B \cap (U_1 \cap U_2) \in L$ this gives $(A \cup B) \cap (U_1 \cap U_2) \in L$, $U_1 \cap U_2 \in N(C(P))$ which contradicts the hypothesis. Hence the equality holds in various cases.

vi) By (iii), we have

$NCA^{*^{*}} = NCcl(NCA^{*})^{*} \subseteq NCcl(NCA^{*}) = NCA^{*}$

Let (X, τ) be a NCTS and L be NCL on X . Let us define the neutrosophic crisp closure operator $NCcl^{*}(A) = A \cup NC(A^{*})$ for any NCS A of X. Clearly, let $NCcl^{*}(A)$ is a neutrosophic crisp operator. Let $NC\tau^{*}(L)$ be NCT generated by $NCcl^{*}$ .i.e $NC\tau^{*}(L) = \{A : NCcl^{*}(A^{c}) = A^{c}\}$ now $L = \{\phi_N\} \Rightarrow NCcl^{*}(A) = A \cup NCA^{*} = A \cup NCcl(A)$ for every neutrosophic crisp set A. So, $N\tau^{*}(\{\phi_N\}) = \tau$. Again $L = \{all \text{ NCSs on X}\} \Longrightarrow NCcl^{*}(A) = A,$ because $NCA^{*} = \phi_N$, for every neutrosophic crisp set A so $NC\tau^{*}(L)$ is the neutrosophic crisp discrete topology on X. So we can conclude by Theorem 4.1.(ii). $NC\tau^{*}(\{\phi_N\}) = NC\tau^{*}(L)$ i.e. $NC\tau \subseteq NC\tau^{*}$, for any neutrosophic ideal L_1 on X. In particular, we have for two topological neutrosophic ideals L_1, and L_2 on X, $L_1 \subseteq L_2 \Rightarrow NC\tau^{*}(L_1) \subseteq NC\tau^{*}(L_2)$.

4.3 Theorem

Let τ_1, τ_2 be two neutrosophic crisp topologies on X. Then for any topological neutrosophic crisp ideal L on X, $\tau_1 \le \tau_2$ implies $NA^{*}(L,\tau_2) \subseteq NA^{*}(L,\tau_1)$, for every A $\in L$ then $NC\tau^{*}_{1} \subseteq NC\tau^{*}_{2}$

Proof

Clear.

A basis $NC\beta(L,\tau)$ for $NC\tau^{*}(L)$ can be described as follows:

$NC\beta(L,\tau) = \{A - B : A \in \tau, B \in L\}$. Then we have the following theorem

4.4 Theorem

$NC\beta(L,\tau) = \{A - B : A \in \tau, B \in L\}$ Forms a basis for the generated NT of the NCT (X, τ) with topological neutrosophic crisp ideal L on X.

Proof

Straight forward.

The relationship between $NC\tau$ and $_{NC}\tau^{*}(L)$ established throughout the following result which have an immediately proof .

4.5 Theorem

Let τ_1, τ_2 be two neutrosophic crisp topologies on X. Then for any topological neutrosophic ideal L on X, $\tau_1 \subseteq \tau_2$ implies $NC\tau^{*}_{1} \subseteq NC\tau^{*}_{2}$.

4.6 Theorem

Let (X, τ) be a NCTS and L_1, L_2 be two neutrosophic crisp ideals on X . Then for any neutrosophic crisp set A in X, we have

i) $NCA^*(L_1 \cup L_2,\tau) = NCA^*\left(L_1, NC\tau^*(L_1)\right) \wedge NCA^*\left(L_2, NC\tau^*(L_2)\right)$ ii) $NC\tau^*(L_1 \cup L_2) = \left(NC\tau^*(L_1)\right)^*(L_2) \wedge \left(NC\tau^*(L_2)\right)^*(L_1)$

Proof

Let $p \notin (L_1 \cup L_2,\tau)$, this means that there exists $U \in NC(P)$ such that $A \cap U_p \in (L_1 \cup L_2)$ i.e. There exists $\ell_1 \in L_1$ and $\ell_2 \in L_2$ such that $A \cap U \in (\ell_1 \vee \ell_2)$ because of the heredity of L_1, and assuming $\ell_1 \wedge \ell_2 = O_N$. Thus we have $(A \cap U) - \ell_1 = \ell_2$ and $(A \cap U_p) - \ell_2 = \ell_1$ therefore $(U - \ell_1) \cap A = \ell_2 \in L_2$ and $(U - \ell_2) \cap A = \ell_1 \in L_1$. Hence $p \notin NCA^*\left(L_2, NC\tau^*(L_1)\right)$, or $P \notin NCA^*\left(L_1, NC\tau^*(L_2)\right)$, because p must belong to either ℓ_1 or ℓ_2 but not to both. This gives $NCA^*(L_1 \cup L_2,\tau) \geq NCA^*\left(L_1, NC\tau^*(L_1)\right) \cap NCA^*\left(L_2, NC\tau^*(L_2)\right)$.

To show the second inclusion, let us assume $P \notin NCA^*\left(L_1, NC\tau^*(L_2)\right)$. This implies that there exist $U \in N(P)$ and $\ell_2 \in L_2$ such that $(U_p - \ell_2) \cap A \in L_1$. By the heredity of L_2, if we assume that $\ell_2 \subseteq A$ and define $\ell_1 = (U - \ell_2) \cap A$. Then we have $A \cap U \in (\ell_1 \cup \ell_2) \in L_1 \cup L_2$. Thus, $NCA^*(L_1 \cup L_2,\tau) \subseteq NCA^*\left(L_1, NC\tau^*(L_1)\right) \cap NCA^*\left(L_2, NC\tau^*(L_2)\right)$ and similarly, we can get $NCA^*(L_1 \cup L_2,\tau) \subseteq NCA^*\left(L_2, \tau^*(L_1)\right)$. This gives the other inclusion, which complete the proof.

4.1 Corollary

.Let (X,τ) be a NCTS with topological neutrosophic crisp ideal L on X. Then

i) $NCA^*(L,\tau) = NCA^*(L,\tau^*)$ and $NC\tau^*(L) = NC(NC\tau^*(L))^*(L)$

ii) $NC\tau^*(L_1 \cup L_2) = \left(NC\tau^*(L_1)\right) \cup \left(NC\tau^*(L_2)\right)$

Proof

Follows by applying the previous statement.

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Soft Neutrosophic Loops and Their Generalization

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Abstract. Soft set theory is a general mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. In this paper we introduced soft neutrosophic loop,soft neutosophic biloop, soft neutrosophic N -loop with the discuission of some of their characteristics. We also introduced a new type of soft neutrophic loop, the so called soft strong neutrosophic loop which is of pure neutrosophic character. This notion also found in all the other corresponding notions of soft neutrosophic thoery. We also given some of their properties of this newly born soft structure related to the strong part of neutrosophic theory.

Keywords: Neutrosophic loop, neutrosophic biloop, neutrosophic N-loop, soft set, soft neutrosophic loop,soft neutrosophic biloop, soft neutrosophic N-loop.

1 Introduction

Florentin Smarandache for the first time intorduced the concept of neutrosophy in 1995, which is basically a new branch of philosophy which actually studies the origion, nature, and scope of neutralities. The neutrosophic logic came into being by neutrosophy. In neutrosophic logic each proposition is approximated to have the percentage of truth in a subset T , the percentage of indeterminacy in a subset I , and the percentage of falsity in a subset F . Neutrosophic logic is an extension of fuzzy logic. In fact the neutrosophic set is the generalization of classical set, fuzzy conventional set, intuitionistic fuzzy set, and interal valued fuzzy set. Neutrosophic logic is used to overcome the problems of imperciseness, indeterminate, and inconsistentness of date etc. The theoy of neutrosophy is so applicable to every field of agebra. W.B Vasantha Kandasamy and Florentin Smarandache introduced neutrosophic fields, neutrosophic rings, neutrosophic vectorspaces, neutrosophic groups, neutrosophic bigroups and neutrosophic N -groups, neutrosophic semigroups, neutrosophic bisemigroups, and neutrsosophic N -semigroups, neutrosophic loops, nuetrosophic biloops, and neutrosophic N -loops, and so on. Mumtaz ali et.al. introduced nuetosophic LA -semigoups.

Molodtsov intorduced the theory of soft set. This mathematical tool is free from parameterization inadequacy, syndrome of fuzzy set theory, rough set theory, probability theory and so on. This theory has been applied successfully in many fields such as smoothness of functions, game theory, operation reaserch, Riemann integration, Perron integration, and probability. Recently soft set theory attained much attention of the researchers since its appearance and the work based on several operations of soft set introduced in $[2,9,10]$. Some properties and algebra may be found in $[1]$. Feng et.al. introduced soft semirings in $[5]$. By means of level soft sets an adjustable approach to fuzzy soft set can be seen in $[6]$. Some other concepts together with fuzzy set and rough set were shown in $[7,8]$.

This paper is about to introduced soft nuetrosophic loop, soft neutrosphic biloop, and soft neutrosophic N -loop and the related strong or pure part of neutrosophy with the notions of soft set theory. In the proceeding section, we define soft neutrosophic loop, soft neutrosophic strong loop, and some of their properties are discuissed. In the next section, soft neutrosophic biloop are presented with their strong neutrosophic part. Also in this section some of their characterization have been made. In the last section soft neutrosophic N -loop and their coresponding strong theory have been constructed with some of their properties.

2 Fundamental Concepts

Neutrosophic Loop

Definition 1. A neutrosophic loop is generated by a loop L and I denoted by $\langle L \cup I \rangle$. A neutrosophic loop in general need not be a loop for $I^2 = I$ and I may not have an inverse but every element in a loop has an inverse.

Definition 2. Let $\langle L \cup I \rangle$ be a neutrosophic loop. A proper subset $\langle P \cup I \rangle$ of $\langle L \cup I \rangle$ is called the neutrosophic subloop, if $\langle P \cup I \rangle$ is itself a neutrosophic loop under the operations of $\langle L \cup I \rangle$.

Definition 3. Let $(\langle L \cup I \rangle, \circ)$ be a neutrosophic loop of finite order. A proper subset P of $\langle L \cup I \rangle$ is said to be Lagrange neutrosophic subloop, if P is a neutrosophic subloop under the operation $\circ$ and $o(P) / \mathrm{o}\langle L \cup I \rangle$.

Definition 4. If every neutrosophic subloop of $\langle L \cup I \rangle$ is Lagrange then we call $\langle L \cup I \rangle$ to be a Lagrange neutrosophic loop.

Definition 5. If $\langle L \cup I \rangle$ has no Lagrange neutrosophic subloop then we call $\langle L \cup I \rangle$ to be a Lagrange free neutrosophic loop.

Definition 6. If $\langle L \cup I \rangle$ has atleast one Lagrange neutrosophic subloop then we call $\langle L \cup I \rangle$ to be a weakly Lagrange neutrosophic loop.

Neutrosophic Biloops

Definition 6. Let $(\langle B \cup I \rangle, *_1, *_2)$ be a non-empty neutrosophic set with two binary operations $*_1, *_2$, $\langle B \cup I \rangle$ is a neutrosophic biloop if the following conditions are satisfied.

1. $\langle B \cup I \rangle = P_1 \cup P_2$ where P_1 and P_2 are proper subsets of $\langle B \cup I \rangle$.
2. $(P_1, *_1)$ is a neutrosophic loop.
3. $(P_2, *_2)$ is a group or a loop.

Definition 7. Let $(\langle B \cup I \rangle, *_1, *_2)$ be a neutrosophic biloop. A proper subset P of $\langle B \cup I \rangle$ is said to be a neutrosophic subbiloop of $\langle B \cup I \rangle$ if $(P_1, *_1, *_2)$ is itself a neutrosophic biloop under the operations of $\langle B \cup I \rangle$.

Definition 8. Let $B = (B_1 \cup B_2, *_1, *_2)$ be a finite neutrosophic biloop. Let $P = (\mathrm{P}_1 \cup P_2, *_1, *_2)$ be a neutrosophic biloop. If $\mathrm{o(P)} / \mathrm{o(B)}$ then we call P, a Lagrange neutrosophic subbiloop of B.

Definition 9. If every neutrosophic subbiloop of B is Lagrange then we call B to be a Lagrange neutrosophic biloop.

Definition 10. If B has atleast one Lagrange neutrosophic subbiloop then we call B to be a weakly Lagrange neutrosophic biloop.

Definition 11. If B has no Lagrange neutrosophic subbiloops then we call B to be a Lagrange free neutrosophic biloop.

Neutrosophic N-loop

Definition 12. Let $S(B) = \{S(B_1) \cup S(B_2) \cup ... \cup S(B_n), *_1, *_2, ..., *_N\}$ be a non-empty neutrosophic set with N -binary operations. $S(B)$ is a neutrosophic N -loop if $S(B) = S(B_1) \cup S(B_2) \cup ... \cup S(B_n)$, $S(B_i)$ are proper subsets of $S(B)$ for $1 \le i \le N$ and some of $S(B_i)$ are neutrosophic loops and some of the $S(B_i)$ are groups.

Definition 13. Let $S(B) = \{S(B_1) \cup S(B_2) \cup ... \cup S(B_n), *_1, *_2, ..., *_N\}$ be a neutrosophic N -loop. A proper subset $(\mathrm{P}, *_1, *_2, ..., *_N)$ of $S(B)$ is said to be a neutrosophic sub N -loop of $S(B)$ if P itself is a neutrosophic N -loop under the operations of $S(B)$.

Definition 14. Let $(L = L_1 \cup L_2 \cup ... \cup L_N, *_1, *_2, ..., *_N)$ be a neutrosophic N -loop of finite order. Suppose P is a proper subset of L, which is a neutrosophic sub N -loop. If $o(P) / o(L)$ then we call P a Lagrange neutrosophic sub N -loop.

Definition 15. If every neutrosophic sub N -loop is Lagrange then we call L to be a Lagrange neutrosophic N -loop.

Definition 16. If L has atleast one Lagrange neutrosophic sub N -loop then we call L to be a weakly Lagrange neutrosophic N -loop.

Definition 17. If L has no Lagrange neutrosophic sub N -loop then we call L to be a Lagrange free neutrosophic N -loop.

Soft Sets

Throughout this subsection U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U , and $A, B \subset E$. Molodtsov defined the soft set in the following manner:

Definition 7. A pair (F, A) is called a soft set over U where F is a mapping given by $F : A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U . For $a \in A$, $F(\mathrm{a})$ may be considered as the set of a -elements of the soft set (F, A) , or as the set of a -approximate elements of the soft set.

Example 1. Suppose that U is the set of shops. E is the set of parameters and each parameter is a word or sentence. Let

$$E = \left\{ \begin{matrix} \text{high rent,normal rent,} \\ \text{in good condition,in bad condition} \end{matrix} \right\}.$$

Let us consider a soft set (F, A) which describes the attractiveness of shops that Mr. Z is taking on rent. Suppose that there are five houses in the universe $U = \{s_1, s_2, s_3, s_4, s_5\}$ under consideration, and that $A = \{a_1, a_2, a_3\}$ be the set of parameters where

a_1 stands for the parameter 'high rent,

a_2 stands for the parameter 'normal rent,

a_3 stands for the parameter 'in good condition.

Suppose that

$$F(a_1) = \{s_1, s_4\} ,$$
$$F(a_2) = \{s_2, s_5\} ,$$
$$F(a_3) = \{s_3\}.$$

The soft set (F, A) is an approximated family $\{F(a_i), i = 1, 2, 3\}$ of subsets of the set U which gives us a collection of approximate description of an object. Then (F, A) is a soft set as a collection of approximations over U , where

$$F(a_1) = high\ rent = \{s_1, s_2\},$$
$$F(a_2) = normal\ rent = \{s_2, s_5\},$$
$$F(a_3) = in\ good\ condition = \{s_3\}.$$

Definition 8. For two soft sets (F, A) and (H, C) over U , (F, A) is called a soft subset of (H, C) if

1. $A \subseteq C$ and
2. $F(a) \subseteq H(a)$, for all $x \in A$.

This relationship is denoted by $(F, A) \subset (H, \mathrm{C})$. Similarly (F, A) is called a soft superset of (H, C) if (H, C) is a soft subset of (F, A) which is denoted by $(F, A) \supset (H, \mathrm{C})$.

Definition 9. Two soft sets (F, A) and (H, C) over U are called soft equal if (F, A) is a soft subset of (H, C) and (H, C) is a soft subset of (F, A) .

Definition 10. Let (F, A) and (K, C) be two soft sets over a common universe U such that $A \cap C \neq \phi$. Then their restricted intersection is denoted by $(F, A) \cap_R (K, \mathrm{C}) = (H, \mathrm{D})$ where (H, D) is defined as $H(c) = F(c) \cap \mathrm{K}(c)$ for all $c \in \mathrm{D} = A \cap C$.

Definition 11. The extended intersection of two soft sets (F, A) and (K, C) over a common universe U is the soft set (H, D) , where $D = A \cup C$, and for all $c \in C$, $H(c)$ is defined as

$$H(c) = \begin{cases} F(c) & \text{if } \mathrm{c} \in A - C, \\ \mathrm{K}(c) & \text{if } \mathrm{c} \in C - A, \\ F(c) \cap \mathrm{K}(c) & \text{if } \mathrm{c} \in A \cap C. \end{cases}$$

We write $(F, A) \cap_\varepsilon (K, C) = (H, \mathrm{D})$.

Definition 12. The restricted union of two soft sets (F, A) and (K, C) over a common universe U is the soft set (H, D), where $D = A \cup C$, and for all $c \in D$, $H(c)$ is defined as $H(c) = F(c) \cup \mathrm{K}(c)$ for all $c \in D$. We write it as

$(F,A)\cup_R (K,\mathrm{C}) = (H,\mathrm{D})$.

Definition 13. The extended union of two soft sets (F,A) and (K,C) over a common universe U is the soft set (H,D), where $D = A \cup C$, and for all $c \in D$, $H(c)$ is defined as

$$H(c) = \begin{cases} F(c) & \text{if } c \in A - C, \\ \mathrm{K}(c) & \text{if } c \in C - A, \\ F(c) \cup K(c) & \text{if } c \in A \cap C. \end{cases}$$

We write $(F,A)\cup_\varepsilon (K,\mathrm{C}) = (H,\mathrm{D})$.

3 Soft Neutrosophic Loop

Definition 14. Let $\langle L \cup I\rangle$ be a neutrosophic loop and (F,A) be a soft set over $\langle L \cup I\rangle$. Then (F,A) is called soft neutrosophic loop if and only if $F(a)$ is neutrosophic subloop of $\langle L \cup I\rangle$ for all $a \in A$.

Example 2. Let $\langle L \cup I\rangle = \langle L_7(4) \cup I\rangle$ be a neutrosophic loop where $L_7(4)$ is a loop. Then (F,A) is a soft neutrosophic loop over $\langle L \cup I\rangle$, where

$$F(a_1) = \{\langle e, eI, 2, 2I\rangle\}, F(a_2) = \{\langle e, 3\rangle\},$$
$$\mathrm{F}(a_3) = \{\langle e, eI\rangle\}.$$

Theorem 1. Every soft neutrosophic loop over $\langle L \cup I\rangle$ contains a soft loop over L.

Proof. The proof is straightforward.

Theorem 2. Let (F,A) and (H,A) be two soft neutrosophic loops over $\langle L \cup I\rangle$. Then their intersection $(F,A)\cap(H,A)$ is again soft neutrosophic loop over $\langle L \cup I\rangle$.

Proof. The proof is staightforward.

Theorem 3. Let (F,A) and (H,C) be two soft neutrosophic loops over $\langle L \cup I\rangle$. If $A \cap C = \phi$, then $(F,A)\cup(H,\mathrm{C})$ is a soft neutrosophic loop over $\langle L \cup I\rangle$.

Remark 1. The extended union of two soft neutrosophic loops (F,A) and (K,C) over $\langle L \cup I\rangle$ is not a soft neutrosophic loop over $\langle L \cup I\rangle$.

With the help of example we can easily check the above remark.

Proposition 1. The extended intersection of two soft neutrosophic loopps over $\langle L \cup I\rangle$ is a soft neutrosophic loop over $\langle L \cup I\rangle$.

Remark 2. The restricted union of two soft neutrosophic loops (F,A) and (K,C) over $\langle L \cup I\rangle$ is not a soft neutrosophic loop over $\langle L \cup I\rangle$.

One can easily check it by the help of example.

Proposition 2. The restricted intersection of two soft neutrosophic loops over $\langle L \cup I\rangle$ is a soft neutrosophic loop over $\langle L \cup I\rangle$.

Proposition 3. The AND operation of two soft neutrosophic loops over $\langle L \cup I\rangle$ is a soft neutrosophic loop over $\langle L \cup I\rangle$.

Remark 3. The OR operation of two soft neutosophic loops over $\langle L \cup I\rangle$ may not be a soft nuetrosophic loop over $\langle L \cup I\rangle$.

Definition 15. Let $\langle L_n(m) \cup I\rangle = \{e, 1, 2, ..., n, eI, 1I, 2I, ..., nI\}$ be a new class of neutrosophic loop and (F,A) be a soft neutrosophic loop over $\langle L_n(m) \cup I\rangle$. Then (F,A) is called soft new class neutrosophic loop if $F(a)$ is a neutrosophic subloop of $\langle L_n(m) \cup I\rangle$ for all $a \in A$.

Example 3. Let

$\langle L_5(3) \cup I \rangle = \{e,1,2,3,4,5,eI,1I,2I,3I,4I,5I\}$ be a new class of neutrosophic loop. Let $A = \{a_1,a_2,a_3,a_4,a_5\}$ be a set of parameters. Then (F,A) is soft new class neutrosophic loop over $\langle L_5(3) \cup I \rangle$, where

$$F(a_1) = \{e,eI,1,1I\}, F(a_2) = \{e,eI,2,2I\},$$
$$F(a_3) = \{e,eI,3,3I\}, F(a_3) = \{e,eI,4,4I\},$$
$$F(a_5) = \{e,eI,5,5I\}.$$

Theorem 4. Every soft new class neutrosophic loop over $\langle L_n(m) \cup I \rangle$ is a soft neutrosophic loop over $\langle L_n(m) \cup I \rangle$ but the converse is not true.

Proposition 4. Let (F,A) and (K,C) be two soft new class neutrosophic loops over $\langle L_n(m) \cup I \rangle$. Then

1) Their extended intersection $(F,A) \cap_E (K,C)$ is a soft new class neutrosophic loop over $\langle L_n(m) \cup I \rangle$.
2) Their restricted intersection $(F,A) \cap_R (K,C)$ is a soft new classes neutrosophic loop over $\langle L_n(m) \cup I \rangle$.
3) Their AND operation $(F,A) \wedge (K,C)$ is a soft new class neutrosophic loop over $\langle L_n(m) \cup I \rangle$.

Remark 4. Let (F,A) and (K,C) be two soft new class neutrosophic loops over $\langle L_n(m) \cup I \rangle$. Then

1) Their extended union $(F,A) \cup_E (K,C)$ is not a soft new class neutrosophic loop over $\langle L_n(m) \cup I \rangle$.
2) Their restricted union $(F,A) \cup_R (K,C)$ is not a soft new class neutrosophic loop over $\langle L_n(m) \cup I \rangle$.
3) Their OR operation $(F,A) \vee (K,C)$ is not a soft new class neutrosophic loop over $\langle L_n(m) \cup I \rangle$.

One can easily verify $(1),(2)$, and (3) by the help of examples.

Definition 16. Let (F,A) be a soft neutrosophic loop over $\langle L \cup I \rangle$. Then (F,A) is called the identity soft neutrosophic loop over $\langle L \cup I \rangle$ if $F(a) = \{e\}$ for all $a \in A$, where e is the identity element of $\langle L \cup I \rangle$.

Definition 17. Let (F,A) be a soft neutrosophic loop over $\langle L \cup I \rangle$. Then (F,A) is called an absolute soft neutrosophic loop over $\langle L \cup I \rangle$ if $F(a) = \langle L \cup I \rangle$ for all $a \in A$.

Definition 18. Let (F,A) and (H,C) be two soft neutrosophic loops over $\langle L \cup I \rangle$. Then (H,C) is callsed soft neutrosophic subloop of (F,A), if

1. $C \subseteq A$.
2. $H(a)$ is a neutrosophic subloop of $F(a)$ for all $a \in A$.

Example 4. Consider the neutrosophic loop

$$\langle L_{15}(2) \cup I \rangle = \{e,1,2,3,4,...,15,eI,1I,2I,...,14I,15I\}$$

of order 32. Let $A = \{a_1,a_2,a_3\}$ be a set of parameters. Then (F,A) is a soft neutrosophic loop over $\langle L_{15}(2) \cup I \rangle$, where

$$F(a_1) = \{e,2,5,8,11,14,eI,2I,5I,8I,11I,14I\},$$
$$F(a_2) = \{e,2,5,8,11,14\},$$
$$F(a_3) = \{e,3,eI,3I\}.$$

Thus (H,C) is a soft neutrosophic subloop of (F,A) over $\langle L_{15}(2) \cup I \rangle$, where

$$H(a_1) = \{e,eI,2I,5I,8I,11I,14I\},$$
$$H(a_2) = \{e,3\}.$$

Theorem 5. Every soft loop over L is a soft neutrosophic subloop over $\langle L \cup I \rangle$.

Definition 19. Let $\langle L \cup I \rangle$ be a neutrosophic loop and (F,A) be a soft set over $\langle L \cup I \rangle$. Then (F,A) is called soft normal neutrosophic loop if and only if $F(a)$ is normal neutrosophic subloop of $\langle L \cup I \rangle$ for all

$a \in A$.

Example 5. Let $\langle L_5(3) \cup I \rangle = \{e,1,2,3,4,5,eI,1I,2I,3I,4I,5I\}$ be a neutrosophic loop. Let $A = \{a_1, a_2, a_3\}$ be a set of parameters. Then clearly (F, A) is soft normal neutrosophic loop over $\langle L_5(3) \cup I \rangle$, where

$$F(a_1) = \{e, eI, 1, 1I\}, F(a_2) = \{\mathrm{e}, \mathrm{eI}, 2, 2\mathrm{I}\},$$
$$F(a_3) = \{\mathrm{e}, \mathrm{eI}, 3, 3\mathrm{I}\}.$$

Theorem 6. Every soft normal neutrosophic loop over $\langle L \cup I \rangle$ is a soft neutrosophic loop over $\langle L \cup I \rangle$ but the converse is not true.

Proposition 5. Let (F, A) and (K, C) be two soft normal neutrosophic loops over $\langle L \cup I \rangle$. Then

1. Their extended intersection $(F, A) \cap_E (K, \mathrm{C})$ is a soft normal neutrosophic loop over $\langle L \cup I \rangle$.
2. Their restricted intersection $(F, A) \cap_R (K, \mathrm{C})$ is a soft normal neutrosophic loop over $\langle L \cup I \rangle$.
3. Their AND operation $(F, A) \wedge (K, \mathrm{C})$ is a soft normal neutrosophic loop over $\langle L \cup I \rangle$.

Remark 5. Let (F, A) and (K, C) be two soft normal neutrosophic loops over $\langle L \cup I \rangle$. Then

1. Their extended union $(F, A) \cup_E (K, \mathrm{C})$ is not a soft normal neutrosophic loop over $\langle L \cup I \rangle$.
2. Their restricted union $(F, A) \cup_R (K, \mathrm{C})$ is not a soft normal neutrosophic loop over $\langle L \cup I \rangle$.
3. Their OR operation $(F, A) \vee (K, \mathrm{C})$ is not a soft normal neutrosophic loop over $\langle L \cup I \rangle$.

One can easily verify $(1), (2)$, and (3) by the help of examples.

Definition 20. Let $\langle L \cup I \rangle$ be a neutrosophic loop and (F, A) be a soft neutrosophic loop over $\langle L \cup I \rangle$. Then (F, A) is called soft Lagrange neutrosophic loop if $F(a)$ is a Lagrange neutrosophic subloop of $\langle L \cup I \rangle$ for all $a \in A$.

Example 6. In Example (1), (F, A) is a soft Lagrange neutrosophic loop over $\langle L \cup I \rangle$.

Theorem 7. Every soft Lagrange neutrosophic loop over $\langle L \cup I \rangle$ is a soft neutrosophic loop over $\langle L \cup I \rangle$ but the converse is not true.

Theorem 8. If $\langle L \cup I \rangle$ is a Lagrange neutrosophic loop, then (F, A) over $\langle L \cup I \rangle$ is a soft Lagrange neutrosophic loop but the converse is not true.

Remark 6. Let (F, A) and (K, C) be two soft Lagrange neutrosophic loops over $\langle L \cup I \rangle$. Then

1. Their extended intersection $(F, A) \cap_E (K, \mathrm{C})$ is not a soft Lagrange neutrosophic loop over $\langle L \cup I \rangle$.
2. Their restricted intersection $(F, A) \cap_R (K, \mathrm{C})$ is not a soft Lagrange neutrosophic loop over $\langle L \cup I \rangle$.
3. Their AND operation $(F, A) \wedge (K, \mathrm{C})$ is not a soft Lagrange neutrosophic loop over $\langle L \cup I \rangle$.
4. Their extended union $(F, A) \cup_E (K, \mathrm{C})$ is not a soft Lagrnage neutrosophic loop over $\langle L \cup I \rangle$.
5. Their restricted union $(F, A) \cup_R (K, \mathrm{C})$ is not a soft Lagrange neutrosophic loop over $\langle L \cup I \rangle$.
6. Their OR operation $(F, A) \vee (K, \mathrm{C})$ is not a soft Lagrange neutrosophic loop over $\langle L \cup I \rangle$.

One can easily verify $(1), (2), (3), (4), (5)$ and (6) by the help of examples.

Definition 21. Let $\langle L \cup I \rangle$ be a neutrosophic loop and

(F,A) be a soft neutrosophic loop over $\langle L \cup I \rangle$. Then (F,A) is called soft weak Lagrange neutrosophic loop if atleast one $F(a)$ is not a Lagrange neutrosophic subloop of $\langle L \cup I \rangle$ for some $a \in A$.

Example 7. Consider the neutrosophic loop

$$\langle L_{15}(2) \cup I \rangle = \{e,1,2,3,4,...,15,eI,1I,2I,...,14I,15I\}$$

of order 32. Let $A = \{a_1, a_2, a_3\}$ be a set of parameters. Then (F,A) is a soft weakly Lagrange neutrosophic loop over $\langle L_{15}(2) \cup I \rangle$, where

$$F(a_1) = \{e,2,5,8,11,14,eI,2I,5I,8I,11I,14I\},$$
$$F(a_2) = \{\mathrm{e},2,5,8,11,14\},$$
$$\mathrm{F}(a_3) = \{e,3,eI,3I\}.$$

Theorem 9. Every soft weak Lagrange neutrosophic loop over $\langle L \cup I \rangle$ is a soft neutrosophic loop over $\langle L \cup I \rangle$ but the converse is not true.

Theorem 10. If $\langle L \cup I \rangle$ is weak Lagrange neutrosophic loop, then (F,A) over $\langle L \cup I \rangle$ is also soft weak Lagrange neutrosophic loop but the converse is not true.

Remark 7. Let (F,A) and (K,C) be two soft weak Lagrange neutrosophic loops over $\langle L \cup I \rangle$. Then

1. Their extended intersection $(F,A) \cap_E (K,\mathrm{C})$ is not a soft weak Lagrange neutrosophic loop over $\langle L \cup I \rangle$.
2. Their restricted intersection $(F,A) \cap_R (K,\mathrm{C})$ is not a soft weak Lagrange neutrosophic loop over $\langle L \cup I \rangle$.
3. Their AND operation $(F,A) \wedge (K,\mathrm{C})$ is not a soft weak Lagrange neutrosophic loop over $\langle L \cup I \rangle$.
4. Their extended union $(F,A) \cup_E (K,\mathrm{C})$ is not a soft weak Lagrnage neutrosophic loop over $\langle L \cup I \rangle$.
5. Their restricted union $(F,A) \cup_R (K,\mathrm{C})$ is not a soft weak Lagrange neutrosophic loop over $\langle L \cup I \rangle$.
6. Their OR operation $(F,A) \vee (K,\mathrm{C})$ is not a soft weak Lagrange neutrosophic loop over $\langle L \cup I \rangle$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 22. Let $\langle L \cup I \rangle$ be a neutrosophic loop and (F,A) be a soft neutrosophic loop over $\langle L \cup I \rangle$. Then (F,A) is called soft Lagrange free neutrosophic loop if $F(a)$ is not a lagrange neutrosophic subloop of $\langle L \cup I \rangle$ for all $a \in A$.

Theorem 11. Every soft Lagrange free neutrosophic loop over $\langle L \cup I \rangle$ is a soft neutrosophic loop over $\langle L \cup I \rangle$ but the converse is not true.

Theorem 12. If $\langle L \cup I \rangle$ is a Lagrange free neutrosophic loop, then (F,A) over $\langle L \cup I \rangle$ is also a soft Lagrange free neutrosophic loop but the converse is not true.

Remark 8. Let (F,A) and (K,C) be two soft Lagrange free neutrosophic loops over $\langle L \cup I \rangle$. Then

1. Their extended intersection $(F,A) \cap_E (K,\mathrm{C})$ is not a soft Lagrange soft neutrosophic loop over $\langle L \cup I \rangle$.
2. Their restricted intersection $(F,A) \cap_R (K,\mathrm{C})$ is not a soft Lagrange soft neutrosophic loop over $\langle L \cup I \rangle$.
3. Their AND operation $(F,A) \wedge (K,\mathrm{C})$ is not a soft Lagrange free neutrosophic loop over $\langle L \cup I \rangle$.
4. Their extended union $(F,A) \cup_E (K,\mathrm{C})$ is not a soft Lagrnage soft neutrosophic loop over $\langle L \cup I \rangle$.
5. Their restricted union $(F,A) \cup_R (K,\mathrm{C})$ is not a soft Lagrange free neutrosophic loop over $\langle L \cup I \rangle$.

6. Their OR operation $(F,A)\vee(K,C)$ is not a soft Lagrange free neutrosophic loop over $\langle L\cup I\rangle$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

4 Soft Neutrosophic Strong Loop

Definition 23. Let $\langle L\cup I\rangle$ be a neutrosophic loop and (F,A) be a soft set over $\langle L\cup I\rangle$. Then (F,A) is called soft neutrosophic strong loop if and only if $F(a)$ is a strong neutrosophic subloop of $\langle L\cup I\rangle$ for all $a\in A$.

Proposition 6. Let (F,A) and (K,C) be two soft neutrosophic strong loops over $\langle L\cup I\rangle$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ is a soft neutrosophic strong loop over $\langle L\cup I\rangle$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ is a soft neutrosophic strong loop over $\langle L\cup I\rangle$.
3. Their AND operation $(F,A)\wedge(K,C)$ is a soft neutrosophic strong loop over $\langle L\cup I\rangle$.

Remark 9. Let (F,A) and (K,C) be two soft neutrosophic strong loops over $\langle L\cup I\rangle$. Then

1. Their extended union $(F,A)\cup_E(K,C)$ is not a soft neutrosophic strong loop over $\langle L\cup I\rangle$.
2. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft neutrosophic strong loop over $\langle L\cup I\rangle$.
3. Their OR operation $(F,A)\vee(K,C)$ is not a soft neutrosophic strong loop over $\langle L\cup I\rangle$.

One can easily verify $(1),(2),$ and (3) by the help of examples.

Definition 24. Let (F,A) and (H,C) be two soft neutrosophic strong loops over $\langle L\cup I\rangle$. Then (H,C) is called soft neutrosophic strong subloop of (F,A), if

1. $C\subseteq A$.
2. $H(a)$ is a neutrosophic strong subloop of $F(a)$ for all $a\in A$.

Definition 25. Let $\langle L\cup I\rangle$ be a neutrosophic strong loop and (F,A) be a soft neutrosophic loop over $\langle L\cup I\rangle$. Then (F,A) is called soft Lagrange neutrosophic strong loop if $F(a)$ is a Lagrange neutrosophic strong subloop of $\langle L\cup I\rangle$ for all $a\in A$.

Theorem 13. Every soft Lagrange neutrosophic strong loop over $\langle L\cup I\rangle$ is a soft neutrosophic loop over $\langle L\cup I\rangle$ but the converse is not true.

Theorem 14. If $\langle L\cup I\rangle$ is a Lagrange neutrosophic strong loop, then (F,A) over $\langle L\cup I\rangle$ is a soft Lagrange neutrosophic loop but the converse is not true.

Remark 10. Let (F,A) and (K,C) be two soft Lagrange neutrosophic strong loops over $\langle L\cup I\rangle$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ is not a soft Lagrange neutrosophic strong loop over $\langle L\cup I\rangle$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ is not a soft Lagrange strong neutrosophic loop over $\langle L\cup I\rangle$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft Lagrange neutrosophic strong loop over $\langle L\cup I\rangle$.
4. Their extended union $(F,A)\cup_E(K,C)$ is not a soft Lagrnage neutrosophic strong loop over $\langle L\cup I\rangle$.
5. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft Lagrange neutrosophic strong loop over $\langle L\cup I\rangle$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft Lagrange neutrosophic strong loop over

$\langle L \cup I \rangle$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 26. Let $\langle L \cup I \rangle$ be a neutrosophic strong loop and (F, A) be a soft neutrosophic loop over $\langle L \cup I \rangle$. Then (F, A) is called soft weak Lagrange neutrosophic strong loop if atleast one $F(a)$ is not a Lagrange neutrosophic strong subloop of $\langle L \cup I \rangle$ for some $a \in A$.

Theorem 15. Every soft weak Lagrange neutrosophic strong loop over $\langle L \cup I \rangle$ is a soft neutrosophic loop over $\langle L \cup I \rangle$ but the converse is not true.

Theorem 16. If $\langle L \cup I \rangle$ is weak Lagrange neutrosophic strong loop, then (F, A) over $\langle L \cup I \rangle$ is also soft weak Lagrange neutrosophic strong loop but the converse is not true.

Remark 11. Let (F, A) and (K, C) be two soft weak Lagrange neutrosophic strong loops over $\langle L \cup I \rangle$. Then

1. Their extended intersection $(F, A) \cap_E (K, \mathrm{C})$ is not a soft weak Lagrange neutrosophic strong loop over $\langle L \cup I \rangle$.
2. Their restricted intersection $(F, A) \cap_R (K, \mathrm{C})$ is not a soft weak Lagrange neutrosophic strong loop over $\langle L \cup I \rangle$.
3. Their AND operation $(F, A) \wedge (K, \mathrm{C})$ is not a soft weak Lagrange neutrosophic strong loop over $\langle L \cup I \rangle$.
4. Their extended union $(F, A) \cup_E (K, \mathrm{C})$ is not a soft weak Lagrnage neutrosophic strong loop over $\langle L \cup I \rangle$.
5. Their restricted union $(F, A) \cup_R (K, \mathrm{C})$ is not a soft weak Lagrange neutrosophic strong loop over $\langle L \cup I \rangle$.
6. Their OR operation $(F, A) \vee (K, \mathrm{C})$ is not a soft weak Lagrange neutrosophic strong loop over

$\langle L \cup I \rangle$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 27. Let $\langle L \cup I \rangle$ be a neutrosophic strong loop and (F, A) be a soft neutrosophic loop over $\langle L \cup I \rangle$. Then (F, A) is called soft Lagrange free neutrosophic strong loop if $F(a)$ is not a lagrange neutrosophic strong subloop of $\langle L \cup I \rangle$ for all $a \in A$.

Theorem 17. Every soft Lagrange free neutrosophic strong loop over $\langle L \cup I \rangle$ is a soft neutrosophic loop over $\langle L \cup I \rangle$ but the converse is not true.

Theorem 18. If $\langle L \cup I \rangle$ is a Lagrange free neutrosophic strong loop, then (F, A) over $\langle L \cup I \rangle$ is also a soft Lagrange free neutrosophic strong loop but the converse is not true.

Remark 12. Let (F, A) and (K, C) be two soft Lagrange free neutrosophic strong loops over $\langle L \cup I \rangle$. Then

1. Their extended intersection $(F, A) \cap_E (K, \mathrm{C})$ is not a soft Lagrange free neutrosophic strong loop over $\langle L \cup I \rangle$.
2. Their restricted intersection $(F, A) \cap_R (K, \mathrm{C})$ is not a soft Lagrange free neutrosophic strong loop over $\langle L \cup I \rangle$.
3. Their AND operation $(F, A) \wedge (K, \mathrm{C})$ is not a soft Lagrange free neutrosophic strong loop over $\langle L \cup I \rangle$.
4. Their extended union $(F, A) \cup_E (K, \mathrm{C})$ is not a soft Lagrnage free strong neutrosophic strong loop over $\langle L \cup I \rangle$.
5. Their restricted union $(F, A) \cup_R (K, \mathrm{C})$ is not a soft Lagrange free neutrosophic loop over $\langle L \cup I \rangle$.
6. Their OR operation $(F, A) \vee (K, \mathrm{C})$ is not a soft Lagrange free neutrosophic strong loop over

$\langle L \cup I \rangle$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Soft Neutrosophic Biloop

Definition 27. Let $(\langle B \cup I \rangle, *_1, *_2)$ be a neutrosophic biloop and (F, A) be a soft set over $(\langle B \cup I \rangle, *_1, *_2)$. Then (F, A) is called soft neutrosophic biloop if and only if $F(a)$ is a neutrosophic subbiloop of $(\langle B \cup I \rangle, *_1, *_2)$ for all $a \in A$.

Example 8. Let $(\langle B \cup I \rangle, *_1, *_2) = (\{e, 1, 2, 3, 4, 5, eI, 1I, 2I, 3I, 4I, 5I\} \cup \{g : g^6 = e\}$ be a neutrosophic biloop. Let $A = \{a_1, a_2\}$ be a set of parameters. Then (F, A) is clearly soft neutrosophic biloop over $(\langle B \cup I \rangle, *_1, *_2)$, where

$$F(a_1) = \{\mathrm{e}, 2, \mathrm{eI}, 2\mathrm{I}\} \cup \{\mathrm{g}^2, g^4, e\},$$
$$F(a_2) = \{\mathrm{e}, 3, \mathrm{eI}, 3\mathrm{I}\} \cup \{\mathrm{g}^3, e\}.$$

Theorem 19. Let (F, A) and (H, A) be two soft neutrosophic biloops over $(\langle B \cup I \rangle, *_1, *_2)$. Then their intersection $(F, A) \cap (H, A)$ is again a soft neutrosophic biloop over $(\langle B \cup I \rangle, *_1, *_2)$.

Proof. Straightforward.

Theorem 20. Let (F, A) and (H, C) be two soft neutrosophic biloops over $(\langle B \cup I \rangle, *_1, *_2)$ such that $A \cap C = \phi$. Then their union is soft neutrosophic biloop over $(\langle B \cup I \rangle, *_1, *_2)$.

Proof. Straightforward.

Proposition 7. Let (F, A) and (K, C) be two soft neutrosophic biloops over $(\langle B \cup I \rangle, *_1, *_2)$. Then

1. Their extended intersection $(F, A) \cap_E (K, \mathrm{C})$ is a soft neutrosophic biloop over $(\langle B \cup I \rangle, *_1, *_2)$.
2. Their restricted intersection $(F, A) \cap_R (K, \mathrm{C})$ is a soft neutrosophic biloop over $(\langle B \cup I \rangle, *_1, *_2)$.
3. Their AND operation $(F, A) \wedge (K, \mathrm{C})$ is a soft neutrosophic biloop over $(\langle B \cup I \rangle, *_1, *_2)$.

Remark 13. Let (F, A) and (K, C) be two soft neutrosophic biloops over $(\langle B \cup I \rangle, *_1, *_2)$. Then

1. Their extended union $(F, A) \cup_E (K, \mathrm{C})$ is not a soft neutrosophic biloop over $(\langle B \cup I \rangle, *_1, *_2)$.
2. Their restricted union $(F, A) \cup_R (K, \mathrm{C})$ is not a soft neutrosophic biloop over $(\langle B \cup I \rangle, *_1, *_2)$.
3. Their OR operation $(F, A) \vee (K, \mathrm{C})$ is not a soft neutrosophic biloop over $(\langle B \cup I \rangle, *_1, *_2)$.

One can easily verify $(1),(2),$ and (3) by the help of examples.

Definition 28. Let $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$ be a new class neutrosophic biloop and (F, A) be a soft set over $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$. Then $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$ is called soft new class neutrosophic subbiloop if and only if $F(a)$ is a neutrosophic subbiloop of $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$ for all $a \in A$.

Example 9. Let $B = (B_1 \cup B_2, *_1, *_2)$ be a new class neutrosophic biloop, where $B_1 = \langle L_5(3) \cup I \rangle = \{e, 1, 2, 3, 4, 5, eI, 2I, 3I, 4I, 5\mathrm{I}\}$ be a new class of neutrosophic loop and $B_2 = \{g : g^{12} = e\}$ is a group.

$\{e, eI, 1, 1I\} \cup \{1, g^6\}$,
$\{e, eI, 2, 2I\} \cup \{1, g^2, g^4, g^6, g^8, g^{10}\}$,
$\{e, eI, 3, 3I\} \cup \{1, g^3, g^6, g^9\}$,

$\{e, eI, 4, 4I\} \cup \{1, g^4, g^8\}$ are neutrosophic subloops of B. Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of parameters. Then (F, A) is soft new class neutrosophic biloop over B, where

$$F(a_1) = \{e, eI, 1, 1I\} \cup \{e, g^6\},$$
$$F(a_2) = \{\mathrm{e}, \mathrm{eI}, 2, 2\mathrm{I}\} \cup \{e, g^2, g^4, g^6, g^8, g^{10}\},$$
$$F(a_3) = \{e, eI, 3, 3I\} \cup \{e, g^3, g^6, g^6\},$$
$$F(a_4) = \{e, eI, 4, 4I\} \cup \{e, g^4, g^8\}.$$

Theorem 21. Every soft new class neutrosophic biloop over $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$ is trivially a soft neutrosophic biloop over but the converse is not true.

Proposition 8. Let (F, A) and (K, C) be two soft new class neutrosophic biloops over $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$. Then

1. Their extended intersection $(F, A) \cap_E (K, \mathrm{C})$ is a soft new class neutrosophic biloop over $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$.
2. Their restricted intersection $(F, A) \cap_R (K, \mathrm{C})$ is a soft new class neutrosophic biloop over $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$.
3. Their AND operation $(F, A) \wedge (K, \mathrm{C})$ is a soft new class neutrosophic biloop over $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$.

Remark 14. Let (F, A) and (K, C) be two soft new class neutrosophic biloops over $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$. Then

1. Their extended union $(F, A) \cup_E (K, \mathrm{C})$ is not a soft new class neutrosophic biloop over $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$.
2. Their restricted union $(F, A) \cup_R (K, \mathrm{C})$ is not a soft new class neutrosophic biloop over $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$.
3. Their OR operation $(F, A) \vee (K, \mathrm{C})$ is not a soft new class neutrosophic biloop over $B = (\langle L_n(m) \cup I \rangle \cup B_2, *_1, *_2)$.

One can easily verify $(1), (2),$ and (3) by the help of examples.

Definition 29. Let (F, A) be a soft neutrosophic biloop over $B = (\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$. Then (F, A) is called the identity soft neutrosophic biloop over $B = (\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$ if $F(a) = \{e_1, e_2\}$ for all $a \in A$, where e_1, e_2 are the identities of $B = (\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$ respectively.

Definition 30. Let (F, A) be a soft neutrosophic biloop over $B = (\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$. Then (F, A) is called an absolute-soft neutrosophic biloop over $B = (\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$ if $F(a) = (\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$ for all $a \in A$.

Definition 31. Let (F, A) and (H, C) be two soft neutrosophic biloops over $B = (\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$. Then (H, C) is called soft neutrosophic subbiloop of (F, A), if

1. $C \subseteq A$.
2. $H(a)$ is a neutrosophic subbiloop of $F(a)$ for all $a \in A$.

Example 10. Let $B = (B_1 \cup B_2, *_1, *_2)$ be a neutrosophic biloop, where $B_1 = \langle L_5(3) \cup I \rangle = \{e, 1, 2, 3, 4, 5, eI, 2I, 3I, 4I, 5\mathrm{I}\}$ be a new class of neutrosophic loop and $B_2 = \{g : g^{12} = e\}$ is a group. Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of parameters. Then (F, A) is soft neutrosophic biloop over B, where

$$F(a_1) = \{e, eI, 1, 1I\} \cup \{e, g^6\},$$
$$F(a_2) = \{\mathrm{e}, \mathrm{eI}, 2, 2\mathrm{I}\} \cup \{e, g^2, g^4, g^6, g^8, g^{10}\},$$
$$F(a_3) = \{e, eI, 3, 3I\} \cup \{e, g^3, g^6, g^6\},$$
$$F(a_4) = \{e, eI, 4, 4I\} \cup \{e, g^4, g^8\}.$$

Then (H, C) is soft neutrosophic subbiloop of (F, A), where

$H(a_1)=\{e,2) \cup \{e,g^2\}$,
$H(a_2)=\{\mathrm{e},\mathrm{eI},3,3I\} \cup \{e,g^6\}$.

Definition 32. Let $(\langle B \cup I \rangle, *_1, *_2)$ be a neutrosophic biloop and (F,A) be a soft set over $(\langle B \cup I \rangle, *_1, *_2)$. Then (F,A) is called soft Lagrange neutrosophic biloop if and only if $F(a)$ is Lagrange neutrosophic subbiloop of $(\langle B \cup I \rangle, *_1, *_2)$ for all $a \in A$.

Example 11. Let $B=(\mathrm{B}_1 \cup B_2, *_1, *_2)$ be a neutrosophic biloop of order 20, where $B_1=\langle L_5(3) \cup I \rangle$ and $B_2=\{g : g^8=e\}$. Then clearly (F,A) is a soft Lagrange soft neutrosophic biloop over $(\langle B \cup I \rangle, *_1, *_2)$, where

$F(a_1)=\{\mathrm{e},\mathrm{eI},2,2\,\mathrm{I}\} \cup \{\mathrm{e}\}$,
$\mathrm{F}(a_2)=\{e,eI,3,3I\} \cup \{e\}$.

Theorem 22. Every soft Lagrange neutrosophic biloop over $B=(\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$ is a soft neutrosophic biloop but the converse is not true.

Remark 15. Let (F,A) and (K,C) be two soft Lagrange neutrosophic biloops over $B=(\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$. Then

1. Their extended intersection $(F,A) \cap_E (K,\mathrm{C})$ is not a soft Lagrange neutrosophic biloop over $B=(\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$.
2. Their restricted intersection $(F,A) \cap_R (K,\mathrm{C})$ is not a soft Lagrange neutrosophic biloop over $B=(\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$.
3. Their AND operation $(F,A) \wedge (K,\mathrm{C})$ is not a soft Lagrange neutrosophic biloop over $B=(\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$.
4. Their extended union $(F,A) \cup_E (K,\mathrm{C})$ is not a soft Lagrnage neutrosophic biloop over $B=(\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$.
5. Their restricted union $(F,A) \cup_R (K,\mathrm{C})$ is not a soft Lagrange neutrosophic biloop over $B=(\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$.
6. Their OR operation $(F,A) \vee (K,\mathrm{C})$ is not a soft Lagrange neutrosophic biloop over $B=(\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 33. Let $(\langle B \cup I \rangle, *_1, *_2)$ be a neutrosophic biloop and (F,A) be a soft set over $(\langle B \cup I \rangle, *_1, *_2)$. Then (F,A) is called soft weakly Lagrange neutrosophic biloop if atleast one $F(a)$ is not a Lagrange neutrosophic subbiloop of $(\langle B \cup I \rangle, *_1, *_2)$ for some $a \in A$.

Example 12. Let $B=(\mathrm{B}_1 \cup B_2, *_1, *_2)$ be a neutrosophic biloop of order 20, where $B_1=\langle L_5(3) \cup I \rangle$ and $B_2=\{g : g^8=e\}$. Then clearly (F,A) is a soft weakly Lagrange neutrosophic biloop over $(\langle B \cup I \rangle, *_1, *_2)$, where

$F(a_1)=\{\mathrm{e},\mathrm{eI},2,2\,\mathrm{I}\} \cup \{\mathrm{e}\}$,
$\mathrm{F}(a_2)=\{e,eI,3,3I\} \cup \{\mathrm{e},\mathrm{g}^4\}$.

Theorem 23. Every soft weakly Lagrange neutrosophic biloop over $B=(\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$ is a soft neutrosophic biloop but the converse is not true.

Theorem 24. If $B=(\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$ is a weakly Lagrange neutrosophic biloop, then (F,A) over B is also soft weakly Lagrange neutrosophic biloop but the converse is not holds.

Remark 16. Let (F,A) and (K,C) be two soft weakly Lagrange neutrosophic biloops over $B=(\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$. Then

1. Their extended intersection $(F,A) \cap_E (K,\mathrm{C})$ is not a soft weakly Lagrange neutrosophic biloop over $B=(\langle B_1 \cup I \rangle \cup B_2, *_1, *_2)$.
2. Their restricted intersection $(F,A) \cap_R (K,\mathrm{C})$

is not a soft weakly Lagrange neutrosophic biloop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$.

3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft weakly Lagrange neutrosophic biloop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$.
4. Their extended union $(F,A)\cup_E(K,C)$ is not a soft weakly Lagrnage neutrosophic biloop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$.
5. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft weakly Lagrange neutrosophic biloop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft weakly Lagrange neutrosophic biloop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 34. Let $(\langle B\cup I\rangle,*_1,*_2)$ be a neutrosophic biloop and (F,A) be a soft set over $(\langle B\cup I\rangle,*_1,*_2)$. Then (F,A) is called soft Lagrange free neutrosophic biloop if and only if $F(a)$ is not a Lagrange neutrosophic subbiloop of $(\langle B\cup I\rangle,*_1,*_2)$ for all $a\in A$.

Example 13. Let $B=(\mathrm{B}_1\cup B_2,*_1,*_2)$ be a neutrosophic biloop of order 20, where $B_1=\langle L_5(3)\cup I\rangle$ and $B_2=\{g:g^8=e\}$. Then clearly (F,A) is a soft Lagrange free neutrosophic biloop over $(\langle B\cup I\rangle,*_1,*_2)$, where

$$F(a_1)=\{\mathrm{e},\mathrm{eI},2,2\mathrm{I}\}\cup\{\mathrm{e},\mathrm{g}^2,\mathrm{g}^4,\mathrm{g}^6\},$$
$$\mathrm{F}(a_2)=\{e,eI,3,3I\}\cup\{\mathrm{e},\mathrm{g}^4\}.$$

Theorem 25. Every soft Lagrange free neutrosophic biloop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$ is a soft neutrosophic biloop but the converse is not true.

Theorem 26. If $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$ is a Lagrange free neutrosophic biloop, then (F,A) over B is also soft Lagrange free neutrosophic biloop but the converse is not holds.

Remark 17. Let (F,A) and (K,C) be two soft Lagrange free neutrosophic biloops over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ is not a soft Lagrange free neutrosophic biloop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ is not a soft Lagrange free neutrosophic biloop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft Lagrange free neutrosophic biloop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$.
4. Their extended union $(F,A)\cup_E(K,C)$ is not a soft Lagrnage free neutrosophic biloop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$.
5. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft Lagrange free neutrosophic biloop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft Lagrange free neutrosophic biloop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Soft Neutrosophic Strong Biloop

Definition 35. Let $B=(B_1\cup B_2,*_1,*_2)$ be a neutrosophic biloop where B_1 is a neutrosopphic biloop and B_2 is a neutrosophic group and (F,A) be soft set over B. Then (F,A) over B is called soft neutrosophic strong biloop if and only if $F(a)$ is a neutrosopchic strong subbiloop of B for all $a\in A$.

Example 14. Let $B=(B_1\cup B_2,*_1,*_2)$ where $B_1=\langle L_5(2)\cup I\rangle$ is a neutrosophic loop and $B_2=\{0,1,2,3,4,,1\mathrm{I},2\mathrm{I},3\mathrm{I},4\mathrm{I}\}$ under multiplication modulo 5 is a neutrosophic group. Let $A=\{a_1,a_2\}$ be a set of parameters. Then (F,A) is soft neutrosophic strong biloop over B, where

$$F(a_1)=\{e,2,eI,2I\}\cup\{1,I,4I\},$$
$$F(a_2)=\{\mathrm{e},3,\mathrm{eI},3\mathrm{I}\}\cup\{1,\mathrm{I},4\mathrm{I}\}.$$

Theorem 27. Every soft neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$ is a soft neutrosophic biloop but the converse is not true.

Theorem 28. If $B=(B_1\cup B_2,*_1,*_2)$ is a neutrosophic strong biloop, then (F,A) over B is also soft neutrosophic strong biloop but the converse is not holds.

Proposition 9. Let (F,A) and (K,C) be two soft neutrosophic strong biloops over $B=(B_1\cup B_2,*_1,*_2)$. Then

1. Their extended intersection $(F,A)\cap_E(K,\mathrm{C})$ is a soft neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
2. Their restricted intersection $(F,A)\cap_R(K,\mathrm{C})$ is a soft neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
3. Their AND operation $(F,A)\wedge(K,\mathrm{C})$ is a soft neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.

Remark 18. Let (F,A) and (K,B) be two soft neutrosophic strong biloops over $B=(B_1\cup B_2,*_1,*_2)$. Then

1. Their extended union $(F,A)\cup_E(K,\mathrm{C})$ is not a soft neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
2. Their restricted union $(F,A)\cup_R(K,\mathrm{C})$ is not a soft neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
3. Their OR operation $(F,A)\vee(K,\mathrm{C})$ is not a soft neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.

One can easily verify $(1),(2),$ and (3) by the help of examples.

Definition 36. Let (F,A) and (H,C) be two soft neutrosophic strong biloops over $B=(B_1\cup B_2,*_1,*_2)$. Then (H,C) is called soft neutrosophic strong subbiloop of (F,A), if

3. $C\subseteq A$.
4. $H(a)$ is a neutrosophic strong subbiloop of $F(a)$ for all $a\in A$.

Definition 37. Let $B=(B_1\cup B_2,*_1,*_2)$ be a neutrosophic biloop and (F,A) be a soft set over $B=(B_1\cup B_2,*_1,*_2)$. Then (F,A) is called soft Lagrange neutrosophic strong biloop if and only if $F(a)$ is a Lagrange neutrosophic strong subbiloop of $B=(B_1\cup B_2,*_1,*_2)$ for all $a\in A$.

Theorem 29. Every soft Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$ is a soft neutrosophic biloop but the converse is not true.

Remark 19. Let (F,A) and (K,C) be two soft Lagrange neutrosophic strong biloops over $B=(B_1\cup B_2,*_1,*_2)$. Then

1. Their extended intersection $(F,A)\cap_E(K,\mathrm{C})$ is not a soft Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
2. Their restricted intersection $(F,A)\cap_R(K,\mathrm{C})$ is not a soft Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
3. Their AND operation $(F,A)\wedge(K,\mathrm{C})$ is not a soft Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
4. Their extended union $(F,A)\cup_E(K,\mathrm{C})$ is not a soft Lagrnage neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
5. Their restricted union $(F,A)\cup_R(K,\mathrm{C})$ is not a soft Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
6. Their OR operation $(F,A)\vee(K,\mathrm{C})$ is not a soft Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 38. Let $B=(B_1\cup B_2,*_1,*_2)$ be a neutrosophic biloop and (F,A) be a soft set over $B=(B_1\cup B_2,*_1,*_2)$. Then (F,A) is called soft weakly Lagrange neutrosophic strong biloop if atleast one $F(a)$ is not a Lagrange neutrosophic strong subbiloop of $B=(B_1\cup B_2,*_1,*_2)$ for some $a\in A$.

Theorem 30. Every soft weakly Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$ is a soft neutrosophic biloop but the converse is not true.

Theorem 31. If $B=(B_1\cup B_2,*_1,*_2)$ is a weakly Lagrange neutrosophic strong biloop, then (F,A) over B is also soft weakly Lagrange neutrosophic strong biloop but the converse does not holds.

Remark 20. Let (F,A) and (K,C) be two soft weakly Lagrange neutrosophic strong biloops over $B=(B_1\cup B_2,*_1,*_2)$. Then

1. Their extended intersection $(F,A)\cap_E(K,\mathrm{C})$ is not a soft weakly Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
2. Their restricted intersection $(F,A)\cap_R(K,\mathrm{C})$ is not a soft weakly Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
3. Their AND operation $(F,A)\wedge(K,\mathrm{C})$ is not a soft weakly Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
4. Their extended union $(F,A)\cup_E(K,\mathrm{C})$ is not a soft weakly Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
5. Their restricted union $(F,A)\cup_R(K,\mathrm{C})$ is not a soft weakly Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
6. Their OR operation $(F,A)\vee(K,\mathrm{C})$ is not a soft weakly Lagrange neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 39. Let $B=(B_1\cup B_2,*_1,*_2)$ be a neutrosophic biloop and (F,A) be a soft set over $B=(B_1\cup B_2,*_1,*_2)$. Then (F,A) is called soft Lagrange free neutrosophic strong biloop if and only if $F(a)$ is not a Lagrange neutrosophic subbiloop of $B=(B_1\cup B_2,*_1,*_2)$ for all $a\in A$.

Theorem 32. Every soft Lagrange free neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$ is a soft neutrosophic biloop but the converse is not true.

Theorem 33. If $B=(B_1\cup B_2,*_1,*_2)$ is a Lagrange free neutrosophic strong biloop, then (F,A) over B is also soft strong lagrange free neutrosophic strong biloop but the converse is not true.

Remark 21. Let (F,A) and (K,C) be two soft Lagrange free neutrosophic strong biloops over $B=(B_1\cup B_2,*_1,*_2)$. Then

1. Their extended intersection $(F,A)\cap_E(K,\mathrm{C})$ is not a soft Lagrange free neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
2. Their restricted intersection $(F,A)\cap_R(K,\mathrm{C})$ is not a soft Lagrange free neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
3. Their AND operation $(F,A)\wedge(K,\mathrm{C})$ is not a soft Lagrange free neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
4. Their extended union $(F,A)\cup_E(K,\mathrm{C})$ is not a soft Lagrnage free neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
5. Their restricted union $(F,A)\cup_R(K,\mathrm{C})$ is not a soft Lagrange free neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.
6. Their OR operation $(F,A)\vee(K,\mathrm{C})$ is not a soft Lagrange free neutrosophic strong biloop over $B=(B_1\cup B_2,*_1,*_2)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Soft Neutrosophic N-loop

Definition 40. Let $S(B) = \{S(B_1) \cup S(B_2) \cup ... \cup S(B_N), *_1, *_2, ..., *_N\}$ be a neutrosophic N-loop and (F, A) be a soft set over $S(B)$. Then (F, A) is called soft neutrosophic N-loop if and only if $F(a)$ is a neutrosopchic sub N-loop of $S(B)$ for all $a \in A$.

Example 15. Let $S(B) = \{S(B_1) \cup S(B_2) \cup S(B_3), *_1, *_2, *_3\}$ be a neutrosophic 3-loop, where $S(B_1) = \langle L_5(3) \cup I \rangle$, $S(B_2) = \{g : g^{12} = e\}$ and $S(B_3) = S_3$. Then (F, A) is sof neutrosophic N-loop over $S(B)$, where

$$F(a_1) = \{\mathrm{e}, \mathrm{eI}, 2, 2\mathrm{I}\} \cup \{\mathrm{e}, \mathrm{g}^6\} \cup \{\mathrm{e}, (12)\},$$
$$\mathrm{F}(a_2) = \{e, eI, 3, 3I\} \cup \{e, g^4, g^8\} \cup \{e, (13)\}.$$

Theorem 34. Let (F, A) and (H, A) be two soft neutrosophic N-loops over $S(B) = \{S(B_1) \cup S(B_2) \cup ... \cup S(B_N), *_1, *_2, ..., *_N\}$. Then their intersection $(F, A) \cap (H, A)$ is again a soft neutrosophic N-loop over $S(B)$.

Proof. Straightforward.

Theorem 35. Let (F, A) and (H, C) be two soft neutrosophic N-loops over $S(B) = \{S(B_1) \cup S(B_2) \cup ... \cup S(B_N), *_1, *_2, ..., *_N\}$ such that $A \cap C = \phi$. Then their union is soft neutrosophic N-loop over $S(B)$.

Proof. Straightforward.

Proposition 10. Let (F, A) and (K, C) be two soft neutrosophic N-loops over $S(B) = \{S(B_1) \cup S(B_2) \cup ... \cup S(B_N), *_1, *_2, ..., *_N\}$. Then

1. Their extended intersection $(F, A) \cap_E (K, \mathrm{C})$ is a soft neutrosophic N-loop over $S(B)$.
2. Their restricted intersection $(F, A) \cap_R (K, \mathrm{C})$ is a soft neutrosophic N-loop over $S(B)$.
3. Their AND operation $(F, A) \wedge (K, \mathrm{C})$ is a soft neutrosophic N-loop over $S(B)$.

Remark 22. Let (F, A) and (H, C) be two soft neutrosophic N-loops over $S(B) = \{S(B_1) \cup S(B_2) \cup ... \cup S(B_N), *_1, *_2, ..., *_N\}$ Then

1. Their extended union $(F, A) \cup_E (K, \mathrm{C})$ is not a soft neutrosophic N-loop over $S(B)$.
2. Their restricted union $(F, A) \cup_R (K, \mathrm{C})$ is not a soft neutrosophic N-loop over $S(B)$.
3. Their OR operation $(F, A) \vee (K, \mathrm{C})$ is not a soft neutrosophic N-loop over $S(B)$.

One can easily verify (1), (2), and (3) by the help of examples.

Definition 41. Let (F, A) be a soft neutrosophic N-loop over $S(B) = \{S(B_1) \cup S(B_2) \cup ... \cup S(B_N), *_1, *_2, ..., *_N\}$. Then (F, A) is called the identity soft neutrosophic N-loop over $S(B)$ if $F(a) = \{e_1, e_2, ..., e_N\}$ for all $a \in A$, where $e_1, e_2, ..., e_N$ are the identities element of $S(B_1), S(B_2), ..., S(B_N)$ respectively.

Definition 42. Let (F, A) be a soft neutrosophic N-loop over $S(B) = \{S(B_1) \cup S(B_2) \cup ... \cup S(B_N), *_1, *_2, ..., *_N\}$. Then (F, A) is called an absolute-soft neutrosophic N-loop over $S(B)$ if $F(a) = S(B)$ for all $a \in A$.

Definition 43. Let (F, A) and (H, C) be two soft neutrosophic N-loops over $S(B) = \{S(B_1) \cup S(B_2) \cup ... \cup S(B_N), *_1, *_2, ..., *_N\}$. Then (H, C) is called soft neutrosophic sub N-loop of (F, A), if

1. $C \subseteq A$.
2. $H(a)$ is a neutrosophic sub N-loop of $F(a)$ for all $a \in A$.

Definition 45. Let $S(B)=\{S(B_1)\cup S(B_2)\cup...\cup S(B_N),*_1,*_2,...,*_N\}$ be a neutrosophic N -loop and (F,A) be a soft set over $S(B)$. Then (F,A) is called soft Lagrange neutrosophic N -loop if and only if $F(a)$ is Lagrange neutrosophic sub N -loop of $S(B)$ for all $a\in A$.

Theorem 36. Every soft Lagrange neutrosophic N -loop over $S(B)=\{S(B_1)\cup S(B_2)\cup...\cup S(B_N),*_1,*_2,...,*_N\}$ is a soft neutrosophic N -loop but the converse is not true.

Remark 23. Let (F,A) and (K,C) be two soft Lagrange neutrosophic N -loops over $S(B)=\{S(B_1)\cup S(B_2)\cup...\cup S(B_N),*_1,*_2,...,*_N\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ is not a soft Lagrange neutrosophic N -loop over $S(B)$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ is not a soft Lagrange neutrosophic N -loop over $S(B)$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft Lagrange neutrosophic N -loop over (B).
4. Their extended union $(F,A)\cup_E(K,C)$ is not a soft Lagrnage neutrosophic N -loop over $S(B)$.
5. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft Lagrange neutrosophic N -loop over $S(B)$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft Lagrange neutrosophic N -loop over $S(B)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 46. Let $S(B)=\{S(B_1)\cup S(B_2)\cup...\cup S(B_N),*_1,*_2,...,*_N\}$ be a neutrosophic N -loop and (F,A) be a soft set over $S(B)$. Then (F,A) is called soft weakly Lagrange neutrosophic biloop if atleast one $F(a)$ is not a Lagrange neutrosophic sub N -loop of $S(B)$ for some $a\in A$.

Theorem 37. Every soft weakly Lagrange neutrosophic N -loop over $S(B)=\{S(B_1)\cup S(B_2)\cup...\cup S(B_N),*_1,*_2,...,*_N\}$ is a soft neutrosophic N -loop but the converse is not true.

Theorem 38. If $S(B)=\{S(B_1)\cup S(B_2)\cup...\cup S(B_N),*_1,*_2,...,*_N\}$ is a weakly Lagrange neutrosophic N -loop, then (F,A) over $S(B)$ is also soft weakly Lagrange neutrosophic N -loop but the converse is not holds.

Remark 24. Let (F,A) and (K,C) be two soft weakly Lagrange neutrosophic N -loops over $S(B)=\{S(B_1)\cup S(B_2)\cup...\cup S(B_N),*_1,*_2,...,*_N\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ is not a soft weakly Lagrange neutrosophic N -loop over $S(B)$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ is not a soft weakly Lagrange neutrosophic N -loop over $S(B)$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft weakly Lagrange neutrosophic N -loop over $S(B)$.
4. Their extended union $(F,A)\cup_E(K,C)$ is not a soft weakly Lagrnage neutrosophic N -loop over $S(B)$.
5. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft weakly Lagrange neutrosophic N -loop over $S(B)$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft weakly Lagrange neutrosophic N -loop over $S(B)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 47. Let $S(B)=\{S(B_1)\cup S(B_2)\cup...\cup S(B_N),*_1,*_2,...,*_N\}$ be a neutrosophic N -loop and (F,A) be a soft set over

$S(B)$. Then (F,A) is called soft Lagrange free neutrosophic N-loop if and only if $F(a)$ is not a Lagrange neutrosophic sub N-loop of $S(B)$ for all $a \in A$.

Theorem 39. Every soft Lagrange free neutrosophic N-loop over $S(B)=\{S(B_1)\cup S(B_2)\cup ...\cup S(B_N),*_1,*_2,...,*_N\}$ is a soft neutrosophic biloop but the converse is not true.

Theorem 40. If $S(B)=\{S(B_1)\cup S(B_2)\cup ...\cup S(B_N),*_1,*_2,...,*_N\}$ is a Lagrange free neutrosophic N-loop, then (F,A) over $S(B)$ is also soft lagrange free neutrosophic N-loop but the converse is not hold.

Remark 25. Let (F,A) and (K,C) be two soft Lagrange free neutrosophic N-loops over $S(B)=\{S(B_1)\cup S(B_2)\cup ...\cup S(B_N),*_1,*_2,...,*_N\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,\mathrm{C})$ is not a soft Lagrange free neutrosophic N-loop over $B=(\langle B_1\cup I\rangle\cup B_2,*_1,*_2)$.
2. Their restricted intersection $(F,A)\cap_R(K,\mathrm{C})$ is not a soft Lagrange free neutrosophic N-loop over $S(B)$.
3. Their AND operation $(F,A)\wedge(K,\mathrm{C})$ is not a soft Lagrange free neutrosophic N-loop over $S(B)$.
4. Their extended union $(F,A)\cup_E(K,\mathrm{C})$ is not a soft Lagrnage free neutrosophic N-loop over $S(B)$.
5. Their restricted union $(F,A)\cup_R(K,\mathrm{C})$ is not a soft Lagrange free neutrosophic N-loop over $S(B)$.
6. Their OR operation $(F,A)\vee(K,\mathrm{C})$ is not a soft Lagrange free neutrosophic N-loop over $S(B)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Soft Neutrosophic Strong N-loop

Definition 48. Let $\langle L\cup I\rangle=\{L_1\cup L_2\cup ...\cup L_N,*_1,*_2,...,*_N\}$ be a neutrosophic N-loop and (F,A) be a soft set over $\langle L\cup I\rangle=\{L_1\cup L_2\cup ...\cup L_N,*_1,*_2,...,*_N\}$. Then (F,A) is called soft neutrosophic strong N-loop if and only if $F(a)$ is a neutrosopchic strong sub N-loop of $\langle L\cup I\rangle=\{L_1\cup L_2\cup ...\cup L_N,*_1,*_2,...,*_N\}$ for all $a\in A$.

Example 16. Let $\langle L\cup I\rangle=\{L_1\cup L_2\cup L_3,*_1,*_2,*_3\}$ where $L_1=\langle L_5(3)\cup I\rangle, L_2=\langle L_7(3)\cup I\rangle$ and $L_3=\{1,2,1I,2I\}$. Then (F,A) is a soft neutrosophic strong N-loop over $\langle L\cup I\rangle$, where

$$F(a_1)=\{\mathrm{e},2,\mathrm{eI},2\mathrm{I}\}\cup\{\mathrm{e},2,\mathrm{eI},2\mathrm{I}\}\cup\{1,\mathrm{I}\},$$
$$\mathrm{F}(a_2)=\{\mathrm{e},3,\mathrm{eI},3\mathrm{I}\}\cup\{\mathrm{e},3,\mathrm{eI},3\mathrm{I}\}\cup\{1,2,2\mathrm{I}\}.$$

Theorem 41. All soft neutrosophic strong N-loops are soft neutrosophic N-loops but the converse is not true.

One can easily see the converse with the help of example.

Proposition 11. Let (F,A) and (K,C) be two soft neutrosophic strong N-loops over $\langle L\cup I\rangle=\{L_1\cup L_2\cup ...\cup L_N,*_1,*_2,...,*_N\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,\mathrm{C})$ is a soft neutrosophic strong N-loop over $\langle L\cup I\rangle$.
2. Their restricted intersection $(F,A)\cap_R(K,\mathrm{C})$ is a soft neutrosophic strong N-loop over $\langle L\cup I\rangle$.
3. Their AND operation $(F,A)\wedge(K,\mathrm{C})$ is a soft neutrosophic strong N-loop over $\langle L\cup I\rangle$.

Remark 26. Let (F,A) and (K,C) be two soft neutrosophic strong N-loops over $\langle L\cup I\rangle=\{L_1\cup L_2\cup ...\cup L_N,*_1,*_2,...,*_N\}$. Then

1. Their extended union $(F,A)\cup_E(K,C)$ is not a soft neutrosophic strong N-loop over $\langle L\cup I\rangle$.
2. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft neutrosophic strong N-loop over $\langle L\cup I\rangle$.
3. Their OR operation $(F,A)\vee(K,C)$ is not a soft neutrosophic strong N-loop over $\langle L\cup I\rangle$.

One can easily verify $(1),(2),$ and (3) by the help of examples.

Definition 49. Let (F,A) and (H,C) be two soft neutrosophic strong N-loops over $\langle L\cup I\rangle=\{L_1\cup L_2\cup\ldots\cup L_N,*_1,*_2,\ldots,*_N\}$. Then (H,C) is called soft neutrosophic strong sub N-loop of (F,A), if

1. $C\subseteq A$.
2. $H(a)$ is a neutrosophic strong sub N-loop of $F(a)$ for all $a\in A$.

Definition 50. Let $\langle L\cup I\rangle=\{L_1\cup L_2\cup\ldots\cup L_N,*_1,*_2,\ldots,*_N\}$ be a neutrosophic strong N-loop and (F,A) be a soft set over $\langle L\cup I\rangle$. Then (F,A) is called soft Lagrange neutrosophic strong N-loop if and only if $F(a)$ is a Lagrange neutrosophic strong sub N-loop of $\langle L\cup I\rangle$ for all $a\in A$.

Theorem 42. Every soft Lagrange neutrosophic strong N-loop over $\langle L\cup I\rangle=\{L_1\cup L_2\cup\ldots\cup L_N,*_1,*_2,\ldots,*_N\}$ is a soft neutrosophic N-loop but the converse is not true.

Remark 27. Let (F,A) and (K,C) be two soft Lagrange neutrosophic strong N-loops over $\langle L\cup I\rangle=\{L_1\cup L_2\cup\ldots\cup L_N,*_1,*_2,\ldots,*_N\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ is not a soft Lagrange neutrosophic strong N-loop over $\langle L\cup I\rangle$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ is not a soft Lagrange neutrosophic strong N-loop over $\langle L\cup I\rangle$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft Lagrange neutrosophic strong N-loop over $\langle L\cup I\rangle$.
4. Their extended union $(F,A)\cup_E(K,C)$ is not a soft Lagrnage neutrosophic strong N-loop over $\langle L\cup I\rangle$.
5. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft Lagrange neutrosophic strong N-loop over $\langle L\cup I\rangle$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft Lagrange neutrosophic strong N-loop over $\langle L\cup I\rangle$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 51. Let $\langle L\cup I\rangle=\{L_1\cup L_2\cup\ldots\cup L_N,*_1,*_2,\ldots,*_N\}$ be a neutrosophic strong N-loop and (F,A) be a soft set over $\langle L\cup I\rangle$. Then (F,A) is called soft weakly Lagrange neutrosophic strong N-loop if atleast one $F(a)$ is not a Lagrange neutrosophic strong sub N-loop of $\langle L\cup I\rangle$ for some $a\in A$.

Theorem 43. Every soft weakly Lagrange neutrosophic strong N-loop over $\langle L\cup I\rangle=\{L_1\cup L_2\cup\ldots\cup L_N,*_1,*_2,\ldots,*_N\}$ is a soft neutrosophic N-loop but the converse is not true.

Theorem 44. If $\langle L\cup I\rangle=\{L_1\cup L_2\cup\ldots\cup L_N,*_1,*_2,\ldots,*_N\}$ is a weakly Lagrange neutrosophic strong N-loop, then (F,A) over $\langle L\cup I\rangle$ is also a soft weakly Lagrange neutrosophic strong N-loop but the converse is not true.

Remark 28. Let (F,A) and (K,C) be two soft weakly Lagrange neutrosophic strong N-loops over $\langle L\cup I\rangle=\{L_1\cup L_2\cup\ldots\cup L_N,*_1,*_2,\ldots,*_N\}$. Then

1. Their extended intersection $(F,A)\cap_E (K,C)$ is not a soft weakly Lagrange neutrosophic strong N -loop over $\langle L\cup I\rangle$.
2. Their restricted intersection $(F,A)\cap_R (K,C)$ is not a soft weakly Lagrange neutrosophic strong N -loop over $\langle L\cup I\rangle$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft weakly Lagrange neutrosophic strong N -loop over $\langle L\cup I\rangle$.
4. Their extended union $(F,A)\cup_E (K,C)$ is not a soft weakly Lagrnage neutrosophic strong N -loop over $\langle L\cup I\rangle$.
5. Their restricted union $(F,A)\cup_R (K,C)$ is not a soft weakly Lagrange neutrosophic strong N -loop over $\langle L\cup I\rangle$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft weakly Lagrange neutrosophic strong N -loop over $\langle L\cup I\rangle$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 52. Let $\langle L\cup I\rangle=\{L_1\cup L_2\cup...\cup L_N,*_1,*_2,...,*_N\}$ be a neutrosophic N -loop and (F,A) be a soft set over $\langle L\cup I\rangle$. Then (F,A) is called soft Lagrange free neutrosophic strong N -loop if and only if $F(a)$ is not a Lagrange neutrosophic strong sub N -loop of $\langle L\cup I\rangle$ for all $a\in A$.

Theorem 45. Every soft Lagrange free neutrosophic strong N -loop over $\langle L\cup I\rangle=\{L_1\cup L_2\cup...\cup L_N,*_1,*_2,...,*_N\}$ is a soft neutrosophic N -loop but the converse is not true.

Theorem 45. If $\langle L\cup I\rangle=\{L_1\cup L_2\cup...\cup L_N,*_1,*_2,...,*_N\}$ is a Lagrange free neutrosophic strong N -loop, then (F,A) over $\langle L\cup I\rangle$ is also a soft Lagrange free neutrosophic strong N -loop but the converse is not true.

Remark 29. Let (F,A) and (K,C) be two soft Lagrange free neutrosophic strong N -loops over $\langle L\cup I\rangle=\{L_1\cup L_2\cup...\cup L_N,*_1,*_2,...,*_N\}$. Then

1. Their extended intersection $(F,A)\cap_E (K,C)$ is not a soft Lagrange free neutrosophic strong N -loop over $\langle L\cup I\rangle$.
2. Their restricted intersection $(F,A)\cap_R (K,C)$ is not a soft Lagrange free neutrosophic strong N -loop over $\langle L\cup I\rangle$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft Lagrange free neutrosophic strong N -loop over $\langle L\cup I\rangle$.
4. Their extended union $(F,A)\cup_E (K,C)$ is not a soft Lagrnage free neutrosophic strong N -loop over $\langle L\cup I\rangle$.
5. Their restricted union $(F,A)\cup_R (K,C)$ is not a soft Lagrange free neutrosophic strong N -loop over $\langle L\cup I\rangle$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft Lagrange free neutrosophic strong N -loop over $\langle L\cup I\rangle$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Conclusion

This paper is an extension of neutrosphic loop to soft neutrosophic loop. We also extend neutrosophic biloop, neutrosophic N -loop to soft neutrosophic biloop, and soft neutrosophic N -loop. Their related properties and results are explained with many illustrative examples. The notions related with strong part of neutrosophy also established within soft neutrosophic loop.

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Generalization of Neutrosophic Rings and Neutrosophic Fields

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Abstract. In this paper we present the generalization of neutrosophic rings and neutrosophic fields. We also extend the neutrosophic ideal to neutrosophic biideal and neutrosophic N-ideal. We also find some new type of notions which are related to the strong or pure part of neutrosophy. We have given sufficient amount of examples to illustrate the theory of neutrosophic birings, neutrosophic N-rings with neutrosophic bifields and neutrosophic N-fields and display many properties of them in this paper.

Keywords: Neutrosophic ring, neutrosophic field, neutrosophic biring, neutrosophic N-ring, neutrosophic bifield neutrosophic N-field.

1 Introduction

Neutrosophy is a new branch of philosophy which studies the origin and features of neutralities in the nature. Florentin Smarandache in 1980 firstly introduced the concept of neutrosophic logic where each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F so that this neutrosophic logic is called an extension of fuzzy logic. In fact neutrosophic set is the generalization of classical sets, conventional fuzzy set $[1]$, intuitionistic fuzzy set $[2]$ and interval valued fuzzy set $[3]$. This mathematical tool is used to handle problems like imprecise, indeterminacy and inconsistent data etc. By utilizing neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache dig out neutrosophic algebraic structures in $[11]$. Some of them are neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, and neutrosophic bigroupoids and so on.

In this paper we have tried to develop the the generalization of neutrosophic ring and neutrosophic field in a logical manner. Firstly, preliminaries and basic concepts are given for neutrosophic rings and neutrosophic fields. Then we presented the newly defined notions and results in neutrosophic birings and neutrosophic N-rings, neutrosophic bifields and neutosophic N-fields. Various types of neutrosophic biideals and neutrosophic N-ideal are defined and elaborated with the help of examples.

2 Fundamental Concepts

In this section, we give a brief description of neutrosophic rings and neutrosophic fields.

Definition: Let R be a ring. The neutrosophic ring $\langle R \cup I \rangle$ is also a ring generated by R and I under the operation of R, where I is called the neutrosophic element with property $I^2 = I$. For an integer n, $n+I$ and nI are neutrosophic elements and $0.I = 0$. I^{-1}, the inverse of I is not defined and hence does not exist.

Definition: Let $\langle R \cup I \rangle$ be a neutrosophic ring. A proper subset P of $\langle R \cup I \rangle$ is called a neutosophic subring if P itself a neutrosophic ring under the operation of $\langle R \cup I \rangle$.

Definition: Let T be a non-empty set with two binary operations $*$ and $\circ$. T is said to be a pseudo neutrosophic ring if

1. T contains element of the form $a+bI$ $(a,b$ are reals and $b \neq 0$ for atleast one value$)$.

2. $(T,*)$ is an abelian group.
3. $(T,\circ)$ is a semigroup.

Definition: Let $\langle R \cup I \rangle$ be a neutrosophic ring. A non-empty set P of $\langle R \cup I \rangle$ is called a neutrosophic ideal of $\langle R \cup I \rangle$ if the following conditions are satisfied.

1. P is a neutrosophic subring of $\langle R \cup I \rangle$, and
2. For every $p \in P$ and $r \in \langle R \cup I \rangle$, pr and $rp \in P$.

Definition: Let K be a field. The neutrosophic field generated by $\langle K \cup I \rangle$ which is denoted by $K(I) = \langle K \cup I \rangle$.

Definition: Let $K(I)$ be a neutrosophic field. A proper subset P of $K(I)$ is called a neutrosophic sufield if P itself a neutrosophic field.

3 **Neutrosophic Biring**

Definition **. Let $(BN(\mathrm{R}),*,\circ)$ be a non-empty set with two binary operations $*$ and $\circ$. $(BN(\mathrm{R}),*,\circ)$ is said to be a neutrosophic biring if $BN(\mathrm{Rs}) = R_1 \cup R_2$ where atleast one of $(\mathrm{R}_1,*,\circ)$ or $(\mathrm{R}_2,*,\circ)$ is a neutrosophic ring and other is just a ring. R_1 and R_2 are proper subsets of $BN(\mathrm{R})$.

Example 2. Let $BN(\mathrm{R}) = (\mathrm{R}_1,*,\circ) \cup (\mathrm{R}_2,*,\circ)$ where $(\mathrm{R}_1,*,\circ) = (\langle \mathbb{Z} \cup I \rangle,+,\times)$ and $(\mathrm{R}_2,*,\circ) = (\mathbb{Q},+,\times)$. Clearly $(\mathrm{R}_1,*,\circ)$ is a neutrosophic ring under addition and multiplication. $(\mathrm{R}_2,*,\circ)$ is just a ring. Thus $(BN(\mathrm{R}),*,\circ)$ is a neutrosophic biring.

Theorem: Every neutrosophic biring contains a corresponding biring.

Definition: Let $BN(\mathrm{R}) = (\mathrm{R}_1,*,\circ) \cup (\mathrm{R}_2,*,\circ)$ be a neutrosophic biring. Then $BN(\mathrm{R})$ is called a commutative neutrosophic biring if each $(\mathrm{R}_1,*,\circ)$ and $(\mathrm{R}_2,*,\circ)$ is a commutative neutrosophic ring.

Example 2. Let $BN(\mathrm{R}) = (\mathrm{R}_1,*,\circ) \cup (\mathrm{R}_2,*,\circ)$ where $(\mathrm{R}_1,*,\circ) = (\langle \mathbb{Z} \cup I \rangle,+,\times)$ and $(\mathrm{R}_2,*,\circ) = (\mathbb{Q},+,\times)$. Clearly $(\mathrm{R}_1,*,\circ)$ is a commutative neutrosophic ring and $(\mathrm{R}_2,*,\circ)$ is also a commutative ring. Thus $(BN(\mathrm{R}),*,\circ)$ is a commutative neutrosophic biring.

Definition: Let $BN(\mathrm{R}) = (\mathrm{R}_1,*,\circ) \cup (\mathrm{R}_2,*,\circ)$ be a neutrosophic biring. Then $BN(\mathrm{R})$ is called a pseudo neutrosophic biring if each $(\mathrm{R}_1,*,\circ)$ and $(\mathrm{R}_2,*,\circ)$ is a pseudo neutrosophic ring.

Example 2. Let $BN(\mathrm{R}) = (\mathrm{R}_1,+,\times) \cup (\mathrm{R}_2,+,\circ)$ where $(\mathrm{R}_1,+,\times) = \{0, I, 2I, 3I\}$ is a pseudo neutrosophic ring under addition and multiplication modulo 4 and $(\mathrm{R}_2,+,\times) = \{0, \pm 1\mathrm{I}, \pm 2\mathrm{I}, \pm 3\mathrm{I}, \ldots\}$ is another pseudo neutrosophic ring. Thus $(BN(\mathrm{R}),+,\times)$ is a pseudo neutrosophic biring.

Theorem: Every pseudo neutrosophic biring is trivially a neutrosophic biring but the converse may not be true.

Definition 8. Let $(BN(\mathrm{R}) = R_1 \cup R_2; *, \circ)$ be a neutrosophic biring. A proper subset $(T,*,\circ)$ is said to be a neutrosophic subbiring of $BN(\mathrm{R})$ if

1) $T = T_1 \cup T_2$ where $T_1 = R_1 \cap T$ and $T_2 = R_2 \cap T$ and
2) At least one of $(T_1,\circ)$ or $(T_2,*)$ is a neutrosophic ring.

Example: Let $BN(\mathrm{R}) = (\mathrm{R}_1,*,\circ) \cup (\mathrm{R}_2,*,\circ)$ where $(\mathrm{R}_1,*,\circ) = (\langle \mathbb{R} \cup I \rangle,+,\times)$ and $(\mathrm{R}_2,*,\circ) = (\mathbb{C},+,\times)$. Let $P = P_1 \cup P_2$ be a proper subset of $BN(R)$, where $P_1 = (\mathbb{Q},+,\times)$ and $P_2 = (\mathbb{R},+,\times)$. Clearly $(P,+,\times)$ is a neutrosophic subbiring of $BN(R)$.

Definition: If both $(\mathrm{R}_1,*)$ and $(\mathrm{R}_2,\circ)$ in the above definition ** are neutrosophic rings then we call

$(BN(\mathrm{R}),*,\circ)$ to be a strong neutrosophic biring.

Example 2. Let $BN(\mathrm{R})=(\mathrm{R}_1,*,\circ)\cup(\mathrm{R}_2,*,\circ)$ where $(\mathrm{R}_1,*,\circ)=(\langle\mathbb{Z}\cup I\rangle,+,\times)$ and $(\mathrm{R}_2,*,\circ)=(\langle\mathbb{Q}\cup I\rangle,+,\times)$. Clearly R_1 and R_2 are neutrosophic rings under addition and multiplication. Thus $(BN(\mathrm{R}),*,\circ)$ is a strong neutrosophic biring.

Theorem. All strong neutrosophic birings are trivially neutrosophic birings but the converse is not true in general.

To see the converse, we take the following Example.

Example 2. Let $BN(\mathrm{R})=(\mathrm{R}_1,*,\circ)\cup(\mathrm{R}_2,*,\circ)$ where $(\mathrm{R}_1,*,\circ)=(\langle\mathbb{Z}\cup I\rangle,+,\times)$ and $(\mathrm{R}_2,*,\circ)=(\mathbb{Q},+,\times)$. Clearly $(\mathrm{R}_1,*,\circ)$ is a neutrosophic ring under addition and multiplication. $(\mathrm{R}_2,*,\circ)$ is just a ring. Thus $(BN(\mathrm{R}),*,\circ)$ is a neutrosophic biring but not a strong neutrosophic biring.

Remark: A neutrosophic biring can have subbirings, neutrosophic subbirings, strong neutrosophic subbirings and pseudo neutrosohic subbirings.

Definition 8. Let $(BN(\mathrm{R})=R_1\cup R_2;*,\circ)$ be a neutrosophic biring and let $(T,*,\circ)$ is a neutrosophic subbiring of $BN(\mathrm{R})$. Then $(T,*,\circ)$ is called a neutrosophic biideal of $BN(R)$ if

1) $T=T_1\cup T_2$ where $T_1=R_1\cap T$ and $T_2=R_2\cap T$ and
2) At least one of $(T_1,*,\circ)$ or $(T_2,*,\circ)$ is a neutrosophic ideal.

If both $(T_1,*,\circ)$ and $(T_2,*,\circ)$ in the above definition are neutrosophic ideals, then we call $(T,*,\circ)$ to be a strong neutrosophic biideal of $BN(R)$.

Example: Let $BN(\mathrm{R})=(\mathrm{R}_1,*,\circ)\cup(\mathrm{R}_2,*,\circ)$ where $(\mathrm{R}_1,*,\circ)=(\langle\mathbb{Z}_{12}\cup I\rangle,+,\times)$ and $(\mathrm{R}_2,*,\circ)=(\mathbb{Z}_{16},+,\times)$. Let $P=P_1\cup P_2$ be a neutrosophic subbiring of $BN(R)$, where $P_1=\{0,6,2I,4I,6I,8I,10I,6+2I,...,6+10I\}$ and $P_2=\{02I,4I,6I,8I,10I,12I,14I\}$. Clearly $(P,+,\times)$ is a neutrosophic biideal of $BN(R)$.

Theorem: Every neutrosophic biideal is trivially a neutrosophic subbiring but the converse may not be true.

Theorem: Every strong neutrosophic biideal is trivially a neutrosophic biideal but the converse may not be true.

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Definition 8. Let $(BN(\mathrm{R})=R_1\cup R_2;*,\circ)$ be a neutrosophic biring and let $(T,*,\circ)$ is a neutrosophic subbiring of $BN(\mathrm{R})$. Then $(T,*,\circ)$ is called a pseudo neutrosophic biideal of $BN(R)$ if

1. $T=T_1\cup T_2$ where $T_1=R_1\cap T$ and $T_2=R_2\cap T$ and
2. $(T_1,*,\circ)$ and $(T_2,*,\circ)$ are pseudo neutrosophic ideals.

Theorem: Every pseudo neutrosophic biideal is trivially a neutrosophic subbiring but the converse may not be true.

Theorem: Every pseudo neutrosophic biideal is trivially a strong neutrosophic subbiring but the converse may not be true.

Theorem: Every pseudo neutrosophic biideal is trivially a neutrosophic biideal but the converse may not be true.

Theorem: Every pseudo neutrosophic biideal is trivially a strong neutrosophic biideal but the converse may not be true.

4 Neutrosophic N-ring

Definition*. Let $\{\mathrm{N}(\mathrm{R}),*_1,...,*_2,\circ_1,\circ_2,...,\circ_N\}$ be a non-empty set with two N-binary operations defined on it. We call $N(R)$ a neutrosophic N-ring (N a positive integer) if the following conditions are satisfied.

1) $\mathrm{N}(\mathrm{R})=R_1\cup R_2\cup...\cup R_N$ where each R_i is a proper subset of $\mathrm{N}(\mathrm{R})$ i.e. $R_i\not\subset R_j$ or $R_j\not\subset R_i$ if $i\neq j$.

2) $(\mathrm{R}_i, *_i, \circ_i)$ is either a neutrosophic ring or a ring for $i = 1,2,3,...,N$.

Example 2. Let $N(\mathrm{R}) = (\mathrm{R}_1, *, \circ) \cup (\mathrm{R}_2, *, \circ) \cup (\mathrm{R}_3, *, \circ)$ where $(\mathrm{R}_1, *, \circ) = (\langle \mathbb{Z} \cup I \rangle, +, \times)$, $(\mathrm{R}_2, *, \circ) = (\mathbb{Q}, +, \times)$ and $(R_3, *, \circ) = (Z_{12}, +, \times)$. Thus $(N(\mathrm{R}), *, \circ)$ is a neutrosophic N-ring.

Theorem: Every neutrosophic N-ring contains a corresponding N-ring.

Definition: Let $\mathrm{N(R)} = \{\mathrm{R}_1 \cup R_2 \cup .... \cup \mathrm{R}_N, *_1, *_2, ..., *_N, \circ_1, \circ_2, ..., \circ_N\}$ be a neutrosophic N-ring. Then $N(\mathrm{R})$ is called a pseudo neutrosophic N-ring if each $(\mathrm{R}_i, *_i)$ is a pseudo neutrosophic ring where $i = 1,2,...,\mathrm{N}$.

Example 2. Let $N(\mathrm{R}) = (\mathrm{R}_1, +, \times) \cup (\mathrm{R}_2, +, \times) \cup (\mathrm{R}_3, +, \times)$ where $(\mathrm{R}_1, +, \times) = \{0, I, 2I, 3I\}$ is a pseudo neutrosophic ring under addition and multiplication modulo 4, $(\mathrm{R}_2, +, \times) = \{0, \pm 1\mathrm{I}, \pm 2\mathrm{I}, \pm 3\mathrm{I}, ...\}$ is a pseudo neutrosophic ring and $(\mathrm{R}_3, +, \times) = \{0, \pm 2\mathrm{I}, \pm 4\mathrm{I}, \pm 6\mathrm{I}...\}$. Thus $(N(\mathrm{R}), +, \times)$ is a pseudo neutrosophic 3-ring.

Theorem: Every pseudo neutrosophic N-ring is trivially a neutrosophic N-ring but the converse may not be true.

Definition. If all the N-rings $(\mathrm{R}_i, *_i)$ in definition * are neutrosophic rings (i.e. for $i = 1,2,3,...,N$) then we call $\mathrm{N(R)}$ to be a neutrosophic strong N-ring.

Example 2. Let $N(\mathrm{R}) = (\mathrm{R}_1, *, \circ) \cup (\mathrm{R}_2, *, \circ) \cup (\mathrm{R}_3, *, \circ)$ where $(\mathrm{R}_1, *, \circ) = (\langle \mathbb{Z} \cup I \rangle, +, \times)$, $(\mathrm{R}_2, *, \circ) = (\langle \mathbb{Q} \cup I \rangle, +, \times)$ and $(R_3, *, \circ) = (\langle \mathbb{Z}_{12} \cup I \rangle, +, \times)$. Thus $(N(\mathrm{R}), *, \circ)$ is a strong neutrosophic N-ring.

Theorem: All strong neutrosophic N-rings are neutrosophic N-rings but the converse may not be true.

Definition 13. Let $\mathrm{N(R)} = \{\mathrm{R}_1 \cup R_2 \cup .... \cup \mathrm{R}_N, *_1, *_2, ..., *_N, \circ_1, \circ_2, ..., \circ_N\}$ be a neutrosophic N-ring. A proper subset $P = \{\mathrm{P}_1 \cup P_2 \cup .... \mathrm{P}_N, *_1, *_2, ..., *_N\}$ of $\mathrm{N(R)}$ is said to be a neutrosophic N-subring if $P_i = P \cap R_i, i = 1,2,...,N$ are subrings of R_i in which atleast some of the subrings are neutrosophic subrings.

Example: Let $N(\mathrm{R}) = (\mathrm{R}_1, *, \circ) \cup (\mathrm{R}_2, *, \circ) \cup (\mathrm{R}_3, *, \circ)$ where $(\mathrm{R}_1, *, \circ) = (\langle \mathbb{R} \cup I \rangle, +, \times)$, $(\mathrm{R}_2, *, \circ) = (\mathbb{C}, +, \times)$ and $(\mathrm{R}_2, *, \circ) = (\mathrm{Z}_{10}, +, \times)$ Let $P = P_1 \cup P_2 \cup P_3$ be a proper subset of $N(R)$, where $P_1 = (\mathbb{Q}, +, \times)$, $P_2 = (\mathbb{R}, +, \times)$ and $(\mathrm{R}_3, *, \circ) = \{0, 2, 4, 6, 8, \mathrm{I}, 2\mathrm{I}, 4\mathrm{I}, 6\mathrm{I}, 8\mathrm{I}\}$. Clearly $(P, +, \times)$ is a neutrosophic sub 3-ring of $N(R)$.

Definition 14. Let $\mathrm{N(R)} = \{\mathrm{R}_1 \cup R_2 \cup .... \cup \mathrm{R}_N, *_1, *_2, ..., *_N, \circ_1, \circ_2, ..., \circ_N\}$ be a neutrosophic N-ring. A proper subset $T = \{\mathrm{T}_1 \cup T_2 \cup .... \cup \mathrm{T}_N, *_1, *_2, ..., *_N, \circ_1, \circ_2, ..., \circ_N\}$ of $N(R)$ is said to be a neutrosophic strong sub N-ring if each $(T_i, *_i)$ is a neutrosophic subring of $(\mathrm{R}_i, *_i, \circ_i)$ for $i = 1,2,...,N$ where $T_i = R_i \cap T$.

Remark: A strong neutrosophic su N-ring is trivially a neutrosophic sub N-ring but the converse is not true.

Remark: A neutrosophic N-ring can have sub N-rings, neutrosophic sub N-rings, strong neutrosophic sub N-rings and pseudo neutrosohic sub N-rings.

Definition 16. Let $\mathrm{N(R)} = \{\mathrm{R}_1 \cup R_2 \cup .... \cup \mathrm{R}_N, *_1, *_2, ..., *_N, \circ_1, \circ_2, ..., \circ_N\}$ be a neutrosophic N-ring. A proper subset $P = \{\mathrm{P}_1 \cup P_2 \cup .... \cup P_N, *_1, *_2, ..., *_N, \circ_1, \circ_2, ..., \circ_N\}$ where $P_t = P \cap R_t$ for $t = 1,2,...,N$ is said to be a neutrosophic N-ideal of $N(R)$ if the following conditions are satisfied.

1) Each it is a neutrosophic subring of $R_t, t = 1,2,...,N$.

2) Each it is a two sided ideal of R_t for $t = 1,2,...,N$.
If $(\mathrm{P}_i, *_i, \circ_i)$ in the above definition are neutrosophic ideals, then we call $(\mathrm{P}_i, *_i, \circ_i)$ to be a strong neutrosophic N-ideal of $N(R)$.

Theorem: Every neutrosophic N-ideal is trivially a neutrosophic sub N-ring but the converse may not be true.

Theorem: Every strong neutrosophic N-ideal is trivially a neutrosophic N-ideal but the converse may not be true.

Theorem: Every strong neutrosophic N-ideal is trivially a neutrosophic sub N-ring but the converse may not be true.

Theorem: Every strong neutrosophic biideal is trivially a strong neutrosophic subbiring but the converse may not be true.

Definition 16. Let
$\mathrm{N(R)} = \{\mathrm{R}_1 \cup R_2 \cup .... \cup \mathrm{R}_N, *_1, *_2, ..., *_N, \circ_1, \circ_2, ..., \circ_N\}$
be a neutrosophic N-ring. A proper subset
$P = \{\mathrm{P}_1 \cup P_2 \cup .... \cup P_N, *_1, *_2, ..., *_N, \circ_1, \circ_2, ..., \circ_N\}$
where $P_t = P \cap R_t$ for $t = 1,2,...,N$ is said to be a pseudo neutrosophic N-ideal of $N(R)$ if the following conditions are satisfied.

1. Each it is a neutrosophic subring of $R_t, t = 1,2,...,N$.
2. Each $(\mathrm{P}_i, *_i, \circ_i)$ is a pseudo neutrosophic ideal.

Theorem: Every pseudo neutrosophic N-ideal is trivially a neutrosophic sub N-ring but the converse may not be true.

Theorem: Every pseudo neutrosophic N-ideal is trivially a strong neutrosophic sub N-ring but the converse may not be true.

Theorem: Every pseudo neutrosophic N-ideal is trivially a neutrosophic N-ideal but the converse may not be true.

Theorem: Every pseudo neutrosophic N-ideal is trivially a strong neutrosophic N-ideal but the converse may not be true.

5 Neutrosophic Bi-Fields and Neutrosophic N-Fields

Definition **. Let $(BN(\mathrm{F}), *, \circ)$ be a non-empty set with two binary operations $*$ and $\circ$. $(BN(\mathrm{F}), *, \circ)$ is said to be a neutrosophic bifiel if $BN(\mathrm{F}) = F_1 \cup F_2$ where atleast one of $(\mathrm{F}_1, *, \circ)$ or $(\mathrm{F}_2, *, \circ)$ is a neutrosophic field and other is just a field. F_1 and F_2 are proper subsets of $BN(\mathrm{F})$.
If in the above definition both $(\mathrm{F}_1, *, \circ)$ and $(\mathrm{F}_2, *, \circ)$ are neutrosophic fields, then we call $(BN(\mathrm{F}), *, \circ)$ to be a neutrosophic strong bifield.

Example 2. Let $BN(\mathrm{F}) = (\mathrm{F}_1, *, \circ) \cup (\mathrm{F}_2, *, \circ)$ where $(\mathrm{F}_1, *, \circ) = (\langle \mathbb{C} \cup I \rangle, +, \times)$ and $(\mathrm{F}_2, *, \circ) = (\mathbb{Q}, +, \times)$. Clearly $(\mathrm{F}_1, *, \circ)$ is a neutrosophic field and $(\mathrm{F}_2, *, \circ)$ is just a field. Thus $(BN(\mathrm{F}), *, \circ)$ is a neutrosophic bifield.

Theorem: All strong neutrosophic bifields are trivially neutrosophic bifields but the converse is not true.

Definition 8. Let $BN(\mathrm{F}) = (\mathrm{F}_1 \cup F_2, *, \circ)$ be a neutrosophic bifield. A proper subset $(T, *, \circ)$ is said to be a neutrosophic subbifield of $BN(\mathrm{F})$ if

3) $T = T_1 \cup T_2$ where $T_1 = F_1 \cap T$ and $T_2 = F_2 \cap T$ and
4) At least one of $(T_1, \circ)$ or $(T_2, *)$ is a neutrosophic field and the other is just a field.

Example: Let $BN(\mathrm{F}) = (\mathrm{F}_1, *, \circ) \cup (\mathrm{F}_2, *, \circ)$ where $(\mathrm{F}_1, *, \circ) = (\langle \mathbb{R} \cup I \rangle, +, \times)$ and $(\mathrm{F}_2, *, \circ) = (\mathbb{C}, +, \times)$. Let $P = P_1 \cup P_2$ be a proper subset of $BN(\mathrm{F})$, where $P_1 = (\mathbb{Q}, +, \times)$ and $P_2 = (\mathbb{R}, +, \times)$. Clearly $(P, +, \times)$ is a neutrosophic subbifield of $BN(\mathrm{F})$.

Definition*. Let $\{\mathrm{N(F)}, *_1, ..., *_2, \circ_1, \circ_2, ..., \circ_N\}$ be a non-empty set with two N-binary operations defined on it. We call $N(R)$ a neutrosophic N-field (N a positive integer) if the following conditions are satisfied.

1. $\mathrm{N(F)} = F_1 \cup F_2 \cup ... \cup F_N$ where each F_i is a proper subset of $\mathrm{N(F)}$ i.e. $R_i \not\subset R_j$ or $R_j \not\subset R_i$ if $i \neq j$.
2. $(\mathrm{R}_i, *_i, \circ_i)$ is either a neutrosophic field or just a field for $i = 1,2,3,...,N$.

If in the above definition each $(R_i, *_i, \circ_i)$ is a neutrosophic field, then we call $N(R)$ to be a strong neutrosophic N-field.

Theorem: Every strong neutrosophic N-field is obviously a neutrosophic field but the converse is not true.

Definition 14. Let $N(F) = \{F_1 \cup F_2 \cup .... \cup F_N, *_1, *_2, ..., *_N, \circ_1, \circ_2, ..., \circ_N\}$ be a neutrosophic N-field. A proper subset $T = \{T_1 \cup T_2 \cup .... \cup T_N, *_1, *_2, ..., *_N, \circ_1, \circ_2, ..., \circ_N\}$ of $N(F)$ is said to be a neutrosophic N-subfield if each $(T_i, *_i)$ is a neutrosophic subfield of $(F_i, *_i, \circ_i)$ for $i = 1, 2, ..., N$ where $T_i = F_i \cap T$.

Conclusion

In this paper we extend neutrosophic ring and neutrosophic field to neutrosophic biring, neutrosophic N-ring and neutrosophic bifield and neutrosophic N-field. The neutrosophic ideal theory is extend to neutrosophic biideal and neutrosophic N-ideal. Some new type of neutrosophic ideals are discovered which is strongly neutrosophic or purely neutrosophic. Related examples are given to illustrate neutrosophic biring, neutrosophic N-ring, neutrosophic bifield and neutrosophic N-field and many theorems and properties are discussed.

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Cosine Similarity Measure of Interval Valued Neutrosophic Sets

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Abstract. In this paper, we define a new cosine similarity between two interval valued neutrosophic sets based on Bhattacharya's distance [19]. The notions of interval valued neutrosophic sets (IVNS, for short) will be used as vector representations in 3D-vector space. Based on the comparative analysis of the existing similarity measures for IVNS, we find that our proposed similarity measure is better and more robust. An illustrative example of the pattern recognition shows that the proposed method is simple and effective.

Keywords: Cosine Similarity Measure; Interval Valued Neutrosophic Sets

1. Introduction

The neutrsophic sets (**NS**), pioneered by F. Smarandache [1], has been studied and applied in different fields, including decision making problems [2, 3, 4 , 5, 23], databases [6-7], medical diagnosis problems [8] , topology [9], control theory [10], Image processing [11,12,13] and so on. The character of NSs is that the values of its membership function, non-membership function and indeterminacy function are subsets. The concept of neutrosophic sets generalizes the following concepts: the classic set, fuzzy set, interval valued fuzzy set, Intuitionistic fuzzy set, and interval valued intuitionistic fuzzy set and so on, from a philosophical point of view. Therefore, Wang et al [14] introduced an instance of neutrosophic sets known as single valued neutrosophic sets (SVNS), which were motivated from the practical point of view and that can be used in real scientific and engineering application, and provide the set theoretic operators and various properties of SVNSs. However, in many applications, due to lack of knowledge or data about the problem domains, the decision information may be provided with intervals, instead of real numbers. Thus, interval valued neutrosophic sets (IVNS), as a useful generation of NS, was introduced by Wang et al [15], which is characterized by a membership function, non-membership function and an indeterminacy function, whose values are intervals rather than real numbers. Also, the interval valued neutrosophic set can represent uncertain, imprecise, incomplete and inconsistent information which exist in the real world. As an important extension of NS, IVNS has many applications in real life [16, 17].

Many methods have been proposed for measuring the degree of similarity between neutrosophic set, S. Broumi and F. Smarandache [22] proposed several definitions of similarity measure between NS. P. Majumdar and S.K. Samanta [21] suggested some new methods for measuring the similarity between neutrosophic set. However, there is a little investigation on the similarity measure of IVNS, although some method on measure of similarity between intervals valued neutrosophic sets have been presented in [5] recently.

Pattern recognition has been one of the fastest growing areas during the last two decades because of its usefulness and fascination. In pattern recognition, on the basis of the knowledge of known pattern, our aim is to classify the unknown pattern. Because of the complex and uncertain nature of the problems. The problem pattern recognition is given in the form of interval valued neutrosophic sets.

In this paper, motivated by the cosine similarity measure based on Bhattacharya's distance [19], we propose a new method called "cosine similarity measure for interval valued neutrosophic sets. Also the proposed and existing similarity measures are compared to show that the proposed similarity measure is more reasonable than some similarity measures. The proposed similarity measure is applied to pattern recognition

This paper is organized as follow: In section 2 some basic definitions of neutrosophic set, single valued neutrosophic set, interval valued neutrosophic set and cosine similarity measure are presented briefly. In section 3, cosine similarity measure of interval valued neutrosophic sets and their proofs are introduced. In section 4, results of the proposed similarity measure and existing similarity measures are compared .In section 5, the proposed similarity measure is applied to deal with the problem related to medical diagnosis. Finally we conclude the paper.

2. Preliminaries

This section gives a brief overview of the concepts of neutrosophic set, single valued neutrosophic set, interval valued neutrosophic set and cosine similarity measure.

2.2 Neutrosophic Sets

Definition 2.1 [1]

Let U be an universe of discourse then the neutrosophic set A is an object having the form

A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$ >, x ∈U}, where the functions T, I, F : U→]−0, 1+[define respectively the degree of membership (or Truth) , the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element x U to the set A with the condition.

$$^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}. \quad \textbf{(1)}$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$. So instead of] −0, 1^{+}[we need to take the interval [0, 1] for technical applications, because $]^{-}0, 1^{+}[$will be difficult to apply in the real applications such as in scientific and engineering problems.

For two NS, A_{NS} = {<x, $T_A(x)$, $I_A(x)$, $F_A(x)$ > | x∈X }

And B_{NS} = {<x, $T_B(x)$, $I_B(x)$, $F_B(x)$ > | x∈X >the two relations are defined as follows:

(1) $A_{NS} \subseteq B_{NS}$ If and only if $T_A(x) \leq T_B(x)$, $I_A(x)$ $I_B(x)$, $F_A(x) \geq F_B(x)$ for any x ∈X.

(2) $A_{NS} = B_{NS}$ if and only if $T_A(x) = T_B(x)$, $I_A(x)$ $= I_B(x)$, $F_A(x) = F_B(x)$ for any x ∈X.

2.3.Single Valued Neutrosophic Sets

Definition 2.3 [14]

Let X be a space of points (objects) with generic elements in X denoted by x. An SVNS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$, for each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x)$ ∈[0, 1].

When X is continuous, an SVNS A can be written as

$$A= \int_X \frac{<T_A(x), I_A(x), F_A(x)>}{x}, x \in X \quad \textbf{(2)}$$

When X is discrete, an SVNS A can be written as

$$A= \sum_i^n \frac{<T_A(x_i), I_A(x_i), F_A(x_i)>}{x_i}, x_i \in X \quad \textbf{(3)}$$

For two SVNS, A_{SVNS} = {<x, $T_A(x)$, $I_A(x)$, $F_A(x)$ > | x ∈X }

And B_{SVNS} ={<x, $T_A(x)$, $I_A(x)$, $F_A(x)$ > | x ∈X }the two relations are defined as follows:

(1) $A_{SVNS} \subseteq B_{SVNS}$ if and only if $T_A(x) \leq T_B(x)$ $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$

(2) $A_{SVNS} = B_{SVNS}$ if and only if $T_A(x) = T_B(x)$, $I_A(x)$ $= I_B(x)$, $F_A(x) = F_B(x)$ for any x ∈X.

2.4 Interval Valued Neutrosophic Sets

Definition 2.4 [15]

Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS) A in X is characterized by truth-membership function $T_A(x)$, indeteminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$. For each point x in X, we have that $T_A(x)$, $I_A(x)$, $F_A(x)$ ∈ [0, 1] .

For two IVNS, A_{INS} ={<x, [$T_A^L(x)$, $T_A^U(x)$] , [$I_A^L(x)$, $I_A^U(x)$],[$F_A^L(x)$, $F_A^U(x)$]> | x ∈X }

And B_{INS} = {<x, ={<x, [$T_B^L(x)$, $T_B^U(x)$] , [$I_B^L(x)$, $I_B^U(x)$],[$F_B^L(x)$, $F_B^U(x)$]> | x ∈X }>|x ∈X } the two relations are defined as follows:

(1) $A_{INS} \subseteq B_{INS}$ if and only if $T_A^L(x) \leq T_B^L(x)$, $T_A^U(x) \leq T_B^U(x)$, $I_A^L(x) \geq I_B^L(x)$, $F_A^L(x) \geq F_B(x)$, $F_A^U(x) \geq F_B^U(x)$.

(2) $A_{INS} = B_{INS}$ if and only if , $T_A^L(x) = T_B^L(x)$, $T_A^U(x) = T_B^U(x)$, $I_A^L(x) = I_B^L(x)$, $I_A^U(x) = I_B^U(x)$, $F_A^L(x) = F_B^L(x)$, $F_A^U(x) = F_B^U(x)$ for any x ∈X.

2.5 Cosine Similarity

Definition 2.5

Cosine similarity is a fundamental angle-based measure of similarity between two vectors of n dimensions using the cosine of the angle between them Candan and Sapino [20]. It measures the similarity between two vectors based only on the direction, ignoring the impact of the distance between them. Given two vectors of attributes, X= ($x_1, x_2, \ldots, x_n$) and Y= ($y_1, y_2, \ldots, y_n$), the cosine similarity, cosθ, is represented using a dot product and magnitude as

$$\textbf{Cos}\theta = \frac{\sum_{i=}^{n} x_i y_i}{\sqrt{\sum_{i=}^{n} x_i^2}\sqrt{\sum_{i=}^{n} y_i^2}} \quad \textbf{(4)}$$

In vector space, a cosine similarity measure based on Bhattacharya's distance [19] between two fuzzy set $\mu_A(x_i)$ and $\mu_B(x_i)$ defined as follows:

$$C_F(A,B) = \frac{\sum_{i=}^{n} \mu_A(x_i)\mu_B(x_i)}{\sqrt{\sum_{i=}^{n} \mu_A(x_i)^2}\sqrt{\sum_{i=}^{n} \mu_B(x_i)^2}} \quad \textbf{(5)}$$

The cosine of the angle between the vectors is within the values between 0 and 1.

In 2-D vector space, J. Ye [18] defines cosine similarity measure between IFS as follows:

$$C_{IFS}(A,B) = \frac{\sum_{i=}^{n} \mu_A(x_i)\mu_B(x_i) + \nu_A(x_i)\nu_B(x_i)}{\sqrt{\sum_{i=}^{n} \mu_A(x_i)^2 + \nu_A(x_i)^2}\sqrt{\sum_{i=}^{n} \mu_B(x_i)^2 + \nu_B(x_i)^2}} \quad \textbf{(6)}$$

III . Cosine Similarity Measure for Interval Valued Neutrosophic Sets.

The existing cosine similarity measure is defined as the inner product of these two vectors divided by the product of their lengths. The cosine similarity measure is the cosine of the angle between the vector representations of the two fuzzy sets. The cosine similarity measure is a classic measure used in information retrieval and is the most widely reported measures of vector similarity [19]. However, to the best of our Knowledge, the existing cosine similarity measures does not deal with interval valued neutrosophic sets. Therefore, to overcome this limitation in this section, a new cosine similarity measure between interval valued neutrosophic sets is proposed in 3-D vector space.

Let A be an interval valued neutrosophic sets in a universe of discourse X ={x}, the interval valued neutrosophic sets is characterized by the interval of membership $[T_A^L, T_A^U]$,the interval degree of non-membership $[F_A^L, F_A^U]$ and the interval degree of indeterminacy $[I_A^L, I_A^U]$ which can be considered as a vector representation with the three elements. Therefore, a cosine similarity measure for interval neutrosophic sets is proposed in an analogous manner to the cosine similarity measure proposed by J. Ye [18].

Definition 3.1 : Assume that there are two interval neutrosophic sets A and B in X $=\{x_1, x_2,..., x_n\}$ Based on the extension measure for fuzzy sets, a cosine similarity measure between interval valued neutrosophic sets A and B is proposed as follows:

$$C_N(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{\Delta T_A(x_i)\Delta T_B(x_i) + \Delta I_A(x_i)\Delta I_B(x_i) + \Delta F_A(x_i)\Delta F_B(x_i)}{\sqrt{(\Delta T_A(x_i))^2 + (\Delta I_A(x_i))^2 + (\Delta F_A(x_i))^2}\sqrt{(\Delta T_B(x_i))^2 + (\Delta I_B(x_i))^2 + (\Delta F_B(x_i))^2}}. \quad \textbf{(7)}$$

Where

$\Delta T_A(x_i) = T_A^L(x_i) + T_A^U(x_i)$, $\Delta T_B(x_i) = T_B^L(x_i) + T_B^U(x_i)$

$\Delta I_A(x_i) = I_A^L(x_i) + I_A^U(x_i)$, $\Delta I_B(x_i) = I_B^L(x_i) + I_B^U(x_i)$

And $\Delta F_A(x_i) = F_A^L(x_i) + F_A^U(x_i)$,

$\Delta F_B(x_i) = F_B^U(x_i) + F_B^U(x_i)$

Proposition 3.2

Let A and B be interval valued neutrosophic sets then

- **i.** $\mathbf{0} \le C_N(A,B) \le \mathbf{1}$
- ii. $C_N(A,B) = C_N(B,A)$
- iii. $C_N(A,B) = 1$ if A= B i.e

$T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$

$I_A^L(x_i) = I_B^L(x_i)$, $I_A^U(x_i) = I_B^U(x_i)$ and

$F_A^L(x_i) = F_B^L(x_i)$, $F_A^U(x_i) = F_B^U(x_i)$ for i=1,2,…., n

Proof : (i) it is obvious that the proposition is true according to the cosine valued

(ii) it is obvious that the proposition is true.

(iii) when A =B, there are

$T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$

$I_A^L(x_i) = I_B^L(x_i)$, $I_A^U(x_i) = I_B^U(x_i)$ and

$F_A^L(x_i) = F_B^L(x_i)$, $F_A^U(x_i) = F_B^U(x_i)$ for i=1,2,…., n

, So there is $C_N(A,B) = 1$

If we consider the weights of each element x_i , a weighted cosine similarity measure between IVNSs A and B is given as follows:

$$C_{WN}(A,B) = \frac{1}{n}\sum_{i=1}^{n} w_i \frac{\Delta T_A(x_i)\Delta T_B(x_i) + \Delta I_A(x_i)\Delta I_B(x_i) + \Delta F_A(x_i)\Delta F_B(x_i)}{\sqrt{(\Delta T_A(x_i))^2 + (\Delta I_A(x_i))^2 + (\Delta F_A(x_i))^2}\sqrt{(\Delta T_B(x_i))^2 + (\Delta I_B(x_i))^2 + (\Delta F_B(x_i))^2}}. \quad \textbf{(8)}$$

Where $w_i \in [0.1]$,i =1,2,...,n ,and $\sum_{i=1}^{n} w_i = 1$.

If we take $w_i = \frac{1}{n}$, i =1,2,…,n , then there is $C_{WN}(A,B) = C_N(A,B)$.

The weighted cosine similarity measure between two IVNSs A and B also satisfies the following properties:

- **i.** $\mathbf{0} \le C_{WN}(A,B) \le \mathbf{1}$
- ii. $C_{WN}(A,B) = C_{WN}(B,A)$
- iii. $C_{WN}(A,B) = 1$ if A= B i.e

$T_A^L(x_i) = T_B^L(x_i)$, $T_A^U(x_i) = T_B^U(x_i)$

$I_A^L(x_i) = I_B^L(x_i)$, $I_A^U(x_i) = I_B^U(x_i)$ and

$F_A^L(x_i) = F_B^L(x_i)$, $F_A^U(x_i) = F_B^U(x_i)$ for i=1,2,…., n

Proposition 3.3

Let the distance measure of the angle as **d(A,B)= arcos** $C_N(A,B)$ **,**then it satisfies the following properties.

- **i.** d(A, B) ≥ 0, if $\mathbf{0} \le C_N(A,B) \le \mathbf{1}$

ii. d(A, B) = **arcos(1) = 0, if** $C_N(A,B)$ **= 1**

iii. d(A, B) = d(B, A) if $C_N(A,B) = C_N(B,A)$

iv. d(A, C) ≤ d(A, B) + d(B, C) if A ⊆ B ⊆ C for any interval valued neutrosophic sets C.

Proof : obviously, d(A,B) satisfies the (i) – (iii). In the following , d(A,B) will be proved to satisfy the (iv).

For any C = { x_i }, A ⊆ B ⊆ C since Eq (7) is the sum of terms. Let us consider the distance measure of the angle between vectors:

d_i (A(x_i), B(x_i)) = **arcos(** C_N (A(x_i), B(x_i)**),**

d_i (B(x_i), C(x_i)) = **arcos(** C_N (B(x_i), C(x_i)**), and**
d_i (A(x_i), C(x_i)) = **arcos(** C_N (A(x_i), C(x_i)**),,** for i=1, 2, .., n, where

$$C_N(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta T_A(x_i)\Delta T_B(x_i)+\Delta I_A(x_i)\Delta I_B(x_i)+\Delta F_A(x_i)\Delta F_B(x_i)}{\sqrt{(\Delta T_A(x_i))^2+(\Delta I_A(x_i))^2+(\Delta F_A(x_i))^2}\sqrt{(\Delta T_B(x_i))^2+(\Delta I_B(x_i))^2+(\Delta F_B(x_i))^2}}. \quad (9)$$

$$C_N(B,C)=\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta T_B(x_i)\Delta T_C(x_i)+\Delta I_B(x_i)\Delta I_C(x_i)+\Delta F_B(x_i)\Delta F_C(x_i)}{\sqrt{(\Delta T_B(x_i))^2+(\Delta I_B(x_i))^2+(\Delta F_B(x_i))^2}\sqrt{(\Delta T_C(x_i))^2+(\Delta I_C(x_i))^2+(\Delta F_C(x_i))^2}}. \quad (10)$$

$$C_N(A,C)=\frac{1}{n}\sum_{i=1}^{n}\frac{\Delta T_A(x_i)\Delta T_C(x_i)+\Delta I_A(x_i)\Delta I_C(x_i)+\Delta F_A(x_i)\Delta F_C(x_i)}{\sqrt{(\Delta T_A(x_i))^2+(\Delta I_A(x_i))^2+(\Delta F_A(x_i))^2}\sqrt{(\Delta T_C(x_i))^2+(\Delta I_C(x_i))^2+(\Delta F_C(x_i))^2}}. \quad (11)$$

For three vectors

A(x_i)= < x_i , [$T_A^L(x_i)$, $T_A^U(x_i)$], [$I_A^L(x_i)$, $I_A^U(x_i)$], [$F_A^L(x_i)$, $F_A^U(x_i)$] >

B(x_i) = < [$T_B^L(x_i)$, $T_B^U(x_i)$], [$I_B^L(x_i)$, $I_b^U(x_i)$], [$F_B^L(x_i)$, $F_B^U(x_i)$] >

C(x_i) = < [$T_C^L(x_i)$, $T_C^U(x_i)$], [$I_C^L(x_i)$, $I_C^U(x_i)$], [$F_C^L(x_i)$, $F_C^U(x_i)$] > in a plane

If A(x_i)⊆ B (x_i)⊆ C (x_i) (i =1, 2,…, n), then it is obvious that d(A(x_i), C(x_i)) ≤ d(A(x_i), B(x_i)) + d(B(x_i), C(x_i)), According to the triangle inequality. Combining the inequality with E.q (7), we can obtain d(A, C) ≤ d(A, B) + d(B, C). Thus, d(A,B) satisfies the property (iv). So we have finished the proof.

IV. Comparison of New Similarity Measure with the Existing Measures.

Let A and B be two interval neutrosophic set in the universe of discourse X={ x_1 , x_2 ,…, x_n }. For the cosine similarity and the existing similarity measures of interval valued neutrosophic sets introduced in [5, 21], they are listed as follows:

Pinaki's similarity I [21]

$$S_{PI} = \frac{\sum_{i=1}^{n}\{\min\{T_A(x_i)\cdot T_B(x_i)\}+\min\{I_A(x_i)\cdot I_B(x_i)\}+\min\{F_A(x_i)\cdot F_B(x_i)\}\}}{\sum_{i=1}^{n}\{\max\{T_A(x_i)\cdot T_B(x_i)\}+\max\{I_A(x_i)\cdot I_B(x_i)\}+\max\{F_A(x_i)\cdot F_B(x_i)\}\}}. \quad (12)$$

Also ,P. Majumdar [21] proposed weighted similarity measure for neutrosophic set as follows:

$$S_{PII} = \frac{\sum_{i=1}^{n} w_i(T_A(x_i)\cdot T_B(x_i)+I_A(x_i)\cdot I_B(x_i)+F_A(x_i)\cdot F_B(x_i))}{Max(w_i\sqrt{T_A(x_i)^2+T_A(x_i)^2+T_A(x_i)^2},w_i\sqrt{T_B(x_i)^2+I_B(x_i)^2+F_B(x_i)^2})} \quad (13)$$

Where, S_{PI} , S_{PII} denotes Pinaki's similarity I and Pinaki's similarity II

Ye's similarity [5] is defined as the following:

$$S_{ye}(A,B) = 1-\sum_{i=1}^{n} w_i\begin{bmatrix}|\inf T_A(x_i)-\inf T_B(x_i)|+|\sup T_A(x_i)-\sup T_B(x_i)| \\ +|\inf I_A(x_i)-\inf i_B(x_i)|+|\sup I_A(x_i)-\sup I_B(x_i)| \\ +|\inf F_A(x_i)-\inf F_B(x_i)|+|\sup F_A(x_i)-\sup F_B(x_i)|\end{bmatrix} \quad (14)$$

Example 1:

Let A = {<x, (0.2, 0.2 0.3)>} and B= {<x, (0.5, 0.2 0.5)>}

Pinaki similarity I = 0.58

Pinaki similarity II (with w_i =1) = 0.29

Ye similarity **(w**ith w_i =1) = 0.83

Cosine similarity $C_N(A,B)$ = **0.95**

Example 2:

Let A= {<x, ([0.2, 0.3], [0.5, 0.6] ,[0.3, 0.5])>} and B{<x, ([0.5, 0.6], [0.3, 0.6] ,[0.5, 0.6])>}

Pinaki similarty I = NA

Pinaki similarty II (with w_i =1) = NA

Ye similarity **(**with w_i =1) =0.81

Cosine similarity $C_N(A,B)$ = **0.92**

On the basis of computational study. J.Ye [5] have shown that their measure is more effective and reasonable .A similar kind of study with the help of the proposed new measure based on the cosine similarity, has been done and it is found that the obtained results are more refined and accurate. It may be observed from the example 1 and 2 that the values of similarity measures are more closer to 1 with $C_N(A,B)$,the proposed similarity measure. This implies that we may be more deterministic for correct diagnosis and proper treatment.

V. Application of Cosine Similarity Measure for Interval Valued Neutrosophic Numbers to Pattern Recognition

In order to demonstrate the application of the proposed cosine similarity measure for interval valued neutrosophic numbers to pattern recognition, we discuss the medical diagnosis problem as follows:
For example the patient reported temperature claiming that the patient has temperature between 0.5 and 0.7 severity /certainty, some how it is between 0.2 and 0.4 indeterminable if temperature is cause or the effect of his current disease. And it between 0.1 and 0.2 sure that temperature has no relation with his main disease. This piece of information about one patient and one symptom may be written as:
(patient , Temperature) = <[0.5, 0.7], [0.2 ,0.4], [0.1, 0.2]>
(patient , Headache) = < [0.2, 0.3], [0.3 ,0.5], [0.3, 0.6]>
(patient , Cough) = <[0.4, 0.5], [0.6 ,0.7], [0.3, 0.4]>
Then, P = {< x_1 , [0.5, 0.7], [0.2 ,0.4], [0.1, 0.2] >, < x_2 , [0.2, 0.3], [0.3, 0.5], [0.3, 0.6] > ,< x_3 , [0.4, 0.5], [0.6 ,0.7], [0.3, 0.4]>}

And each diagnosis A_i (i=1, 2, 3) can also be represented by interval valued neutrosophic numbers with respect to all the symptoms as follows:

= {< x_1 , [0.5, 0.6], [0.2 ,0.3], [0.4, 0.5] >, < x_2 , [0.2 , 0.6], [0.3 ,0.4], [0.6 , 0.7]>,< x_3 , [0.1, 0.2], [0.3 ,0.6], [0.7, 0.8]>}

= {< x_1 , [0.4, 0.5], [0.3, 0.4], [0.5, 0.6] >, < x_2 , [0.3, 0.5], [0.4 ,0.6], [0.2, 0.4]> , < x_3 , [0.3, 0.6], [0.1, 0.2], [0.5, 0.6]>}

= {< x_1 , [0.6, 0.8], [0.4 ,0.5], [0.3, 0.4]>, < x_2 , [0.3, 0.7], [0.2, 0.3], [0.4, 0.7]> ,< x_3 , [0.3, 0.5], [0.4, 0.7], [0.2, 0.6]>}

Our aim is to classify the pattern P in one of the classes A_1 , A_2 , A_3 .According to the recognition principle of maximum degree of similarity measure between interval valued neutrosophic numbers, the process of diagnosis A_k to patient P is derived according to

k = arg Max{ $C_N(A_i,P)$)}

from the previous formula (7) , we can compute the cosine similarity between A_i (i=1, 2, 3) and P as follows;

$C_N(A_1,P)$ **=0.8988,** $C_N(A_2,P)$ **=0.8560,** $C_N(A_3,P)$ **=0.9654**

Then, we can assign the patient to diagnosis A_3 (Typoid) according to recognition of principal.

VI. Conclusions.

In this paper a cosine similarity measure between two and weighted interval valued neutrosophic sets is proposed. The results of the proposed similarity measure and existing similarity measure are compared. Finally, the proposed cosine similarity measure is applied to pattern recognition.

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Neutrosophic Crisp Set Theory

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Abstract. The purpose of this paper is to introduce new types of neutrosophic crisp sets with three types 1, 2, 3. After given the fundamental definitions and operations, we obtain several properties, and discussed the relationship between neutrosophic crisp sets and others. Also, we introduce and study the neutrosophic crisp point and neutrosophic crisp relations. Possible applications to database are touched upon.

Keywords: Neutrosophic Set, Neutrosophic Crisp Sets; Neutrosophic Crisp Relations; Generalized Neutrosophic Sets; Intuitionistic Neutrosophic Sets.

1 Introduction

Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. The fundamental concepts of neutrosophic set, introduced by Smarandache in [16, 17, 18] and Salama et al. in [4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 19,20, 21], provides a natural foundation for treating mathematically the neutrosophic phenomena which exist pervasively in our real world and for building new branches of neutrosophic mathematics. Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts [1, 2, 3, 4, 23] such as a neutrosophic set theory. In this paper we introduce new types of neutrosophic crisp set. After given the fundamental definitions and operations, we obtain several properties, and discussed the relationship between neutrosophic crisp sets and others. Also, we introduce and study the neutrosophic crisp points and relation between two new neutrosophic crisp notions. Finally, we introduce and study the notion of neutrosophic crisp relations.

2 Terminologies

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [16, 17, 18], and Salama et al. [7, 11, 12, 20]. Smarandache introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where $]^-0, 1^+[$ is nonstandard unit interval.

Definition 2.1 [7]

A neutrosophic crisp set (NCS for short) $A = \langle A_1, A_2, A_3 \rangle$ can be identified to an ordered triple $\langle A_1, A_2, A_3 \rangle$ are subsets on X and every crisp set in X is obviously a NCS having the form $\langle A_1, A_2, A_3 \rangle$,

Salama et al. constructed the tools for developed neutrosophic crisp set, and introduced the NCS ϕ_N, X_N in X as follows:

ϕ_N may be defined as four types:

i) Type1: $\phi_N = \langle \phi, \phi, X \rangle$, or

ii) Type2: $\phi_N = \langle \phi, X, X \rangle$, or

iii) Type3: $\phi_N = \langle \phi, X, \phi \rangle$, or

iv) Type4: $\phi_N = \langle \phi, \phi, \phi \rangle$

1) X_N may be defined as four types

i) Type1: $X_N = \langle X, \phi, \phi \rangle$,

ii) Type2: $X_N = \langle X, X, \phi \rangle$,

iii) Type3: $X_N = \langle X, X, \phi \rangle$,

iv) Type4: $X_N = \langle X, X, X \rangle$,

Definition 2.2 [6, 7]

Let $A = \langle A_1, A_2, A_3 \rangle$ a NCS on X, then the complement of the set A (A^c, for short) may be defined as three kinds

(C_1) Type1: $A^c = \langle A^c{}_1, A^c{}_2, A^c{}_3 \rangle$,

(C_2) Type2: $A^c = \langle A_3, A_2, A_1 \rangle$

(C_3) Type3: $A^c = \langle A_3, A^c{}_2, A_1 \rangle$

Definition 2.3 [6, 7]

Let X be a non-empty set, and NCSS A and B in the form $A = \langle A_1, A_2, A_3 \rangle$, $B = \langle B_1, B_2, B_3 \rangle$, then we may consider two possible definitions for subsets ($A \subseteq B$)

($A \subseteq B$) may be defined as two types:

1) Type1: $A \subseteq B \Leftrightarrow A_1 \subseteq B_1, A_2 \subseteq B_2$ and $A_3 \supseteq B_3$ or
2) Type2: $A \subseteq B \Leftrightarrow A_1 \subseteq B_1, A_2 \supseteq B_2$ and $A_3 \supseteq B_3$.

Definition 2.5 [6, 7]

Let X be a non-empty set, and NCSs A and B in the form $A = \langle A_1, A_2, A_3 \rangle$, $B = \langle B_1, B_2, B_3 \rangle$ are NCSS Then

1) $A \cap B$ may be defined as two types:
   i. Type1: $A \cap B = \langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3 \rangle$ or
   ii. Type2: $A \cap B = \langle A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3 \rangle$
2) $A \cup B$ may be defined as two types:
   i) Type1: $A \cup B = \langle A_1 \cup B_1, A_2 \cap B_2, A_3 \cup B_3 \rangle$ or
   ii) Type2: $A \cup B = \langle A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3 \rangle$.

3 Some Types of Neutrosophic Crisp Sets

We shall now consider some possible definitions for some types of neutrosophic crisp sets

Definition 3.1

The object having the form $A = \langle A_1, A_2, A_3 \rangle$ is called

1) **(Neutrosophic Crisp Set with Type 1)** If satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$. (NCS-Type1 for short).
2) **(Neutrosophic Crisp Set with Type 2)** If satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$ and $A_1 \cup A_2 \cup A_3 = X$. (NCS-Type2 for short).
3) **(Neutrosophic Crisp Set with Type 3)** If satisfying $A_1 \cap A_2 \cap A_3 = \phi$ and $A_1 \cup A_2 \cup A_3 = X$. (NCS-Type3 for short).

Definition 3.3

1) **(Neutrosophic Set [9, 16, 17]):** Let X be a non-empty fixed set. A neutrosophic set (NS for short) A is an object having the form $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$ where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ which represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-member ship (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set A where $0^- \le \mu_A(x), \sigma_A(x), \nu_A(x) \le 1^+$ and $0^- \le \mu_A(x) + \sigma_A(x) + \nu_A(x) \le 3^+$.
2) **(Generalized Neutrosophic Set [8]):** Let X be a non-empty fixed set. A generalized neutrosophic (GNS for short) set A is an object having the form $A = \langle x, \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$ where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ which represent the degree of member ship function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-member ship (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set A where $0^- \le \mu_A(x), \sigma_A(x), \nu_A(x) \le 1^+$ and the functions satisfy the condition $\mu_A(x) \wedge \sigma_A(x) \wedge \nu_A(x) \le 0.5$ and $0^- \le \mu_A(x) + \sigma_A(x) + \nu_A(x) \le 3^+$.
3) **(Intuitionistic Neutrosophic Set [22]).** Let X be a non-empty fixed set. An intuitionistic neutrosophic set A (INS for short) is an object having the form $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$ where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ which represent the degree of member ship function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-member ship (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set A where $0.5 \le \mu_A(x), \sigma_A(x), \nu_A(x)$ and the functions satisfy the condition $\mu_A(x) \wedge \sigma_A(x) \le 0.5$, $\mu_A(x) \wedge \nu_A(x) \le 0.5$, $\sigma_A(x) \wedge \nu_A(x) \le 0.5$, and $^-0 \le \mu_A(x) + \sigma_A(x) + \nu_A(x) \le 2^+$. A neutrosophic crisp with three types the object $A = \langle A_1, A_2, A_3 \rangle$ can be identified to an ordered triple $\langle A_1, A_2, A_3 \rangle$ are subsets on X, and every crisp set in X is obviously a NCS having the form $\langle A_1, A_2, A_3 \rangle$.

Every neutrosophic set $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$ on X is obviously on NS having the form $\langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$.

Remark 3.1

1) The neutrosophic set not to be generalized neutrosophic set in general.
2) The generalized neutrosophic set in general not intuitionistic NS but the intuitionistic NS is generalized NS.

Intuitionistic NS $\longrightarrow$ Generalized NS $\longrightarrow$ NS

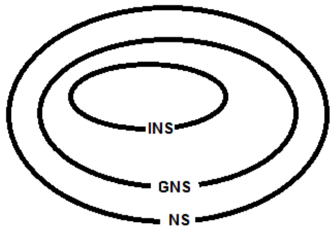

Fig. 1. Represents the relation between types of NS

Corollary 3.1

Let X non-empty fixed set and $A=\langle \mu_A(x),\sigma_A(x),\nu_A(x)\rangle$ be INS on X Then:

1) Type1- A^c of INS be a GNS.
2) Type2- A^c of INS be a INS.
3) Type3- A^c of INS be a GNS.

Proof

Since A INS then $\mu_A(x),\sigma_A(x),\nu_A(x)$, and
$\mu_A(x)\wedge\sigma_A(x)\le 0.5, \nu_A(x)\wedge\mu_A(x)\le 0.5$
$\nu_A(x)\wedge\sigma_A(x)\le 0.5$ Implies
$\mu^c{}_A(x),\sigma^c{}_A(x),\nu^c{}_A(x)\le 0.5$ then is not to be Type1- A^c INS. On other hand the Type 2- A^c,
$A^c=\langle \nu_A(x),\sigma_A(x),\mu_A(x)\rangle$ be INS and Type3- A^c,
$A^c=\langle \nu_A(x),\sigma^c{}_A(x),\mu_A(x)\rangle$ and $\sigma^c{}_A(x)\le 0.5$ implies to
$A^c=\langle \nu_A(x),\sigma^c{}_A(x),\mu_A(x)\rangle$ GNS and not to be INS

Example 3.1

Let $X=\{a,b,c\}$, and A,B,C are neutrosophic sets on X, $A=\langle(0.7,0.9,0.8)\setminus a,(0.6,0.7,0.6)\setminus b,(0.9,0.7,0.8\setminus c\rangle$,
$B=\langle(0.7,0.9,0.5)\setminus a,(0.6,0.4,0.5)\setminus b,(0.9,0.5,0.8\setminus c\rangle$
$C=\langle(0.7,0.9,0.5)\setminus a,(0.6,0.8,0.5)\setminus b,(0.9,0.5,0.8\setminus c\rangle$ By the Definition 3.3 no.3 $\mu_A(x)\wedge\sigma_A(x)\wedge\nu_A(x)\ge 0.5$, A be not GNS and INS,
$B=\langle(0.7,0.9,0.5)\setminus a,(0.6,0.4,0.5)\setminus b,(0.9,0.5,0.8\setminus c\rangle$ not INS, where $\sigma_A(b)=0.4<0.5$. Since
$\mu_B(x)\wedge\sigma_B(x)\wedge\nu_B(x)\le 0.5$ then B is a GNS but not INS.
$A^c=\langle(0.3,0.1,0.2)\setminus a,(0.4,0.3,0.4)\setminus b,(0.1,0.3,0.2\setminus c\rangle$
Be a GNS, but not INS.
$B^c=\langle(0.3,0.1,0.5)\setminus a,(0.4,0.6,0.5)\setminus b,(0.1,0.5,0.2\setminus c\rangle$
Be a GNS, but not INS, C be INS and GNS,
$C^c=\langle(0.3,0.1,0.5)\setminus a,(0.4,0.2,0.5)\setminus b,(0.1,0.5,0.2\setminus c\rangle$
Be a GNS but not INS.

Definition 3.2

A NCS-Type1 ϕ_{N_1}, X_{N_1} in X as follows:

1) ϕ_{N1} may be defined as three types:
   i) Type1: $\phi_{N_1}=\langle\phi,\phi,X\rangle$, or
   ii) Type2: $\phi_{N_1}=\langle\phi,X,\phi\rangle$, or
   iii) Type3: $\phi_N=\langle\phi,\phi,\phi\rangle$.
2) X_{N1} may be defined as one type
   Type1: $X_{N_1}=\langle X,\phi,\phi\rangle$.

Definition 3.3

A NCS-Type2, ϕ_{N_2}, X_{N2} in X as follows:

1) ϕ_{N2} may be defined as two types:
   i) Type1: $\phi_{N_2}=\langle\phi,\phi,X\rangle$, or
   ii) Type2: $\phi_{N_2}=\langle\phi,X,\phi\rangle$
2) X_{N2} may be defined as one type
   Type1: $X_{N_2}=\langle X,\phi,\phi\rangle$

Definition 3.4

A NCS-Type 3, ϕ_{N3}, X_{N3} in X as follows:

1) ϕ_{N3} may be defined as three types:
   i) Type1: $\phi_{N3}=\langle\phi,\phi,X\rangle$, or
   ii) Type2: $\phi_{N3}=\langle\phi,X,\phi\rangle$, or
   iii) Type3: $\phi_{N3}=\langle\phi,X,X\rangle$.
2) X_{N3} may be defined as three types
   i) Type1: $X_{N3}=\langle X,\phi,\phi\rangle$,
   ii) Type2: $X_{N3}=\langle X,X,\phi\rangle$,
   iii) Type3: $X_{N3}=\langle X,\phi,X\rangle$,

Corollary 3.2

In general

1- Every NCS-Type 1, 2, 3 are NCS.
2- Every NCS-Type 1 not to be NCS-Type2, 3.
3- Every NCS-Type 2 not to be NCS-Type1, 3.
4- Every NCS-Type 3 not to be NCS-Type2, 1, 2.
5- Every crisp set be NCS.

The following Venn diagram represents the relation between NCSs

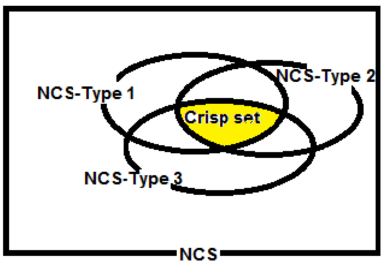

Fig 1. Venn diagram represents the relation between NCSs

Example 3.2

Let $X=\{a,b,c,d,e,f\}$, $A=\langle\{a,b,c,d\},\{e\},\{f\}\rangle$, $D=\langle\{a,b\},\{e,c\},\{f,d\}\rangle$ be a NCS-Type 2,

$B = \langle \{a,b,c\},\{d\},\{e\} \rangle$ be a NCT-Type1 but not NCS-Type 2, 3. $C = \langle \{a,b\},\{c,d\},\{e,f,a\} \rangle$ be a NCS-Type 3.but not NCS-Type1, 2.

Definition 3.5

Let X be a non-empty set, $A = \langle A_1, A_2, A_3 \rangle$

1)If A be a NCS-Type1 on X, then the complement of the set A (A^c, for short) maybe defined as one kind of complement Type1: $A^c = \langle A_3, A_2, A_1 \rangle$.

2) If A be a NCS-Type 2 on X, then the complement of the set A (A^c, for short) may be defined as one kind of complement $A^c = \langle A_3, A_2, A_1 \rangle$.

3) If A be NCS-Type3 on X, then the complement of the set A (A^c, for short) maybe defined as one kind of complement defined as three kinds of complements

(C_1) Type1: $A^c = \langle A^c{}_1, A^c{}_2, A^c{}_3 \rangle$,

(C_2) Type2: $A^c = \langle A_3, A_2, A_1 \rangle$

(C_3) Type3: $A^c = \langle A_3, A^c{}_2, A_1 \rangle$

Example 3.3

Let $X = \{a,b,c,d,e,f\}$, $A = \langle \{a,b,c,d\},\{e\},\{f\} \rangle$ be a NCS-Type 2, $B = \langle \{a,b,c\},\{\phi\},\{d,e\} \rangle$ be a NCS-Type1., $C = \langle \{a,b\},\{c,d\},\{e,f\} \rangle$ NCS-Type 3, then the complement $A = \langle \{a,b,c,d\},\{e\},\{f\} \rangle$,

$A^c = \langle \{f\},\{e\},\{a,b,c,d\} \rangle$ NCS-Type 2, the complement of $B = \langle \{a,b,c\},\{\phi\},\{d,e\} \rangle$, $B^c = \langle \{d,e\},\{\phi\},\{a,b,c\} \rangle$ NCS-Type1. The complement of $C = \langle \{a,b\},\{c,d\},\{e,f\} \rangle$ may be defined as three types:

Type 1: $C^c = \langle \{c,d,e,f\},\{a,b,e,f\},\{a,b,c,d\} \rangle$.

Type 2: $C^c = \langle \{e,f\},\{a,b,e,f\},\{a,b\} \rangle$,

Type 3: $C^c = \langle \{e,f\},\{c,d\},\{a,b\} \rangle$,

Proposition 3.1

Let $\{A_j : j \in J\}$ be arbitrary family of neutrosophic crisp subsets on X, then

1) $\cap A_j$ may be defined two types as :

i) Type1: $\cap A_j = \langle \cap Aj_1, \cap A_{j_2}, \cup A_{j_3} \rangle$,or

ii) Type2: $\cap A_j = \langle \cap Aj_1, \cup A_{j_2}, \cup A_{j_3} \rangle$.

2) $\cup A_j$ may be defined two types as :

1) Type1: $\cup A_j = \langle \cup Aj_1, \cap A_{j_2}, \cap A_{j_3} \rangle$ or

2) Type2: $\cup A_j = \langle \cup Aj_1, \cup A_{j_2}, \cap A_{j_3} \rangle$.

Definition 3.6

(a) If $B = \langle B_1, B_2, B_3 \rangle$ is a NCS in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is a NCS in X defined by $f^{-1}(B) = \langle f^{-1}(B_1), f^{-1}(B_2), f^{-1}(B_3) \rangle$.

(b) If $A = \langle A_1, A_2, A_3 \rangle$ is a NCS in X, then the image of A under f, denoted by $f(A)$, is the a NCS in Y defined by $f(A) = \langle f(A_1), f(A_2), f(A_3)^c) \rangle$.

Here we introduce the properties of images and preimages some of which we shall frequently use in the following.

Corollary 3.3

Let A, $\{A_i : i \in J\}$, be a family of NCS in X, and B, $\{B_j : j \in K\}$ NCS in Y, and $f : X \to Y$ a function. Then

(a) $A_1 \subseteq A_2 \Leftrightarrow f(A_1) \subseteq f(A_2)$,

$B_1 \subseteq B_2 \Leftrightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$,

(b) $A \subseteq f^{-1}(f(A))$ and if f is injective, then $A = f^{-1}(f(A))$,

(c) $f^{-1}(f(B)) \subseteq B$ and if f is surjective, then $f^{-1}(f(B)) = B$,

(d) $f^{-1}(\cup B_i) = f^{-1}(B_i)$, $f^{-1}(\cap B_i) = \cap f^{-1}(B_i)$,

(e) $f(\cup A_{i_i}) = \cup f(A_{ii})$; $f(\cap A_{i_i}) \subseteq \cap f(A_{ii})$; and if f is injective, then $f(\cap A_{i_i}) = \cap f(A_{ii})$;

(f) $f^{-1}(Y_N) = X_N$, $f^{-1}(\phi_N) = \phi_N$.

(g) $f(\phi_N) = \phi_N$, $f(X_N) = Y_N$, if f is subjective.

Proof

Obvious

4 Neutrosophic Crisp Points

One can easily define a nature neutrosophic crisp set in X, called "neutrosophic crisp point" in X, corresponding to an element X:

Definition 4.1

Let $A = \langle A_1, A_2, A_3 \rangle$,be a neutrosophic crisp set on a set X, then $p = \langle \{p_1\},\{p_2\},\{p_3\} \rangle$, $p_1 \neq p_2 \neq p_3 \in X$ is called a neutrosophic crisp point on A.

A NCP $p = \langle \{p_1\}, \{p_2\}, \{p_3\} \rangle$, is said to be belong to a neutrosophic crisp set $A = \langle A_1, A_2, A_3 \rangle$, of X, denoted by $p \in A$, if may be defined by two types

Type 1: $\{p_1\} \subseteq A_1, \{p_2\} \subseteq A_2$ and $\{p_3\} \subseteq A_3$ or

Type 2: $\{p_1\} \subseteq A_1, \{p_2\} \supseteq A_2$ and $\{p_3\} \subseteq A_3$

Theorem 4.1

Let $A = \langle A_1, A_2, A_3 \rangle$ and $B = \langle B_1, B_2, B_3 \rangle$, be neutrosophic crisp subsets of X. Then $A \subseteq B$ iff $p \in A$ implies $p \in B$ for any neutrosophic crisp point p in X.

Proof

Let $A \subseteq B$ and $p \in A$, Type 1: $\{p_1\} \subseteq A_1, \{p_2\} \subseteq A_2$ and $\{p_3\} \subseteq A_3$ or Type 2: $\{p_1\} \subseteq A_1, \{p_2\} \supseteq A_2$ and $\{p_3\} \subseteq A_3$ Thus $p \in B$. Conversely, take any point in X. Let $p_1 \in A_1$ and $p_2 \in A_2$ and $p_3 \in A_3$. Then p is a neutrosophic crisp point in X. and $p \in A$. By the hypothesis $p \in B$. Thus $p_1 \in B_1$ or Type1: $\{p_1\} \subseteq B_1, \{p_2\} \subseteq B_2$ and $\{p_3\} \subseteq B_3$ or Type 2: $\{p_1\} \subseteq B_1, \{p_2\} \supseteq B_2$ and $\{p_3\} \subseteq B_3$. Hence $A \subseteq B$.

Theorem 4.2

Let $A = \langle A_1, A_2, A_3 \rangle$, be a neutrosophic crisp subset of X. Then $A = \cup \{p : p \in A\}$.

Proof

Obvious

Proposition 4.1

Let $\{A_j : j \in J\}$ is a family of NCSs in X. Then

(a_1) $p = \langle \{p_1\}, \{p_2\}, \{p_3\} \rangle \in \bigcap_{j \in J} A_j$ iff $p \in A_j$ for each $j \in J$.

(a_2) $p \in \bigcup_{j \in J} A_j$ iff $\exists j \in J$ such that $p \in A_j$.

Proposition 4.2

Let $A = \langle A_1, A_2, A_3 \rangle$ and $B = \langle B_1, B_2, B_3 \rangle$ be two neutrosophic crisp sets in X. Then $A \subseteq B$ iff for each p we have $p \in A \Leftrightarrow p \in B$ and for each p we have $p \in A \Rightarrow p \in B$. iff $A = B$ for each p we have $p \in A \Rightarrow p \in B$ and for each p we have $p \in A \Leftrightarrow p \in B$.

Proposition 4.3

Let $A = \langle A_1, A_2, A_3 \rangle$ be a neutrosophic crisp set in X. Then $A = \cup < \{p_1 : p_1 \in A_1\}, \{p_2 : p_2 \in A_2\}, \{p_3 : p_3 \in A_3\}$.

Definition 4.2

Let $f : X \to Y$ be a function and p be a neutrosophic crisp point in X. Then the image of p under f, denoted by $f(p)$, is defined by $f(p) = \langle \{q_1\}, \{q_2\}, \{q_3\} \rangle$, where $q_1 = f(p_1), q_2 = f(p_2)$ and $q_3 = f(p_3)$. It is easy to see that $f(p)$ is indeed a NCP in Y, namely $f(p) = q$, where $q = f(p)$, and it is exactly the same meaning of the image of a NCP under the function f.

Definition 4.3

Let X be a nonempty set and $p \in X$. Then the neutrosophic crisp point p_N defined by $p_N = \langle \{p\}, \phi, \{p\}^c \rangle$ is called a neutrosophic crisp point (NCP for short) in X, where NCP is a triple ({only element in X}, empty set,{the complement of the same element in X }). Neutrosophic crisp points in X can sometimes be inconvenient when express neutrosophic crisp set in X in terms of neutrosophic crisp points. This situation will occur if $A = \langle A_1, A_2, A_3 \rangle$ NCS-Type1, $p \notin A_1$. Therefore we shall define "vanishing" neutrosophic crisp points as follows:

Definition 4.4

Let X be a nonempty set and $p \in X$ a fixed element in X Then the neutrosophic crisp set $p_{N_N} = \langle \phi, \{p\}, \{p\}^c \rangle$ is called vanishing" neutrosophic crisp point (VNCP for short) in X. where VNCP is a triple (empty set,{only element in X}, { the complement of the same element in X}).

Example 4.1

Let $X = \{a, b, c, d\}$ and $p = b \in X$. Then $p_N = \langle \{b\}, \phi, \{a, c, d\} \rangle$, $p_{N_N} = \langle \phi, \{b\}, \{a, c, d\} \rangle$, $P = \langle \{b\}, \{a\}, \{d\} \rangle$.

Now we shall present some types of inclusion of a neutrosophic crisp point to a neutrosophic crisp set:

Definition 4.5

Let $p_N = \langle \{p\}, \phi, \{p\}^c \rangle$ is a NCP in X and $A = \langle A_1, A_2, A_3 \rangle$ a neutrosophic crisp set in X.

(a) p_N is said to be contained in A ($p_N \in A$ for short) iff $p \in A_1$.

(b) p_{N_N} be VNCP in X and $A = \langle A_1, A_2, A_3 \rangle$ a neutrosophic crisp set in X. Then p_{N_N} is said to be contained in A ($p_{N_N} \in A$ for short) iff $p \notin A_3$.

Remark 4.2

p_N and p_{N_N} are NCS-Type1

Proposition 4.4

Let $\{A_j : j \in J\}$ is a family of NCSs in X. Then

(a_1) $p_N \in \bigcap_{j\in J} A_j$ iff $p_N \in A_j$ for each $j \in J$.

(a_2) $p_{N_N} \in \bigcap_{j\in J} A_j$ iff $p_{N_N} \in A_j$ for each $j \in J$.

(b_1) $p_N \in \bigcup_{j\in J} A_j$ iff $\exists j \in J$ such that $p_N \in A_j$.

(b_2) $p_{N_N} \in \bigcap_{j\in J} A_j$ iff $\exists j \in J$ such that $p_{N_N} \in A_j$.

Proof

Straightforward.

Proposition 4.5

Let $A = \langle A_1, A_2, A_3 \rangle$ and $B = \langle B_1, B_2, B_3 \rangle$ are two neutrosophic crisp sets in X. Then $A \subseteq B$ iff for each p_N we have $p_N \in A \Leftrightarrow p_N \in B$ and for each p_{N_N} we have $p_N \in A \Rightarrow p_{N_N} \in B$. $A = B$ iff for each p_N we have $p_N \in A \Rightarrow p_N \in B$ and for each p_{N_N} we have $p_{N_N} \in A \Leftrightarrow p_{N_N} \in B$.

Proof

Obvious

Proposition 4.6

Let $A = \langle A_1, A_2, A_3 \rangle$ be a neutrosophic crisp set in X. Then $A = (\cup\{p_N : p_N \in A\}) \cup (\cup\{p_{NN} : p_{NN} \in A\})$.

Proof

It is sufficient to show the following equalities: $A_1 = (\cup\{p\ \} : p_N \in A\}) \cup (\cup\{\phi : p_{NN} \in A\})$, $A_3 = \phi$ and $A_3 = (\cap\{\{p\ \}^c : p_N \in A\}) \cap (\cap\{\{p\}^c : p_{NN} \in A\})$ which are fairly obvious.

Definition 4.6

Let $f : X \to Y$ be a function and p_N be a nutrosophic crisp point in X. Then the image of p_N under f, denoted by $f(p_N)$ is defined by $f(p_N) = \langle \{q\}, \phi, \{q\}^c \rangle$ where $q = f(p\ \ )$.

Let p_{NN} be a VNCP in X. Then the image of p_{NN} under f, denoted by $f(p_{NN})$, is defined by $f(p_{NN}) = \langle \phi, \{q\}, \{q\}^c \rangle$ where $q = f(p\ \ )$.

It is easy to see that $f(p_N)$ is indeed a NCP in Y, namely $f(p_N) = q_N$ where $q = f(p\ \ )$, and it is exactly the same meaning of the image of a NCP under he function f . $f(p_{NN})$, is also a VNCP in Y, namely $f(p_{NN}) = q_{NN}$, where $q = f(p\ \ )$.

Proposition 4.7

States that any NCS A in X can be written in the form $A = \underset{N}{A} \cup \underset{NN}{A} \cup \underset{NNN}{A}$, where $\underset{N}{A} = \cup\{p_N : p_N \in A\}$, $\underset{N}{A} = \phi_N$ and $\underset{NNN}{A} = \cup\{p_{NN} : p_{NN} \in A\}$. It is easy to show that, if $A = \langle A_1, A_2, A_3 \rangle$, then $\underset{N}{A} = \langle A_1, \phi, A_1^c \rangle$ and $\underset{NN}{A} = \langle \phi, A_2, A_3 \rangle$.

Proposition 4.8

Let $f : X \to Y$ be a function and $A = \langle A_1, A_2, A_3 \rangle$ be a neutrosophic crisp set in X. Then we have $f(A) = f(\underset{N}{A}) \cup f(\underset{NN}{A}) \cup f(\underset{NNN}{A})$.

Proof

This is obvious from $A = \underset{N}{A} \cup \underset{NN}{A} \cup \underset{NNN}{A}$.

Proposition 4.9

Let $A = \langle A_1, A_2, A_3 \rangle$ and $B = \langle B_1, B_2, B_3 \rangle$ be two neutrosophic crisp sets in X. Then

a) $A \subseteq B$ iff for each p_N we have $p_N \in A \Leftrightarrow p_N \in B$ and for each p_{N_N} we have $p_N \in A \Rightarrow p_{N_N} \in B$.

b) $A = B$ iff for each p_N we have $p_N \in A \Rightarrow p_N \in B$ and for each p_{N_N} we have $p_{N_N} \in A \Leftrightarrow p_{N_N} \in B$.

Proof

Obvious

Proposition 4.10

Let $A = \langle A_1, A_2, A_3 \rangle$ be a neutrosophic crisp set in X. Then $A = (\cup\{p_N : p_N \in A\}) \cup (\cup\{p_{NN} : p_{NN} \in A\})$.

Proof

It is sufficient to show the following equalities:
$A_1 = (\cup\{p\ \} : p_N \in A\}) \cup (\cup\{\phi : p_{NN} \in A\})\ A_3 = \phi$
and $A_3 = \left(\cap\left\{\{p\ \}^c : p_N \in A\right\}\right) \cap \left(\cap\left\{\{p\}^c : p_{NN} \in A\right\}\right)$,
which are fairly obvious.

Definition 4.7

Let $f : X \to Y$ be a function.

(a) Let p_N be a neutrosophic crisp point in X. Then the image of p_N under f, denoted by $f(p_N)$, is defined by $f(p_N) = \left\langle \{q\}, \phi, \{q\}^c \right\rangle$, where $q = f(p\ )$.

(b) Let p_{NN} be a VNCP in X. Then the image of p_{NN} under f, denoted by $f(p_{NN})$, is defined by $f(p_{NN}) = \left\langle \phi, \{q\}, \{q\}^c \right\rangle$, where $q = f(p\ )$. It is easy to see that $f(p_N)$ is indeed a NCP in Y, namely $f(p_N) = q_N$, where $q = f(p\ )$, and it is exactly the same meaning of the image of a NCP under the function f. $f(p_{NN})$ is also a VNCP in Y, namely $f(p_{NN}) = q_{NN}$, where $q = f(p\ )$.

Proposition 4.11

Any NCS A in X can be written in the form $A = \underset{N}{A} \cup \underset{NN}{A} \cup \underset{NNN}{A}$, where $\underset{N}{A} = \cup\{p_N : p_N \in A\}$, $\underset{N}{A} = \phi_N$ and $\underset{NNN}{A} = \cup\{p_{NN} : p_{NN} \in A\}$. It is easy to show that, if $A = \left\langle A_1, A_2, A_3 \right\rangle$, then $\underset{N}{A} = \left\langle x, A_1, \phi, A_1^c \right\rangle$ and $\underset{NN}{A} = \left\langle x, \phi, A_2, A_3 \right\rangle$.

Proposition 4.12

Let $f : X \to Y$ be a function and $A = \left\langle A_1, A_2, A_3 \right\rangle$ be a neutrosophic crisp set in X. Then we have $f(A) = f(\underset{N}{A}) \cup f(\underset{NN}{A}) \cup f(\underset{NNN}{A})$.

Proof

This is obvious from $A = \underset{N}{A} \cup \underset{NN}{A} \cup \underset{NNN}{A}$.

5 Neutrosophic Crisp Set Relations

Here we give the definition relation on neutrosophic crisp sets and study of its properties.

Let X, Y and Z be three crisp nonempty sets

Definition 5.1

Let X and Y are two non-empty crisp sets and NCSS A and B in the form $A = \left\langle A_1, A_2, A_3 \right\rangle$ on X, $B = \left\langle B_1, B_2, B_3 \right\rangle$ on Y. Then

i) The product of two neutrosophic crisp sets A and B is a neutrosophic crisp set $A \times B$ given by

$A \times B = \left\langle A_1 \times B_1, A_2 \times B_2, A_3 \times B_3 \right\rangle$ on $X \times Y$.

ii) We will call a neutrosophic crisp relation $R \subseteq A \times B$ on the direct product $X \times Y$.

The collection of all neutrosophic crisp relations on $X \times Y$ is denoted as $NCR(X \times Y)$

Definition 5.2

Let R be a neutrosophic crisp relation on $X \times Y$, then the inverse of R is donated by R^{-1} where $R \subseteq A \times B$ on $X \times Y$ then $R^{-1} \subseteq B \times A$ on $Y \times X$.

Example 5.1

Let $X = \{a,b,c,d\}$, $A = \left\langle \{a,b\}, \{c\}, \{d\} \right\rangle$ and $B = \left\langle \{a\}, \{c\}, \{d,b\} \right\rangle$ then the product of two neutrosophic crisp sets given by

$A \times B = \left\langle \{(a,a),(b,a)\}, \{(c,c)\}, \{(d,d),(d,b)\} \right\rangle$ and

$B \times A = \left\langle \{(a,a),(a,b)\}, \{(c,c)\}, \{(d,d),(b,d)\} \right\rangle$, and

$R_1 = \left\langle \{(a,a)\}, \{(c,c)\}, \{(d,d)\} \right\rangle$, $R_1 \subseteq A \times B$ on $X \times X$,

$R_2 = \left\langle \{(a,b)\}, \{(c,c)\}, \{(d,d),(b,d)\} \right\rangle$ $R_2 \subseteq B \times A$ on $X \times X$,

$R_1^{-1} = \left\langle \{(a,a)\}, \{(c,c)\}, \{(d,d)\} \right\rangle \subseteq B \times A$ and

$R_2^{-1} = \left\langle \{(b,a)\}, \{(c,c)\}, \{(d,d),(d,b)\} \right\rangle \subseteq B \times A$.

Example 5.2

Let $X = \{a,b,c,d,e,f\}$, $A = \left\langle \{a,b,c,d\}, \{e\}, \{f\} \right\rangle$,

$D = \left\langle \{a,b\}, \{e,c\}, \{f,d\} \right\rangle$ be a NCS-Type 2,

$B = \left\langle \{a,b,c\}, \{\phi\}, \{d,e\} \right\rangle$ be a NCS-Type1.

$C = \left\langle \{a,b\}, \{c,d\}, \{e,f\} \right\rangle$ be a NCS-Type 3.Then

$A \times D = \left\langle \{(a,a),(a,b),(b,a),(b,b),(b,b),(c,a),(c,b),(d,a),(d,b)\}, \{(e,e),(e,c)\}, \{(f,f),(f,d)\} \right\rangle$

$D \times C = \left\langle \{(a,a),(a,b),(b,a),(b,b)\}, \{(e,c),(e,d),(c,c),(c,d)\}, \{(f,e),(f,f),(d,e),(d,f)\} \right\rangle$

We can construct many types of relations on products.

We can define the operations of neutrosophic crisp relation.

Definition 5.3

Let R and S be two neutrosophic crisp relations between X and Y for every $(x,y) \in X \times Y$ and NCSS A and B in the form $A = \left\langle A_1, A_2, A_3 \right\rangle$ on X, $B = \left\langle B_1, B_2, B_3 \right\rangle$ on Y Then we can defined the following operations

i) $R \subseteq S$ may be defined as two types

a) Type1: $R \subseteq S \Leftrightarrow A_{1_R} \subseteq B_{1_S}$, $A_2 \subseteq B_2$, $A_{3R} \supseteq B_{3S}$

b) Type2: $R \subseteq S \Leftrightarrow A_{1_R} \subseteq B_{1_S}$, $A_{2R} \supseteq B_{2S}$, $B_{3S} \subseteq A_{3R}$

ii) $R \cup S$ may be defined as two types

a) Type1: $R \cup S = \left\langle A_{1R} \cup B_{1S}, A_{2R} \cup B_{2S}, A_{3R} \cap B_{3S} \right\rangle$,

b) Type2:
$R \cup S = \langle A_{1R} \cup B_{1S}, A_{2R} \cap B_{2S}, A_{3R} \cap B_{3S} \rangle$.

iii) $R \cap S$ may be defined as two types

a) Type1: $R \cap S = \langle A_{1R} \cap B_{1S}, A_{2R} \cup B_{2S}, A_{3R} \cup B_{3S} \rangle$,

b) Type2:
$R \cap S = \langle A_{1R} \cap B_{1S}, A_{2R} \cap B_{2S}, A_{3R} \cup B_{3S} \rangle$.

Theorem 5.1

Let R, S and Q be three neutrosophic crisp relations between X and Y for every $(x, y) \in X \times Y$, then

i) $R \subseteq S \Rightarrow R^{-1} \subseteq S^{-1}$.

ii) $(R \cup S)^{-1} \Rightarrow R^{-1} \cup S^{-1}$.

iii) $(R \cap S)^{-1} \Rightarrow R^{-1} \cap S^{-1}$.

iv) $(R^{-1})^{-1} = R$.

v) $R \cap (S \cup Q) = (R \cap S) \cup (R \cap Q)$.

vi) $R \cup (S \cap Q) = (R \cup S) \cap (R \cup Q)$.

vii) If $S \subseteq R$, $Q \subseteq R$, then $S \cup Q \subseteq R$

Proof

Clear

Definition 5.4

The neutrosophic crisp relation $I \in NCR(X \times X)$, the neutrosophic crisp relation of identity may be defined as two types

i) Type1: $I = \{< \{A \times A\}, \{A \times A\}, \phi >\}$

ii) Type2: $I = \{< \{A \times A\}, \phi, \phi >\}$

Now we define two composite relations of neutrosophic crisp sets.

Definition 5.5

Let R be a neutrosophic crisp relation in $X \times Y$, and S be a neutrosophic crisp relation in $Y \times Z$. Then the composition of R and S, $R \circ S$ be a neutrosophic crisp relation in $X \times Z$ as a definition may be defined as two types

i) Type 1:
$R \circ S \leftrightarrow (R \circ S)(x, z) = \cup\{< \{(A_1 \times B_1)_R \cap (A_2 \times B_2)_S\}$, $\{(A_2 \times B_2)_R \cap (A_2 \times B_2)_S\}$, $\{(A_3 \times B_3)_R \cap (A_3 \times B_3)_S\} >$.

ii) Type 2:
$R \circ S \leftrightarrow (R \circ S)(x, z) = \cap\{< \{(A_1 \times B_1)_R \cup (A_2 \times B_2)_S\}$, $\{(A_2 \times B_2)_R \cup (A_2 \times B_2)_S\}$, $\{(A_3 \times B_3)_R \cup (A_3 \times B_3)_S\} >$.

Example 5.3

Let $X = \{a, b, c, d\}$, $A = \langle \{a, b\}, \{c\}, \{d\} \rangle$ and $B = \langle \{a\}, \{c\}, \{d, b\} \rangle$ then the product of two events given by $A \times B = \langle \{(a,a),(b,a)\}, \{(c,c)\}, \{(d,d),(d,b)\} \rangle$, and $B \times A = \langle \{(a,a),(a,b)\}, \{(c,c)\}, \{(d,d),(b,d)\} \rangle$, and

$R_1 = \langle \{(a,a)\}, \{(c,c)\}, \{(d,d)\} \rangle$, $R_1 \subseteq A \times B$ on $X \times X$,

$R_2 = \langle \{(a,b)\}, \{(c,c)\}, \{(d,d),(b,d)\} \rangle$ $R_2 \subseteq B \times A$ on $X \times X$.

$R_1 \circ R_2 = \cup \langle \{(a,a)\} \cap \{(a,b)\}, \{(c,c)\}, \{(d,d)\} \rangle$
$= \langle \{\phi\}, \{(c,c)\}, \{(d,d)\} \rangle$ and

$I_{A1} = \langle \{(a,a).(a,b).(b.a)\}, \{(a,a).(a,b).(b,a)\}, \{\phi\} \rangle$,

$I_{A2} = \langle \{(a,a).(a,b).(b.a)\}, \{\phi\}, \{\phi\} \rangle$

Theorem 5.2

Let R be a neutrosophic crisp relation in $X \times Y$, and S be a neutrosophic crisp relation in $Y \times Z$ then $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.

Proof

Let $R \subseteq A \times B$ on $X \times Y$ then $R^{-1} \subseteq B \times A$, $S \subseteq B \times D$ on $Y \times Z$ then $S^{-1} \subseteq D \times B$, from Definition 5.4 and similarly we can $I_{(R \circ S)^{-1}}(x, z) = I_{S^{-1}}(x, z)$ and $I_{R^{-1}}(x, z)$ then $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

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Introduction to Image Processing via Neutrosophic Techniques

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Abstract. This paper is an attempt of proposing the processing approach of neutrosophic technique in image processing. As neutrosophic sets is a suitable tool to cope with imperfectly defined images, the properties, basic operations distance measure, entropy measures, of the neutrosophic sets method are presented here. İn this paper we, introduce the distances between neutrosophic sets: the Hamming distance, the normalized Hamming distance, the Euclidean distance and normalized Euclidean distance. We will extend the concepts of distances to the case of neutrosophic hesitancy degree. Entropy plays an important role in image processing. In our further considertions on entropy for neutrosophic sets the concept of cardinality of a neutrosophic set will also be useful. Possible applications to image processing are touched upon.

Keywords: Neutrosophic sets; Hamming distance; Euclidean distance; Normalized Euclidean distance; Image processing.

1. Introduction

Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. Smarandache [9, 10] and Salama et al [4, 5, 6, 7, 8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 45]. Entropy plays an important role in image processing. İn this paper we, introduce the distances between neutrosophic sets: the Hamming distance. İn this paper we, introduce the distances between neutrosophic sets: the Hamming distance, The normalized Hamming distance, the Euclidean distance and normalized Euclidean distance. We will extend the concepts of distances to the case of neutrosophic hesitancy degree. In our further considertions on entropy for neutrosophic sets the concept of cardinality of a neutrosophic set will also be useful.

2. Terminologies

Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts [1, 2, 3, 11, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 46] such as a neutrosophic set theory. We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [9, 10] and Salama et al. [4, 5, 6, 7, 8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 45]. Smarandache introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where $]0^-,1^+[$ is nonstandard unit interval. Salama et al. introduced the following:

Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x) \rangle$ where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ which represent the degree of member ship function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-member ship (namely $\nu_A(x)$) respectively of each element $x \in X$ to the set A where

$0^- \le \mu_A(x), \sigma_A(x), \nu_A(x) \le 1^+$ and

$0^- \le \mu_A(x) + \sigma_A(x) + \nu_A(x) \le 3^+$ .Smarandache introduced the following: Let T, I,F be real standard or nonstandard subsets of $\left]0^-, 1^+\right[$, with

Sup_T=t_sup, inf_T=t_inf
Sup_I=i_sup, inf_I=i_inf
Sup_F=f_sup, inf_F=f_inf
n-sup=t_sup+i_sup+f_sup
n-inf=t_inf+i_inf+f_inf,
T, I, F are called neutrosophic components

3. Distances Betoween Neutrosophic Sets

We will now extend the concepts of distances presented in [11] to the case of neutrosophic sets.

Definition 3.1

Let $A = \{(\mu_A(x), \nu_A(x), \gamma_A(x)), x \in X\}$ and $B = \{(\mu_B(x), \nu_B(x), \gamma_B(x)), x \in X\}$ in $X = \{x_1, x_2, x_3, ..., x_n\}$ then

i) The Hamming distance is equal to

$$d_{Ns}(A,B) = \sum_{i=1}^{n} \left(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\gamma_A(x_i) - \gamma_B(x_i)| \right)$$

.

ii) The Euclidean distance is equal to

$$e_{Ns}(A,B) = \sqrt{\sum_{i=1}^{n} \left((\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\gamma_A(x_i) - \gamma_B(x_i))^2 \right)}$$

iii) The normalized Hamming distance is equal to

$$NH_{Ns}(A,B) = \frac{1}{2n} \sum_{i=1}^{n} \left(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\gamma_A(x_i) - \gamma_B(x_i)| \right)$$

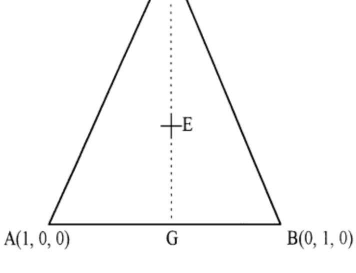
iv) The normalized Euclidean distance is equal to

$$NE_{Ns}(A,B) = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} \left((\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\gamma_A(x_i) - \gamma_B(x_i))^2 \right)}$$

Example 3.1

Let us consider for simplicity degenrated neutrosophic sets A, B, D, G, F in $X = \{a\}$. A full description of each neutrosophic set i.e. $A = \{(\mu_A(x), \nu_A(x), \gamma_A(x)), a \in X\}$, may be exemplified by $A = \{\langle 1,0,0 \rangle, a \in X\}$, $B = \{\langle 0,1,0 \rangle, a \in X\}$, $D = \{\langle 0,0,1 \rangle, a \in X\}$, $G = \{\langle 0.5,0.5,0 \rangle, a \in X\}$, $E = \{\langle 0.25,0.25,0.0.5 \rangle, a \in X\}$, .

Let us calculate four distances between the above neutrosophic sets using i), ii), iii) and iv) formulas ,



(Fig.1) A geometrical interpretation of the neutrosophic considered in Example 5.1 .

We obtain $e_{Ns}(A,D) = \frac{1}{2}$, $e_{Ns}(B,D) = \frac{1}{2}$, $e_{Ns}(A,B) = \frac{1}{2}$, $e_{Ns}(A,G) = \frac{1}{2}$, $e_{Ns}(B,G) = \frac{1}{2}$, $e_{Ns}(E,G) = \frac{1}{4}$, $e_{Ns}(D,G) = \frac{1}{4}$, $NE_{Ns}(A,B) = 1$, $NE_{Ns}(A,D) = 1$, $NE_{Ns}(B,D) = 1$, $NE_{Ns}(A,G) = \frac{1}{2}$, $NE_{Ns}(B,G) = \frac{1}{2}$, $NE_{Ns}(B,G) = \frac{1}{2}$, $NE_{Ns}(E,G) = \frac{\sqrt{3}}{4}$, a nd $NE_{Ns}(D,G) = \frac{\sqrt{3}}{2}$,

From the above results the triangle ABD (Fig.1) has edges equal to $\sqrt{2}$ and $e_{Ns}(A,D) = e_{Ns}(B,D) = e_{Ns}(A,B) = \frac{1}{2}$ and $NE_{Ns}(A,B) = NE_{Ns}(A,D) = NE_{Ns}(B,D) = 2NE_{Ns}(A,G) = 2NE_{Ns}(B,G) = 1$, and $NE_{Ns}(E,G)$ is equal to half of the height of triangle with all edges equal to $\sqrt{2}$ multiplied by, $\frac{1}{\sqrt{2}}$ i.e. $\frac{\sqrt{3}}{4}$.

Example 3.2

Let us consider the following neutrosophic sets A and B in $X = \{a, b, c, d, e\}$. ,

$A = \{\langle 0.5,0.3,0.2 \rangle, \langle 0.2,0.6,0.2 \rangle, \langle 0.3,0.2,0.5 \rangle, \langle 0.2,0.2,0.6 \rangle, \langle 1,0,0 \rangle\}$

$B = \{\langle 0.2,0.6,0.2 \rangle, \langle 0.3,0.2,0.5 \rangle, \langle 0.5,0.2,0.3 \rangle, \langle 0.9,0,0.1 \rangle, \langle 0,0,0 \rangle\}$

.Then

$d_{Ns}(A,B) = 3$, $NH_{Ns}(A,B) = 0.43$, $e_{Ns}(A,B) = 1.49$ and $NE_{Ns}(A,B) = 0.55$.

Remark 3.1

Clearly these distances satisfy the conditions of metric space.

Remark 3.2

It is easy to notice that for formulas i), ii), iii) and iv) the following is valid:

a) $0 \le d_{Ns}(A,B) \le n$
b) $0 \le NH_{Ns}(A,B) \le 1$
c) $0 \le e_{Ns}(A,B) \le \sqrt{n}$
d) $0 \le NE_{Ns}(A,B) \le 1$.

This representation of a neutrosophic set (Fig. 2) will be a point of departure for neutrosophic crisp distances, and entropy of neutrosophic sets.

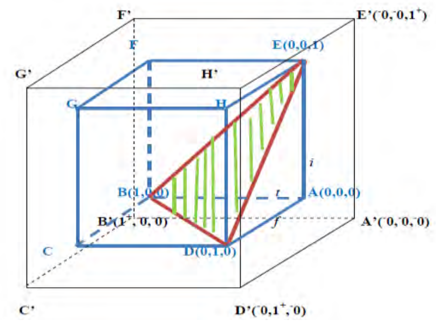


Fig. 2. A three-dimension representation of a neutrosophic set[9, 10].

4. Hesitancy Degree and Cardinality for Neutrosophic Sets

We will now extend the concepts of distances to the case of neutrosophic hesitancy degree. By taking into account the four parameters characterization of neutrosophic sets i.e. $A=\{<\mu_A(x),\nu_A(x),\gamma_A(x),\pi_A(x)>, x\in X\}$

Definition4.1

Let $A=\{(\mu_A(x),\nu_A(x),\gamma_A(x)), x\in X\}$ and $B=\{(\mu_B(x),\nu_B(x),\gamma_B(x)), x\in X\}$ on $X=\{x_1,x_2,x_3,\dots,x_n\}$

For a neutrosophic set $A=\{(\mu_A(x),\nu_A(x),\gamma_A(x)), x\in X\}$ in X, we call $\pi_A(x)=3-\mu_A(x)-\nu_A(x)-\gamma_A(x)$, the neutrosophic index of x in A. İt is a hesitancy degree of x to A it is obvtous that $0\le\pi_A(x)\le 3$.

Definition 4.2

Let $A=\{(\mu_A(x),\nu_A(x),\gamma_A(x)), x\in X\}$ and $B=\{(\mu_B(x),\nu_B(x),\gamma_B(x)), x\in X\}$ in $X=\{x_1,x_2,x_3,\dots,x_n\}$ then

i) The Hamming distance is equal to

$$d_{Ns}(A,B)=\sum_{i=1}^{n}\left(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\gamma_A(x_i)-\gamma_B(x_i)|+|\pi_A(x_i)-\pi_B(x_i)|\right)$$

. Taking into account that

$\pi_A(x_i)=3-\mu_A(x_i)-\nu_A(x_i)-\gamma_A(x_i)$ and

$\pi_B(x_i)=3-\mu_B(x_i)-\nu_B(x_i)-\gamma_B(x_i)$

we have

$$|\pi_A(x_i)-\pi_B(x_i)|=|3-\mu_A(x_i)-\nu_A(x_i)-\gamma_A(x_i)-3+\mu_A(x_i)+\nu_B(x_i)+\gamma_B(x_i)|$$
$$\le|\mu_B(x_i)-\mu_A(x_i)|+|\nu_B(x_i)-\nu_A(x_i)|+|\gamma_B(x_i)-\gamma_A(x_i)|$$

.

ii) The Euclidean distance is equal to

$$e_{Ns}(A,B)=\sqrt{\sum_{i=1}^{n}\left((\mu_A(x_i)-\mu_B(x_i))^2+(\nu_A(x_i)-\nu_B(x_i))^2+(\gamma_A(x_i)-\gamma_B(x_i))^2+(\pi_A(x_i)-\pi_B(x_i))^2\right)}$$

we have

$$(\pi_A(x_i)-\pi_B(x_i))^2=$$
$$(-\mu_A(x_i)-\nu_A(x_i)-\gamma_A(x_i)+\mu_B(x_i)+\nu_B(x_i)+\gamma_B(x_i))^2$$
$$=(\mu_B(x_i)-\mu_A(x_i))^2+(\nu_A(x_i)-\nu_B(x_i))^2+$$
$$(\gamma_A(x_i)-\gamma_B(x_i))^2$$
$$+2(\mu_B(x_i)-\mu_A(x_i))(\nu_A(x_i)-\nu_B(x_i))$$
$$(\gamma_B(x_i)-\gamma_A(x_i))$$

iii) The normalized Hamming distance is equal to

$$NH_{Ns}(A,B)=\frac{1}{2n}\sum_{i=1}^{n}\left(|\mu_A(x_i)-\mu_B(x_i)|+|\nu_A(x_i)-\nu_B(x_i)|+|\gamma_A(x_i)-\gamma_B(x_i)|+|\pi_A(x_i)-\pi_B(x_1)|\right)$$

iv) The normalized Euclidean distance is equal to

$$NE_{Ns}(A,B)=\sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left((\mu_A(x_i)-\mu_B(x_i))^2+(\nu_A(x_i)-\nu_B(x_i))^2+(\gamma_A(x_i)-\gamma_B(x_i))^2+(\pi_A(x_i)-\pi_B(x_i))^2\right)}$$

5.2 Remark

It is easy to notice that for formulas i), ii), iii) and the following is valid:

a) $0\le d_{Ns}(A,B)\le 2n$

b) $0\le NH_{Ns}(A,B)\le 2$

c) $0\le e_{Ns}(A,B)\le\sqrt{2n}$

d) $0\le NE_{Ns}(A,B)\le\sqrt{2}$.

5. from Images to Neutrosophic Sets, and Entropy

Given the definitions of the previous section several possible contributions are discussed. Neutrosophic sets may be used to solve some of the problems of data causes problems in the classification of pixels. Hesitancy in images originates from various factors, which in their majority are due to the inherent weaknesses of the acquisition and the imaging mechanisms. Limitations of the acquisition chain, such as the quantization noise, the suppression of the dynamic range, or the nonlinear behavior of the mapping system, affect our certainty on deciding whether a pixel is "*gray*" or "*edgy*" and therefore introduce a degree of hesitancy associated with the corresponding pixel. Therefore, hesitancy should encapsulate the aforementioned sources of indeterminacy that characterize digital images. Defining the membership component of the A–NS that describes the brightness of pixels in an image, is a more straightforward task that can be carried out in a similar manner as in traditional fuzzy image processing systems. In the presented heuristic framework, we consider the membership value of a gray level *g* to be its normalized intensity level; that is $\mu_A(g) = \frac{g}{L-1_x}$ where $g \in \{0,..., L-1\}$.It should be mentioned that any other method for calculating $\mu_A(g)$ can also be applied.

In the image is A being (x, y) the coordinates of each pixel and the $g(x, y)$ be the gray level of the pixel (x, y) implies $0 \le g(x, y) \le L-1$. Each image pixel is associated with four numerical values:

- A value representing the membership $\mu_A(x)$, obtained by means of membership function associated with the set that represents the expert's knowledge of the image.
- A value representing the indeterminacy $\nu_A(x)$, obtained by means of the indeterminacy function associated with the set that represents the ignorance of the expert's decision.
- A value representing the non-membership $\gamma_A(x)$, obtained by means of the non -membership function associated with the set that represents the ignorance of the expert's decision.
- A value representing the hesitation measure $\pi_A(x)$, obtained by means of the $\pi_A(x) = 3 - \mu_A(x) - \nu_A(x) - \gamma_A(x)$.

Let an image A of size $M \times N$ pixels having L gray levels ranging between 0 and *L-1*. The image in the neutrosophic domain is considered as an array of neutrosophic singletons. Here, each element denoted the degree of the membership, indeterminacy and non-membership according to a pixel with respect to an image considered. An image A in neutrosophic set is $A = \{< \mu_A(g_{ij}), \nu_A(g_{ij}), \gamma_A(g_{ij}) >, g_{ij} \in \{0,..., L-1\}\}$ where $\mu_A(g_{ij}), \nu_A(g_{ij}), \gamma_A(g_{ij})$ denote the degrees of membership indeterminacy and non-membership of the $(i, j)-th$ pixel to the set A associated with an image property $\mu_A(g) = \frac{g - g_{\min}}{g - g_{\max}}$ where $g_{\min}$ and $g_{\max}$ are the minimum and the maximum gray levels of the image. Entropy plays an important role in image processing. In our further considertions on entropy for neutrosophic sets the concept of cardinality of a neutrosophic set will also be useful

Definition 5.1

Let $A = \langle(\mu_A(x), \nu_A(x), \gamma_A(x)), x \in X\rangle$ a neutrosophic set in X, first, we define two cardinalities of a neutrosophic set

- The least (sure) cadinality of A is equal to so is called segma-count, and is called here the $\min \sum cont(A) = \sum_{i=1} \mu_A(x_i) + \sum_{i=1} \nu_A(x_i)$
- The bigesst cadinality of A , which is possible due to $\pi_A(x)$ is equal to $\max \sum cont(A) = \sum_{i=1} (\mu_A(x_i) + \pi_A(x_i)) + \sum_{i=1} \nu_A(x_i) + \pi_A(x_i))$

and , clearly for A^c we have

$\min \sum cont(A^c) = \sum_{i=1} \gamma_A(x_i) + \sum_{i=1} \nu_A(x_i)$,

$\max \sum cont(A^c) = \sum_{i=1} (\gamma_A(x_i) + \pi_A(x_i)) + \sum_{i=1} \nu_A(x_i) + \pi_A(x_i))$

. Then the cadinality of neutrosophic set is defined as the interval

$Card(A) = [\min \sum Cont(A), \max \sum Cont(A)]$

Definition 5.2

An entropy on $NS(X)$ is a real-valued functional $E : NS(X) \to [0,1]$, satisfying the following axiomatic requirements:

E$_1$: $E(A) = 0$ iff A is a neutrosophic crisp set; that is $\mu_A(x_i) = 0$ or $\mu_A(x_i) = 1$ for all $x_i \in X$.

E$_2$: $E(A) = 1$ iff $\mu_A(x_i) = \nu_A(x_i) = \gamma_A(x_i)$ for all $x_i \in X$. that is $A = A^c$.

E$_3$: $E(A) \le E(B)$ if A refine B ; i.e. $A \le B$.

E$_4$: $E(A) = E(A^c)$

Where a neutrosophic entropy measure be define as

$E(A) = \frac{1}{n}\sum_{i=1}^{n} \frac{\max Count(A_i \cap A_i^c)}{\max Count(A_i \cup A_i^c)}$ where $n = Cardinal(X)$ and A_i denotes the single-element A–NS corresponding to the i[th] element of the universe X and is described as

$A_i = \{(\mu_A(x_i), \nu_A(x_i), \gamma_A(x_i)), x_i \in X\}$.

In other words, A_i is the i[th] "component" of A.

Moreover, $\max Count(A)$ denotes the biggest cardinality of A and is given by :

$\max \sum cont(A) = \sum_{i=1} (\mu_A(x_i) + \pi_A(x_i)) + \sum_{i=1} \nu_A(x_i) + \pi_A(x_i))$

Conclusion

Some of the properties of the neutrosophic sets, Distance measures, Hesitancy Degree, Cardinality and Entropy measures are briefed in this paper. These measures can be used effectively in image processing and pattern recognition. The future work will cover the application of these measures.

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Interval Valued Neutrosophic Soft Topological Spaces

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Abstract. In this paper we introduce the concept of interval valued neutrosophic soft topological space together with interval valued neutrosophic soft finer and interval valued neutrosophic soft coarser topology. We also define interval valued neutrosophic interior and closer of an interval valued neutrosophic soft set. Some theorems and examples are cites. Interval valued neutrosophic soft subspace topology are studied. Some examples and theorems regarding this concept are presented..

Keywords: Soft set, interval valued neutrosophic set, interval valued neutrosophic soft set, interval valued neutrosophic soft topological space.

1 Introduction

In 1999, Molodtsov [9] introduced the concept of soft set theory which is completely new approach for modeling uncertainty. In this paper [9] Molodtsov established the fundamental results of this new theory and successfully applied the soft set theory into several directions. Maji et al. [7] defined and studied several basic notions of soft set theory in 2003. Pie and Miao [11], Aktas and Cagman [1] and Ali et. al. [2] improved the work of Maji et al [7]. The intuitionistic fuzzy set is introduced by Atanaasov [4] as a generalization of fuzzy set [15] where he added degree of non-membership with degree of membership. Neutrosophic set introduced by F. Smarandache in 1995 [12]. Smarandache [13] introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconstant data. Maji [8] combined neutrosophic set and soft set and established some operations on these sets. Wang et al. [14] introduced interval neutrosophic sets. Deli [6] introduced the concept of interval-valued neutrosophic soft sets.

In this paper we form a topological structure on interval valued neutrosophic soft sets and establish some properties of interval valued neutrosophic soft topological space with supporting proofs and examples.

2 Preliminaries

In this section we recall some basic notions relevant to soft sets, interval-valued neutrosophic sets and interval-valued neutrosophic soft sets.

Definition 2.1: [9] Let U be an initial universe and E be a set of parameters. Let $P(U)$ denotes the power set of U and $A \subseteq E$. Then the pair (f, A) is called a *soft set* over U, where f is a mapping given by $f : A \to P(U)$.

Definition 2.2: [13] A neutrosophic set A on the universe of discourse U is defined as $A = \{(x, \mu_A(x), \gamma_A(x), \delta_A(x)) : x \in U\}$, where $\mu_A, \gamma_A, \delta_A : U \to]^-0, 1^+[$ are functions such that the condition: $\forall x \in U$, $^-0 \le \mu_A(x) + \gamma_A(x) + \delta_A(x) \le 3^+$ is satisfied.

Here $\mu_A(x), \gamma_A(x), \delta_A(x)$ represent the truth-membership, indeterminacy-membership and falsity-membership respectively of the element $x \in U$. From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0, 1]$.

Definition 2.3: [14] An *interval valued neutrosophic set* A on the universe of discourse U is defined as $A = \{(x, \mu_A(x), \gamma_A(x), \delta_A(x)) : x \in U\}$, where $\mu_A, \gamma_A, \delta_A : U \to Int]^-0, 1^+[$ are functions such that the

condition:
$\forall x \in U$, $^{-}0 \le sup\mu_A(x) + sup\gamma_A(x) + sup\delta_A(x) \le 3^{+}$ is satisfied.

In real life applications it is difficult to use interval valued neutrosophic set with interval-value from real standard or non-standard subset of $Int(]^{-}0,1^{+}[)$. Hence we consider the interval valued neutrosophic set which takes the interval-value from the subset of $Int([0,1])$ (where $Int([0,1])$ denotes the set of all closed sub intervals of $[0,1]$). The set of all interval valued neutrosophic sets on U is denoted by *IVNS(U)*.

Definition 2.4: [6] Let U be an universe set, E be a set of parameters and $A \subseteq E$. Let *IVNs(U)* denotes the set of all interval valued neutrosophic sets of U. Then the pair (f, A) is called an *interval valued neutrosophic soft set* (*IVNSs* in short) over U, where f is a mapping given by $f : A \to IVNs(U)$. The collection of all interval valued neutrosophic soft sets over U is denoted by *IVNSs(U)*.

Definition 2.5: [6] Let U be a universe set and E be a set of parameters. Let $(f, A), (g, B) \in IVNSs(U)$, where $f : A \to IVNs(U)$ is defined by

$$f(a) = \{(x, \mu_{f(a)}(x), \gamma_{f(a)}(x), \delta_{f(a)}(x)) : x \in U\}$$

and $g : B \to IVNs(U)$ is defined by

$$g(b) = \{(x, \mu_{g(b)}(x), \gamma_{g(b)}(x), \delta_{g(b)}(x)) : x \in U\}$$

where

$\mu_{f(a)}(x), \gamma_{f(a)}(x), \delta_{f(a)}(x), \mu_{g(b)}(x), \gamma_{g(b)}(x), \delta_{g(b)}(x) \in Int([0,1])$

for $x \in U$. Then

(i) (f, A) is called *interval valued neutrosophic subset* of (g, B) (denoted by $(f, A) \subseteq (g, B)$) if $A \subseteq B$ and $\mu_{f(e)}(x) \le \mu_{g(e)}(x)$, $\gamma_{f(e)}(x) \ge \gamma_{g(e)}(x)$, $\delta_{f(e)}(x) \ge \delta_{g(e)}(x)$ $\forall e \in A, \forall x \in U$. Where

$\mu_{f(e)}(x) \le \mu_{g(e)}(x)$ iff $inf\,\mu_{f(e)} \le inf\,\mu_{g(e)}$ and $sup\mu_{f(e)} \le sup\mu_{g(e)}$

$\gamma_{f(e)}(x) \ge \gamma_{g(e)}(x)$ iff $inf\,\gamma_{f(e)} \ge inf\,\gamma_{g(e)}$ and $sup\gamma_{f(e)} \ge sup\gamma_{g(e)}$

$\delta_{f(e)}(x) \ge \delta_{g(e)}(x)$ iff $inf\,\delta_{f(e)} \ge inf\,\delta_{g(e)}$ and $sup\delta_{f(e)} \ge sup\delta_{g(e)}$.

(ii) Their *union*, denoted by $(f, A) \cup (g, B) = (h, C)$ (say), is an interval valued neutrosophic soft set over U, where $C = A \cup B$ and for $e \in C$, $h : C \to IVNS(U)$ is defined by

$h(e) = \{(x, \mu_{h(e)}(x), \gamma_{h(e)}(x), \delta_{h(e)}(x)) : x \in U\}$, where for $x \in U$,

$$\mu_{h(e)}(x) = \begin{cases} \mu_{f(e)}(x) & if\ e \in A - B \\ \mu_{g(e)}(x) & if\ e \in B - A \\ \mu_{f(e)}(x) \vee \mu_{g(e)}(x) & if\ e \in A \cap B \end{cases}$$

$$\gamma_{h(e)}(x) = \begin{cases} \gamma_{f(e)}(x) & if\ e \in A - B \\ \gamma_{g(e)}(x) & if\ e \in B - A \\ \gamma_{f(e)}(x) \wedge \gamma_{g(e)}(x) & if\ e \in A \cap B \end{cases}$$

$$\delta_{h(e)}(x) = \begin{cases} \delta_{f(e)}(x) & if\ e \in A - B \\ \delta_{g(e)}(x) & if\ e \in B - A \\ \delta_{f(e)}(x) \wedge \delta_{g(e)}(x) & if\ e \in A \cap B \end{cases}$$

(iii) Their *intersection*, denoted by $(f, A) \cap (g, B) = (h, C)$ (say), is an interval valued neutrosophic soft set of over U, where $C = A \cap B$ and for $e \in C$, $h : C \to IVNS(U)$ is defined by

$h(e) = \{(x, \mu_{h(e)}(x), \gamma_{h(e)}(x), \delta_{h(e)}(x)) : x \in U\}$, where for $x \in U$ and $e \in C$,

$\mu_{h(e)}(x) = \mu_{f(e)}(x) \wedge \mu_{g(e)}(x), \gamma_{h(e)}(x) = \gamma_{f(e)}(x) \vee \gamma_{g(e)}(x)$

and $\delta_{h(e)}(x) = \delta_{f(e)}(x) \vee \delta_{g(e)}(x)$.

(iv) The *complement* of (f, A), denoted by $(f, A)^c$ is an interval valued neutrosophic soft set over U and is defined as $(f, A)^c = (f^c, \rceil A)$, where $f^c : \rceil A \to IVNS(U)$ is defined by

$$f^c(a) = \{(x, \delta_{f(a)}(x), [1 - sup\gamma_{f(a)}(x), 1 - inf\,\gamma_{f(a)}(x)], \mu_{f(a)}(x)) : x \in U\}$$

for $a \in A$.

Definition 2.6:[5,6] An *IVNSs* (f, A) over the universe U is said to be universe *IVNSs* with respect to A if $\mu_{f(a)}(x) = [1,1]$, $\gamma_{f(a)}(x) = [0,0]$, $\delta_{f(a)}(x) = [0,0]$ $\forall x \in U, \forall a \in A$. It is denoted by I.

Definition 2.7: An *IVNSs* (f, A) over the universe U is said to be null *IVNSs* with respect to A if $\mu_{f(a)}(x) = [0,0]$, $\gamma_{f(a)}(x) = [1,1]$, $\delta_{f(a)}(x) = [1,1]$ $\forall x \in U, \forall a \in A$. It is denoted by ϕ.

3 Interval Valued Neutrosophic Soft Topological Spaces

In this section, we give the definition of interval valued neutrosophic soft topological spaces with some examples and results. We also define discrete and indiscrete interval valued neutrosophic soft topological space along with interval valued neutrosophic soft finer and coarser topology.

Let U be an universe set, E be the set of parameters, $\wp(U)$ be the set of all subsets of U, *IVNs(U)* be the set of all interval valued neutrosophic sets in U and *IVSNs(U;E)* be the family of all interval valued neutrosophic soft sets over U via parameters in E.

Definition 3.1: Let (ζ_A, E) be an element of *IVNSs(U;E)*, $\wp(\zeta_A, E)$ be the collection of all interval valued neutrosophic soft subsets of (ζ_A, E). A sub family τ of $\wp(\zeta_A, E)$ is called an interval valued neutrosophic soft topology (in short *IVNS*-topology) on (ζ_A, E) if the following axioms are satisfied:

(i) $(\phi_{\zeta_A}, E), (\zeta_A, E) \in \tau$

(ii) $\{(f_A^k, E) : k \in K\} \subseteq \tau \Rightarrow \bigcup_{k\in K} (f_A^k, E) \in \tau$

(iii) If $(g_A, E), (h_A, E) \in \tau$ then $(g_A, E) \cap (h_A, E) \in \tau$

The triplet (ζ_A, E, τ) is called interval valued neutrosophic soft topological space (in short *IVNS*-topological space) over (ζ_A, E). The members of τ are called τ –open *IVNS* sets (or simply open sets). Here $\phi_{\zeta_A} : A \to IVNS(U)$ is defined as $\phi_{\zeta_A}(e) = \{(x, [0,0], [1,1], [1,1]) : x \in U\}$ $\forall e \in A$.

Example 3.2: Let $U = \{u_1, u_2, u_3\}$, $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2, e_3\}$. The tabular representation of (ζ_A, E) given by

U	e_1	e_2	e_3
u_1	([.5,.8],[.3,.5],[.2,.7])	([.4,.7],[.2,.3],[.1,.3])	([.3,.9],[0,.1],[0,.2])
u_2	([.4,.7],[.3,.4],[.1,.2])	([.6,.9],[.1,.2],[.1,.2])	([.4,.8],[.1,.2],[0,.5])
u_3	([.5,1],[0,.1],[.3,.6])	([.6,.8],[.2,.4],[.1,.3])	([.4,.9],[.1,.3],[.2,.4])

Table1:Tabular representation of (ζ_A, E)

The tabular representation of (ϕ_{ζ_A}, E) is given by

U	e_1	e_2	e_3
u_1	([0,0],[1,1],[1,1])	([0,0],[1,1],[1,1])	*([0,0],[1,1],[1,1])*
u_2	([0,0],[1,1],[1,1])	([0,0],[1,1],[1,1])	*([0,0],[1,1],[1,1])*
u_3	([0,0],[1,1],[1,1])	([0,0],[1,1],[1,1])	*([0,0],[1,1],[1,1])*

Table2:Tabular representation of (ϕ_{ζ_A}, E)

The tabular representation of (f_A^1, E) is given by

U	e_1	e_2	e_3
u_1	*([.1,.7],[.4,.8],[.3,1])*	*([.1,.3],[.4,.6],[.2,.6])*	*([.2,.5],[.8,.9],[.4,.9])*
u_2	*([.1,.3],[.6,.7],[.2,.8])*	*([0,.5],[.5,.8],[.4,1])*	*([0,.3],[.6,.9],[.1,.7])*
u_3	*([.4,.8],[.6,.7],[.6,.9])*	*([0,.3],[.4,.7],[.2,.8])*	*([.1,.3],[.6,.8],[.3,.7])*

Table3:Tabular representation of (f_A^1, E)

The tabular representation of (f_A^2, E) is given by

U	e_1	e_2	e_3
u_1	([.4,.7],[.5,.7],[.4,.9])	([.2,.3],[.4,.5],[.7,.9])	([.3,.7],[.5,.8],[.1,.2])
u_2	([.3,.5],[.4,.8],[.1,.4])	([.4,.6],[.3,.5],[.2,.5])	([.1,.3],[.3,.5],[.6,.8])
u_3	([.3,.9],[.1,.2],[.6,.7])	([.5,.7],[.6,.7],[.3,.4])	([.2,.6],[.3,.5],[.5,.8])

Table4: Tabular representation of (f_A^2, E)

Let $(f_A^3, E) = (f_A^1, E) \cap (f_A^2, E)$ then the tabular representation of (f_A^3, E) is given by

U	e_1	e_2
u_1	([.1,.7],[.5,.8],[.4,1])	([.1,.3],[.4,.6],[.7,.9])
u_2	([.1,.3],[.6,.8],[.2,.8])	([0,.5],[.5,.8],[.4,1])
u_3	([.3,.8],[.6,.7],[.6,.9])	([0,.3],[.6,.7],[.3,.8])

e_3
([.2,.5],[.8,.9],[.4,.9])
([0,.3],[.6,.9],[.6,.8])
([.1,.3],[.6,.8],[.5,.8])

Table5:Tabular representation of $\left(f_A^3,E\right)$

Let $\left(f_A^4,E\right)=\left(f_A^1,E\right)\cup\left(f_A^2,E\right)$ then the tabular representation of $\left(f_A^4,E\right)$ is given by

U	e_1	e_2
u_1	([.4,.7],[.4,.7],[.3,.9])	([.2,.3],[.4,.5],[.2,.6])
u_2	([.3,.5],[.4,.7],[.1,.4])	([.4,.6],[.3,.5],[.2,.5])
u_3	([.4,.9],[.1,.2],[.6,.7])	([.5,.7],[.4,.7],[.2,.4])

e_3
([.3,.7],[.5,.8],[.1,.2])
([.1,.3],[.3,.5],[.1,.7])
([.2,.6],[.3,.5],[.3,.7])

Table6:Tabular representation of $\left(f_A^4,E\right)$

Here we observe that the sub-family $\tau_1=\left\{\left(\phi_{\zeta_A},E\right),\left(\zeta_A,E\right),\left(f_A^1,E\right),\left(f_A^2,E\right),\left(f_A^3,E\right),\left(f_A^4,E\right)\right\}$ of $\wp\left(\zeta_A,E\right)$ is a *IVNS*-topology on $\left(\zeta_A,E\right)$, as it satisfies the necessary three axioms of topology and $\left(\zeta_A,E,\tau\right)$ is a *IVNS*-topological space. But the sub-family $\tau_2=\left\{\left(\phi_{\zeta_A},E\right),\left(\zeta_A,E\right),\left(f_A^1,E\right),\left(f_A^2,E\right)\right\}$ of $\wp\left(\zeta_A,E\right)$ is not an *IVNS*-topology on $\left(\zeta_A,E\right)$, as the union $\left(f_A^4,E\right)=\left(f_A^1,E\right)\cup\left(f_A^2,E\right)$ does not belong to τ_2.

Definition 3.3: As every *IVNS*-topology on $\left(\zeta_A,E\right)$ must contains the sets $\left(\phi_{\zeta_A},E\right)$ and $\left(\zeta_A,E\right)$, so the family $\vartheta=\left\{\left(\phi_{\zeta_A},E\right),\left(\zeta_A,E\right)\right\}$ forms a *IVNS*-topology on $\left(\zeta_A,E\right)$. The topology is called indiscrete *IVNS*-topology and the triplet $\left(\zeta_A,E,\vartheta\right)$ is called an indiscrete interval valued neutrosophic soft topological space (or simply indiscrete *IVNS*-topological space).

Definition 3.4: Let ξ denotes the family of all *IVNS*-subsets of $\left(\zeta_A,E\right)$. Then we observe that ξ satisfies all the axioms of topology on $\left(\zeta_A,E\right)$. This topology is called discrete interval valued neutrosophic soft topology and the triplet $\left(\zeta_A,E,\xi\right)$ is called discrete interval valued neutrosophic soft topological space (or simply discrete *IVNS*-topological space).

Theorem 3.5: Let $\left\{\tau_i : i\in I\right\}$ be any collection of *IVNS*-topology on $\left(\zeta_A,E\right)$. Then their intersection $\bigcap_{i\in I}\tau_i$ is also a *IVNS*-topology on $\left(\zeta_A,E\right)$.

Proof: (i) Since $\left(\phi_{\zeta_A},E\right),\left(\zeta_A,E\right)\in\tau_i$ for each $i\in I$. Hence $\left(\phi_{\zeta_A},E\right),\left(\zeta_A,E\right)\in\bigcap_{i\in I}\tau_i$.

(ii) Let $\left\{\left(f_A^k,E\right):k\in K\right\}$ be an arbitrary family of interval valued neutrosophic soft sets where $\left(f_A^k,E\right)\in\bigcap_{i\in I}\tau_i$ for each $k\in K$. Then for each $i\in I$, $\left(f_A^k,E\right)\in\tau_i$ for $k\in K$ and since for each $i\in I$, τ_i ia a *IVNS*-topology, therefore $\bigcup_{k\in K}\left(f_A^k,E\right)\in\tau_i$ for each $i\in I$. Hence $\bigcup_{k\in K}\left(f_A^k,E\right)\in\bigcap_{i\in I}\tau_i$.

(iii) Let $\left(f_A^1,E\right),\left(f_A^2,E\right)\in\bigcap_{i\in I}\tau_i$, then $\left(f_A^1,E\right),\left(f_A^2,E\right)\in\tau_i$ for each $i\in I$. Since for each $i\in I$, τ_i is an *IVNS*-topology, therefore $\left(f_A^1,E\right)\cap\left(f_A^2,E\right)\in\tau_i$ for each $i\in I$. Hence $\left(f_A^1,E\right)\cap\left(f_A^2,E\right)\in\bigcap_{i\in I}\tau_i$.

Thus $\bigcap_{i\in I}\tau_i$ satisfies all the axioms of topology. Hence $\bigcap_{i\in I}\tau_i$ forms a *IVNS*-topology. But union of *IVNS*-topologies need not be a *IVNS*-topology. Let us show this with the following example.

Example 3.6: In example 3.2, the sub families $\tau_3=\left\{\left(\phi_{\zeta_A},E\right),\left(\zeta_A,E\right),\left(f_A^1,E\right)\right\}$ and $\tau_4=\left\{\left(\phi_{\zeta_A},E\right),\left(\zeta_A,E\right),\left(f_A^2,E\right)\right\}$ are IVNS-topologies in $\left(\zeta_A,E\right)$. But their union $\tau_3\cup\tau_4=\left\{\left(\phi_{\zeta_A},E\right),\left(\zeta_A,E\right),\left(f_A^1,E\right),\left(f_A^2,E\right)\right\}$ is not a IVNS-topology in $\left(\zeta_A,E\right)$.

Definition 3.7: Let $\left(\zeta_A,E,\tau\right)$ be an *IVNS*-topological space over $\left(\zeta_A,E\right)$. An interval valued neutrosophic soft subset $\left(f_A,E\right)$ of $\left(\zeta_A,E\right)$ is called interval valued neutrosophic soft closed set (in short *IVNS*-closed set) if its complement $\left(f_A,E\right)^c$ is a member of τ.

Example 3.8: Let us consider example 3.2. then the IVNS-closed sets in $\left(\zeta_A,E,\tau_1\right)$ are

U	e_1	e_2	e_3
u_1	([.2,.7],[.5,.7],[.5,.8])	([.1,.3],[.7,.8],[.4,.7])	([0,.2],[.9,1],[.3,.9])
u_2	([.1,.2],[.6,.7],[.4,.7])	([.1,.2],[.8,.9],[.6,.9])	([0,.5],[.8,.9],[.4,.8])
u_3	([.3,.6],[.9,1],[.5,1])	([.1,.3],[.6,.8],[.6,.8])	([.2,.4],[.7,.9],[.4,.9])

Table7:Tabular representation of $\left(\zeta_A,E\right)^c$

U	e_1	e_2	e_3
u_1	([1,1], [0,0],[0,0])	([1,1], [0,0],[0,0])	([1,1], [0,0],[0,0])
u_2	([1,1], [0,0],[0,0])	([1,1], [0,0],[0,0])	([1,1], [0,0],[0,0])
u_3	([1,1], [0,0],[0,0])	([1,1], [0,0],[0,0])	([1,1], [0,0],[0,0])

Table8:Tabular representation of $\left(\phi_{\zeta_A},E\right)^c$

U	e_1	e_2	e_3
u_1	([.3,1],[.2,.6],[.1,.7])	([.2,.6],[.4,.6],[.1,.3])	([.4,.9],[.1,.2],[.2,.5])
u_2	([.2,.8],[.3,.4],[.1,.3])	([.4,1],[.2,.5],[0,.5])	([.1,.6],[.1,.4],[0,.3])
u_3	([.6,.9,[.3,.4],[.4,.8])	([.2,.8],[.3,.6],[0,.3])	([.3,.7],[.2,.4],[.1,.3])

Table9:Tabular representation of $\left(f_A^1,E\right)^c$

U	e_1	e_2	e_3
u_1	([.4,.9],[.3,.5],[.4,.7])	([.7,.9],[.5,.6],[.2,.3])	([.1,.2],[.2,.5],[.3,.7])
u_2	([.1,.4],[.2,.6],[.3,.5])	([.2,.5],[.5,.7],[.4,.6])	([.6,.8],[.5,.7],[.1,.3])
u_3	([.6,.7],[.8,.9],[.3,.9])	([.3,.4],[.3,.4],[.5,.7])	([.5,.8],[.5,.7],[.2,.6])

Table10:Tabular representation of $\left(f_A^2,E\right)^c$

U	e_1	e_2	e_3
u_1	([.4,1],[.2,.5],[.1,.7])	([.7,.9],[.4,.6],[.1,.3])	([.4,.9],[.1,.2],[.2,.5])
u_2	([.2,.8],[.2,.4],[.1,.3])	([.4,1],[.2,.5],[0,.5])	([.6,.8],[.1,.4],[0,.3])
u_3	([.6,.9],[.3,.4],[.3,.8])	([.3,.8],[.3,.4],[0,.3])	([.5,.8],[.2,.4],[.1,.3])

Table11:Tabular representation of $\left(f_A^3,E\right)^c$

U	e_1	e_2	e_3
u_1	([.3,.9],[.3,.6],[.4,.7])	([.2,.6],[.5,.6],[.2,.3])	([.1,.2],[.2,.5],[.3,.7])
u_2	([.1,.4],[.3,.6],[.3,.5])	([.2,.5],[.5,.7],[.4,.6])	([.1,.7],[.5,.7],[.1,.3])
u_3	([.6,.7],[.8,.9],[.4,.9])	([.2,.4],[.3,.6],[.5,.7])	([.3,.7],[.5,.7],[.2,.6])

Table12:Tabular representation of $\left(f_A^4,E\right)^c$

are the IVNS-closed sets in $\left(\zeta_A,E,\tau_1\right)$.

Theorem 3.9: Let $\left(\zeta_A,E,\tau\right)$ be an *IVNS*-topological space over $\left(\zeta_A,E\right)$. Then

1. $\left(\phi_{\zeta_A},E\right)^c$, $\left(\zeta_A,E\right)^c$ are *IVNS*-closed sets.
2. Arbitrary intersection of *IVNS*-closed sets is *IVNS*-closed set.
3. Finite union of IVNS-closed sets is *IVNS*-closed set.

Proof: 1. Since $\left(\phi_{\zeta_A},E\right),\left(\zeta_A,E\right)\in\tau$, therefore $\left(\phi_{\zeta_A},E\right)^c,\left(\zeta_A,E\right)^c$ are *IVNS*-closed sets.

2. Let $\left\{\left(f_A^k,E\right):k\in K\right\}$ be an arbitrary family of *IVNS*-closed sets in $\left(\zeta_A,E,\tau\right)$ and let $\left(f_A,E\right)=\bigcap_{k\in K}\left(f_A^k,E\right)$.

Now $(f_A,E)^c = \left(\bigcap_{k\in K}(f_A^k,E)\right)^c = \bigcup_{k\in K}(f_A^k,E)^c$ and $(f_A^k,E)^c \in \tau$ for each $k\in K$, so $\bigcup_{k\in K}(f_A^k,E)^c \in \tau$. Hence $(f_A,E)^c\in\tau$. Thus (f_A,E) is *IVNS*-closed set.

3. Let $\{(f_A^i,E): i=1,2,3,...,n\}$ be a family of *IVNS*-closed sets in (ζ_A,E,τ) and let $(g_A,E)=\bigcup_{i=1}^{n}(f_A^i,E)$.

Now $(g_A,E)^c = \left(\bigcup_{i=1}^{n}(f_A^i,E)\right)^c = \bigcap_{i=1}^{n}(f_A^i,E)^c$ and $(f_A^i,E)^c\in\tau$ for $i=1,2,3,...,n$, so $\bigcap_{i=1}^{n}(f_A^i,E)^c\in\tau$. Hence $(g_A,E)^c\in\tau$. Thus (g_A,E) is *IVNS*-closed set.

Definition 3.10: Let (ζ_A,E,τ_1) and (ζ_A,E,τ_2) be two *IVNS*-topological spaces over (ζ_A,E). If each $(f_A,E)\in\tau_2$ implies $(f_A,E)\in\tau_1$, then τ_1 is called interval valued neutrosophic soft finer topology than τ_2 and τ_2 is called interval valued neutrosophic soft coarser topology than τ_1.

Example 3.11: In example 3.2 and 3.6, τ_1 is interval valued neutrosophic soft finer topology than τ_3 and τ_3 is called interval valued neutrosophic soft coarser topology than τ_1.

Definition 3.12: Let (ζ_A,E,τ) be a *IVNS*-topological space over (ζ_A,E) and ß be a subfamily of τ. If every element of τ can be express as the arbitrary interval valued neutrosophic soft union of some elements of ß, then ß is called an interval valued neutrosophic soft basis for the *IVNS*-topology τ.

Example 3.13: In example 3.2, for the *IVNS*-topology $\tau_1=\{(\phi_{\zeta_A},E),(\zeta_A,E),(f_A^1,E),(f_A^2,E),(f_A^3,E),(f_A^4,E)\}$, the subfamily ß$=\{(\phi_{\zeta_A},E),(\zeta_A,E),(f_A^1,E),(f_A^2,E),(f_A^3,E)\}$ of $\wp(\zeta_A,E)$ is a interval valued neutrosophic soft basis for the *IVNS*-topology τ_1.

4 Some Properties of Interval Valued Neutrosophic Soft Topological Spaces

In this section some properties of interval valued neutrosophic soft topological spaces are introduced. Some results on IVNSInt and IVNSCl are also intoduced.

Definition 4.1: Let (ζ_A,E,τ) be a *IVNS*-topological space and let $(f_A,E)\in IVNSS(U;E)$. The interval valued neutrosophic soft interior and closer of (f_A,E) is denoted by *IVNSInt(fA,E)* and *IVNSCl(fA,E)* are defined as $IVNSInt(f_A,E)=\bigcup\{(g_A,E)\in\tau:(g_A,E)\subseteq(f_A,E)\}$ and $INVNSCl(f_A,E)=$ $\bigcap\{(g_A,E)\in\tau^c:(f_A,E)\subseteq(g_A,E)\}$ respectively.

Example 4.2: Let us consider example 3.2 and take an IVNSS (f_A^5,E) as

U	e_1	e_2	e_3
u_1	([.2,.8],[.3,.6],[.2,.8])	([.2,.4],[.4,.6],[.2,.4])	([.2,.6],[.7,.8],[.3,.4])
u_2	([.1,.6],[.4,.5],[.2,.7])	([.2,.6],[.5,.7],[.1,.7])	([.1,.4],[.2,.5],[.1,.5])
u_3	([.5,.8],[.5,.6],[.5,.8])	([.1,.4],[.4,.6],[.1,.5])	([.2,.5],[.5,.8],[.2,.4])

Table13:Tabular representation of (f_A^5,E)

Now $IVNSInt(f_A^5,E)=(f_A^1,E)$ and $IVNSCl(f_A^5,E)=(f_A^1,E)^c$.

Theorem 4.3: Let (ζ_A,E,τ) be a *IVNS*-topological space and (f_A,E), $(g_A,E)\in IVNSS(U;E)$ then the following properties hold

1. $IVNSInt(f_A,E)\subseteq(f_A,E)$
2. $(f_A,E)\subseteq(g_A,E)\Rightarrow IVNSInt(f_A,E)\subseteq IVNSInt(g_A,E)$
3. $IVNSInt(f_A,E)\in\tau$
4. $(f_A,E)\in\tau \Leftrightarrow IVNSInt(f_A,E)=(f_A,E)$
5. $IVNSInt(IVNSInt(f_A,E))=IVNSInt(f_A,E)$
6. $IVNSInt(\phi_A,E)=\phi_A$, $IVNSInt(U_A,E)=U_A$

Proof:

1. Straight forward.

2. $(f_A,E)\subseteq(g_A,E)$ implies all the *IVNS*-open sets contained in (f_A,E) also contained in (g_A,E).

i.e.

$$\{(f_A^*,E)\in\tau:(f_A^*,E)\subseteq(f_A,E)\}\subseteq\{(g_A^*,E)\in\tau:(g_A^*,E)\subseteq(g_A,E)\}$$

i.e.
$\bigcup\{(f_A^*,E)\in\tau:(f_A^*,E)\subseteq(f_A,E)\}\subseteq\bigcup\{(g_A^*,E)\in\tau:(g_A^*,E)\subseteq(g_A,E)\}$
i.e. $IVNSInt(f_A,E)\subseteq IVNSInt(g_A,E)$

3. $IVNSInt(f_A,E)=\bigcup\{(f_A^*,E)\in\tau:(f_A^*,E)\subseteq(f_A,E)\}$
It is clear that $\bigcup\{(f_A^*,E)\in\tau:(f_A^*,E)\subseteq(f_A,E)\}\in\tau$
So, $IVNSInt(f_A,E)\in\tau$.

4. Let $(f_A,E)\in\tau$, then by (1) $IVNSInt(f_A,E)\subseteq(f_A,E)$.
Now since $(f_A,E)\in\tau$ and $(f_A,E)\subseteq(f_A,E)$, Therefore
$(f_A,E)\subseteq\bigcup\{(g_A^*,E)\in\tau:(g_A^*,E)\subseteq(g_A,E)\}=IVNSInt(f_A,E)$
i.e, $(f_A,E)\subseteq IVNSInt(f_A,E)$
Thus $IVNSInt(f_A,E)=(f_A,E)$
Conversly, let $IVNSInt(f_A,E)=(f_A,E)$
Since by (3) $IVNSInt(f_A,E)\in\tau$
Therefore $(f_A,E)\in\tau$

5. By (3) $IVNSInt(f_A,E)\in\tau$
$\therefore$ By (4) $IVNSInt(IVNSInt(f_A,E))=IVNSInt(f_A,E)$.

6. We know that $(\phi_A,E),(U_A,E)\in\tau$
$\therefore$ By (4) $IVNSInt(\phi_A,E)=\phi_A$, $IVNSInt(U_A,E)=U_A$

Theorem 4.4: Let (ζ_A,E,τ) be a *IVNS*-topological space and (f_A,E), $(g_A,E)\in IVNSs(U;E)$ then the following properties hold

1. $(f_A,E)\subseteq IVNSCl(f_A,E)$
2. $(f_A,E)\subseteq(g_A,E)\Rightarrow IVNSCl(f_A,E)\subseteq IVNSCl(g_A,E)$
3. $(IVNSCl(f_A,E))^c\in\tau$
4. $(f_A,E)^c\in\tau\Leftrightarrow IVNSCl(f_A,E)=(f_A,E)$
5. $IVNSCl(IVNSCl(f_A,E))=IVNSCl(f_A,E)$
6. $IVNSCl(\phi_A,E)=\phi_A$, $IVNSCl(U_A,E)=U_A$

Proof: straight forward.

Theorem 4.5: Let (ζ_A,E,τ) be an *IVNS*-topological space on (ζ_A,E) and let $(f_A,E),(g_A,E)\in IVNSs(U;E)$. Then the following properties hold

1. $IVNSInt((f_A,E)\cap(g_A,E))=IVNSInt(f_A,E)\cap IVNSInt(g_A,E)$
2. $IVNSInt((f_A,E)\cup(g_A,E))\supseteq IVNSInt(f_A,E)\cup IVNSInt(g_A,E)$
3. $IVNSCl((f_A,E)\cup(g_A,E))=IVNSCl(f_A,E)\cup IVNSCl(g_A,E)$
4. $IVNSCl((f_A,E)\cap(g_A,E))\subseteq IVNSCl(f_A,E)\cap IVNSCl(g_A,E)$
5. $(IVNSInt(f_A,E))^c=IVNSCl(f_A,E)^c$
6. $(IVNSCl(f_A,E))^c=IVNSInt(f_A,E)^c$

Proof:

1. By theorem 4.2 (1), $IVNSInt(f_A,E)\subseteq(f_A,E)$ and $IVNSInt(g_A,E)\subseteq(g_A,E)$. Thus $IVNSInt(f_A,E)\cap IVNSInt(g_A,E)\subseteq(f_A,E)\cap(g_A,E)$.
Hence
$IVNSInt(f_A,E)\cap IVNSInt(g_A,E)\subseteq IVNSInt((f_A,E)\cap(g_A,E))$
…………… (i)
Again since $(f_A,E)\cap(g_A,E)\subseteq(f_A,E)$. By theorem 4.2 (2), $IVNSInt((f_A,E)\cap(g_A,E))\subseteq IVNSInt(f_A,E)$.
Similarly
$IVNSInt((f_A,E)\cap(g_A,E))\subseteq IVNSInt(g_A,E)$
Hence
$IVNSInt((f_A,E)\cap(g_A,E))\subseteq IVNSInt(f_A,E)\cap IVNSInt(g_A,E)$ …
.…………. (ii)
Using (i) and (ii) we get, $IVNSInt((f_A,E)\cap(g_A,E))=IVNSInt(f_A,E)\cap IVNSInt(g_A,E)$.

2. Since $(f_A,E)\subseteq(f_A,E)\cup(g_A,E)$.
By theorem 4.2 (2), $IVNSInt(f_A,E)\subseteq IVNSInt((f_A,E)\cup(g_A,E))$.
Similarly, $IVNSInt(g_A,E)\subseteq IVNSInt((f_A,E)\cup(g_A,E))$.
Hence $IVNSInt((f_A,E)\cup(g_A,E))\supseteq IVNSInt(f_A,E)\cup IVNSInt(g_A,E)$.

3. Similar to 1.
4. Similar to 2.
5. $(IVNSInt(f_A,E))^c=(\cup\{(g_A,E)\in\tau:(g_A,E)\subseteq(f_A,E)\})^c$

$=\cap\{(g_A,E)\in\tau^c:(f_A,E)^c\subseteq(g_A,E)\}$
$=IVNSCl(f_A,E)^c$

6. Similar to 5.

Equality does not hold in theorem 4.4 (2), (4). Let us show this by an example.

Example 4.6: Let $U=\{u_1,u_2\}$, $E=\{e_1,e_2,e_3\}$, $A=\{e_1,e_2\}$. The tabular representation of (ζ_A,E) is given by

U	e1	e2
u1	([.5,.8],[.3,.5],[.2,.7])	([.3,.9],[.1,.2],[0,.1])
u2	([.4,.6],[.3,.4],[.1,.2])	([.4,.8],[.1,.3],[.1,.2])

Table14:Tabular representation of (ζ_A,E)

The tabular representation of (ϕ_{ζ_A},E) is given by

U	e1	e2
u1	([0,0], [1,1], [1,1])	([0,0], [1,1], [1,1])
u2	([0,0], [1,1], [1,1])	([0,0], [1,1], [1,1])

Table15:Tabular representation of (ϕ_{ζ_A},E)

The tabular representation of (f_A,E) is given by

U	e1	e2
u1	([.1,.7],[.4,.8],[.3,1])	([.2,.5],[.7,.9],[.3,.7])
u2	([.1,.2],[.6,.7],[.2,.7])	([0,.3],[.5,.8],[.4,1])

Table16:Tabular representation of (f_A,E)

Clearly $\tau=\{(\phi_{\zeta_A},E),(\zeta_A,E),(f_A,E)\}$ is a IVNS-topology on (ζ_A,E). Let us now take two interval valued neutrosophic soft sets (g_A,E) and (h_A,E) as

U	e1	e2
u1	([.1,.6],[.4,.9],[.4,1])	([.1,.5],[.7,.9],[.3,.8])
u2	([.1,.2],[.6,.7],[.2,.8])	([0,.2],[.5,.9],[.4,1])

Table17:Tabular representation of (g_A,E)

U	e1	e2
u1	([0,.7],[.5,.8],[.3,1])	([.2,.5],[.8,1],[.6,.7])
u2	([.1,.2],[.6,.8],[.3,.7])	([0,.3],[.6,.8],[.5,1])

Table18:Tabular representation of (h_A,E)

Now $(g_A,E)\cup(h_A,E)=(f_A,E)$

$\therefore$

$IVNSInt((g_A,E)\cup(h_A,E))=IVNSInt(f_A,E)=(f_A,E)$

Also $IVNSInt(g_A,E)=(\phi_{\zeta_A},E)$, $IVNSInt(h_A,E)=(\phi_{\zeta_A},E)$

$\therefore$

$IVNSInt(g_A,E)\cup IVNSInt(h_A,E)=(\phi_{\zeta_A},E)\cup(\phi_{\zeta_A},E)=(\phi_{\zeta_A},E)$

Thus

$IVNSInt((f_A,E)\cup(g_A,E))\neq IVNSInt(f_A,E)\cup IVNSInt(g_A,E)$.

Therefore equality does not hold for (2).

By theorem 4.4 (5),

$IVNSCl(g_A,E)^c=(IVNScl(g_A,E))^c=(\phi_{\zeta_A},E)^c=(\zeta_A,E)$.

Similarly $IVNScl(h_A,E)^c=(\zeta_A,E)$.

Therefore

$IVNSCl(g_A,E)^c\cap IVNSCl(h_A,E)^c=(\zeta_A,E)\cap(\zeta_A,E)=(\zeta_A,E)$. Also

$$
\begin{aligned}
IVNSCl((g_A,E)^c\cap(h_A,E)^c)&=IVNSCl((g_A,E)\cup(h_A,E))^c\\
&=(IVNSInt((g_A,E)\cup(h_A,E)))^c\\
&=(IVNSInt(f_A,E))^c\\
&=(f_A,E)^c
\end{aligned}
$$

Thus

$IVNSCl((f_A,E)\cap(g_A,E))\neq IVNSCl(f_A,E)\cap IVNSCl(g_A,E)$. Therefore equality doesnot hold in (4).

5 Interval Valued Neutrosophic Soft Subspace Topology

In this section we introduce the concept of interval valued neutrosophic soft subspace topology along with some examples and results.

Theorem 5.1: Let (ζ_A,E,τ) be an *IVNS*-topological space on (ζ_A,E) and $(f_A,E)\in\wp(\zeta_A,E)$. Then the collection $\tau_{(f_A,E)}=\{(f_A,E)\cap(g_A,E):(g_A,E)\in\tau\}$ is an *IVNS*-topology on (ζ_A,E).

Proof:

(i) Since $(\phi_{\zeta_A},E),(\zeta_A,E)\in\tau$, therefore $(f_A,E)\cap(\phi_{\zeta_A},E)=(\phi_{f_A},E)\in\tau_{(f_A,E)}$ and $(f_A,E)\cap(\zeta_A,E)=(f_A,E)\in\tau_{(f_A,E)}$.

(ii) Let $(f_A^k,E)\in\tau_{(f_A,E)},\forall k\in K$. Then $(f_A^k,E)=(f_A,E)\cap(g_A^k,E)$ where $(g_A^k,E)\in\tau$ for each $k\in K$.

Now

$\bigcup_{k\in K}(f_A^k,E)=\bigcup_{k\in K}((f_A,E)\cap(g_A^k,E))=(f_A,E)\cap\left(\bigcup_{k\in K}(g_A^k,E)\right)\in\tau_{(f_A,E)}$

(since $\bigcup_{k\in K}(g_A^k,E)\in\tau$ as each $(g_A^k,E)\in\tau$.

(iii) Let $(f_A^1,E),(f_A^2,E)\in\tau_{(f_A,E)}$ then

$(f_A^1,E)=(f_A,E)\cap(g_A^1,E)$ and

$(f_A^2,E)=(f_A,E)\cap(g_A^2,E)$ where $(g_A^1,E),(g_A^2,E)\in\tau$.

Now

$$\left(f_A^1,E\right)\cap\left(f_A^2,E\right)=\left((f_A,E)\cap\left(g_A^1,E\right)\right)\cap\left((f_A,E)\cap\left(g_A^2,E\right)\right)$$
$$=(f_A,E)\cap\left(\left(g_A^1,E\right)\cap\left(g_A^2,E\right)\right)\in\tau_{(f_A,E)}$$

(since $\left(g_A^1,E\right)\cap\left(g_A^2,E\right)\in\tau$ as $\left(g_A^1,E\right),\left(g_A^2,E\right)\in\tau$).

Definition 5.2: Let (ζ_A,E,τ) be an *IVNS*-topological space on (ζ_A,E) and $(f_A,E)\in\wp(\zeta_A,E)$. Then the IVNS-topology $\tau_{(f_A,E)}=\{(f_A,E)\cap(g_A,E):(g_A,E)\in\tau\}$ is called interval valued neutrosophic soft subspace topology and $\left(f_A,E,\tau_{(f_A,E)}\right)$ is called interval valued neutrosophic soft subspace of (ζ_A,E,τ) .

Example 5.3: Let us consider the IVNS-topology $\tau_1=\left\{\left(\phi_{\zeta_A},E\right),(\zeta_A,E),\left(f_A^1,E\right),\left(f_A^2,E\right),\left(f_A^3,E\right),\left(f_A^4,E\right)\right\}$ as in example 3.2 and an *IVNSS* (f_A,E):

U	e_1	e_2	e_3
u_1	([.4,.6],[.6,.7],[.3,.5])	([.5,.7],[.4,.6],[0,.3])	([.3,.5],[.5,.8],[.2,.3])
u_2	([.2,.3],[.3,.6],[.5,.7])	([.6,.8],[.4,.5],[.2,.3])	([.5,.8],[.5,.7],[.2,.3])
u_3	([.5,.7],[.4,.6],[.3,.4])	([.4,.5],[.7,.9],[.6,.7])	([.1,.3],[.7,.9],[.5,.7])

Table19:Tabular representation of $\left(f_A^1,E\right)$

Then $\left(\phi_{f_A},E\right)=(f_A,E)\cap\left(\phi_{\zeta_A},E\right)$:

U	e_1	e_2	e_3
u_1	([0,0],[1,1],[1,1])	([0,0],[1,1],[1,1])	([0,0],[1,1],[1,1])
u_2	([0,0],[1,1],[1,1])	([0,0],[1,1],[1,1])	([0,0],[1,1],[1,1])
u_3	([0,0],[1,1],[1,1])	([0,0],[1,1],[1,1])	([0,0],[1,1],[1,1])

Table20:Tabular representation of $\left(\phi_{f_A},E\right)$

$\left(g_A^1,E\right)=(f_A,E)\cap\left(f_A^1,E\right)$:

U	e_1	e_2	e_3
u_1	([.1,.6],[.6,.7],[.3,1])	([.1,.3],[.4,.6],[.2,.6])	([.2,.5],[.5,.8],[.4,.9])
u_2	([.1,.3],[.6,.7],[.5,.8])	([0,.5],[.4,.5],[.4,1])	([0,.3],[.6,.9],[.2,.7])
u_3	([.4,.7],[.4,.6],[.6,.9])	([0,.3],[.7,.9],[.6,.8])	([.1,.3],[.7,.9],[.5,.7])

Table21:Tabular representation of $\left(g_A^1,E\right)$

$\left(g_A^2,E\right)=(f_A,E)\cap\left(f_A^2,E\right)$:

U	e_1	e_2	e_3
u_1	([.4,.6],[.6,.7],[.4,.9])	([.2,.3],[.4,.6],[.7,.9])	([.3,.5],[.5,.8],[.2,.3])
u_2	([.2,.3],[.4,.8],[.5,.7])	([.4,.6],[.4,.5],[.2,.5])	([.1,.3],[.5,.7],[.6,.8])
u_3	([.3,.7],[.4,.6],[.6,.7])	([.4,.5],[.7,.9],[.6,.7])	([.1,.3],[.7,.9],[.3,.8])

Table22:Tabular representation of $\left(g_A^2,E\right)$

$\left(g_A^3,E\right)=(f_A,E)\cap\left(f_A^3,E\right)$:

U	e_1	e_2	e_3
u_1	([.1,.6],[.6,.8],[.4,1])	([.1,.3],[.4,.6],[.7,.9])	([.2,.5],[.5,.8],[.4,.9])
u_2	([.1,.3],[.6,.8],[.5,.8])	([0,.5],[.4,.5],[.4,1])	([0,.3],[.6,.9],[.6,.8])
u_3	([.3,.7],[.4,.6],[.6,.9])	([0,.3],[.7,.9],[.6,.8])	([.1,.3],[.7,.9],[.5,.8])

Table23:Tabular representation of $\left(g_A^3,E\right)$

$\left(g_A^4,E\right)=(f_A,E)\cap\left(f_A^4,E\right)$:

U	e_1	e_2	e_3
u_1	([.2,.5],[.5,.8],[.4,.9])	([.2,.5],[.5,.8],[.4,.9])	([.3,.5],[.5,.8],[.2,.3])
u_2	([0,.3],[.6,.9],[.6,.8])	([0,.3],[.6,.9],[.6,.8])	([.1,.3],[.5,.7],[.2,.7])
u_3	([.1,.3],[.7,.9],[.5,.8])	([.1,.3],[.7,.9],[.5,.8])	([.1,.3],[.7,.9],[.5,.7])

Table24:Tabular representation of $\left(g_A^4,E\right)$

Then $\tau_{(f_A,E)}=\left\{\left(\phi_{f_A},E\right),(f_A,E),\left(g_A^1,E\right),\left(g_A^2,E\right),\left(g_A^4,E\right)\right\}$ is an interval valued neutrosophic soft subspace topology for τ_1 and $\left(f_A, E, \tau_{(f_A,E)}\right)$ is called interval valued neutrosophic soft subspace of $\left(\zeta_A, E, \tau_1\right)$.

Theorem 5.4: Let $\left(\zeta_A, E, \tau\right)$ be an *IVNS*-topological space on $\left(\zeta_A, E\right)$, ß be an *IVNS*-basis for τ and $\left(f_A, E\right) \in \wp\left(\zeta_A, E\right)$. Then the family $ß_{(f_A,E)} = \left\{\left(f_A, E\right) \cap \left(g_A, E\right) : \left(g_A, E\right) \in ß\right\}$ is an *IVNS*-basis for subspace topology $\tau_{(f_A,E)}$.

Proof: Let $\left(h_A, E\right) \in \tau_{(f_A,E)}$ be arbitrary, then there exists an *IVNSS* $\left(g_A, E\right) \in \tau$ such that $\left(h_A, E\right) = \left(f_A, E\right) \cap \left(g_A, E\right)$. Since ß is a basis for τ, therefore there exists a sub collection $\left\{\left(\chi_A^i, E\right) : i \in I\right\}$ of ß such that $\left(g_A, E\right) = \bigcup_{i \in I}\left(\chi_A^i, E\right)$.

Now
$$\left(h_A, E\right) = \left(f_A, E\right) \cap \left(g_A, E\right) = \bigcup_{i \in I}\left(\chi_A^i, E\right) = \bigcup_{i \in I}\left(\left(f_A, E\right) \cap \left(\chi_A^i, E\right)\right)$$
. Since $\left(f_A, E\right) \cap \left(\chi_A^i, E\right) \in ß_{(f_A,E)}$, therefore $ß_{(f_A,E)}$ is an *IVNS*-basis for the subspace topology $\tau_{(f_A,E)}$.

Conclusion

In this paper we introduce the concept of interval valued neutrosophic soft topology. Some basic theorem and properties of the above concept are also studied. IVN interior and IVN closer of an interval valued neutrosophic soft set are also defined. Interval valued neutrosophic soft subspace topology is also studied.

In future there will be more research work in this concept, taking the basic definitions and results from this article.

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Generalization of Soft Neutrosophic Rings and Soft Neutrosophic Fields

Mumtaz Ali, Florentin Smarandache, Luige Vlădăreanu, Muhammad Shabir

Mumtaz Ali, Florentin Smarandache, Luige Vladareanu, Muhammad Shabir. Generalization of Soft Neutrosophic Rings and Soft Neutrosophic Fields. *Neutrosophic Sets and Systems* 6, 35-41

Abstract. In this paper we extend soft neutrosophic rings and soft neutrosophic fields to soft neutrosophic birings, soft neutrosophic N-rings and soft neutrosophic bifields and soft neutrosophic N-fields. We also extend soft neutrosophic ideal theory to form soft neutrosophic biideal and soft neutrosophic N-ideals over a neutrosophic biring and soft neutrosophic N-ring . We have given examples to illustrate the theory of soft neutrosophic birings, soft neutrosophic N-rings and soft neutrosophic fields and soft neutrosophic N-fields and display many properties of these.

Keywords: Neutrosophic biring, neutrosophic N-ring, neutrosophic bifield,neutrosophic N-field, soft set, soft neutrosophic biring, soft neutrosophic N-ring, soft neutrosophic bifield, soft neutrosophic N-field.

1 Introduction

Neutrosophy is a new branch of philosophy which studies the origin and features of neutralities in the nature. Florentin Smarandache in 1980 firstly introduced the concept of neutrosophic logic where each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F so that this neutrosophic logic is called an extension of fuzzy logic. In fact neutrosophic set is the generalization of classical sets, conventional fuzzy set, intuitionistic fuzzy set and interval valued fuzzy set. This mathematical tool is used to handle problems like imprecise, indeterminacy and inconsistent data etc. By utilizing neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache dig out neutrosophic algebraic structures. Some of them are neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophip groupoids, and neutrosophic bigroupoids and so on.

Molodtsov in $[11]$ laid down the stone foundation of a richer structure called soft set theory which is free from the parameterization inadequacy, syndrome of fuzzy se theory, rough set theory, probability theory and so on. In many areas it has been successfully applied such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, and probability. Recently soft set theory has attained much attention since its appearance and the work based on several operations of soft sets introduced in $[2,9,10]$. Some more exciting properties and algebra may be found in $[1]$. Feng et al. introduced the soft semirings $[5]$. By means of level soft sets an adjustable approach to fuzzy soft sets based decision making can be seen in $[6]$. Some other new concept combined with fuzzy sets and rough sets was presented in $[7,8]$. AygÄunoglu et al. introduced the Fuzzy soft groups $[4]$.

Firstly, fundamental and basic concepts are given for neutrosophic birings, neutrosophic N-rings, neutrosohic bifields and soft neutrosophic N-fields . In the next section we presents the newly defined notions and results in soft neutrosophic birings, soft neutrosophic N-rings and soft neutrosophic bifields and soft neutrosophic N-fields. Various types of soft neutrosophic biideals and N-ideals of birings and N-rings are defined and elaborated with the help of examples.

2 Fundamental Concepts

In this section, we give a brief description of neutrosophic birings, neutrosophic N-rings, neutrosophic bifields and neutrosophic N-fields respectively.

Definition 2.1. Let $(BN(\mathrm{R}),*,\circ)$ be a non-empty set with two binary operations $*$ and $\circ$. $(BN(\mathrm{R}),*,\circ)$ is said to be a neutrosophic biring if $BN(\mathrm{Rs}) = R_1 \cup R_2$ where atleast one of $(\mathrm{R}_1,*,\circ)$ or $(\mathrm{R}_2,*,\circ)$ is a neutrosophic ring and other is just a ring. R_1 and R_2 are proper subsets of $BN(\mathrm{R})$.

Definition 2.2: Let $BN(\mathrm{R}) = (\mathrm{R}_1,*,\circ) \cup (\mathrm{R}_2,*,\circ)$ be a neutrosophic biring. Then $BN(\mathrm{R})$ is called a commutative neutrosophic biring if each $(\mathrm{R}_1,*,\circ)$ and $(\mathrm{R}_2,*,\circ)$ is a commutative neutrosophic ring.

Definition 2.3: Let $BN(\mathrm{R}) = (\mathrm{R}_1,*,\circ) \cup (\mathrm{R}_2,*,\circ)$ be a neutrosophic biring. Then $BN(\mathrm{R})$ is called a pseudo neutrosophic biring if each $(\mathrm{R}_1,*,\circ)$ and $(\mathrm{R}_2,*,\circ)$ is a pseudo neutrosophic ring.

Definition 2.4 Let $(BN(\mathrm{R}) = R_1 \cup R_2;*,\circ)$ be a neutrosophic biring. A proper subset $(T,*,\circ)$ is said to be a neutrosophic subbiring of $BN(\mathrm{R})$ if

1) $T = T_1 \cup T_2$ where $T_1 = R_1 \cap T$ and $T_2 = R_2 \cap T$ and
2) At least one of $(T_1,\circ)$ or $(T_2,*)$ is a neutrosophic ring.

Definition 2.5: If both $(\mathrm{R}_1,*)$ and $(\mathrm{R}_2,\circ)$ in the above definition 2.1 are neutrosophic rings then we call $(BN(\mathrm{R}),*,\circ)$ to be a strong neutrosophic biring.

Definition 2.6 Let $(BN(\mathrm{R}) = R_1 \cup R_2;*,\circ)$ be a neutrosophic biring and let $(T,*,\circ)$ is a neutrosophic subbiring of $BN(\mathrm{R})$. Then $(T,*,\circ)$ is called a neutrosophic biideal of $BN(R)$ if

1) $T = T_1 \cup T_2$ where $T_1 = R_1 \cap T$ and $T_2 = R_2 \cap T$ and
2) At least one of $(T_1,*,\circ)$ or $(T_2,*,\circ)$ is a neutrosophic ideal.

If both $(T_1,*,\circ)$ and $(T_2,*,\circ)$ in the above definition are neutrosophic ideals, then we call $(T,*,\circ)$ to be a strong neutrosophic biideal of $BN(R)$.

Definition 2.7: Let $\{\mathrm{N(R)},*_1,...,*_2,\circ_1,\circ_2,...,\circ_N\}$ be a non-empty set with two N-binary operations defined on it. We call $N(R)$ a neutrosophic N-ring (N a positive integer) if the following conditions are satisfied.

1) $\mathrm{N(R)} = R_1 \cup R_2 \cup ... \cup R_N$ where each R_i is a proper subset of $\mathrm{N(R)}$ i.e. $R_i \not\subset R_j$ or $R_j \not\subset R_i$ if $i \neq j$.
2) $(\mathrm{R}_i,*_i,\circ_i)$ is either a neutrosophic ring or a ring for $i = 1,2,3,...,N$.

Definition 2.8: If all the N-rings $(\mathrm{R}_i,*_i)$ in definition 2.7 are neutrosophic rings (i.e. for $i = 1,2,3,...,N$) then we call $\mathrm{N(R)}$ to be a neutrosophic strong N-ring.

Definition 2.9: Let $\mathrm{N(R)} = \{\mathrm{R}_1 \cup R_2 \cup .... \cup \mathrm{R}_N,*_1,*_2,...,*_N,\circ_1,\circ_2,...,\circ_N\}$ be a neutrosophic N-ring. A proper subset $P = \{\mathrm{P}_1 \cup P_2 \cup .... \mathrm{P}_N,*_1,*_2,...,*_N\}$ of $\mathrm{N(R)}$ is said to be a neutrosophic N-subring if $P_i = P \cap R_i, i = 1,2,...,N$ are subrings of R_i in which atleast some of the subrings are neutrosophic subrings.

Definition 2.10: Let $\mathrm{N(R)} = \{\mathrm{R}_1 \cup R_2 \cup .... \cup \mathrm{R}_N,*_1,*_2,...,*_N,\circ_1,\circ_2,...,\circ_N\}$ be a neutrosophic N-ring. A proper subset $P = \{\mathrm{P}_1 \cup P_2 \cup .... \cup P_N,*_1,*_2,...,*_N,\circ_1,\circ_2,...,\circ_N\}$ where $P_t = P \cap R_t$ for $t = 1,2,...,N$ is said to be a neutrosophic N-ideal of $N(R)$ if the following conditions are satisfied.

1) Each it is a neutrosophic subring of $R_t, t = 1,2,...,N$.
2) Each it is a two sided ideal of R_t for $t = 1,2,...,N$.

If $(\mathrm{P}_i,*_i,\circ_i)$ in the above definition are neutrosophic ideals, then we call $(\mathrm{P}_i,*_i,\circ_i)$ to be a strong neutrosophic N-ideal of $N(R)$.

Definition 2.11: Let $(BN(\mathrm{F}),*,\circ)$ be a non-empty set with two binary operations $*$ and $\circ$. $(BN(\mathrm{F}),*,\circ)$ is said to be a neutrosophic bifiel if $BN(\mathrm{F}) = F_1 \cup F_2$ where atleast one of $(\mathrm{F}_1,*,\circ)$ or $(\mathrm{F}_2,*,\circ)$ is a neutrosophic field and other is just a field. F_1 and F_2 are proper subsets of $BN(\mathrm{F})$.

If in the above definition both $(\mathrm{F}_1,*,\circ)$ and $(\mathrm{F}_2,*,\circ)$ are neutrosophic fields, then we call $(BN(\mathrm{F}),*,\circ)$ to be a neutrosophic strong bifield.

Definition 2.12: Let $BN(\mathrm{F}) = (\mathrm{F}_1 \cup F_2,*,\circ)$ be a neutrosophic bifield. A proper subset $(T,*,\circ)$ is said to be a neutrosophic subbifield of $BN(\mathrm{F})$ if

1. $T = T_1 \cup T_2$ where $T_1 = F_1 \cap T$ and $T_2 = F_2 \cap T$ and
2. At least one of $(T_1,\circ)$ or $(T_2,*)$ is a neutrosophic field and the other is just a field.

Definition 2.13: Let $\{\mathrm{N(F)},*_1,...,*_2,\circ_1,\circ_2,...,\circ_N\}$ be a non-empty set with two N -binary operations defined on it. We call $N(R)$ a neutrosophic N -field (N a positive integer) if the following conditions are satisfied.

1. $\mathrm{N(F)} = F_1 \cup F_2 \cup ... \cup F_N$ where each F_i is a proper subset of $\mathrm{N(F)}$ i.e. $R_i \not\subset R_j$ or $R_j \not\subset R_i$ if $i \neq j$.
2. $(\mathrm{R}_i,*_i,\circ_i)$ is either a neutrosophic field or just a field for $i = 1,2,3,...,N$.

If in the above definition each $(\mathrm{R}_i,*_i,\circ_i)$ is a neutrosophic field, then we call $N(R)$ to be a strong neutrosophic N-field.

Definition 2.14: Let $\mathrm{N(F)} = \{\mathrm{F}_1 \cup F_2 \cup .... \cup F_N,*_1,*_2,...,*_N,\circ_1,\circ_2,...,\circ_N\}$ be a neutrosophic N -field. A proper subset $T = \{\mathrm{T}_1 \cup T_2 \cup .... \cup \mathrm{T}_N,*_1,*_2,...,*_N,\circ_1,\circ_2,...,\circ_N\}$ of $N(\mathrm{F})$ is said to be a neutrosophic N -subfield if each $(T_i,*_i)$ is a neutrosophic subfield of $(\mathrm{F}_i,*_i,\circ_i)$ for $i = 1,2,...,N$ where $T_i = F_i \cap T$.

3 Soft Neutrosophic Birings

Definition 3.1: Let $(BN(\mathrm{R}),*,\circ)$ be a neutrosophic biring and (F,A) be a soft set over $(BN(\mathrm{R}),*,\circ)$. Then (F,A) is called soft neutrosophic biring if and only if $F(a)$ is a neutrosophic subbiring of $(BN(\mathrm{R}),*,\circ)$ for all $a \in A$.

Example 3.2: Let $BN(\mathrm{R}) = (\mathrm{R}_1,*,\circ) \cup (\mathrm{R}_2,*,\circ)$ be a neutrosophic biring, where $(\mathrm{R}_1,*,\circ) = (\langle \mathbb{Z} \cup I \rangle,+,\times)$ and $(\mathrm{R}_2,*,\circ) = (\mathbb{Q},+,\times)$. Let $A = \{a_1,a_2,a_3,a_4\}$ be a set of parameters. Then clearly (F,A) is a soft neutrosophic biring over $BN(R)$, where

$$F(a_1) = \langle 2\mathbb{Z} \cup I \rangle \cup \mathbb{R}, F(a_2) = \langle 3\mathbb{Z} \cup I \rangle \cup \mathbb{Q},$$

$$F(a_3) = \langle 5\mathbb{Z} \cup I \rangle \cup \mathbb{Z}, F(a_4) = \langle 6\mathbb{Z} \cup I \rangle \cup 2\mathbb{Z}.$$

Theorem 3.3: Let F,A and (H,A) be two soft neutrosophic birings over $BN(R)$. Then their intersection $F,A \cap H,A$ is again a soft neutrosophic biring over $BN(R)$.

Proof. The proof is straightforward.

Theorem 3.4: Let F,A and H,B be two soft neutrosophic birings over $BN(R)$. If $A \cap B = \phi$, then $F,A \cup H,B$ is a soft neutrosophic biring over $BN(R)$.

Proof. This is straightforward.

Remark 3.5: The extended union of two soft neutrosophic birings F,A and K,B over $BN(R)$ is not a soft neutrosophic ring over $BN(R)$.

We check this by the help of Examples.

Remark 3.6: The restricted union of two soft neutrosophic rings F,A and K,B over $\langle R \cup I \rangle$ is not a soft neutrosophic ring over $\langle R \cup I \rangle$.

Theorem 3.7: The OR operation of two soft neutrosophic rings over $\langle R \cup I \rangle$ may not be a soft neutrosophic ring over $\langle R \cup I \rangle$.

One can easily check these remarks with the help of Examples.

Theorem 3.8: The extended intersection of two soft neu trosophic birings over $BN(R)$ is soft neutrosophic biring over $BN(R)$.

Proof. The proof is straightforward.

Theorem 3.9: The restricted intersection of two soft neutrosophic birings over $BN(R)$ is soft neutrosophic biring over $BN(R)$.

Theorem 3.10: The AND operation of two soft neutrosophic birings over $BN(R)$ is soft neutrosophic biring over $BN(R)$.

Definition 3.11: Let F, A be a soft set over a neutrosophic biring over $BN(R)$. Then (F, A) is called an absolute soft neutrosophic biring if $F(a) = BN(R)$ for all $a \in A$.

Definition 3.12: Let (F, A) be a soft set over a neutrosophic ring $BN(R)$. Then (F, A) is called soft neutrosophic biideal over $BN(R)$ if and only if $F(a)$ is a neutrosophic biideal of $BN(R)$.

Theorem 3.1.3: Every soft neutrosophic biideal (F, A) over a neutrosophic biring $BN(R)$ is trivially a soft neutrosophic biring but the converse may not be true.

Proposition 3.14: Let (F, A) and (K, B) be two soft neutosophic biideals over a neutrosophic biring $BN(R)$. Then

1. Their extended union $(F, A) \cup_E (K, B)$ is again a soft neutrosophic biideal over $BN(R)$.
2. Their extended intersection $(F, A) \cap_E (K, B)$ is again a soft neutrosophic biideal over $BN(R)$.
3. Their restricted union $(F, A) \cup_R (K, B)$ is again a soft neutrosophic biideal over $BN(R)$.
4. Their restricted intersection $(F, A) \cap_R (K, B)$ is again a soft neutrosophic biideal over $BN(R)$.
5. Their OR operation $(F, A) \vee (K, B)$ is again a soft neutrosophic biideal over $BN(R)$.
6. Their AND operation $(F, A) \vee (K, B)$ is again a soft neutrosophic biideal over $BN(R)$.

Definition 3.15: Let (F, A) and (K, B) be two soft neutrosophic birings over $BN(R)$. Then (K, B) is called soft neutrosophic subbiring of (F, A), if

1. $B \subseteq A$, and
2. $K(a)$ is a neutrosophic subbiring of $F(a)$ for all $a \in A$.

Theorem 3.16: Every soft biring over a biring is a soft neutrosophic subbiring of a soft

neutrosophic biring over the corresponding neutrosophic biring if $B \subseteq A$.

Definition 3.16: Let (F, A) and (K, B) be two soft neutrosophic birings over $BN(R)$. Then (K, B) is called a soft neutrosophic biideal of (F, A), if

1. $B \subseteq A$, and
2. $K(a)$ is a neutrosophic biideal of $F(a)$ for all $a \in A$.

Proposition 3.17: All soft neutrosophic biideals are trivially soft neutrosophic subbirings.

4 Soft Neutrosophic N-Ring

Definition 4.1: Let $(N(\mathrm{R}), *_1, *_2, \ldots, *_N)$ be a neutrosophic N-ring and (F, A) be a soft set over $N(R)$ Then (F, A) is called soft neutrosophic N-ring if and only if $F(a)$ is a neutrosophic sub N-ring of $N(R)$ for all $a \in A$.

Example 4.2: Let $N(\mathrm{R}) = (\mathrm{R}_1, *, \circ) \cup (\mathrm{R}_2, *, \circ) \cup (\mathrm{R}_3, *, \circ)$ be aneutrosophic 3-ring, where $(\mathrm{R}_1, *, \circ) = (\langle \mathbb{Z} \cup I \rangle, +, \times)$, $(\mathrm{R}_2, *, \circ) = (\mathbb{C}, +, \times)$ and $(\mathrm{R}_3, *, \circ) = (\mathbb{R}, +, \times)$. Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of parameters. Then clearly (F, A) is a soft neutrosophic N-ring over $N(R)$, where

$$F(a_1) = \langle 2\mathbb{Z} \cup I \rangle \cup \mathbb{R} \cup \mathbb{Q}, F(a_2) = \langle 3\mathbb{Z} \cup I \rangle \cup \mathbb{Q} \cup \mathbb{Z},$$

$$F(a_3) = \langle 5\mathbb{Z} \cup I \rangle \cup \mathbb{Z} \cup 2\mathbb{Z}, F(a_4) = \langle 6\mathbb{Z} \cup I \rangle \cup 2\mathbb{Z} \cup \mathbb{R}$$

.

Theorem 4.3: Let F, A and (H,A) be two soft neutrosophic N-rings over $N(R)$. Then their intersection $F, A \cap H, A$ is again a soft neutrosophic N-ring over $N(R)$.

Proof. The proof is straightforward.

Theorem 4.4: Let F, A and H, B be two soft neutrosophic N-rings over $N(R)$. If $A \cap B = \phi$, then $F, A \cup H, B$ is a soft neutrosophic N-ring over $N(R)$.

Proof. This is straightforward.

Remark 4.5: The extended union of two soft neutrosophic N-rings F, A and K, B over $BN(R)$ is not a soft neutrosophic ring over $N(R)$.

We can check this by the help of Examples.

Remark 4.6: The restricted union of two soft neutrosophic N-rings F, A and K, B over $N(R)$ is not a soft neutrosophic N-ring over $BN(R)$

Theorem 4.7: The OR operation of two soft neutrosophic N-rings over $N(R)$ may not be a soft neutrosophic N-ring over $N(R)$.

One can easily check these remarks with the help of Examples.

Theorem 4.8: The extended intersection of two soft neutrosophic N-rings over $N(R)$ is soft neutrosophic Nring over $N(R)$.

Proof. The proof is straightforward.

Theorem. The restricted intersection of two soft neutrosophic N-rings over $N(R)$ is soft neutrosophic N-ring over $N(\mathrm{R})$.

Proof. It is obvious.

Theorem 4.9: The AND operation of two soft neutrosophic N-rings over $N(R)$ is soft neutrosophic N-ring over $N(R)$.

Definition 4.10: Let F, A be a soft set over a neutrosophic N-ring over $N(R)$. Then (F, A) is called an absolute soft neutrosophic N-ring if $F(a) = N(R)$ for all $a \in A$.

Definition 4.11: Let (F, A) be a soft set over a neutrosophic N-ring $N(R)$. Then (F, A) is called soft neutrosophic N-ideal over $N(R)$ if and only if $F(a)$ is a neutosophic N-ideal of $N(R)$.

Theorem 4.12: Every soft neutrosophic N-ideal (F, A) over a neutrosophic N-ring $N(R)$ is trivially a soft neutrosophic N-ring but the converse may not be true.

Proposition 4.13: Let (F, A) and (K, B) be two soft neutosophic N-ideals over a neutrosophic N-ring $N(R)$. Then

1. Their extended intersection $(F, A) \cap_E (K, B)$ is again a soft neutrosophic N-ideal over $N(R)$.
2. Their restricted intersection $(F, A) \cap_R (K, B)$ is again a soft neutrosophic N-ideal over $N(R)$.
3. Their AND operation $(F, A) \vee (K, B)$ is again a soft neutrosophic N-ideal over $N(R)$.

Remark 4.14: Let (F, A) and (K, B) be two soft neutosophic N-ideals over a neutrosophic N-ring $N(R)$. Then

1. Their extended union $(F, A) \cup_E (K, B)$ is not a soft neutrosophic N-ideal over $N(R)$.
2. Their restricted union $(F, A) \cup_R (K, B)$ is not a soft neutrosophic N-ideal over $N(R)$.
3. Their OR operation $(F, A) \vee (K, B)$ is not a soft neutrosophic N-ideal over $N(R)$.

One can easily see these by the help of examples.

Definition. 4.15: Let (F, A) and (K, B) be two soft neutrosophic N-rings over $N(R)$. Then (K, B) is called soft neutrosophic sub N-ring of (F, A), if

1. $B \subseteq A$, and
2. $K(a)$ is a neutrosophic sub N-ring of $F(a)$ for

all $a \in A$.

Theorem 4.16: Every soft N-ring over a N-ring is a soft neutrosophic sub N-ring of a soft

neutrosophic N-ring over the corresponding neutrosophic N-ring if $B \subseteq A$.

Proof. Straightforward.

Definition 4.17: Let (F, A) and (K, B) be two soft neutrosophic N-rings over $N(R)$. Then (K, B) is called a soft neutrosophic N-ideal of (F, A), if

1. $B \subseteq A$, and
2. $K(a)$ is a neutrosophic N-ideal of $F(a)$ for all $a \in A$.

Proposition 4.18: All soft neutrosophic N-ideals are trivially soft neutrosophic sub N-rings.

5 Soft Neutrosophic Bifield

Defintion 5.1: Let $BN(K)$ be a neutrosophic bifield and let (F, A) be a soft set over $BN(K)$. Then (F, A) is said to be soft neutrosophic bifield if and only if $F(a)$ is a neutrosophic subbifield of $BN(K)$ for all $a \in A$.

Example 5.2: Let $BN(K) = \langle \mathbb{C} \cup I \rangle \cup \mathbb{R}$ be a neutrosophic bifield of complex numbers. Let $A = \{a_1, a_2\}$ be a set of parameters and let (F, A) be a soft set of $BN(K)$. Then (F, A) is a soft neutrosophic bifield over $BN(K)$, where

$$F(a_1) = \langle \mathbb{R} \cup I \rangle \cup \mathbb{Q}, F(a_2) = \langle \mathbb{Q} \cup I \rangle \cup \mathbb{Q}.$$

Where $\langle \mathbb{R} \cup I \rangle$ and $\langle \mathbb{Q} \cup I \rangle$ are the neutosophic fields of real numbers and rational numbers.

Proposition 5.3: Every soft neutrosophic bifield is trivially a soft neutrosophic biring.

Proof. The proof is trivial.

Definition 5.4: Let (F, A) be a soft neutrosophic bifield over a neutrosophic bifield $BN(K)$. Then (F, A) is called an absolute soft neutrosophic bifield if $F(a) = BN(K)$, for all $a \in A$.

Soft Neutrosophic N-field

Defintion 5.4: Let $N(K)$ be a neutrosophic N-field and let (F, A) be a soft set over $N(K)$. Then (F, A) is said to be soft neutrosophic N-field if and only if $F(a)$ is a neutrosophic sub N-field of $N(K)$ for all $a \in A$.

Proposition 5.5: Every soft neutrosophic N-field is trivially a soft neutrosophic N-ring.

Proof. The proof is trivial.

Definition 5.6: Let (F, A) be a soft neutrosophic N-field over a neutrosophic N-field $N(K)$. Then (F, A) is called an absolute soft neutrosophic N-field if $F(a) = N(K)$, for all $a \in A$.

Conclusion

In this paper we extend neutrosophicb rings, neutrosophic N-rings, Neutrosophic bifields and neutrosophic N-fields to soft neutrosophic birings, soft neutrosophic N-rings and soft neutrosophic bifields and soft neutrosophic N-fields respectively. The neutrosophic ideal theory is extend to soft neutrosophic biideal and soft neutrosophic N-ideal. Some new types of soft neutrosophic ideals are discovered which is strongly neutrosophic or purely neutrosophic. Related examples are given to illustrate soft neutrosophic biring, soft neutrosophic N-ring, soft neutrosophic bifield and soft neutrosophic N-field and many theorems and properties are discussed.

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Neutrosophic Refined Similarity Measure Based on Cosine Function

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Abstract: In this paper, the cosine similarity measure of neutrosophic refined (multi-) sets is proposed and its properties are studied. The concept of this cosine similarity measure of neutrosophic refined sets is the extension of improved cosine similarity measure of single valued neutrosophic. Finally, using this cosine similarity measure of neutrosophic refined set, the application of medical diagnosis is presented.

Keywords: Neutrosophic set, neutrosophic refined set, cosine similarity measure.

1.Introduction:

The neutrsophic sets (NS), proposed by F. Smarandache [7], has been studied and applied in different fields, including decision making problems [1,15], databases [21,22], medical diagnosis problems [2], topology [6], control theory [40], image processing [9,22,44] and so on. The concept of neutrosophic sets generalizes the following concepts: the classic set, fuzzy set [20],intuitionistic fuzzy set [19], and interval valued intuitionistic fuzzy set [18] and so on. The character of NSs is that the values of its membership function, non-membership function and indeterminacy function are subsets. Therefore, H.Wang et al [10] introduced an instance of neutrosophic sets known as single valued neutrosophic sets (SVNS), which were motivated from the practical point of view and that can be used in real scientific and engineering application, and provide the set theoretic operators and various properties of SVNSs. However, in many applications, due to lack of knowledge or data about the problem domains, the decision information may be provided with intervals, instead of real numbers. Thus, interval valued neutrosophic sets (IVNS), as a useful generation of NS, was introduced by H.Wang et al [11], which is characterized by a membership function, non-membership function and an indeterminacy function, whose values are intervals rather than real numbers. Also, the interval valued neutrosophic set can represent uncertain, imprecise, incomplete and inconsistent information which exist in the real world. As an important extension of NS, SVNS and IVNS has many applications in real life [13,14,15,16, 17,25,32,33,34,35,36,37,38,39]

Several similarity measures have been proposed by some researchers. Broumi and Smarandache [35] defined the Hausdorff distance between neutrosophic sets and some similarity measures based on the distance, set theoretic approach, and matching function to calculate the similarity degree between neutrosophic sets. In the same year, Broumi and Smarandache [32] also proposed the correlation coefficient between interval neutrosphic sets. Majumdar and Smanta [24] introduced several similarity measures of single valued neutrosophic sets(SVNs) based on distances, a maching function, memebership grades, and then proposed an entropy measure for a SVNS. J.Ye[13] also presented the Hamming and Euclidean distances between interval neutrosophic sets(INSs) an their similarity measures and applied them to multiple attribute decision –making problems with interval neutrosophic information. J.Ye [15] further proposed the distance-based similarity measure of SVNSs and applied it to the group decision making problems with single valued neutrosophic information. In other research, J.Ye [16] proposed three vector similarity measure for SNSs,an instance of SVNS and INS, including the Jaccard, Dice, and cosine similarity measures for SVNS and INSs, and applied them to multicriteria decision-making problems with simplified neutrosophic information. Recently, A.Salama [4], introduced and studied the concepts of correlation and correlation coefficient of neutrosophic data in probability spaces and study some of their properties.

The cosine similarity measure, based on Bhttacharya's distance [3] is the inner product of the two vectors divided by the product of their lengths. As the cosine similarity measure is the cosine of the angle between the vector representations of fuzzy sets, it is extended to cosine similarity measures between SVNSs by J.Ye [15,17] and also to cosine similarity measures between INSs by Broumi and Smarandache [36].

The notion of multisets was formulated first in [31] by Yager as generalization of the concept of set theory. Several authors from time to time made a number of generalization of set theory. For example, Sebastian and Ramakrishnan [42] introduced a new notion called multifuzzy sets, which is a generalization of multiset. Since then, Sebastian and Ramakrishnan [41,42] discussed more properties on multi fuzzy set. Later on, T. K. Shinoj and S. J. John [43] made an extension of the concept of fuzzy multisets by an intuitionistic fuzzy set, which called intuitionistic fuzzy multisets(IFMS). Since then in the study on IFMS, a lot of excellent results have been achieved by researchers [26,27,28,29,30]. An element of a multi fuzzy sets can occur more than once with possibly the same or different membership values, whereas an element of intuitionistic fuzzy multisets allows the repeated occurrences of membership and non--membership values. The concepts of FMS and IFMS fails to deal with indeterminatcy. In 2013, Smarandache [8] extended the classical neutrosophic logic to n-valued refined neutrosophic logic, by refining each neutrosophic component T, I, F into respectively, $T_1, T_2, ..., T_m$ and $I_1, I_2, ..., I_p$ and $F_1, F_2, ..., F_r$. Recently, I.Deli et al .[12] introduced the concept of neutrosophic refined sets and studied some of their basic properties. The concept of neutrosophic refined set (NRS) is a generalization of fuzzy multisets and intuitionistic fuzzy multisets.

In this paper, motivated by the cosine similarity measure based on Bhattacharya's distance and the improved cosine similarity measure of single valued neutrosophic proposed by J.Ye [17]. we propose a new method called "cosine similarity measure for neutrosophic refined sets. The proposed cosine similarity measure is applied to medical diagnosis problems. The paper is structured as follows. In Section 2, we first recall the necessary background on cosine similarity measure and neutrosophic refined sets. In Section 3,we present cosine similarity measure for neutrosophic refined sets and examines their respective properties. In section 4, we present a medical diagnosis using NRS –cosine similarity measure. Finally we conclude the paper.

2.Preliminaries

This section gives a brief overview of the concepts of neutrosophic set, single valued neutrosophic set, cosine similarity measure and neutrosophic refined sets.

2.1 Neutrosophic Sets

Definition 2.1 *[7]*

Let U be an universe of discourse then the neutrosophic set A is an object having the form

A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ U}, where the functions T, I, F : U→]−0, 1+[define respectively the degree of membership (or Truth) , the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element x ∈ U to the set A with the condition.

$$^-0 \le sup\ (x) + supI\ (x) + supF\ (x) \le 3^+. \quad \textbf{(1)}$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. So instead of $]-0, 1^+[$ we need to take the interval [0, 1] for technical applications, because $]^-0, 1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems.

For two NS, A = {<x,$T_A(x)$, (x), (x)> | X}

And = {<x, $T_B(x)$, $I_B(x)$, (x)> | X} the two relations are defined as follows:

(1) If and only if (x) (x), (x) (x), (x) (x)

(2) if, (x)=$T_B(x)$, (x) =$I_B(x)$, (x) =$F_B(x)$

2.2Single Valued Neutrosophic Sets

Definition 2.2 [10]

Let X be a space of points (objects) with generic elements in X denoted by x. An SVNS A in X is characterized by a truth-membership function (x), an indeterminacy-membership function (x), and a falsity-membership function (x), for each point x in X, $T_A(x)$, (x), $F_A(x)$ ∈ [0, 1].

When X is continuous, an SVNS A can be written as

$$A=\int \frac{\ _A(x),\ _A(x),\ _A(x),>}{} \quad \textbf{(2)}$$

When X is discrete, an SVNS A can be written as

$$A= \sum \frac{{}_{A}(x_i),\ {}_{A}(x_i), F_A(x_i), >}{}$$ **(3)**

For two SVNS, $= \{<x, T_A(x), I_A(x), (x)> \mid X\}$

And $B_{SVNS} = \{ <x, (x), I_B(x), (x)> \mid X\}$ the two relations are defined as follows:

(1) if and only if (x) (x), $I_A(x)$ (x), $F_A(x)$ (x)

(2) A , (x) $=T_B(x)$, (x) $=I_B(x)$, $F_A(x) = F_B(x)$ for any X.

2.3 Cosine Similarity

Definition 2.3 [5]

Cosine similarity is a fundamental angle-based measure of similarity between two vectors of n dimensions using the cosine of the angle between them. It measures the similarity between two vectors based only on the direction, ignoring the impact of the distance between them. Given two vectors of attributes, $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$, the cosine similarity, cosθ, is represented using a dot product and magnitude as

$$\mathbf{Cos\theta} = \frac{\sum_i^n}{\sqrt{\sum}\ \sqrt{\sum}} \qquad \textbf{(4)}$$

In vector space, a cosine similarity measure based on Bhattacharya's distance [3] between two fuzzy set $\mu_A(x_i)$ and $\mu_B(x_i)$ defined as follows:

$$(A\ B) = \frac{\sum_i^n\ {}_{A}(x_i)\ {}_{B}(x_i)}{\sqrt{\sum\ {}_{A}(x_i)^2}\sqrt{\sum_i^n\ {}_{B}(x_i)^2}} \qquad \textbf{(5)}$$

The cosine of the angle between the vectors is within the values between 0 and 1.

In 3-D vector space, J. Ye [15] defines cosine similarity measure between SVNS as follows:

$$(A\ B) = \frac{\sum_i^n\ {}_{A}(x_i)\ {}_{B}(x_i) + {}_{A}(x_i) I_B(x_i) + {}_{A}(x_i) F_B(x_i)}{\sqrt{\sum\ {}_{A}(x_i)^2\ {}_{A}(x_i)^2\ {}_{A}(x_i)^2}\sqrt{\sum_i^n\ {}_{B}(x_i)^2\ {}_{B}(x_i)^2\ {}_{A}(x_i)^2}}$$ **(6)**

2.4. Neutrosophic Refined Sets.

Definition 2.4 [12]

Let A and be two neutrosophic refined sets.

$A = \{<x, (T_A^1(x), T_A^2(x), \dots, T_A^p(x)), (I\ (x), I_A^2(x), \dots, I_A^p(x)), (F\ (x), F_A^2(x), \dots, F_A^p(x))>: x \in X\}$

where $T_A^1(x), T_A^2(x), \dots, T_A^p(x) : E \to [0, 1]$,

$, I_A^1(x), I_A^2(x), \dots, I_A^p(x): E \to [0, 1]$, and

$(x), F_A^2(x), \dots, F_A^p(x): E \to [0, 1]$ such that $0 \le$ (x) + (x) + (x) ≤ 3 for i=1,2,…,p for any $x \in X$ $,(T_A^1(x), T_A^2(x), \dots, T_A^p(x))$, $(I_A^1(x), I_A^2(x), \dots, I_A^p(x))$ and $(F\ (x), F_A^2(x), \dots, F_A^p(x))$ is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x, respectively. Also, P is called the dimension of neutrosophic refined sets (NRS) A.

3. Cosine similarity measure for Neutrosophic refined Sets.

Based on the improved cosine similarity measure of single valued neutrosophic sets proposed by J.Ye [17] which consists of membership, indeterminacy and non membership functions defined as follow:

$$(A,B) = -\sum \left[\frac{\pi(|\ {}_{A}(x_i) -\ {}_{B}(x_i)| + |I_A(x_i) -\ {}_{B}(x_i)| + |F_A(x_i) -\ {}_{B}(x_i)|)}{}\right]$$

(7)

And the cosine similarity measure of neutrosophic refined sets consisting of the multiple membership, indetrrminacy, and non-membership function is

$$(A,B) = -\sum \left\{\frac{1}{}\sum\ cos\left[\frac{\pi\left(\left|T_A^j(x_i) -\ (x_i)\right| + \left|I_A^j(x_i) -\ (x_i)\right| + \left|F_A^j(x_i) -\ (x_i)\right|\right)}{}\right]\right\}$$

(8)

Proposition 3.1. The defined cosine similarity measure (A,B) between NRS A and B satisfies the following properties

1. 0 (A, ≤1
2. (A,B=1 if and only if A= B
3. (A,B)= (B,A)
4. If C is a NRS in X and A⊂B⊂C ,then (A,C) (A,B) and (A,C) (B,C)

Proof:

(1)

As the membership, indeterminacy and non-membership functions of the NRSs and the value of the cosine function are within [0 ,1],the similarity measure based on cosine function also is within [0.1]. Hence $0 \le C_{NRS}$(A,B) ≤1.

(2)

For any two NRSs A and B , if A= B , this implies (x_i) = $(x_i), I_A^j(x_i) = I_B^j(x_i), F_A^j(x_i) =$ (x_i) for i= 1,2,…,n and j=1,2….,p and X. Hence $\left|T_A^j(x_i) \quad T \ (x_i)\right| = 0$, $\left|I_A^j(x_i) \quad (x_i)\right|$ =0, and $\left|F_A^j(x_i) \quad (x_i)\right|$=0 .Thus (A,B)=1.

If C_{NRS}A,B)=1 this refers that $\left|T_A^j(x_i) \quad (x_i)\right| = 0$, $\left|I_A^j(x_i) - I_B^j(x_i)\right|$ =0, and $\left|F_A^j(x_i) \quad (x_i)\right|$=0 since cos(0)=1.Then ,these equalities indicates (x_i) = $(x_i), I_A^j(x_i) = I_B^j(x_i), F_A^j(x_i) =$ (x_i) for all i,j values and $x_i \in$ X. Hence A= B

(3)

Proof is straightforward

(4)

If A⊂B⊂C. then there are (x_i) (x_i) (x_i), $(x_i) \geq I_B^j(x_i)$ (x_i), and (x_i) (x_i) (x_i) for all i,j values and X.Then we have the following inequalities

$\left|T_A^j(x_i) - T_B^j(x_i)\right|$ $\left|T_A^j(x_i) \quad (x_i)\right|$, $\left|T_B^j(x_i) \quad (x_i)\right|$ $\left|T_A^j(x_i) \quad (x_i)\right|$,

$\left|I_A^j(x_i) \quad (x_i)\right| \leq \left|I_A^j(x_i) \quad (x_i)\right|$, $\left|I_B^j(x_i) \quad (x_i)\right|$ $\left|I_A^j(x_i) \quad (x_i)\right|$,

$\left|F_A^j(x_i) \quad (x_i)\right|$ $\left|F_A^j(x_i) \quad (x_i)\right|$, $\left|F_B^j(x_i) \quad (x_i)\right|$ $\left|F_A^j(x_i) \quad (x_i)\right|$,

Hence, (A,C) (A,B) and C_{NRS}(A,C) (B,C) for k=1,2, since the cosine function is a decreasing function within the interval $[0, -]$.

4 Application

In this section, we give some applications of NRS in medical diagnosis problems using the cosine similarity measure. Some of it is quoted from [29,30,41].

From now on, we use

A = {<x,(T_A^1(x), I_A^1(x), F_A^1(x)),(T_A^2(x), I_A^2(x), F_A^2(x)),.., ,(T_A^p(x), I_A^p(x), F_A^p(x))>: x ∈ X}

Instead of

A = {<x,(T_A^1(x), T_A^2(x),..., T_A^p(x)), (I (x), I_A^2(x),..., I_A^p(x)), (F (x), F_A^2(x),…, F_A^p(x))>: x ∈ X}

4.1. Medical Diagnosis using NRS –cosine similarity measure

In what follows, let us consider an illustrative example adopted from Rajarajeswari and Uma [29] with minor changes and typically considered in [30,43]. Obviously, the application is an extension of intuitionistic fuzzy multi sets [29].

"As Medical diagnosis contains lots of uncertainties and increased volume of information available to physicians from new medical technologies, the process of classifying different set of symptoms under a single name of disease becomes difficult. In some practical situations, there is the possibility of each element having different truth membership, indeterminate and false membership functions. The proposed similarity measure among the patients Vs symptoms and symptoms Vs diseases gives the proper medical diagnosis. The unique feature of this proposed method is that it considers multi truth membership, indeterminate and false membership. By taking one time inspection, there may be error in diagnosis. Hence, this multi time inspection, by taking the samples of the same patient at different times gives best diagnosis" [29].

Now, an example of a medical diagnosis will be presented.

Example: Let P={P_1,P_2,P_3} be a set of patients, D={Viral Fever, Tuberculosis, Typhoid, Throat disease} be a set of diseases and S={Temperature, cough, throat pain, headache, body pain} be a set of symptoms. Our solution is to examine the patient at different time intervals (three times a day), which in turn give arise to different truth membership, indeterminate and false membership function for each patient.

Table I: Q (the relation Between Patient and Symptoms)

	Temperature	Cough	Throat pain	Headache	Body Pain
P_1	(0.4,0.3,0.4) (0.3,0.4,0.6) (0.2,0.5,0.5)	(0.5,0.4,0.4) (0.4,0.1,0.3) (0.3,0.4,0.5)	(0.3,0.5,0.5) (0.2,0.6,0.4) (0.1,0.6,0.3)	(0.5,0.3,0.4) (0.5,0.4,0.7) (0.3,0.3,0.6)	(0.5,0.2,0.4) (0.2,0.3,0.5) (0.1,0.4,0.3)
P_2	(0.6,0.3,0.5) (0.5,0.5,0.2) (0.4,0.4,0.5)	(0.6,0.3,0.7) (0.4,0.4,0.2) (0.2,0.4,0.5)	(0.6,0.3,0.3) (0.3,0.5,0.4) (0.1,0.4,0.5)	(0.6,0.3,0.1) (0.4,0.5,0.8) (0.2,0.4,0.3)	(0.4,0.4,0.5) (0.3,0.2,0.7) (0.1,0.5,0.5)
P_3	(0.8,0.3,0.5) (0.7,0.5,0.4) (0.6,0.4,0.4)	(0.5,0.5,0.3) (0.1,0.6,0.4) (0.3,0.4,0.3)	(0.3,0.3,0.6) (0.2,0.5,0.7) (0.1,0.4,0.5)	(0.6,0.2,0.5) (0.5,0.3,0.6) (0.2,0.2,0.6)	(0.6,0.4,0.5) (0.3,0.3,0.4) (0.2,0.2,0.6)

Let the samples be taken at three different timings in a day (in 08:00,16:00,24:00)

Remark :At three different timings in a day (in 08:00,16:00,24:00)

P_1 upon the Temperature may have the disease 1 with chance (0.4, 0.3 , 0.4) at 08:00
P_1 upon the Temperature may have the disease 2 with chance (0.3, 0.4 , 0.6) at 16:00
P_1 upon the Temperature may have the disease 3 with chance (0.2, 0.5 , 0.5) at 24:00

Table II: R (the relation among Symptoms and Diseases)

R	Viral Fever	Tuberculosis	Typhoid	Throat disease
Temperature	(0.2,0.5,0.6)	(0.4,0.6,0.5)	(0.6,0.4,0.5)	(0.3,0.7,0.8)
Cough	(0.6,0.4,0.6)	(0.8,0.2,0.3)	(0.3,0.2,0.6)	(0.2,0.4,0.1)
Throat Pain	(0.5,0.2,0.3)	(0.4,0.5,0.3)	(0.4,0.5,0.5)	(0.2,0.6,0.2)
Headache	(0.6,0.8,0.2)	(0.2,0.3,0.6)	(0.1,0.6,0.3)	(0.2,0.5,0.5)
Body Pain	(0.7,0.4,0.4)	(0.2,0.3,0.4)	(0.2,0.3,0.4)	(0.2,0.2,0.3)

Table III: The Correlation Measure between NRS Q and R

Cosine similarity measure	Viral Fever	Tuberculosis	Typhoid	Throat diseas
P_1	0.9793	0.9915	**0.9896**	0.9794
P_2	0.9831	0.9900	**0.9870**	0.9723
P_3	0.9811	0.9931	**0.9917**	0.9822

The highest correlation measure from the Table III gives the proper medical diagnosis. Therefore, patient P_1, P_2 and P_3 suffers from Tuberculosis

5.Conclusion

In this paper, we have extended the improved cosine similarity of single valued neutrosophic set proposed by J.Ye [17] to the case of neutrosophic refined sets and proved some of their basic properties. We have present an application of cosine similarity measure of neutrosophic refined sets in medical diagnosis problems. In The future work, we will extend this cosine similarity measure to the case of interval neutrosophic refined sets.

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Soft Neutrosophic Groupoids and Their Generalization

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Abstract. Soft set theory is a general mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. In this paper we introduced soft neutrosophic groupoid and their generalization with the discuissionf of some of their characteristics. We also introduced a new type of soft neutrophic groupoid, the so called soft strong neutrosophic goupoid which is of pure neutrosophic character. This notion also found in all the other corresponding notions of soft neutrosophic thoery. We also given some of their properties of this newly born soft structure related to the strong part of neutrosophic theory.

Keywords: Neutrosophic groupoid, neutrosophic bigroupoid, neutrosophic N -groupoid, soft set, soft neutrosophic groupoid, soft neutrosophic bigroupoid, soft neutrosophic N -groupoid.

1 Introduction

Florentine Smarandache for the first time introduced the concept of neutrosophy in 1995, which is basically a new branch of philosophy which actually studies the origin, nature, and scope of neutralities. The neutrosophic logic came into being by neutrosophy. In neutrosophic logic each proposition is approximated to have the percentage of truth in a subset T , the percentage of indeterminacy in a subset I , and the percentage of falsity in a subset F . Neutrosophic logic is an extension of fuzzy logic. In fact the neutrosophic set is the generalization of classical set, fuzzy conventional set, intuitionistic fuzzy set, and interval valued fuzzy set. Neutrosophic logic is used to overcome the problems of impreciseness, indeterminate, and inconsistencies of date etc. The theory of neutrosophy is so applicable to every field of algebra. W.B. Vasantha Kandasamy and Florentin Smarandache introduced neutrosophic fields, neutrosophic rings, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups and neutrosophic N -groups, neutrosophic semigroups, neutrosophic bisemigroups, and neutrosophic N -semigroups, neutrosophic loops, nuetrosophic biloops, and neutrosophic N -loops, and so on. Mumtaz ali et al. introduced nuetrosophic LA -semigroups.

Molodtsov introduced the theory of soft set. This mathematical tool is free from parameterization inadequacy, syndrome of fuzzy set theory, rough set theory, probability theory and so on. This theory has been applied successfully in many fields such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration, and probability. Recently soft set theory attained much attention of the researchers since its appearance and the work based on several operations of soft set introduced in $[2,9,10]$. Some properties and algebra may be found in $[1]$. Feng et al. introduced soft semirings in $[5]$. By means of level soft sets an adjustable approach to fuzzy soft set can be seen in $[6]$. Some other concepts together with fuzzy set and rough set were shown in $[7,8]$.

This paper is about to introduced soft nuetrosophic groupoid, soft neutrosophic bigroupoid, and soft neutrosophic N -groupoid and the related strong or pure part of neutrosophy with the notions of soft set theory. In the proceeding section, we define soft neutrosophic groupoid, soft neutrosophic strong groupoid, and some of their properties are discussed. In the next section, soft neutrosophic bigroupoid are presented with their strong neutrosophic part. Also in this section some of their characterization have been made. In the last section soft neutrosophic N -groupoid and their corresponding strong theory have been constructed with some of their properties.

2 Fundamental Concepts

2.1 Neutrosophic Groupoid

Definition 2.1.1. Let G be a groupoid, the groupoid generated by G and I i.e. $G \cup I$ is denoted by $\langle G \cup I \rangle$ is defined to be a neutrosophic groupoid where I is the indeterminacy element and termed as neutrosophic element.

Definition 2.1.2. Let $\langle G \cup I \rangle$ be a neutrosophic groupoid. A proper subset P of $\langle G \cup I \rangle$ is said to be a neutrosophic subgroupoid, if P is a neutrosophic groupoid under the operations of $\langle G \cup I \rangle$. A neutrosophic groupoid $\langle G \cup I \rangle$ is said to have a subgroupoid if $\langle G \cup I \rangle$ has a proper subset which is a groupoid under the operations of $\langle G \cup I \rangle$.

Theorem 2.1.3. Let $\langle G \cup I \rangle$ be a neutrosophic groupoid. Suppose P_1 and P_2 be any two neutrosophic subgroupoids of $\langle G \cup I \rangle$, then $P_1 \cup P_2$, the union of two neutrosophic subgroupoids in general need not be a neutrosophic subgroupoid.

Definition 2.1.4. Let $\langle G \cup I \rangle$ be a neutrosophic groupoid under a binary operation $*$. P be a proper subset of $\langle G \cup I \rangle$. P is said to be a neutrosophic ideal of $\langle G \cup I \rangle$ if the following conditions are satisfied.

1. P is a neutrosophic groupoid.
2. For all $p \in P$ and for all $s \in \langle G \cup I \rangle$ we have $p * s$ and $s * p$ are in P.

2.2 Neutrosophic Bigroupoid

Definition 2.2.1. Let $(BN(\mathrm{G}), *, \circ)$ be a non-empty set with two binary operations $*$ and $\circ$. $(BN(\mathrm{G}), *, \circ)$ is said to be a neutrosophic bigroupoid if $BN(\mathrm{G}) = P_1 \cup P_2$ where atleast one of $(P_1, *)$ or $(P_2, \circ)$ is a neutrosophic groupoid and other is just a groupoid. P_1 and P_2 are proper subsets of $BN(\mathrm{G})$.

If both $(P_1, *)$ and $(P_2, \circ)$ in the above definition are neutrosophic groupoids then we call $(BN(\mathrm{G}), *, \circ)$ a strong neutrosophic bigroupoid. All strong neutrosophic bigroupoids are trivially neutrosophic bigroupoids.

Definition 2.2.2. Let $(BN(\mathrm{G}) = \mathrm{P}_1 \cup P;: *, \circ)$ be a neutrosophic bigroupoid. A proper subset $(T, \circ, *)$ is said to be a neutrosophic subbigroupoid of $BN(\mathrm{G})$ if

1) $T = T_1 \cup T_2$ where $T_1 = P_1 \cap T$ and $T_2 = P_2 \cap T$ and
2) At least one of $(T_1, \circ)$ or $(T_2, *)$ is a neutrosophic groupoid.

Definition 2.2.3. Let $(BN(\mathrm{G}) = \mathrm{P}_1 \cup P, *, \circ)$ be a neutrosophic strong bigroupoid. A proper subset T of $BN(S)$ is called the strong neutrosophic subbigroupoid if $T = T_1 \cup T_2$ with $T_1 = P_1 \cap T$ and $T_2 = P_2 \cap T$ and if both $(T_1, *)$ and $(T_2, \circ)$ are neutrosophic subgroupoids of $(P_1, *)$ and $(P_2, \circ)$ respectively. We call $T = T_1 \cup T_2$ to be a neutrosophic strong subbigroupoid, if atleast one of $(T_1, *)$ or $(T_2, \circ)$ is a groupoid then $T = T_1 \cup T_2$ is only a neutrosophic subgroupoid.

Definition 2.2.4. Let $(BN(\mathrm{G}) = \mathrm{P}_1 \cup P_2, *, \circ)$ be any neutrosophic bigroupoid. Let J be a proper subset of $BN(\mathrm{J})$ such that $J_1 = J \cap P_1$ and $J_2 = J \cap P_2$ are ideals of P_1 and P_2 respectively. Then J is called the neutrosophic biideal of $BN(\mathrm{G})$.

Definition 2.2.5. Let $(BN(\mathrm{G}), *, \circ)$ be a strong neutrosophic bigroupoid where $BN(S) = \mathrm{P}_1 \cup P_2$ with $(P_1, *)$ and $(P_2, \circ)$ be any two neutrosophic groupoids. Let J be a proper subset of $BN(\mathrm{G})$ where $I = I_1 \cup I_2$ with $I_1 = I \cap P_1$ and $I_2 = I \cap P_2$ are neutrosophic ideals of the neutrosophic groupoids P_1 and P_2 respectively. Then I is called or defined as the strong neutrosophic biideal of $BN(\mathrm{G})$.

Union of any two neutrosophic biideals in general is not a neutrosophic biideal. This is true of neutrosophic strong biideals.

2.3 Neutrosophic N -groupoid

Definition 2.3.1. Let $\{\mathrm{N}(\mathrm{G}), *_1, \ldots, *_2\}$ be a non-empty set with N -binary operations defined on it. We call $N(G)$ a neutrosophic N -groupoid (N a positive integer) if the following conditions are satisfied.

1) $\mathrm{N}(\mathrm{G}) = G_1 \cup \ldots \cup \mathrm{G}_N$ where each G_i is a proper subset of $N(G)$ i.e. $G_i \subset G_j$ or $G_j \subset G_i$ if $i \neq j$.
2) $(\mathrm{G}_i, *_i)$ is either a neutrosophic groupoid or a groupoid for $i = 1, 2, 3, \ldots, N$.

If all the N -groupoids $(G_i, *_i)$ are neutrosophic groupoids (i.e. for $i = 1,2,3,...,N$) then we call $N(G)$ to be a neutrosophic strong N -groupoid.

Definition 2.3.2. Let $N(G) = \{G_1 \cup G_2 \cup .... \cup G_N, *_1, *_2, ..., *_N\}$ be a neutrosophic N -groupoid. A proper subset $P = \{P_1 \cup P_2 \cup .... P_N, *_1, *_2, ..., *_N\}$ of $N(G)$ is said to be a neutrosophic N -subgroupoid if $P_i = P \cap G_i, i = 1,2,...,N$ are subgroupiids of G_i in which atleast some of the subgroupoids are neutrosophic subgroupoids.

Definition 2.3.3. Let $N(G) = \{G_1 \cup G_2 \cup .... \cup G_N, *_1, *_2, ..., *_N\}$ be a neutrosophic strong N -groupoid. A proper subset $T = \{T_1 \cup T_2 \cup .... \cup T_N, *_1, *_2, ..., *_N\}$ of $N(G)$ is said to be a neutrosophic strong sub N -groupoid if each $(T_i, *_i)$ is a neutrosophic subgroupoid of $(G_i, *_i)$ for $i = 1,2,...,N$ where $T_i = G_i \cap T$.

If only a few of the $(T_i, *_i)$ in T are just subgroupoids of $(G_i, *_i)$, (i.e. $(T_i, *_i)$ are not neutrosophic subgroupoids then we call T to be a sub N -groupoid of $N(G)$.

Definition 2.3.4. Let $N(G) = \{G_1 \cup G_2 \cup .... \cup G_N, *_1, *_2, ..., *_N\}$ be a neutrosophic N -groupoid. A proper subset $P = \{P_1 \cup P_2 \cup .... \cup P_N, *_1, *_2, ..., *_N\}$ of $N(G)$ is said to be a neutrosophic N -subgroupoid, if the following conditions are true,

1. P is a neutrosophic sub N -groupoid of $N(G)$.
2. Each $P_i = G \cap P_i, i = 1,2,...,N$ is an ideal of G_i.

Then P is called or defined as the neutrosophic N -ideal of the neutrosophic N -groupoid $N(G)$.

Definition 2.3.5. *Let* $N(G) = \{G_1 \cup G_2 \cup .... G_N, *_1, *_2, ..., *_N\}$ *be a neutrosophic strong* N *-groupoid. A proper subset* $J = \{J_1 \cup J_2 \cup .... J_N, *_1, *_2, ..., *_N\}$ *where* $J_t = J \cap G_t$ *for* $t = 1,2,...,N$ *is said to be a neutrosophic strong* N *-ideal of* $N(G)$ *if the following conditions are satisfied.*

1) Each it is a neutrosophic subgroupoid of $G_t, t = 1,2,...,N$ i.e. It is a neutrosophic strong N-subgroupoid of $N(G)$.
2) Each it is a two sided ideal of G_t for $t = 1,2,...,N$.

Similarly one can define neutrosophic strong N -left ideal or neutrosophic strong right ideal of $N(G)$.

A neutrosophic strong N -ideal is one which is both a neutrosophic strong N -left ideal and N -right ideal of $S(N)$.

2.4 Soft Sets

Throughout this subsection U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U, and $A, B \subset E$. Molodtsov defined the soft set in the following manner:

Definition 2.4.1. A pair (F, A) is called a soft set over U where F is a mapping given by $F : A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $a \in A$, $F(\mathrm{a})$ may be considered as the set of a -elements of the soft set (F, A), or as the set of a -approximate elements of the soft set.

Definition 2.4.2. For two soft sets (F, A) and (H, B) over U, (F, A) is called a soft subset of (H, B) if

1. $A \subseteq B$ and
2. $F(a) \subseteq H(a)$, for all $x \in A$.

This relationship is denoted by $(F, A) \subset (H, B)$. Similarly (F, A) is called a soft superset of (H, B) if (H, B) is a soft subset of (F, A) which is denoted by $(F, A) \supset (H, B)$.

Definition 2.4.3. Two soft sets (F, A) and (H, B) over U are called soft equal if (F, A) is a soft subset of (H, B) and (H, B) is a soft subset of (F, A).

Definition 2.4.4. Let (F, A) and (K, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$. Then their restricted intersection is denoted by $(F, A) \cap_R (K, B) = (H, C)$ where (H, C) is defined as $H(c) = F(c) \cap \mathrm{K}(c)$ for all $c \in C = A \cap B$.

Definition 2.4.5. The extended intersection of two soft sets (F, A) and (K, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as

$$H(c) = \begin{cases} F(c) & \text{if c} \in A - B, \\ G(c) & \text{if c} \in B - A, \\ F(c) \cap G(c) & \text{if c} \in A \cap B. \end{cases}$$

We write $(F, A) \cap_{\varepsilon} (K, B) = (H, C)$.

Definition 2.4.6. The restricted union of two soft sets (F, A) and (K, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as $H(c) = F(c) \cup G(c)$ for all $c \in C$. We write it as

$(F, A) \cup_R (K, B) = (H, C)$.

Definition 2.4.7. The extended union of two soft sets (F, A) and (K, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as

$$H(c) = \begin{cases} F(c) & \text{if c} \in A - B, \\ G(c) & \text{if c} \in B - A, \\ F(c) \cup G(c) & \text{if c} \in A \cap B. \end{cases}$$

We write $(F, A) \cup_{\varepsilon} (K, B) = (H, C)$.

3 Soft Neutrosophic Groupoid and Their Properties

3.1 Soft Neutrosophic Groupoid

Definition 3.1.1. Let $\{\langle G \cup I \rangle, *\}$ be a neutrosophic groupoid and (F, A) be a soft set over $\{\langle G \cup I \rangle, *\}$. Then (F, A) is called soft neutrosophic groupoid if and only if $F(a)$ is neutrosophic subgroupoid of $\{\langle G \cup I \rangle, *\}$ for all $a \in A$.

Example 3.1.2. Let

$$\langle Z_{10} \cup I \rangle = \begin{Bmatrix} 0,\ 1,\ 2,\ 3,\ \ldots,\ 9,\ I,\ 2I,\ \ldots,\ 9I, \\ 1+ I,\ 2\ +\ I,\ \ldots,\ 9\ +\ 9I \end{Bmatrix}$$

be a neutrosophic groupoid where $*$ is defined on $\langle Z_{10} \cup I \rangle$ by $a * b = 3a + 2b (\text{mod} 10)$ for all $a, b \in \langle Z10 \cup I \rangle$. Let $A = \{a_1, a_2\}$ be a set of parameters. Then (F, A) is a soft neutrosophic groupoid over $\{\langle Z10 \cup I \rangle, *\}$, where

$$F(a_1) = \{0, 5, 5\mathrm{I}, 5 + 5\mathrm{I}\},$$

$$F(a_2) = (\mathrm{Z}_{10}, *).$$

Theorem 3.1.3. A soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$ always contain a soft groupoid over $(G, *)$.

Proof. The proof of this theorem is straightforward.

Theorem 3.1.4. Let (F, A) and (H, A) be two soft neutrosophic groupoids over $\{\langle G \cup I \rangle, *\}$. Then their intersection $(F, A) \cap (H, A)$ is again a soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.

Proof. The proof is straightforward.

Theorem 3.1.5. Let (F, A) and (H, B) be two soft neutrosophic groupoids over $\{\langle G \cup I \rangle, *\}$. If $A \cap B = \phi$, then $(F, A) \cup (H, B)$ is a soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.

Remark 3.1.6. The extended union of two soft neutrosophic groupoids (F, A) and (K, B) over a neutrosophic groupoid $\{\langle G \cup I \rangle, *\}$ is not a soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.

Proposition 3.1.7. The extended intersection of two soft neutrosophic groupoids over a neutrosophic groupoid $\{\langle G \cup I \rangle, *\}$ is a soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.

Remark 3.1.8. The restricted union of two soft neutrosophic groupoids (F, A) and (K, B) over $\{\langle G \cup I \rangle, *\}$ is not a soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.

Proposition 3.1.9. The restricted intersection of two soft neutrosophic groupoids over $\{\langle G \cup I \rangle, *\}$ is a soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.

Proposition 3.1.10. The AND operation of two soft neutrosophic groupoids over $\{\langle G \cup I \rangle, *\}$ is a soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.

Remark 3.1.11. The OR operation of two soft neutosophic groupoids over $\{\langle G \cup I \rangle, *\}$ is not a soft nuetrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.

Definition 3.1.12. Let (F, A) be a soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$. Then (F, A) is called an absolute-soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$ if $F(a) = \{\langle G \cup I \rangle, *\}$, for all $a \in A$.

Theorem 3.1.13. Every absolute-soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$ always contain absolute soft groupoid over $\{G,*\}$.

Definition 3.1.14. Let (F,A) and (H,B) be two soft neutrosophic groupoids over $\{\langle G\cup I\rangle,*\}$. Then (H,B) is a soft neutrosophic subgroupoid of (F,A), if

1. $B\subseteq A$.
2. $H(a)$ is neutrosophic subgroupoid of $F(a)$, for all $a\in B$.

Example 3.1.15. Let

$$\langle Z_4\cup I\rangle=\left\{\begin{matrix}0,1,2,3,I,2I,3I,1+I,1+2I,1+3I\\ 2+I,2+2I,2+3I,3+I,3+2I,3+3I\end{matrix}\right\}$$

be a neutrosophic groupoid with respect to the operation $*$ where $*$ is defined as $a*b=2a+b(\bmod 4)$ for all $a,b\in\langle Z_4\cup I\rangle$. Let $A=\{a_1,a_2,a_3\}$ be a set of parameters. Then (F,A) is a soft neutrosophic groupoid over $\langle Z_4\cup I\rangle$, where

$$F(a_1)=\{0,2,2I,2+2I\},$$

$$F(a_2)=\{0,2,2+2\mathrm{I}\},$$

$$F(a_3)=\{0,2+2I\}.$$

Let $B=\{a_1,a_2\}\subseteq \mathrm{A}$. Then (H,B) is a soft neutrosophic subgroupoid of (F,A), where

$$H(a_1)=\{0,2+2I\},$$
$$H(a_2)=\{0,2+2I\}.$$

Definition 3.1.16. Let $\{\langle G\cup I\rangle,*\}$ be a neutrosophic groupoid and (F,A) be a soft neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$. Then (F,A) is called soft Lagrange neutrosophic groupoid if and only if $F(a)$ is a Lagrange neutrosophic subgroupoid of $\{\langle G\cup I\rangle,*\}$ for all $a\in A$.

Example 3.1.17. Let

$$\langle Z_4\cup I\rangle=\left\{\begin{matrix}0,1,2,3,I,2I,3I,1+I,1+2I,1+3I\\ 2+I,2+2I,2+3I,3+I,3+2I,3+3I\end{matrix}\right\}$$

be a neutrosophic groupoid of order 16 with respect to the operation $*$ where $*$ is defined as $a*b=2a+b(\bmod 4)$ for all $a,b\in\langle Z_4\cup I\rangle$. Let $A=\{a_1,a_2\}$ be a set of parameters. Then (F,A) is a soft Lagrange neutrosophic groupoid over $\langle Z_4\cup I\rangle$, where

$$F(a_1)=\{0,2,2I,2+2I\},$$
$$F(a_2)=\{0,2+2I\}.$$

Theorem 3.1.18. Every soft Lagrange neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$ is a soft neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$ but the converse is not true.

We can easily show the converse by the help of example.

Theorem 3.1.19. If $\{\langle G\cup I\rangle,*\}$ is a Lagrange neutrosophic groupoid, then (F,A) over $\{\langle G\cup I\rangle,*\}$ is a soft Lagrange neutrosophic groupoid but the converse is not true.

Remark 3.1.20. Let (F,A) and (K,C) be two soft Lagrange neutrosophic groupoids over $\{\langle G\cup I\rangle,*\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,\mathrm{C})$ may not be a soft Lagrange neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$.
2. Their restricted intersection $(F,A)\cap_R(K,\mathrm{C})$ may not be a soft Lagrange neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$.
3. Their *AND* operation $(F,A)\wedge(K,\mathrm{C})$ may not be a soft Lagrange neutrosophic groipoid over $\{\langle G\cup I\rangle,*\}$.
4. Their extended union $(F,A)\cup_E(K,\mathrm{C})$ may not be a soft Lagrange neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$.
5. Their restricted union $(F,A)\cup_R(K,\mathrm{C})$ may not be a soft Lagrange neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$.
6. Their *OR* operation $(F,A)\vee(K,\mathrm{C})$ may not be a soft Lagrange neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 3.1.21. Let $\{\langle G\cup I\rangle,*\}$ be a neutrosophic groipoid and (F,A) be a soft neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$. Then (F,A) is called soft weak Lagrange neutrosophic groupoid if atleast one $F(a)$ is not a Lagrange neutrosophic subgroupoid of $\{\langle G \cup I \rangle, *\}$ for some $a \in A$.

Example 3.1.22. Let $\langle Z_4 \cup I \rangle = \begin{Bmatrix} 0,1,2,3,I,2I,3I,1+I,1+2I,1+3I \\ 2+I,2+2I,2+3I,3+I,3+2I,3+3I \end{Bmatrix}$ be a neutrosophic groupoid of order 16 with respect to the operation $*$ where $*$ is defined as $a*b = 2a+b(\bmod 4)$ for all $a,b \in \langle Z_4 \cup I \rangle$. Let $A = \{a_1, a_2, a_3\}$ be a set of parameters. Then (F,A) is a soft weak Lagrange neutrosophic groupoid over $\langle Z_4 \cup I \rangle$, where

$$F(a_1) = \{0,2,2I,2+2I\},$$

$$F(a_2) = \{0,2,2+2I\},$$

$$F(a_3) = \{0,2+2I\}.$$

Theorem 3.1.23. Every soft weak Lagrange neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$ is a soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$ but the converse is not true.

Theorem 3.1.24. If $\{\langle G \cup I \rangle, *\}$ is weak Lagrange neutrosophic groupoid, then (F,A) over $\{\langle G \cup I \rangle, *\}$ is also soft weak Lagrange neutrosophic groupoid but the converse is not true.

Remark 3.1.25. Let (F,A) and (K,C) be two soft weak Lagrange neutrosophic groupoids over $\{\langle G \cup I \rangle, *\}$. Then

1. Their extended intersection $(F,A) \cap_E (K,C)$ is not a soft weak Lagrange neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.
2. Their restricted intersection $(F,A) \cap_R (K,C)$ is not a soft weak Lagrange neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.
3. Their AND operation $(F,A) \wedge (K,C)$ is not a soft weak Lagrange neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.
4. Their extended union $(F,A) \cup_E (K,C)$ is not a soft weak Lagrnage neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.
5. Their restricted union $(F,A) \cup_R (K,C)$ is not a soft weak Lagrange neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.
6. Their OR operation $(F,A) \vee (K,C)$ is not a soft weak Lagrange neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 3.126. Let $\{\langle G \cup I \rangle, *\}$ be a neutrosophic groupoid and (F,A) be a soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$. Then (F,A) is called soft Lagrange free neutrosophic groupoid if $F(a)$ is not a lagrange neutrosophic subgroupoid of $\{\langle G \cup I \rangle, *\}$ for all $a \in A$.

Example 3.1.27. Let $\langle Z_4 \cup I \rangle = \begin{Bmatrix} 0,1,2,3,I,2I,3I,1+I,1+2I,1+3I \\ 2+I,2+2I,2+3I,3+I,3+2I,3+3I \end{Bmatrix}$ be a neutrosophic groupoid of order 16 with respect to the operation $*$ where $*$ is defined as $a*b = 2a+b(\bmod 4)$ for all $a,b \in \langle Z_4 \cup I \rangle$. Let $A = \{a_1, a_2, a_3\}$ be a set of parameters. Then (F,A) is a soft Lagrange free neutrosophic groupoid over $\langle Z_4 \cup I \rangle$, where

$$F(a_1) = \{0,2I,2+2I\},$$
$$F(a_2) = \{0,2,2+2I\}.$$

Theorem 3.1.28. Every soft Lagrange free neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$ is trivially a soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$ but the converse is not true.

Theorem 3.1.29. If $\{\langle G \cup I \rangle, *\}$ is a Lagrange free neutrosophic groupoid, then (F,A) over $\{\langle G \cup I \rangle, *\}$ is also a soft Lagrange free neutrosophic groupoid but the converse is not true.

Remark 3.1.30. Let (F,A) and (K,C) be two soft Lagrange free neutrosophic groupoids over $\{\langle G \cup I \rangle, *\}$. Then

1. Their extended intersection $(F,A) \cap_E (K,C)$ is not a soft Lagrange free neutrosophic groupoid

over $\{\langle G\cup I\rangle,*\}$.

2. Their restricted intersection $(F,A)\cap_R(K,C)$ is not a soft Lagrange free neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft Lagrange free neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$.
4. Their extended union $(F,A)\cup_E(K,C)$ is not a soft Lagrnage free neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$.
5. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft Lagrange free neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft Lagrange free neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 3.1.31. (F,A) is called soft neutrosophic ideal over $\{\langle G\cup I\rangle,*\}$ if $F(a)$ is a neutrosophic ideal of $\{\langle G\cup I\rangle,*\}$, for all $a\in A$.

Theorem 3.1.32. Every soft neutrosophic ideal (F,A) over $\{\langle G\cup I\rangle,*\}$ is trivially a soft neutrosophic subgroupid but the converse may not be true.

Proposition 3.1.33. Let (F,A) and (K,B) be two soft neutrosophic ideals over $\{\langle G\cup I\rangle,*\}$. Then

1) Their extended intersection $(F,A)\cap_E(K,B)$ is soft neutrosophic ideal over $\{\langle G\cup I\rangle,*\}$.
2) Their restricted intersection $(F,A)\cap_R(K,B)$ is soft neutrosophic ideal over $\{\langle G\cup I\rangle,*\}$.
3) Their AND operation $(F,A)\wedge(K,B)$ is soft neutrosophic ideal over $\{\langle G\cup I\rangle,*\}$.

Remark 3.1.34. Let (F,A) and (K,B) be two soft neutrosophic ideal over $\{\langle G\cup I\rangle,*\}$. Then

1) Their extended union $(F,A)\cup_E(K,B)$ is not soft neutrosophic ideal over $\{\langle G\cup I\rangle,*\}$.
2) Their restricted union $(F,A)\cup_R(K,B)$ is not soft neutrosophic ideal over $\{\langle G\cup I\rangle,*\}$.
3) Their OR operation $(F,A)\vee(K,B)$ is not soft neutrosophic ideal over $\{\langle G\cup I\rangle,*\}$.

One can easily proved $(1),(2),$ and (3) by the help of examples.

Theorem 3.1.35. Let (F,A) be a soft neutrosophic ideal over $\{\langle G\cup I\rangle,*\}$ and $\{(H_i,B_i):\mathrm{i}\in\mathrm{J}\}$ is a non-empty family of soft neutrosophic ideals of (F,A). Then

1. $\underset{i\in J}{\cap}(H_i,B_i)$ is a soft neutrosophic ideal of (F,A).
2. $\underset{i\in J}{\wedge}(H_i,B_i)$ is a soft neutrosophic ideal of $\underset{i\in J}{\wedge}(F,A)$.

3.2 Soft Neutrosophic Strong Groupoid

Definition 3.2.1. Let $\{\langle G\cup I\rangle,*\}$ be a neutrosophic groupoid and (F,A) be a soft set over $\{\langle G\cup I\rangle,*\}$. Then (F,A) is called soft neutrosophic strong groupoid if and only if $F(a)$ is a neutrosophic strong subgroupoid of $\{\langle G\cup I\rangle,*\}$ for all $a\in A$.

Example 3.2.2. Let
$$\langle Z_4\cup I\rangle=\begin{Bmatrix}0,1,2,3,I,2I,3I,1+I,1+2I,1+3I\\ 2+I,2+2I,2+3I,3+I,3+2I,3+3I\end{Bmatrix}$$
be a neutrosophic groupoid with respect to the operation $*$ where $*$ is defined as $a*b=2a+b(\bmod 4)$ for all $a,b\in\langle Z_4\cup I\rangle$. Let $A=\{a_1,a_2,a_3\}$ be a set of parameters. Then (F,A) is a soft neutrosophic strong groupoid over $\langle Z_4\cup I\rangle$, where
$$F(a_1)=\{0,2I,2+2I\},$$
$$F(a_2)=\{0,2+2\mathrm{I}\}.$$

Proposition 3.2.3. Let (F,A) and (K,C) be two soft neutrosophic strong groupoids over $\{\langle G\cup I\rangle,*\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ is a soft neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.

2. Their restricted intersection $(F,A)\cap_R(K,C)$ is a soft neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
3. Their AND operation $(F,A)\wedge(K,C)$ is a soft neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.

Remark 3.2.4. Let (F,A) and (K,C) be two soft neutrosophic strong groupoids over $\{\langle G\cup I\rangle,*\}$. Then

1. Their extended union $(F,A)\cup_E(K,C)$ is a soft neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
2. Their restricted union $(F,A)\cup_R(K,C)$ is a soft neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
3. Their OR operation $(F,A)\vee(K,C)$ is a soft neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.

Definition 3.2.5. Let (F,A) and (H,C) be two soft neutrosophic strong groupoids over $\{\langle G\cup I\rangle,*\}$. Then (H,C) is called soft neutrosophic strong sublgroupoid of (F,A), if

1. $C\subseteq A$.
2. $H(a)$ is a neutrosophic strong subgroupoid of $F(a)$ for all $a\in A$.

Definition 3.2.6. Let $\{\langle G\cup I\rangle,*\}$ be a neutrosophic strong groupoid and (F,A) be a soft neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$. Then (F,A) is called soft Lagrange neutrosophic strong groupoid if and only if $F(a)$ is a Lagrange neutrosophic strong subgroupoid of $\{\langle G\cup I\rangle,*\}$ for all $a\in A$.

Theorem 3.2.7. Every soft Lagrange neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$ is a soft neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$ but the converse is not true.

Theorem 3.2.8. If $\{\langle G\cup I\rangle,*\}$ is a Lagrange neutrosophic strong groupoid, then (F,A) over $\{\langle G\cup I\rangle,*\}$ is a soft Lagrange neutrosophic groupoid but the converse is not true.

Remark 3.2.9. Let (F,A) and (K,C) be two soft Lagrange neutrosophic strong groupoids over $\{\langle G\cup I\rangle,*\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ may not be a soft Lagrange neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ may not be a soft Lagrange strong neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$.
3. Their AND operation $(F,A)\wedge(K,C)$ may not be a soft Lagrange neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
4. Their extended union $(F,A)\cup_E(K,C)$ may not be a soft Lagrange neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
5. Their restricted union $(F,A)\cup_R(K,C)$ may not be a soft Lagrange neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
6. Their OR operation $(F,A)\vee(K,C)$ may not be a soft Lagrange neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 3.2.10. Let $\{\langle G\cup I\rangle,*\}$ be a neutrosophic strong groupoid and (F,A) be a soft neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$. Then (F,A) is called soft weak Lagrange neutrosophic strong groupoid if atleast one $F(a)$ is not a Lagrange neutrosophic strong subgroupoid of $\{\langle G\cup I\rangle,*\}$ for some $a\in A$.

Theorem 3.2.11. Every soft weak Lagrange neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$ is a soft neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$ but the converse is not true.

Theorem 3.2.12. If $\{\langle G\cup I\rangle,*\}$ is weak Lagrange neutrosophic strong groupoid, then (F,A) over $\{\langle G\cup I\rangle,*\}$ is also soft weak Lagrange neutrosophic strong groupoid but the converse is not true.

Remark 3.2.13. Let (F,A) and (K,C) be two soft weak Lagrange neutrosophic strong groupoids over $\{\langle G\cup I\rangle,*\}$. Then

1. Their extended intersection $(F,A)\cap_E (K,C)$ is not a soft weak Lagrange neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
2. Their restricted intersection $(F,A)\cap_R (K,C)$ is not a soft weak Lagrange neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft weak Lagrange neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
4. Their extended union $(F,A)\cup_E (K,C)$ is not a soft weak Lagrnage neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
5. Their restricted union $(F,A)\cup_R (K,C)$ is not a soft weak Lagrange neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft weak Lagrange neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 3.2.14. Let $\langle L\cup I\rangle$ be a neutrosophic strong groupoid and (F,A) be a soft neutrosophic groupoid over $\langle L\cup I\rangle$. Then (F,A) is called soft Lagrange free neutrosophic strong groupoid if $F(a)$ is not a Lagrange neutrosophic strong subgroupoid of $\{\langle G\cup I\rangle,*\}$ for all $a\in A$.

Theorem 3.2.14. Every soft Lagrange free neutrosophic strong groupoid over $\langle L\cup I\rangle$ is a soft neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$ but the converse is not true.

Theorem 3.2.15. If $\{\langle G\cup I\rangle,*\}$ is a Lagrange free neutrosophic strong groupoid, then (F,A) over $\{\langle G\cup I\rangle,*\}$ is also a soft Lagrange free neutrosophic strong groupoid but the converse is not true.

Remark 3.2.16. Let (F,A) and (K,C) be two soft Lagrange free neutrosophic strong groupoids over $\langle L\cup I\rangle$. Then

1. Their extended intersection $(F,A)\cap_E (K,C)$ is not a soft Lagrange free neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
2. Their restricted intersection $(F,A)\cap_R (K,C)$ is not a soft Lagrange free neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft Lagrange free neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
4. Their extended union $(F,A)\cup_E (K,C)$ is not a soft Lagrange free neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.
5. Their restricted union $(F,A)\cup_R (K,C)$ is not a soft Lagrange free neutrosophic groupoid over $\{\langle G\cup I\rangle,*\}$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft Lagrange free neutrosophic strong groupoid over $\{\langle G\cup I\rangle,*\}$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 3.2.17. (F,A) is called soft neutrosophic strong ideal over $\{\langle G\cup I\rangle,*\}$ if $F(a)$ is a neutrosophic strong ideal of $\{\langle G\cup I\rangle,*\}$, for all $a\in A$.

Theorem 3.2.18. Every soft neutrosophic strong ideal (F,A) over $\{\langle G\cup I\rangle,*\}$ is trivially a soft neutrosophic strong groupoid.

Theorem 3.2.19. Every soft neutrosophic strong ideal (F,A) over $\{\langle G\cup I\rangle,*\}$ is trivially a soft neutrosophic ideal.

Proposition 3.2.20. Let (F,A) and (K,B) be two soft neutrosophic strong ideals over $\{\langle G\cup I\rangle,*\}$. Then

1. Their extended intersection $(F,A)\cap_E (K,B)$ is soft neutrosophic strong ideal over $\{\langle G\cup I\rangle,*\}$.
2. Their restricted intersection $(F,A)\cap_R (K,B)$ is soft neutrosophic strong ideal over $\{\langle G\cup I\rangle,*\}$.
3. Their AND operation $(F,A)\wedge(K,B)$ is soft

neutrosophic strong ideal over $\{\langle G \cup I \rangle, *\}$.

Remark 3.2.21. Let (F, A) and (K, B) be two soft neutrosophic strong ideal over $\{\langle G \cup I \rangle, *\}$. Then

1. Their extended union $(F, A) \cup_E (K, B)$ is not soft neutrosophic strong ideal over $\{\langle G \cup I \rangle, *\}$.
2. Their restricted union $(F, A) \cup_R (K, B)$ is not soft neutrosophic strong ideal over $\{\langle G \cup I \rangle, *\}$.
3. Their OR operation $(F, A) \vee (K, B)$ is not soft neutrosophic strong ideal over $\{\langle G \cup I \rangle, *\}$.

One can easily proved $(1), (2),$ and (3) by the help of examples.

Theorem 3.2.22. Let (F, A) be a soft neutrosophic strong ideal over $\{\langle G \cup I \rangle, *\}$ and $\{(H_i, B_i) : \mathrm{i} \in \mathrm{J}\}$ is a non-empty family of soft neutrosophic strong ideals of (F, A). Then

1. $\bigcap_{i \in J}(H_i, B_i)$ is a soft neutrosophic strong ideal of (F, A).
2. $\bigwedge_{i \in J}(H_i, B_i)$ is a soft neutrosophic strong ideal of $\bigwedge_{i \in J}(F, A)$.

4 Soft Neutrosophic Bigroupoid and Their Properties

4.1 Soft Neutrosophic Bigroupoid

Definition 4.1.1. Let $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$ be a neutrosophic bigroupoid and (F, A) be a soft set over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$. Then (F, A) is called soft neutrosophic bigroupoid if and only if $F(a)$ is neutrosophic sub bigroupoid of $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$ for all $a \in A$.

Example 4.1.2. Let $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$ be a neutrosophic groupoid with $\mathrm{B}_N(\mathrm{G}) = G_1 \cup G_2$, where $G_1 = \{\langle Z_{10} \cup I \rangle \mid a * b = 2a + 3b (\mathrm{mod}\, 10); a, b \in \langle Z_{10} \cup I \rangle\}$ and $G_2 = \{\langle Z_4 \cup I \rangle \mid a \circ b = 2a + b (\mathrm{mod}\, 4); a, b \in \langle Z_4 \cup I \rangle\}$. Let $A = \{a_1, a_2\}$ be a set of parameters. Then (F, A) is a soft neutrosophic bigroupoid over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$, where

$$F(a_1) = \{0, 5, 5I, 5 + 5I\} \cup \{0, 2, 2I, 2 + 2I\},$$
$$F(a_2) = (Z_{10}, *) \cup \{0, 2 + 2\mathrm{I}\}.$$

Theorem 4.1.3. Let (F, A) and (H, A) be two soft neutrosophic bigroupoids over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$. Then their intersection $(F, A) \cap (H, A)$ is again a soft neutrosophic groupoid over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$.

Proof. The proof is staightforward.

Theorem 4.1.4. Let (F, A) and (H, B) be two soft neutrosophic groupoids over $\{\langle G \cup I \rangle, *\}$. If $A \cap B = \phi$, then $(F, A) \cup (H, B)$ is a soft neutrosophic groupoid over $\{\langle G \cup I \rangle, *\}$.

Proposition 4.1.5. Let (F, A) and (K, C) be two soft neutrosophic bigroupoids over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$. Then

1. Their extended intersection $(F, A) \cap_E (K, \mathrm{C})$ is a soft neutrosophic bigroupoid over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$.
2. Their restricted intersection $(F, A) \cap_R (K, \mathrm{C})$ is a soft neutrosophic bigroupoid over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$.
3. Their AND operation $(F, A) \wedge (K, \mathrm{C})$ is a soft neutrosophic bigroupoid over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$.

Remark 4.1.6. Let (F, A) and (K, C) be two soft neutrosophic biloops over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$. Then

1. Their extended union $(F, A) \cup_E (K, \mathrm{C})$ is not a soft neutrosophic bigroupoid over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$.
2. Their restricted union $(F, A) \cup_R (K, \mathrm{C})$ is not a soft neutrosophic bigroupoid over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$.
3. Their OR operation $(F, A) \vee (K, \mathrm{C})$ is not a soft neutrosophic bigroupoid over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$.

One can easily verify $(1), (2),$ and (3) by the help of examples.

Definition 4.1.7. Let (F, A) be a soft neutrosophic bigroupoid over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$. Then (F, A) is called an absolute soft neutrosophic bigroupoid over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$ if $F(a) = \{\mathrm{B}_N(\mathrm{G}), *, \circ\}$ for all $a \in A$.

Definition 4.1.8. Let (F, A) and (H, C) be two soft neutrosophic bigroupoids over $\{\mathrm{B}_N(\mathrm{G}), *, \circ\}$. Then (H, C) is called soft neutrosophic sub bigroupoid of

(F, A), if

1. $C \subseteq A$.
2. $H(a)$ is a neutrosophic sub bigroupoid of $F(a)$ for all $a \in A$.

Example 4.1.9. Let $\{B_N(G), *, \circ\}$ be a neutrosophic groupoid with $B_N(G) = G_1 \cup G_2$, where $G_1 = \{\langle Z_{10} \cup I\rangle \mid a * b = 2a + 3b (\text{mod} 10); a, b \in \langle Z_{10} \cup I\rangle\}$ and $G_2 = \{\langle Z_4 \cup I\rangle \mid a \circ b = 2a + b (\text{mod} 4); a, b \in \langle Z_4 \cup I\rangle\}$. Let $A = \{a_1, a_2\}$ be a set of parameters. Let (F, A) is a soft neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$, where

$$F(a_1) = \{0, 5, 5I, 5 + 5I\} \cup \{0, 2, 2I, 2 + 2I\},$$
$$F(a_2) = (Z_{10}, *) \cup \{0, 2 + 2I\}.$$

Let $B = \{a_1\} \subseteq A$. Then (H, B) is a soft neutrosophic sub bigroupoid of (F, A), where

$$H(a_1) = \{0, 5\} \cup \{0, 2 + 2I\}.$$

Definition 4.1.10. Let $\{B_N(G), *, \circ\}$ be a neutrosophic strong bigroupoid and (F, A) be a soft neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$. Then (F, A) is called soft Lagrange neutrosophic bigroupoid if and only if $F(a)$ is a Lagrange neutrosophic sub bigroupoid of $\{B_N(G), *, \circ\}$ for all $a \in A$.

Theorem 4.1.11. Every soft Lagrange neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$ is a soft neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$ but the converse is not true.

One can easily see the converse by the help of examples.

Theorem 4.1.12. If $\{B_N(G), *, \circ\}$ is a Lagrange neutrosophic bigroupoid, then (F, A) over $\{B_N(G), *, \circ\}$ is a soft Lagrange neutrosophic bigroupoid but the converse is not true.

Remark 4.1.13. Let (F, A) and (K, C) be two soft Lagrange neutrosophic bigroupoids over $\{B_N(G), *, \circ\}$. Then

1. Their extended intersection $(F, A) \cap_E (K, C)$ may not be a soft Lagrange neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$.
2. Their restricted intersection $(F, A) \cap_R (K, C)$ may not be a soft Lagrange neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$.
3. Their AND operation $(F, A) \wedge (K, C)$ may not be a soft Lagrange neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$.
4. Their extended union $(F, A) \cup_E (K, C)$ may not be a soft Lagrange neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$.
5. Their restricted union $(F, A) \cup_R (K, C)$ may not be a soft Lagrange neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$.
6. Their OR operation $(F, A) \vee (K, C)$ may not be a soft Lagrange neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$.

One can easily verify $(1), (2), (3), (4), (5)$ and (6) by the help of examples.

Definition 4.1.14. Let $\{B_N(G), *, \circ\}$ be a neutrosophic bigroupoid and (F, A) be a soft neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$. Then (F, A) is called soft weak Lagrange neutrosophic bigroupoid if atleast one $F(a)$ is not a Lagrange neutrosophic sub bigroupoid of $\{B_N(G), *, \circ\}$ for some $a \in A$.

Theorem 4.1.15. Every soft weak Lagrange neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$ is a soft neutrosophic groupoid over $\{B_N(G), *, \circ\}$ but the converse is not true.

Theorem 4.1.16. If $\{B_N(G), *, \circ\}$ is weak Lagrange neutrosophic bigroupoid, then (F, A) over $\{B_N(G), *, \circ\}$ is also soft weak Lagrange neutrosophic bigroupoid but the converse is not true.

Remark 4.1.17. Let (F, A) and (K, C) be two soft weak Lagrange neutrosophic bigroupoids over $\{B_N(G), *, \circ\}$. Then

1. Their extended intersection $(F, A) \cap_E (K, C)$ is not a soft weak Lagrange neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$.
2. Their restricted intersection $(F, A) \cap_R (K, C)$ is not a soft weak Lagrange neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$.
3. Their AND operation $(F, A) \wedge (K, C)$ is not a soft weak Lagrange neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$.
4. Their extended union $(F, A) \cup_E (K, C)$ is not a soft weak Lagrnage neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$.
5. Their restricted union $(F, A) \cup_R (K, C)$ is not a soft weak Lagrange neutrosophic bigroupoid over $\{B_N(G), *, \circ\}$.
6. Their OR operation $(F, A) \vee (K, C)$ is not a

soft weak Lagrange neutrosophic bigroupoid over $\{B_N(G),*,\circ\}$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 4.1.18. Let $\{B_N(G),*,\circ\}$ be a neutrosophic bigroupoid and (F,A) be a soft neutrosophic groupoid over $\{B_N(G),*,\circ\}$. Then (F,A) is called soft Lagrange free neutrosophic bigroupoid if $F(a)$ is not a Lagrange neutrosophic sub bigroupoid of $\{B_N(G),*,\circ\}$ for all $a \in A$.

Theorem 4.1.19. Every soft Lagrange free neutrosophic bigroupoid over $\{B_N(G),*,\circ\}$ is a soft neutrosophic bigroupoid over $\{B_N(G),*,\circ\}$ but the converse is not true.

Theorem 4.1.20. If $\{B_N(G),*,\circ\}$ is a Lagrange free neutrosophic bigroupoid, then (F,A) over $\{B_N(G),*,\circ\}$ is also a soft Lagrange free neutrosophic bigroupoid but the converse is not true.

Remark 4.1.21. Let (F,A) and (K,C) be two soft Lagrange free neutrosophic bigroupoids over $\{B_N(G),*,\circ\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ is not a soft Lagrange free neutrosophic bigroupoid over $\{B_N(G),*,\circ\}$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ is not a soft Lagrange free neutrosophic bigroupoid over $\{B_N(G),*,\circ\}$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft Lagrange free neutrosophic bigroupoid over $\{B_N(G),*,\circ\}$.
4. Their extended union $(F,A)\cup_E(K,C)$ is not a soft Lagrange free neutrosophic bigroupoid over $\{B_N(G),*,\circ\}$.
5. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft Lagrange free neutrosophic bigroupoid over $\{B_N(G),*,\circ\}$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft Lagrange free neutrosophic bigroupoid over $\{B_N(G),*,\circ\}$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 4.1.22. (F,A) is called soft neutrosophic biideal over $\{B_N(G),*,\circ\}$ if $F(a)$ is a neutrosophic biideal of $\{B_N(G),*,\circ\}$, for all $a \in A$.

Theorem 4.1.23. Every soft neutrosophic biideal (F,A) over $\{B_N(G),*,\circ\}$ is a soft neutrosophic bigroupoid.

Proposition 4.1.24. Let (F,A) and (K,B) be two soft neutrosophic biideals over $\{B_N(G),*,\circ\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,B)$ is soft neutrosophic biideal over $\{B_N(G),*,\circ\}$.
2. Their restricted intersection $(F,A)\cap_R(K,B)$ is soft neutrosophic biideal over $\{B_N(G),*,\circ\}$.
3. Their AND operation $(F,A)\wedge(K,B)$ is soft neutrosophic biideal over $\{B_N(G),*,\circ\}$.

Remark 4.1.25. Let (F,A) and (K,B) be two soft neutrosophic biideals over $\{B_N(G),*,\circ\}$. Then

1. Their extended union $(F,A)\cup_E(K,B)$ is not soft neutrosophic biideals over $\{B_N(G),*,\circ\}$.
2. Their restricted union $(F,A)\cup_R(K,B)$ is not soft neutrosophic biidleals over $\{B_N(G),*,\circ\}$.
3. Their OR operation $(F,A)\vee(K,B)$ is not soft neutrosophic biideals over $\{B_N(G),*,\circ\}$.

One can easily proved $(1),(2),$ and (3) by the help of examples

Theorem 4.1.26. Let (F,A) be a soft neutrosophic biideal over $\{B_N(G),*,\circ\}$ and $\{(H_i,B_i): i\in J\}$ is a non-empty family of soft neutrosophic biideals of (F,A). Then

1. $\underset{i\in J}{\cap}(H_i,B_i)$ is a soft neutrosophic biideal of (F,A).
2. $\underset{i\in J}{\wedge}(H_i,B_i)$ is a soft neutrosophic biideal of $\underset{i\in J}{\wedge}(F,A)$.

4.2 Soft Neutrosophic Strong Bigroupoid

Definition 4.2.1. Let $\{B_N(G),*,\circ\}$ be a neutrosophic bigroupoid and (F,A) be a soft set over $\{B_N(G),*,\circ\}$. Then (F,A) is called soft neutrosophic strong bigroupoid if and only if $F(a)$ is neutrosophic strong sub bigroupoid of $\{B_N(G),*,\circ\}$ for all $a \in A$.

Example 4.2.2. Let $\{B_N(G),*,\circ\}$ be a neutrosophic groupoid with $B_N(G)=G_1\cup G_2$, where
$G_1=\{\langle Z_{10}\cup I\rangle \mid a*b=2a+3b(\bmod 10); a,b\in\langle Z_{10}\cup I\rangle\}$
and
$G_2=\{\langle Z_4\cup I\rangle \mid a\circ b=2a+b(\bmod 4); a,b\in\langle Z_4\cup I\rangle\}$.
Let $A=\{a_1,a_2\}$ be a set of parameters. Then (F,A) is a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$, where

$$F(a_1)=\{0,5+5I\}\cup\{0,2+2I\},$$
$$F(a_2)=\{0,5I\}\cup\{0,2+2I\}.$$

Theorem 4.2.3. Let (F,A) and (H,A) be two soft neutrosophic strong bigroupoids over $\{B_N(G),*,\circ\}$. Then their intersection $(F,A)\cap(H,A)$ is again a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.

Proof. The proof is staightforward.

Theorem 4.2.4. Let (F,A) and (H,B) be two soft neutrosophic strong bigroupoids over $\{B_N(G),*,\circ\}$. If $A\cap B=\phi$, then $(F,A)\cup(H,B)$ is a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.

Proposition 4.2.5. Let (F,A) and (K,C) be two soft neutrosophic strong bigroupoids over $\{B_N(G),*,\circ\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ is a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ is a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
3. Their AND operation $(F,A)\wedge(K,C)$ is a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.

Remark 4.2.6. Let (F,A) and (K,C) be two soft neutrosophic strong bigroupoids over $\{B_N(G),*,\circ\}$. Then

1. Their extended union $(F,A)\cup_E(K,C)$ is not a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
2. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
3. Their OR operation $(F,A)\vee(K,C)$ is not a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.

One can easily verify $(1),(2)$, and (3) by the help of examples.

Definition 4.2.7. Let (F,A) and (H,C) be two soft neutrosophic strong bigroupoids over $\{B_N(G),*,\circ\}$. Then (H,C) is called soft neutrosophic strong sub bigroupoid of (F,A), if

1. $C\subseteq A$.
2. $H(a)$ is a neutrosophic strong sub bigroupoid of $F(a)$ for all $a\in A$.

Definition 4.2.8. Let $\{B_N(G),*,\circ\}$ be a neutrosophic strong bigroupoid and (F,A) be a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$. Then (F,A) is called soft Lagrange neutrosophic strong bigroupoid if and only if $F(a)$ is a Lagrange neutrosophic strong sub bigroupoid of $\{B_N(G),*,\circ\}$ for all $a\in A$.

Theorem 4.2.9. Every soft Lagrange neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$ is a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$ but the converse is not true.

One can easily see the converse by the help of examples.

Theorem 4.2.10. If $\{B_N(G),*,\circ\}$ is a Lagrange neutrosophic strong bigroupoid, then (F,A) over $\{B_N(G),*,\circ\}$ is a soft Lagrange neutrosophic strong bigroupoid but the converse is not true.

Remark 4.2.11. Let (F,A) and (K,C) be two soft Lagrange neutrosophic strong bigroupoids over $\{B_N(G),*,\circ\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ may not be a soft Lagrange neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ may not be a soft Lagrange neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
3. Their AND operation $(F,A)\wedge(K,C)$ may not be a soft Lagrange neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.Their extended union $(F,A)\cup_E(K,C)$ may not be a soft Lagrange neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
4. Their restricted union $(F,A)\cup_R(K,C)$ may not be a soft Lagrange neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.

5. Their OR operation $(F,A)\vee(K,C)$ may not be a soft Lagrange neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 4.2.12. Let $\{B_N(G),*,\circ\}$ be a neutrosophic strong bigroupoid and (F,A) be a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$. Then (F,A) is called soft weak Lagrange neutrosophic strong bigroupoid if atleast one $F(a)$ is not a Lagrange neutrosophic strong sub bigroupoid of $\{B_N(G),*,\circ\}$ for some $a\in A$.

Theorem 4.2.13. Every soft weak Lagrange neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$ is a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$ but the converse is not true.

Theorem 4.2.14. If $\{B_N(G),*,\circ\}$ is weak Lagrange neutrosophic strong bigroupoid, then (F,A) over $\{B_N(G),*,\circ\}$ is also soft weak Lagrange neutrosophic strong bigroupoid but the converse is not true.

Remark 4.2.15. Let (F,A) and (K,C) be two soft weak Lagrange neutrosophic strong bigroupoids over $\{B_N(G),*,\circ\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ is not a soft weak Lagrange neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ is not a soft weak Lagrange neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft weak Lagrange neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
4. Their extended union $(F,A)\cup_E(K,C)$ is not a soft weak Lagrnage neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
5. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft weak Lagrange neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft weak Lagrange neutrosophi strong bigroupoid over $\{B_N(G),*,\circ\}$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 4.2.16. Let $\{B_N(G),*,\circ\}$ be a neutrosophic strong bigroupoid and (F,A) be a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$. Then (F,A) is called soft Lagrange free neutrosophic strong bigroupoid if $F(a)$ is not a Lagrange neutrosophic strong sub bigroupoid of $\{B_N(G),*,\circ\}$ for all $a\in A$.

Theorem 4.2.17. Every soft Lagrange free neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$ is a soft neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$ but the converse is not true.

Theorem 4.2.18. If $\{B_N(G),*,\circ\}$ is a Lagrange free neutrosophic strong bigroupoid, then (F,A) over $\{B_N(G),*,\circ\}$ is also a soft Lagrange free neutrosophic strong bigroupoid but the converse is not true.

Remark 4.2.19. Let (F,A) and (K,C) be two soft Lagrange free neutrosophic strong bigroupoids over $\{B_N(G),*,\circ\}$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ is not a soft Lagrange free neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ is not a soft Lagrange free neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft Lagrange free neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
4. Their extended union $(F,A)\cup_E(K,C)$ is not a soft Lagrange free neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
5. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft Lagrange free neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft Lagrange free neutrosophic strong bigroupoid over $\{B_N(G),*,\circ\}$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 4.2.20. (F,A) is called soft neutrosophic strong biideal over $\{B_N(G),*,\circ\}$ if $F(a)$ is a neutrosophic strong biideal of $\{B_N(G),*,\circ\}$, for all $a \in A$.

Theorem 4.2.21. Every soft neutrosophic strong biideal (F,A) over $\{B_N(G),*,\circ\}$ is a soft neutrosophic strong bigroupoid.

Proposition 4.2.22. Let (F,A) and (K,B) be two soft neutrosophic strong biideals over $\{B_N(G),*,\circ\}$. Then

1. Their extended intersection $(F,A)\cap_E (K,B)$ is soft neutrosophic strong biideal over $\{B_N(G),*,\circ\}$.
2. Their restricted intersection $(F,A)\cap_R (K,B)$ is soft neutrosophic strong biideal over $\{B_N(G),*,\circ\}$.
3. Their AND operation $(F,A)\wedge(K,B)$ is soft neutrosophic strong biideal over $\{B_N(G),*,\circ\}$.

Remark 4.2.23. Let (F,A) and (K,B) be two soft neutrosophic strong biideals over $\{B_N(G),*,\circ\}$. Then

1. Their extended union $(F,A)\cup_E (K,B)$ is not soft neutrosophic strong biideals over $\{B_N(G),*,\circ\}$.
2. Their restricted union $(F,A)\cup_R (K,B)$ is not soft neutrosophic strong biidleals over $\{B_N(G),*,\circ\}$.
3. Their OR operation $(F,A)\vee(K,B)$ is not soft neutrosophic strong biideals over $\{B_N(G),*,\circ\}$.

One can easily proved $(1),(2),$ and (3) by the help of examples

Theorem 4.2.24. Let (F,A) be a soft neutrosophic strong biideal over $\{B_N(G),*,\circ\}$ and $\{(H_i,B_i): i\in J\}$ is a non-empty family of soft neutrosophic strong biideals of (F,A). Then

1. $\underset{i\in J}{\cap}(H_i,B_i)$ is a soft neutrosophic strong biideal of (F,A).
2. $\underset{i\in J}{\wedge}(H_i,B_i)$ is a soft neutrosophic strong biideal of $\underset{i\in J}{\wedge}(F,A)$.

5 Soft Neutrosophic N-groupoid and Their Properties

5.1 Soft Neutrosophic N-groupoid

Definition 5.1.1. Let $N(G)=\{G_1\cup G_2\cup\ldots\cup G_N,*_1,*_2,\ldots,*_N\}$ be a neutrosophic N-groupoid and (F,A) be a soft set over $N(G)=\{G_1\cup G_2\cup\ldots\cup G_N,*_1,*_2,\ldots,*_N\}$. Then (F,A) is called soft neutrosophic N-groupoid if and only if $F(a)$ is neutrosophic sub N-groupoid of $N(G)=\{G_1\cup G_2\cup\ldots\cup G_N,*_1,*_2,\ldots,*_N\}$ for all $a\in A$.

Example 5.1.2. Let $N(G)=\{G_1\cup G_2\cup G_3,*_1,*_2,*_3\}$ be a neutrosophic 3-groupoid, where $G_1=\{\langle Z_{10}\cup I\rangle \mid a*b=2a+3b(\text{mod}10); a,b\in\langle Z_{10}\cup I\rangle\}$, $G_2=\{\langle Z_4\cup I\rangle \mid a\circ b=2a+b(\text{mod}4); a,b\in\langle Z_4\cup I\rangle\}$ and $G_3=\{\langle Z_{12}\cup I\rangle \mid a*b=8a+4b(\text{mod}12); a,b\in\langle Z_{12}\cup I\rangle\}$.

Let $A=\{a_1,a_2\}$ be a set of parameters. Then (F,A) is a soft neutrosophic N-groupoid over $N(G)=\{G_1\cup G_2\cup G_3,*_1,*_2,*_3\}$, where $F(a_1)=\{0,5,5I,5+5I\}\cup\{0,2,2I,2+2I\}\cup\{0,2\}$,

$$F(a_2)=(Z_{10},*)\cup\{0,2+2I\}\cup\{0,2I\}.$$

Theorem 5.1.3. Let (F,A) and (H,A) be two soft neutrosophic N-groupoids over $N(G)$. Then their intersection $(F,A)\cap(H,A)$ is again a soft neutrosophic N-groupoid over $N(G)$.

Theorem 5.1.4. Let (F,A) and (H,B) be two soft neutrosophic N-groupoids over $N(G)$. If $A\cap B=\phi$, then $(F,A)\cup(H,B)$ is a soft neutrosophic N-groupoid over $N(G)$.

Proposition 5.1.5. Let (F,A) and (K,C) be two soft neutrosophic N-groupoids over $N(G)$. Then

1. Their extended intersection $(F,A)\cap_E (K,C)$ is a soft neutrosophic N-groupoid over $N(G)$.
2. Their restricted intersection $(F,A)\cap_R (K,C)$ is a soft neutrosophic N-groupoid over $N(G)$.
3. Their AND operation $(F,A)\wedge(K,C)$ is a soft neutrosophic N-groupoid over $N(G)$.

Remark 5.1.4. Let (F,A) and (K,C) be two soft neutrosophic N-groupoids over $N(G)$. Then

1. Their extended union $(F,A)\cup_E (K,C)$ is not a soft neutrosophic N-groupoid over $N(G)$.
2. Their restricted union $(F,A)\cup_R (K,C)$ is not a soft neutrosophic N-groupoid over $N(G)$.
3. Their OR operation $(F,A)\vee(K,C)$ is not a soft neutrosophic N-groupoid over $N(G)$.

One can easily verify $(1),(2),$ and (3) by the help of examples.

Definition 5.1.5. Let (F,A) be a soft neutrosophic N-groupoid over $N(G)$. Then (F,A) is called an absolute soft neutrosophic N-groupoid over $N(G)$ if $F(a)=N(G)$ for all $a\in A$.

Definition 5.1.6. Let (F,A) and (H,C) be two soft neutrosophic N-groupoids over $N(G)$. Then (H,C) is called soft neutrosophic sub N-groupoid of (F,A), if

1. $C\subseteq A$.
2. $H(a)$ is a neutrosophic sub bigroupoid of $F(a)$ for all $a\in A$.

Example 5.1.7. Let $N(G)=\{G_1\cup G_2\cup G_3,*_1,*_2,*_3\}$ be a neutrosophic 3-groupoid, where
$G_1=\{\langle Z_{10}\cup I\rangle \mid a*b=2a+3b(\bmod 10);a,b\in\langle Z_{10}\cup I\rangle\}$
,
$G_2=\{\langle Z_4\cup I\rangle \mid a\circ b=2a+b(\bmod 4);a,b\in\langle Z_4\cup I\rangle\}$
and
$G_3=\{\langle Z_{12}\cup I\rangle \mid a*b=8a+4b(\bmod 12);a,b\in\langle Z_{12}\cup I\rangle\}$.

Let $A=\{a_1,a_2\}$ be a set of parameters. Then (F,A) is a soft neutrosophic N-groupoid over $N(G)=\{G_1\cup G_2\cup G_3,*_1,*_2,*_3\}$, where

$$F(a_1)=\{0,5,5I,5+5I\}\cup\{0,2,2I,2+2I\}\cup\{0,2\},$$

$$F(a_2)=(Z_{10},*)\cup\{0,2+2I\}\cup\{0,2I\}.$$

Let $B=\{a_1\}\subseteq A$. Then (H,B) is a soft neutrosophic sub N-groupoid of (F,A), where

$$H(a_1)=\{0,5\}\cup\{0,2+2I\}\cup\{0,2\}.$$

Definition 5.1.8. Let $N(G)$ be a neutrosophic N-groupoid and (F,A) be a soft neutrosophic N-groupoid over $N(G)$. Then (F,A) is called soft Lagrange neutrosophic N-groupoid if and only if $F(a)$ is a Lagrange neutrosophic sub N-groupoid of $N(G)$ for all $a\in A$.

Theorem 5.1.9. Every soft Lagrange neutrosophic N-groupoid over $N(G)$ is a soft neutrosophic N-groupoid over $N(G)$ but the converse may not be true.

One can easily see the converse by the help of examples.

Theorem 5.1.10. If $N(G)$ is a Lagrange neutrosophic N-groupoid, then (F,A) over $N(G)$ is a soft Lagrange neutrosophic N-groupoid but the converse is not true.

Remark 5.1.11. Let (F,A) and (K,C) be two soft Lagrange neutrosophic N-groupoids over $N(G)$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ may not be a soft Lagrange neutrosophic N-groupoid over $N(G)$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ may not be a soft Lagrange neutrosophic N-groupoid over $N(G)$.
3. Their AND operation $(F,A)\wedge(K,C)$ may not be a soft Lagrange neutrosophic N-groupoid over $N(G)$.
4. Their extended union $(F,A)\cup_E(K,C)$ may not be a soft Lagrange neutrosophic N-groupoid over $N(G)$.
7. Their restricted union $(F,A)\cup_R(K,C)$ may not be a soft Lagrange neutrosophic N-groupoid over $N(G)$.
8. Their OR operation $(F,A)\vee(K,C)$ may not be a soft Lagrange neutrosophic N-groupoid over $N(G)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 5.1.12. Let $N(G)$ be a neutrosophic N-groupoid and (F,A) be a soft neutrosophic N-groupoid over $N(G)$. Then (F,A) is called soft weak Lagrange neutrosophic N-groupoid if atleast one $F(a)$ is not a Lagrange neutrosophic sub N-groupoid of $N(G)$ for some $a\in A$.

Theorem 5.1.13. Every soft weak Lagrange neutrosophic N-groupoid over $N(G)$ is a soft neutrosophic N-groupoid over $N(G)$ but the converse is not true.

Theorem 5.1.14. If $N(G)$ is weak Lagrange neutrosophic N-groupoid, then (F,A) over $N(G)$ is also a soft weak Lagrange neutrosophic bigroupoid but the converse is not true.

Remark 5.1.15. Let (F,A) and (K,C) be two soft weak Lagrange neutrosophic N-groupoids over $N(G)$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ may not be a soft weak Lagrange neutrosophic N-groupoid over $N(G)$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ may not be a soft weak Lagrange neutrosophic N-groupoid over $N(G)$.
3. Their AND operation $(F,A)\wedge(K,C)$ may not be a soft weak Lagrange neutrosophic N-groupoid over $N(G)$.

4. Their extended union $(F,A)\cup_E(K,C)$ may not be a soft weak Lagrnage neutrosophic N-groupoid over $N(G)$.
5. Their restricted union $(F,A)\cup_R(K,C)$ may not be a soft weak Lagrange neutrosophic N-groupoid over $N(G)$.
6. Their OR operation $(F,A)\vee(K,C)$ may not be a soft weak Lagrange neutrosophic N-groupoid over $N(G)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 5.1.16. Let $N(G)$ be a neutrosophic N-groupoid and (F,A) be a soft neutrosophic N-groupoid over $N(G)$. Then (F,A) is called soft Lagrange free neutrosophic N-groupoid if $F(a)$ is not a Lagrange neutrosophic sub N-groupoid of $N(G)$ for all $a\in A$.

Theorem 5.1.17. Every soft Lagrange free neutrosophic N-groupoid over $N(G)$ is a soft neutrosophic N-groupoid over $N(G)$ but the converse is not true.

Theorem 5.1.18. If $N(G)$ is a Lagrange free neutrosophic N-groupoid, then (F,A) over $N(G)$ is also a soft Lagrange free neutrosophic N-groupoid but the converse is not true.

Remark 5.1.19. Let (F,A) and (K,C) be two soft Lagrange free neutrosophic N-groupoids over $N(G)$. Then

1. Their extended intersection $(F,A)\cap_E(K,C)$ is not a soft Lagrange free neutrosophic N-groupoid over $N(G)$.
2. Their restricted intersection $(F,A)\cap_R(K,C)$ is not a soft Lagrange free neutrosophic N-groupoid over $N(G)$.
3. Their AND operation $(F,A)\wedge(K,C)$ is not a soft Lagrange free neutrosophic N-groupoid over $N(G)$.
4. Their extended union $(F,A)\cup_E(K,C)$ is not a soft Lagrange free neutrosophic N-groupoid over $N(G)$.
5. Their restricted union $(F,A)\cup_R(K,C)$ is not a soft Lagrange free neutrosophic N-groupoid over $N(G)$.
6. Their OR operation $(F,A)\vee(K,C)$ is not a soft Lagrange free neutrosophic N-groupoid over $N(G)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 5.1.20. (F,A) is called soft neutrosophic N-ideal over $N(G)$ if and only if $F(a)$ is a neutrosophic N-ideal of $N(G)$, for all $a\in A$.

Theorem 5.1.21. Every soft neutrosophic N-ideal (F,A) over $N(G)$ is a soft neutrosophic N-groupoid.

Proposition 5.1.22. Let (F,A) and (K,B) be two soft neutrosophic N-ideals over $N(G)$. Then

1. Their extended intersection $(F,A)\cap_E(K,B)$ is soft neutrosophic N-ideal over $N(G)$.
2. Their restricted intersection $(F,A)\cap_R(K,B)$ is soft neutrosophic N-ideal over $N(G)$.
3. Their AND operation $(F,A)\wedge(K,B)$ is soft neutrosophic N-ideal over $N(G)$.

Remark 5.1.23. Let (F,A) and (K,B) be two soft neutrosophic N-ideals over $N(G)$. Then

1. Their extended union $(F,A)\cup_E(K,B)$ is not a soft neutrosophic N-ideal over $N(G)$.
2. Their restricted union $(F,A)\cup_R(K,B)$ is not a soft neutrosophic N-idleal over $N(G)$.
3. Their OR operation $(F,A)\vee(K,B)$ is not a soft neutrosophic N-ideal over $N(G)$.

One can easily proved $(1),(2),$ and (3) by the help of examples

Theorem 5.1.24. Let (F,A) be a soft neutrosophic N-ideal over $N(G)$ and $\{(H_i,B_i):\mathrm{i}\in \mathrm{J}\}$ be a non-empty family of soft neutrosophic N-ideals of (F,A). Then

1. $\underset{i\in J}{\cap}(H_i,B_i)$ is a soft neutrosophic N-ideal of (F,A).
2. $\underset{i\in J}{\wedge}(H_i,B_i)$ is a soft neutrosophic N-ideal of $\underset{i\in J}{\wedge}(F,A)$.

5.2 Soft Neutrosophic Strong N-groupoid

Definition 5.2.1. Let $\mathrm{N(G)}=\{G_1\cup G_2\cup...\cup \mathrm{G}_N,*_1,*_2,...,*_N\}$ be a neutrosophic N-groupoid and (F,A) be a soft set over $\mathrm{N(G)}=\{G_1\cup G_2\cup...\cup \mathrm{G}_N,*_1,*_2,...,*_N\}$. Then (F,A) is called soft neutrosophic strong N-groupoid if and only if $F(a)$ is neutrosophic strong sub N-groupoid of $\mathrm{N(G)}=\{G_1\cup G_2\cup...\cup \mathrm{G}_N,*_1,*_2,...,*_N\}$ for all $a\in A$.

Example 5.2.2. Let $N(G)=\{G_1 \cup G_2 \cup G_3, *_1, *_2, *_3\}$ be a neutrosophic 3-groupoid, where
$G_1=\{\langle Z_{10} \cup I\rangle \mid a*b=2a+3b(\bmod 10); a,b \in \langle Z_{10} \cup I\rangle\}$,
$G_2=\{\langle Z_4 \cup I\rangle \mid a \circ b=2a+b(\bmod 4); a,b \in \langle Z_4 \cup I\rangle\}$ and
$G_3=\{\langle Z_{12} \cup I\rangle \mid a*b=8a+4b(\bmod 12); a,b \in \langle Z_{12} \cup I\rangle\}$.
Let $A=\{a_1,a_2\}$ be a set of parameters. Then (F,A) is a soft neutrosophic N-groupoid over $N(G)=\{G_1 \cup G_2 \cup G_3, *_1, *_2, *_3\}$, where

$$F(a_1)=\{0,5I\} \cup \{0,2I\} \cup \{0,2I\},$$

$$F(a_2)=\{0,5+5I\} \cup \{0,2+2I\} \cup \{0,2+2I\}.$$

Theorem 5.2.3. Let (F,A) and (H,A) be two soft neutrosophic strong N-groupoids over $N(G)$. Then their intersection $(F,A) \cap (H,A)$ is again a soft neutrosophic strong N-groupoid over $N(G)$.

Theorem 5.2.4. Let (F,A) and (H,B) be two soft neutrosophic strong N-groupoids over $N(G)$. If $A \cap B=\phi$, then $(F,A) \cup (H,B)$ is a soft neutrosophic strong N-groupoid over $N(G)$.

Theorem 5.2.5. If $N(G)$ is a neutrosophic strong N-groupoid, then (F,A) over $N(G)$ is also a soft neutrosophic strong N-groupoid.

Proposition 5.2.6. Let (F,A) and (K,C) be two soft neutrosophic strong N-groupoids over $N(G)$. Then

1. Their extended intersection $(F,A) \cap_E (K,C)$ is a soft neutrosophic strong N-groupoid over $N(G)$.
2. Their restricted intersection $(F,A) \cap_R (K,C)$ is a soft neutrosophic strong N-groupoid over $N(G)$.
3. Their AND operation $(F,A) \wedge (K,C)$ is a soft neutrosophic strong N-groupoid over $N(G)$.

Remark 5.2.7. Let (F,A) and (K,C) be two soft neutrosophic strong N-groupoids over $N(G)$. Then

1. Their extended union $(F,A) \cup_E (K,C)$ is not a soft neutrosophic strong N-groupoid over $N(G)$.
2. Their restricted union $(F,A) \cup_R (K,C)$ is not a soft neutrosophic strong N-groupoid over $N(G)$.
3. Their OR operation $(F,A) \vee (K,C)$ is not a soft neutrosophic strong N-groupoid over $N(G)$.

One can easily verify $(1),(2)$, and (3) by the help of examples.

Definition 5.2.8. Let (F,A) and (H,C) be two soft neutrosophic strong N-groupoids over $N(G)$. Then (H,C) is called soft neutrosophic strong sub N-groupoid of (F,A), if

1. $C \subseteq A$.
2. $H(a)$ is a neutrosophic sub bigroupoid of $F(a)$ for all $a \in A$.

Definition 5.2.9. Let $N(G)$ be a neutrosophic strong N-groupoid and (F,A) be a soft neutrosophic strong N-groupoid over $N(G)$. Then (F,A) is called soft Lagrange neutrosophic strong N-groupoid if and only if $F(a)$ is a Lagrange neutrosophic sub N-groupoid of $N(G)$ for all $a \in A$.

Theorem 5.2.10. Every soft Lagrange neutrosophic strong N-groupoid over $N(G)$ is a soft neutrosophic N-groupoid over $N(G)$ but the converse may not be true.
One can easily see the converse by the help of examples.

Theorem 5.2.11. Every soft Lagrange neutrosophic strong N-groupoid over $N(G)$ is a soft Lagrange neutrosophic N-groupoid over $N(G)$ but the converse may not be true.

Theorem 5.2.12. If $N(G)$ is a Lagrange neutrosophic strong N-groupoid, then (F,A) over $N(G)$ is a soft Lagrange neutrosophic strong N-groupoid but the converse is not true.

Remark 5.2.13. Let (F,A) and (K,C) be two soft Lagrange neutrosophic strong N-groupoids over $N(G)$. Then

1. Their extended intersection $(F,A) \cap_E (K,C)$ may not be a soft Lagrange neutrosophic strong N-groupoid over $N(G)$.
2. Their restricted intersection $(F,A) \cap_R (K,C)$ may not be a soft Lagrange neutrosophic strong N-groupoid over $N(G)$.
3. Their AND operation $(F,A) \wedge (K,C)$ may not be a soft Lagrange neutrosophic strong N-groupoid over $N(G)$.
4. Their extended union $(F,A) \cup_E (K,C)$ may not be a soft Lagrange neutrosophic strong N-

groupoid over $N(G)$.

5. Their restricted union $(F, A) \cup_R (K, C)$ may not be a soft Lagrange neutrosophic strong N-groupoid over $N(G)$.
6. Their OR operation $(F, A) \vee (K, C)$ may not be a soft Lagrange neutrosophic strong N-groupoid over $N(G)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 5.2.14. Let $N(G)$ be a neutrosophic strong N-groupoid and (F, A) be a soft neutrosophic strong N-groupoid over $N(G)$. Then (F, A) is called soft weak Lagrange neutrosophic strong N-groupoid if atleast one $F(a)$ is not a Lagrange neutrosophic sub N-groupoid of $N(G)$ for some $a \in A$.

Theorem 5.2.15. Every soft weak Lagrange neutrosophic strong N-groupoid over $N(G)$ is a soft neutrosophic N-groupoid over $N(G)$ but the converse is not true.

Theorem 5.2.16. Every soft weak Lagrange neutrosophic strong N-groupoid over $N(G)$ is a soft weak Lagrange neutrosophic N-groupoid over $N(G)$ but the converse is not true.

Remark 5.2.17. Let (F, A) and (K, C) be two soft weak Lagrange neutrosophic strong N-groupoids over $N(G)$. Then

1. Their extended intersection $(F, A) \cap_E (K, C)$ may not be a soft weak Lagrange neutrosophic strong N-groupoid over $N(G)$.
2. Their restricted intersection $(F, A) \cap_R (K, C)$ may not be a soft weak Lagrange neutrosophic strong N-groupoid over $N(G)$.
3. Their AND operation $(F, A) \wedge (K, C)$ may not be a soft weak Lagrange neutrosophic strong N-groupoid over $N(G)$.
4. Their extended union $(F, A) \cup_E (K, C)$ may not be a soft weak Lagrnage neutrosophic strong N-groupoid over $N(G)$.
5. Their restricted union $(F, A) \cup_R (K, C)$ may not be a soft weak Lagrange neutrosophic strong N-groupoid over $N(G)$.
6. Their OR operation $(F, A) \vee (K, C)$ may not be a soft weak Lagrange neutrosophic strong N-groupoid over $N(G)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 5.2.18. Let $N(G)$ be a neutrosophic strong N-groupoid and (F, A) be a soft neutrosophic strong N-groupoid over $N(G)$. Then (F, A) is called soft Lagrange free neutrosophic strong N-groupoid if $F(a)$ is not a Lagrange neutrosophic sub N-groupoid of $N(G)$ for all $a \in A$.

Theorem 5.2.19. Every soft Lagrange free neutrosophic strong N-groupoid over $N(G)$ is a soft neutrosophic N-groupoid over $N(G)$ but the converse is not true.

Theorem 5.2.20. Every soft Lagrange free neutrosophic strong N-groupoid over $N(G)$ is a soft Lagrange neutrosophic N-groupoid over $N(G)$ but the converse is not true.

Theorem 5.2.21. If $N(G)$ is a Lagrange free neutrosophic strong N-groupoid, then (F, A) over $N(G)$ is also a soft Lagrange free neutrosophic strong N-groupoid but the converse is not true.

Remark 5.2.22. Let (F, A) and (K, C) be two soft Lagrange free neutrosophic N-groupoids over $N(G)$. Then

1. Their extended intersection $(F, A) \cap_E (K, C)$ is not a soft Lagrange free neutrosophic strong N-groupoid over $N(G)$.
2. Their restricted intersection $(F, A) \cap_R (K, C)$ is not a soft Lagrange free neutrosophic strong N-groupoid over $N(G)$.
3. Their AND operation $(F, A) \wedge (K, C)$ is not a soft Lagrange free neutrosophic strong N-groupoid over $N(G)$.
4. Their extended union $(F, A) \cup_E (K, C)$ is not a soft Lagrange free neutrosophic strong N-groupoid over $N(G)$.
5. Their restricted union $(F, A) \cup_R (K, C)$ is not a soft Lagrange free neutrosophic strong N-groupoid over $N(G)$.
6. Their OR operation $(F, A) \vee (K, C)$ is not a soft Lagrange free neutrosophic stong N-groupoid over $N(G)$.

One can easily verify $(1),(2),(3),(4),(5)$ and (6) by the help of examples.

Definition 5.2.23. (F, A) is called soft neutrosophic strong N-ideal over $N(G)$ if and only if $F(a)$ is a neutrosophic strong N-ideal of $N(G)$, for all $a \in A$.

Theorem 5.2.24. Every soft neutrosophic strong N-ideal

(F, A) over $N(G)$ is a soft neutrosophic N-groupoid.

Theorem 5.2.25. Every soft neutrosophic strong N-ideal (F, A) over $N(G)$ is a soft neutrosophic N-ideal but the converse is not true.

Proposition 15. Let (F, A) and (K, B) be two soft neutrosophic strong N-ideals over $N(G)$. Then

1. Their extended intersection $(F, A) \cap_E (K, B)$ is soft neutrosophic strong N-ideal over $N(G)$.
2. Their restricted intersection $(F, A) \cap_R (K, B)$ is soft neutrosophic strong N-ideal over $N(G)$.
3. Their AND operation $(F, A) \wedge (K, B)$ is soft neutrosophicstrong N-ideal over $N(G)$.

Remark 5.2.26. Let (F, A) and (K, B) be two soft neutrosophic strong N-ideals over $N(G)$. Then

1. Their extended union $(F, A) \cup_E (K, B)$ is not a soft neutrosophic strong N-ideal over $N(G)$.
2. Their restricted union $(F, A) \cup_R (K, B)$ is not a soft neutrosophic strong N-idleal over $N(G)$.
3. Their OR operation $(F, A) \vee (K, B)$ is not a soft neutrosophic strong N-ideal over $N(G)$.

One can easily proved $(1),(2),$ and (3) by the help of examples

Theorem 5.2.27. Let (F, A) be a soft neutrosophic strong N-ideal over $N(G)$ and $\{(H_i, B_i) : \mathrm{i} \in \mathrm{J}\}$ be a non-empty family of soft neutrosophic strong N-ideals of (F, A). Then

1. $\underset{i \in J}{\cap}(H_i, B_i)$ is a soft neutrosophic strong N-ideal of (F, A).
2. $\underset{i \in J}{\wedge}(H_i, B_i)$ is a soft neutrosophic strong N-ideal of $\underset{i \in J}{\wedge}(F, A)$.

Conclusion

This paper is an extension of neutrosphic groupoids to soft neutrosophic groupoids. We also extend neutrosophic bigroupoid, neutrosophic N -groupoid to soft neutrosoph ic bigroupoid, and soft neutrosophic N -groupoid. Their related properties and results are explained with many illustrative examples. The notions related with strong part of neutrosophy also established within soft neutrosophic groupoids.

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Interval Neutrosophic Rough Set

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Abstract: This Paper combines interval- valued neutrouphic sets and rough sets. It studies roughness in interval- valued neutrosophic sets and some of its properties. Finally we propose a Hamming distance between lower and upper approximations of interval valued neutrosophic sets.

Keywords: interval valued neutrosophic sets, rough sets, interval valued neutrosophic sets.

1.Introduction

Neutrosophic set (NS for short), a part of neutrosophy introduced by Smarandache [1] as a new branch of philosophy, is a mathematical tool dealing with problems involving imprecise, indeterminacy and inconsistent knowledge. Contrary to fuzzy sets and intuitionistic fuzzy sets, a neutrosophic set consists of three basic membership functions independently of each other, which are truth, indeterminacy and falsity. This theory has been well developed in both theories and applications. After the pioneering work of Smarandache, In 2005, Wang [2] introduced the notion of interval neutrosophic sets (INS for short) which is another extension of neutrosophic sets. INS can be described by a membership interval, a non-membership interval and indeterminate interval, thus the interval neutrosophic (INS) has the virtue of complementing NS, which is more flexible and practical than neutrosophic set, and Interval Neutrosophic Set (INS) provides a more reasonable mathematical framework to deal with indeterminate and inconsistent information. The interval neutrosophic set generalize, the classical set ,fuzzy set [3] , the interval valued fuzzy set [4], intuitionistic fuzzy set [5] , interval valued intuitionstic fuzzy set [6] and so on. Many scholars have performed studies on neutrosophic sets , interval neutrosophic sets and their properties [7,8,9,10,11,12,13]. Interval neutrosophic sets have also been widely applied to many fields [14,15,16,17,18,19].

The rough set theory was introduced by Pawlak [20] in 1982, which is a technique for managing the uncertainty and imperfection, can analyze incomplete information effectively. Therefore, many models have been built upon different aspect, i.e, univers, relations, object, operators by many scholars [21,22,23,24,25,26] such as rough fuzzy sets, fuzzy rough sets, generalized fuzzy rough, rough intuitionistic fuzzy set. intuitionistic fuzzy rough sets [27]. It has been successfully applied in many fields such as attribute reduction [28,29,30,31], feature selection [32,33,34], rule extraction [35,36,37,38] and so on. The rough sets theory approximates any subset of objects of the universe by two sets, called the lower and upper approximations. It focuses on the ambiguity caused by the limited discernibility of objects in the universe of discourse.

More recently, S.Broumi et al [39] combined neutrosophic sets with rough sets in a new hybrid mathematical structure called “rough neutrosophic sets” handling incomplete and indeterminate information . The concept of rough neutrosophic sets generalizes fuzzy rough sets and intuitionistic fuzzy rough sets. Based on the equivalence relation on the universe of discourse, A.Mukherjee et al [40] introduced lower and upper approximation of interval valued intuitionistic fuzzy set in Pawlak’s approximation space . Motivated by this ,we extend the interval intuitionistic fuzzy lower and upper approximations to the case of interval valued neutrosophic set. The concept of interval valued neutrosophic rough set is introduced by coupling both interval neutrosophic sets and rough sets.

The organization of this paper is as follow : In section 2, we briefly present some basic definitions and preliminary results are given which will be used in the rest of the paper. In section 3 , basic concept of rough approximation of an interval valued neutrosophic sets and their properties are presented. In section 4, Hamming distance between lower approximation and upper approximation of interval neutrosophic set is introduced, Finally, we concludes the paper.

2.Preliminaries

Throughout this paper, We now recall some basic notions of neutrosophic sets , interval valued neutrosophic sets , rough set theory and intuitionistic fuzzy rough sets. More can found in ref [1, 2,20,27].

Definition 1 [1]

Let U be an universe of discourse then the neutrosophic set A is an object having the form A= {< x: $\mu_{A(x)}$, $\nu_{A(x)}$, $\omega_{A(x)}$ >,x ∈ U}, where the functions μ, ν, ω : $U \rightarrow]^-0,1^+[$ define respectively the degree of membership , the degree of indeterminacy, and the degree of non-membership of the element x ∈ X to the set A with the condition.

$$^-0 \leq \mu_{A(x)} + \nu_{A(x)} + \omega_{A(x)} \leq 3^+. \quad (1)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$.so instead of $]^-0,1^+[$ we need to take the interval [0,1] for technical applications, because $]^-0,1^+[$will be difficult to apply in the real applications such as in scientific and engineering problems.

Definition 2 [2]

Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS) A in X is characterized by truth-membership function $\mu_A(x)$, indeteminacy-membership function $\nu_A(x)$ and falsity-membership function $\omega_A(x)$. For each point x in X, we have that $\mu_A(x)$, $\nu_A(x)$, $\omega_A(x) \in [0,1]$.

For two IVNS, A= {<x , $[\mu_A^L(x), \mu_A^U(x)]$, $[\nu_A^L(x), \nu_A^U(x)]$, $[\omega_A^L(x), \omega_A^U(x)]$ > | x ∈ X } (2)

And B= {<x , $[\mu_B^L(x), \mu_B^U(x)]$, $[\nu_B^L(x), \nu_B^U(x)]$, $[\omega_B^L(x), \omega_B^U(x)]$> | x ∈ X } the two relations are defined as follows:

(1)$A \subseteq B$if and only if $\mu_A^L(x) \leq \mu_B^L(x)$,$\mu_A^U(x) \leq \mu_B^U(x)$, $\nu_A^L(x) \geq \nu_B^L(x)$,$\omega_A^U(x) \geq \omega_B^U(x)$, $\omega_A^L(x) \geq \omega_B^L(x)$,$\omega_A^U(x) \geq \omega_B^U(x)$

(2)$A = B$ if and only if , $\mu_A(x) = \mu_B(x)$,$\nu_A(x) = \nu_B(x)$,$\omega_A(x) = \omega_B(x)$ for any x ∈ X

The complement of A_{IVNS} is denoted by A_{IVNS}^o and is defined by

A^o={ <x , $[\omega_A^L(x), \omega_A^U(x)]$> , $[1-\nu_A^U(x), 1-\nu_A^L(x)]$, $[\mu_A^L(x), \mu_A^U(x)]$ | x ∈ X }

A∩B ={ <x , [min($\mu_A^L(x)$,$\mu_B^L(x)$), min($\mu_A^U(x)$,$\mu_B^U(x)$)], [max($\nu_A^L(x)$,$\nu_B^L(x)$), max($\nu_A^U(x)$,$\nu_B^U(x)$], [max($\omega_A^L(x)$,$\omega_B^L(x)$), max($\omega_A^U(x)$,$\omega_B^U(x)$)] >: x ∈ X }

A∪B ={ <x , [max($\mu_A^L(x)$,$\mu_B^L(x)$), max($\mu_A^U(x)$,$\mu_B^U(x)$)], [min($\nu_A^L(x)$,$\nu_B^L(x)$), min($\nu_A^U(x)$,$\nu_B^U(x)$], [min($\omega_A^L(x)$,$\omega_B^L(x)$), min($\omega_A^U(x)$,$\omega_B^U(x)$)] >: x ∈ X }

0_N = {<x, [0, 0] ,[1 , 1], [1 ,1] >| x ∈ X}, denote the neutrosophic empty set ϕ

1_N = {<x, [0, 0] ,[0 , 0], [1 ,1] >| x ∈ X}, denote the neutrosophic universe set U

As an illustration, let us consider the following example.

Example 1. Assume that the universe of discourse U={x1, x2, x3}, where x1characterizes the capability, x2characterizes the trustworthiness and x3 indicates the prices of the objects. It may be further assumed that the values of x1, x2 and x3 are in [0, 1] and they are obtained from some questionnaires of some experts. The experts may impose their opinion in three components viz. the degree of goodness, the degree of indeterminacy and that of poorness to explain the characteristics of the objects. Suppose A is an interval neutrosophic set (INS) of U, such that,

A = {< x1,[0.3 0.4],[0.5 0.6],[0.4 0.5] >,< x2, ,[0.1 0.2],[0.3 0.4],[0.6 0.7]>,< x3, [0.2 0.4],[0.4 0.5],[0.4 0.6] >}, where the degree of goodness of capability is 0.3, degree of indeterminacy of capability is 0.5 and degree of falsity of capability is 0.4 etc.

Definition 3 [20]

Let R be an equivalence relation on the universal set U. Then the pair (U, R) is called a Pawlak approximation space. An equivalence class of R containing x will be denoted by $[x]_R$. Now for $X \subseteq U$, the lower and upper approximation of X with respect to (U, R) are denoted by respectively R^*X and $R_* X$ and are defined by

$R_* X = \{x \in U: [x]_R \subseteq X\}$,

$R^*X = \{ x \in U: [x]_R \cap X \neq \emptyset\}$.

Now if $R^*X = R_* X$, then X is called definable; otherwise X is called a rough set.

Definition 4 [27]

Let U be a universe and X , a rough set in U. An IF rough set A in U is characterized by a membership function μ_A :U→ [0, 1] and non-membership function ν_A :U→ [0 , 1] such that

$\mu_A(\underline{R} X) = 1$, $\nu_A(\underline{R} X) = 0$

Or $[\mu_A(x), \nu_A(x)] = [1, 0]$ if $x \in (\underline{R} X)$ and $\mu_A(U - \overline{R} X) = 0$, $\nu_A(U - \overline{R} X) = 1$

Or $[\mu_A(x), \nu_A(x)] = [0, 1]$ if $x \in U - \overline{R} X$,

$0 \leq \mu_A(\overline{R} X - \underline{R} X) + \nu_A(\overline{R} X - \underline{R} X) \leq 1$

Example 2: Example of IF Rough Sets

Let U= {Child, Pre-Teen, Teen, Youth, Teenager, Young-Adult, Adult, Senior, Elderly} be a universe.

Let the equivalence relation R be defined as follows:

R*= {[Child, Pre-Teen], [Teen, Youth, Teenager], [Young-Adult, Adult],[Senior, Elderly]}.

Let X = {Child, Pre-Teen, Youth, Young-Adult} be a subset of univers U.

We can define X in terms of its lower and upper approximations:

$\underline{R} X$ = {Child, Pre-Teen}, and $\overline{R} X$ = {Child, Pre-Teen, Teen, Youth, Teenager, Young-Adult, Adult}.

The membership and non-membership functions μ_A:U→] 1 , 0 [and ν_A: U→] 1 , 0 [on a set A are defined as follows:

μ_AChild) = 1, μ_A (Pre-Teen) = 1 and μ_A (Child) = 0, μ_A(Pre-Teen) = 0

μ_A (Young-Adult) = 0, μ_A (Adult) = 0, μ_A(Senior) = 0, μ_A (Elderly) = 0

3.Basic Concept of Rough Approximations of an Interval Valued Neutrosophic Set and their Properties.

In this section we define the notion of interval valued neutrosophic rough sets (in brief ivn- rough set) by combining both rough sets and interval neutrosophic sets. IVN- rough sets are the generalizations of interval valued intuitionistic fuzzy rough sets, that give more information about uncertain or boundary region.

Definition 5 : Let (U,R) be a pawlak approximation space ,for an interval valued neutrosophic set

$A = \{<x , [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)], [\omega_A^L(x), \omega_A^U(x)] > | \; x \in X \}$ be interval neutrosophic set. The lower approximation $\underline{A}_R$ and $\overline{A}_R$ upper approximations of A in the pawlak approwimation space (U, R) are defined as:

$\underline{A}_R$**={<x,** $[\wedge_{y \in [x]_R}\{\mu_A^L(y)\}, \wedge_{y \in [x]_R}\{\mu_A^U(y)\}]$, $[\vee_{y \in [x]_R}\{\nu_A^L(y)\}, \vee_{y \in [x]_R}\{\nu_A^U(y)\}]$, $[\vee_{y \in [x]_R}\{\omega_A^L(y)\}, \vee_{y \in [x]_R}\{\omega_A^U(y)\}]$>:x ∈ U}.

$\overline{A}_R$**={<x,** $[\vee_{y \in [x]_R}\{\mu_A^L(y)\}, \vee_{y \in [x]_R}\{\mu_A^U(y)\}]$, $[\wedge_{y \in [x]_R}\{\nu_A^L(y)\}, \wedge_{y \in [x]_R}\{\nu_A^U(y)\}]$, $[\wedge_{y \in [x]_R}\{\omega_A^L(y)\}, \wedge_{y \in [x]_R}\{\omega_A^U(y)\}]$:x ∈ U}.

Where " ∧ " means " min" and " ∨ " means " max", R denote an equivalence relation for interval valued neutrosophic set A.

Here $[x]_R$ is the equivalence class of the element x. It is easy to see that

$[\wedge_{y \in [x]_R}\{\mu_A^L(y)\}, \wedge_{y \in [x]_R}\{\mu_A^U(y)\}] \subset [0,1]$

$[\vee_{y \in [x]_R}\{\nu_A^L(y)\}, \vee_{y \in [x]_R}\{\nu_A^U(y)\}] \subset [0,1]$

$[\vee_{y \in [x]_R}\{\omega_A^L(y)\}, \vee_{y \in [x]_R}\{\omega_A^U(y)\}] \subset [0,1]$

And

$\mathbf{0} \leq \wedge_{y \in [x]_R}\{\mu_A^U(y)\} + \vee_{y \in [x]_R}\{\nu_A^U(y)\} + \vee_{y \in [x]_R}\{\omega_A^U(y)\} \leq \mathbf{3}$

Then, $\underline{A}_R$ is an interval neutrosophic set

Similarly , we have

$[\vee_{y\in[x]_R}\{\mu_A^L(y)\},\ \vee_{y\in[x]_R}\{\mu_A^U(y)\}] \subset [0,1]$

$[\wedge_{y\in[x]_R}\{\nu_A^L(y)\},\ \wedge_{y\in[x]_R}\{\nu_A^U(y)\}] \subset [0,1]$

$[\wedge_{y\in[x]_R}\{\omega_A^L(y)\},\ \wedge_{y\in[x]_R}\{\omega_A^U(y)\}] \subset [0,1]$

And

$\mathbf{0} \leq \vee_{y\in[x]_R}\{\mu_A^U(y)\} + \wedge_{y\in[x]_R}\{\nu_A^U(y)\} + \wedge_{y\in[x]_R}\{\omega_A^U(y)\} \leq \mathbf{3}$

Then, $\overline{A}_R$ is an interval neutrosophic set

If $\underline{A}_R = \overline{A}_R$,then A is a definable set, otherwise A is an interval valued neutrosophic rough set, $\underline{A}_R$ and $\overline{A}_R$ are called the lower and upper approximations of interval valued neutrosophic set with respect to approximation space (U, R), respectively. $\underline{A}_R$ and $\overline{A}_R$ are simply denoted by $\underline{A}$ and $\overline{A}$.

In the following , we employ an example to illustrate the above concepts

Example:

Theorem 1. Let A, B be interval neutrosophic sets and $\underline{A}$ and $\overline{A}$ the lower and upper approximation of interval –valued neutrosophic set A with respect to approximation space (U, R) ,respectively. $\underline{B}$ and $\overline{B}$ the lower and upper approximation of interval –valued neutrosophic set B with respect to approximation space (U,R) ,respectively.Then we have

i. $\underline{A} \subseteq A \subseteq \overline{A}$
ii. $\overline{A\cup B} = \overline{A} \cup \overline{B}$, $\underline{A\cap B} = \underline{A} \cap \underline{B}$
iii. $\underline{A} \cup \underline{B} = \underline{A\cup B}$, $\overline{A\cap B} = \overline{A} \cap \overline{B}$
iv. $\underline{(\overline{A})} = \overline{(\overline{A})} = \overline{A}$, $\underline{(\underline{A})} = \overline{(\underline{A})} = \underline{A}$
v. $\underline{U} = U$; $\overline{\phi} = \phi$
vi. If $A \subseteq B$,then $\underline{A} \subseteq \underline{B}$ and $\overline{A} \subseteq \overline{B}$
vii. $\underline{A^c} = (\overline{A})^c$, $\overline{A^c} = (\underline{A})^c$

Proof: we prove only i,ii,iii, the others are trivial

(i)

Let $A = \{<x , [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)], [\omega_A^L(x), \omega_A^U(x)] >| x \in X\}$ be interval neutrosophic set

From definition of $\underline{A}_R$ and $\overline{A}_R$, we have

Which implies that

$\mu_{\underline{A}}^L(x) \leq \mu_A^L(x) \leq \mu_{\overline{A}}^L(x)$; $\mu_{\underline{A}}^U(x) \leq \mu_A^U(x) \leq \mu_{\overline{A}}^U(x)$ for all $x \in X$

$\nu_{\underline{A}}^L(x) \geq \nu_A^L(x) \geq \nu_{\overline{A}}^L(x)$; $\nu_{\underline{A}}^U(x) \geq \nu_A^U(x) \geq \nu_{\overline{A}}^U(x)$ for all $x \in X$

$\omega_{\underline{A}}^L(x) \geq \omega_A^L(x) \geq \omega_{\overline{A}}^L(x)$; $\omega_{\underline{A}}^U(x) \geq \omega_A^U(x) \geq \omega_{\overline{A}}^U(x)$ for all $x \in X$

$([\mu_{\underline{A}}^L , \mu_{\underline{A}}^U], [\nu_{\underline{A}}^L , \nu_{\underline{A}}^U], [\omega_{\underline{A}}^L , \omega_{\underline{A}}^U]) \subseteq ([\mu_A^L , \mu_A^U], [\nu_A^L , \nu_A^U], [\omega_A^L , \omega_A^U]) \subseteq ([\mu_{\overline{A}}^L , \mu_{\overline{A}}^U], [\nu_{\overline{A}}^L , \nu_{\overline{A}}^U], [\omega_{\overline{A}}^L , \omega_{\overline{A}}^U])$ .Hence $\underline{A}_R \subseteq A \subseteq \overline{A}_R$

(ii) Let $A = \{<x , [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)], [\omega_A^L(x), \omega_A^U(x)] >| x \in X\}$ and

$B = \{<x, [\mu_B^L(x), \mu_B^U(x)], [\nu_B^L(x), \nu_B^U(x)] , [\omega_B^L(x), \omega_B^U(x)] >| x \in X\}$ are two intervalvalued neutrosophic set and

$\overline{A\cup B} = \{<x , [\mu_{\overline{A\cup B}}^L(x), \mu_{\overline{A\cup B}}^U(x)] , [\nu_{\overline{A\cup B}}^L(x), \nu_{\overline{A\cup B}}^U(x)] , [\omega_{\overline{A\cup B}}^L(x), \omega_{\overline{A\cup B}}^U(x)] >| x \in X\}$

$\overline{A} \cup \overline{B} = \{x, [\max(\mu_{\overline{A}}^L(x) , \mu_{\overline{B}}^L(x)) ,\max(\mu_{\overline{A}}^U(x) , \mu_{\overline{B}}^U(x))],[\min(\nu_{\overline{A}}^L(x) , \nu_{\overline{B}}^L(x)) ,\min(\nu_{\overline{A}}^U(x) , \nu_{\overline{B}}^U(x))],[\min(\omega_{\overline{A}}^L(x) , \omega_{\overline{B}}^L(x)) ,\min(\omega_{\overline{A}}^U(x) , \omega_{\overline{B}}^U(x))]$

for all $x \in X$

$\mu_{\overline{A\cup B}}^L(x) = \vee\{\mu_{A\cup B}^L(y) | y \in [x]_R\}$

$= \vee\{\mu_A^L(y) \vee \mu_B^L(y) | y \in [x]_R\}$

$= (\vee \mu_A^L(y) | y \in [x]_R) \vee (\vee \mu_A^L(y) | y \in [x]_R)$

$= (\mu_{\overline{A}}^L \vee \mu_{\overline{B}}^L)(x)$

$\mu_{\overline{A\cup B}}^U(x) = \vee\{\mu_{A\cup B}^U(y) | y \in [x]_R\}$

$= \vee\{\mu_A^U(y) \vee \mu_B^U(y) | y \in [x]_R\}$

$= (\vee\ \mu_A^u(y) \mid y \in [x]_R) \vee (\vee\ \mu_A^U(y) \mid y \in [x]_R)$

$=(\mu_{\overline{A}}^U \vee \mu_{\overline{B}}^U)(x)$

$\nu_{\overline{A \cup B}}^L(x)=\wedge\{ \nu_{A \cup B}^L(y) \mid y \in [x]_R\}$

$= \wedge \{\nu_A^L(y) \wedge \nu_B^L(y) \mid y \in [x]_R\}$

$= (\wedge\ \nu_A^L(y) \mid y \in [x]_R) \wedge (\wedge\ \nu_B^L(y) \mid y \in [x]_R)$

$=(\nu_{\overline{A}}^L \wedge \nu_{\overline{B}}^L)(x)$

$\nu_{\overline{A \cup B}}^U(x)=\wedge\{ \nu_{A \cup B}^U(y) \mid y \in [x]_R\}$

$= \wedge \{\nu_A^U(y) \wedge \nu_B^U(y) \mid y \in [x]_R\}$

$= (\wedge\ \nu_A^U(y) \mid y \in [x]_R) \wedge (\wedge\ \nu_B^U(y) \mid y \in [x]_R)$

$=(\nu_{\overline{A}}^U(y) \wedge \nu_{\overline{B}}^U(y))(x)$

$\omega_{\overline{A \cup B}}^L(x)=\wedge\{ \omega_{A \cup B}^L(y) \mid y \in [x]_R\}$

$= \wedge \{\omega_A^L(y) \wedge \omega_B^L(y) \mid y \in [x]_R\}$

$= (\wedge\ \omega_A^L(y) \mid y \in [x]_R) \wedge (\wedge\ \omega_B^L(y) \mid y \in [x]_R)$

$=(\omega_{\overline{A}}^L \wedge \omega_{\overline{B}}^L)(x)$

$\omega_{\overline{A \cup B}}^U(x)=\wedge\{ \omega_{A \cup B}^U(y) \mid y \in [x]_R\}$

$= \wedge \{\omega_A^U(y) \wedge \nu_B^U(y) \mid y \in [x]_R\}$

$= (\wedge\ \omega_A^U(y) \mid y \in [x]_R) \wedge (\wedge\ \omega_B^U(y) \mid y \in [x]_R)$

$=(\omega_{\overline{A}}^U \wedge \omega_{\overline{B}}^U)(x)$

Hence, $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Also for $\underline{A \cap B} = \underline{A} \cap \underline{B}$ for all x ∈ A

$\mu_{\underline{A \cap B}}^L(x) =\wedge\{ \mu_{A \cap B}^L(y) \mid y \in [x]_R\}$

$= \wedge \{\mu_A^L(y) \wedge \mu_B^L(y) \mid y \in [x]_R\}$

$= \wedge\ (\mu_A^L(y) \mid y \in [x]_R) \wedge (\vee\ \mu_B^L(y) \mid y \in [x]_R)$

$=\mu_{\underline{A}}^L(x) \wedge \mu_{\underline{B}}^L(x)$

$=(\mu_{\underline{A}}^L \wedge \mu_{\underline{B}}^L)(x)$

Also

$\mu_{\underline{A \cap B}}^U(x) =\wedge\{ \mu_{A \cap B}^U(y) \mid y \in [x]_R\}$

$= \wedge \{\mu_A^U(y) \wedge \mu_B^U(y) \mid y \in [x]_R\}$

$= \wedge\ (\mu_A^U(y) \mid y \in [x]_R) \wedge (\vee\ \mu_B^U(y) \mid y \in [x]_R)$

$=\mu_{\underline{A}}^U(x) \wedge \mu_{\underline{B}}^U(x)$

$=(\mu_{\underline{A}}^U \wedge \mu_{\underline{B}}^U)(x)$

$\nu_{\underline{A \cap B}}^L(x) =\vee\{ \nu_{A \cap B}^L(y) \mid y \in [x]_R\}$

$= \vee \{\nu_A^L(y) \vee \nu_B^L(y) \mid y \in [x]_R\}$

$= \vee\ (\nu_A^L(y) \mid y \in [x]_R) \vee (\vee\ \nu_B^L(y) \mid y \in [x]_R)$

$=\nu_{\underline{A}}^L(x) \vee \nu_{\underline{B}}^L(x)$

$=(\nu_{\underline{A}}^L \vee \nu_{\underline{B}}^L)(x)$

$\nu_{\underline{A \cap B}}^U(x) =\vee\{ \nu_{A \cap B}^U(y) \mid y \in [x]_R\}$

$= \vee \{\nu_A^U(y) \vee \nu_B^U(y) \mid y \in [x]_R\}$

$= \vee\ (\nu_A^U(y) \mid y \in [x]_R) \vee (\vee\ \nu_B^U(y) \mid y \in [x]_R)$

$=\nu_{\underline{A}}^U(x) \vee \nu_{\underline{B}}^U(x)$

$=(\nu_{\underline{A}}^U \vee \nu_{\underline{B}}^U)(x)$

$\omega_{\underline{A \cap B}}^L(x) =\vee\{ \omega_{A \cap B}^L(y) \mid y \in [x]_R\}$

$= \vee \{\omega_A^L(y) \vee \omega_B^L(y) \mid y \in [x]_R\}$

$= \vee\ (\omega_A^L(y) \mid y \in [x]_R) \vee (\vee\ \omega_B^L(y) \mid y \in [x]_R)$

$=\omega_{\underline{A}}^L(x) \vee \nu\omega_{\underline{B}}^L(x)$

$=(\omega_{\underline{A}}^L \vee \omega_{\underline{B}}^L)(x)$

$\omega_{\underline{A \cap B}}^U(x) =\vee\{ \omega_{A \cap B}^U(y) \mid y \in [x]_R\}$

$= \vee \{\omega_A^U(y) \vee \omega_B^U(y) \mid y \in [x]_R\}$

$= \vee\ (\omega_A^U(y) \mid y \in [x]_R) \vee (\vee\ \omega_B^U(y) \mid y \in [x]_R)$

$=\omega_{\underline{A}}^U(x) \vee \omega_{\underline{B}}^U(x)$

$=(\omega^{U}_{\underline{A}} \vee \omega^{U}_{\underline{B}})(x)$

(iii)

$\mu^{U}_{\overline{A \cap B}}(x) = \vee\{ \mu^{U}_{A \cap B}(y) \mid y \in [x]_R\}$

$= \vee \{\mu^{U}_{A}(y) \wedge \mu^{U}_{B}(y) \mid y \in [x]_R\}$

$=(\vee (\mu^{U}_{A}(y) \mid y \in [x]_R)) \wedge (\vee (\mu^{U}_{A}(y) \mid y \in [x]_R))$

$= \mu^{U}_{\overline{A}}(x) \vee \mu^{U}_{\overline{B}}(x)$

$=(\mu^{U}_{\overline{A}} \vee \mu^{U}_{\overline{B}})(x)$

$\nu^{U}_{\overline{A \cap B}}(x) = \wedge\{ \nu^{U}_{A \cap B}(y) \mid y \in [x]_R\}$

$= \wedge \{\nu^{U}_{A}(y) \wedge \nu^{U}_{B}(y) \mid y \in [x]_R\}$

$=(\wedge (\nu^{U}_{A}(y) \mid y \in [x]_R)) \vee (\wedge (\nu^{U}_{A}(y) \mid y \in [x]_R))$

$= \nu^{U}_{\overline{A}}(x) \vee \nu^{U}_{\overline{B}}(x)$

$=(\nu^{U}_{\overline{A}} \vee \nu^{U}_{\overline{B}})(x)$

$\omega^{U}_{\overline{A \cap B}}(x) = \wedge\{ \omega^{U}_{A \cap B}(y) \mid y \in [x]_R\}$

$= \wedge \{\omega^{U}_{A}(y) \wedge \omega\nu^{U}_{B}(y) \mid y \in [x]_R\}$

$=(\wedge (\omega^{U}_{A}(y) \mid y \in [x]_R)) \vee (\wedge (\omega^{U}_{A}(y) \mid y \in [x]_R))$

$= \omega^{U}_{\overline{A}}(x) \vee \omega^{U}_{\overline{B}}(x)$

$=(\omega^{U}_{\overline{A}} \vee \omega^{U}_{\overline{B}})(x)$

Hence follow that $\overline{A \cap B} = \overline{A} \cap \overline{B}$ .we get $\underline{A} \cup \underline{B} = \underline{A \cup B}$ by following the same procedure as above.

Definition 6:

Let (U,R) be a pawlak approximation space ,and A and B two interval valued neutrosophic sets over U.

If $\underline{A} = \underline{B}$,then A and B are called interval valued neutrosophic lower rough equal.

If $\overline{A} = \overline{B}$, then A and B are called interval valued neutrosophic upper rough equal.

If $\underline{A} = \underline{B}$, $\overline{A} = \overline{B}$, then A and B are called interval valued neutrosophic rough equal.

Theorem 2 .

Let (U,R) be a pawlak approximation space ,and A and B two interval valued neutrosophic sets over U. then

1. $\underline{A} = \underline{B} \Leftrightarrow \underline{A \cap B} = \underline{A}$, $\underline{A \cap B} = \underline{B}$
2. $\overline{A} = \overline{B} \Leftrightarrow \overline{A \cup B} = \overline{A}$, $\overline{A \cup B} = \overline{B}$
3. If $\overline{A} = \overline{A'}$ and $\overline{B} = \overline{B'}$,then $\overline{A \cup B} = \overline{A' \cup B'}$
4. If $\underline{A} = \underline{A'}$ and $\underline{B} = \underline{B'}$,Then
5. If $A \subseteq B$ and $\underline{B} = \underline{\phi}$,then $\underline{A} = \underline{\phi}$
6. If $A \subseteq B$ and $\underline{B} = \underline{U}$,then $\underline{A} = \underline{U}$
7. If $\underline{A} = \underline{\phi}$ or $\underline{B} = \underline{\phi}$ or then $\underline{A \cap B} = \underline{\phi}$
8. If $\overline{A} = \overline{U}$ or $\overline{B} = \overline{U}$,then $\overline{A \cup B} = \overline{U}$
9. $\overline{A} = \overline{U} \Leftrightarrow A = U$
10. $\overline{A} = \overline{\phi} \Leftrightarrow A = \phi$

Proof: the proof is trial

4.Hamming distance between Lower Approximation and Upper Approximation of IVNS

In this section , we will compute the Hamming distance between lower and upper approximations of interval neutrosophic sets based on Hamming distance introduced by **Ye [41]** of interval neutrosophic sets.

Based on Hamming distance between two interval neutrosophic set A and B as follow:

$$d(A,B)=\frac{1}{6}\sum_{i=1}^{n}\left[\left|\mu^{L}_{A}(x_i) - \mu^{L}_{B}(x_i)\right| + \left|\mu^{U}_{A}(x_i) - \mu^{U}_{B}(x_i)\right| + \left|\nu^{L}_{A}(x_i) - \nu^{L}_{B}(x_i)\right| + \left|\nu^{U}_{A}(x_i) - \nu^{U}_{B}(x_i)\right| + \left|\omega^{L}_{A}(x_i) - \omega^{L}_{B}(x_i)\right| + \left|\omega^{L}_{A}(x_i) - \nu^{U}_{B}(x_i)\right|\right]$$

we can obtain the standard hamming distance of $\underline{A}$ and $\overline{A}$ from

$$d_H(\underline{A}, \overline{A}) = \frac{1}{6}\sum_{i=1}^{n}\left[\left|\mu^{L}_{\underline{A}}(x_j) - \mu^{L}_{\overline{A}}(x_j)\right| + \left|\mu^{U}_{\underline{A}}(x_j) - \mu^{U}_{\overline{A}}(x_j)\right| + \left|\nu^{L}_{\underline{A}}(x_j) - \nu^{L}_{\overline{A}}(x_j)\right| + \left|\nu^{U}_{\underline{A}}(x_j) - \nu^{U}_{\overline{A}}(x_j)\right| + \left|\omega^{L}_{\underline{A}}(x_j) - \omega^{L}_{\overline{A}}(x_j)\right| + \left|\omega^{U}_{\underline{A}}(x_j) - \omega^{U}_{\overline{A}}(x_j)\right|\right]$$

Where

$\underline{A}_R$**={<x, [**$\wedge_{y\in[x]_R}\{\mu_A^L(y)\}$, $\wedge_{y\in[x]_R}\{\mu_A^U(y)\}$], [$\vee_{y\in[x]_R}\{\nu_A^L(y)\}$, $\vee_{y\in[x]_R}\{\nu_A^U(y)\}$], [$\vee_{y\in[x]_R}\{\omega_A^L(y)\}$, $\vee_{y\in[x]_R}\{\omega_A^U(y)\}$]>:x ∈ U}.

$\overline{A}_R$**={<x, [**$\vee_{y\in[x]_R}\{\mu_A^L(y)\}$, $\vee_{y\in[x]_R}\{\mu_A^U(y)\}$], [$\wedge_{y\in[x]_R}\{\nu_A^L(y)\}$, $\wedge_{y\in[x]_R}\{\nu_A^U(y)\}$], [$\wedge_{y\in[x]_R}\{\omega_A^L(y)\}$, $\wedge_{y\in[x]_R}\{\omega_A^U(y)\}$]:x ∈ U}.

$\mu_{\underline{A}}^L(x_j) = \wedge_{y\in[x]_R}\{\mu_A^L(y)\}$; $\mu_{\underline{A}}^U(x_j) = \wedge_{y\in[x]_R}\{\mu_A^U(y)\}$

$\nu_{\underline{A}}^L(x_j) = \vee_{y\in[x]_R}\{\nu_A^L(y)\}$; $\nu_{\underline{A}}^U(x_j) = \vee_{y\in[x]_R}\{\nu_A^U(y)\}$

$\omega_{\underline{A}}^L(x_j) = \vee_{y\in[x]_R}\{\omega_A^L(y)\}$; $\omega_{\underline{A}}^U(x_j) = \vee_{y\in[x]_R}\{\omega_A^U(y)\}$

$\mu_{\overline{A}}^L(x_j) = \vee_{y\in[x]_R}\{\mu_A^L(y)\}$; $\mu_{\overline{A}}^U(x_j) = \vee_{y\in[x]_R}\{\mu_A^U(y)\}$

$\mu_{\overline{A}}^L(x_j) = \wedge_{y\in[x]_R}\{\nu_A^L(y)\}$; $\mu_{\overline{A}}^U(x_j) = \wedge_{y\in[x]_R}\{\nu_A^U(y)\}$

$\omega_{\overline{A}}^L(x_j) = \wedge_{y\in[x]_R}\{\omega_A^L(y)\}$; $\omega_{\overline{A}}^U(x_j) = \wedge_{y\in[x]_R}\{\omega_A^U(y)\}$

Theorem 3. Let (U, R) be approximation space, A be an interval valued neutrosophic set over U . Then

(1) If d ($\underline{A}$, $\overline{A}$) = 0, then A is a definable set.

(2) If 0 < d($\underline{A}$, $\overline{A}$) < 1, then A is an interval-valued neutrosophic rough set.

Theorem 4. Let (U, R) be a Pawlak approximation space, and A and B two interval-valued neutrosophic sets over U . Then

1. d ($\underline{A}$, $\overline{A}$) ≥ d ($\underline{A}$, A) and d ($\underline{A}$, $\overline{A}$) ≥ d (A , $\overline{A}$);
2. d ($\overline{A \cup B}$, $\overline{A} \cup \overline{B}$) = 0, d ($\underline{A \cap B}$, $\underline{A} \cap \underline{B}$) = 0.
3. d ($\underline{A} \cup \underline{B}$, A ∪ B) ≥ d($\underline{A} \cup \underline{B}$, $\underline{A \cup B}$) and d($\underline{A} \cup \underline{B}$, A ∪ B) ≥ d($\underline{A \cup B}$, A ∪ B) ; and d(A ∩ B, $\overline{A} \cap \overline{B}$) ≥ d(A ∩ B, $\overline{A \cap B}$) and d(A ∩ B, $\overline{A} \cap \overline{B}$) ≥ $d(\overline{A \cap B}, \overline{A} \cap \overline{B})$
4. d($\overline{(\overline{A})}$, $\underline{(\overline{A})}$)= 0 , d($\overline{(\overline{A})}$, $\overline{A}$) = 0 , d($\underline{(\overline{A})}$, $\overline{A}$)= 0; d($\underline{(\underline{A})}$, $\overline{(\underline{A})}$) = 0 , d($\underline{(\underline{A})}$, , $\overline{A}$) = 0 , d($\underline{(\overline{A})}$, $\underline{A}$) = 0,
5. d ($\underline{U}$, U) =0 , d($\overline{\phi}$, ϕ) = 0
6. if A B ,then d($\overline{A}$,B) ≥ d($\underline{A}$, $\underline{B}$) and d($\underline{A}$, B) ≥ d($\underline{B}$,B) d(A , $\overline{B}$) ≥d(A, $\overline{A}$) and d(A, $\overline{B}$)= ≥d($\overline{A}$, $\overline{B}$)
7. d($\underline{A^c}$,$(\overline{A})^c$)= 0, d($\overline{A^c}$,$(\underline{A})^c$) = 0

5-Conclusion

In this paper we have defined the notion of interval valued neutrosophic rough sets. We have also studied some properties on them and proved some propositions. The concept combines two different theories which are rough sets theory and interval valued neutrosophic set theory. Further, we have introduced the Hamming distance between two interval neutrosophic rough sets. We hope that our results can also be extended to other algebraic system.

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A New Approach to Multi-spaces Through the Application of Soft Sets

Mumtaz Ali, Florentin Smarandache, Said Broumi, Muhammad Shabir

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Abstract. Multi-space is the notion combining different fields in to a unifying field, which is more applicable in our daily life. In this paper, we introduced the notion of multi-soft space which is the approximated collection of the multi-subspaces of a multi-space . Further, we defined some basic operations such as union, intersection, AND, OR etc. We also investigated some properties of multi-soft spaces.

Keywords: Multi-space, soft set, multi-soft space.

1. Introduction

Multi-spaces [24] were introduced by Smarandache in 1969 under the idea of hybrid structures: combining different fields into a unifying field [23] that are very effective in our real life. This idea has a wide range of acceptance in the world of sciences. In any domain of knowledge a Smarandache multi-space is the union of n different spaces with some different for an integer $n \geq 2$. Smarandache multi-space is a qualitative notion as it is too huge which include both metric and non-metric spaces. This multi-space can be used for both discrete or connected spaces specially in spacetimes and geometries in theoretical physics. Multi-space theory has applied in physics successfully in the Unified Field Theory which unite the gravitational, electromagnetic, weak and strong interactions or in the parallel quantum computing or in the mu-bit theory etc. Several multi-algebraic structures have been introduced such as multi-groups, multi-rings, multi-vector spaces, multi-metric spaces etc. Literature on multi-algebraic structures can be found in [17].

Molodtsov [20] proposed the theory of soft sets. This mathematical framework is free from parameterization inadequacy, syndrome of fuzzy set theory, rough set theory, probability theory and so on. Soft set theory has been applied successfully in many areas such as smoothness of functions, game theory, operation research, Riemann integration, Perron integration, and probability thoery. Soft sets gained much attention of the researchers recently from its appearance and some literature on soft sets can be seen in [1]-[16]. Some other properties and algebras may be found in [18,19,20]. Some other concepts together with fuzzy set and rough set were shown in [21,22,23].

In section 2, we review some basic concepts and notions on multi-spaces and soft sets. In section 3, we define multi-subspac. Then multi-soft spaces has been introduced in the current section. Multi-soft space is a parameterized collection of multi-subspaces. We also investigated some properties and other notions of multi-soft spaces.

2. Basic Concepts

In this section, we review some basic material of multi-spaces and soft sets.

Definition 2.1 [24]. For any integer i, $1 \leq i \leq n$, let M_i be a set with ensemble of law L_i, and the intersection of k sets $M_{i_1}, M_{i_2}, \ldots, M_{i_k}$ of them constrains the law $I\ M_{i_1}, M_{i_2}, \ldots, M_{i_k}$. Then the union of M_i, $1 \leq i \leq n$

$$M = \bigcup_{i=1}^{n} M_i$$

is called a multi-space.

Let U be an initial universe, E is a set of parameters, $P(U)$ is the power set of U, and $A, B \subset E$. Molodtsov defined the soft set in the following manner:

Definition 2.2 [20]. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $a \in A$, $F\ \ a$ may be considered as the set of a-elements of the soft set (F, A), or as the set of a-approximate elements of the soft set.

Example 2.3. Suppose that U is the set of shops. E is the set of parameters and each parameter is a word or sentence. Let

$$E = \left\{ \begin{matrix} \text{high rent,normal rent,} \\ \text{in good condition,in bad condition} \end{matrix} \right\}$$

Let us consider a soft set (F, A) which describes the attractiveness of shops that Mr. Z is taking on rent. Suppose that there are five houses in the universe $U = \{s_1, s_2, s_3, s_4, s_5\}$ under consideration, and that $A = \{a_1, a_2, a_3\}$ be the set of parameters where

a_1 stands for the parameter 'high rent,

a_2 stands for the parameter 'normal rent,

a_3 stands for the parameter 'in good condition.

Suppose that

$$F(a_1) = \{s_1, s_4\},$$
$$F(a_2) = \{s_2, s_5\},$$
$$F(a_3) = \{s_3\}.$$

The soft set (F, A) is an approximated family $\{F(a_i), i = 1, 2, 3\}$ of subsets of the set U which gives us a collection of approximate description of an object. Then (F, A) is a soft set as a collection of approximations over U, where

$$F(a_1) = \textit{high rent} = \{s_1, s_2\},$$
$$F(a_2) = \textit{normal rent} = \{s_2, s_5\},$$
$$F(a_3) = \textit{in good condition} = \{s_3\}.$$

Definition 2.4 [19]. For two soft sets (F, A) and (H, B) over U, (F, A) is called a soft subset of (H, B) if

1. $A \subseteq B$ and
2. $F(a) \subseteq H(a)$, for all $x \in A$.

This relationship is denoted by $(F, A) \subset (H, \mathrm{B})$. Similarly (F, A) is called a soft superset of (H, B) if (H, B) is a soft subset of (F, A) which is denoted by $(F, A) \supset (H, \mathrm{B})$.

Definition 2.5 [19]. Two soft sets (F, A) and (H, B) over U are called soft equal if (F, A) is a soft subset of (H, B) and (H, B) is a soft subset of (F, A).

Definition 2.6 [19]. Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \phi$. Then their restricted intersection is denoted by $(F, A) \cap_R (\mathrm{G}, \mathrm{B}) = (H, \mathrm{C})$ where (H, C) is defined as $H(c) = F(c) \cap \mathrm{K}(c)$ for all $c \in C = A \cap \mathrm{B}$.

Definition 2.7 [12]. The extended intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$, and for all $c \in C$, $H(c)$ is defined as

$$H(c)=\begin{cases} F(c) & \text{if } c\in A-B,\\ G(c) & \text{if } c\in B-A,\\ F(c)\cap G(c) & \text{if } c\in A\cap B.\end{cases}$$

We write $(F,A)\cap_{\varepsilon}(G,B)=(H,C)$.

Definition 2.8 [19]. The restricted union of two soft sets (F,A) and (G,B) over a common universe U is the soft set (H,B), where $C=A\cup B$, and for all $c\in C$, $H(c)$ is defined as $H(c)=F(c)\cup G(c)$ for all $c\in C$. We write it as $(F,A)\cup_R(G,B)=(H,C)$.

Definition 2.9 [12]. The extended union of two soft sets (F,A) and (G,B) over a common universe U is the soft set (H,B), where $C=A\cup B$, and for all $c\in C$, $H(c)$ is defined as

$$H(c)=\begin{cases} F(c) & \text{if } c\in A-B\\ G(c) & \text{if } c\in B-A,\\ F(c)\cup G(c) & \text{if } c\in A\cap B.\end{cases}$$

We write $(F,A)\cup_{\varepsilon}(G,B)=(H,C)$.

In the next section, we introduced multi-soft spaces.

3. Multi-Soft Space and Its Properties

In this section, first we introduced the definition of multi-subspace. Further, we introduced multi-soft spaces and their core properties.

Definition 3.1. Let M be a multi-space and $M^{'}\subseteq M$. Then $M^{'}$ is called a multi-subspace if $M^{'}$ is a multi-space under the operations and constaints of M.

Definition 3.2. Let $A_1=\{a_j : j\in J\}$, $A_2=\{a_k : k\in K\}$,..., $A_n=\{a_n : n\in L\}$ be n-set of parameters. Let $(F_1,A_1),(F_2,A_2),...,(F_n,A_n)$ are soft set over the distinct universes $M_1,M_2,...,M_n$ respectively. Then (H,C) is called a multi-soft space over $M=M_1\cup M_2\cup...\cup M_n$, where $(H,C)=(F_1,A_1)\cup_E(F_2,A_2)\cup_E,...,\cup_E(F_n,A_n)$ such that $C=A_1\cup A_2\cup....\cup A_n$ and for all $c\in C$, $H(c)$ is defined by

$$H(c)=F_{i_1}(c)\cup F_{i_2}(c)\cup...\cup F_{i_k}(c)$$

if

$$c\in(A_{i1}\cap A_{i_2}\cap...\cap A_{ik})-(A_{i_{k+1}}\cup A_{i_{k+2}}\cup...\cup A_{i_n}),$$

where $(i_1,i_2,...,i_k,i_{k+1},...,i_n)$ are all possible permutations of the indexes $(1,2,...,n)$ *k = 1, 2, ..., n.* There are 2^{n-1} pieces of the piece-wise function (H,C).

Proposition 3.3. Let M be a universe of discourse and (F,A) is a soft set over M. Then (F,A) is a multi-soft space over M if and only if M is a multi-space.

Proof: Suppose that M is a multi-space and $F:A\rightarrow P(M)$ be a mapping. Then clearly for each $a\in A$, then $F(a)$ is a subset of M which is a multi-subspace. Thus each $F(a)$ is a multi-subspace of M and so the soft set (F,A) is the parameterized collection of multi-subspaces of M. Hence (F,A) is a multi-soft space over M.

For converse, suppose that (F,A) is a multi-soft space over M. This implies that $F(a)$ is a multi-subspace of M for all $a\in A$. Therefore, M is a mutli-space.

This situation can be illustrated in the following Example.

Example 3.4. Let $M=\{u_1,u_2,u_3,u_4,u_5,u_6,u_7\}$ be an initial universe such that M is a multi-space. Let $A_1=\{a_1,a_2,a_3,a_8\}$, $A_2=\{a_2,a_4,a_5,a_6,a_8\}$ and $A_3=\{a_5,a_7,a_8\}$ are set of parameters. Let $(F_1,A_1),(F_2,A_2)$ and (F_3,A_3) respectively be the soft sets over M as following:

$$F_1(a_1)=\{m_1,m_2,m_3\},$$

$$F_1(a_2)=\{m_4,m_5\},$$

$$F_1(a_3)=\{m_1,m_4,m_6,m_7\},$$

$$F_1(a_8)=\{m_2,m_4,m_6,m_7\}.$$

and

$$F_2(a_2)=\{m_1,m_2,m_3,m_6,m_7\},$$

$$F_2(a_4)=\{m_3,m_4,m_5,m_6\},$$

$$F_2(a_5)=\{m_2,m_4,m_5\},$$

$$\mathrm{F}_2(a_6)=\{\mathrm{m}_1,\mathrm{m}_7\},$$

$$F_2(a_8)=\{\mathrm{m}_1,\mathrm{m}_3,\mathrm{m}_5,\mathrm{m}_7\}.$$

Also

$$F_3(a_5)=\{\mathrm{m}_1,\mathrm{m}_2,\mathrm{m}_3,\mathrm{m}_4,\mathrm{m}_5,\},$$

$$F_3(a_7)=\{\mathrm{m}_4,\mathrm{m}_5,\mathrm{m}_7\},$$

$$F_3(a_8)=\{\mathrm{m}_2\}.$$

Let $A=A_1\cup A_2\cup A_3=\{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8\}$. Then the multi-soft space of $(F_1,A_1),(\mathrm{F}_2,A_2)$ and (F_3,A_3) is (F,A), where $(F,A)=(F_1,A_1)\cup_E(F_2,A_2)\cup_E(F_3,A_3)$ such that

$$F(a_1)=F_1(a_1)=\{\mathrm{m}_1,\mathrm{m}_2,\mathrm{m}_3\},\text{ as } a_1\in A_1-A_2\cup A_3,$$

$$F(a_2)=F_1(a_2)\cup\mathrm{F}_2(a_2)=\{\mathrm{m}_1,\mathrm{m}_2,\mathrm{m}_3,\mathrm{m}_4,\mathrm{m}_5,\mathrm{m}_6,\mathrm{m}_7\},\text{ as } a_2\in A_1\cap A_2-A_3,$$

$$F(a_3)=F_1(a_3)=\{\mathrm{m}_1,\mathrm{m}_4,\mathrm{m}_6,\mathrm{m}_7\}\text{ as } a_3\in A_1-A_2\cup A_3,$$

$$F(a_4)=F_2(a_4)=\{\mathrm{m}_3,\mathrm{m}_4,\mathrm{m}_5,\mathrm{m}_6\},\text{ as } a_4\in A_2-A_1\cup A_3,$$

$$F(a_5)=F_2(a_5)\cup F_3(a_5)=\{\mathrm{m}_1,\mathrm{m}_2,\mathrm{m}_3,\mathrm{m}_4,\mathrm{m}_5,\},\text{ as } a_5\in A_2\cap A_3-A_1,$$

$$\mathrm{F}(a_6)=\mathrm{F}_2(a_6)=\{\mathrm{m}_1,\mathrm{m}_7\}\text{ as } a_6\in A_2-A_1\cup A_3,$$

$$F(a_7)=F_3(a_7)=\{\mathrm{m}_4,\mathrm{m}_5,\mathrm{m}_7\}\text{ as } a_7\in A_3-A_1\cup A_2,$$

$$F(a_8)=F_1(a_8)\cup F_2(a_8)\cup F_3(a_8)=\{\mathrm{m}_1,\mathrm{m}_2,\mathrm{m}_3,\mathrm{m}_4,\mathrm{m}_5,\mathrm{m}_6,\mathrm{m}_7\}\text{ as } a_8\in A_1\cap A_2\cap A_3.$$

Definition 3.5. Let (F,A) and (H,B) be two multi-soft spaces over $M_1\cup M_2\cup\ldots\cup M_n$. Then (F,A) is called a multi-soft subspace of (H,B) if

1. $A\subseteq B$ and
2. $F(a)\subseteq H(a)$, for all $a\in A$.

This can be denoted by $(F,A)\subset(H,B)$.

Similarly (F,A) is called a multi-soft superspace of (F,A) if (F,A) is a multi-soft subspace of (F,A) which is denoted by $(F,A)\supset(H,B)$.

Definition 3.6. Two multi-soft spaces (F,A) and (H,B) over $M_1\cup M_2\cup\ldots\cup M_n$ are called multi-soft multi-equal if (F,A) is a multi-soft subspace of (H,B) and (H,B) is a multi-soft subspace of (F,A).

Proposition 3.6. Let (F,A) and (K,B) be two multi-soft spaces over $M_1\cup M_2\cup\ldots\cup M_n$ such that $A\cap B\neq\phi$. Then their restricted intersection $(F,A)\cap_R(K,B)=(H,C)$ is also a multi-soft space over $M_1\cup M_2\cup\ldots\cup M_n$.

Proposition 3.7. The extended intersection of two multi-soft multi-spaces (F,A) and (K,B) over $M_1\cup M_2\cup\ldots\cup M_n$ is again a multi-soft multi-space over $M_1\cup M_2\cup\ldots\cup M_n$.

Proposition 3.8. Let (F,A) and (K,B) be two multi-soft multi-spaces over $M_1\cup M_2\cup\ldots\cup M_n$ such that $A\cap B\neq\phi$. Then their restricted union $(F,A)\cup_R(K,B)=(H,C)$ is also a multi-soft muti-space over $M_1\cup M_2\cup\ldots\cup M_n$.

Proposition 3.9. The extended union of two multi-soft multi-spaces (F,A) and (K,B) over $M_1\cup M_2\cup\ldots\cup M_n$ is again a multi-soft multi-space over $M_1\cup M_2\cup\ldots\cup M_n$.

Proposition 3.10. The AND operation of two multi-soft multi-spaces (F,A) and (K,B) over $M_1 \cup M_2 \cup ... \cup M_n$ is again a multi-soft mulit-space over $M_1 \cup M_2 \cup ... \cup M_n$.

Proposition 3.11. The OR operation of two multi-soft multi-spaces (F,A) and (K,B) over $M_1 \cup M_2 \cup ... \cup M_n$ is again a multi-soft multi-space over $M_1 \cup M_2 \cup ... \cup M_n$.

Proposition 3.12. The complement of a multi-soft space over a multi-space M is again a multi-soft space over M.

Prof. This is straightforward.

Definition 3.13. A multi-soft multi-space (F,A) over $M_1 \cup M_2 \cup ... \cup M_n$ is called absolute multi-soft multi-space if $F(a) = M_1 \cup M_2 \cup ... \cup M_n$ for all $a \in A$.

Proposition 3.14. Let (F,A), (G,B) and (H,C) are three multi-soft multi-spaces over $M_1 \cup M_2 \cup ... \cup M_n$. Then

1. $(F,A) \cup_E (G,B) \cup_E (H,C) = (F,A) \cup_E (G,B) \cup_E (H,C)$,
2. $(F,A) \cap_R (G,B) \cap_R (H,C) = (F,A) \cap_R (G,B) \cap_R (H,C)$.

Proposition 3.15. Let (F,A), (G,B) and (H,C) are three multi-soft multi-spaces over $M_1 \cup M_2 \cup ... \cup M_n$. Then

1. $(F,A) \wedge (G,B) \wedge (H,C) = (F,A) \wedge (G,B) \wedge (H,C)$,
2. $(F,A) \vee (G,B) \vee (H,C) = (F,A) \vee (G,B) \vee (H,C)$.

Conclusion

In this paper, we introduced multi-soft spaces which is a first attempt to study the multi-spaces in the context of soft sets. Multi-soft spaces are more rich structure than the multi-spaces which represent different fields in an approximated unifying field. We also studied some properties of multi-soft spaces. A lot of further research can do in the future in this area. In the future, one can define the algebraic structures of multi-soft spaces.

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Soft Interval-Valued Neutrosophic Rough Sets

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Abstract: In this paper, we first defined soft interval-valued neutrosophic rough sets(SIVN- rough sets for short) which combines interval valued neutrosophic soft set and rough sets and studied some of its basic properties. This concept is an extension of soft interval valued intuitionistic fuzzy rough sets(SIVIF- rough sets). Finally an illustartive example is given to verfy the developped algorithm and to demonstrate its practicality and effectiveness.

Keywords: Interval valued neutrosophic soft sets, rough set, soft Interval valued neutrosophic rough sets

1. Introduction

In 1999, Florentin Smarandache introduced the concept of neutrosophic set (NS) [13] which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. The concept of neutrosophic set is the generalization of the classical sets, conventional fuzzy set [27], intuitionistic fuzzy set [24] and interval valued fuzzy set [45] and so on. A neutrosophic sets is defined on universe U. x= x(T, I, F) ∈ A with T, I and F being the real standard or non –standard subset of $]0^{-},1^{+}[$, T is the degree of truth membership of A, I is the degree of indeterminacy membership of A and F is the degree of falsity membership of A. In the neutrosophic set, indeterminacy is quantified explicitly and truth-membership, indeterminacy membership and false –membership are independent.

Recently, works on the neutrosophic set theory is progressing rapidly. M. Bhowmik and M. Pal [28, 29] defined the concept "intuitionistic neutrosophic set". Later on A. A. Salam and S. A.Alblowi [1] introduced another concept called "generalized neutrosophic set". Wang et al [18] proposed another extension of neutrosophic set called "single valued neutrosophic sets". Also, H.Wang et al. [17] introduced the notion of interval valued neutrosophic sets (IVNSs) which is an instance of neutrosophic set. The IVNSs is characterized by an interval membership degree,interval indeterminacy degree and interval non-membership degree. K.Geogiev [25] explored some properties of the neutrosophic logic and proposed a general simplification of the neutrosophic sets into a subclass of theirs, comprising of elements of R^3. Ye [20, 21] defined similarity measures between interval neutrosophic sets and their multicriteria decision-making method. P. Majumdar and S.K. Samant [34] proposed some types of similarity and entropy of neutrosophic sets. S.Broumi and F. Smarandache [38,39,40] proposed several similarity measures of neutrosophic sets. P. Chi and L. Peid [33] extended TOPSIS to interval neutrosophic sets.

In 1999, Molodtsov [8]initiated the concept of soft set theory as proposed a new mathematical for dealing with uncertainties. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applied to many different fields including game theory, operations research, Riemmann integration, Perron integration. Recently, I. Deli [10] combined the concept of soft set and interval valued neutrosophic sets together by introducing anew concept called " interval valued neutrosophic soft sets" and gave an application of interval valued neutrosophic soft sets in decision making. This concept generalizes the concept of the soft sets, fuzzy soft sets [35], intuitionistic fuzzy soft sets [36], interval valued intuitionistic fuzzy soft sets [22], the concept of neutrosophic soft sets [37] and intuitionistic neutrosophic soft sets [41].

The concept of rough set was originally proposed by Pawlak [50] as a formal tool for modeling and processing incomplete information in information systems. Rough set theory has been conceived as a tool to conceptualize, organize and analyze various types of data, in particular, to deal with inexact, uncertain or vague knowledge in applications related to artificial intelligence technique. Therefore, many models have been built upon different aspect, i.e, universe, relations, object, operators by many scholars [6, 9, 23, 48, 49, 51] such as rough fuzzy sets, fuzzy rough sets, generalized fuzzy rough, rough intuitionistic fuzzy set, intuitionistic fuzzy rough sets [26]. The rough sets has been successfully applied in many fields such as attribute reduction [19, 30, 31, 46], feature selection [11, 18, 44], rule extraction [5, 7, 12, 47] and so on. The rough sets theory approximates any subset of objects of the universe by two sets, called the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non empty intersection with the set.

Moreover, many new rough set models have also been established by combining the Pawlak rough set with other uncertainty theories such as soft set theory. Feng et al [14] provided a framework to combine fuzzy sets, rough sets, and soft sets all together, which gives rise to several interesting new concepts such as rough soft sets, soft rough sets, and soft rough fuzzy sets. The combination of hybrid structures of soft sets and rough sets models was also discussed by some researchers [15,32,43]. Later on, J. Zhang, L. Shu, and S. Liao [22] proposed the notions of soft rough intuitionistic fuzzy sets and intuitionistic fuzzy soft rough sets, which can be seen as two new generalized soft rough set models, and investigated some properties of soft rough intuitionistic fuzzy sets and intuitionistic fuzzy soft rough sets in detail. A.Mukherjee and A. Saha [3] proposed the concept of interval valued intuitionistic fuzzy soft rough sets. Also A. Saha and A. Mukherjee [4] introduced the concept of Soft interval valued intuitionistic fuzzy rough sets.

More recently, S.Broumi et al. [42] combined neutrosophic sets with rough sets in a new hybrid mathematical structure called "rough neutrosophic sets" handling incomplete and indeterminate information . The concept of rough neutrosophic sets generalizes rough fuzzy sets and rough intuitionistic fuzzy sets. Based on the equivalence relation on the universe of discourse, A. Mukherjee et al. [3] introduced soft lower and upper approximation of interval valued intuitionistic fuzzy set in Pawlak's approximation space. Motivated by the idea of soft interval valued intuitionistic fuzzy rough sets introduced in [4], we extend the soft interval intuitionistic fuzzy rough to the case of an interval valued neutrosophic set. The concept of soft interval valued neutrosophic rough set is introduced by coupling both the interval valued neutrosophic soft sets and rough sets.

The paper is structured as follows. In Section 2, we first recall the necessary background on soft sets, interval neutrosophic sets, interval neutrosophic soft sets, rough set, rough neutrosophic sets and soft interval valued intuitionistic fuzzy rough sets. Section 3 presents the concept of soft interval neutrosophic rough sets and examines their respective properties. Section 4 presents a multiciteria group decision making scheme under soft interval –valued neutrosophic rough sets. Section 5 presents an application of multiciteria group decision making scheme regarding the candidate selection problem . Finally we concludes the paper.

2. Preliminaries

Throughout this paper, let U be a universal set and E be the set of all possible parameters under consideration with respect to U, usually, parameters are attributes, characteristics, or properties of objects in U. We now recall some basic notions of soft sets, interval neutrosophic setsinterval neutrosophic soft set, rough set, rough neutrosophic sets and soft interval valued intuitionistic fuzzy rough sets. For more details the reader may refer to [4, 8, 10, 13, 17, 50, 42].

Definition 2.1 [13] : Let U be an universe of discourse then the neutrosophic set A is an object having the form A = {< x: $\mu_A(x)$, $\nu_A(x)$, $\omega_A(x)$>,x $\in$ U}, where the functions $\mathbf{\mu_A(x), \nu_A(x), \omega_A(x)}$: U→]$^-$0,1$^+$[define respectively the degree of membership , the degree of indeterminacy, and the degree of non-membership of the element x $\in$ X to the set A with the condition.

$$^-0 \leq sup\ \mu_A(x)+ \sup\nu_A(x)+ sup\omega_A(x) \leq 3^+. \quad (1)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of]$^-$0,1$^+$[. So instead of]$^-$0,1$^+$[we need to take the interval [0,1] for technical applications, because]$^-$0,1$^+$[will be difficult to apply in the real applications such as in scientific and engineering problems.

Definition 2.3 [13]

Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS) A in X is characterized by truth-membership function $\mu_A(x)$, indeterminacy-membership function $\nu_A(x)$ and falsity-membership function $\omega_A(x)$. For each point x in X, we have that $\mu_A(x), \nu_A(x), \omega_A(x) \in$ int([0 ,1]).

For two IVNS, A_{IVNS}= {<x , $[\mu_A^L(x), \mu_A^U(x)]$, $[\nu_A^L(x), \nu_A^U(x)]$, $[\omega_A^L(x), \omega_A^U(x)]$ > | x ∈ X } (2)

And B_{IVNS}= {<x , $[\mu_B^L(x), \mu_B^U(x)]$, $[\nu_B^L(x), \nu_B^U(x)]$, $[\omega_B^L(x), \omega_B^U(x)]$> | x ∈ X } the two relations are defined as follows:

(1)$A_{IVNS} \subseteq B_{IVNS}$if and only if $\mu_A^L(x) \leq \mu_B^L(x)$, $\mu_A^U(x) \leq \mu_B^U(x)$,$\nu_A^L(x) \geq \nu_B^L(x)$, $\omega_A^U(x) \geq \omega_B^U(x)$, $\omega_A^L(x) \geq \omega_B^L(x)$,$\omega_A^U(x) \geq \omega_B^U(x)$.

(2)$A_{\mathrm{IVNS}} = B_{\mathrm{IVNS}}$ if and only if , $\mu_A(x) = \mu_B(x)$,$\nu_A(x) = \nu_B(x)$,$\omega_A(x) = \omega_B(x)$ for any x ∈ X

The complement of A_{IVNS} is denoted by A^o_{IVNS} and is defined by

A^o_{IVNS}={ <x , $[\omega^L_A(x), \omega^U_A(x)]$, $[1-\nu^U_A(x), 1-\nu^L_A(x)]$, $[\mu^L_A(x), \mu^U_A(x)]$ | x ∈ X }

A∩B ={ <x , [min($\mu^L_A(x)$,$\mu^L_B(x)$), min($\mu^U_A(x)$,$\mu^U_B(x)$)], [max($\nu^L_A(x)$,$\nu^L_B(x)$), max($\nu^U_A(x)$,$\nu^U_B(x)$)], [max($\omega^L_A(x)$,$\omega^L_B(x)$), max($\omega^U_A(x)$,$\omega^U_B(x)$)] >: x ∈ X }

A∪B ={ <x , [max($\mu^L_A(x)$,$\mu^L_B(x)$), max($\mu^U_A(x)$,$\mu^U_B(x)$)], [min($\nu^L_A(x)$,$\nu^L_B(x)$),min($\nu^U_A(x)$,$\nu^U_B(x)$)], [min($\omega^L_A(x)$,$\omega^L_B(x)$), min($\omega^U_A(x)$,$\omega^U_B(x)$)] >: x ∈ X }

As an illustration, let us consider the following example.

Example 2.4. Assume that the universe of discourse U={x_1, x_2, x_3}, where x_1 characterizes the capability, x_2 characterizes the trustworthiness and x_3 indicates the prices of the objects. It may be further assumed that the values of x_1, x_2 and x_3 are in [0, 1] and they are obtained from some questionnaires of some experts. The experts may impose their opinion in three components viz. the degree of goodness, the degree of indeterminacy and that of poorness to explain the characteristics of the objects. Suppose A is an interval valued neutrosophic set (IVNS) of U, such that, A = {< x_1,[0.3 0.4],[0.5 0.6],[0.4 0.5] >,< x_2, ,[0.1 0.2],[0.3 0.4],[0.6 0.7]>,< x_3, [0.2 0.4],[0.4 0.5],[0.4 0.6] >}, where the degree of goodness of capability is [0.3, 0.4], degree of indeterminacy of capability is[0.5, 0.6] and degree of falsity of capability is [0.4, 0.5] etc.

Definition 2.5 . [8]

Let U be an initial universe set and E be a set of parameters. Let P(U) denote the power set of U. Consider a nonempty set A, A ⊂ E. A pair (K, A) is called a soft set over U, where K is a mapping given by K : A → P(U).

As an illustration, let us consider the following example.

Example 2.6 . Suppose that U is the set of houses under consideration, say U = {h_1, h_2, . . ., h_5}. Let E be the set of some attributes of such houses, say E = {e_1, e_2, . . ., e_8}, where e_1, e_2, . . ., e_8 stand for the attributes "beautiful", "costly", "in the green surroundings'", "moderate", respectively.

In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. For example, the soft set (K, A) that describes the "attractiveness of the houses" in the opinion of a buyer, say Thomas, may be defined like this:

A={e_1,e_2,e_3,e_4,e_5};

K(e_1) = {h_2, h_3, h_5}, K(e_2) = {h_2, h_4}, K(e_3) = {h_1}, K(e_4) = U, K(e_5) = {h_3, h_5}.

Definition 2.7. [10]

Let U be an initial universe set and A ⊂ E be a set of parameters. Let IVNS (U) denote the set of all interval valued neutrosophic subsets of U. The collection (K, A) is termed to be the soft interval neutrosophic set over U, where F is a mapping given by K: A → IVNS(U).

The interval valued neutrosophic soft set defined over an universe is denoted by IVNSS.

Here,

1. ϒ is an ivn-soft subset of Ψ, denoted by ϒ ⋐ Ψ, if K(e) ⊆L(e) for all e∈E.
2. ϒ is an ivn-soft equals to Ψ, denoted by ϒ = Ψ, if K(e)=L(e) for all e∈E.
3. The complement of ϒ is denoted by $ϒ^c$, and is defined by $ϒ^c$ = {(x, K^o (x)): x∈E}
4. The union of ϒ and Ψ is denoted by ϒ ∪̈ Ψ, if K(e) ∪L(e) for all e∈E.
5. The intersection of ϒand Ψ is denoted by ϒ ∩̈ Ψ,if K(e) ∪ L(e) for all e∈E.

Example 2.8 :

Let U be the set of houses under consideration and E is the set of parameters (or qualities). Each parameter is an interval neutrosophic word or sentence involving interval neutrosophic words. Consider E = { beautiful, costly, moderate, expensive }. In this case, to define an interval neutrosophic soft set means to point out beautiful houses, costly houses, and so on. Suppose that, there are four houses in the universe U given by, U = {h_1,h_2,h_3,h_4 } and the set of parameters A = {e_1,e_2,e_3}, where each e_i is a specific criterion for houses:

e_1 stands for 'beautiful',

e_2 stands for 'costly',

e_3 stands for 'moderate',

Suppose that,

K(beautiful)={< h_1,[0.5, 0.6], [0.6, 0.7], [0.3, 0.4]>,< h_2,[0.4, 0.5], [0.7 ,0.8], [0.2, 0.3] >, < h_3,[0.6, 0.7],[0.2 ,0.3],[0.3, 0.5] >,< h_4,[0.7 ,0.8],[0.3, 0.4],[0.2, 0.4] >}

.K(costly)={< h_1,[0.3, 0.6], [0.2 0.7], [0.1, 0.4]>,< h_2,[0.3, 0.5], [0.6 ,0.8], [0.2, 0.6] >, < h_3,[0.3, 0.7],[0.1 ,0.3],[0.3, 0.6] >,< h_4,[0.6 ,0.8],[0.2, 0.4],[0.2, 0.5 >}

K(moderate)={< h_1,[0.5, 0.8], [0.4, 0.7], [0.3, 0.6]>,< h_2,[0.3, 0.5], [0.7 ,0.9], [0.2, 0.4] >, < h_3,[0.1, 0.7],[0.3 ,0.3],[0.3, 0.6] >,< h_4,[0.3,0.8],[0.2, 0.4],[0.3, 0.6] >}.

Defintion.2.9 [50]
Let R be an equivalence relation on the universal set U. Then the pair (U, R) is called a Pawlak approximation space. An equivalence class of R containing x will be denoted by $[x]_R$. Now for X ⊆ U, the lower and upper approximation of X with respect to (U, R) are denoted by respectively R_*X and $\boldsymbol{R}^*X$ and are defined by

$R_*X=\{x \in U: [x]_R \subseteq X\}$,

$R^*X=\{ x \in U: [x]_R \cap X \neq \emptyset\}$.

Now if $R_*X = R^* X$, then X is called definable; otherwise X is called a rough set.

Definition 2.10 [42]
Let U be a non-null set and R be an equivalence relation on U. Let F be neutrosophic set in U with the membership function μ_F, indeterminacy function ν_F and non-membership function ω_F. Then, the lower and upper rough approximations of F in (U, R) are denoted by $\underline{R}$ (F) and $\overline{R}$(F) and respectively defined as follows:

$\overline{R}(F) =\{ <x, \mu_{\overline{R}(F)}(x), \nu_{\overline{R}(F)}(x), \omega_{\overline{R}(F)}(x)> \mid x\in U\}$,

$\underline{R}(F) =\{ <x, \mu_{\underline{R}(F)}(x), \nu_{\underline{R}(F)}(x), \omega_{\underline{R}(F)}(x)> \mid x\in U\}$,

Where:

$\mu_{\overline{R}(F)}(x) =\vee_{y \in [x]_R} \mu_F(y)$, $\nu_{\overline{R}(F)}(x)=\wedge_{y \in [x]_R} \nu_F(y)$, $\omega_{\overline{R}(F)} =\wedge_{y \in [x]_R} \omega_F(y)$,

$\mu_{\underline{R}(F)}(x)=\wedge_{y \in [x]_R} \mu_F(y)$, $\nu_{\underline{R}(F)}(x)=\vee_{y \in [x]_R} \nu_F(y)$, $\omega_{\underline{R}(F)} =\vee_{y \in [x]_R} \omega_F(y)$,

It is easy to observe that $\overline{R}$(F) and $\underline{R}$(F) are two neutrosophic sets in U, thus NS mapping $\overline{R}$, $\underline{R}$:R(U) → R(U) are, respectively, referred to as the upper and lower rough NS approximation operators, and the pair ($\underline{R}$(F), $\overline{R}$(F)) is called the rough neutrosophic set.

Definition 2.11[4] . Let us consider an interval-valued intuitionstic fuzzy set σ defined by

$\sigma = \{x, \mu_\sigma(x), \nu_\sigma(x): x \in U\}$ where $\mu_\sigma(x), \nu_\sigma(x), \in$ int ([0, 1]) for each x ∈ U and

$0 \leq \mu_\sigma(x)+ \nu_\sigma(x) \leq 1$

Now Let Θ=(f,A) be an interval-valued intuitionstic fuzzy soft set over U and the pair SIVIF= (U, Θ) be the soft interval-valued intuitionistic fuzzy approximation space.

Let f:A→ IVIFS^U be defined f(a) ={ x, $\mu_{f(a)}(x)$, $\nu_{f(a)}(x)$: x ∈ U } for each a ∈ A. Then , the lower and upper soft interval-valued intuitionistic fuzzy rough approximations of σ with respect to SIVIF are denoted by $\downarrow \mathrm{Apr}_{\mathrm{SIVIF}}(\sigma)$ and $\uparrow \mathrm{Apr}_{\mathrm{SIVIF}}(\sigma)$ respectively, which are interval valued intuitionistic fuzzy sets in U given by:

$\downarrow \mathrm{Apr}_{\mathrm{SIVIF}}(\sigma) =\{< x,$
$[\wedge_{a \in A} (\inf \mu_{f(a)}(x) \wedge \inf \mu_\sigma(x)),\ \wedge_{a \in A} (\sup \mu_{f(a)}(x) \wedge \sup \mu_\sigma(x)],\ [\wedge_{a \in A} (\inf \nu_{f(a)}(x) \vee \inf \nu_\sigma(x)),$
$\wedge_{a \in A} (\sup \nu_{f(a)}(x) \vee \sup \nu_\sigma(x)]>: x \in U \}$

$\uparrow \mathrm{Apr}_{\mathrm{SIVIF}}(\sigma) =\{< x, [\wedge_{a \in A} (\inf \mu_{f(a)}(x) \vee \inf \mu_\sigma(x)),$
$\wedge_{a \in A} (\sup \mu_{f(a)}(x) \vee \sup \mu_\sigma(x)],\ [\wedge_{a \in A} (\inf \nu_{f(a)}(x) \wedge \inf \nu_\sigma(x)),\ \wedge_{a \in A} (\sup \nu_{f(a)}(x) \wedge \sup \nu_\sigma(x)] >: x \in U\}$

The operators $\downarrow \mathrm{Apr}_{\mathrm{SIVIF}}(\sigma)$ and $\uparrow \mathrm{Apr}_{\mathrm{SIVIF}}(\sigma)$ are called the lower and upper soft interval-valued intuitionistic fuzzy rough approximation operators on interval valued intuitionistic fuzzy sets. If $\downarrow \mathrm{Apr}_{\mathrm{SIVIF}}(\sigma) = \uparrow \mathrm{Apr}_{\mathrm{SIVIF}}(\sigma)$, then σ is said to be soft interval valued intuitionistic fuzzy definable; otherwise is called a soft interval valued intuitionistic fuzzy rough set.

Example 3.3 . Let U={x, y) and A={a, b}. Let (f, A) be an interval –valued intuitionstic fuzzy soft set over U where f:A→ IVIFS^U be defined

f(a)= { <x,[0.2, 0. 5], [0.3, 0.4]>, <y, [0.6, 0.7],[0.1, 0.2] >}

f(b)= { <x,[0.1, 0. 3], [0.4, 0.5>, <y, [0.5, 0.8],[0.1, 0.2] >}

Let σ = { <x,[0.3, 0.4], [0.3, 0.4]>, <y, [0.2, 0.4],[0.4, 0.5] >}. Then

$\downarrow \mathrm{Apr}_{\mathrm{SIVIF}}(\sigma)$= { <$x$,[0.1, 0.3],[0.3, 0.4] >, <y,[0.2, 0.4],[0.4, 0.5]>}

$\uparrow \mathrm{Apr}_{\mathrm{SIVIF}}(\sigma)$ = { <x,[0.3, 0.4],[0.3, 0.4] >, <y,[0.5, 0.7],[0.1, 0.2]>}. Then σ is a soft interval-valued intuitionstic fuzzy rough set.

3. Soft Interval Neutrosophic Rough Set.

A. Saha and A. Mukherjee [4] used the interval valued intuitioinstic fuzzy soft set to granulate the universe of discourse and obtained a mathematical model called soft interval –valued intuitionistic fuzzy rough set. Because the soft interval –valued intuitionistic fuzzy rough set cannot deal with indeterminate and inconsistent data, in this section, we attempt to develop an new concept called soft interval –valued neutrosophic rough sets.

Definition 3.1. Let us consider an interval-valued neutrosophic set σ defined by

$\sigma = \{x, \mu_\sigma(x), \nu_\sigma(x), \omega_\sigma(x) : x \in U\}$ where $\mu_\sigma(x)$, $\nu_\sigma(x), \omega_\sigma(x) \in$ int ([0, 1]) for each x ∈ U and

$$0 \leq \mu_\sigma(x)+ \nu_\sigma(x) + \omega_\sigma(x) \leq 3$$

Now Let Θ=(f,A) be an interval-valued neutrosophic soft set over U and the pair SIVN= (U, Θ) be the soft interval-valued neutrosophic approximation space.

Let f:A→ $IVNS^U$ be defined f(a) ={ x, $\mu_{f(a)}(x)$, $\nu_{f(a)}(x)$, $\omega_{f(a)}(x)$: x ∈ U } for each a ∈ A. Then , the lower and upper soft interval-valued neutrosophic rough approximations of σ with respect to SIVN are denoted by $\downarrow \mathrm{Apr_{SIVN}}(\sigma)$ and $\uparrow \mathrm{Apr_{SIVN}}(\sigma)$ respectively, which are interval valued neutrosophic sets in U given by:

$\downarrow \mathrm{Apr_{SIVN}}(\sigma)$ ={<x, [$\wedge_{a\in A}$ (inf $\mu_{f(a)}$(x) ∧ inf μ_σ(x)), $\wedge_{a\in A}$ (sup $\mu_{f(a)}$(x) ∧ sup μ_σ(x)], [$\wedge_{a\in A}$ (inf $\nu_{f(a)}$(x) ∨ inf ν_σ(x)), $\wedge_{a\in A}$ (sup $\nu_{f(a)}$(x) ∨ sup ν_σ(x)], [$\wedge_{a\in A}$ (inf $\omega_{f(a)}$(x) ∨ inf ω_σ(x)) , $\wedge_{a\in A}$ (sup $\omega_{f(a)}$(x) ∨ sup ω_σ(x)]>: x ∈ U }

$\uparrow \mathrm{Apr_{SIVN}}(\sigma)$ ={< x, [$\wedge_{a\in A}$ (inf $\mu_{f(a)}$(x) ∨ inf μ_σ(x)) , $\wedge_{a\in A}$ (sup $\mu_{f(a)}$(x) ∨ sup μ_σ(x)], [$\wedge_{a\in A}$ (inf $\nu_{f(a)}$(x) ∧ inf ν_σ(x)) , $\wedge_{a\in A}$ (sup $\nu_{f(a)}$(x) ∧ sup ν_σ(x)], [$\wedge_{a\in A}$ (inf $\omega_{f(a)}$(x) ∧ inf ω_σ(x)) , $\wedge_{a\in A}$ (sup $\omega_{f(a)}$(x) ∧ sup ω_σ(x)]>: x ∈ U}

The operators $\downarrow \mathrm{Apr_{SIVN}}(\sigma)$ and $\uparrow \mathrm{Apr_{SIVN}}(\sigma)$ are called the lower and upper soft interval-valued neutrosophic rough approximation operators on interval valued neutrosophic sets. If $\downarrow \mathrm{Apr_{SIVN}}(\sigma) = \uparrow \mathrm{Apr_{SIVN}}(\sigma)$, then σ is said to be soft interval valued neutrosophic definable; otherwise is called a soft interval valued neutrosophic rough set.

Remark 3.2: it is to be noted that if μ_σ(x), ν_σ(x), ω_σ(x) ∈ int ([0, 1]) and 0 ≤ μ_σ(x)+ ν_σ(x) + ω_σ(x) ≤ 1, then soft interval valued neutrosophic rough sets becomes soft interval valued intuitionistic fuzzy rough sets.

Example 3.3 . Let U={x, y) and A={a, b}. Let (f, A) be an interval –valued neutrosophic soft se over U where f:A→ $IVNS^U$ be defined

f(a)= {<x,[0.2, 0. 5],[0.3, 0.4],[0.4, 0.5]>,<y,[0.6, 0.7],[0.1, 0.2],[0.3 0.4]>}

f(b)={<x,[0.1, 0. 3],[0.4, 0.5],[0.1, 0.2]>, <y,[0.5, 0.8],[0.1, 0.2],[0.1 0.2]>}

Let σ ={<x,[0.3, 0.4],[0.3, 0.4],[0.1, 0.2]>, <y,[0.2, 0.4],[0.4, 0.5],[0.2 0.3]>}. Then

$\downarrow \mathrm{Apr_{SIVN}}(\sigma)$= { <$x$,[0.1, 0.3],[0.3, 0.4],[0.1, 0.2]>, <y,[0.2, 0.4],[0.4, 0.5],[0.2, 0.3]>}

$\uparrow \mathrm{Apr_{SIVN}}(\sigma)$ = { <x,[0.3, 0.4],[0.3, 0.4],[0.1, 0.2]>, <y,[0.5, 0.7],[0.1, 0.2],[0.1, 0.2]>}. Then σ is a soft interval-valued neutrosophic rough set.

Theorem 3.4

Let Θ=(f,A) be an interval-valued neutrosophic soft set over U and SIVN= (U, Θ) be the soft interval-valued neutrosophic approximation space. Then for σ, $\lambda \in \mathrm{IVNS^U}$, we have

1) $\downarrow \mathrm{Apr_{SIVN}}(\emptyset) = \emptyset = \uparrow \mathrm{Apr_{SIVN}}(\emptyset)$
2) $\downarrow \mathrm{Apr_{SIVN}}(U) = U = \uparrow \mathrm{Apr_{SIVN}}(U)$
3) $\sigma \subseteq \lambda \Rightarrow \downarrow \mathrm{Apr_{SIVN}}(\sigma) \subseteq \downarrow \mathrm{Apr_{SIVN}}(\lambda)$
4) $\sigma \subseteq \lambda \Rightarrow \uparrow \mathrm{Apr_{SIVN}}(\sigma) \subseteq \uparrow \mathrm{Apr_{SIVN}}(\lambda)$
5) $\downarrow \mathrm{Apr_{SIVN}}(\sigma \cap \lambda) \subseteq \downarrow \mathrm{Apr_{SIVN}}(\sigma) \cap \downarrow \mathrm{Apr_{SIVN}}(\lambda)$.
6) $\uparrow \mathrm{Apr_{SIVN}}(\sigma \cap \lambda) \subseteq \uparrow \mathrm{Apr_{SIVN}}(\sigma) \cap \uparrow \mathrm{Apr_{SIVN}}(\lambda)$**.**
7) $\downarrow \mathrm{Apr_{SIVN}}(\sigma) \cup \downarrow \mathrm{Apr_{SIVN}}(\lambda) \subseteq \downarrow \mathrm{Apr_{SIVN}}(\sigma \cup \lambda)$.
8) $\uparrow \mathrm{Apr_{SIVN}}(\sigma) \cup \uparrow \mathrm{Apr_{SIVN}}(\lambda) \subseteq \uparrow \mathrm{Apr_{SIVN}}(\sigma \cup \lambda)$

Proof .(1)-(4) are straight forward.

(5) We have

σ={<x,[inf μ_σ(x), sup μ_σ(x)], [inf ν_σ(x), sup ν_σ(x)], [inf ω_σ(x), sup ω_σ(x)]>:x∈ U},

λ ={<x,[inf μ_λ(x), sup μ_λ(x)], [inf ν_λ(x), sup ν_λ(x)], [inf ω_λ(x), sup ω_λ(x)]>:x∈ U}

and

$\sigma \cap \lambda$= {<x,[inf $\mu_{\sigma\cap\lambda}$(x), sup $\mu_{\sigma\cap\lambda}$(x)], [inf $\nu_{\sigma\cap\lambda}$(x), sup $\nu_{\sigma\cap\lambda}$(x)], [inf $\omega_{\sigma\cap\lambda}$(x), sup $\omega_{\sigma\cap\lambda}$(x)]>:x∈ U},

Now

$\downarrow \mathrm{Apr_{SIVN}}(\sigma \cap \lambda)$={< x, [$\wedge_{a\in A}$ (inf $\mu_{f(a)}$(x) ∧ inf $\mu_{\sigma\cap\lambda}$(x)) , $\wedge_{a\in A}$ (sup $\mu_{f(a)}$(x) ∧ sup $\mu_{\sigma\cap\lambda}$(x)], [$\wedge_{a\in A}$ (inf $\nu_{f(a)}$(x) ∨ inf $\nu_{\sigma\cap\lambda}$(x)) , $\wedge_{a\in A}$ (sup $\nu_{f(a)}$(x) ∨ sup $\nu_{\sigma\cap\lambda}$(x)], [$\wedge_{a\in A}$ (inf $\omega_{f(a)}$(x) ∨ inf $\omega_{\sigma\cap\lambda}$(x)) , $\wedge_{a\in A}$ (sup $\omega_{f(a)}$(x) ∨ sup $\omega_{\sigma\cap\lambda}$(x)]>: x ∈ U }

={< x, [$\wedge_{a\in A}$ (inf $\mu_{f(a)}$(x) ∧ min(inf μ_σ (x) , inf μ_λ(x)) , $\wedge_{a\in A}$ (sup $\mu_{f(a)}$(x) ∧ min(sup μ_σ (x) , sup μ_λ(x))], [$\wedge_{a\in A}$ (inf $\nu_{f(a)}$(x) ∨ max(inf ν_σ (x) , inf ν_λ(x))) , $\wedge_{a\in A}$ (sup $\nu_{f(a)}$(x) ∨ max(sup ν_σ (x) , sup ν_λ(x))], [$\wedge_{a\in A}$ (inf $\omega_{f(a)}$(x) ∨ max(inf ω_σ (x) , inf ω_λ(x))) , $\wedge_{a\in A}$ (sup $\omega_{f(a)}$(x) ∨ max(sup ω_σ (x) , sup ω_λ(x)]>: x ∈ U }

Now $\downarrow \mathrm{Apr_{SIVN}}(\sigma) \cap \downarrow \mathrm{Apr_{SIVN}}(\lambda)$.

= {< x, [**min (** $\wedge_{a\in A}$ (inf $\mu_{f(a)}$(x) ∧ inf μ_σ (x)) , $\wedge_{a\in A}$ (inf $\mu_{f(a)}$(x) ∧ inf μ_λ (x))), **min (** $\wedge_{a\in A}$ (sup $\mu_{f(a)}$(x) ∧ sup μ_σ (x)) , $\wedge_{a\in A}$ (sup $\mu_{f(a)}$(x) ∧ sup μ_λ (x)))] ,[**max (** $\wedge_{a\in A}$ (inf $\nu_{f(a)}$(x) ∨ inf ν_σ (x)) , $\wedge_{a\in A}$ (inf $\nu_{f(a)}$(x) ∨ inf ν_λ (x))), **max(** $\wedge_{a\in A}$ (sup $\nu_{f(a)}$(x) ∨ sup ν_σ (x)) , $\wedge_{a\in A}$ (sup $\nu_{f(a)}$(x) ∨ sup ν_λ (x)))], [**max (** $\wedge_{a\in A}$ (inf $\omega_{f(a)}$(x) ∨ inf ω_σ (x))

, $\wedge_{a\in A}(\inf \omega_{f(a)}(x) \vee \inf \omega_{\lambda}(x))$), **max(** $\wedge_{a\in A}(\sup \omega_{f(a)}(x) \vee \sup \omega_{\sigma}(x))$, $\wedge_{a\in A}(\sup \omega_{f(a)}(x) \vee \sup \omega_{\lambda}(x))$)]> : x∈ U}.

Since $\min(\inf \mu_{\sigma}(y), \inf\mu_{\lambda}(y)) \leq \inf \mu_{\sigma}(y)$
and $\min(\inf \mu_{\sigma}(y), \inf\mu_{\lambda}(y)) \leq \inf \mu_{\lambda}(y)$
we have
$\wedge_{a\in A}(\inf \mu_{f(a)}(x) \wedge \min(\inf \mu_{\sigma}(x), \inf \mu_{\lambda}(x)) \leq \wedge_{a\in A}(\inf \mu_{f(a)}(x) \wedge \inf \mu_{\sigma}(x))$
and $\wedge_{a\in A}(\inf \mu_{f(a)}(x) \wedge \min(\inf \mu_{\sigma}(x), \inf \mu_{\lambda}(x)) \leq \wedge_{a\in A}(\inf \mu_{f(a)}(x) \wedge \inf \mu_{\lambda}(x))$

Hence $\wedge_{a\in A}(\inf \mu_{f(a)}(x) \wedge \min(\inf \mu_{\sigma}(x), \inf \mu_{\lambda}(x)) \leq$ **min** $(\wedge_{a\in A}(\inf \mu_{f(a)}(x) \wedge \inf \mu_{\sigma}(x)), \wedge_{a\in A}(\inf \mu_{f(a)}(x) \wedge \inf \mu_{\lambda}(x)))$

Similarly
$\wedge_{a\in A}(\sup \mu_{f(a)}(x) \wedge \min(\sup \mu_{\sigma}(x), \sup \mu_{\lambda}(x)) \leq$ **min** $(\wedge_{a\in A}(\sup \mu_{f(a)}(x) \wedge \sup \mu_{\sigma}(x)), \wedge_{a\in A}(\sup \mu_{f(a)}(x) \wedge \sup \mu_{\lambda}(x)))$
Again since
$\max(\inf \nu_{\sigma}(y), \inf\nu_{\lambda}(y)) \geq \inf \nu_{\sigma}(y)$
and $\max(\inf \nu_{\sigma}(y), \inf\nu_{\lambda}(y)) \geq \inf \nu_{\lambda}(y)$

we have
$\wedge_{a\in A}(\inf \nu_{f(a)}(x) \vee \max(\inf \nu_{\sigma}(x), \inf \nu_{\lambda}(x)) \geq \wedge_{a\in A}(\inf \nu_{f(a)}(x) \vee \inf \nu_{\sigma}(x))$
and $\wedge_{a\in A}(\inf \nu_{f(a)}(x) \vee \max(\inf \nu_{\sigma}(x), \inf \nu_{\lambda}(x)) \geq \wedge_{a\in A}(\inf \nu_{f(a)}(x) \vee \inf \nu_{\lambda}(x))$

Hence $\wedge_{a\in A}(\inf \nu_{f(a)}(x) \vee \max(\inf \nu_{\sigma}(x), \inf \nu_{\lambda}(x)) \geq$ **max** $(\wedge_{a\in A}(\inf \nu_{f(a)}(x) \vee \inf \nu_{\sigma}(x)), \wedge_{a\in A}(\inf \nu_{f(a)}(x) \vee \inf \nu_{\lambda}(x)))$

Similarly
$\wedge_{a\in A}(\sup \nu_{f(a)}(x) \vee \max(\sup \nu_{\sigma}(x), \sup \nu_{\lambda}(x)) \geq$ **max** $(\wedge_{a\in A}(\sup \nu_{f(a)}(x) \vee \sup \nu_{\sigma}(x)), \wedge_{a\in A}(\sup \nu_{f(a)}(x) \vee \sup \nu_{\lambda}(x)))$

Again since
$\max(\inf \omega_{\sigma}(y), \inf\omega_{\lambda}(y)) \geq \inf \omega_{\sigma}(y)$
And $\max(\inf \omega_{\sigma}(y), \inf\omega_{\lambda}(y)) \geq \inf \omega_{\lambda}(y)$

we have
$\wedge_{a\in A}(\inf \omega_{f(a)}(x) \vee \max(\inf \omega_{\sigma}(x), \inf \omega_{\lambda}(x)) \geq \wedge_{a\in A}(\inf \nu_{\omega f(a)}(x) \vee \inf \omega_{\sigma}(x))$
and $\wedge_{a\in A}(\inf \omega_{f(a)}(x) \vee \max(\inf \omega_{\sigma}(x), \inf \omega_{\lambda}(x)) \geq \wedge_{a\in A}(\inf \omega_{f(a)}(x) \wedge \inf \omega_{\lambda}(x))$

Hence
$\wedge_{a\in A}(\inf \omega_{f(a)}(x) \vee \max(\inf \omega_{\sigma}(x), \inf \nu_{\lambda}(x)) \geq$ **max** $(\wedge_{a\in A}(\inf \omega_{f(a)}(x) \vee \inf \omega_{\sigma}(x)), \wedge_{a\in A}(\inf \omega_{f(a)}(x) \vee \inf \omega_{\lambda}(x)))$

Similarly
$\wedge_{a\in A}(\sup \omega_{f(a)}(x) \vee \max(\sup \omega_{\sigma}(x), \sup \omega_{\lambda}(x)) \geq$ **max** $(\wedge_{a\in A}(\sup \omega_{f(a)}(x) \vee \sup \omega_{\sigma}(x)), \wedge_{a\in A}(\sup \omega_{f(a)}(x) \vee \sup \omega_{\lambda}(x)))$
Consequently,
$\downarrow \mathrm{Apr}_{\mathrm{SIVN}}(\sigma \cap \lambda) \subseteq \downarrow \mathrm{Apr}_{\mathrm{SIVN}}(\sigma) \cap \downarrow \mathrm{Apr}_{\mathrm{SIVN}}(\lambda)$.

(6) Proof is similar to (5).
(7) we have
σ={<x,[$\inf \mu_{\sigma}(x)$, $\sup \mu_{\sigma}(x)$] ,[$\inf \nu_{\sigma}(x)$, $\sup \nu_{\sigma}(x)$] ,[$\inf \omega_{\sigma}(x)$, $\sup \omega_{\sigma}(x)$]>:x∈ U},
λ ={<x,[$\inf \mu_{\lambda}(x)$, $\sup \mu_{\lambda}(x)$],[$\inf \nu_{\lambda}(x)$, $\sup \nu_{\lambda}(x)$] ,[$\inf \omega_{\lambda}(x)$, $\sup \omega_{\lambda}(x)$]>:x∈ U}
And

$\sigma \cup \lambda$= {<x,[$\inf \mu_{\sigma \cup \lambda}(x)$, $\sup \mu_{\sigma \cup \lambda}(x)$], [$\inf \nu_{\sigma \cup \lambda}(x)$, $\sup \nu_{\sigma \cup \lambda}(x)$], [$\inf \omega_{\sigma \cup \lambda}(x)$, $\sup \omega_{\sigma \cup \lambda}(x)$]>:x∈ U},
$\downarrow \mathrm{Apr}_{\mathrm{SIVN}}(\sigma \cup \lambda)$={< **x, [** $\wedge_{a \in A} (\inf \mu_{f(a)}(x) \wedge \inf \mu_{\sigma \cup \lambda}(x))$, $\wedge_{a \in A} (\sup \mu_{f(a)}(x) \wedge \sup \mu_{\sigma \cup \lambda}(x)]$,
[$\wedge_{a \in A} (\inf \nu_{f(a)}(x) \vee \inf \nu_{\sigma \cup \lambda}(x))$, $\wedge_{a \in A} (\sup \nu_{f(a)}(x) \vee \sup \nu_{\sigma \cup \lambda}(x)]$, **[** $\wedge_{a \in A} (\inf \omega_{f(a)}(x) \vee \inf \omega_{\sigma \cup \lambda}(x))$,
$\wedge_{a \in A} (\sup \omega_{f(a)}(x) \vee \sup \omega_{\sigma \cup \lambda}(x)]$>: x ∈ U }
={< **x, [** $\wedge_{a \in A} (\inf \mu_{f(a)}(x) \wedge \max(\inf \mu_{\sigma}(x), \inf \mu_{\lambda}(x))$, $\wedge_{a \in A} (\sup \mu_{f(a)}(x) \wedge \max(\sup \mu_{\sigma}(x), \sup \mu_{\lambda}(x))]$,
[$\wedge_{a \in A} (\inf \nu_{f(a)}(x) \vee \min(\inf \nu_{\sigma}(x), \inf \nu_{\lambda}(x)))$, $\wedge_{a \in A} (\sup \nu_{f(a)}(x) \vee \min(\sup \nu_{\sigma}(x), \sup \nu_{\lambda}(x))]$,
[$\wedge_{a \in A} (\inf \omega_{f(a)}(x) \vee \min(\inf \omega_{\sigma}(x), \inf \omega_{\lambda}(x)))$, $\wedge_{a \in A} (\sup \omega_{f(a)}(x) \vee \min(\sup \omega_{\sigma}(x), \sup \omega_{\lambda}(x)]$>: x ∈ U }

Now $\downarrow \mathrm{Apr}_{\mathrm{SIVN}}(\sigma) \cup \downarrow \mathrm{Apr}_{\mathrm{SIVN}}(\lambda)$.

= {< **x, [max (** $\wedge_{a \in A} (\inf \mu_{f(a)}(x) \wedge \inf \mu_{\sigma}(x))$, $\wedge_{a \in A} (\inf \mu_{f(a)}(x) \wedge \inf \mu_{\lambda}(x))$), **max(** $\wedge_{a \in A} (\sup \mu_{f(a)}(x) \wedge \sup \mu_{\sigma}(x))$, $\wedge_{a \in A} (\sup \mu_{f(a)}(x) \wedge \sup \mu_{\lambda}(x))$)], [**min (** $\wedge_{a \in A} (\inf \nu_{f(a)}(x) \vee \inf \nu_{\sigma}(x))$, $\wedge_{a \in A} (\inf \nu_{f(a)}(x) \vee \inf \nu_{\lambda}(x))$), **min (** $\wedge_{a \in A} (\sup \nu_{f(a)}(x) \vee \sup \nu_{\sigma}(x))$, $\wedge_{a \in A} (\sup \nu_{f(a)}(x) \vee \sup \nu_{\lambda}(x))$)], [**min (** $\wedge_{a \in A} (\inf \omega_{f(a)}(x) \vee \inf \omega_{\sigma}(x))$, $\wedge_{a \in A} (\inf \omega_{f(a)}(x) \vee \inf \omega_{\lambda}(x))$), **min (** $\wedge_{a \in A} (\sup \omega_{f(a)}(x) \vee \sup \omega_{\sigma}(x))$, $\wedge_{a \in A} (\sup \omega_{f(a)}(x) \vee \sup \omega_{\lambda}(x))$)]> : x∈ U}

Since $\max(\inf \mu_{\sigma}(y), \inf \mu_{\lambda}(y)) \geq \inf \mu_{\sigma}(y)$
and $\max(\inf \mu_{\sigma}(y), \inf \mu_{\lambda}(y)) \geq \inf \mu_{\lambda}(y)$
we have
$\wedge_{a \in A} (\inf \mu_{f(a)}(x) \wedge \max(\inf \mu_{\sigma}(x), \inf \mu_{\lambda}(x)) \geq \wedge_{a \in A} (\inf \mu_{f(a)}(x) \wedge \inf \mu_{\sigma}(x))$
and $\wedge_{a \in A} (\inf \mu_{f(a)}(x) \wedge \max(\inf \mu_{\sigma}(x), \inf \mu_{\lambda}(x)) \geq \wedge_{a \in A} (\inf \mu_{f(a)}(x) \wedge \inf \mu_{\lambda}(x))$

Hence $\wedge_{a \in A} (\inf \mu_{f(a)}(x) \wedge \max(\inf \mu_{\sigma}(x), \inf \mu_{\lambda}(x))$ ≥ **max** ($\wedge_{a \in A} (\inf \mu_{f(a)}(x) \wedge \inf \mu_{\sigma}(x))$, $\wedge_{a \in A} (\inf \mu_{f(a)}(x) \wedge \inf \mu_{\lambda}(x))$)

Similarly
$\wedge_{a \in A} (\sup \mu_{f(a)}(x) \wedge \max(\sup \mu_{\sigma}(x), \sup \mu_{\lambda}(x))$ ≥ **max** ($\wedge_{a \in A} (\sup \mu_{f(a)}(x) \wedge \sup \mu_{\sigma}(x))$, $\wedge_{a \in A} (\sup \mu_{f(a)}(x) \wedge \sup \mu_{\lambda}(x))$)
Again since
$\min(\inf \nu_{\sigma}(y), \inf \nu_{\lambda}(y)) \leq \inf \nu_{\sigma}(y)$
and $\min(\inf \nu_{\sigma}(y), \inf \nu_{\lambda}(y)) \leq \inf \nu_{\lambda}(y)$

we have
$\wedge_{a \in A} (\inf \nu_{f(a)}(x) \vee \min(\inf \nu_{\sigma}(x), \inf \nu_{\lambda}(x))$ ≤ $\wedge_{a \in A} (\inf \nu_{f(a)}(x) \vee \inf \nu_{\sigma}(x))$
and $\wedge_{a \in A} (\inf \nu_{f(a)}(x) \vee \min(\inf \nu_{\sigma}(x), \inf \nu_{\lambda}(x))$ ≤ $\wedge_{a \in A} (\inf \nu_{f(a)}(x) \vee \inf \nu_{\lambda}(x))$

Hence $\wedge_{a \in A} (\inf \nu_{f(a)}(x) \vee \min(\inf \nu_{\sigma}(x), \inf \nu_{\lambda}(x))$ ≤ **min** ($\wedge_{a \in A} (\inf \nu_{f(a)}(x) \vee \inf \nu_{\sigma}(x))$, $\wedge_{a \in A} (\inf \nu_{f(a)}(x) \vee \inf \nu_{\lambda}(x))$)

Similarly
$\wedge_{a \in A} (\sup \nu_{f(a)}(x) \vee \min(\sup \nu_{\sigma}(x), \sup \nu_{\lambda}(x))$ ≤ **minx** ($\wedge_{a \in A} (\sup \nu_{f(a)}(x) \vee \sup \nu_{\sigma}(x))$, $\wedge_{a \in A} (\sup \nu_{f(a)}(x) \vee \sup \nu_{\lambda}(x))$)

Again since
$\min(\inf \omega_{\sigma}(y), \inf \omega_{\lambda}(y)) \leq \inf \omega_{\sigma}(y)$
And $\min(\inf \omega_{\sigma}(y), \inf \omega_{\lambda}(y)) \leq \inf \omega_{\lambda}(y)$

we have
$\wedge_{a \in A} (\inf \omega_{f(a)}(x) \vee \min(\inf \omega_{\sigma}(x), \inf \omega_{\lambda}(x)) \leq \wedge_{a \in A} (\inf \nu_{\omega f(a)}(x) \vee \inf \omega_{\sigma}(x))$
and $\wedge_{a \in A} (\inf \omega_{f(a)}(x) \vee \min(\inf \omega_{\sigma}(x), \inf \omega_{\lambda}(x) \leq \wedge_{a \in A} (\inf \omega_{f(a)}(x) \vee \inf \omega_{\lambda}(x))$

Hence $\wedge_{a\in A}(\inf\omega_{f(a)}(x) \vee \min(\inf\omega_{\sigma}(x), \inf\nu_{\lambda}(x)) \le \mathbf{min}(\wedge_{a\in A}(\inf\omega_{f(a)}(x) \vee \inf\omega_{\sigma}(x)), \wedge_{a\in A}(\inf\omega_{f(a)}(x) \vee \inf\omega_{\lambda}(x)))$

Similarly
$\wedge_{a\in A}(\sup\omega_{f(a)}(x) \vee \min(\sup\omega_{\sigma}(x), \sup\omega_{\lambda}(x)) \le \mathbf{min}(\wedge_{a\in A}(\sup\omega_{f(a)}(x) \wedge \sup\omega_{\sigma}(x)), \wedge_{a\in A}(\sup\omega_{f(a)}(x) \wedge \sup\omega_{\lambda}(x)))$
Consequently,
$\downarrow \mathrm{Apr}_{\mathrm{SIVN}}(\sigma) \cup \downarrow \mathrm{Apr}_{\mathrm{SIVN}}(\lambda) \subseteq \downarrow \mathrm{Apr}_{\mathrm{SIVN}}(\sigma \cup \lambda)$
(8) Proof is similar to (7).

Theorem 3.5. Every soft interval-valued neutrosophic rough set is an interval valued neutrosophic soft set.

Proof. Let Θ=(f,A) be an interval-valued neutrosophic soft set over U and SIVN=(U, Θ) be the soft interval-valued neutrosophic approximation space. Let σ be a soft interval-valued neutrosophic rough set. Let us define an interval-valued neutrosophic set χ by:

$$\chi=\{(x, [\frac{\wedge_{a\in A}(\inf\mu_{f(a)}(x)\wedge\inf\mu_{\sigma}(x))}{\wedge_{a\in A}(\inf\mu_{f(a)}(x)\vee\inf\mu_{\sigma}(x))}, \frac{\wedge_{a\in A}(\sup\mu_{f(a)}(x)\wedge\sup\mu_{\sigma}(x))}{\wedge_{a\in A}(\sup\mu_{f(a)}(x)\vee\sup\mu_{\sigma}(x))}], [\frac{\wedge_{a\in A}(\inf\nu_{f(a)}(x)\wedge\inf\nu_{\sigma}(x))}{\wedge_{a\in A}(\inf\nu_{f(a)}(x)\vee\inf\nu_{\sigma}(x))}, \frac{\wedge_{a\in A}(\sup\nu_{f(a)}(x)\wedge\sup\nu_{\sigma}(x))}{\wedge_{a\in A}(\sup\nu_{f(a)}(x)\vee\sup\nu_{\sigma}(x))}], [\frac{\wedge_{a\in A}(\inf\omega_{f(a)}(x)\wedge\inf\omega_{\sigma}(x))}{\wedge_{a\in A}(\inf\omega_{f(a)}(x)\vee\inf\omega_{\sigma}(x))}, \frac{\wedge_{a\in A}(\sup\omega_{f(a)}(x)\wedge\sup\mu\omega_{\sigma}(x))}{\wedge_{a\in A}(\sup\omega_{f(a)}(x)\vee\sup\omega_{\sigma}(x))}]): x\in U\}$$

Now, for $\theta \in [0, 1]$, we consider the following six sets:

$$F_1(\theta)=\{x\in U : \frac{\wedge_{a\in A}(\inf\mu_{f(a)}(x)\wedge\inf\mu_{\sigma}(x))}{\wedge_{a\in A}(\inf\mu_{f(a)}(x)\vee\inf\mu_{\sigma}(x))} \ge \theta\}$$

$$F_2(\theta)=\{x\in U : \frac{\wedge_{a\in A}(\sup\mu_{f(a)}(x)\wedge\sup\mu_{\sigma}(x))}{\wedge_{a\in A}(\sup\mu_{f(a)}(x)\vee\sup\mu_{\sigma}(x))} \ge \theta\}$$

$$F_3(\theta)=\{x\in U : \frac{\wedge_{a\in A}(\inf\nu_{f(a)}(x)\wedge\inf\nu_{\sigma}(x))}{\wedge_{a\in A}(\inf\nu_{f(a)}(x)\vee\inf\nu_{\sigma}(x))} \ge \theta\}$$

$$F_4(\theta)=\{x\in U : \frac{\wedge_{a\in A}(\sup\nu_{f(a)}(x)\wedge\sup\nu_{\sigma}(x))}{\wedge_{a\in A}(\sup\nu_{f(a)}(x)\vee\sup\nu_{\sigma}(x))} \ge \theta\}$$

$$F_5(\theta)=\{x\in U : \frac{\wedge_{a\in A}(\inf\omega_{f(a)}(x)\wedge\inf\omega_{\sigma}(x))}{\wedge_{a\in A}(\inf\omega_{f(a)}(x)\vee\inf\omega_{\sigma}(x))} \ge \theta\}$$

$$F_6(\theta)=\{x\in U : \frac{\wedge_{a\in A}(\sup\omega_{f(a)}(x)\wedge\sup\mu\omega_{\sigma}(x))}{\wedge_{a\in A}(\sup\omega_{f(a)}(x)\vee\sup\omega_{\sigma}(x))} \ge \theta\}$$

Then $\psi(\theta)$= { (x, [inf{ θ: x ∈ $F_1(\theta)$}, inf{ θ: x ∈ $F_2(\theta)$}], [inf{ θ: x ∈ $F_3(\theta)$}, inf{ θ: x ∈ $F_4(\theta)$}], [inf{ θ: x ∈ $F_5(\theta)$}, inf{ θ: x ∈ $F_6(\theta)$}]) :x ∈ U}is an interval –valued neutrosophic set over U for each $\theta \in [0, 1]$. Consequently (ψ, θ) is an interval-valued neutrosophic soft set over U.

4.A Multi-criteria Group Decision Making Problem

In this section, we extend the soft interval –valued intuitionistic fuzzy rough set based multi-criteria group decision making scheme [4] to the case of the soft interval-valued neutrosophic rough set.

Let U={$o_1, o_2, o_3,\ldots, o_r$} be a set of objects and E be a set of parameters and A = $\{e_1, e_2, e_3,\ldots, e_m\}$ ⊆ E and S=(F, A) be an interval- neutrosophic soft set over U. Let us assume that we have an expert group G = $\{T_1, T_2, T_3,\ldots, T_n\}$ consisting of n specialists to evaluate the objects in U. Each specialist will examine all the objects in U and will point out his/her evaluation result. Let X_i denote the primary evaluation result of the specialist T_i. It is easy to see that the primary evaluation result of the whole expert group G can be represented as an interval valued neutrosophic evaluation soft set S^* = (F^*, G) over U, where $F^*: G \longrightarrow IVNS^U$ is given by $F^*(T_i)= X_i$, for i=1,2,..n.

Now we consider the soft interval valued neutrosophic rough approximations of the specialist T_i's primary evaluation result X_i w.r.t the soft interval valued neutrosophic approximation space SIVN = (U, S). Then we obtain two other interval valued neutrosophic soft sets $\downarrow S^*$= ($\downarrow F^*$,G) and $\uparrow S^*$= ($\uparrow F^*$,G) over U, where $\downarrow S^*$: $G \longrightarrow IVNS^U$ is given by $\downarrow F^* =\downarrow X_i$ and $\uparrow F^*: G \longrightarrow IVNS^U$ is given by $\uparrow F^*(T_i)= =\uparrow X_i$, for i=1,2,..n. Here $\downarrow S^*$ can be considered as the evaluation result for the whole expert group G with 'low confidence', $\uparrow S^*$ can be considered as the evaluation result for the whole expert group G with 'high confidence' and S^* can be considered as the evaluation result for the whole expert group G with 'middle confidence' Let us define two interval valued neutrosophic sets $IVNS_{\downarrow S^*}$ **and** $IVNS_{\uparrow S^*}$ by

$$IVNS_{\downarrow S^*}=\{\langle o_k, [\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{inf}\,\mu_{\downarrow F^*(T_j)}(o_k), \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{sup}\,\mu_{\downarrow F^*(T_j)}(o_k)], [\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{inf}\,\nu_{\downarrow F^*(T_j)}(o_k), \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{sup}\,\nu_{\downarrow F^*(T_j)}(o_k)], [\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{inf}\,\omega_{\downarrow F^*(T_j)}(o_k), \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{sup}\,\omega_{\downarrow F^*(T_j)}(o_k)]>: k=1,2,..r\}$$

And

$$IVNS_{\uparrow S^*}=\{\langle o_k, [\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{inf}\,\mu_{\uparrow F^*(T_i)}(o_k), \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{sup}\,\mu_{\uparrow F^*(T_i)}(o_k)], [\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{inf}\,\nu_{\uparrow F^*(T_i)}(o_k), \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{sup}\,\nu_{\uparrow F^*(T_i)}(o_k)], [\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{inf}\,\omega_{\uparrow F^*(T_i)}(o_k), \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{sup}\,\omega_{\uparrow F^*(T_i)}(o_k)]>: k=1,2,..r\}$$

Now we define another interval valued neutrosophic set $IVNS_{S^*}$ by

$IVNS_{S^*}$ ={$\langle o_k$, $[\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{inf}\,\mu_{F^*(T_j)}(o_k)$, $\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{sup}\,\mu_{F^*(T_j)}(o_k)]$, $[\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{inf}\,\nu_{F^*(T_j)}(o_k)$, $\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{sup}\,\nu_{F^*(T_j)}(o_k)]$, $[\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{inf}\,\omega_{F^*(T_j)}(o_k)$, $\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{sup}\,\omega_{F^*(T_j)}(o_k)]\rangle$: $k = 1,2,..r$}

Then clearly,

$IVNS_{\downarrow S^*} \subseteq IVNS_{S^*} \subseteq IVNS_{\uparrow S^*}$

Let C={L (low confidence), M (middle confidence), H (high confidence)} be a set of parameters. Let us consider the interval valued neutrosophic soft set S^{**}= (f, C) over U, where f: $C \longrightarrow IVNS^U$ is given by f(L)= $IVNS_{\downarrow S^*}$, f(M)= $IVNS_{S^*}$, f(H)= $IVNS_{\uparrow S^*}$. Now given a weighting vector W= (ω_L , ω_M , ω_H) such that ω_L , ω_M , $\omega_H \in$ [0, 1], we define $\alpha: U \longrightarrow P(U) by$ $\alpha(o_k) = \omega_L \diamond s_{f(L)}(o_k) + \omega_M \diamond s_{f(M)}(o_k) + \diamond s_{f(H)}(o_k)$, $o_k \in$ U ($\diamond$ represents ordinary multiplication) where

$$s_{f(L)}(o_k) = \frac{\boldsymbol{inf}\mu_{\downarrow F^*(T_j)} + \boldsymbol{sup}\mu_{\downarrow F^*(T_j)} - \boldsymbol{inf}\,\nu_{\downarrow F^*(T_j)} . \boldsymbol{sup}\,\nu_{\downarrow F^*(T_j)} - \boldsymbol{inf}\omega_{\downarrow F^*(T_j)} . \boldsymbol{sup}\omega_{\downarrow F^*(T_j)}}{2}$$

denotes the score function, the same as $s_{f(M)}(o_k)$ and $s_{f(H)}(o_k)$. Here $\alpha(o_k)$ is called the weighted evaluation value of the alternative $o_k \in$ U. Finally, we can select the object o_p =max{ $\alpha(o_k)$}:k=1,2,…,r} as the most preferred alternative.

Algorithm:

(1) Input the original description Interval valued neutrosophic soft set (F, A).

(2) Construct the interval valued neutrosophic evaluation soft set S^* =(F^*, G)

(3) Compute the soft interval valued neutrosophic rough approximations and then construct the interval valued neutrosophic soft sets $\downarrow S^*$ **and** $\uparrow S^*$

(4) Construct the interval valued neutrosophic $IVNS_{\downarrow S^*}$, $IVNS_{S^*}$, $IVNS_{\uparrow S^*}$

(5) Construct the interval valued neutrosophic soft set S^{**}.

(6) Input the weighting vector W and compute the weighted evaluation values of each alternative $\alpha(o_k)$ of each alternative $o_k \in$ U.

(7) Select the object $o_p$such that object o_p =max{ $\alpha(o_k)$}:k=1,2,…,r} as the most preferred alternative.

5.An illustrative example

The following example is adapted from **[4]** with minor changes.

Let us consider a staff selection problem to fill a position in a private company.

Let U = {c_1, c_2, c_3, c_4, c_5} is the universe set consisting of five candidates. Let us consider the soft set S=(F, A), which describes the "quality of the candidates", where A={e_1 (experience), e_2 (computer knowledge), e_3 (young and efficient), e_4 (good communication skill)}. Let the tabular representation of the interval valued neutrosophicsoft set (F, A) be:

	c_1	c_2	c_3	c_4	c_5
e_1	([.2, .3],[.4, .5],[.3, .4])	([.5, .7],[.1, .3],[.2, .3])	([.4, .5],[.2, .4],[.2, .5])	([.1, .2],[.1, .3],[.1, .2])	([.3, .5],[.3, .4],[.1, .2])
e_2	([.3, .6],[.1, .2],[.2, .3])	([.1, .3],[.2, .3],[.2, .4])	([.3, .6],[.2, .4],[.2, .4])	([.5, .6],[.2, .3],[.2, .4])	([.1, .3],[.3, .6],[.2, .5])
e_3	([.4, .5],[.2, .3],[.4, .5])	([.2, .4],[.2, .5],[.1, .2])	([1, .3],[.4, .6],[.3, .5])	([.3, .4],[.3, .4],[.4, .6])	([.4, .6],[.1, .3],[.2, .3])
e_4	([.2, .4],[.6, .7],[.6, .7])	([.6, .7],[.1, .2],[.4, .5])	([.3, .4],[.3, .4],[.1, .2])	([.2, .4],[.4, .6],[.1, .2])	([.5, .7],[.1, .2],[.1, .5])

Let G = {T_1, T_2, T_3, T_4, T_4} be the set of interviewers to judge the quality of the candidate in U. Now if X_i denote the primary evaluation result of the interviewer T_i (for i=1, 2, 3, 4,5), then the primary evaluation result of the whole expert group G can be represented as an interval valued neutrosophic evaluation soft set S^*= (F^*,G) over U, where $F^*: G \longrightarrow IVNS^U$ is given by F^* (T_i) = X_i for i=1, 2, 3, 4,5.

Let the tabular representation of S^* be given as:

	c_1	c_2	c_3	c_4	c_5
T_1	([.4, .6],[.4, .5],[.3, .4])	([.3, .4],[.1, .2],[.2, .3])	([.2, 3],[.2, .3],[.2, .5])	([.6, .8],[.1, .2],[.1, .2])	([.1, .4],[.2, .3],[.1, .2])
T_2	([.3, .5],[.2, .4],[.2, .3])	([.5, .7],[.1, .3],[.2, .4])	([.4, .6],[.1, .3],[.2, .4])	([.3, .5],[.1, .3],[.2, .4])	([.4, .5],[.2, .3],[.2, .5])
T_3	([.1, .3],[.5, .6],[.4, .5])	([.2, .3],[.4, .5],[.1, .2])	([.1, .4],[.2, .4],[.3, .5])	([.2, .3],[.5, .6],[.4, .6])	([.3, .6],[.2, .3],[.2, .3])
T_4	([.2, .3],[.3, .4],[.6, .7])	([.4, .7],[.1, .2],[.4, .5])	([.3, .5],[.4, .5],[.1, .2])	([.4, .5],[.2, .4],[.1, .2])	([.5, .7],[.1, .2],[.1, .5])
T_5	([.6, .7],[.1, .2],[.6, .7])	([.3, .5],[.3, .4],[.4, .6])	([.5, .6],[.3, .4],[.2, .3])	([.1, .3],[.3, .6],[.4, .6])	([.1, .2],[.6, .8],[.2, .5])

Let us choose P=(U, S) as the soft interval valued neutrosophic approximation space. Let us consider the interval valued neutrosophic evaluation soft sets.

$\downarrow S^* = (\downarrow F^*, G)$ and $\uparrow S^* = (\uparrow F^*, G)$ over U.
Then the tabular representation of these sets are:

$\downarrow S^* = (\downarrow F^*, G)$:

	c_1	c_2	c_3	c_4	c_5
T_1	([.2, .3],[.1, .2],[.3, .4])	([.1, .3],[.3, .4],[.2, .3])	([.1, .3],[.2, .4],[.2, .5])	([.1, .2],[.1, .3],[.1, .2])	([.1, .3],[.2, .4],[.1, .2])
T_2	([.2, .3],[.2, .4],[.2, .3])	([.1, .3],[.1, .3],[.2, .4])	([.1, 3],[.2, .4],[.2, .4])	([.1, .2],[.1, .3],[.2, .4])	([.1, .3],[.2, .3],[.2, .5])
T_3	([.1, .3],[.5, .6],[.4, .5])	([.1, .3],[.4, .5],[.1, .2])	([.1, .3],[.2, .4],[.3, .5])	([.1, .2],[.5, .6],[.4, .6])	([.1, .3],[.2, .3],[.2, .3])
T_4	([.2, .3],[.3, .4],[.6, .7])	([.1, .3],[.1, .2],[.4, .5])	([.1, .3],[.4, .5],[.1, .2])	([.1, .2],[.2, .4],[.1, .2])	([.1, .3],[.1, .2],[.1, .5])
T_5	([.2, .3],[.1, .2],[.6, .7])	([.1, .3],[.2, .5],[.4, .6])	([.1, .3],[.3, .4],[.2, .3])	([.1, .2],[.3, 6],[.4, .6])	([.1, .2],[.6, .8],[.2, .5])

$\uparrow S^* = (\uparrow F^*, G)$

	c_1	c_2	c_3	c_4	c_5
T_1	([.4, .6],[.1, .2],[.2, .3])	([.3, .4],[.1, .2],[.1, .2])	([.2, .3],[.2, .3],[.1, .2])	([.6, .8],[.1, .2],[.1, .2])	([.1, .4],[.1, .2],[.1, .2])
T_2	([.3, .5],[1, .2],[.2, .3])	([.5, .7],[.1, .2],[.1, .2])	([.4, .6],[.1, .3],[.1, .2])	([.3, .5],[.1, .3,[.1, .2])	([.4, .5],[.1, .2],[.1, .2])
T_3	([.2, .3],[.1, .2],[.2, .3])	([.2, .3],[.1, .2],[.1, .2])	([.1, .4],[.2, .4],[.1, .2])	([.2, .3],[.1 .3],[.1, .2])	([.3, .6],[.1, .2],[.1, .2])
T_4	([.2, .3],[.1, .2],[.2, .3])	([.4, .7],[.1, .2],[.1, .2])	([.3, .5],[.2, .4],[.1, .2])	([.4, .5],[.1, .3],[.1, .2])	([.5, .7],[.1, .2],[.1, .2])
T_5	([.6, .7],[.1, .2],[.2, .3])	([.3, .5],[.1, .2],[.1, .2])	([.5, .6],[.2, .4],[.1, .2])	([.1, .3],[.1, 3],[.1, .2])	([.1, .3],[.1, .2],[.1, .2])

Here, $\downarrow S^* \subseteq S^* \subseteq \uparrow S^*$

$IVNS_{\downarrow S^*}$ = { <c_1,[0.15, 0.35],[0.4, 0.625],[0.42, 0.52]> <c_2,[0.175, 0.325],[0.375, 0.575],[0.26, 0.4]>, <c_3,[0.175, 0.375],[0.375, 0.575],[0.2, 0.38]>, <c_4,[0.175, 0.375],[0.375, 0.575],[0.24, 0.4]>, <c_5,[0.175, 0.375],[0.375, 0.575],[0.16, 0.4]>}.

$IVNS_{\uparrow S^*}$= { <c_1,[0.575, 0.75],[0.125, 0.225],[0.2, 0.3]> <c_2,[0.575, 0.75],[0.125, 0.225], [0.1, 0.2]>, <c_3,[0.575, 0.725],[0.125, 0.225],[0.1, 0.2]>, <c_4,[0.525, 0.700],[0.125, 0.225],[0.1, 0.2]>, <c_5,[0.55, 0.700],[0.125, 0.225],[0.1, 0.2]>}.

$IVNS_{S^*}$= { <c_1,[0.25, 0.45],[0.375, 0.475],[0.42, 0.52]> <c_2,[0.375, 0.525],[0.225, 0.35], [0.26, 0.4]>, <c_3,[0.350, 0.525],[0.2, 0.4],[0.2, 0.38]>, <c_4,[0.4, 0.6],[0.20, 0.35],[0.24, 0.4]>, <c_5,[0.35, 0.55],[0.15, 0.375],[0.16, 0.4]>}.

Here, $IVNS_{\downarrow S^*} \subseteq IVNS_{S^*} \subseteq IVNS_{\uparrow S^*}$ **. Let** C={ L (low confidence), M (middle confidence),H(high confidence)} be a set of parameters. Let us consider the interval valued neutrosophic soft set S^{**}= (f, C) over U, where f: $C \longrightarrow IVNS^U$ is given by f(L) = $IVNS_{\downarrow S^*}$, f(M) = $IVNS_{S^*}$, f(H) = $IVNS_{\uparrow S^*}$. Now assuming the weighting vector W =(ω_L, ω_M, ω_H) such that ω_L= 0.7 ω_M=0.6, ω_H=0.8, we have ,

$\alpha(c_1)$ = 0.7 ⋄ 0.0158 +0.6 ⋄ 0.15174 +0.8 ⋄ 0.6184
=0.5968
$\alpha(c_2)$= 0.7 ⋄ 0.0901 +0.6 ⋄ 0.3586 +0.8 ⋄ 0.6384
= 0.7890
$\alpha(c_3)$= 0.7 ⋄ 0.1041 +0.6 ⋄ 0.3595 +0.8 ⋄ 0.6384
=0.7993
$\alpha(c_4)$= 0.7 ⋄ 0.1191 +0.6 ⋄ 0.4170 +0.8 ⋄ 0.6134
=0.8243
$\alpha(c_5)$= 0.7 ⋄ 0.1351 +0.6 ⋄ 0.3898 +0.8 ⋄ 0.600
=0.8093

Since max($\alpha(c_1)$, $\alpha(c_2)$, $\alpha(c_3)$, $\alpha(c_4)$, $\alpha(c_5)$} = 0.8243, so the candidate c_4 will be selected as the most preferred alternative.

5.Conclusions

In this paper we have defined, for the first time, the notion of soft interval valued neutrosophic rough sets which is a combination of interval valued neutrosophic rough sets and soft sets. We have studied some of their basic properties. Thus our work is a generalization of SIVIF-rough sets. We hope that this paper will promote the future study on soft interval valued neutrosophic rough sets to carry out a general framework for their application in practical life.

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An Extended TOPSIS Method for Multiple Attribute Decision Making based on Interval Neutrosophic Uncertain Linguistic Variables

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Abstract: The interval neutrosophic uncertain linguistic variables can easily express the indeterminate and inconsistent information in real world, and TOPSIS is a very effective decision making method more and more extensive applications. In this paper, we will extend the TOPSIS method to deal with the interval neutrosophic uncertain linguistic information, and propose an extended TOPSIS method to solve the multiple attribute decision making problems in which the attribute value takes the form of the interval neutrosophic uncertain linguistic variables and attribute weight is unknown. Firstly, the operational rules and properties for the interval neutrosophic variables are introduced. Then the distance between two interval neutrosophic uncertain linguistic variables is proposed and the attribute weight is calculated by the maximizing deviation method, and the closeness coefficients to the ideal solution for each alternatives. Finally, an illustrative example is given to illustrate the decision making steps and the effectiveness of the proposed method.

Keywords: The interval neutrosophic linguistic, multiple attribute decision making, TOPSIS, maximizing deviation method

I-Introduction

F. Smarandache [7] proposed the neutrosophic set (NS) by adding an independent indeterminacy-membership function. The concept of neutrosophic set is generalization of classic set, fuzzy set [25], intuitionistic fuzzy set [22], interval intuitionistic fuzzy set [23,24] and so on. In NS, the indeterminacy is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are completely independent. From scientific or engineering point of view, the neutrosophic set and set-theoretic view, operators need to be specified .Otherwise, it will be difficult to apply in the real applications. Therefore, H. Wang et al [8] defined a single valued neutrosophic set (SVNS) and then provided the set theoretic operations and various properties of single valued neutrosophic sets. Furthermore, H. Wang et al.[9] proposed the set theoretic operations on an instance of neutrosophic set called interval valued neutrosophic set (IVNS) which is more flexible and practical than NS. The works on neutrosophic set (NS) and interval valued neutrosophic set (IVNS), in theories and application have been progressing rapidly (e.g, [1,2,4,6,7,8,9,10,11,12,13,14,15,16,17, ,18,19,20,21,27,28,29,30,31,32,33,35,36,37,38,39,40,41,42 ,43,44,45,46,47,48,53].

Multiple attribute decision making (MADM) problem are of importance in most kinds of fields such as engineering, economics, and management. In many situations decision makers have incomplete , indeterminate and inconsistent information about alternatives with respect to attributes. It is well known that the conventional and fuzzy or intuitionistic fuzzy decision making analysis [26, 50, 51,] using different techniques tools have been found to be inadequate to handle indeterminate an inconsistent data. So, Recently, neutrosophic multicriteria decision making problems have been proposed to deal with such situation.

TOPSIS (Technique for Order Performance by Similarity to Ideal Solution) method, initially introduced by C. L. Hwang and Yoon [3], is a widely used method for dealing with MADM problems, which focuses on choosing the alternative with the shortest distance from the positive ideal solution (PIS) and the farthest distance from the negative ideal solution (NIS). The traditional TOPSIS is only used to solve the decision making problems with crisp numbers, and many extended TOPSIS were proposed to deal with fuzzy information. Z. Yue [55] extended TOPSIS to deal with interval numbers, G. Lee et al.[5] extend TOPSIS to deal wit fuzzy numbers, P. D. Liu and Su [34], Y. Q. Wei and Liu [49] extended TOPSIS to linguistic information environments, Recently, Z. Zhang and C. Wu [53] proposed the single valued neutrosophic or interval neutrosophic TOPSIS method to calculate the relative closeness coefficient of each alternative to the single valued neutrosophic or interval neutrosophic positive ideal solution, based on which the considered alternatives are ranked and then the most desirable one is selected. P. Biswas et al. [32] introduced single –valued neutrosophic multiple attribute decision making problem with incompletely known or completely unknown attribute weight information based on modified GRA.

Based on the linguistic variable and the concept of interval neutrosophic sets, J. Ye [19] defined interval neutrosophic linguistic variable, as well as its operation principles, and developed some new aggregation operators for the interval neutrosophic linguistic information, including interval neutrosophic linguistic arithmetic weighted average (INLAWA) operator, linguistic geometric weighted average(INLGWA) operator and discuss some properties. Furthermore, he proposed the decision making method for multiple attribute decision making (MADM) problems with an illustrated example to show the process of decision making and the effectiveness of the proposed method. In order to process incomplete, indeterminate and inconsistent information more efficiency and precisely J. Ye [20] further proposed the interval neutrosophic uncertain linguistic variables by combining uncertain linguistic variables and interval neutrosophic sets, and proposed the operational rules, score function , accuracy functions ,and certainty function of interval neutrosophic uncertain linguistic variables. Then the interval neutrosophic uncertain linguistic weighted arithmetic averaging (INULWAA) and the interval neutrosophic uncertain linguistic weighted arithmetic averaging (INULWGA) operator are developed, and a multiple attribute decision method with interval neutrosphic uncertain linguistic information was developed.

To do so, the remainder of this paper is set out as follows. Section 2 briefly recall some basic concepts of neutrosphic sets, single valued neutrosophic sets (SVNSs), interval neutrosophic sets(INSs), interval neutrosophic linguistic variables and interval neutrosophic uncertain linguistic variables. In section 3, we develop an extended TOPSIS method for the interval neutrosophic uncertain linguistic variables, In section 4, we give an application example to show the decision making steps, In section 5, a comparison with existing methods are presented. Finally, section 6 concludes the paper.

II-Preliminaries

In the following, we shall introduce some basic concepts related to uncertain linguistic variables, single valued neutrosophic set, interval neutrosophic sets, interval neutrosophic uncertain linguistic sets, and interval neutrosophic uncertain linguistic set.

2.1 Neutrosophic sets

Definition 2.1 [7]

Let U be a universe of discourse then the neutrosophic set A is an object having the form

$A = \{< x: T_A(x), I_A(x), F_A(x) >, x \in X \}$,

Where the functions $T_A(x)$, $I_A(x)$, $F_A(x)$: U→]$^-$0,1+[define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element x ∈ X to the set A with the condition.

$$^-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+. \quad \textbf{(1)}$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. So instead of $]^-0,1^+[$ we need to take the interval [0,1] for technical applications, because $]^-0,1^+[$will be difficult to apply in the real applications such as in scientific and engineering problems.

2.2 Single valued Neutrosophic Sets

Definition 2.2 [8]

Let X be an universe of discourse, then the neutrosophic set A is an object having the form

$A = \{< x: T_A(x), I_A(x), F_A(x) >, x \in X \}$,

where the functions $T_A(x)$,$I_A(x)$, $F_A(x)$: U→[0,1]define respectively the degree of membership , the degree of indeterminacy, and the degree of non-membership of the element x ∈ X to the set A with the condition.

$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ **(2)**

Definition 2.3 [8]

A single valued neutrosophic set A is contained in another single valued neutrosophic set B i.e. A ⊆ B if ∀x ∈ U, $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$. **(3)**

2.3 Interval Neutrosophic Sets

Definition 2.4[9]

Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS) A in X is characterized by truth-membership function $T_A(x)$, indeteminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$. For each point x in X, we have that $T_A(x)$, $I_A(x)$, $F_A(x) \subseteq [0, 1]$.

For two IVNS, A_{IVNS}= {<x, $[T_A^L(x), T_A^U(x)]$, $[I_A^L(x), I_A^U(x)]$, $[F_A^L(x), F_A^U(x)]$ >| x ∈ X } **(4)**

And B_{IVNS}= {<x, $[T_B^L(x), T_B^U(x)]$, $[I_B^L(x), I_B^U(x)]$, $[F_B^L(x), F_B^U(x)]$>| x ∈ X } the two relations are defined as follows:

(1) $A_{IVNS} \subseteq B_{IVNS}$ If and only if $T_A^L(x) \leq T_B^L(x)$, $T_A^U(x) \leq T_B^U(x)$, $I_A^L(x) \geq I_B^L(x)$, $I_A^U(x) \geq I_B^U(x)$, $F_A^L(x) \geq F_B^L(x)$, $F_A^U(x) \geq F_B^U(x)$

(2) $A_{IVNS} = B_{IVNS}$ if and only if, $T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$ for any x ∈ X

The complement of A_{IVNS} is denoted by A_{IVNS}^o and is defined by

A_{IVNS}^o= {<x, $[F_A^L(x), F_A^U(x)]$>, $[1 - I_A^U(x), 1 - I_A^L(x)]$, $[T_A^L(x), T_A^U(x)]$ | x ∈ X }

A∩B ={ <x , [min($T_A^L(x)$,$T_B^L(x)$), min($T_A^U(x)$,$T_B^U(x)$)], [max($I_A^L(x)$,$I_B^L(x)$), max($I_A^U(x)$,$I_B^U(x)$], [max($F_A^L(x)$,$F_B^L(x)$), max(F(x),$F_B^U(x)$)] >: x ∈ X }

A∪B ={ <x , [max($T_A^L(x)$,$T_B^L(x)$), max($T_A^U(x)$,$T_B^U(x)$)], [min($I_A^L(x)$,$I_B^L(x)$), min($I_A^U(x)$,$I_B^U(x)$], [min($F_A^L(x)$,$F_B^L(x)$), min($F_A^U(x)$,$F_B^U(x)$)] >: x ∈ X }

2.4 Uncertain linguistic variable.

A linguistic set is defined as a finite and completely ordered discreet term set,

$S=(s_0, s_1, \ldots, s_{l-1})$, where l is the odd value. For example, when l=7, the linguistic term set S can be defined as follows: S={s_0(extremely low); s_1(very low); s_2(low); s_3(medium); s_4(high); s_5(very high); s_6(extermley high)}

Definition 2.5. Suppose $\tilde{s} = [s_a, s_b]$, where $s_a, s_b \in \tilde{S}$ with a ≤ b are the lower limit and the upper limit of S, respectively. Then $\tilde{s}$ is called an uncertain linguitic varaible.

Definition 2.6. Suppose $\tilde{s}_1 = [s_{a_1}, s_{b_1}]$ and $\tilde{s}_2 = [s_{a_2}, s_{b_2}]$ are two uncertain linguistic variable ,then the distance between $\tilde{s}_1$ and $\tilde{s}_2$ is defined as follows.

$d(\tilde{s}_1, \tilde{s}_2) = \frac{1}{2(l-1)} (|a_2 - a_1| + |b_2 - b_1|)$ **(5)**

2.5 Interval neutrosophic linguistic set

Based on interval neutrosophic set and linguistic variables, J. Ye [18] presented the extension form of the linguistic set, i.e, interval neutroosphic linguistic set, which is shown as follows:

Definition 2.7 :[19] An interval neutrosophic linguistic set A in X can be defined as

A ={<x, $s_{\theta(x)}$, (T_A(x), I_A(x), F_A(x))>| x ∈ X} **(6)**

Where $s_{\theta(x)} \in \hat{s}$, T_A(x) = $[T_A^L(x), T_A^U(x)] \subseteq [0.1]$, I_A(x) = $[I_A^L(x), I_A^U(x)] \subseteq [0.1]$, and F_A(x) = $[F_A^L(x), F_A^U(x)] \subseteq [0.1]$ with the condition $0 \leq T_A^U(x) + I_A^U(x) + F_A^U(x) \leq 3$ for any x ∈ X. The function $T_A(x)$, $I_A(x)$ and $F_A(x)$ express, respectively, the truth-membership degree, the indeterminacy –membership degree, and the falsity-membership degree with interval values of the element x in X to the linguistic variable $s_{\theta(x)}$.

2.6 Interval neutrosophic uncertain linguistic set.

Based on interval neutrosophic set and uncertain linguistic variables, J.Ye [20] presented the extension form of the uncertain linguistic set, i.e, interval neutrosphic uncertain linguistic set, which is shown as follows:

Definition 2.8 :[20] An interval neutrosophic uncertain linguistic set A in X can be defined as

A ={<x,[$s_{\theta(x)}$, $s_{\rho(x)}$], (T_A(x), I_A(x), F_A(x))>| x ∈ X} **(7)**

Where $s_{\theta(x)} \in \hat{s}$, T_A(x) = $[T_A^L(x), T_A^U(x)] \subseteq [0.1]$, I_A(x) = $[I_A^L(x), I_A^U(x)] \subseteq [0.1]$, and F_A(x) = $[F_A^L(x), F_A^U(x)] \subseteq [0.1]$ with the condition $0 \leq T_A^U(x) + I_A^U(x) + F_A^U(x) \leq 3$ for any x ∈ X. The function $T_A(x)$, $I_A(x)$ and $F_A(x)$ express, respectively, the truth-membership degree, the indeterminacy–membership degree, and the falsity-membership degree with interval values of the element x in X to the uncertain linguistic variable [$s_{\theta(x)}$, $s_{\rho(x)}$].

Definition 2.9 Let $\tilde{a}_1$=< [$s_{\theta(\tilde{a}_1)}$, $s_{\rho(\tilde{a}_1)}$], ([$T^L(\tilde{a}_1)$,$T^U(\tilde{a}_1)$], [$I^L(\tilde{a}_1)$,$I^U(\tilde{a}_1)$], [$F^L(\tilde{a}_1)$,$F^U(\tilde{a}_1)$])> and $\tilde{a}_2$={<x, [$s_{\theta(\tilde{a}_2)}$, $s_{\rho(\tilde{a}_2)}$], ([$T^L(\tilde{a}_2)$,$T^U(\tilde{a}_2)$], [$I^L(\tilde{a}_2)$,$I^U(\tilde{a}_2)$], [$F^L(\tilde{a}_2)$,$F^U(\tilde{a}_2)$])>

be two INULVs and λ ≥ 0, then the operational laws of INULVs are defined as follows:

$\tilde{a}_1 \oplus \tilde{a}_2$ =< $[s_{\theta(\tilde{a}_1)+\theta(\tilde{a}_2)}, s_{\rho(\tilde{a}_1)+\rho(\tilde{a}_2)}]$, ([$T^L(\tilde{a}_1)$+ $T^L(\tilde{a}_2)$- $T^L(\tilde{a}_1)$ $T^L(\tilde{a}_2)$,$T^U(\tilde{a}_1)$+ $T^U(\tilde{a}_2)$- $T^U(\tilde{a}_1)$ $T^U(\tilde{a}_2)$], [$I^L(\tilde{a}_1)$ $I^L(\tilde{a}_2)$,$I^U(\tilde{a}_1)$ $I^U(\tilde{a}_2)$], [$F^L(\tilde{a}_1)$ $F(\tilde{a}_2)$,$F^U(\tilde{a}_1)$ $F^L(\tilde{a}_2)$])> **(8)**

$\tilde{a}_1 \otimes \tilde{a}_2$ =< $[s_{\theta(\tilde{a}_1)\times\theta(\tilde{a}_2)}]$, ([$T^L(\tilde{a}_1)$ $T^L(\tilde{a}_2)$, $T^U(\tilde{a}_1)$ $T^U(\tilde{a}_2)$], [$I^L(\tilde{a}_1)$+ $I^L(\tilde{a}_2)$ - $I^L(\tilde{a}_1)$ $I^L(\tilde{a}_2)$, $I^U(\tilde{a}_1)$+ $I^U(\tilde{a}_2)$- $I^U(\tilde{a}_1)$ $I^U(\tilde{a}_2)$], [$F^L(\tilde{a}_1)$+ $F^L(\tilde{a}_2)$ - $F^L(\tilde{a}_1)$ $F(\tilde{a}_2)$, $F^U(\tilde{a}_1)$+ $F^U(\tilde{a}_2)$ - $F^U(\tilde{a}_1)$ $F^U(\tilde{a}_2)$])> **(9)**

$\lambda\tilde{a}_1$=<$[s_{\lambda\theta(\tilde{a}_1)}, s_{\lambda\rho(\tilde{a}_1)}]$,([1-$(1-T^L(\tilde{a}_1))^\lambda$,1-$(1-T^U(\tilde{a}_1))^\lambda$], [$(I^L(\tilde{a}_1))^\lambda$,$(I^U(\tilde{a}_1))^\lambda$], [$(F^L(\tilde{a}_1))^\lambda$,$(F^U(\tilde{a}_1))^\lambda$]>

(10)

$\tilde{a}_1^\lambda$=< $[s_{\theta^\lambda(\tilde{a}_1)}, s_{\rho^\lambda(\tilde{a}_1)}]$, ([$(T^L(\tilde{a}_1))^\lambda$,$(T^U(\tilde{a}_1))^\lambda$], [1-$(1-I^L(\tilde{a}_1))^\lambda$, 1-$(1-I^U(\tilde{a}_1))^\lambda$], [1-$(1-F^L(\tilde{a}_1))^\lambda$, 1-$(1-F^U(\tilde{a}_1))^\lambda$]> **(11)**

Obviously, the above operational results are still INULVs.

III. The Extended TOPSIS for the Interval Neutrosophic Uncertain Linguistic Variables

A. The description of decision making problems with interval neutrosphic uncertain linguistic information.

For the MADM problems with interval neutrosophic uncertain variables, there are m alternatives A= $(A_1, A_2,\ldots, A_m)$ which can be evaluated by n attributes C=$(C_1, C_2,\ldots, C_n)$ and the weight of attributes A_i is w_i, and meets the conditions $0 \le w_i \le 1$, $\sum_{j=1}^{n} w_j$=1.Suppose z_{ij} (i=1, 2,…, n; j=1, 2,…, m) is the evaluation values of alternative A_i with respect to attribute C_j

And it can be represented by interval neutrosophic uncertain linguistic variable z_{ij}= <$[x_{ij}^L, x_{ij}^U]$,([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U])>, where $[x_{ij}^L, x_{ij}^U]$ is the uncertain linguistic variable, and $x_{ij}^L, x_{ij}^U \in$ S, S =$(s_0, s_1,\ldots, s_{l-1})$, T_{ij}^L, T_{ij}^U, I_{ij}^L, I_{ij}^U and F_{ij}^L, F_{ij}^U ∈ [0, 1] and $0 \le T_{ij}^U + I_{ij}^U + F_{ij}^U \le 3$. Suppose attribute weight vector W=$(w_1, w_2,\ldots w_n)$ is completely unknown, according to these condition, we can rank the alternatives $(A_1, A_2,\ldots, A_m)$

B. Obtain the attribute weight vector by the maximizing deviation.

In order to obtain the attribute weight vector, we firstly define the distance between two interval neutrosophic uncertain variables.

Definition 3.1

Let $\tilde{s}_1$ = <$[s_{a_1}, s_{b_1}]$,([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U])>, $\tilde{s}_2$ = <$[s_{a_2}, s_{b_2}]$,([T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U])> and $\tilde{s}_3$ = <$[s_{a_3}, s_{b_3}]$,([T_C^L, T_C^U], [I_C^L, I_C^U], [F_C^L, F_C^U])>, be any three interval neutrosophic uncertain linguistic variables, and $\tilde{S}$ be the set of linguistic variables, f is a map, and $f: \tilde{S} \times \tilde{S} \longrightarrow$ R. If $d(\tilde{s}_1, \tilde{s}_2)$ meets the following conditions

(1) $0 \le d_{INULV}(\tilde{s}_1, \tilde{s}_2) \le 1$, $d_{INULV}(\tilde{s}_1, \tilde{s}_1)$= 0

(2) $d_{INULV}(\tilde{s}_1, \tilde{s}_2) = d_{INULV}(\tilde{s}_2, \tilde{s}_1)$

(3) $d_{IVNS}(\tilde{s}_1, \tilde{s}_2) + d_{INULV}(\tilde{s}_2, \tilde{s}_3) \ge d_{INULV}(\tilde{s}_1, \tilde{s}_3)$

then $d_{INULV}(\tilde{s}_1, \tilde{s}_2)$ is called the distance between two interval neutrosophic uncertain linguistic variables $\tilde{s}_1$

Definition 3.2:

Let $\tilde{s}_1$ = <$[s_{a_1}, s_{b_1}]$,([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U])>, and $\tilde{s}_2$ = <$[s_{a_2}, s_{b_2}]$,([T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U])>, be any two interval neutrosophic uncertain linguistic variables, then the Hamming distance between $\tilde{s}_1$ and $\tilde{s}_2$ can be defined as follows.

$$d_{INULV}(\tilde{s}_1, \tilde{s}_2) = \frac{1}{12(l-1)} (|a_1 \times T_A^L - a_2 \times T_B^L| + |a_1 \times T_A^U - a_2 \times T_B^U| + |a_1 \times I_A^L - a_2 \times I_B^L| + |a_1 \times I_A^U - a_2 \times I_B^U| + |a_1 \times F_A^L - a_2 \times F_B^L| + |a_1 \times F_A^U - a_2 \times F_B^U| + |b_1 \times T_A^L - b_2 \times T_B^L| + |b_1 \times T_A^U - b_2 \times T_B^U| + |b_1 \times I_A^L - b_2 \times I_B^L| + |b_1 \times I_A^U - b_2 \times I_B^U| + |b_1 \times F_A^L - b_2 \times F_B^L| + |b_1 \times F_A^U - b_2 \times F_B^U|) \quad \textbf{(12)}$$

In order to illustrate the effectiveness of definition 3.2, the distance defined above must meet the three conditions in definition 3.1

Proof

Obviously, the distance defined in (12) can meets the conditions (1) and (2) in definition 3.1

In the following, we will prove that the distance defined in (12) can also meet the condition (3) in definition 3.1

For any one interval neutrosophic uncertain linguistic variable $\tilde{s}_3$ = <$[s_{a_3}, s_{b_3}]$,([T_C^L, T_C^U], [I_C^L, I_C^U], [F_C^L, F_C^U])>,

$$d_{IVNS}(\tilde{s}_1, \tilde{s}_3) = \frac{1}{12(l-1)} (|a_1 \times T_A^L - a_3 \times T_C^L| + |a_1 \times T_A^U - a_3 \times T_C^U| + |a_1 \times I_A^L - a_3 \times I_C^L| + |a_1 \times I_A^U - a_3 \times I_C^U| + |a_1 \times F_A^L - a_3 \times F_C^L| + |a_1 \times F_A^U - a_3 \times F_C^U| + |b_1 \times T_A^L - b_3 \times T_C^L| + |b_1 \times T_A^U - b_3 \times T_C^U| + |b_1 \times I_A^L - b_3 \times I_C^L| + |b_1 \times I_A^U - b_3 \times I_C^U| + |b_1 \times F_A^L - b_3 \times F_C^L| + |b_1 \times F_A^U - b_3 \times F_C^U|)$$

$=\frac{1}{12(l-1)}$ $(|a_1 \times T_A^L - a_2 \times T_B^L + a_2 \times T_B^L - a_3 \times T_C^L|+|a_1 \times T_A^U - a_2 \times T_B^U + a_2 \times T_B^U - a_3 \times T_C^U|+|a_1 \times I_A^L - a_2 \times I_B^L + a_2 \times I_B^L - a_3 \times I_C^L|+|a_1 \times I_A^U - a_2 \times I_B^U + a_2 \times I_B^U - a_3 \times I_C^U|$
$+|a_1 \times F_A^L - a_2 \times F_B^L + a_2 \times F_B^L - a_3 \times F_C^L|+|a_1 \times F_A^U - a_2 \times F_B^U + a_2 \times F_B^U - a_3 \times F_C^U|$

$+|b_1 \times T_A^L - b_2 \times T_B^L + b_2 \times T_B^L - b_3 \times T_C^L|+|b_1 \times T_A^U - b_2 \times T_B^U + b_2 \times T_B^U - b_3 \times T_C^U|+|b_1 \times I_A^L - b_2 \times I_B^L + b_2 \times I_B^L - b_3 \times I_C^L|+|b_1 \times I_A^U - b_2 \times I_B^U + b_2 \times I_B^U - b_3 \times I_C^U|$
$+|b_1 \times F_A^L - b_2 \times F_B^L + b_2 \times F_B^L - a_3 \times F_C^L|+|b_1 \times F_A^U - b_2 \times F_B^U + b_2 \times F_B^U - b_3 \times F_C^U|$

And
$\frac{1}{12(l-1)}(|a_1 \times T_A^L - a_2 \times T_B^L|+|a_2 \times T_B^L - a_3 \times T_C^L|+|a_1 \times T_A^U - a_2 \times T_B^U|+|a_2 \times T_B^U - a_3 \times T_C^U|+|a_1 \times I_A^L - a_2 \times I_B^L|+|a_2 \times I_B^L - a_3 \times I_C^L|+|a_1 \times I_A^U - a_2 \times I_B^U|+|a_2 \times I_B^U - a_3 \times I_C^U|+$

$|a_1 \times F_A^L - a_2 \times F_B^L|+|a_2 \times F_B^L - a_3 \times F_C^L|+|a_1 \times F_A^U - a_2 \times F_B^U|+|a_2 \times F_B^U - a_3 \times F_C^U|+$

$+|b_2 \times T_B^L - b_3 \times T_C^L|+|b_1 \times T_A^U - b_2 \times T_B^U|+|b_2 \times T_B^U - b_3 \times T_C^U|+|b_1 \times I_A^L - b_2 \times I_B^L|+|b_2 \times I_B^L - b_3 \times I_C^L|+|b_1 \times I_A^U - b_2 \times I_B^U|+|b_2 \times I_B^U - b_3 \times I_C^U|+|b_1 \times F_A^L - b_2 \times F_B^L|+|b_2 \times F_B^L - b_3 \times F_C^L|+|b_1 \times F_A^U - b_2 \times F_B^U|+|b_2 \times F_B^U - b_3 \times F_C^U|)$

$=\frac{1}{12(l-1)}(|a_1 \times T_A^L - a_2 \times T_B^L|+|a_1 \times T_A^U - a_2 \times T_B^U|+|a_1 \times I_A^L - a_2 \times I_B^L|+|a_1 \times I_A^U - a_2 \times I_B^U|+|a_1 \times F_A^L - a_2 \times F_B^L|$
$+|a_1 \times F_A^U - a_2 \times F_B^U|+|b_1 \times T_A^L - b_2 \times T_B^L|+|b_1 \times T_A^U - b_2 \times T_B^U|+|b_1 \times I_A^L - b_2 \times I_B^L|+|b_1 \times I_A^U - b_2 \times I_B^U| +|b_1 \times F_A^L - b_2 \times F_B^L|+|b_1 \times F_A^U - b_2 \times F_B^U|+$
$|a_2 \times T_B^L - a_3 \times T_C^L|+|a_2 \times T_B^U - a_3 \times T_C^U|+|a_2 \times I_B^L - a_3 \times I_C^L|+|a_2 \times I_B^U - a_3 \times I_C^U|+|a_2 \times F_B^L - a_3 \times F_C^L|+|a_2 \times F_B^U - a_3 \times F_C^U|+|b_2 \times T_B^L - b_3 \times T_C^L|+|b_2 \times T_B^U - b_3 \times T_C^U|+|b_2 \times I_B^L - b_3 \times I_C^L|+|b_2 \times I_B^U - b_3 \times I_C^U|+|b_2 \times F_B^L - b_3 \times F_C^L|+|b_2 \times F_B^U - b_3 \times F_C^U|)$

$=\frac{1}{12(l-1)}(|a_1 \times T_A^L - a_2 \times T_B^L|+|a_1 \times T_A^U - a_2 \times T_B^U|+|a_1 \times I_A^L - a_2 \times I_B^L|+|a_1 \times I_A^U - a_2 \times I_B^U|+|a_1 \times F_A^L - a_2 \times F_B^L|$
$+|a_1 \times F_A^U - a_2 \times F_B^U|+|b_1 \times T_A^L - b_2 \times T_B^L|+|b_1 \times T_A^U - b_2 \times T_B^U|+|b_1 \times I_A^L - b_2 \times I_B^L|+|b_1 \times I_A^U - b_2 \times I_B^U| +|b_1 \times F_A^L - b_2 \times F_B^L|+|b_1 \times F_A^U - b_2 \times F_B^U|)+$
$\frac{1}{12(l-1)}(|a_2 \times T_B^L - a_3 \times T_C^L|+|a_2 \times T_B^U - a_3 \times T_C^U|+|a_2 \times I_B^L - a_3 \times I_C^L|+|a_2 \times I_B^U - a_3 \times I_C^U|+|a_2 \times F_B^L - a_3 \times F_C^L|+|a_2 \times F_B^U - a_3 \times F_C^U|+|b_2 \times T_B^L - b_3 \times T_C^L|+|b_2 \times T_B^U - b_3 \times T_C^U|+|b_2 \times I_B^L - b_3 \times I_C^L|+|b_2 \times I_B^U - b_3 \times I_C^U|+|b_2 \times F_B^L - b_3 \times F_C^L|+|b_2 \times F_B^U - b_3 \times F_C^U|)$

$=d_{INULV}(\tilde{s}_1, \tilde{s}_2) + d_{INULV}(\tilde{s}_2, \tilde{s}_3)$
So , $d_{INULV}(\tilde{s}_1, \tilde{s}_2) + d_{INULV}(\tilde{s}_2, \tilde{s}_3) \geq d_{INULV}(\tilde{s}_1, \tilde{s}_3)$

Especially, when T_A^L=T_A^U, I_A^L=I_A^U, F_A^L=F_A^U,and T_B^L=T_B^U, I_B^L=I_B^U, and F_B^L=F_B^U the interval neutrosophic uncertain linguistic variables $\tilde{s}_1, \tilde{s}_2$ can be reduced to single valued uncertain linguistic variables. So the single valued neutrosophic uncertain linguistic variables are the special case of the interval neutrosophic uncertain linguistic variables.

Because the attribute weight is fully unknown, we can obtain the attribute weight vector by the maximizing deviation method. Its main idea can be described as follows. If all attribute values z_{ij} (j=1, 2,…, n) in the attribute C_j have a small difference for all alternatives, it shows that the attribute C_j has a small importance in ranking all alternatives, and it can be assigned a small attribute weight, especially, if all attribute values z_{ij} (j=1, 2,…,n) in the attribute C_j are equal, then the attribute C_j has no effect on sorting, and we can set zero to the weight of attribute C_j. On the contrary, if all attribute values z_{ij} (j=1, 2,…, n) in the attribute C_j have a big difference, the attribute C_j will have a big importance in ranking all alternatives, and its weight can be assigned a big value.
Here, based on the maximizing deviation method, we construct an optimization model to determine the optimal relative weights of criteria under interval neutrosophic uncertain linguistic environment. For the criterion $C_i \in$ C, we can use the distance $d(z_{ij}, z_{kj})$ to represent the deviation between attribute values z_{ij} and z_{kj}, and D_{ij} $=\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j$ can present the weighted deviation sum for the alternative A_i to all alternatives, then

D_j (w_j)=$\sum_{i=1}^{m} D_{ij}(w_j)$= $\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j$ presents the weighted deviation sum for all alternatives, D (w_j)= $\sum_{j=1}^{n} D_j(w_j)$= $\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j$, presents total weighted deviations for all alternatives with respect to all attributes.

Based on the above analysis, we can construct a non linear programming model to select the weight vector w by maximizing D (w),as follow:

$$\begin{cases} \text{Max } D(w_j) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j \\ s.t\ \sum_{j=1}^{n} w_j^2,\ w_j \in [0,1], j = 1,2,\dots,n \end{cases} \quad (13)$$

Then we can build Lagrange multiplier function, and get

$$L(w_j,\lambda)= \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j + \lambda\,(\sum_{j=1}^{n} w_j^2 - 1)$$

$$\text{Let} \begin{cases} \frac{\partial L(w_j,\lambda)}{\partial w_j} = \sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j + 2\lambda w_j = 0 \\ \frac{\partial L(w_j,\lambda)}{\partial w_j} = \sum_{j=1}^{n} w_j^2 - 1 = 0 \end{cases}$$

We can get

$$\begin{cases} 2\lambda = \sqrt{\sum_j^n (\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj}))^2} \\ w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})}{\sqrt{\sum_j^n (\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj}))^2}} \end{cases} \quad (14)$$

Then we can get the normalized attribute weight, and have

$$w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})} \quad (15)$$

C. The Extended TOPSIS Method for the Interval Neutrosophic Uncertain linguistic Information.

The standard TOPSIS method can only process the real numbers, and cannot deal with the interval neutrosophic uncertain linguistic information. In the following, we will extend TOPSIS to process the interval neutrosophic uncertain linguistic variables. The steps are shown as follows

(1) Normalize the decision matrix

Considering the benefit or cost type of the attribute values, we can give the normalized matrix R=(r_{ij}), where r_{ij}=<[r_{ij}^L, r_{ij}^U],],([$\dot{T}_{ij}^L$, $\dot{T}_{ij}^U$], [$\dot{I}_{ij}^L$, $\dot{I}_{ij}^U$], [$\dot{F}_{ij}^L$, $\dot{F}_{ij}^U$])>,The normalization can be made shown as follows.

(i) For benefit type,

$$\begin{cases} r_{ij}^L = x_{ij}^L, r_{ij}^U = x_{ij}^U \quad \text{for } (1 \le i \le m,\ 1 \le j \le n) \\ \dot{T}_{ij}^L = T_{ij}^L,\ \dot{T}_{ij}^U = T_{ij}^U,\ \dot{I}_{ij}^L = I_{ij}^L,\ \dot{I}_{ij}^U = I_{ij}^U,\ \ \dot{F}_{ij}^L = F_{ij}^L, \dot{F}_{ij}^U = F_{ij}^U \end{cases} \quad (16)$$

(ii) For cost type,

$$\begin{cases} r_{ij}^L = \text{neg}(x_{ij}^U), r_{ij}^U = \text{neg}(x_{ij}^L) \quad \text{for } (1 \le i \le m,\ 1 \le j \le n) \\ \dot{T}_{ij}^L = T_{ij}^L,\ \dot{T}_{ij}^U = T_{ij}^U,\ \dot{I}_{ij}^L = I_{ij}^L,\ \dot{I}_{ij}^U = I_{ij}^U,\ \ \dot{F}_{ij}^L = F_{ij}^L, \dot{F}_{ij}^U = F_{ij}^U \end{cases} \quad (17)$$

(2) Construct the weighted normalize matrix

Y=$[y_{ij}]_{m\times n}$

$$\begin{bmatrix} <[y_{11}^L, y_{11}^U],], ([\ddot{T}_{11}^L, \ddot{T}_{11}^U], [\ddot{I}_{11}^L, \ddot{I}_{11}^U], [\ddot{F}_{11}^L, \ddot{F}_{11}^U])> & \dots & <[y_{11}^L, y_{11}^U],], ([\ddot{T}_{1n}^L, \ddot{T}_{1n}^U], [\ddot{I}_{1n}^L, \ddot{I}_{1n}^U], [\ddot{F}_{1n}^L, \ddot{F}_{1n}^U])> \\ <[y_{21}^L, y_{21}^U],], ([\ddot{T}_{21}^L, \ddot{T}_{21}^U], [\ddot{I}_{21}^L, \ddot{I}_{21}^U], [\ddot{F}_{21}^L, \ddot{F}_{21}^U])> & \dots. & <[y_{2n}^L, y_{2n}^U],], ([\ddot{T}_{2n}^L, \ddot{T}_{2n}^U], [\ddot{I}_{2n}^L, \ddot{I}_{2n}^U], [\ddot{F}_{2n}^L, \ddot{F}_{2n}^U])> \\ \dots & & \dots\dots \\ <[y_{mn}^L, y_{mn}^U],], ([\ddot{T}_{mn}^L, \ddot{T}_{mn}^U], [\ddot{I}_{mn}^L, \ddot{I}_{mn}^U], [\ddot{F}_{mn}^L, \ddot{F}_{mn}^U])> & \dots & <[y_{mn}^L, y_{mn}^U],], ([\ddot{T}_{mn}^L, \ddot{T}_{mn}^U], [\ddot{I}_{mn}^L, \ddot{I}_{mn}^U], [\ddot{F}_{mn}^L, \ddot{F}_{mn}^U])> \end{bmatrix}$$

Where

$$\begin{cases} y_{ij}^L = w_j r_{ij}^L, y_{ij}^U = w_j r_{ij}^U \\ \ddot{T}_{ij}^L = 1-(1-\dot{T}_{ij}^L)^{w_j}, \ddot{T}_{ij}^U = 1-(1-\dot{T}_{ij}^U)^{w_j}, \ddot{I}_{ij}^L = (\dot{I}_{ij}^L)^{w_j}, \ddot{I}_{ij}^U = (\dot{I}_{ij}^U)^{w_j},\ \ddot{F}_{ij}^L = (\dot{F}_{ij}^L)^{w_j}, \ddot{F}_{ij}^U = (\dot{F}_{ij}^U)^{w_j} \end{cases} \quad (18)$$

(3) Identify, the sets of the positive ideal solution Y^+= ($y_1^+, y_2^+, \dots, y_m^+$) and the negative ideal solution Y^-= ($y_1^-, y_2^-, \dots, y_m^-$) , then we can get

$$Y^+ = (y_1^+, y_2^+, \dots, y_m^+) = (<[y_1^{L+}, y_1^{U+}], ([\ddot{T}_1^{L+}, \ddot{T}_1^{U+}], [\ddot{I}_1^{L+}, \ddot{I}_1^{U+}], [\ddot{F}_1^{L+}, \ddot{F}_1^{U+}])>, <[y_2^{L+}, y_2^{U+}], ([\ddot{T}_2^{L+}, \ddot{T}_2^{U+}], [\ddot{I}_2^{L+}, \ddot{I}_2^{U+}], [\ddot{F}_2^{L+}, \ddot{F}_2^{U+}])>, \dots, <[y_n^{L+}, y_n^{U+}], ([\ddot{T}_n^{L+}, \ddot{T}_n^{U+}], [\ddot{I}_n^{L+}, \ddot{I}_n^{U+}], [\ddot{F}_n^{L+}, \ddot{F}_n^{U+}])> \quad (19)$$

$Y^{-}= (y_1^{-}, y_2^{-},\ldots, y_m^{-})=$

$$)=(<[\boldsymbol{y_1^{L-}}, \boldsymbol{y_1^{U-}}], ([\ddot{\boldsymbol{T}}_1^{L-}, \ddot{\boldsymbol{T}}_1^{U-}], [\ddot{\boldsymbol{I}}_1^{L-}, \ddot{\boldsymbol{I}}_1^{U-}], [\ddot{\boldsymbol{F}}_1^{L-}, \ddot{\boldsymbol{F}}_1^{U-}])>, <[\boldsymbol{y_2^{L-}}, \boldsymbol{y_2^{U-}}], ([\ddot{\boldsymbol{T}}_2^{L-}, \ddot{\boldsymbol{T}}_2^{U-}], [\ddot{\boldsymbol{I}}_2^{L-}, \ddot{\boldsymbol{I}}_2^{U-}], [\ddot{\boldsymbol{F}}_2^{L-}, \ddot{\boldsymbol{F}}_2^{U-}])>, \ldots, <[\boldsymbol{y_n^{L-}}, \boldsymbol{y_n^{U-}}], ([\ddot{\boldsymbol{T}}_n^{L-}, \ddot{\boldsymbol{T}}_n^{U-}], [\ddot{\boldsymbol{I}}_n^{L-}, \ddot{\boldsymbol{I}}_n^{U-}], [\ddot{\boldsymbol{F}}_n^{L-}, \ddot{\boldsymbol{F}}_n^{U-}])> \quad \textbf{(20)}$$

Where

$$\boldsymbol{y_j^{L+}} = \mathbf{max}_i(\boldsymbol{y_{ij}^{L}}), \boldsymbol{y_j^{U+}} = \mathbf{max}_i(\boldsymbol{y_{ij}^{U}}),$$
$$\ddot{\boldsymbol{T}}_j^{L+} = \mathbf{max}_i(\ddot{\boldsymbol{T}}_{ij}^{L}), \ddot{\boldsymbol{T}}_j^{U+} = \mathbf{max}_i(\ddot{\boldsymbol{T}}_{ij}^{U}), \ddot{\boldsymbol{I}}_j^{L+} = \mathbf{min}_i(\ddot{\boldsymbol{I}}_{ij}^{L}), \ddot{\boldsymbol{I}}_j^{U+} = \mathbf{min}_i(\ddot{\boldsymbol{I}}_{ij}^{U}), \ddot{\boldsymbol{F}}_j^{L+} = \mathbf{min}_i(\ddot{\boldsymbol{F}}_{ij}^{L}), \ddot{\boldsymbol{F}}_j^{U+} = \mathbf{min}_i(\ddot{\boldsymbol{F}}_{ij}^{U}),$$
$$\boldsymbol{y_j^{L-}} = \mathbf{min}_i(\boldsymbol{y_{ij}^{L}}), \boldsymbol{y_j^{U-}} = \mathbf{min}_i(\boldsymbol{y_{ij}^{U}}),$$
$$\ddot{\boldsymbol{T}}_j^{L-} = \mathbf{min}_i(\ddot{\boldsymbol{T}}_{ij}^{L}), \ddot{\boldsymbol{T}}_j^{U-} = \mathbf{min}_i(\ddot{\boldsymbol{T}}_{ij}^{U}), \ddot{\boldsymbol{I}}_j^{L-} = \mathbf{max}_i(\ddot{\boldsymbol{I}}_{ij}^{L}), \ddot{\boldsymbol{I}}_j^{U-} = \mathbf{max}_i(\ddot{\boldsymbol{I}}_{ij}^{U}), \ddot{\boldsymbol{F}}_j^{L-} = \mathbf{max}_i(\ddot{\boldsymbol{F}}_{ij}^{L}), \ddot{\boldsymbol{F}}_j^{U-} = \mathbf{max}_i(\ddot{\boldsymbol{F}}_{ij}^{U}),$$
(21)

(4) Obtain the distance between each alternative and the positive ideal solution, and between each alternative and the negative ideal solution, then we can get

$$D^{+}= (d_1^{+}, d_2^{+},\ldots, d_m^{+})$$
$$D^{-}= (d_1^{-}, d_2^{-},\ldots, d_m^{-}) \qquad \textbf{(22)}$$

Where,

$$\begin{cases} d_i^{+} = \left[\sum_{j=1}^{n}\left(d(y_{ij}, y_j^{+})\right)^2\right]^{\frac{1}{2}} \\ d_i^{-} = \left[\sum_{j=1}^{n}\left(d(y_{ij}, y_j^{-})\right)^2\right]^{\frac{1}{2}} \end{cases} \qquad \textbf{(23)}$$

Where , $d(y_{ij}, y_j^{+})$is the distance between the interval valued neutrosophic uncertain linguistic variables y_{ij} and y_j^{+} and $d(y_{ij}, y_j^{-})$ is the distance between the interval valued neutrosophic uncertain linguistic variables y_{ij} and y_j^{-} which can be calculated by (12)

(5) Obtain the closeness coefficients of each alternative to the ideal solution, and then we can get

$$cc_i=\frac{d_i^{+}}{d_i^{+}+d_i^{-}} \quad \text{(i=1,2,…,m)} \qquad \textbf{(24)}$$

(6) Rank the alternatives

According to the closeness coefficient above, we can choose an alternative with minimum cc_i or rank alternatives according to cc_i in ascending order

IV. An illustrative example

In this part, we give an illustrative example adapted from J. Ye [20] for the extended TOPSIS method to multiple attribute decision making problems in which the attribute values are the interval neutrosophic uncertain linguistic variables.

Suppose that an investment company, wants to invest a sum of money in the best option. To invest the money, there is a panel with four possible alternatives: (1) A_1 is car company; (2) A_2 is food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investement company must take a decision according to the three attributes: (1) C_1 is the risk; (2) C_2 is the growth; (3) C_3 is a the environmental impact. The weight vector of the attributes is ω= $(0.35, 0.25, 0.4)^{\mathrm{T}}$.The expert evaluates the four possible alternatives of A_i (i=1,2,3,4) with respect to the three attributes of C_j (i=1,2,3), where the evaluation information is expressed by the form of INULV values under the linguistic term set S={s_0=extremely poor, s_1=very poor, s_2= poor, s_3= medium, s_4= good, s_5= very good, s_6= extermely good}.

The evaluation information of an alternative A_i (i=1, 2, 3) with respect to an attribute C_j (j=1, 2, 3) can be given by the expert. For example, the INUL value of an alternative A_1 with respect to an attribute C_1 is given as <[s_4, s_5], ([0.4, 0.5],[0.2, 0.3], [0.3, 0.4])> by the expert, which indicates that the mark of the alternative A_1 with respect to the attribute C_1 is about the uncertain linguistic value [s_4, s_5,] with the satisfaction degree interval [0.4 ,0.5], indeterminacy degree interval [0.2, 0.3], and dissatisfaction degree interval [0.3, 0.4]. similarly, the four possible alternatives with respect to the three attributes can be evaluated by the expert, thus we can obtain the following interval neutrosophic uncertain linguistic decision matrix:

$(R)_{\mathrm{m}\times\mathrm{n}}=$

$$\begin{bmatrix} <([s_4,s_5],([0.4,0.5],[0.2,0.3],[0.3,0.4])> & <([s_5,s_6],([0.4,0.6],[0.1,0.2],[0.2,0.4])> & <([s_4,s_5],([0.2,0.3],[0.1,0.2],[0.5,0.6])> \\ <([s_5,s_6],([0.5,0.7],[0.1,0.2],[0.2,0.3])> & <([s_4,s_5],([0.6,0.7],[0.1,0.2],[0.2,0.3])> & <([s_4,s_5],([0.5,0.7],[0.2,0.2],[0.1,0.2])> \\ <([s_5,s_6],([0.3,0.5],[0.1,0.2],[0.3,0.4])> & <([s_5,s_6],([0.5,0.6],[0.1,0.3],[0.3,0.4])> & <([s_4,s_4],([0.5,0.6],[0.1,0.3],[0.1,0.3])> \\ <([s_3,s_4],([0.7,0.8],[0.0,0.1],[0.1,0.2])> & <([s_3,s_4],([0.5,0.7],[0.1,0.2],[0.2,0.3])> & <([s_5,s_6],([0.3,0.4],[0.1,0.2],[0.1,0.2])> \end{bmatrix}$$

A. Decision steps

To get the best an alternatives, the following steps are involved:

Step 1: Normalization
Because the attributes are all the benefit types, we don't need the normalization of the decision matrix X
Step 2: Determine the attribute weight vector W, by formula (24), we can get
w_1= 0.337 , w_2= 0.244 , w_3=0.379
Step 3: Construct the weighted normalized matrix, by formula (18), we can get

$$Y = \begin{bmatrix} <([s_{1.508}, s_{1.885}], ([0.175, 0.229], [0.545, 0.635], [0.635, 0.708])> & <([s_{1.225}, s_{1.467}], ([0.117, 0.201], [0.570, 0.675], [0.675, 0.800])> & <([s_{1.508}, s_{1.885}], ([0.081, 0.126], [0.420, 0.545], [0.77, 0.825])> \\ <([s_{1.885}, s_{2.262}], ([0.229, 0.365], [0.42, 0.545], [0.545, 0.635])> & <([s_{0.98},, s_{1.225}], ([0.201, 0.255], [0.570, 0.675], [0.675, 0.745])> & <([s_{1.508}, s_{1.885}], ([0.231, 0.365], [0.545, 0.545], [0.420, 0.545])> \\ <([s_{1.885}, s_{2.262}], ([0.125, 0.23], [0.42, 0.545], [0.635, 0.708])> & <([s_{0.98}, s_{1.225}], ([0.156, 0.201], [0.570, 0.745], [0.745, 0.800])> & <([s_{1.508}, s_{1.508}], ([0.231, 0.292], [0.420, 0.635], [0.420, 0.635])> \\ <([s_{1.131}, s_{1.508}], ([0.364, 0.455], [0.0, 0.42], [0.42, 0.545])> & <([s_{0.735}, s_{0.98}], ([0.156, 0.255], [0.570, 0.674], [0.675, 0.745])> & <([s_{1.885}, s_{2.262}], ([0.126, 0.175], [0.420, 0.545], [0.420, 0.545])> \end{bmatrix}$$

Step 4: Identify the sets of the positive ideal solution $Y^+ = (y_1^+, y_2^+, y_3^+)$ and the negative ideal solution $Y^- = (y_1^-, y_2^-, y_3^-)$, by formulas (19)- (21), we can get then we can get

$Y^+ = (<([s_{1.885}, s_{2.262}], ([0.365, 0.455], [0, 0.42], [0.42, 0.545])>$
$, <([s_{1.225}, s_{1.47}], ([0.201, 0.255], [0.569, 0.674], [0.674, 0.745])>,$
$<([s_{1.885}, s_{2.262}], ([0.230, 0.365], [0.420, 0.545], [0.420, 0.545])>)$

$Y^- = (<([s_{1.131}, s_{1.508}], ([0.126, 0.230], [0.545, 0.635], [0.635, 0.708])>$
$, <([s_{0.735}, s_{0.98}], ([0.117, 0.201], [0.569, 0.745], [0.745, 0.799])>, <$
$([s_{1.508}, s_{1.508}], ([0.081, 0.126], [0.545, 0.635], [0.770, 0.825])>)$

Step 5: Obtain the distance between each alternative and the positive ideal solution, and between each alternative and the negative ideal solution, by formulas (22)-(23), we can get
D^+= (0.402, 0.065, 0.089, 0.066)

D^-= (0.052, 0.073, 0.080, 0.065)

Step 6: Calculate the closeness coefficients of each alternative to the ideal solution, by formula (24) and then we can get

cc_i = (0.885, 0.472, 0.527, 0.503)
Step 7: Rank the alternatives
According to the closeness coefficient above, we can choose an alternative with minimum to cc_i in ascending order. We can get

$$A_2 \geq A_4 \geq A_3 \geq A_1$$

So, the most desirable alternative is A_2

V-Comparison analysis with the existing interval neutrosophic uncertain linguistic multicriteria decision making method.

Recently, J. Ye [20] developed a new method for solving the MCDM problems with interval neutrosophic uncertain linguistic information. In this section, we will perform a comparison analysis between our new method and the existing method, and then highlight the advantages of the new method over the existing method.

(1) Compared with method proposed proposed by J. Ye [20], the method in this paper can solve the MADM problems with unknown weight, and rank the alternatives by the closeness coefficients. However, the method proposed by J. Ye [20] cannot deal with the unknown weight It can be seen that the result of the proposed method is same to the method proposed in [20].

(2) Compared with other extended TOPSIS method
Because the interval neutrosophic uncertain linguistic variables are the generalization of interval neutrosophic linguistic variables (INLV), interval neutrosophic variables (INV),and intuitionistic uncertain linguistic variable. Obviously, the extended TOPSIS method proposed by J. Ye [19], Z. Wei [54], Z. Zhang and C. Wu [3], are the special cases of the proposed method in this paper.
In a word, the method proposed in this paper is more generalized. At the same time, it is also simple and easy to use.

VI-**Conclusion**

In real decision making, there is great deal of qualitative information which can be expressed by uncertain linguistic variables. The interval neutrosophic uncertain linguistic variables were produced by combining the uncertain linguistic variables and interval neutrosophic set, and could easily express the indeterminate and inconsistent information in real world. TOPSIS had been proved to be a very effective decision making method and has been achieved more and more extensive applications. However, the standard TOPSIS method can only process the real numbers. In this paper, we extended TOPSIS method to deal with the interval neutrosophic uncertain linguistic variables information, and proposed an extended TOPSIS method with respect to the MADM problems in which the attribute values take the form of the interval neutrosophic and attribute weight unknown. Firstly, the operational rules and properties for the interval neutrosophic uncertain linguistic variables were presented. Then the distance between two interval neutrosophic uncertain linguistic variables was proposed and the attribute weight was calculated by the maximizing deviation method, and the closeness coefficient to the ideal solution for each alternative used to rank the alternatives. Finally, an illustrative example was given to illustrate the decision making steps, and compared with the existing method and proved the effectiveness of the proposed method. However, we hope that the concept presented here will create new avenue of research in current neutrosophic decision making area.

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Thesis-Antithesis-Neutrothesis, and Neutrosynthesis

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Abstract. In this short paper we extend the dialectical triad thesis-antithesis-synthesis (dynamics of <A> and <antiA>, to get a synthesis) to the neutrosophic tetrad thesis-antithesis-neutrothesis-neutrosynthesis (dynamics of <A>, <antiA>, and <neutA>, in order to get a neutrosynthesis). We do this for better reflecting our world, since the neutralities between opposites play an important role. The neutrosophic synthesis (neutrosynthesis) is more refined that the dialectical synthesis. It carries on the unification and synthesis regarding the opposites and their neutrals too.

Keywords: Thesis, Antithesis, Synthesis, Thesis-Antithesis-Neutrothesis, and Neutrosynthesis.

1. Introduction.

In neutrosophy, <A>, <antiA>, and <neutA> combined two by two, and also all three of them together form the NeutroSynthesis. Neutrosophy establishes the universal relations between <A>, <antiA>, and <neutA>.
<A> is the thesis, <antiA> the antithesis, and <neutA> the neutrothesis (neither <A> nor <antiA>, but the neutrality in between them).
In the neutrosophic notation, <nonA> (not <A>, outside of <A>) is the union of <antiA> and <neutA>.
<neutA> may be from no middle (excluded middle), to one middle (included middle), to many finite discrete middles (finite multiple included-middles), and to an infinitude of discrete or continuous middles (infinite multiple included-middles) [for example, as in color for the last one, let's say between black and white there is an infinite spectrum of middle/intermediate colors].

2. Thesis, Antithesis, Synthesis.

The classical reasoning development about evidences, popularly known as thesis-antithesis-synthesis from dialectics, was attributed to the renowned philosopher Georg Wilhelm Friedrich Hegel (1770-1831) and later it was used by Karl Marx (1818-1883) and Friedrich Engels (1820-1895). About thesis and antithesis have also written Immanuel Kant (1724-1804), Johann Gottlieb Fichte (1762-1814), and Thomas Schelling (born 1921). While in ancient Chinese philosophy the opposites yin [feminine, the moon] and yang [masculine, the sun] were considered complementary.

Thesis, Antithesis, Neutrothesis, Neutrosynthesis.

Neutrosophy is a generalization of dialectics (which is based on contradictions only, <A> and <antiA>), because neutrosophy is based on contradictions and on the neutralities between them (<A>, <antiA>, and <neutA>). Therefore, the dialectical triad thesis-antithesis-synthesis is extended to the neutrosophic tetrad thesis-antithesis-neutrothesis-neutrosynthesis. We do this not for the sake of generalization, but for better reflecting our world. A neutrosophic synthesis (neutrosynthesis) is more refined that the dialectical synthesis. It carries on the unification and synthesis regarding the opposites and their neutrals too.

Neutrosophic Dynamicity.

We have extended in [1] the Principle of Dynamic Opposition [opposition between <A> and <antiA>] to the **Principle of Dynamic Neutropposition** [which means oppositions among <A>, <antiA>, and <neutA>]. Etymologically "neutropposition" means "neutrosophic opposition".
This reasoning style is not a neutrosophic scheme, but it is based on reality, because if an idea (or notion) <A> arises, then multiple versions of this idea are spread out, let's denote them by <A>$_1$, <A>$_2$, …, <A>$_m$. Afterwards, the opposites (in a smaller or higher degree) ideas are born, as reactions to <A> and its versions <A>$_i$. Let's denote these versions of opposites by <antiA>$_1$, <antiA>$_2$, …, <antiA>$_n$. The neutrality <neutA> between these contradictories ideas may embrace various forms, let's denote them by <neutA>$_1$, <neutA>$_2$, …, <neutA>$_p$, where m, n, p are integers greater than or equal to 1.

In general, for each <A> there may be corresponding many <antiA>'s and many <neutA>'s. Also, each <A> may be interpreted in many different versions of <A>'s too.
Neutrosophic Dynamicity means the interactions among all these multi-versions of <A>'s
with their multi-<antiA>'s and their multi-<neutA>'s, which will result in a new thesis, let's call it <A'> at a superior level. And a new cycle of <A'>, <antiA'>, and <neutA'> restarts its neutrosophic dynamicity.

Practical Example

Let's say <A> is a country that goes to war with another country, which can be named <antiA> since it is antagonistic to the first country. But many neutral countries <neutA> can interfere, either supporting or aggressing one of them, in a smaller or bigger degree. Other neutral countries <neutA> can still remain neutral in this war. Yet, there is a continuous dynamicity between the three categories (<A>, <antiA>, <neutA.), for countries changing sides (moving from a coalition to another coalition), or simply retreating from any coalition.
In our easy example, we only wanted to emphasize the fact that <neutA> plays a role in the conflict between the opposites <A> and <antiA>, role which was ignored by dialectics.
So, the dialectical synthesis is extended to a neutrosophic synthesis, called neutrosynthesis, which combines thesis, antithesis, and neutrothesis.

Theoretical Example.

Suppose <A> is a philosophical school, and its opposite philosophical school is <antiA>. In the dispute between <A> and <antiA>, philosophers from the two contradictory groups may bring arguments against the other philosophical school from various neutral philosophical schools' ideas (<neutA>, which were neither for <A> nor <antiA>) as well.

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Refined Literal Indeterminacy and the Multiplication Law of Sub-Indeterminacies

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Abstract. In this paper, we make a short history about: the neutrosophic set, neutrosophic numerical components and neutrosophic literal components, neutrosophic numbers, neutrosophic intervals, neutrosophic hypercomplex numbers of dimension n, and elementary neutrosophic algebraic structures. Afterwards, their generalizations to refined neutrosophic set, respectively refined neutrosophic numerical and literal components, then refined neutrosophic numbers and refined neutrosophic algebraic structures. The aim of this paper is to construct examples of splitting the literal indeterminacy ($\boldsymbol{I}$) into literal sub-indeterminacies ($\boldsymbol{I_1}, \boldsymbol{I_2}, \dots, \boldsymbol{I_r}$), and to define a multiplication law of these literal sub-indeterminacies in order to be able to build refined $\boldsymbol{I}-$neutrosophic algebraic structures. Also, examples of splitting the numerical indeterminacy ($\boldsymbol{i}$) into numerical sub-indeterminacies, and examples of splitting neutrosophic numerical components into neutrosophic numerical sub-components are given.

Keywords: neutrosophic set, elementary neutrosophic algebraic structures, neutrosophic numerical components, neutrosophic literal components, neutrosophic numbers, refined neutrosophic set, refined elementary neutrosophic algebraic structures, refined neutrosophic numerical components, refined neutrosophic literal components, refined neutrosophic numbers, literal indeterminacy, literal sub-indeterminacies, $\boldsymbol{I}$-neutrosophic algebraic structures.

1 Introduction

Neutrosophic Set was introduced in 1995 by Florentin Smarandache, who coined the words "neutrosophy" and its derivative „neutrosophic". The first published work on neutrosophics was in 1998 see [3].

There exist two types of neutrosophic components: numerical and literal.

2 Neutrosophic Numerical Components

Of course, the *neutrosophic numerical components* (t, i, f) are crisp numbers, intervals, or in general subsets of the unitary standard or nonstandard unit interval.

Let $\mathcal{U}$ be a universe of discourse, and M a set included in $\mathcal{U}$. A generic element x from $\mathcal{U}$ belongs to the set M in the following way: $x(t, i, f) \in M$, meaning that x's degree of membership/truth with respect to the set M is t, x's degree of indeterminacy with respect to the set M is i, and x's degree of non-membership/falsehood with respect to the set M is f, where t, i, f are independent standard subsets of the interval $[0, 1]$, or non-standard subsets of the non-standard interval $]^-0, 1^+[$ in the case when one needs to make distinctions between *absolute and relative* truth, indeterminacy, or falsehood.

Many papers and books have been published for the cases when t, i, f were single values (crisp numbers), or t, i, f were intervals.

3 Neutrosophic Literal Components

In 2003, W. B. Vasantha Kandasamy and Florentin Smarandache [4] introduced the *literal indeterminacy* "I", such that $I^2 = I$ (whence $I^n = I$ for $n \geq 1$, n integer). They extended this to *neutrosophic numbers* of the form: $a + bI$, where a, b are real or complex numbers, and

$$(a_1 + b_1 I) + (a_2 + b_2 I) = (a_1 + a_2) + (b_1 + b_2)I \quad (1)$$

$$(a_1 + b_1 I)(a_2 + b_2 I) = (a_1 a_2) + (a_1 b_2 + a_2 b_1 + b_1 b_2)I \quad (2)$$

and developed many I-neutrosophic algebraic structures based on sets formed of neutrosophic numbers.

Working with imprecisions, Vasantha Kandasamy & Smarandache have proposed (approximated) I^2 *by* I; yet different approaches may be investigated by the interested researchers where $I^2 \neq I$ (in accordance with their believe and with the practice), and thus a new field would arise in the neutrosophic theory.

The neutrosophic number $N = a + bI$ can be interpreted as: "a" represents the determinate part of number N, while "bI" the indeterminate part of number N.

For example, $\sqrt{7} = 2.6457...$ that is irrational has infinitely many decimals. We cannot work with this exact number in our real life, we need to approximate it. Hence, we may write it as $2 + I$ with $I \in (0.6, 0.7)$, or as $2.6 + 3I$ with $I \in (0.01, 0.02)$, or $2.64 + 2I$ with $I \in (0.002, 0.004)$, etc. depending on the problem to be solved and on the needed accuracy.

Jun Ye [9] applied the neutrosophic numbers to decision making in *2014*.

4 Neutrosophic Intervals

We now for the first time extend the neutrosophic number to (open, closed, or half-open half-closed) neutrosophic interval. A *neutrosophic interval A* is an (open, closed, or half-open half-closed) interval that has some indeterminacy in one of its extremes, i.e. it has the form $A = [a, b] \cup \{cI\}$, or $A = \{cI\} \cup [a, b]$, where $[a, b]$ is the determinate part of the neutrosophic interval A, and I is the indeterminate part of it (while *a, b, c* are real numbers, and $\cup$ means union). (Herein I is an interval.)

We may even have neutrosophic intervals with double indeterminacy (or refined indeterminacy): one to the left (I_1), and one to the right (I_2):

$$A = \{c_1 I_1\} \cup [a, b] \cup \{c_2 I_2\}. \tag{3}$$

A classical real interval that has a neutrosophic number as one of its extremes becomes a neutrosophic interval. For example: $[0, \sqrt{7}]$ can be represented as $[0, 2] \cup I$ with $I = (2.0, 2.7)$, or $[0, 2] \cup \{10I\}$ with $I = (0.20, 0.27)$, or $[0, 2.6] \cup \{10I\}$ with $I = (0.26, 0.27)$, or $[0, 2.64] \cup \{10I\}$ with $I = (0.264, 0.265)$, etc. in the same way depending on the problem to be solved and on the needed accuracy.

We gave examples of closed neutrosophic intervals, but the open and half-open half-closed neutrosophic intervals are similar.

5 Notations

In order to make distinctions between the numerical and literal neutrosophic components, we start denoting the *numerical indeterminacy* by lower case letter "i" (whence consequently similar notations for *numerical truth* "t", and for *numerical falsehood* "f"), and *literal indeterminacy* by upper case letter "I" (whence consequently similar notations for *literal truth* "T", and for *literal falsehood* "F").

6 Refined Neutrosophic Components

In *2013*, F. Smarandache [3] introduced the refined neutrosophic components in the following way: the neutrosophic numerical components t, i, f can be refined (split) into respectively the following refined neutrosophic numerical sub-components:

$$\langle t_1, t_2, \dots t_p;\ i_1, i_2, \dots\ i_r;\ f_1, f_2, \dots f_s \rangle, \tag{4}$$

where p, r, s are integers ≥ 1 and $\max\{p, r, s\} \geq 2$, meaning that at least one of p, r, s is ≥ 2; and t_j represents types of numeral truths, i_k represents types of numeral indeterminacies, and f_l represents types of numeral falsehoods, for $j = 1, 2, \dots, p$; $k = 1, 2, \dots, r$; $l = 1, 2, \dots, s$.

t_j, i_k, f_l are called numerical subcomponents, or respectively *numerical sub-truths*, *numerical sub-indeterminacies*, and *numerical sub-falsehoods*.

Similarly, the neutrosophic literal components T, I, F can be refined (split) into respectively the following neutrosophic literal subcomponents:

$$\langle T_1, T_2, \dots T_p;\ I_1, I_2, \dots I_r;\ F_1, F_2, \dots F_s \rangle, \tag{5}$$

where p, r, s are integers ≥ 1 too, and $\max\{p, r, s\} \geq 2$, meaning that at least one of p, r, s is ≥ 2; and similarly T_j represent types of literal truths, I_k represent types of literal indeterminacies, and F_l represent types of literal falsehoods, for $j = 1, 2, \dots, p$; $k = 1, 2, \dots, r$; $l = 1, 2, \dots, s$.

T_j, I_k, F_l are called literal subcomponents, or respectively *literal sub-truths*, *literal sub-indeterminacies*, and *literal sub-falsehoods*.

Let consider a *simple example of refined numerical components*.

Suppose that a country C is composed of two districts D_1 and D_2, and a candidate John Doe competes for the position of president of this country C. Per whole country, $NL(\text{Joe Doe}) = (0.6, 0.1, 0.3)$, meaning that 60% of people voted for him, 10% of people were indeterminate or neutral – i.e. didn't vote, or gave a black vote, or a blank vote –, and 30% of people voted against him, where NL means the neutrosophic logic values.

But a political analyst does some research to find out what happened to each district separately. So, he does a refinement and he gets:

$$\begin{pmatrix} 0.40 & 0.20 & ; & 0.08 & 0.02 & ; & 0.05 & 0.25 \\ t_1 & t_2 & & i_1 & i_2 & & f_1 & f_2 \end{pmatrix} \tag{6}$$

which means that 40% of people that voted for Joe Doe were from district D_1, and 20% of people that voted for Joe Doe were from district D_2; similarly, 8% from D_1 and 2% from D_2 were indeterminate (neutral), and 5% from D_1 and 25% from D_2 were against Joe Doe.

It is possible, in the same example, to refine (split) it in a different way, considering another criterion, namely: what percentage of people did not vote (i_1), what percentage of people gave a blank vote – cutting all candidates on the ballot – (i_2), and what percentage of people gave a blank vote – not selecting any candidate on the ballot (i_3). Thus, the numerical indeterminacy (i) is refined into i_1, i_2, and i_3:

$$\begin{pmatrix} 0.60 & ; & 0.05 & 0.04 & 0.01 & ; & 0.30 \\ t & & i_1 & i_2 & i_3 & & f \end{pmatrix} \tag{7}$$

7 Refined Neutrosophic Numbers

In 2015, F. Smarandache [6] introduced the *refined literal indeterminacy* (I), which was split (refined) as $I_1, I_2, \ldots, I_r$, with $r \geq 2$, where I_k, for $k = 1, 2, \ldots, r$ represent types of literal sub-indeterminacies. A refined neutrosophic number has the general form:

$$N_r = a + b_1 I_1 + b_2 I_2 + \cdots + b_r I_r, \quad (8)$$

where $a, b_1, b_2, \ldots, b_r$ are real numbers, and in this case N_r is called a *refined neutrosophic real number*; and if at least one of $a, b_1, b_2, \ldots, b_r$ is a complex number (i.e. of the form $\alpha + \beta\sqrt{-1}$, with $\beta \neq 0$, and α, β real numbers), then N_r is called a *refined neutrosophic complex number*.

An example of refined neutrosophic number, with three types of indeterminacies resulted from the cubic root (I_1), from Euler's constant e (I_2), and from number π (I_3):

$$N_3 = -6 + \sqrt[3]{59} - 2e + 11\pi \quad (9)$$

Roughly

$N_3 = -6 + (3 + I_1) - 2(2 + I_2) + 11(3 + I_3)$

$= (-6 + 3 - 4 + 33) + I_1 - 2I_2 + 11I_3 = 26 + I_1 - 2I_2 + 11I_3$

where $I_1 \in (0.8, 0.9)$, $I_2 \in (0.7, 0.8)$, and $I_3 \in (0.1, 0.2)$, since $\sqrt[3]{59} = 3.8929\ldots$, $e = 2.7182\ldots$, $\pi = 3.1415\ldots$.

Of course, other *3*-valued refined neutrosophic number representations of N_3 could be done depending on accuracy.

Then F. Smarandache [6] defined the *refined I-neutrosophic algebraic structures* in 2015 as algebraic structures based on sets of refined neutrosophic numbers.

Soon after this definition, Dr. Adesina Agboola wrote a paper on refined *I*-neutrosophic algebraic structures [7].

They were called "I-neutrosophic" because the refinement is done with respect to the literal indeterminacy (I), in order to distinguish them from the refined (t, i, f)-neutrosophic algebraic structures, where "(t, i, f)-neutrosophic" is referred to as refinement of the neutrosophic numerical components t, i, f.

Said Broumi and F. Smarandache published a paper [8] on refined neutrosophic numerical components in 2014.

8 Neutrosophic Hypercomplex Numbers of Dimension n

The *Hypercomplex Number of Dimension n* (or *n-Complex Number*) was defined by S. Olariu [10] as a number of the form:

$$u = x_o + h_1x_1 + h_2x_2 + \ldots + h_{n-1}x_{n-1} \quad (10)$$

where $n \geq 2$, and the variables $x_0, x_1, x_2, \ldots, x_{n-1}$ are real numbers, while $h_1, h_2, \ldots, h_{n-1}$ are the complex units, $h_o = 1$, and they are multiplied as follows:

$$h_jh_k = h_{j+k} \text{ if } 0 \leq j+k \leq n-1, \text{ and } h_jh_k = h_{j+k-n} \text{ if } n \leq j+k \leq 2n-2. \quad (11)$$

We think that the above (11) complex unit multiplication formulas can be written in a simpler way as:

$$h_jh_k = h_{j+k \ (mod\ n)} \quad (12)$$

where *mod n* means *modulo n.*

For example, if $n = 5$, then $h_3h_4 = h_{3+4(mod\ 5)} = h_{7(mod5)} = h_2$.

Even more, formula (12) allows us to multiply many complex units at once, as follows:

$$h_{j1}h_{j2}\ldots h_{jp} = h_{j1+j2+\ldots+jp\ (mod\ n)}, \text{ for } p \geq 1. \quad (13)$$

We now define for the first time the *Neutrosophic Hypercomplex Number of Dimension n* (or *Neutrosophic n-Complex Number*), which is a number of the form:

$$u+vI, \quad (14)$$

where u and v are n-complex numbers and I = indeterminacy.

We also introduce now the *Refined Neutrosophic Hypercomplex Number of Dimension n* (or *Refined Neutrosophic n-Complex Number*) as a number of the form:

$$u+v_1I_1+v_2I_2+\ldots+v_rI_r \quad (15)$$

where $u, v_1, v_2, \ldots, v_r$ are n-complex numbers, and $I_1, I_2, \ldots, I_r$ are sub-indeterminacies, for $r \geq 2$.

Combining these, we may define a *Hybrid Neutrosophic Hypercomplex Number* (or *Hybrid Neutrosophic n-Complex Number*), which is a number of the form $u+vI$, where either u or v is a n-complex number while the other one is different (may be an m-complex number, with $m \neq n$, or a real number, or another type of number).

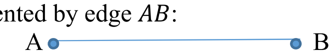
And a *Hybrid Refined Neutrosophic Hypercomplex Number* (or *Hybrid Refined Neutrosophic n-Complex Number*), which is a number of the form $u+v_1I_1+v_2I_2+\ldots+v_rI_r$, where at least one of $u, v_1, v_2, \ldots, v_r$ is a n-complex number, while the others are different (may be m-complex numbers, with $m \neq n$, and/or a real numbers, and/or other types of numbers).

9 Neutrosophic Graphs

We now introduce for the first time the general definition of a *neutrosophic graph* [12], which is a (directed or undirected) graph that has some indeterminacy with respect to its edges, or with respect to its vertexes (nodes), or with respect to both (edges and vertexes simultaneously). We have four main categories of neutrosophic graphs:

1) The *(t, i, f)-Edge Neutrosophic Graph.*

In such a graph, the connection between two vertexes A and B, represented by edge AB:



has the neutroosphic value of (t, i, f).

2) *I-Edge Neutrosophic Graph.*

This one was introduced in 2003 in the book "Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps", by Dr. Vasantha Kandasamy and F. Smarandache, that used a different approach for the edge:

which can be just I = literal indeterminacy of the edge, with $I^2 = I$ (as in I-Neutrosophic algebraic structures). Therefore, simply we say that the connection between vertex A and vertex B is indeterminate.

3) *Orientation-Edge Neutrosophic Graph.*

At least one edge, let's say AB, has an unknown orientation (i.e. we do not know if it is from A to B, or from B to A).

4) *I-Vertex Neutrosophic Graph.*

Or at least one literal indeterminate vertex, meaning we do not know what this vertex represents.

5) (t, i, f)*-Vertex Neutrosophic Graph.*

We can also have at least one neutrosophic vertex, for example vertex A only partially belongs to the graph (t), indeterminate appurtenance to the graph (i), does not partially belong to the graph (f), we can say $A(t, i, f)$.

And combinations of any two, three, four, or five of the above five possibilities of neutrosophic graphs.

If (t, i, f) or the literal I are refined, we can get corresponding *refined neurosophic graphs*.

10 Example of Refined Indeterminacy and Multiplication Law of Sub-Indeterminacies

Discussing the development of Refined I-Neutrosophic Structures with Dr. W.B. Vasantha Kandasamy, Dr. A.A.A. Agboola, Mumtaz Ali, and Said Broumi, a question has arisen: if I is refined into $I_1, I_2, \ldots, I_r$, with $r \geq 2$, how to define (or compute) $I_j * I_k$, for $j \neq k$?

We need to design a Sub-Indeterminacy $*$ Law Table.

Of course, this depends on the way one defines the algebraic binary multiplication law $*$ on the set:

$$\{N_r = a + b_1 I_1 + b_2 I_2 + \cdots + b_r I_r | a, b_1, b_2, \ldots, b_r \in M\}, \quad (16)$$

where M can be $\mathbb{R}$ (the set of real numbers), or $\mathbb{C}$ (the set of complex numbers).

We present the below example.

But, first, let's present several (possible) interconnections between logic, set, and algebra.

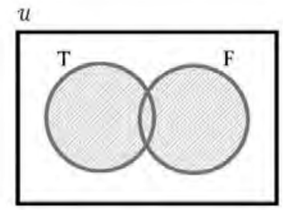
	Logic	*Set*	*Algebra*
operators	Disjunction (or) $\vee$	Union $\cup$	Addition $+$
	Conjunction (and) $\wedge$	Intersection $\cap$	Multiplication $\cdot$
	Negation $\neg$	Complement $\complement$	Subtraction $-$
	Implication $\rightarrow$	Inclusion $\subseteq$	Subtraction, Addition $-, +$
	Equivalence $\leftrightarrow$	Identity $\equiv$	Equality $=$

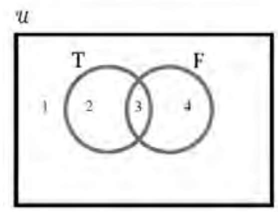
Table 1: Interconnections between logic, set, and algebra.

In general, if a Venn Diagram has n sets, with $n \geq 1$, the number of disjoint parts formed is 2^n. Then, if one combines the 2^n parts either by none, or by one, or by 2, ..., or by 2^n, one gets:

$$C_{2^n}^0 + C'_{2^n} + C_{2^n}^2 + \cdots + C_{2^n}^{2^n} = (1+1)^{2^n} = 2^{2^n}. \quad (17)$$

Hence, for $n = 2$, the Venn diagram, with literal truth (T), and literal falsehood (F), will make $2^2 = 4$ disjoint parts, where the whole rectangle represents the whole universe of discourse $(\mathcal{U})$.



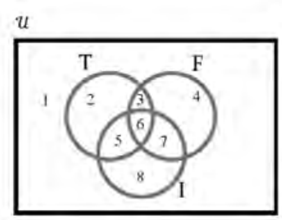


Venn Diagram for n =2.

Then, combining the four disjoint parts by none, by one, by two, by three, and by four, one gets

$$C_4^0 + C_4^1 + C_4^2 + C_4^3 + C_4^4 = (1+1)^4 = 2^4 = 16 = 2^{2^2}. \quad (18)$$

For $n = 3$, one has $2^3 = 8$ disjoint parts,

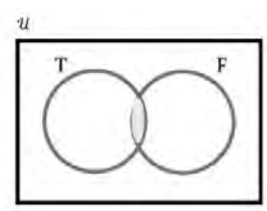


Venn Diagram for n = 3.

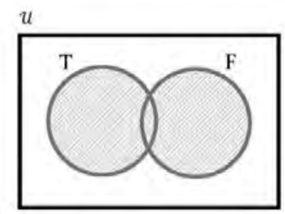
and combining them by none, by one, by two, and so on, by eight, one gets $2^8 = 256$, or $2^{2^3} = 256$.

For the case when $n = 2 = \{T, F\}$ one can make up to *16* sub-indeterminacies, such as:

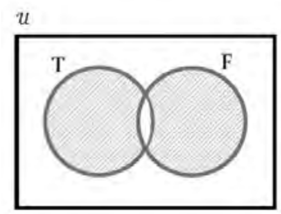
$$I_1 = C = \textbf{contradiction} = \text{True and False} = T \wedge F$$



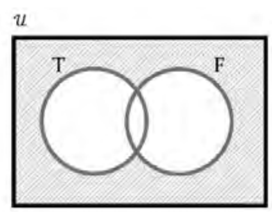
$I_2 = Y =$ **uncertainty** $=$ True or False $= T \vee F$



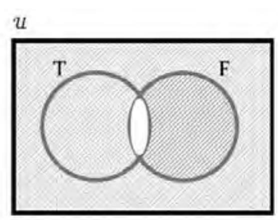
$I_3 = S =$ **unsureness** $=$ either True or False $= T \underline{\vee} F$



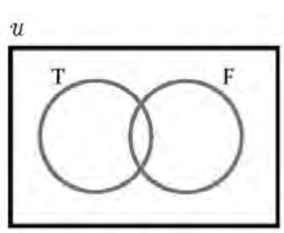
$I_4 = H =$ **nihilness** $=$ neither True nor False $= \neg T \wedge \neg F$



$I_5 = V =$ **vagueness** $=$ not True or not False $= \neg T \vee \neg F$



$I_6 = E =$ **emptiness** $=$ neither True nor not True $= \neg T \wedge \neg(\neg T) = \neg T \wedge T$



Let's consider the literal indeterminacy (I) refined into only six literal sub-indeterminacies as above.

The binary multiplication law

$$*: \{I_1, I_2, I_3, I_4, I_5, I_6\}^2 \rightarrow \{I_1, I_2, I_3, I_4, I_5, I_6\} \quad (19)$$

defined as:

$I_j * I_k$ = intersections of their Venn diagram representations;

or $I_j * I_k$ = application of $\wedge$ operator, i.e. $I_j \wedge I_k$.

We make the following:

$*$	I_1	I_2	I_3	I_4	I_5	I_6
I_1	I_1	I_1	I_6	I_6	I_6	I_6
I_2	I_1	I_2	I_3	I_6	I_3	I_6
I_3	I_6	I_3	I_3	I_6	I_3	I_6
I_4	I_6	I_6	I_6	I_4	I_4	I_6
I_5	I_6	I_3	I_3	I_4	I_5	I_6
I_6	I_6	I_6	I_6	I_6	I_6	I_6

Table 2: Sub-Indeterminacies Multiplication Law

11 Remark on the Variety of Sub-Indeterminacies Diagrams

One can construct in various ways the diagrams that represent the sub-indeterminacies and similarly one can define in many ways the $*$ algebraic multiplication law, $I_j * I_k$, depending on the problem or application to solve.

What we constructed above is just an example, not a general procedure.

Let's present below several calculations, so the reader gets familiar:

$I_1 * I_2 =$ (shaded area of I_1) $\cap$ (shaded area of I_2) $=$ shaded area of I_1,
or $I_1 * I_2 = (T \wedge F) \wedge (T \vee F) = T \wedge F = I_1$.
$I_3 * I_4 =$ (shaded area of I_3) $\cap$ (shaded area of I_4) $=$ empty set $= I_6$,
or $I_3 * I_4 = (T \underline{\vee} F) \wedge (\neg T \wedge \neg F) = [T \wedge (\neg T \wedge \neg F)] \underline{\vee} [F \wedge (\neg T \wedge \neg F)] = (T \wedge \neg T \wedge \neg F) \underline{\vee} (F \wedge \neg T \wedge \neg F) =$ (impossible) $\underline{\vee}$ (impossible)
because of $T \wedge \neg T$ in the first pair of parentheses and because of $F \wedge \neg F$ in the second pair of parentheses
$=$ (impossible) $= I_6$.
$I_5 * I_5 =$ (shaded area of I_5) $\cap$ (shaded area of I_5) $=$ (shaded area of I_5) $= I_5$,
or $I_5 * I_5 = (\neg T \vee \neg F) \wedge (\neg T \vee \neg F) = \neg T \vee \neg F = I_5$.

Now we are able to build refined I-neutrosophic algebraic structures on the set

$$S_6 = \{a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_6 I_6, \text{for } a_0, a_1, a_2, \ldots a_6 \in \mathbb{R}\}, \quad (20)$$

by defining the addition of refined I-neutrosophic numbers:

$$(a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_6 I_6) + (b_0 + b_1 I_1 + b_2 I_2 + \cdots + b_6 I_6) = (a_0 + b_0) + (a_1 + b_1) I_1 + (a_2 + b_2) I_2 + \cdots + (a_6 + b_6) I_6 \in S_6. \quad (21)$$

And the multiplication of refined neutrosophic numbers:

$$(a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_6 I_6) \cdot (b_0 + b_1 I_1 + b_2 I_2 + \cdots + b_6 I_6) = a_0 b_0 + (a_0 b_1 + a_1 b_0) I_1 + (a_0 b_2 + a_2 b_0) I_2 + \cdots + (a_0 b_6 + a_6 b_0) I_6 + \\ + \sum_{j,k=1}^{6} a_j b_k \left(I_j * I_k\right) = a_0 b_0 + \sum_{k=1}^{6} (a_0 b_k + a_k b_0) I_k + \sum_{j,k=1}^{6} a_j b_k \left(I_j * I_k\right) \in S_6, \qquad (22)$$

where the coefficients (scalars) $a_m \cdot b_n$, for $m = 0, 1, 2, \ldots, 6$ and $n = 0, 1, 2, \ldots, 6$, are multiplied as any real numbers, while $I_j * I_k$ are calculated according to the previous Sub-Indeterminacies Multiplication Law (Table 2).

Clearly, both operators (addition and multiplication of refined neutrosophic numbers) are well-defined on the set S_6.

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Taylor Series Approximation to Solve Neutrosophic Multiobjective Programming Problem

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Abstract. In this paper, Taylor series is used to solve neutrosophic multi-objective programming problem (NMOPP). In the proposed approach, the truth membership, Indeterminacy membership, falsity membership functions associated with each objective of multi-objective programming problems are transformed into a single objective linear programming problem by using a first order Taylor polynomial series. Finally, to illustrate the efficiency of the proposed method, a numerical experiment for supplier selection is given as an application of Taylor series method for solving neutrosophic multi-objective programming problem at end of this paper.

Keywords: Taylor series; Neutrosophic optimization; Multiobjective programming problem.

1 Introduction

In 1995,Smarandache [1] starting from philosophy (when he fretted to distinguish between *absolute truth* and *relative truth* or between *absolute falsehood* and *relative falsehood* in logics, and respectively between *absolute membership* and *relative membership* or *absolute non-membership* and *relative non-membership* in set theory) [1] began to use the non-standard analysis. Also, inspired from the sport games (winning, defeating, or tie scores), from votes (pro, contra, null/black votes), from positive/negative/zero numbers, from yes/no/NA, from decision making and control theory (making a decision, not making, or hesitating), from accepted/rejected/pending, etc. and guided by the fact that the law of excluded middle did not work any longer in the modern logics. [1] combined the non-standard analysis with a tri-component logic/set/probability theory and with philosophy .How to deal with all of them at once, is it possible to unity them? [1].
The words “neutrosophy” and “neutrosophic” were invented by F. Smarandache in his 1998 book [1]. Etymologically, “neutro-sophy” (noun) [French *neutre* < Latin *neuter*, neutral, and Greek *sophia*, skill / wisdom] means knowledge of neutral thought. While “neutrosophic” (adjective), means having the nature of, or having the characteristic of Neutrosophy.
Netrosophic theory means Neutrosophy applied in many fields in order to solve problems related to indeterminacy. Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. This theory considers every entity <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. entities supporting neither <A> nor<antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.
Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only). According to this theory every entity <A> tends to be neutralized and balanced by <antiA> and <nonA> entities - as a state of equilibrium. In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well. Hence, in one hand, the Neutrosophic Theory is based on the triad <A>, <neutA>, and <antiA>. In the other hand, Neutrosophic Theory studies the indeterminacy, labeled as I, with $I^n = I$ for $n \geq 1$, and $mI + nI = (m+n)I$, in neutrosophic structures developed in algebra, geometry, topology etc.
The most developed fields of Neutrosophic theory are Neutrosophic Set, Neutrosophic Logic, Neutrosophic Probability, and Neutrosophic Statistics - that started in 1995, and recently Neutrosophic Precalculus and

Neutrosophic Calculus, together with their applications in practice. Neutrosophic Set and Neutrosophic Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsity (*F*), where *T,I,F* are standard or non-standard subsets of $]^-0, 1^+[$.
Multi-objective linear programming problem (MOLPP) a prominent tool for solving many real decision making problems like game theory, inventory problems, agriculture based management systems, financial and corporate planning, production planning, marketing and media selection, university planning and student admission, health care and hospital planning, air force maintenance units, bank branches etc.

Our objective in this paper is to propose an algorithm to the solution of neutrosophic multi-objective programming problem (NMOPP) with the help of the first order Taylor's theorem. Thus, neutrosophic multi-objective linear programming problem is reduced to an equivalent multi-objective linear programming problem. An algorithm is proposed to determine a global optimum to the problem in a finite number of steps. The feasible region is a bounded set. In the proposed approach, we have attempted to reduce computational complexity in the solution of (NMOPP). The proposed algorithm is applied to supplier selection problem .

The rest of this article is organized as follows. Section 2 gives brief some preliminaries. Section 3 describes the formation of the problem. Section 4 presents the implementation and validation of the algorithm with practical application. Finally, Section 5 presents the conclusion and proposals for future work.

2 Some preliminaries

Definition 1. [1] A triangular fuzzy number $\tilde{J}$ is a continuous fuzzy subset from the real line R whose triangular membership function $\mu_{\tilde{J}}(J)$ is defined by a continuous mapping from R to the closed interval [0,1], where

(1) $\mu_{\tilde{J}}(J)=0$ for all $J \in (-\infty, a_1]$,

(2) $\mu_{\tilde{J}}(J)$ is strictly increasing on $J \in [a_1, m]$,

(3) $\mu_{\tilde{J}}(J)=1$ for $J = m$,

(4) $\mu_{\tilde{J}}(J)$ is strictly decreasing on $J \in [m, a_2]$,

(5) $\mu_{\tilde{J}}(J)=0$ for all $J \in [a_2, +\infty)$.

This will be elicited by:

$$\mu_{\tilde{J}}(J)=\begin{cases} \dfrac{J-a_1}{m-a_1}, & a_1 \le J \le m, \\ \dfrac{a_2-J}{a_2-m}, & m \le J \le a_2, \\ 0, & otherwise. \end{cases} \tag{1}$$

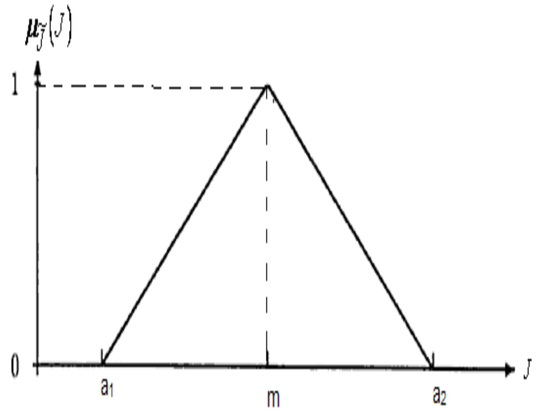

Figure 1: Membership Function of Fuzzy Number *J*.

where m is a given value and a_1 , a_2 denote the lower and upper bounds. Sometimes, it is more convenient to use the notation explicitly highlighting the membership function parameters. In this case, we obtain

$$\mu(J; a_1, m, a_2) = \text{Max}\left\{\text{Min}\left[\frac{J-a_1}{m-a_1}, \frac{a_2-J}{a_2-m}\right], 0\right\} \tag{2}$$

In what follows, the definition of the α-level set or α-cut of the fuzzy number $\tilde{J}$ is introduced.

Definition 2. [1] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed non-empty universe. An intuitionistic fuzzy set IFS A in X is defined as

$$A = \{\langle x, \mu_A(x), \upsilon_A(x)\rangle | x \in X\} \tag{3}$$

which is characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function

$\upsilon_A : X \to [0,1]$ with the condition $0 \le \mu_A(x) + \upsilon_A(x) \le 1$ for all $x \in X$ where μ_A and υ_A represent ,respectively, the degree of membership and non-membership of the element x to the set A. In addition, for each IFS A in X, $\pi_A(x) = 1 - \mu_A(x) - \upsilon_A(x)$ for all $x \in X$ is called the degree of hesitation of the element x to the set A . Especially, if $\pi_A(x)=0,$, then the IFS A is degraded to a fuzzy set.

Definition 3. [4] The α-level set of the fuzzy parameters $\tilde{J}$ in problem (1) is defined as the ordinary set $L_\alpha(\tilde{J})$ for which the degree of membership function exceeds the level, α, $\alpha \in [0,1]$, where:

$$L_\alpha(\tilde{J}) = \{ J \in R \mid \mu_{\tilde{J}}(J) \geq \alpha \} \tag{4}$$

For certain values α_j^* to be in the unit interval,

Definition 4. [1] Let X be a space of points (objects) and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. It has been shown in figure 2. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]0^-,1^+[$. That is $T_A(x):X\rightarrow]0^-,1^+[$, $I_A(x):X\rightarrow]0^-,1^+[$ and $F_A(x):X\rightarrow]0^-,1^+[$. There is not restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^-\leq \sup T_A(x) \leq \sup I_A(x) \leq F_A(x) \leq 3^+$.
In the following, we adopt the notations $\mu_A(x)$, $\sigma_A(x)$ and $\nu_A(x)$ instead of $T_A(x)$, $I_A(x)$ and $F_A(x)$, respectively. Also we write SVN numbers instead of single valued neutrosophic numbers.

Definition 5. [10] Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form
$A=\{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$
where $\mu_A(x):X\rightarrow[0,1]$, $\sigma_A(x):X\rightarrow[0,1]$ and $\nu_A(x):X\rightarrow[0,1]$ with $0\leq\mu_A(x)+\sigma_A(x)+\nu_A(x)\leq 3$ for all $x \in X$. The intervals $\mu_A(x)$, $\sigma_A(x)$ and $\nu_A(x)$ denote the truth- membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively.
For convenience, a SVN number is denoted by $A=(a,b,c)$, where $a,b,c \in [0,1]$ and $a+b+c\leq 3$.

Definition 6

Let $\tilde{J}$ be a neutrosophic triangular number in the set of real numbers R, then its truth-membership function is defined as

$$T_{\tilde{J}}(J) = \begin{cases} \dfrac{J-a_1}{a_2-a} & a_1 \leq J \leq a_2, \\ \dfrac{a_2-}{a_3-a} & a_2 \leq J \leq a_3, \\ 0, & otherwise. \end{cases} \tag{5}$$

its indeterminacy-membership function is defined as

$$I_{\tilde{J}}(J) = \begin{cases} \dfrac{J-b_1}{b_2-b} & b_1 \leq J \leq b_2, \\ \dfrac{b_2-}{b_3-b} & b_2 \leq J \leq b_3, \\ 0, & otherwise. \end{cases} \tag{6}$$

and its falsity-membership function is defined as

$$F_{\tilde{J}}(J) = \begin{cases} \dfrac{J-\ _1}{c_2-c}, & c_1 \leq J \leq c_2, \\ \dfrac{c_2-}{c_3-c} & c_2 \leq J \leq c_3, \\ 1, & otherwise. \end{cases} \tag{7}$$

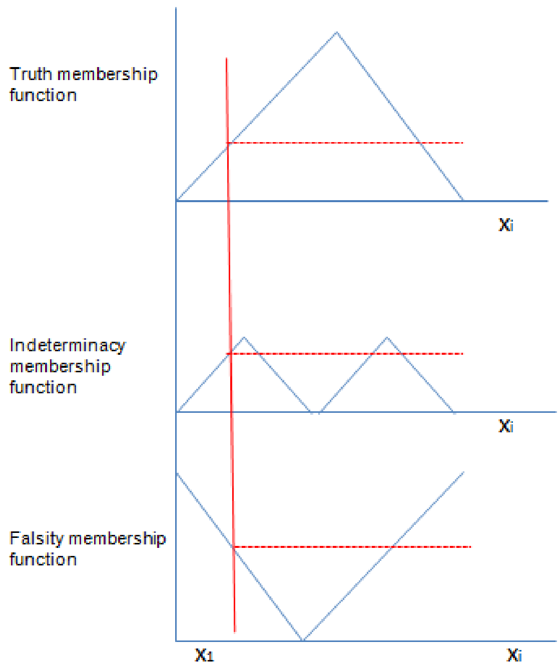

Figure 2: Neutrosophication process [11]

3 Formation of The Problem

The multi-objective linear programming problem and the multi- objective neutrosophic linear programming problem are described in this section.

A. Multi-objective Programming Problem (MOPP)

In this paper, the general mathematical model of the MOPP is as follows[6]:

$$\min/\max \left[z_1(x_1,...,x_n), z_2(x_1,...,x_n),...,z_p(x_1,...,x_n) \right] \tag{8}$$

subject to $x \in S, x$

$$S = \left\{ x \in R^n \middle| AX \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} b, \quad X \ge 0. \right\} \tag{9}$$

B. Neutrosophic Multi-objective Programming Problem (NMOPP)

If an imprecise aspiration level is introduced to each of the objectives of MOPP, then these neutrosophic objectives are termed as neutrosophic goals.

Let $z_i \in \left\lfloor z_i, z_i^U \right\rceil$ denote the imprecise lower and upper bounds respectively for the i^{th} neutrosophic objective function.

For maximizing objective function, the truth membership, indeterminacy membership, falsity membership functions can be expressed as follows:

$$\mu_i^I(z_i) = \begin{cases} 1, & \text{if } z_i \ge z_i^U, \\ \dfrac{z_i - z^L}{z_i^U - z_i}, & \text{if } z_i^L \le z_i \le z_i^U, \\ 0, & \text{if } z_i \le z^L \end{cases} \tag{10}$$

$$\sigma_i^I(z_i) = \begin{cases} 1, & \text{if } z_i \ge z_i^U, \\ \dfrac{z_i - z^L}{z_i^U - z_i}, & \text{if } z_i^L \le z_i \le z_i^U, \\ 0, & \text{if } z_i \le z^L \end{cases} \tag{11}$$

$$\upsilon_i^I(z_i) = \begin{cases} 0, & \text{if } z_i \ge z_i^U, \\ \dfrac{z_i - z^L}{z_i^U - z_i}, & \text{if } z_i^L \le z_i \le z_i^U, \\ 1, & \text{if } z_i \le z^L \end{cases} \tag{12}$$

for minimizing objective function, the truth membership, Indeterminacy membership, falsity membership functions can be expressed as follows:

$$\mu_i^I(z_i) = \begin{cases} 1, & \text{if } z_i \le z_i^L, \\ \dfrac{z_i^U - z_i}{z_i^U - z_i}, & \text{if } z_i^L \le z_i \le z_i^U, \\ 0, & \text{if } z_i \ge z_i^U \end{cases} \tag{13}$$

$$\sigma_i^I(z_i) = \begin{cases} 1, & \text{if } z_i \le z_i^L, \\ \dfrac{z_i^U - z_i}{z_i^U - z_i}, & \text{if } z_i^L \le z_i \le z_i^U, \\ 0, & \text{if } z_i \ge z_i^U \end{cases} \tag{14}$$

$$\upsilon_i^I(z_i) = \begin{cases} 0, & \text{if } z_i \le z_i^L, \\ \dfrac{z_i^U - z_i}{z_i^U - z_i}, & \text{if } z_i^L \le z_i \le z_i^U, \\ 1, & \text{if } z_i \ge z_i^U \end{cases} \tag{15}$$

4 Algorithm for Neutrosophic Multi-Objective Programming Problem

The computational procedure and proposed algorithm of presented model is given as follows:

Step 1. Determine $x_i^* = \left(x_{i1}^*, x_{i2}^*, ..., x_{in}^* \right)$ that is used to maximize or minimize the i^{th} truth membership function $\mu_i^I(X)$, the indeterminacy membership $\sigma_i^I(X)$, and the falsity membership functions $\upsilon_i^I(X)$. $i=1,2,..,p$ and n is the number of variables.

Step 2. Transform the truth membership, indeterminacy membership, falsity membership functions by using first-order Taylor polynomial series

$$\mu_i^I(x) \cong \mu_i^I(x_i^*) + \sum_{j=1}^{n} (x_j - x_{ij}) \frac{\partial \mu^I(x_i^*)}{\partial x_j} \tag{16}$$

$$\sigma_i^I(x) \cong \sigma_i^I(x_i^*) + \sum_{j=1}^{n} (x_j - x_{ij}) \frac{\partial \sigma^I(x_i^*)}{\partial x_j} \tag{17}$$

$$\upsilon_i^I(x) \cong \upsilon_i^I(x_i^*) + \sum_{j=1}^{n} (x_j - x_{ij}) \frac{\partial \upsilon^I(x_i^*)}{\partial x_j} \tag{18}$$

Step 3. Find satisfactory $x_i^* = (x_{i1}^*, x_{i2}^*, ..., x_{in}^*)$ by solving the reduced problem to a single objective for the truth membership, indeterminacy membership, falsity membership functions respectively.

$$p(x) = \sum_{i=1}^{p} \left[\mu_i^I(x_i^*) + \sum_{j=1}^{n} (x_j - x_{ij}^*) \frac{\partial \mu_i^I(x_i^*)}{\partial x_j} \right]$$

$$q(x) = \sum_{i=1}^{p} \left[\sigma_i^I(x_i) + \sum_{j=1}^{n} (x_j - x_{ij}^*) \frac{\partial \sigma_i^I(x_i^*)}{\partial x_j} \right] \tag{19}$$

$$h(x) = \sum_{i=1}^{p} \left[\upsilon_i^I(x_i) + \sum_{j=1}^{n} (x_j - x_{ij}^*) \frac{\partial \upsilon_i^I(x_i^*)}{\partial x_j} \right]$$

Thus neutrosophic multiobjective linear programming problem is converted into a new mathematical model and is given below:

Maximize or Minimize *p(x)*

Maximize or Minimize *q(x)*

Maximize or Minimize *h(x)*

Where $\mu_i^I(X)$, $\sigma_i^I(X)$ and $\upsilon_i^I(X)$ calculate using equations (10), (11), and (12) or equations (13), (14), and (15) according to type functions maximum or minimum respectively.

4.1 Illustrative Example

A multi-criteria supplier selection is selected from [2]. For supplying a new product to a market assume that three suppliers should be managed. The purchasing criteria are net price, quality and service. The capacity constraints of suppliers are also considered.

It is assumed that the input data from suppliers' performance on these criteria are not known precisely. The neutrosophic values of their cost, quality and service level are presented in Table 1.

The multi-objective linear formulation of numerical example is presented as :

$$\begin{aligned}
&\min\ z_1 = 5x_1 + 7x_2 + 4x_3, \\
&\max\ z_2 = 0.80x_1 + 0.90x_2 + 0.85x_3, \\
&\max\ z_3 = 0.90x_1 + 0.80x_2 + 0.85x_3, \\
&s.t.: \\
&x_1 + x_2 + x_3 = 800, \\
&x_1 \le 400, \\
&x_2 \le 450, \\
&x_3 \le 450, \\
&x_i \ge 0, \quad = 1,2,3.
\end{aligned}$$

Table 1: Suppliers quantitative information

	Z1 Cost	Z2Quality (%)	Z3 Service (%)	Capacity
Supplier 1	5	0.80	0.90	400
Supplier 2	7	0.90	0.80	450
Supplier 3	4	0.85	0.85	450

The truth membership, Indeterminacy membership, falsity membership functions were considered to be neutrosophic triangular. When they depend on three scalar parameters (a1,m,a2). z_1 depends on neutrosophic aspiration levels (3550,4225,4900), when z_2 depends on neutrosophic aspiration levels (660,681.5,702.5), and z3 depends on neutrosophic aspiration levels (657.5,678.75,700).

The truth membership functions of the goals are obtained as follows:

$$\mu_1^I(z_1) = \begin{cases} 0, & if \quad z_1 \le 3550, \\ \dfrac{4225 - z_1}{4225 - 3550}, & if \quad 3550 \le z_1 \le 4225, \\ \dfrac{4900 - z_1}{4900 - 4225}, & if \quad 4225 \le z_1 \le 4900, \\ 0, & if \quad z_1 \ge 4900 \end{cases}$$

$$\mu_2^I(z_2) = \begin{cases} 0, & if \quad z_2 \ge 702.5, \\ \dfrac{z_2 - 681.5}{702.5 - 681.5}, & if \quad 681.5 \le z_2 \le 702.5, \\ \dfrac{z_2 - 660}{681.5 - 660}, & if \quad 660 \le z_2 \le 681.5, \\ 0, & if \quad z_2 \le 660. \end{cases}$$

$$\mu_3^I(z_3)=\begin{cases}0, & if \quad z_3\ge 700,\\ \dfrac{z_3-678.75}{700-678.75}, & if \quad 678.75\le z_3\le 700,\\ \dfrac{z_3-657.5}{678.75-657.5}, & if \quad 657.5\le z_3\le 678.75,\\ 0, & if \quad z_3\le 657.5.\end{cases}$$

If

$$\mu_1^I(z_1)=\max\left(\min\left(\frac{4225-(5x_1+7x_2+4x_3)}{675},\frac{4900-(5x_1+7x_2+4x_3)}{675},0\right)\right)$$

$$\mu_2^I(z_2)=\min(\max(\frac{(0.8x_1+0.9x_2+0.85x_3)-681.5}{21},\frac{(0.8x_1+0.9x_2+0.85x_3)-660}{21},1))$$

$$\mu_3^I(z_3)=\min(\max(\frac{(0.9x_1+0.8x_2+0.85x_3)-678.75}{21.25},\frac{(0.9x_1+0.8x_2+0.85x_3)-657.5}{21.25},1))$$

Then

$\mu_1^{I*}(350,0,450)$, $\mu_2^{I*}(0,450,350)$, $\mu_3^{I*}(400,0,400)$

The truth membership functions are transformed by using first-order Taylor polynomial series

$$\widehat{\mu}_1^I(x)=\mu_1^I(350,0,450)+\left[(x_1-350)\frac{\partial\mu_1^I(350,0,450)}{\partial x_1}\right]+\left[(x_2-0)\frac{\partial\mu_1^I(350,0,450)}{\partial x_2}\right]+\left[(x_3-450)\frac{\partial\mu_1^I(350,0,450)}{\partial x_3}\right]$$

$$\widehat{\mu}_1^I(x)\cong -0.00741x_1-0.0104x_2-0.00593x_3+5.2611$$

In the similar way, we get

$$\widehat{\mu}_2^I(x)\cong 0.0381x_1+0.0429x_2+0.0405x_3-33.405$$

$$\widehat{\mu}_3^I(x)\cong 0.042x_1+0.037x_2+0.0395x_3-32.512$$

The the p(x) is

$$p(x)=\widehat{\mu}_1^I(x)+\widehat{\mu}_2^I(x)+\widehat{\mu}_3^I(x)$$

$$p(x)\cong 0.07259x_1+0.0695x_2+0.0741x_3-60.6559$$

$s.t.$:

$x_1+x_2+x_3=800,$

$x_1\le 400,$

$x_2\le 450,$

$x_3\le 450,$

$x_i\ge 0,\ i=1,2,3.$

The linear programming software LINGO 15.0 is used to solve this problem. The problem is solved and the optimal solution for the truth membership model is obtained is as follows: (x_1, x_2, x_3) = (350,0,450) z_1=3550, z_2=662. 5, z_3=697. 5.

The truth membership values are $\mu_1=1,\ \mu_2=0.1163,\ \mu_3=0.894.$ The truth membership function values show that both goals z_1 , $z_3$and z_2 are satisfied with 100% , 11.63% and 89.4% respectively for the obtained solution which is $x1$=350; $x2$= 0, x3=450.

In the similar way, we get $\sigma_i^I(X)$, *q(x)* Consequently we get the optimal solution for the Indeterminacy membership model is obtained is as follows: (x_1,x_2,x_3)=(350,0,450) z_1=3550, z_2=662.5, z_3=697.5

and the Indeterminacy membership values are $\mu_1=1,\ \mu_2=0.1163,\ \mu_3=0.894.$ The Indeterminacy membership function values show that both goals z_1 , $z_3$and z_2 are satisfied with 100% , 11.63% and 89.4% respectively for the obtained solution which is $x1$=350; $x2$= 0, x3=450.

In the similar way, we get $\upsilon_i^I(X)$ *and h(x)* Consequently we get the optimal solution for the falsity membership model is obtained is as follows: (x_1,x_2,x_3)=(350,0,450) z_1=3550, z_2=662.5, z_3=697.5

and the falsity membership values are $\mu_1=0,\ \mu_2=0.8837,\ \mu_3=0.106.$ The falsity membership function values show that both goals z_1 , $z_3$and z_2 are satisfied with 0% , 88.37% and 10.6% respectively for the obtained solution which is $x1$=350; $x2$= 0, x3=450.

5 Conclusions and Future Work

In this paper, we have proposed a solution to Neutrosophic Multiobjective programming problem (NMOPP). The truth membership, Indeterminacy membership, falsity membership functions associated with each objective of the problem are transformed by using the first order Taylor polynomial series. The neutrosophic multi-objective programming problem is reduced to an equivalent multiobjective programming problem by the proposed method. The solution obtained from this method is very near to the solution of MOPP. Hence this method gives a more accurate solution as compared with other methods. Therefore the complexity in solving NMOPP, has reduced to easy computation. In the future studies, the proposed algorithm can be solved by metaheuristic algorithms.

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Neutrosophic Quadruple Numbers, Refined Neutrosophic Quadruple Numbers, Absorbance Law, and the Multiplication of Neutrosophic Quadruple Numbers

Florentin Smarandache

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Abstract. In this paper, we introduce for the first time the neutrosophic quadruple numbers (of the form $\boldsymbol{a+bT+cI+dF}$) and the refined neutrosophic quadruple numbers.
Then we define an absorbance law, based on a prevalence order, both of them in order to multiply the neutrosophic components $\boldsymbol{T, I, F}$ or their sub-components $\boldsymbol{T_j, I_k, F_l}$ and thus to construct the multiplication of neutrosophic quadruple numbers.

Keywords: neutrosophic quadruple numbers, refined neutrosophic quadruple numbers, absorbance law, multiplication of neutrosophic quadruple numbers, multiplication of refined neutrosophic quadruple numbers.

1 Neutrosophic Quadruple Numbers

Let's consider an entity (i.e. a number, an idea, an object, etc.) which is represented by a known part (a) and an unknown part $(bT + cI + dF)$.

Numbers of the form:

$$NQ = a + bT + cI + dF, \quad (1)$$

where a, b, c, d are real (or complex) numbers (or intervals or in general subsets), and

T = truth / membership / probability,

I = indeterminacy,

F = false / membership / improbability,

are called Neutrosophic Quadruple (Real respectively Complex) Numbers (or Intervals, or in general Subsets).

"a" is called the known part of NQ, while "$bT + cI + dF$" is called the unknown part of NQ.

2 Operations

Let

$$NQ_1 = a_1 + b_1T + c_1I + d_1F, \quad (2)$$

$$NQ_2 = a_2 + b_2T + c_2I + d_2F \quad (3)$$

and $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{C}$) a real (or complex) scalar.

Then:

2.1 Addition

$$NQ_1 + NQ_2 = (a_1 + a_2) + (b_1 + b_2)T + (c_1 + c_2)I + (d_1 + d_2)F. \quad (4)$$

2.2 Substraction

$$NQ_1 - NQ_2 = (a_1 - a_2) + (b_1 - b_2)T + (c_1 - c_2)I + (d_1 - d_2)F. \quad (5)$$

2.3 Scalar Multiplication

$$\alpha \cdot NQ = NQ \cdot \alpha = \alpha a + \alpha bT + \alpha cI + \alpha dF. \quad (6)$$

One has:

$$0 \cdot T = 0 \cdot I = 0 \cdot F = 0, \quad (7)$$

and

$$mT + nT = (m + n)T, \quad (8)$$

$$mI + nI = (m + n)I, \quad (9)$$

$$mF + nF = (m + n)F. \quad (10)$$

3 Refined Neutrosophic Quadruple Numbers

Let us consider that Refined Neutrosophic Quadruple Numbers are numbers of the form:

$$RNQ = a + \sum_{i=1}^{p} b_i\, T_i + \sum_{j=1}^{r} c_j\, I_j + \sum_{k=1}^{s} d_k\, F_k, \quad (11)$$

where a, all b_i, all c_j, and all d_k are real (or complex) numbers, intervals, or, in general, subsets,

while $T_1, T_2, \dots, T_p$ are refinements of T;

$I_1, I_2, \dots, I_r$ are refinements of I;

and $F_1, F_2, \dots, F_s$ are refinements of F.

There are cases when the known part *(a)* can be refined as well as $a_1, a_2, \dots$.

The operations are defined similarly.

Let

$$RNQ^{(u)} = a^{(u)} + \sum_{i=1}^{p} b_i^{(u)} T_i + \sum_{j=1}^{r} c_j^{(u)} I_j + \sum_{k=1}^{s} d_k^{(u)} F_k, \quad (12)$$

for $u = 1$ or 2.
Then:

3.1 Addition

$$RNQ^{(1)} + RNQ^{(2)} = \left[a^{(1)} + a^{(2)}\right] + \sum_{i=1}^{p} \left[b_i^{(1)} + b_i^{(2)}\right] T_i + \sum_{j=1}^{r} \left[c_j^{(1)} + c_j^{(2)}\right] I_j + \sum_{k=1}^{s} \left[d_k^{(1)} + d_k^{(2)}\right] F_k. \quad (13)$$

3.2 Substraction

$$RNQ^{(1)} - RNQ^{(2)} = \left[a^{(1)} - a^{(2)}\right] + \sum_{i=1}^{p} \left[b_i^{(1)} - b_i^{(2)}\right] T_i + \sum_{j=1}^{r} \left[c_j^{(1)} - c_j^{(2)}\right] I_j + \sum_{k=1}^{s} \left[d_k^{(1)} - d_k^{(2)}\right] F_k. \quad (14)$$

3.3 Scalar Multiplication

For $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{C}$) one has:

$$\alpha \cdot RNQ^{(1)} = \alpha \cdot a^{(1)} + \alpha \cdot \sum_{i=1}^{p} b_i^{(1)} T_i + \alpha \cdot \sum_{j=1}^{r} c_j^{(1)} I_j + \alpha \cdot \sum_{k=1}^{s} d_k^{(1)} F_k. \quad (15)$$

4 Absorbance Law

Let S be a set, endowed with a total order $x \prec y$, named "x prevailed by y" or "x less stronger than y" or "x less preferred than y". We consider $x \preccurlyeq y$ as "x prevailed by or equal to y" "x less stronger than or equal to y", or "x less preferred than or equal to y".

For any elements $x, y \in S$, with $x \preccurlyeq y$, one has the absorbance law:

$$\boxed{x \cdot y = y \cdot x = \text{absorb } (x, y) = \max\{x, y\} = y,} \quad (16)$$

which means that the bigger element absorbs the smaller element (the big fish eats the small fish!).

Clearly,

$$x \cdot x = x^2 = \text{absorb } (x, x) = \max\{x, x\} = x, \quad (17)$$

and

$$x_1 \cdot x_2 \cdot \ldots \cdot x_n = \text{absorb}(\ldots \text{absorb}(\text{absorb}(x_1, x_2), x_3) \ldots, x_n) = \max\{\ldots \max\{\max\{x_1, x_2\}, x_3\} \ldots, x_n\} = \max\{x_1, x_2, \ldots, x_n\}. \quad (18)$$

Analougously, we say that "$x \succ y$" and we read: "x prevails to y" or "x is stronger than y" or "x is preferred to y".

Also, $x \succcurlyeq y$, and we read: "x prevails or is equal to y" "x is stronger than or equal to y", or "x is preferred or equal to y".

5 Multiplication of Neutrosophic Quadruple Numbers

It depends on the prevalence order defined on $\{T, I, F\}$.

Suppose in an optimistic way the neutrosophic expert considers the prevalence order $T \succ I \succ F$. Then:

$$NQ_1 \cdot NQ_2 = (a_1 + b_1T + c_1I + d_1F) \cdot (a_2 + b_2T + c_2I + d_2F) = a_1a_2 + (a_1b_2 + a_2b_1 + b_1b_2 + b_1c_2 + c_1b_2 + b_1d_2 + d_1b_2)T + (a_1c_2 + a_2c_1 + c_1d_2 + c_2d_1)I + (a_1d_2 + a_2d_1 + d_1d_2)F, \quad (19)$$

since $TI = IT = T, TF = FT = T, IF = FI = I$,
while $T^2 = T, I^2 = I, F^2 = F$.

Suppose in an pessimistic way the neutrosophic expert considers the prevalence order $F \succ I \succ T$. Then:

$$NQ_1 \cdot NQ_2 = (a_1 + b_1T + c_1I + d_1F) \cdot (a_2 + b_2T + c_2I + d_2F) = a_1a_2 + (a_1b_2 + a_2b_1 + b_1b_2)T + (a_1c_2 + a_2c_1 + b_1c_2 + b_2c_1 + c_1c_2)I + (a_1d_2 + a_2d_1 + b_1d_2 + b_2d_1 + c_1d_2 + c_2d_1 + d_1d_2)F, \quad (20)$$

since

$$F \cdot I = I \cdot F = F, F \cdot T = T \cdot F = F, I \cdot T = T \cdot I = I$$

while similarly $F^2 = F, I^2 = I, T^2 = T$.

5.1 Remark

Other prevalence orders on $\{T, I, F\}$ can be proposed, depending on the application/problem to solve, and on other conditions.

6 Multiplication of Refined Neutrosophic Quadruple Numbers

Besides a neutrosophic prevalence order defined on $\{T, I, F\}$, we also need a sub-prevalence order on $\{T_1, T_2, \dots, T_p\}$, a sub-prevalence order on $\{I_1, I_2, \dots, I_r\}$, and another sub-prevalence order on $\{F_1, F_2, \dots, F_s\}$.

We assume that, for example, if $T \succ I \succ F$, then $T_j \succ I_k \succ F_l$ for any $j \in \{1, 2, \dots, p\}$, $k \in \{1, 2, \dots, r\}$, and $l \in \{1, 2, \dots, s\}$. Therefore, any prevalence order on $\{T, I, F\}$ imposes a prevalence suborder on their corresponding refined components.

Without loss of generality, we may assume that

$$T_1 \succ T_2 \succ \cdots \succ T_p$$

(if this was not the case, we re-number the subcomponents in a decreasing order).

Similarly, we assume without loss of generality that:

$I_1 \succ I_2 \succ \cdots \succ I_r$, and

$F_1 \succ F_2 \succ \cdots \succ F_s$.

6.1 Exercise for the Reader

Let's have the neutrosophic refined space

$$NS = \{T_1, T_2, T_3, I, F_1, F_2\},$$

with the prevalence order $T_1 \succ T_2 \succ T_3 \succ I \succ F_1 \succ F_2$.

Let's consider the refined neutrosophic quadruples

$NA = 2 - 3T_1 + 2T_2 + T_3 - I + 5F_1 - 3F_2$, and

$NB = 0 + T_1 - T_2 + 0 \cdot T_3 + 5I - 8F_1 + 5F_2$.

By multiplication of sub-components, the bigger absorbs the smaller. For example:

$T_2 \cdot T_3 = T_2$,

$T_1 \cdot F_1 = T_1$,

$I \cdot F_2 = I$,

$T_2 \cdot F_1 = T_2$, etc.

Multiply NA with NB.

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Isolated Single Valued Neutrosophic Graphs

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Abstract: Many results have been obtained on isolated graphs and complete graphs. In this paper, a necessary and sufficient condition will be proved for a single valued neutrosophic graph to be an isolated single valued neutrosophic graph.

Keywords: Single valued neutrosophic graphs, complete single valued neutrosophic graphs, isolated single valued neutrosophic graphs.

1. Introduction

The notion of neutrosophic sets (NSs) was proposed by Smarandache [8] as a generalization of the fuzzy sets [14], intuitionistic fuzzy sets [12], interval valued fuzzy set [11] and interval-valued intuitionistic fuzzy sets [13] theories. The neutrosophic set is a powerful mathematical tool for dealing with incomplete, indeterminate and inconsistent information in real world. The neutrosophic sets are characterized by a truth-membership function (*t*), an indeterminacy-membership function (*i*) and a falsity-membership function (*f*) independently, which are within the real standard or nonstandard unit interval]−0, 1+[. In order to conveniently use NS in real life applications, Wang et al. [9] introduced the concept of the single-valued neutrosophic set (SVNS), a subclass of the neutrosophic sets. The same authors [10] introduced the concept of the interval valued neutrosophic set (IVNS), which is more precise and flexible than the single valued neutrosophic set. The IVNS is a generalization of the single valued neutrosophic set, in which the three membership functions are independent and their value belong to the unit interval [0, 1]. More works on single valued neutrosophic sets, interval valued neutrosophic sets and their applications can be found on *http://fs.gallup.unm.edu/NSS/* [38].

Graph theory has now become a major branch of applied mathematics and it is generally regarded as a branch of combinatorics. Graph is a widely used tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, optimization and computer science.

If one has uncertainty regarding either the set of vertices or edges, or both, the model becomes a fuzzy graph. The extension of fuzzy graph [2, 4, 25] theory have been developed by several researchers, e.g. vague graphs [27], considering the vertex sets and edge sets as vague sets; intuitionistic fuzzy graphs [3, 15, 26], considering the vertex sets and edge sets as intuitionistic fuzzy sets; interval valued fuzzy graphs [16, 17, 23, 24], considering the vertex sets and edge sets as interval valued fuzzy sets; interval valued intuitionistic fuzzy graphs [35], considering the vertex sets and edge sets as interval valued intuitionistic fuzzy sets; bipolar fuzzy graphs [18, 19, 21, 22], considering the vertex sets and edge sets as bipolar fuzzy sets; m-polar fuzzy graphs [20], considering the vertex sets and edge sets as m-polar fuzzy sets.

But, if the relations between nodes (or vertices) in problems are indeterminate, the fuzzy graphs and their extensions fail. For this purpose, Smarandache [5, 6, 7, 37] defined four main categories of neutrosophic graphs; two are based on literal indeterminacy (*I*), called: *I*-edge neutrosophic graph and *I*-vertex neutrosophic graph, deeply studied and gaining popularity among the researchers due to their applications via real world problems [1, 38]; the two others are based on (*t, i, f*) components, called: (*t, i, f*)-edge neutrosophic graph and (*t, i, f*)-vertex neutrosophic graph, concepts not developed at all by now.

Later on, Broumi et al. [29] introduced a third neutrosophic graph model, which allows the attachment of truth-membership (*t*), indeterminacy-membership (*i*) and falsity-membership degrees (*f*) both to vertices and edges, and investigated some of their properties. The third neutrosophic graph model is called the single valued neutrosophic graph (SVNG for short). The single valued neutrosophic graph is a generalization of fuzzy graph and intuitionistic fuzzy graph. Also, the same authors [28] introduced neighborhood degree of a vertex and closed neighborhood degree of a vertex in single valued neutrosophic graph as a generalization of neighborhood degree of a vertex and closed neighborhood degree of a vertex in fuzzy graph and intuitionistic fuzzy graph. Recently, Broumi et al. [31, 33, 34] introduced the concept of interval valued neutrosophic graph as a generalization of fuzzy graph, intuitionistic fuzzy graph and single valued neutrosophic graph and discussed some of their properties with proof and examples.

The aim of this paper is to prove a necessary and sufficient condition for a single valued neutrosophic graph to be a single valued neutrosophic graph.

2. Preliminaries

In this section, we mainly recall some notions related to neutrosophic sets, single valued neutrosophic sets, single valued neutrosophic graphs, relevant to the present article. See [8, 9] for further details and background.

Definition 2.1 [8]

Let X be a space of points (objects) with generic elements in X denoted by x; then, the neutrosophic set A (NS A) is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ X}, where the functions T, I, F: X → $]^-0,1^+[$ define respectively a truth-membership function, an indeter-minacy-membership function and a falsity-membership function of the element x ∈ X to the set A with the condition:

$$^-0 \leq T_A(x)+ I_A(x)+ F_A(x)\leq 3^+. \quad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]^-0,1^+[$.

Since it is difficult to apply NSs to practical problems, Wang et al. [9] introduced the concept of SVNS, which is an instance of a NS, and can be used in real scientific and engineering applications.

Definition 2.2 [9]

Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X $T_A(x)$, $I_A(x)$, $F_A(x)$ ∈ [0, 1]. A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\}. \quad (2)$$

Definition 2.3 [29]

A single valued neutrosophic graph (SVN-graph) with underlying set V is defined to be a pair G= (A, B), where:

1. The functions T_A:V→[0, 1], I_A:V→[0, 1] and F_A:V→[0, 1] denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element v_i ∈ V, respectively, and:

$$0\leq T_A(v_i) + I_A(v_i) +F_A(v_i) \leq 3,$$

for all v_i ∈ V.

2. The functions T_B: E ⊆ V x V →[0, 1], I_B:E ⊆ V x V →[0, 1] and F_B: E ⊆ V x V →[0, 1] are defined by $T_B(v_i, v_j) \leq \min [T_A(v_i), T_A(v_j)]$, $I_B(v_i, v_j) \geq \max [I_A(v_i), I_A(v_j)]$ and $F_B(v_i, v_j) \geq \max [F_A(v_i), F_A(v_j)]$, denoting the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge (v_i, v_j) ∈ E respectively, where:

$$0\leq T_B(v_i, v_j) + I_B(v_i, v_j)+ F_B(v_i, v_j) \leq 3,$$

for all (v_i, v_j) ∈ E (i, j = 1, 2, …, n)

We call *A* the single valued neutrosophic vertex set of *V*, and *B* the single valued neutrosophic edge set of *E*, respectively.

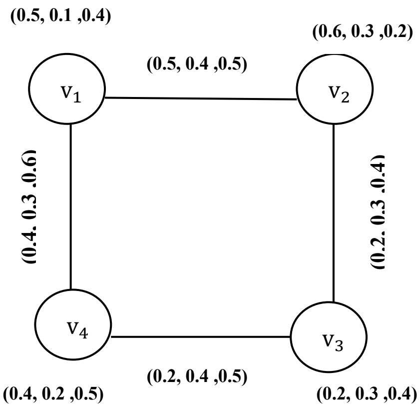

Figure 1: Single valued neutrosophic graph.

Definition 2.4 [29]

A partial SVN-subgraph of SVN-graph G= (A, B) is a SVN-graph H = (V′, E′), such that:

- V′ ⊆ V,

where $T'_A(v_i) \leq T_A(v_i)$, $I'_A(v_i) \geq I_A(v_i)$, $F'_A(v_i) \geq F_A(v_i)$, for all v_i ∈ V;

- E′ ⊆ E,

where $T'_B(v_i, v_j) \leq T_B(v_i, v_j)$, $I'_{Bij} \geq I_B(v_i, v_j)$, $F'_B(v_i, v_j) \geq F_B(v_i, v_j)$, for all $(v_i\ v_j)$ ∈ E.

Definition 2.8 [29]

A single valued neutrosophic graph G = (A, B) of G* = (V, E) is called complete single valued neutrosophic graph, if:

$$T_B(v_i, v_j) = \min [T_A(v_i), T_A(v_j)],$$
$$I_B(v_i, v_j) = \max [I_A(v_i), I_A(v_j)],$$
$$F_B(v_i, v_j) = \max [F_A(v_i), F_A(v_j)],$$

for all $v_i, v_j \in V$.

Definition 2.9 [29]

The complement of a single valued neutrosophic graph G (A, B) on G^* is a single valued neutrosophic graph $\overline{G}$ on G^*, where:

1. $\overline{A} = A = (T_A, I_A, F_A)$;

2. $\overline{T_A}(v_i) = T_A(v_i)$, $\overline{I_A}(v_i) = I_A(v_i)$, $\overline{F_A}(v_i) = F_A(v_i)$,

for all $v_j \in V$.

3. $\overline{T_B}(v_i, v_j) = \min\left[T_A(v_i), T_A(v_j)\right] - T_B(v_i, v_j)$,

$\overline{I_B}(v_i, v_j) = \max\left[I_A(v_i), I_A(v_j)\right] - I_B(v_i, v_j)$

and

$\overline{F_B}(v_i, v_j) = \max\left[F_A(v_i), F_A(v_j)\right] - F_B(v_i, v_j)$,

for all $(v_i, v_j) \in E$.

3. Main Result

Theorem 3.1

A single valued neutrosophic graph G = (A, B) is an isolated single valued graph if and only if its complement is a complete single valued neutrosophic graph.

Proof

Let G : (A, B) be a single valued neutrosophic graph, $\bar{G} = (A, \bar{B})$ be its complement, and G : (A, B) be an isolated single valued neutrosophic graph.

Then,

$T_B(u, v) = 0$,

$I_B(u, v) = 0$

and

$F_B(u, v) = 0$,

for all $(u, v) \in V \times V$.

Since

$\overline{T_B}(u, v) = \min(T_A(u), T_A(v)) - T_B(u, v)$,

for all $(u, v) \in V \times V$,

$\overline{T_B}(u, v) = \min(T_A(u), T_A(v))$

and

$\overline{I_B}(u, v) = \max(I_A(u), I_A(v)) - I_B(u, v)$,

for all $(u, v) \in V \times V$,

$\overline{I_B}(u, v) = \max(I_A(u), I_A(v))$

and

$\overline{F_B}(u, v) = \max(F_A(u), F_A(v)) - F_B(u, v)$,

for all $(u, v) \in V \times V$,

$\overline{F_B}(u, v) = \max(F_A(u), F_A(v))$,

hence $\bar{G} = (A, \bar{B})$ is a complete single valued neutrosophic graph.

Conversely, let $\bar{G} = (A, \bar{B})$ be a complete single valued neutrosophic graph

$\overline{T_B}(u, v) = \min(T_A(u), T_A(v))$,

for all $(u, v) \in V \times V$.

Since

$\overline{T_B}(u, v) = \min(T_A(u), T_A(v)) - \overline{T_B}(u, v)$,

for all $(u, v) \in V \times V$,

$= \overline{T_B}(u, v) - \overline{T_B}(u, v)$,

for all $(u, v) \in V \times V$,

$= 0$,

for all $(u, v) \in V \times V$,

$T_B(u, v) = 0$,

for all $(u, v) \in V \times V$.

$\overline{I_B}(u, v) = \max(I_A(u), I_A(v))$,

for all $(u, v) \in V \times V$.

Since

$\overline{I_B}(u, v) = \max(I_A(u), I_A(v)) - \overline{I_B}(u, v)$,

for all $(u, v) \in V \times V$

$= \overline{I_B}(u, v) - \overline{I_B}(u, v)$,

for all $(u, v) \in V \times V$

$= 0$,

for all $(u, v) \in V \times V$,

$I_B(u, v) = 0$,

for all $(u, v) \in V \times V$.

Also,

$\overline{F_B}(u, v) = \max(F_A(u), F_A(v))$,

for all $(u, v) \in V \times V$.

Since

$\overline{F_B}(u, v) = \max(F_A(u), F_A(v)) - \overline{F_B}(u, v)$,

for all $(u, v) \in V \times V$,

$= \overline{F_B}(u, v) - \overline{F_B}(u, v)$,

for all $(u, v) \in V \times V$

$= 0$,

for all $(u, v) \in V \times V$

$F_B(\mathrm{u}, \mathrm{v}) = 0$ for all (u, v) ∈ V× V,

hence G = (A, B) is an isolated single valued neutrosophic graph.

4. Conclusion

Many problems of practical interest can be represented by graphs. In general, graph theory has a wide range of applications in various fields. In this paper, we defined for the first time the notion of an isolated single valued neutrosophic graph. In future works, we plan to study the concept of an isolated interval valued neutrosophic graph.

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Degree of Dependence and Independence of the (Sub)Components of Fuzzy Set and Neutrosophic Set

Florentin Smarandache

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Abstract. We have introduced for the first time the *degree of dependence* (and consequently the *degree of independence*) between the components of the fuzzy set, and also between the components of the neutrosophic set in our 2006 book's fifth edition [1]. Now we extend it for the first time to the refined neutrosophic set considering the *degree of dependence or independence of subcomponets*.

Keywords: neutrosophy, neutrosophic set, fuzzy set, degree of dependence of (sub)components, degree of independence of (sub)components.

1 Refined Neutrosophic Set.

We start with the most general definition, that of a *n-valued refined neutrosophic set A*. An element x from A belongs to the set in the following way:

$$x(T_1, T_2, \dots, T_p;\ I_1, I_2, \dots, I_r;\ F_1, F_2, \dots, F_s) \in A, \quad (1)$$

where $p, r, s \geq 1$ are integers, and $p + r + s = n \geq 3$, where

$$T_1, T_2, \dots, T_p;\ I_1, I_2, \dots, I_r;\ F_1, F_2, \dots, F_s \quad (2)$$

are respectively sub-membership degrees, sub-indeterminacy degrees, and sub-nonmembership degrees of element x with respect to the n-valued refined neutrosophic set A.

Therefore, one has n (sub)components.

Let's consider all of them being crisp numbers in the interval $[0, 1]$.

2 General case.

Now, in general, let's consider n crisp-components (variables):

$$y_1, y_2, \dots, y_n \in [0, 1]. \quad (3)$$

If all of them are 100% independent two by two, then their sum:

$$0 \leq y_1 + y_2 + \dots + y_n \leq n. \quad (4)$$

But if all of them are 100% dependent (totally interconnected), then

$$0 \leq y_1 + y_2 + \dots + y_n \leq 1. \quad (5)$$

When some of them are partially dependent and partially independent, then

$$y_1 + y_2 + \dots + y_n \in (1, n). \quad (6)$$

For example, if y_1 and y_2 are 100% dependent, then

$$0 \leq y_1 + y_2 \leq 1, \quad (7)$$

while other variables $y_3, \dots, y_n$ are 100% independent of each other and also with respect to y_1 and y_2, then

$$0 \leq y\_3 + \dots + y\_n \leq n - 2, \quad (8)$$

thus

$$0 \leq y_1 + y_2 + y_3 + \dots + y_n \leq n - 1. \quad (9)$$

3 Fuzzy Set.

Let T and F be the membership and respectively the nonmembership of an element $x(T, F)$ with respect to a fuzzy set A, where T, F are crisp numbers in $[0, 1]$.

If T and F are 100% dependent of each other, then one has as in classical fuzzy set theory

$$0 \leq T + F \leq 1. \quad (10)$$

But if T and F are 100% independent of each other (that we define now for the first time in the domain of fuzzy setand logic), then

$$0 \leq T + F \leq 2. \quad (11)$$

We consider that the sum $T + F = 1$ if the information about the components is complete, and $T + F < 1$ if the information about the components is incomplete.

Similarly, $T + F = 2$ for complete information, and $T + F < 2$ for incomplete information.

For complete information on T and F, one has $T + F \in [1, 2]$.

4 Degree of Dependence and Degree of Independence for two Components.

In general (see [1], 2006, pp. 91-92), the sum of two components x and y that vary in the unitary interval [0, 1] is:

$$0 \leq x + y \leq 2 - d^{\circ}(x, y), \qquad (12)$$

where $d^{\circ}(x, y)$ is the *degree of dependence* between x and y.

Therefore $2 - d^{\circ}(x, y)$ is the *degree of independence* between x and y.

Of course, $d^{\circ}(x, y) \in [0, 1]$, and it is zero when x and y are 100% independent, and 1 when x and y are 100% dependent.

In general, if T and F are d% dependent [and consequently $(100 - d)$% independent], then

$$0 \leq T + F \leq 2 - d/100. \qquad (13)$$

5 Example of Fuzzy Set with Partially Dependent and Partially Independent Components.

As an example, if T and F are 75% (= 0.75) dependent, then

$$0 \leq T + F \leq 2 - 0.75 = 1.25. \qquad (14)$$

6 Neutrosophic Set

Neutrosophic set is a general framework for unification of many existing sets, such as fuzzy set (especially intuitionistic fuzzy set), paraconsistent set, intuitionistic set, etc. The main idea of NS is to characterize each value statement in a 3D-Neutrosophic Space, where each dimension of the space represents respectively the membership/truth (T), the nonmembership/falsehood (F), and the indeterminacy with respect to membership/nonmembership (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of $]^-0, 1^+[$ with not necessarily any connection between them.

For software engineering proposals the classical unit interval [0, 1] is used.

For single valued neutrosophic set, the sum of the components (T+I+F) is (see [1], p. 91):

$$0 \leq T+I+F \leq 3, \qquad (15)$$

when all three components are independent;

$$0 \leq T+I+F \leq 2, \qquad (16)$$

when two components are dependent, while the third one is independent from them;

$$0 \leq T+I+F \leq 1, \qquad (17)$$

when all three components are dependent.

When three or two of the components T, I, F are independent, one leaves room for incomplete information (sum < 1), paraconsistent and contradictory information (sum > 1), or complete information (sum = 1).

If all three components T, I, F are dependent, then similarly one leaves room for incomplete information (sum < 1), or complete information (sum = 1).

The dependent components are tied together.

Three sources that provide information on T, I, and F respectively are independent if they do not communicate with each other and do not influence each other.

Therefore, max{T+I+F} is in between 1 (when the degree of independence is zero) and 3 (when the degree of independence is 1).

7 Examples of Neutrosophic Set with Partially Dependent and Partially Independent Components.

The max{T+I+F} may also get any value in (1, 3).

a) For example, suppose that T and F are 30% dependent and 70% independent (hence T + F ≤ 2-0.3 = 1.7), while I and F are 60% dependent and 40% independent (hence I + F ≤ 2-0.6 = 1.4). Then max{T + I + F} = 2.4 and occurs for T = 1, I = 0.7, F = 0.7.

b) Second example: suppose T and I are 100% dependent, but I and F are 100% independent. Therefore T + I ≤ 1 and I + F ≤ 2, then T + I + F ≤ 2.

8 More on Refined Neutrosophic Set

The Refined Neutrosophic Set [4], introduced for the first time in 2013. In this set the neutrosophic component (T) is split into the subcomponents (T_1, T_2, …, T_p) which represent types of truths (or sub-truths), the neutrosophic component (I) is split into the subcomponents (I_1, I_2, …, I_r) which represents types of indeterminacies (or sub-indeterminacies), and the neutrosophic components (F) is split into the subcomponents (F_1, F_2, …, F_s) which represent types of falsehoods (or sub-falsehoods), such that p, r, s are integers ≥ 1 and p + r + s = n ≥ 4. (18)

When n = 3, one gets the non-refined neutrosophic set. All T_j, I_k, and F_l subcomponents are subsets of [0, 1].

Let's consider the case of refined single-valued neutrosophic set, i.e. when all n subcomponents are crisp numbers in [0, 1].

Let the sum of all subcomponents be:

$$S = \sum_{1}^{p} T_j + \sum_{1}^{r} I_k + \sum_{1}^{s} F_l \qquad (19)$$

When all subcomponents are independent two by two, then

$$0 \leq S \leq n. \qquad (20)$$

If *m* subcomponents are 100% dependent, $2 \leq m \leq n$, no matter if they are among T_j, I_k, F_l or mixed, then

$$0 \leq S \leq n - m + 1 \quad (21)$$

and one has $S = n - m + 1$ when the information is complete, while $S < n - m + 1$ when the information is incomplete.

9 Examples of Refined Neutrosophic Set with Partially Dependent and Partially Independent Components.

Suppose T is split into T_1, T_2, T_3, and I is not split, while F is split into F_1, F_2. Hence one has:

$$\{T_1, T_2, T_3; I; F_1, F_2\}. \quad (22)$$

Therefore a total of 6 (sub)components.

a) If all 6 components are 100% independent two by two, then:

$$0 \leq T_1 + T_2 + T_3 + I + F_1 + F_2 \leq 6 \quad (23)$$

b) Suppose the subcomponets T_1, T_2, and F_1 are 100% dependent all together, while the others are totally independent two by two and independent from T_1, T_2, F_1, therefore:

$$0 \leq T_1 + T_2 + F_1 \leq 1 \quad (24)$$

whence

$$0 \leq T_1 + T_2 + T_3 + I + F_1 + F_2 \leq 6 - 3 + 1 = 4. \quad (25)$$

One gets equality to 4 when the information is complete, or strictly less than 4 when the information is incomplete.

c) Suppose in another case that T_1 and I are 20% dependent, or $d°(T_1, I) = 20\%$, while the others similarly totally independent two by two and independent from T_1 and I, hence

$$0 \leq T_1 + I \leq 2 - 0.2 = 1.8 \quad (26)$$

whence

$$0 \leq T_1 + T_2 + T_3 + I + F_1 + F_2 \leq 1.8 + 4 = 5.8, \quad (27)$$

since $0 \leq T_2 + T_3 + F_1 + F_2 \leq 4$. (28)

Similarly, to the right one has equality for complete information, and strict inequality for incomplete information.

Conclusion.

We have introduced for the first time the degree of dependence/independence between the components of fuzzy set and neutrosophic set. We have given easy examples about the range of the sum of components, and how to represent the degrees of dependence and independence of the components. Then we extended it to the refined neutrosophic set considering the degree of dependence or independence of subcomponets.

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Degrees of Membership > 1 and < 0 of the Elements with Respect to a Neutrosophic OffSet

Florentin Smarandache

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Abstract. We have defined the Neutrosophic Over-/Under-/Off-Set and -Logic for the first time in 1995 and published in 2007. During 1995-2016 we presented them to various national and international conferences and seminars ([16]-[37]) and did more publishing during 2007-2016 ([1]-[15]). These new notions are totally different from other sets/logics/probabilities.
We extended the neutrosophic set respectively to Neutrosophic Overset {when some neutrosophic component is > 1}, to Neutrosophic Underset {when some neutrosophic component is < 0}, and to Neutrosophic Offset {when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and other neutrosophic component < 0}. This is no surprise since our real-world has numerous examples and applications of over-/under-/off-neutrosophic components.

Keywords: Neutrosophic overset, neutrosophic underset, neutrosophic offset, neutrosophic overlogic, neutrosophic underlogic, neutrosophic offlogic, neutrosophic overprobability, neutrosophic underprobability, neutrosophic offprobability, overmembership (membership degree > 1), undermembership (membership degree < 0), offmembership (membership degree off the interval [0, 1]).

1. Introduction

In the classical set and logic theories, in the fuzzy set and logic, and in intuitionistic fuzzy set and logic, the degree of membership and degree of nonmembership have to belong to, or be included in, the interval [0, 1]. Similarly, in the classical probability and in imprecise probability the probability of an event has to belong to, or respectively be included in, the interval [0, 1].

Yet, we have observed and presented to many conferences and seminars around the globe {see [16]-[37]} and published {see [1]-[15]} that in our real world there are many cases when the degree of membership is greater than 1. The set, which has elements whose membership is over 1, we called it *Overset*.

Even worst, we observed elements whose membership with respect to a set is under 0, and we called it *Underset*.

In general, a set that has elements whose membership is above 1 and elements whose membership is below 0, we called it *Offset* (i.e. there are elements whose memberships are off (over and under) the interval [0, 1]).

"Neutrosophic" means based on three components *T (truth-membership)*, *I (indeterminacy)*, and *F (falsehood-nonmembership)*. And "over" means above 1, "under" means below 0, while "offset" means behind/beside the set on both sides of the interval [0, 1], over and under, more and less, supra and below, out of, off the set. Similarly, for "offlogic", "offmeasure", "offprobability", "offstatistics" etc.

It is like a pot with boiling liquid, on a gas stove, when the liquid swells up and leaks out of pot. The pot (the interval [0, 1]) can no longer contain all liquid (i.e., all neutrosophic truth / indeterminate / falsehood values), and therefore some of them fall out of the pot (i.e., one gets neutrosophic truth / indeterminate / falsehood values which are > 1), or the pot cracks on the bottom and the liquid pours down (i.e., one gets neutrosophic truth / indeterminate / falsehood values which are < 0).

Mathematically, they mean getting values off the interval [0, 1].

The American aphorism "think outside the box" has a perfect resonance to the neutrosophic offset, where the box is the interval [0, 1], yet values outside of this interval are permitted.

2. Example of Overmembership and Undermembership.

In a given company a full-time employer works 40 hours per week. Let's consider the last week period. Helen worked part-time, only 30 hours, and the other 10 hours she was absent without payment; hence, her membership degree was $30/40 = 0.75 < 1$.

John worked full-time, 40 hours, so he had the membership degree 40/40 = 1, with respect to this company.

But George worked overtime 5 hours, so his membership degree was (40+5)/40 = 45/40 = 1.125 > 1. Thus, we need to make distinction between employees who work overtime, and those who work full-time or part-time. That's why we need to associate a degree of membership strictly greater than 1 to the overtime workers.

Now, another employee, Jane, was absent without pay for the whole week, so her degree of membership was 0/40 = 0.

Yet, Richard, who was also hired as a full-time, not only didn't come to work last week at all (0 worked hours), but he produced, by accidentally starting a devastating fire, much damage to the company, which was estimated at a value half of his salary (i.e. as he would have gotten for working 20 hours that week). Therefore, his membership degree has to be less that Jane's (since Jane produced no damage). Whence, Richard's degree of membership, with respect to this company, was - 20/40 = - 0.50 < 0.

Consequently, we need to make distinction between employees who produce damage, and those who produce profit, or produce neither damage no profit to the company.

Therefore, the membership degrees > 1 and < 0 are real in our world, so we have to take them into consideration.

Then, similarly, the Neutrosophic Logic/Measure/Probability/Statistics etc. were extended to respectively *Neutrosophic Over-/Under-/Off-Logic, -Measure, -Probability, -Statistics* etc. [Smarandache, 2007].

Another Example of Membership Above 1 and Membership Below 0.

Let's consider a spy agency S = $\{S_1, S_2, \ldots, S_{1000}\}$ of a country Atara against its enemy country Batara. Each agent S_j, $j \in \{1, 2, \ldots, 1000\}$, was required last week to accomplish 5 missions, which represent the full-time contribution/membership.

Last week agent S_{27} has successfully accomplished his 5 missions, so his membership was $T(A_{27})$ = 5/5 = 1 = 100% (full-time membership).

Agent S_{32} has accomplished only 3 missions, so his membership is $T(S_{32})$ = 3/5 = 0.6 = 60% (part-time membership).

Agent S_{41} was absent, without pay, due to his health problems; thus $T(S_{41})$ = 0/5 = 0 = 0% (null-membership).

Agent S_{53} has successfully accomplished his 5 required missions, plus an extra mission of another agent that was absent due to sickness, therefore $T(S_{53})$ = (5+1)/5 = 6/5 = 1.2 > 1 (therefore, he has membership above 1, called over-membership).

Yet, agent S_{75} is a double-agent, and he leaks highly confidential information about country Atara to the enemy country Batara, while simultaneously providing misleading information to the country Atara about the enemy country Batara. Therefore S_{75} is a negative agent with respect to his country Atara, since he produces damage to Atara, he was estimated to having intentionally done wrongly all his 5 missions, in addition of compromising a mission of another agent of country Atara, thus his membership $T(S_{75})$ = - (5+1)/5 = - 6/5 = -1.2 < 0 (therefore, he has a membership below 0, called under-membership).

3. Definitions and the main work

1. **Definition of Single-Valued Neutrosophic Overset.**

Let U be a universe of discourse and the neutrosophic set $A_1 \subset U$.

Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U$, with respect to the neutrosophic set A_1:

$T(x), I(x), F(x) : U \rightarrow [0, \Omega]$

where $0 < 1 < \Omega$, and Ω is called overlimit,

$T(x), I(x), F(x) \in [0, \Omega]$.

A Single-Valued Neutrosophic Overset A_1 is defined as:

$A_1 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,

such that there exists at least one element in A_1 that has at least one neutrosophic component that is > 1, and no element has neutrosophic components that are < 0.

For **example**: $A_1 = \{(x_1, <1.3, 0.5, 0.1>), (x_2, <0.2, 1.1, 0.2>)\}$, since $T(x_1) = 1.3 > 1$, $I(x_2) = 1.1 > 1$, and no neutrosophic component is < 0.

Also $O_2 = \{(a, <0.3, -0.1, 1.1>)\}$, since I(a) = - 0.1 < 0 and F(a) = 1.1 > 1.

2. **Definition of Single-Valued Neutrosophic Underset.**

Let U be a universe of discourse and the neutrosophic set $A_2 \subset U$.

Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U$, with respect to the neutrosophic set A_2:

$T(x), I(x), F(x) : U \rightarrow [\Psi, 1]$

where $\Psi < 0 < 1$, and Ψ is called underlimit,

$T(x), I(x), F(x) \in [\Psi, 1]$.

A Single-Valued Neutrosophic Underset A_2 is defined as:
$A_2 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,
such that there exists at least one element in A_2 that has at least one neutrosophic component that is < 0, and no element has neutrosophic components that are > 1.
For **example**: $A_2 = \{(x_1, <-0.4, 0.5, 0.3>), (x_2, <0.2, 0.5, -0.2>)\}$, since $T(x_1) = -0.4 < 0$, $F(x_2) = -0.2 < 0$, and no neutrosophic component is > 1.

3. **Definition of Single-Valued Neutrosophic Offset.**

Let U be a universe of discourse and the neutrosophic set $A_3 \subset U$.
Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U$, with respect to the set A_3:
$T(x), I(x), F(x) : U \rightarrow [\Psi, \Omega]$
where $\Psi < 0 < 1 < \Omega$, and Ψ is called underlimit, while Ω is called overlimit,
$T(x), I(x), F(x) \in [\Psi, \Omega]$.
A Single-Valued Neutrosophic Offset A_3 is defined as:
$A_3 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,
such that there exist some elements in A_3 that have at least one neutrosophic component that is > 1, and at least another neutrosophic component that is < 0.
For **examples**: $A_3 = \{(x_1, <1.2, 0.4, 0.1>), (x_2, <0.2, 0.3, -0.7>)\}$, since $T(x_1) = 1.2 > 1$ and $F(x_2) = -0.7 < 0$.
Also $B_3 = \{(a, <0.3, -0.1, 1.1>)\}$, since $I(a) = -0.1 < 0$ and $F(a) = 1.1 > 1$.

4. **Single Valued Neutrosophic Overset / Underset / Offset Operators.**

Let U be a universe of discourse and $A = \{(x, <T_A(x), I_A(x), F_A(x)>), x \in U\}$ and
and $B = \{(x, <T_B(x), I_B(x), F_B(x)>), x \in U\}$ be two single-valued neutrosophic oversets / undersets / offsets.
$T_A(x), I_A(x), F_A(x), T_B(x), I_B(x), F_B(x): U \rightarrow [\Psi, \Omega]$
where $\Psi \le 0 < 1 \le \Omega$, and Ψ is called underlimit, while Ω is called overlimit,
$T_A(x), I_A(x), F_A(x), T_B(x), I_B(x), F_B(x) \in [\Psi, \Omega]$.
We take the inequality sign $\le$ instead of < on both extremes above, in order to comprise all three cases: overset {when $\Psi = 0$, and $1 < \Omega$}, underset {when $\Psi < 0$, and $1 = \Omega$}, and offset {when $\Psi < 0$, and $1 < \Omega$}.

4.1. Single Valued Neutrosophic Overset / Underset / Offset Union.
Then $A \cup B = \{(x, <\max\{T_A(x), T_B(x)\}, \min\{I_A(x), I_B(x)\}, \min\{F_A(x), F_B(x)\}>), x \in U\}$

4.2. Single Valued Neutrosophic Overset / Underset / Offset Intersection.
Then $A \cap B = \{(x, <\min\{T_A(x), T_B(x)\}, \max\{I_A(x), I_B(x)\}, \max\{F_A(x), F_B(x)\}>), x \in U\}$

4.3. Single Valued Neutrosophic Overset / Underset / Offset Complement.
The neutrosophic complement of the neutrosophic set A is
$C(A) = \{(x, <F_A(x), \Psi + \Omega - I_A(x), T_A(x)>), x \in U\}$.

5. **Definition of Interval-Valued Neutrosophic Overset.**

Let U be a universe of discourse and the neutrosophic set $A_1 \subset U$.
Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U$, with respect to the neutrosophic set A_1:
$T(x), I(x), F(x) : U \rightarrow P([0, \Omega])$,
where $0 < 1 < \Omega$, and Ω is called overlimit,
$T(x), I(x), F(x) \subseteq [0, \Omega]$, and $P([0, \Omega])$ is the set of all subsets of $[0, \Omega]$.
An Interval-Valued Neutrosophic Overset A_1 is defined as:
$A_1 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,
such that there exists at least one element in A_1 that has at least one neutrosophic component that is partially or totally above 1, and no element has neutrosophic components that is partially or totally below 0.
For **example**: $A_1 = \{(x_1, <(1, 1.4], 0.1, 0.2>), (x_2, <0.2, [0.9, 1.1], 0.2>)\}$, since $T(x_1) = (1, 1.4]$ is totally above 1, $I(x_2) = [0.9, 1.1]$ is partially above 1, and no neutrosophic component is partially or totally below 0.

6. **Definition of Interval-Valued Neutrosophic Underset.**

Let U be a universe of discourse and the neutrosophic set $A_2 \subset U$.
Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U$, with respect to the neutrosophic set A_2:
$T(x), I(x), F(x) : U \rightarrow [\Psi, 1]$,
where $\Psi < 0 < 1$, and Ψ is called underlimit,
$T(x), I(x), F(x) \subseteq [\Psi, 1]$, and $P([\Psi, 1])$ is the set of all subsets of $[\Psi, 1]$.
An Interval-Valued Neutrosophic Underset A_2 is defined as:
$A_2 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,
such that there exists at least one element in A_2 that has at least one neutrosophic component that is partially or totally below 0, and no element has neutrosophic components that are partially or totally above 1.
For **example**: $A_2 = \{(x_1, <(-0.5,-0.4), 0.6, 0.3>), (x_2, <0.2, 0.5, [-0.2, 0.2]>)\}$, since $T(x_1) = (-0.5, -0.4)$ is totally below 0, $F(x_2) = [-0.2, 0.2]$ is partially below 0, and no neutrosophic component is partially or totally above 1.

7. **Definition of Interval-Valued Neutrosophic Offset.**

Let U be a universe of discourse and the neutrosophic set $A_3 \subset U$.

Let T(x), I(x), F(x) be the functions that describe the degrees of membership, indeterminate-membership, and nonmembership respectively, of a generic element $x \in U$, with respect to the set A_3:
$T(x), I(x), F(x) : U \rightarrow P([\Psi,\Omega])$,
where $\Psi < 0 < 1 < \Omega$, and Ψ is called underlimit, while Ω is called overlimit,
$T(x), I(x), F(x) \subseteq [\Psi,\Omega]$, and $P([\Psi,\Omega])$ is the set of all subsets of $[\Psi,\Omega]$.
An Interval-Valued Neutrosophic Offset A_3 is defined as:
$A_3 = \{(x, <T(x), I(x), F(x)>), x \in U\}$,
such that there exist some elements in A_3 that have at least one neutrosophic component that is partially or totally above 1, and at least another neutrosophic component that is partially or totally below 0.
For **examples**: $A_3 = \{(x_1, <[1.1, 1.2], 0.4, 0.1>), (x_2, <0.2, 0.3, (-0.7, -0.3)>)\}$, since $T(x_1) = [1.1, 1.2]$ that is totally above 1, and $F(x_2) = (-0.7, -0.3)$ that is totally below 0.
Also $B_3 = \{(a, <0.3, [-0.1, 0.1], [1.05, 1.10]>)\}$, since $I(a) = [-0.1, 0.1]$ that is partially below 0, and $F(a) = [1.05, 1.10]$ that is totally above 1.

8. Interval-Valued Neutrosophic Overset / Underset / Offset Operators.

Let U be a universe of discourse and $A = \{(x, <T_A(x), I_A(x), F_A(x)>), x \in U\}$
and $B = \{(x, <T_B(x), I_B(x), F_B(x)>), x \in U\}$ be two interval-valued neutrosophic oversets / undersets / offsets.
$T_A(x), I_A(x), F_A(x), T_B(x), I_B(x), F_B(x): U \rightarrow P([\Psi,\Omega])$,
where $P([\Psi,\Omega])$ means the set of all subsets of $[\Psi,\Omega]$,
and $T_A(x), I_A(x), F_A(x), T_B(x), I_B(x), F_B(x) \subseteq [\Psi,\Omega]$,
with $\Psi \le 0 < 1 \le \Omega$, and Ψ is called underlimit, while Ω is called overlimit.
We take the inequality sign $\le$ instead of $<$ on both extremes above, in order to comprise all three cases: overset {when $\Psi = 0$, and $1 < \Omega$}, underset {when $\Psi < 0$, and $1 = \Omega$}, and offset {when $\Psi < 0$, and $1 < \Omega$}.

8.1. Interval-Valued Neutrosophic Overset / Underset / Offset Union.

Then $A \cup B =$
$\{(x, <[\max\{\inf(T_A(x)), \inf(T_B(x))\}, \max\{\sup(T_A(x)), \sup(T_B(x))\}],$
$[\min\{\inf(I_A(x)), \inf(I_B(x))\}, \min\{\sup(I_A(x)), \sup(I_B(x))\}],$
$[\min\{\inf(F_A(x)), \inf(F_B(x))\}, \min\{\sup(F_A(x)), \sup(F_B(x))\}]>, x \in U\}$.

8.2. Interval-Valued Neutrosophic Overset / Underset / Offset Intersection.

Then $A \cap B =$
$\{(x, <[\min\{\inf(T_A(x)), \inf(T_B(x))\}, \min\{\sup(T_A(x)), \sup(T_B(x))\}],$
$[\max\{\inf(I_A(x)), \inf(I_B(x))\}, \max\{\sup(I_A(x)), \sup(I_B(x))\}],$
$[\max\{\inf(F_A(x)), \inf(F_B(x))\}, \max\{\sup(F_A(x)), \sup(F_B(x))\}]>, x \in U\}$.

8.3. Interval-Valued Neutrosophic Overset / Underset / Offset Complement.

The complement of the neutrosophic set A is
$C(A) = \{(x, <F_A(x), [\Psi + \Omega - \sup\{I_A(x)\}, \Psi + \Omega - \inf\{I_A(x)\}], T_A(x)>), x \in U\}$.

Conclusion

The membership degrees over 1 (overmembership), or below 0 (undermembership) are part of our real world, so they deserve more study in the future.

The neutrosophic overset / underset / offset together with neutrosophic overlogic / underlogic / offlogic and especially neutrosophic overprobability / underprobability / and offprobability have many applications in technology, social science, economics and so on that the readers may be interested in exploring.

After designing the neutrosophic operators for single-valued neutrosophic overset/underset/offset, we extended them to interval-valued neutrosophic overset/underset/offset operators. We also presented another example of membership above 1 and membership below 0.

Of course, in many real world problems the neutrosophic union, neutrosophic intersection, and neutrosophic complement for interval-valued neutrosophic overset/underset/offset can be used. Future research will be focused on practical applications.

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Author's Presentations at Seminars and National and International Conferences

The author has presented the

- neutrosophic overset, neutrosophic underset, neutrosophic offset;
- neutrosophic overlogic, neutrosophic underlogic, neutrosophic offlogic;
- neutrosophic overmeasure, neutrosophic undermeasure, neutrosophic offmeasure;
- neutrosophic overprobability, neutrosophic underprobability, neutrosophic offprobability;
- neutrosophic overstatistics, neutrosophic understatistics, neutrosophic offstatistics; as follows:

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A Neutrosophic Binomial Factorial Theorem with their Refrains

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Abstract. The Neutrosophic Precalculus and the Neutrosophic Calculus can be developed in many ways, depending on the types of indeterminacy one has and on the method used to deal with such indeterminacy. This article is innovative since the form of neutrosophic binomial factorial theorem was constructed in addition to its refrains. Two other important theorems were proven with their corollaries, and numerical examples as well. As a conjecture, we use ten (indeterminate) forms in neutrosophic calculus taking an important role in limits. To serve article's aim, some important questions had been answered.

Keyword: Neutrosophic Calculus, Binomial Factorial Theorem, Neutrosophic Limits, Indeterminate forms in Neutrosophic Logic, Indeterminate forms in Classical Logic.

1 Introduction (Important questions)

Q 1 *What are the types of indeterminacy?*

There exist two types of indeterminacy

a. Literal indeterminacy (I).

As example:

$$2+3I \tag{1}$$

b. Numerical indeterminacy.

As example:

$$x(0.6,0.3,0.4) \in A, \tag{2}$$

meaning that the indeterminacy membership = 0.3.

Other examples for the indeterminacy component can be seen in functions: $f(0) = 7 \; or \; 9$ or $f(0 \; or \; 1) = 5$ or $f(x) = [0.2,\; 0.3]\, x^2$ … etc.

Q 2 *What is the values of I to the rational power?*

1. Let

$$\sqrt{I} = x + y\, I$$
$$0 + I = x^2 + (2xy + y^2)I$$
$$x = 0, y = \pm 1. \tag{3}$$

In general,

$$\sqrt[2k]{I} = \pm I \tag{4}$$

where $k \in z^+ = \{1,2,3, \dots\}$.

2. Let

$$\sqrt[3]{I} = x + y\, I$$
$$0 + I = x^3 + 3x^2 y\, I + 3xy^2\, I^2 + y^3 I^3$$
$$0 + I = x^3 + (3x^2 y + 3xy^2 + y^3)I$$
$$x = 0, y = 1 \rightarrow \sqrt[3]{I} = I. \tag{5}$$

In general,

$$\sqrt[2k+1]{I} = I, \tag{6}$$

where $k \in z^+ = \{1,2,3, \dots\}$.

Basic Notes

1. A component I to the zero power is undefined value, (i.e. I^0 is undefined), since $I^0 = I^{1+(-1)} = I^1 * I^{-1} = \frac{I}{I}$ which is impossible case (avoid to divide by I).
2. The value of I to the negative power is undefined value (i.e. I^{-n} , $n > 0$ is undefined).

Q 3 *What are the indeterminacy forms in neutrosophic calculus?*

In classical calculus, the indeterminate forms are [4]:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty^0, 0^0, 1^\infty, \infty - \infty. \tag{7}$$

The form 0 to the power I (i.e. 0^I) is an indeterminate form in Neutrosophic calculus; it is tempting to argue that an indeterminate form of type 0^I has zero value since "zero to any power is zero". However, this is fallacious, since 0^I is not a power of number, but rather a statement about limits.

Q 4 *What about the form* 1^I*?*

The base "one" pushes the form 1^I to one while the power I pushes the form 1^I to I, so 1^I is an indeterminate form in neutrosophic calculus. Indeed, the form a^I, $a \in R$ is always an indeterminate form.

Q 5 *What is the value of* a^I *, where* $a \in R$?

Let $y_1 = 2^x, x \in R, y_2 = 2^I$; it is obvious that $\lim_{x\to\infty} 2^x = \infty$, $\lim_{x\to-\infty} 2^x = 0$, $\lim_{x\to 0} 2^x = 1$; while we cannot determine if $2^I \to \infty \; or \; 0 \; or \; 1$, therefore we can say that $y_2 = 2^I$ indeterminate form in Neutrosophic calculus. The same for a^I, where $a \in R$ [2].

2 Indeterminate forms in Neutrosophic Logic

It is obvious that there are seven types of indeterminate forms in classical calculus [4],

$$\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty, 0^0, \infty^0, 1^\infty, \infty-\infty.$$

As a conjecture, we can say that there are ten forms of the indeterminate forms in Neutrosophic calculus

$$I^0, 0^I, \frac{I}{0}, I\cdot\infty, \frac{\infty}{I}, \infty^I, I^\infty, I^I,$$
$$a^I (a \in R), \infty \pm a\cdot I \; .$$

Note that:

$$\frac{I}{0} = I\cdot\frac{1}{0} = I\cdot\infty = \infty\cdot I.$$

3 Various Examples

Numerical examples on neutrosophic limits would be necessary to demonstrate the aims of this work.

Example (3.1) [1], [3]

The neutrosophic (numerical indeterminate) values can be seen in the following function:

Find $\lim_{x\to 0} f(x)$, where $f(x) = x^{[2.1,2.5]}$.

Solution:

Let $y = x^{[2.1,2.5]} \;\; \to \ln y = [2.1, 2.5] \; \ln x$

$$\therefore \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{[2.1,2.5]}{\frac{1}{\ln x}} = \frac{[2.1,2.5]}{\frac{1}{\ln 0}}$$
$$= \frac{[2.1,2.5]}{\frac{1}{-\infty}} = \frac{[2.1,2.5]}{-0}$$
$$= \left[\frac{2.1}{-0}, \frac{2.5}{-0}\right] = (-\infty, -\infty)$$
$$= -\infty$$

Hence $y = e^{-\infty} = 0$

OR it can be solved briefly by

$y = x^{[2.1,2.5]} = [0^{2.1}, 0^{2.5}] = [0,0] = 0.$

Example (3.2)

$\lim_{x\to[9,11]} [3.5,5.9] x^{[1,2]} = [3.5,5.9] \; [9,11]^{[1,2]} = [3.5,5.9] \; [9^1, 11^2] = [(3.5)(9), (5.9)(121)] = [31.5,713.9].$

Example (3.3)

$$\lim_{x\to\infty} [3.5,5.9] \; x^{[1,2]} = [3.5,5.9] \;\; \infty^{[1,2]}$$
$$= [3.5,5.9] \; [\infty^1, \infty^2]$$
$$= [3.5\cdot(\infty), 5.9\cdot(\infty)]$$
$$= (\infty, \infty) = \infty.$$

Example (3.4)

Find the following limit using more than one technique $\lim_{x\to 0} \frac{\sqrt{[4,5]\cdot x+1}-1}{x}$.

Solution:

The above limit will be solved firstly by using the L'Hôpital's rule and secondly by using the rationalizing technique.

<u>*Using L'Hôpital's rule*</u>

$$\lim_{x\to 0} \frac{1}{2} ([4,5]\cdot x+1)^{-1/2} \; [4,5]$$
$$= \lim_{x\to 0} \frac{[4,5]}{2\sqrt{([4,5]\cdot x+1)}}$$
$$= \frac{[4,5]}{2} = \left[\frac{4}{2}, \frac{5}{2}\right] = [2,2.5]$$

<u>*Rationalizing technique [3]*</u>

$$\lim_{x\to 0} \frac{\sqrt{[4,5]\cdot x+1}-1}{x} = \frac{\sqrt{[4,5]\cdot 0+1}-1}{0}$$
$$= \frac{\sqrt{[4\cdot 0, 5\cdot 0]+1}-1}{0} = \frac{\sqrt{[0,0]+1}-1}{0}$$
$$= \frac{\sqrt{0+1}-1}{0} = \frac{0}{0}$$
$$= \text{undefined}.$$

Multiply with the conjugate of the numerator:

$$\lim_{x\to 0}\frac{\sqrt{[4,5]x+1}-1}{x}\cdot\frac{\sqrt{[4,5]x+1}+1}{\sqrt{[4,5]x+1}+1}$$
$$=\lim_{x\to 0}\frac{\left(\sqrt{[4,5]x+1}\right)^2-(1)^2}{x\left(\sqrt{[4,5]x+1}+1\right)}$$
$$=\lim_{x\to 0}\frac{[4,5]\cdot x\cancel{+1-1}}{x\cdot\left(\sqrt{[4,5]x+1}+1\right)}$$
$$=\lim_{x\to 0}\frac{[4,5]\cancel{\cdot x}}{\cancel{x\cdot}\left(\sqrt{[4,5]x+1}+1\right)}$$
$$=\lim_{x\to 0}\frac{[4,5]}{\left(\sqrt{[4,5]x+1}+1\right)}$$
$$=\frac{[4,5]}{\left(\sqrt{[4,5]\cdot 0+1}+1\right)}=\frac{[4,5]}{\sqrt{1}+1}$$
$$=\frac{[4,5]}{2}=\left[\frac{4}{2},\frac{5}{2}\right]=[2,2.5].$$

Identical results.

Example (3.5)

Find the value of the following neutrosophic limit $\lim_{x\to -3}\frac{x^2+3x-[1,2]x-[3,6]}{x+3}$ using more than one technique .

Analytical technique [1], [3]

$\lim_{x\to -3}\frac{x^2+3x-[1,2]x-[3,6]}{x+3}$

By substituting x= -3 ,

$$\lim_{x\to -3}\frac{(-3)^2+3\cdot(-3)-[1,2]\cdot(-3)-[3,6]}{-3+3}$$
$$=\frac{9-9-[1\cdot(-3),2\cdot(-3)]-[3,6]}{0}$$
$$=\frac{0-[-6,-3]-[3,6]}{0}=\frac{[3,6]-[3,6]}{0}$$
$$=\frac{[3-6,6-3]}{0}=\frac{[-3,3]}{0},$$

which has undefined operation $\frac{0}{0}$, since $0\in[-3,3]$. Then we factor out the numerator, and simplify:

$$\lim_{x\to -3}\frac{x^2+3x-[1,2]x-[3,6]}{x+3}=$$
$$\lim_{x\to -3}\frac{(x-[1,2])\cdot(x+3)}{(x+3)}=\lim_{x\to -3}(x-[1,2])$$
$$=-3-[1,2]=[-3,-3]-[1,2]$$
$$=-([3,3]+[1,2])=[-5,-4].$$

Again, Solving by using L'Hôpital's rule

$$\lim_{x\to -3}\frac{x^2+3x-[1,2]x-[3,6]}{x+3}$$
$$=\lim_{x\to -3}\frac{2\,x+3-[1,2]}{1}$$
$$=\lim_{x\to -3}\frac{2\,(-3)+3-[1,2]}{1}$$
$$=-6+3-[1,2]$$
$$=-3-[1,2]$$
$$=[-3-1,-3-2]$$
$$=[-5,-4]$$

The above two methods are identical in results.

4 New Theorems in Neutrosophic Limits

Theorem (4.1) (Binomial Factorial)

$\lim_{x\to\infty}(I+\frac{1}{x})^x=Ie$; I is the literal indeterminacy, e = 2.7182828

Proof

$$\left(I+\frac{1}{x}\right)^x=\binom{x}{0}I^X\left(\frac{1}{x}\right)^0+\binom{x}{1}I^{X-1}\left(\frac{1}{x}\right)^1$$
$$+\binom{x}{2}I^{X-2}\left(\frac{1}{x}\right)^2+\binom{x}{3}I^{X-3}\left(\frac{1}{x}\right)^3$$
$$+\binom{x}{4}I^{X-4}\left(\frac{1}{x}\right)^4+\cdots$$
$$=I+x.I.\frac{1}{x}+\frac{I}{2!}\left(1-\frac{1}{x}\right)$$
$$+\frac{I}{3!}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)+\frac{I}{4!}\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)$$
$$\left(1-\frac{3}{x}\right)+\cdots$$

It is clear that $\frac{1}{x}\to 0\ \ as\ \ x\ \to\infty$

$\therefore\lim_{x\to\infty}(I-\frac{1}{x})^x=I+I+\frac{I}{2!}+\frac{I}{3!}+\frac{I}{4!}+\cdots=I+\sum_{n=1}^{\infty}\frac{I^n}{n!}$

$\therefore\lim_{x\to\infty}(I+\frac{1}{x})^x=Ie$, where $e=1+\sum_{n=1}^{\infty}\frac{1}{n!}$, I is the literal indeterminacy.

Corollary (4.1.1)

$\lim_{x\to 0}(I+x)^{\frac{1}{x}}=Ie$

Proof:-

Put $y=\frac{1}{x}$

It is obvious that $y\to\infty$, as $x\to 0$

$\therefore\lim_{x\to 0}(I+x)^{\frac{1}{x}}=\lim_{y\to\infty}(I+\frac{1}{y})^y=Ie$

(using Th. 4.1)

Corollary (4.1.2)

$\lim_{x\to\infty}(I+\frac{k}{x})^x=Ie^k$, where k > 0 & $k\neq 0$, I is the literal indeterminacy.

Proof

$$\lim_{x\to\infty}(I+\frac{k}{x})^{x} = \lim_{x\to\infty}\left[(I+\frac{k}{x})^{\frac{x}{k}}\right]^{k}$$

Put $y=\frac{k}{x}\to xy=k\to x=\frac{k}{y}$

Note that $y\to 0\ as\ x\to\infty$

$$\therefore \lim_{x\to\infty}\left(I+\frac{k}{x}\right)^{x} = \lim_{y\to 0}\left[(I+y)^{\frac{1}{y}}\right]^{k}$$

(using corollary 4.1.1).

$$=\left[\lim_{y\to 0}(I+y)^{\frac{1}{y}}\right]^{k} = (Ie)^{k} = I^{k}e^{k} = Ie^{k}$$

Corollary (4.1.3)

$\lim_{x\to 0}(I+\frac{x}{k})^{\frac{1}{x}} = (Ie)^{\frac{1}{k}} = \sqrt[k]{Ie}$,

where $k\neq 1$ & $k>0$.

Proof

The immediate substitution of the value of x in the above limit gives indeterminate form I^{∞},

i.e. $\lim_{x\to 0}(I+\frac{x}{k})^{\frac{1}{x}} = \lim_{x\to 0}(I+\frac{0}{k})^{\frac{1}{0}} = I^{\infty}$

So we need to treat this value as follow:-

$$\lim_{x\to 0}(I+\frac{x}{k})^{\frac{1}{x}} = \lim_{x\to 0}\left[(I+\frac{x}{k})^{\frac{k}{x}}\right]^{\frac{1}{k}} = \left[\lim_{x\to 0}(I+\frac{x}{k})^{\frac{k}{x}}\right]^{\frac{1}{k}}$$

put $y=\frac{x}{k}\to x=ky\to\frac{1}{x}=\frac{1}{ky}$

As $x\to 0$, $y\to 0$

$$\lim_{x\to 0}\left(I+\frac{x}{k}\right)^{\frac{1}{x}} = \lim_{y\to 0}\left[(I+y)^{\frac{1}{y}}\right]^{\frac{1}{k}}$$

$$=\left[\lim_{y\to 0}(I+y)^{\frac{1}{y}}\right]^{\frac{1}{k}}$$

Using corollary (4.1.1)

$$=(Ie)^{\frac{I}{k}} = \sqrt[k]{Ie}$$

Theorem (4.2)

$\lim_{x\to 0}\frac{(lna)[Ia^{x}-I]}{xlna+lnI} = \frac{lna}{1+lnI}$

Where $a>0, a\neq 1$

Note that $\lim_{x\to 0}\frac{(lna)[Ia^{x}-I]}{xlna+lnI} = \lim_{x\to 0}\frac{Ia^{x}-I}{x+\frac{lnI}{lna}}$

Proof

Let $y=Ia^{x}-I\ \to y+I=Ia^{x}\ \to\ \ln(y+I) = \ln I+\ln a^{x}$

$\to \ln(y+I) = \ln I + xlna \to$

$$x=\frac{\ln(y+I)-lnI}{lna}$$

$$\frac{(\ln a)(Ia^{x}-I)}{xlna+lnI} = \frac{(Ia^{x}-I)}{x+\frac{lnI}{lna}}$$

$$=\frac{y}{\frac{\ln(y+I)-lnI}{lna}+\frac{lnI}{lna}}$$

$$=lna.\frac{y}{\ln(y+I)} = lna.\frac{1}{\frac{1}{y}\ln(y+I)}$$

$$=lna.\frac{1}{\ln(y+I)^{\frac{1}{y}}}$$

$$\therefore \lim_{x\to 0}\frac{(\ln a)(Ia^{x}-I)}{xlna+lnI} = lna\ \frac{1}{\lim_{y\to 0} ln(y+I)^{\frac{1}{y}}}$$

$$=lna\,.\frac{1}{ln\lim_{y\to 0}(y+I)^{\frac{1}{y}}}$$

$=lna\ \frac{1}{ln(Ie)}$ using corollary (4.1.1)

$$=\frac{lna}{lnI+\ lne} = \frac{lna}{lnI+1}$$

Corollary (4.2.1)

$$\lim_{x\to 0}\frac{Ia^{kx}-I}{x+\frac{lnI}{lna^{k}}} = \frac{k\ lna}{1+lnI}$$

Proof

Put $y=kx\to x=\frac{y}{k}$

$y\to 0\ \ as\ x\to 0$

$$\lim_{x\to 0}\frac{Ia^{kx}-I}{x+\frac{lnI}{lna^{k}}} = \lim_{y\to 0}\frac{Ia^{y}-I}{\frac{y}{k}+\frac{lnI}{k\,lna}} = k.\lim_{y\to 0}\frac{Ia^{y}-I}{y+\frac{lnI}{lna}}$$

using Th. (4.2)

$$=k.(\frac{lna}{1+lnI})$$

Corollary (4.2.2)

$$\lim_{x\to 0}\frac{Ie^{x}-I}{x+lnI} = \frac{1}{1+lnI}$$

Proof

Let $y=\ Ie^{x}-I$ $\ ,y\to 0\ \ as\ x\to 0$

$y+I=\ Ie^{x}\to \ln(y+I) = lnI + x\ lne$

$x=\ln(y+I)-lnI$

$$\therefore\ \frac{Ie^{x}-I}{x+\ lnI} = \frac{y}{\ln(y+I)-lnI+lnI}$$

$$=\frac{1}{\frac{1}{y}\ln(y+I)}$$

$$=\frac{1}{\ln(y+I)^{\frac{1}{y}}}$$

$$\therefore \lim_{x\to 0}\frac{Ie^{x}-I}{x+lnI} = \lim_{y\to 0}\frac{1}{\ln(y+I)^{\frac{1}{y}}}$$

$$=\frac{1}{\ln\ \lim_{y\to 0}(y+I)^{\frac{1}{y}}}$$

using corollary (4.1.1)

$$\frac{1}{\ln(Ie)} = \frac{1}{lnI+lne} = \frac{1}{lnI+1}$$

Corollary (4.2.3)

$$\lim_{x\to 0}\frac{Ie^{kx}-I}{x+\frac{lnI}{k}}=\frac{k}{1+lnI}$$

Proof

let $y=kx\to x=\frac{y}{k}$

$y\to 0\ \ as\, x\to 0$

$\lim_{x\to 0}\frac{Ie^{kx}-I}{x+\frac{lnI}{k}}=\lim_{y\to 0}\frac{Ie^{y}-I}{\frac{y}{k}+\frac{lnI}{k}}=k.\lim_{y\to 0}\frac{Ie^{y}-I}{y+lnI}$ using Corollary (4.2.2) to get

$$=k.\left(\frac{1}{1+lnI}\right)=\frac{k}{1+lnI}$$

Theorem (4.3)

$$\lim_{x\to 0}\frac{\ln(I+kx)}{x}=k(1+lnI)$$

Proof

$$\lim_{x\to 0}\frac{\ln(I+kx)}{x}=\lim_{x\to 0}\frac{\ln(I+kx)-lnI+lnI}{x}$$

Let $y=\ln(I+kx)-lnI\to y+lnI=\ln(I+kx)$

$$e^{y+lnI}=I+kx\to x=\frac{e^{y}e^{lnI}-I}{k}=\frac{I\,e^{y}-I}{k}$$

$y\to 0\ \ as\, x\to 0$

$$\lim_{x\to 0}\frac{\ln(I+kx)-lnI+lnI}{x}$$

$$=\lim_{y\to 0}\frac{y+lnI}{\frac{I\,e^{y}-I}{k}}$$

$$\lim_{y\to 0}\frac{k}{\frac{I\,e^{y}-I}{y+lnI}}=\frac{k}{\lim_{y\to 0}\left(\frac{I\,e^{y}-I}{y+lnI}\right)}$$

using corollary (4.2.2) to get the result

$$=\frac{k}{\frac{1}{1+lnI}}=k(1+lnI)$$

Theorem (4.4)

Prove that, for any two real numbers a,b

$\lim_{x\to 0}\frac{I\mathrm{a}^{x}-I}{I\mathrm{b}^{x}-I}=1$, where $a,b>0$ & $a,b\neq 1$

Proof

The direct substitution of the value x in the above limit conclude that $\frac{0}{0}$,so we need to treat it as follow:

$$\lim_{x\to 0}\frac{I\mathrm{a}^{x}-I}{I\mathrm{b}^{x}-I}=\lim_{x\to 0}\frac{\frac{ln\mathrm{a}[I\mathrm{a}^{x}-I]}{xln\mathrm{a}+lnI}*\frac{xln\mathrm{a}+lnI}{ln\mathrm{a}}}{\frac{ln\mathrm{b}[I\mathrm{b}^{x}-I]}{xln\mathrm{b}+lnI}*\frac{xln\mathrm{b}+lnI}{ln\mathrm{b}}}$$

$$=\frac{\lim_{x\to x}\frac{ln\mathrm{a}[I\mathrm{a}^{x}-I]}{xln\mathrm{a}+lnI}}{\lim_{x\to x}\frac{ln\mathrm{b}[I\mathrm{b}^{x}-I]}{xln\mathrm{b}+lnI}}*\frac{\lim_{x\to 0}(xln\mathrm{a}+lnI)}{\lim_{x\to 0}(xln\mathrm{b}+lnI)}*\frac{ln\mathrm{b}}{ln\mathrm{a}}$$

(using Th.(4.2) twice (first in numerator second in denominator))

$$=\frac{\frac{ln\mathrm{a}}{1+lnI}}{\frac{ln\mathrm{b}}{1+lnI}}*\frac{lnI}{lnI}*\frac{ln\mathrm{b}}{ln\mathrm{a}}=1.$$

5 Numerical Examples

Example (5.1)

Evaluate the limit $\lim_{x\to 0}\frac{I5^{4x}-I}{x+\frac{lnI}{ln5^{4}}}$

Solution

$\lim_{x\to 0}\frac{I5^{4x}-I}{x+\frac{lnI}{ln5^{4}}}=\frac{4ln5}{1+lnI}$ (using corollary 4. 2.1)

Example (5.2)

Evaluate the limit $\lim_{x\to 0}\frac{Ie^{4x}-I}{I3^{2x}-I}$

Solution

$$\lim_{x\to 0}\frac{Ie^{4x}-I}{I3^{2x}-I}=\lim_{x\to 0}\frac{\frac{ln3[Ie^{4x}-I]}{(x+\frac{lnI}{4})}*(x+\frac{lnI}{4})}{\frac{ln3[I3^{2x}-I]}{(x+\frac{lnI}{ln3^{2}})}*(x+\frac{lnI}{ln3^{2}})}$$

$$=\frac{\lim_{x\to 0}\frac{ln3[Ie^{4x}-I]}{\left(x+\frac{lnI}{4}\right)}}{\lim_{x\to 0}\frac{ln3[I3^{2x}-I]}{\left(x+\frac{lnI}{ln3^{2}}\right)}}*\frac{\lim_{x\to 0}\left(x+\frac{lnI}{4}\right)}{\lim_{x\to 0}\left(x+\frac{lnI}{ln3^{2}}\right)}$$

(using corollary (4.2.3) on numerator & corollary (4.2.1) on denominator)

$$=\frac{\frac{4}{1+lnI}}{\frac{2ln3}{1+lnI}}*\frac{\frac{lnI}{4}}{\frac{lnI}{ln3^{2}}}=1.$$

5 Conclusion

In this article, we introduced for the first time a new version of binomial factorial theorem containing the literal indeterminacy *(I).* This theorem enhances three corollaries. As a conjecture for indeterminate forms in classical calculus, ten of new indeterminate forms in Neutrosophic calculus had been constructed. Finally, various examples had been solved.

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Neutrosophic Integer Programming Problem

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Abstract. In this paper, we introduce the integer programming in neutrosophic environment, by considering coffecients of problem as a triangulare neutrosophic numbers. The degrees of acceptance, indeterminacy and rejection of objectives are simultaneously considered. The Neutrosophic Integer Programming Problem (NIP) is transformed into a crisp programming model, using truth membership (T), indeterminacy membership (I), and falsity membership (F) functions as well as single valued triangular neutrosophic numbers. To measure the efficiency of the model, we solved several numerical examples.

Keywords : Neutrosophic; integer programming; single valued triangular neutrosophic number.

1 Introduction

In linear programming models, decision variables are allowed to be fractional. For example, it is reasonable to accept a solution giving an hourly production of automobiles at $64\frac{1}{2}$, if the model were based upon average hourly production. However, fractional solutions are not realistic in many situations and to deal with this matter, integer programming problems are introduced. We can define integer programming problem as a linear programming problem with integer restrictions on decision variables. When some, but not all decision variables are restricted to be integer, this problem called a mixed integer problem and when all decision variables are integers, it's a pure integer program. Integer programming plays an important role in supporting managerial decisions. In integer programming problems the decision maker may not be able to specify the objective function and/or constraints functions precisely. In 1995, Smarandache [1-3] introduce neutrosophy which is the study of neutralities as an extension of dialectics. Neutrosophic is the derivative of neutrosophy and it includes neutrosophic set, neutrosophic probability, neutrosophic statistics and neutrosophic logic. Neutrosophic theory means neutrosophy applied in many fields of sciences, in order to solve problems related to indeterminacy. Although intuitionistic fuzzy sets can only handle incomplete information not indeterminate, the neutrosophic set can handle both incomplete and indeterminate information.[4] Neutrosophic sets characterized by three independent degrees as in Fig.1., namely truth-membership degree (*T*), indeterminacy-membership degree(*I*), and falsity-membership degree (*F*), where *T,I,F* are standard or non-standard subsets of $]^-0, 1^+[$. The decision makers in neutrosophic set want to increase the degree of truth-membership and decrease the degree of indeterminacy and falsity membership.

The structure of the paper is as follows: the next section is a preliminary discussion; the third section describes the formulation of integer programing problem using the proposed model; the fourth section presents some illustrative examples to put on view how the approach can be applied; the last section summarizes the conclusions and gives an outlook for future research.

2 Some Preliminaries

2.1 Neutrosophic Set [4]

Let X be a space of points (objects) and $x\in X$. A neutrosophic set A in X is defined by a truth-membership function (x), an indeterminacy-membership function (x) and a falsity-membership function $F_A(x)$. (x), $I(x)$ and $F(x)$ are real standard or real nonstandard subsets of $]0^-,1^+[$. That is $T_A(x):X\to]0^-,1^+[$, $I_A(x):X\to]0^-,1^+[$ and $F_A(x):X\to]0^-,1^+[$. There is no restriction on the sum of (x), (x) and $F_A(x)$, so

$0^-\leq \sup(x)\leq \sup I_A(x)\leq F_A(x)\leq 3^+$.

2.2 Single Valued Neutrosophic Sets (SVNS) [3-4]

Let X be a universe of discourse. A single valued neutrosophic set A over X is an object having the form

$A=\{\langle x, T(x), I_A(x), F_A(x)\rangle : x\in X\}$,

where $T_A(x):X\to[0,1]$, $I_A(x):X\to[0,1]$ and $F_A(x):X\to[0,1]$ with $0\leq T_A(x)+I_A(x)+F_A(x)\leq 3$ for all $x\in X$. The intervals $T(x)$,

$I(x)$ and $F_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively.

In the following, we write SVN numbers instead of single valued neutrosophic numbers. For convenience, a SVN number is denoted by $A=(a,b,c)$, where $a,b,c \in [0,1]$ and $a+b+c \leq 3$.

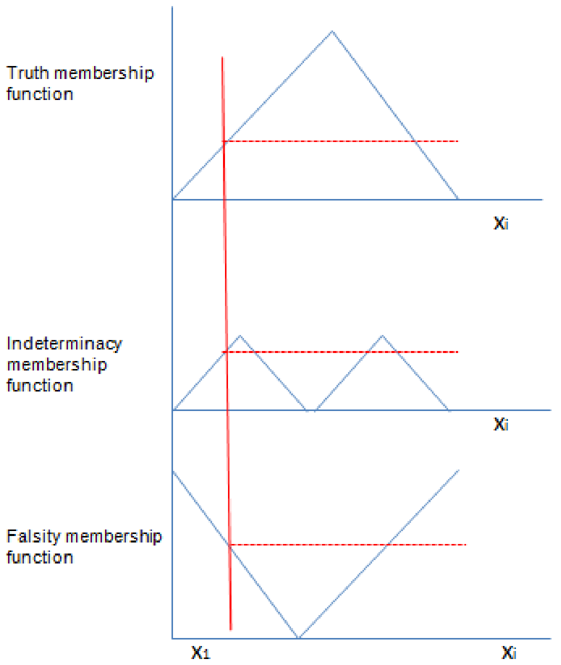

Figure 1: Neutrosophication process

2.3 Complement [5]

The complement of a single valued neutrosophic set A is denoted by $c\,(A)$ and is defined by

$T_c(A)(x) = F(A)(x)$,

$I_c(A)(x) = 1 - I(A)(x)$,

$F_c(A)(x) = T(A)(x)$ for all x *in* X

2.4 Union [5]

The union of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as C = AUB, whose truth-membership, indeterminacy membership and falsity-membership functions are given by

$T(C)(x) = max\ (\ T(A)(x)\ ,T(B)(x)\ )$,
$I(C)(x) = max\ (\ I(A)(x)\ ,I(B)(x)\ )$,
$F(C)(x) = \ min((A)(x)\ ,F(B)(x)\ )$ *for all* x *in* X

2.5 Intersection [5]

The intersection of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as C = A∩B, whose truth-membership, indeterminacy membership and falsity-membership functions are given by

$T(C)(x) = min\ (\ T(A)(x)\ ,T(B)(x)\ )$,
$I(C)(x) = min\ (\ I(A)(x)\ ,I(B)(x)\ )$,
$F(C)(x) = \ max((A)(x)\ ,F(B)(x)\ )$ *for all* x *in* X

3 Neutrosophic Integer Programming Problems

Integer programming problem with neutrosophic coefficients (NIPP) is defined as the following:

Maximize $Z= \sum_{j=1}^{n} \tilde{c}_j x_j$

Subject to

$$\sum_{j=1}^{n} a_{ij}^{\sim n} x_j \leq b_i \quad i = 1, \dots, m\,, \tag{1}$$

$x_j \geq 0, \qquad j = 1, \dots n\,,$

x_j integer for $j \in \{0,1, \dots n\}$.

Where $\tilde{c}_j$, $a_{ij}^{\sim n}$ are neutrosophic numbres.

The single valued neutrosophic number ($a_{ij}^{\sim n}$) is donated by A=(a,b,c) where a,b,c ∈ [0,1] And a,b,c ≤ 3

The truth- membership function of neutrosophic number $a_{ij}^{\sim n}$ is defined as:

$$T\,a_{ij}^{\sim n}(x)=\begin{cases} \frac{x-a_1}{a_2-a_1} & a_1 \leq x \leq a_2 \\ \frac{a_2-x}{a_3-a_2} & a_2 \leq x \leq a_3 \\ 0 & otherwise \end{cases} \tag{2}$$

The indeterminacy- membership function of *neutrosophic number* $a_{ij}^{\sim n}$ is defined as:

$$I\,a_{ij}^{\sim n}(x)=\begin{cases} \frac{x-b_1}{b_2-b_1} & b_1 \leq x \leq b_2 \\ \frac{b_2-x}{b_3-b_2} & b_2 \leq x \leq b_3 \\ 0 & otherwise \end{cases} \tag{3}$$

And its falsity- membership function of *neutrosophic number* $a_{ij}^{\sim n}$ is defined as:

$$F\,a_{ij}^{\sim n}(x)=\begin{cases} \frac{x-C_1}{C_2-C_1} & C_1 \leq x \leq C_2 \\ \frac{b_2-x}{b_3-b_2} & C_2 \leq x \leq C_3 \\ 1 & otherwise \end{cases} \tag{4}$$

Then we find the maximum and minimum values of the objective function for truth-membership, indeterminacand falsity membership as follows:

$f^{max} = \max\{f(x_i^*)\}$ and $f^{min} = \min\{f(x_i^*)\}$ where $1 \leq i \leq k$

$f_{min}^{F} = f_{min}^{T}$ and $f_{max}^{F} = f_{max}^{T} - R(f_{max}^{T} - f_{min}^{T})$

$f^I_{max}=f^I_{max}$ *and* $f^I_{min}=f^I_{min}-S(f^T_{max}-f^T_{min})$
Where R ,S are predetermined real number in (0,1)
The truth membership, indeterminacy membership, falsity membership of objective function as follows:

$$T^f(x)=\begin{cases}1 & if\ \ f\le f^{min}\\ \frac{f^{max}-f(x)}{f^{max}-f^{min}} & if\ \ f^{min}<f(x)\le f^{max}\\ 0 & if\ f(x)\ >f^{max}\end{cases} \quad (5)$$

$$I^f(x)=\begin{cases}0 & if\ \ f\le f^{min}\\ \frac{f(x)-f^{max}}{f^{max}-f^{min}} & if\ \ f^{min}<f(x)\le f^{max}\\ 0 & if\ f(x)\ >f^{max}\end{cases} \quad (6)$$

$$F^f(x)=\begin{cases}0 & if\ \ f\le f^{min}\\ \frac{f(x)-f^{min}}{f^{max}-f^{min}} & if\ \ f^{min}<f(x)\le f^{max}\\ 1 & if\ f(x)\ >f^{max}\end{cases} \quad (7)$$

The neutrosophic set of the j^{th} decision variable x_j is defined as:

$$T^{(x)}_{x_j}=\begin{cases}1 & if\ \ x_j\le 0\\ \frac{d_j-x_j}{d_j} & if\ \ 0<x_j\le d_j\\ 0 & if\ \ x_j\ >d_j\end{cases} \quad (8)$$

$$F^{(x)}_{x_j}=\begin{cases}0 & if\ \ x_j\le 0\\ \frac{x_j}{d_j+b_j} & if\ \ 0<x_j\le d_j\\ 1 & if\ \ x_j\ >d_j\end{cases} \quad (9)$$

$$I^{(x)}_j=\begin{cases}0 & if\ \ x_j\le 0\\ \frac{x_j-d_j}{d_j+b_j} & if\ \ 0<x_j\le d_j\\ 0 & if\ \ x_j\ >d_j\end{cases} \quad (10)$$

Where d_j, b_j are integer numbers.

4 Neutrosophic Optimization Model of integer programming problem

In our neutrosophic model we want to maximize the degree of acceptance and minimize the degree of rejection and indeterminacy of the neutrosophic objective function and constraints. Neutrosophic optimization model can be defined as:

$$\begin{aligned}&maxT_{(x)}\\&minF_{(x)}\\&minI_{(x)}\\&\textit{Subject to}\\&\quad T_{(X)}\ge F_{(x)}\\&\quad T_{(X)}\ge I_{(x)}\\&\quad 0\le T_{(X)}+I_{(x)}+F_{(x)}\le 3\\&\quad T_{(X)},\ \ I_{(X)},\ \ F_{(X)}\ge 0\\&\quad x\ge 0\ \text{, integer.}\end{aligned} \quad (11)$$

Where $T_{(x)}.F_{(x)},I_{(x)}$ denotes the degree of acceptance, rejection and indeterminacy of x respectively.

The above problem is equivalent to the following:

$$\begin{aligned}&max\ \alpha,\ min\ \beta\ ,min\ \theta\\&\textit{Subject to}\\&\alpha\le T_{(x)}\\&\beta\le F_{(x)}\\&\theta\le I_{(x)}\\&\alpha\ge\beta\\&\alpha\ge\theta\\&0\le\ \alpha+\beta+\theta\le 3\\&x\ge 0\ \text{, integer.}\end{aligned} \quad (12)$$

Where α denotes the minimal acceptable degree, β denote the maximal degree of rejection and θ denote maximal degree of indeterminacy.

The neutrosophic optimization model can be changed into the following optimization model:

$$\begin{aligned}&max(\alpha-\beta-\theta)\\&\textit{Subject to}\\&\alpha\le T_{(x)}\\&\beta\ge F_{(x)}\\&\theta\ge I_{(x)}\\&\alpha\ge\beta\\&\alpha\ge\theta\\&0\le\ \alpha+\beta+\theta\le 3\\&\alpha,\beta,\theta\ge 0\\&x\ge 0\ \text{, integer.}\end{aligned} \quad (13)$$

The previous model can be written as:

$$\begin{aligned}&min\ (1-\ \alpha)\,\beta\,\theta\\&\textit{Subject to}\\&\alpha\le T_{(x)}\\&\beta\ge F_{(x)}\\&\theta\ge I_{(x)}\\&\alpha\ge\beta\\&\alpha\ge\theta\\&0\le\ \alpha+\beta+\theta\le 3\\&x\ge 0\ \text{, integer.}\end{aligned} \quad (14)$$

5 The Algorithms for Solving Neutrosophic integer Programming Problem (NIPP)

5.1 Neutrosophic Cutting Plane Algorithm

Step 1: Convert neutrosophic integer programming problem to its crisp model by using the following method:
By defining a method to compare any two single valued triangular neutrosophic numbers which is based on the score function and the accuracy function. Let $\tilde{a} = \langle (a_1, b_1, c_1), w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ be a single valued triangular neutrosophic number, then

$$S(\tilde{a}) = \frac{1}{16}[a+b+c]\times(2+\mu_{\tilde{a}} - v_{\tilde{a}} - \lambda_{\tilde{a}}) \qquad (15)$$

and

$$A(\tilde{a}) = \frac{1}{16}[a+b+c]\times(2+\mu_{\tilde{a}} - v_{\tilde{a}} + \lambda_{\tilde{a}}) \qquad (16)$$

is called the score and accuracy degrees of $\tilde{a}$, respectively. The neutrosophic integer programming NIP can be represented by crisp programming model using truth membership, indeterminacy membership, and falsity membership functions and the score and accuracy degrees of ã, at equations (15) or (16).

Step 2: Create the decision set which include the highest degree of truth-membership and the least degree of falsity and indeterminacy memberships.

Step 3: Solve the problem as a linear programming problem and ignore integrality.

Step 4: If the optimal solution is integer, then it's right. Otherwise, go to the next step.

Step 5: Generate a constraint which is satisfied by all integer solutions and add this constraint to the problem.

Step 6: Go to step 1.

5.2 Neutrosophic Branch and Bound Algorithm

Step 1: Convert neutrosophic integer programming problem to its crisp model by using the following method:
By defining a method to compare any two single valued triangular neutrosophic numbers which is based on the score function and the accuracy function. Let $\tilde{a} = \langle (a_1, b_1, c_1), w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ be a single valued triangular neutrosophic number, then

$$S(\tilde{a}) = \frac{1}{16}[a+b+c]\times(2+\mu_{\tilde{a}} - v_{\tilde{a}} - \lambda_{\tilde{a}}) \qquad (15)$$

and

$$A(\tilde{a}) = \frac{1}{16}[a+b+c]\times(2+\mu_{\tilde{a}} - v_{\tilde{a}} + \lambda_{\tilde{a}}) \qquad (16)$$

is called the score and accuracy degrees of $\tilde{a}$, respectively. The neutrosophic integer programming NIP can be represented by crisp programming model using truth membership, indeterminacy membership, and falsity membership functions and the score and accuracy degrees of ã, at equations (15) or (16).

Step 2: Create the decision set which include the highest degree of truth-membership and the least degree of falsity and indeterminacy memberships.

Step 3: At the first node let the solution of linear programming model with integer restriction as an upper bound and the rounded-down integer solution as a lower bound.

Step 4: For branching process, we select the variable with the largest fractional part. Two constrains are obtained after the branching process, one for$\le$ and the other is $\ge$ constraint.

Step 5: Create two nodes for the two new constraints.

Step 6: Solve the model again, after adding new constraints at each node.

Step 7: The optimal integer solution has been reached, if the feasible integer solution has the largest upper bound value of any ending node. Otherwise return to step 4.

The previous algorithm is for a maximization model. For a minimization model, the solution of linear programming problem with integer restrictions are rounded up and upper and lower bounds are reversed.

6 Numerical Examples

To measure the efficiency of our proposed model we solved many numerical examples.

6.1 Illustrative Example #1

$$\begin{aligned} &max \quad \tilde{5}x_1 + \tilde{3}x_2 \\ &subject\ to \quad \tilde{4}x_1 + \tilde{3}x_2 \le \widetilde{12} \\ &\qquad\qquad \tilde{1}x_1 + \tilde{3}x_2 \le \tilde{6} \\ &\qquad\qquad x_1, x_2 \ge 0\ and\ integer \end{aligned}$$

where

$\tilde{5} = \langle (4,5,6), 0.8, 0.6, 0.4 \rangle$
$\tilde{3} = \langle (2.5,3,3.5), 0.75, 0.5, 0.3 \rangle$
$\tilde{4} = \langle (3.5,4,4.1), 1, 0.5, 0.0 \rangle$
$\tilde{3} = \langle (2.5,3,3.5), 0.75, 0.5, 0.25 \rangle$
$\tilde{1} = \langle (0,1,2), 1, 0.5, 0 \rangle$
$\tilde{3} = \langle (2.8,3,3.2), 0.75, 0.5, 0.25 \rangle$
$\widetilde{12} = \langle (11,12,13), 1, 0.5, 0 \rangle$
$\tilde{6} = \langle (5.5,6,7.5), 0.8, 0.6, 0.4 \rangle$

Then the neutrosophic model converted to the crisp model by using Eq.15 , Eq.16.as follows :

$$\max \quad 5.6875x_1 + 3.5968x_2$$
$$\text{subject to} \quad \begin{aligned} 4.3125x_1 + 3.625x_2 &\leq 14.375 \\ 0.2815x_1 + 3.925x_2 &\leq 7.6375 \\ x_1, x_2 &\geq 0 \textit{ and integer} \end{aligned}$$

The optimal solution of the problem is $x^* = (3,0)$ with optimal objective value 17.06250.

6.2 Illustrative Example #2

$$max \quad \widetilde{25}x_1 + \widetilde{48}x_2$$
$$subject\ to \quad \begin{aligned} 15x_1 + 30x_2 &\leq 45000 \\ 24x_1 + 6x_2 &\leq 24000 \\ 21x_1 + 14x_2 &\leq 28000 \\ x_1, x_2 &\geq 0 \textit{ and integer} \end{aligned}$$

where

$\widetilde{25} = \langle (19,25,33), 0.8, 0.5, 0 \rangle$;

$\widetilde{48} = \langle (44,48,54), 0.9, 0.5, 0 \rangle$

Then the neutrosophic model converted to the crisp model as :

$$\max \quad 27.8875x1 + 55.3x_2$$
$$\text{subject to} \quad \begin{aligned} 15x_1 + 30x_2 &\leq 45000 \\ 24x_1 + 6x_2 &\leq 24000 \\ 21x_1 + 14x_2 &\leq 28000 \\ x_1, x_2 &\geq 0 \textit{ and integer} \end{aligned}$$

The optimal solution of the problem is $x^* = (500,1250)$ with optimal objective value 83068.75.

7 Conclusions and Future Work

In this paper, we proposed an integer programming model based on neutrosophic environment, simultaneously considering the degrees of acceptance, indeterminacy and rejection of objectives, by proposed model for solving neutrosophic integer programming problems (NIPP). In the model, we maximize the degrees of acceptance and minimize indeterminacy and rejection of objectives. NIPP was transformed into a crisp programming model using truth membership, indeterminacy membership, falsity membership and score functions. We also give numerical examples to show the efficiency of the proposed method. Future research directs to studying the duality theory of integer programming problems based on Neutrosophic.

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Neutrosophic Modal Logic

Florentin Smarandache

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Abstract: We introduce now for the first time the neutrosophic modal logic. The Neutrosophic Modal Logic includes the neutrosophic operators that express the modalities. It is an extension of neutrosophic predicate logic and of neutrosophic propositional logic.

Applications of neutrosophic modal logic are to neutrosophic modal metaphysics. Similarly to classical modal logic, there is a plethora of neutrosophic modal logics. Neutrosophic modal logics is governed by a set of neutrosophic axioms and neutrosophic rules.

Keywords: neutrosophic operators, neutrosophic predicate logic, neutrosophic propositional logic, neutrosophic epistemology, neutrosophic mereology.

1 Introduction.

The paper extends the fuzzy modal logic [1, 2, and 4], fuzzy environment [3] and neutrosophic sets, numbers and operators [5 – 12], together with the last developments of the neutrosophic environment {including (t, i, f)-neutrosophic algebraic structures, neutrosophic triplet structures, and neutrosophic overset / underset / offset} [13 - 15] passing through the symbolic neutrosophic logic [16], ultimately to neutrosophic modal logic.

All definitions, sections, and notions introduced in this paper were never done before, neither in my previous work nor in other researchers'.

Therefore, we introduce now the Neutrosophic Modal Logic and the Refined Neutrosophic Modal Logic.

Then we can extend them to Symbolic Neutrosophic Modal Logic and Refined Symbolic Neutrosophic Modal Logic, using labels instead of numerical values.

There is a large variety of neutrosophic modal logics, as actually happens in classical modal logic too. Similarly, the neutrosophic accessibility relation and possible neutrosophic worlds have many interpretations, depending on each particular application. Several neutrosophic modal applications are also listed.

Due to numerous applications of neutrosophic modal logic (see the examples throughout the paper), the introduction of the neutrosophic modal logic was needed.

Neutrosophic Modal Logic is a logic where some neutrosophic modalities have been included.

Let $\mathcal{P}$ be a neutrosophic proposition. We have the following types of **neutrosophic modalities**:

A) Neutrosophic Alethic Modalities (related to *truth*) has three neutrosophic operators:

i. ***Neutrosophic Possibility***: It is neutrosophically possible that $\mathcal{P}$.

ii. ***Neutrosophic Necessity***: It is neutrosophically necessary that $\mathcal{P}$.

iii. ***Neutrosophic Impossibility***: It is neutrosophically impossible that $\mathcal{P}$.

B) Neutrosophic Temporal Modalities (related to *time*)

It was the neutrosophic case that $\mathcal{P}$.

It will neutrosophically be that $\mathcal{P}$.

And similarly:

It has always neutrosophically been that $\mathcal{P}$.

It will always neutrosophically be that $\mathcal{P}$.

C) Neutrosophic Epistemic Modalities (related to *knowledge*):

It is neutrosophically known that $\mathcal{P}$.

D) Neutrosophic Doxastic Modalities (related to *belief*):

It is neutrosophically believed that $\mathcal{P}$.

E) Neutrosophic Deontic Modalities:

It is neutrosophically obligatory that $\mathcal{P}$.

It is neutrosophically permissible that $\mathcal{P}$.

2 Neutrosophic Alethic Modal Operators

The modalities used in classical (alethic) modal logic can be neutrosophicated by inserting the indeterminacy. We insert the **degrees of possibility** and **degrees of necessity**, as refinement of classical modal operators.

3 Neutrosophic Possibility Operator

The classical Possibility Modal Operator «$\lozenge P$» meaning «It is possible that P» is extended to **Neutrosophic Possibility Operator**: $\lozenge_N \mathcal{P}$ meaning

«It is (*t, i, f*)-possible that $\mathcal{P}$ », using Neutrosophic Probability, where «(t, i, f)-possible» means *t* % possible (chance that $\mathcal{P}$ occurs), *i* % indeterminate (indeterminate-chance that $\mathcal{P}$ occurs), and *f* % impossible (chance that $\mathcal{P}$ does not occur).

If $\mathcal{P}(t_p, i_p, f_p)$ is a neutrosophic proposition, with t_p, i_p, f_p subsets of [0, 1], then the neutrosophic truth-value of the neutrosophic possibility operator is:

$$\lozenge_N \mathcal{P} = \left(\sup(t_p), \inf(i_p), \inf(f_p)\right),$$

which means that if a proposition P is t_p true, i_p indeterminate, and f_p false, then the value of the neutrosophic possibility operator $\lozenge_N \mathcal{P}$ is: $\sup(t_p)$ possibility, $\inf(i_p)$ indeterminate-possibility, and $\inf(f_p)$ impossibility.

For ***example***.

Let *P* = «It will be snowing tomorrow».

According to the meteorological center, the neutrosophic truth-value of $\mathcal{P}$ is:

$\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$,

i.e. $[0.5, 0.6]$ true, $(0.2, 0.4)$ indeterminate, and $\{0.3, 0.5\}$ false.

Then the neutrosophic possibility operator is:

$\lozenge_N \mathcal{P} =$
$(\sup[0.5, 0.6], \inf(0.2, 0.4), \inf\{0.3, 0.5\}) =$
$(0.6, 0.2, 0.3)$,

i.e. 0.6 possible, 0.2 indeterminate-possibility, and 0.3 impossible.

4 Neutrosophic Necessity Operator

The classical Necessity Modal Operator « $\square P$ » meaning «It is necessary that *P*» is extended to **Neutrosophic Necessity Operator**: $\square_N \mathcal{P}$ meaning «It is (*t, i, f*)-necessary that $\mathcal{P}$ », using again the Neutrosophic Probability, where similarly «(t, i, f)-necessity» means *t* % necessary (surety that $\mathcal{P}$ occurs), *i* % indeterminate (indeterminate-surety that $\mathcal{P}$ occurs), and *f* % unnecessary (unsurely that $\mathcal{P}$ occurs).

If $\mathcal{P}(t_p, i_p, f_p)$ is a neutrosophic proposition, with t_p, i_p, f_p subsets of [0, 1], then the neutrosophic truth value of the neutrosophic necessity operator is:

$$\square_N \mathcal{P} = \left(\inf(t_p), \sup(i_p), \sup(f_p)\right),$$

which means that if a proposition $\mathcal{P}$ is t_p true, i_p indeterminate, and f_p false, then the value of the neutrosophic necessity operator $\square_N \mathcal{P}$ is: $\inf(t_p)$ necessary, $\sup(i_p)$ indeterminate-necessity, and $\sup(f_p)$ unnecessary.

Taking the previous *example*:

$\mathcal{P}$ = «It will be snowing tomorrow», with $\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$, then the neutrosophic necessity operator is:

$\square_N \mathcal{P} =$
$(\inf[0.5, 0.6], \sup(0.2, 0.4), \sup\{0.3, 0.5\}) =$
$(0.5, 0.4, 0.5)$,

i.e. 0.5 necessary, 0.4 indeterminate-necessity, and 0.5 unnecessary.

5 Connection between *Neutrosophic Possibility Operator* and *Neutrosophic Necessity Operator*.

In classical modal logic, a modal operator is equivalent to the negation of the other:

$\lozenge P \leftrightarrow \neg\square\neg P$,

$\square P \leftrightarrow \neg \lozenge \neg P$.

In neutrosophic logic one has a class of neutrosophic negation operators. The most used one is:

$\underset{N}{\neg} P(t, i, f) = \bar{P}(f, 1 - i, t)$,

where *t, i, f* are real subsets of the interval [0, 1].

Let's check what's happening in the neutrosophic modal logic, using the previous *example*.

One had:

$\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$,

then

$\underset{N}{\neg} \mathcal{P} = \bar{\mathcal{P}}(\{0.3, 0.5\}, 1 - (0.2, 0.4), [0.5, 0.6]) =$
$\bar{\mathcal{P}}(\{0.3, 0.5\}, 1 - (0.2, 0.4), [0.5, 0.6]) =$
$\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$.

Therefore, denoting by $\underset{N}{\leftrightarrow}$ the neutrosophic equivalence, one has:

$\underset{N}{\neg}\underset{N}{\square}\underset{N}{\neg} \mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\}) \underset{N}{\leftrightarrow}$

$\underset{N}{\leftrightarrow}$ It is not neutrosophically necessary that «It will not be snowing tomorrow»

$\underset{N}{\leftrightarrow}$ It is not neutrosophically necessary that $\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$

$\underset{N}{\leftrightarrow}$ It is neutrosophically possible that $\underset{N}{\neg}\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$

$\underset{N}{\leftrightarrow}$ It is neutrosophically possible that $\mathcal{P}([0.5, 0.6], 1 - (0.6, 0.8), \{0.3, 0.5\})$

$\underset{N}{\leftrightarrow}$ It is neutrosophically possible that $\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$

$\underset{N}{\leftrightarrow}\underset{N}{\lozenge} \mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\}) =$
$(0.6, 0.2, 0.3)$.

Let's check the second neutrosophic equivalence.

$\underset{N}{\neg}\underset{N}{\lozenge}\underset{N}{\neg}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})\underset{N}{\leftrightarrow}$

$\underset{N}{\leftrightarrow}$ It is not neutrosophically possible that «It will not be snowing tomorrow»

$\underset{N}{\leftrightarrow}$ It is not neutrosophically possible that $\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$

$\underset{N}{\leftrightarrow}$ It is neutrosophically necessary that $\underset{N}{\neg}\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$

$\underset{N}{\leftrightarrow}$ It is neutrosophically necessary that $\mathcal{P}([0.5, 0.6], 1 - (0.6, 0.8), \{0.3, 0.5\})$

$\underset{N}{\leftrightarrow}$ It is neutrosophically necessary that $\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$

$\underset{N}{\leftrightarrow}\underset{N}{\square}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\}) = (0.6, 0.2, 0.3)$.

6 Neutrosophic Modal Equivalences

Neutrosophic Modal Equivalences hold within a certain accuracy, depending on the definitions of neutrosophic possibility operator and neutrosophic necessity operator, as well as on the definition of the neutrosophic negation – employed by the experts depending on each application. Under these conditions, one may have the following neutrosophic modal equivalences:

$$\lozenge_N \mathcal{P}(t_p, i_p, f_p) \underset{N}{\leftrightarrow} \underset{N}{\neg}\underset{N}{\square}\underset{N}{\neg}\mathcal{P}(t_p, i_p, f_p)$$

$$\square_N \mathcal{P}(t_p, i_p, f_p) \underset{N}{\leftrightarrow} \underset{N}{\neg}\underset{N}{\lozenge}\underset{N}{\neg}\mathcal{P}(t_p, i_p, f_p)$$

For example, other definitions for the **neutrosophic modal operators** may be:

$\lozenge_N \mathcal{P}(t_p, i_p, f_p) = \left(\sup(t_p), \sup(i_p), \inf(f_p)\right)$, or

$\lozenge_N \mathcal{P}(t_p, i_p, f_p) = \left(\sup(t_p), \frac{i_p}{2}, \inf(f_p)\right)$ etc., while

$\square_N \mathcal{P}(t_p, i_p, f_p) = \left(\inf(t_p), \inf(i_p), \sup(f_p)\right)$, or

$\square_N \mathcal{P}(t_p, i_p, f_p) = \left(\inf(t_p), 2i_p \cap [0,1], \sup(f_p)\right)$ etc.

7 Neutrosophic Truth Threshold

In neutrosophic logic, first we have to introduce a **neutrosophic truth threshold**, $TH = \langle T_{th}, I_{th}, F_{th}\rangle$, where T_{th}, I_{th}, F_{th} are subsets of [0, 1]. We use upper-case letters (T, I, F) in order to distinguish the neutrosophic components of the threshold from those of a proposition in general.

We can say that the proposition $\mathcal{P}(t_p, i_p, f_p)$ is **neutrosophically true** if:

$\inf(t_p) \geq \inf(T_{th})$ and $\sup(t_p) \geq \sup(T_{th})$;

$\inf(i_p) \leq \inf(I_{th})$ and $\sup(t_p) \leq \sup(I_{th})$;

$\inf(f_p) \leq \inf(F_{th})$ and $\sup(f_p) \leq \sup(F_{th})$.

For the particular case when all T_{th}, I_{th}, F_{th} and t_p, i_p, f_p are single-valued numbers from the interval [0, 1], then one has:

The proposition $\mathcal{P}(t_p, i_p, f_p)$ is **neutrosophically true** if:

$t_p \geq T_{th}$;

$i_p \leq I_{th}$;

$f_p \leq F_{th}$.

The neutrosophic truth threshold is established by experts in accordance to each applications.

8 Neutrosophic Semantics

Neutrosophic Semantics of the Neutrosophic Modal Logic is formed by a **neutrosophic frame** G_N, which is a non-empty neutrosophic set, whose elements are called **possible neutrosophic worlds**, and a **neutrosophic binary relation** $\mathcal{R}_N$, called **neutrosophic accesibility relation**, between the possible neutrosophic worlds. By notation, one has:

$$\langle G_N, \mathcal{R}_N\rangle.$$

A neutrosophic world w'_N that is neutrosophically accessible from the neutrosophic world w_N is symbolized as:

$$w_N \mathcal{R}_N w'_N.$$

In a **neutrosophic model** each neutrosophic proposition $\mathcal{P}$ has a **neutrosophic truth-value** $(t_{w_N}, i_{w_N}, f_{w_N})$ respectively to each neutrosophic world $w_N \in G_N$, where $t_{w_N}, i_{w_N}, f_{w_N}$ are subsets of [0, 1].

A **neutrosophic actual world** can be similarly noted as in classical modal logic as $w_N *$.

Formalization.

Let S_N be a set of neutrosophic propositional variables.

9 Neutrosophic Formulas

1) Every neutrosophic propositional variable $\mathcal{P} \in S_N$ is a neutrosophic formula.

2) If *A, B* are neutrosophic formulas, then $\underset{N}{\neg}A$, $A\underset{N}{\wedge}B$, $A\underset{N}{\vee}B$, $A\underset{N}{\rightarrow}B$, $A\underset{N}{\leftrightarrow}B$, and $\underset{N}{\lozenge}A$, $\underset{N}{\square}A$, are also neutrosophic formulas, where $\underset{N}{\neg}$, $\underset{N}{\wedge}$, $\underset{N}{\vee}$, $\underset{N}{\rightarrow}$, $\underset{N}{\leftrightarrow}$, and $\underset{N}{\lozenge}$,

$\underset{N}{\square}$ represent the neutrosophic negation, neutrosophic intersection, neutrosophic union, neutrosophic implication, neutrosophic equivalence, and neutrosophic possibility operator, neutrosophic necessity operator respectively.

10 Accesibility Relation in a Neutrosophic Theory

Let G_N be a set of neutrosophic worlds w_N such that each w_N chracterizes the propositions (formulas) of a given neutrosophic theory τ.

We say that the neutrosophic world w'_N is accesible from the neutrosophic world w_N, and we write: $w_N \mathcal{R}_N w'_N$ or $\mathcal{R}_N(w_N, w'_N)$, if for any proposition (formula) $\mathcal{P} \in w_N$, meaning the neutrosophic truth-value of $\mathcal{P}$ with respect to w_N is

$$\mathcal{P}\left(t_p^{w_N}, i_p^{w_N}, f_p^{w_N}\right),$$

one has the neutrophic truth-value of $\mathcal{P}$ with respect to w'_N

$$\mathcal{P}\left(t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}\right),$$

where

$\inf\left(t_p^{w'_N}\right) \geq \inf\left(t_p^{w_N}\right)$ and $\sup\left(t_p^{w'_N}\right) \geq \sup\left(t_p^{w_N}\right)$;

$\inf\left(i_p^{w'_N}\right) \leq \inf\left(i_p^{w_N}\right)$ and $\sup\left(i_p^{w'_N}\right) \leq \sup\left(i_p^{w_N}\right)$;

$\inf\left(f_p^{w'_N}\right) \leq \inf\left(f_p^{w_N}\right)$ and $\sup\left(f_p^{w'_N}\right) \leq \sup\left(f_p^{w_N}\right)$

(in the general case when $t_p^{w_N}, i_p^{w_N}, f_p^{w_N}$ and $t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}$ are subsets of the interval [0, 1]).

But in the instant of $t_p^{w_N}, i_p^{w_N}, f_p^{w_N}$ and $t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}$ as single-values in [0, 1], the above inequalities become:

$$t_p^{w'_N} \geq t_p^{w_N},$$

$$i_p^{w'_N} \leq i_p^{w_N},$$

$$f_p^{w'_N} \leq f_p^{w_N}.$$

11 Applications

If the neutrosophic theory τ is the Neutrosophic Mereology, or Neutrosophic Gnosisology, or Neutrosophic Epistemology etc., the neutrosophic accesibility relation is defined as above.

12 Neutrosophic n-ary Accesibility Relation

We can also extend the classical **binary** accesibility relation $\mathcal{R}$ to a **neutrosophic n-ary accesibility relation**

$\mathcal{R}_N^{(n)}$, for n integer ≥ 2.

Instead of the classical $R(w, w')$, which means that the world w' is accesible from the world w, we generalize it to:

$$\mathcal{R}_N^{(n)}\left(w_{1_N}, w_{2_N}, \dots, w_{n_N}; w'_N\right),$$

which means that the neutrosophic world w'_N is accesible from the neutrosophic worlds $w_{1_N}, w_{2_N}, \dots, w_{n_N}$ all together.

13 Neutrosophic Kripke Frame

$k_N = \langle G_N, R_N \rangle$ is a neutrosophic Kripke frame, since:

$i.$ G_N is an arbitrary non-empty neutrosophic set of **neutrosophic worlds**, or **neutrosophic states**, or **neutrosophic situations**.

$ii.$ $R_N \subseteq G_N \times G_N$ is a **neutrosophic accesibility relation** of the neutrosophic Kripke frame. Actually, one has a degree of accesibility, degree of indeterminacy, and a degree of non-accesibility.

14 Neutrosophic (t, i, f)-Assignement

The Neutrosophic (t, i, f)-Assignement is a neutrosophic mapping

$$v_N: S_N \times G_N \rightarrow [0,1] \times [0,1] \times [0,1]$$

where, for any neutrosophic proposition $\mathcal{P} \in S_N$ and for any neutrosophic world w_N, one defines:

$$v_N(\mathcal{P}, w_N) = \left(t_p^{w_N}, i_p^{w_N}, f_p^{w_N}\right) \in [0,1] \times [0,1] \times [0,1]$$

which is the neutrosophical logical truth value of the neutrosophic proposition $\mathcal{P}$ in the neutrosophic world w_N.

15 Neutrosophic Deducibility

We say that the neutrosophic formula $\mathcal{P}$ is neutrosophically deducible from the neutrosophic Kripke frame k_N, the neutrosophic *(t, i, f)* – assignment v_N, and the neutrosophic world w_N, and we write as:

$$k_N, v_N, w_N \underset{N}{\vDash} \mathcal{P}.$$

Let's make the notation:

$$\alpha_N(\mathcal{P};\ k_N, v_N, w_N)$$

that denotes the neutrosophic logical value that the formula $\mathcal{P}$ takes with respect to the neutrosophic Kripke frame k_N, the neutrosophic *(t, i, f)*-assignement v_N, and the neutrosphic world w_N.

We define α_N by neutrosophic induction:

1. $\alpha_N(\mathcal{P};\ k_N, v_N, w_N) \overset{def}{=} v_N(\mathcal{P}, w_N)$ if $\mathcal{P} \in S_N$ and $w_N \in G_N$.
2. $\alpha_N\left(\underset{N}{\neg}\mathcal{P};\ k_N, v_N, w_N\right) \overset{def}{=} \underset{N}{\neg}[\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]$.
3. $\alpha_N\left(\mathcal{P} \underset{N}{\wedge} Q;\ k_N, v_N, w_N\right) \overset{def}{=} [\alpha_N(\mathcal{P};\ k_N, v_N, w_N)] \underset{N}{\wedge} [\alpha_N(Q;\ k_N, v_N, w_N)]$

4. $\alpha_N\left(\mathcal{P}\underset{N}{\vee}Q;\ k_N, v_N, w_N\right) \overset{def}{=} [\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]\underset{N}{\vee}[\alpha_N(Q;\ k_N, v_N, w_N)]$

5. $\alpha_N\left(\mathcal{P}\underset{N}{\rightarrow}Q;\ k_N, v_N, w_N\right) \overset{def}{=} [\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]\underset{N}{\rightarrow}[\alpha_N(Q;\ k_N, v_N, w_N)]$

6. $\alpha_N\left(\underset{N}{\Diamond}\mathcal{P};\ k_N, v_N, w_N\right) \overset{def}{=} \langle\sup, \inf, \inf\rangle\{\alpha_N(\mathcal{P};\ k_N, v_N, w'_N), w' \in G_N \text{ and } w_N R_N w'_N\}$.

7. $\alpha_N\left(\underset{N}{\Box}\mathcal{P};\ k_N, v_N, w_N\right) \overset{def}{=} \langle\inf, \sup, \sup\rangle\{\alpha_N(\mathcal{P};\ k_N, v_N, w'_N), w'_N \in G_N \text{ and } w_N R_N w'_N\}$.

8. $\underset{N}{\vDash}\mathcal{P}$ if and only if $w_N * \vDash \mathcal{P}$ (a formula $\mathcal{P}$ is neutrosophically deducible if and only if $\mathcal{P}$ is neutrosophically deducible in the actual neutrosophic world).

We should remark that α_N has a degree of truth (t_{α_N}), a degree of indeterminacy (i_{α_N}), and a degree of falsehood (f_{α_N}), which are in the general case subsets of the interval [0, 1].

Applying $\langle\sup, \inf, \inf\rangle$ to α_N is equivalent to calculating:

$\langle\sup(t_{\alpha_N}), \inf(i_{\alpha_N}), \inf(f_{\alpha_N})\rangle$,

and similarly

$\langle\inf, \sup, \sup\rangle\alpha_N = \langle\inf(t_{\alpha_N}), \sup(i_{\alpha_N}), \sup(f_{\alpha_N})\rangle$.

16 Refined Neutrosophic Modal Single-Valued Logic

Using neutrosophic (*t, i, f*) - thresholds, we refine for the first time the neutrosophic modal logic as:

a) Refined Neutrosophic Possibility Operator.

$\underset{N}{\Diamond_1}\mathcal{P}_{(t,i,f)}$ = «It is very little possible (degree of possibility t_1) that $\mathcal{P}$», corresponding to the threshold (t_1, i_1, f_1), i.e. $0 \le t \le t_1$, $i \ge i_1$, $f \ge f_1$, for t_1 a very little number in [0, 1];

$\underset{N}{\Diamond_2}\mathcal{P}_{(t,i,f)}$ = «It is little possible (degree of possibility t_2) that $\mathcal{P}$», corresponding to the threshold (t_2, i_2, f_2), i.e. $t_1 < t \le t_2$, $i \ge i_2 > i_1$, $f \ge f_2 > f_1$;

… … …

and so on;

$\underset{N}{\Diamond_m}\mathcal{P}_{(t,i,f)}$ = «It is possible (with a degree of possibility t_m) that $\mathcal{P}$», corresponding to the threshold (t_m, i_m, f_m), i.e. $t_{m-1} < t \le t_m$, $i \ge i_m > i_{m-1}$, $f \ge f_m > f_{m-1}$.

b) Refined Neutrosophic Necessity Operator.

$\underset{N}{\Box_1}\mathcal{P}_{(t,i,f)}$ = «It is a small necessity (degree of necessity t_{m+1}) that $\mathcal{P}$», i.e. $t_m < t \le t_{m+1}$, $i \ge i_{m+1} \ge i_m$, $f \ge f_{m+1} > f_m$;

$\underset{N}{\Box_2}\mathcal{P}_{(t,i,f)}$ = «It is a little bigger necessity (degree of necessity t_{m+2}) that $\mathcal{P}$», i.e. $t_{m+1} < t \le t_{m+2}$, $i \ge i_{m+2} > i_{m+1}$, $f \ge f_{m+2} > f_{m+1}$;

… … …

and so on;

$\underset{N}{\Box_k}\mathcal{P}_{(t,i,f)}$ = «It is a very high necessity (degree of necessity t_{m+k}) that $\mathcal{P}$», i.e. $t_{m+k-1} < t \le t_{m+k} = 1$, $i \ge i_{m+k} > i_{m+k-1}$, $f \ge f_{m+k} > f_{m+k-1}$.

17 Application of the Neutrosophic Threshold

We have introduced the term of (*t, i, f*)-physical law, meaning that a physical law has a degree of truth (*t*), a degree of indeterminacy (*i*), and a degree of falsehood (*f*). A physical law is 100% true, 0% indeterminate, and 0% false in perfect (ideal) conditions only, maybe in laboratory.

But our actual world $(w_N *)$ is not perfect and not steady, but continously changing, varying, fluctuating.

For example, there are physicists that have proved a universal constant (*c*) is not quite universal (i.e. there are special conditions where it does not apply, or its value varies between $(c - \varepsilon, c + \varepsilon)$, for $\varepsilon > 0$ that can be a tiny or even a bigger number).

Thus, we can say that a proposition $\mathcal{P}$ is **neutrosophically nomological necessary**, if $\mathcal{P}$ is neutrosophically true at all possible neutrosophic worlds that obey the (*t, i, f*)-physical laws of the actual neutrosophic world $w_N *$.

In other words, at each possible neutrosophic world w_N, neutrosophically accesible from $w_N *$, one has:

$\mathcal{P}(t_p^{w_N}, i_p^{w_N}, f_p^{w_N}) \ge TH(T_{th}, I_{th}, F_{th})$,

i.e. $t_p^{w_N} \ge T_{th}$, $i_p^{w_N} \le I_{th}$, and $f_p^{w_N} \ge F_{th}$.

18 Neutrosophic Mereology

Neutrosophic Mereology means the theory of the neutrosophic relations among the parts of a whole, and the neutrosophic relations between the parts and the whole.

A neutrosophic relation between two parts, and similarly a neutrosophic relation between a part and the whole, has a degree of connectibility (*t*), a degree of indeterminacy (*i*), and a degree of disconnectibility (*f*).

19 Neutrosophic Mereological Threshold

Neutrosophic Mereological Threshold is defined as:

$$TH_M = (\min(t_M), \max(i_M), \max(f_M))$$

where t_M is the set of all degrees of connectibility between the parts, and between the parts and the whole;

i_M is the set of all degrees of indeterminacy between the parts, and between the parts and the whole;

f_M is the set of all degrees of disconnectibility between the parts, and between the parts and the whole.

We have considered all degrees as single-valued numbers.

20 Neutrosophic Gnosisology

Neutrosophic Gnosisology is the theory of (*t, i, f*)-knowledge, because in many cases we are not able to completely (100%) find whole knowledge, but only a part of it (*t* %), another part remaining unknown (*f* %), and a third part indeterminate (unclear, vague, contradictory) (*i* %), where *t, i, f* are subsets of the interval [0, 1].

21 Neutrosophic Gnosisological Threshold

Neutrosophic Gnosisological Threshold is defined, similarly, as:

$$TH_G = (\min(t_G), \max(i_G), \max(f_G)),$$

where t_G is the set of all degrees of knowledge of all theories, ideas, propositions etc.,
i_G is the set of all degrees of indeterminate-knowledge of all theories, ideas, propositions etc.,
f_G is the set of all degrees of non-knowledge of all theories, ideas, propositions etc.
We have considered all degrees as single-valued numbers.

22 Neutrosophic Epistemology

And **Neutrosophic Epistemology**, as part of the Neutrosophic Gnosisology, is the theory of (*t, i, f*)-scientific knowledge.

Science is infinite. We know only a small part of it (*t* %), another big part is yet to be discovered (*f* %), and a third part indeterminate (unclear, vague, contradictort) (*i* %).

Of course, *t, i, f* are subsets of [0, 1].

23 Neutrosophic Epistemological Threshold

It is defined as:

$$TH_E = (\min(t_E), \max(i_E), \max(f_E))$$

where t_E is the set of all degrees of scientific knowledge of all scientific theories, ideas, propositions etc.,
i_E is the set of all degrees of indeterminate scientific knowledge of all scientific theories, ideas, propositions etc.,
f_E is the set of all degrees of non-scientific knowledge of all scientific theories, ideas, propositions etc.
We have considered all degrees as single-valued numbers.

24 Conclusions

We have introduced for the first time the Neutrosophic Modal Logic and the Refined Neutrosophic Modal Logic.

Symbolic Neutrosophic Logic can be connected to the neutrosophic modal logic too, where instead of numbers we may use labels, or instead of quantitative neutrosophic logic we may have a quantitative neutrosophic logic. As an extension, we may introduce **Symbolic Neutrosophic Modal Logic** and **Refined Symbolic Neutrosophic Modal Logic**, where the symbolic neutrosophic modal operators (and the symbolic neutrosophic accessibility relation) have qualitative values (labels) instead on numerical values (subsets of the interval [0, 1]).

Applications of neutrosophic modal logic are to neutrosophic modal metaphysics. Similarly to classical modal logic, there is a plethora of neutrosophic modal logics. Neutrosophic modal logics is governed by a set of neutrosophic axioms and neutrosophic rules. The neutrosophic accessibility relation has various interpretations, depending on the applications. Similarly, the notion of possible neutrosophic worlds has many interpretations, as part of possible neutrosophic semantics.

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Uniform Single Valued Neutrosophic Graphs

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Abstract. In this paper, we propose a new concept named the uniform single valued neutrosophic graph. An illustrative example and some properties are examined. Next, we develop an algorithmic approach for computing the complement of the single valued neutrosophic graph. A numerical example is demonstrated for computing the complement of single valued neutrosophic graphs and uniform single valued neutrosophic graph.

Keywords: **S**ingle valued neutrosophic sets; Uniform single valued neutrosophic graph; Complement operators

1 Introduction

In 1965, Zadeh [7] originally introduced the concept of fuzzy set(FSs) which is characterized by a membership degree in [0, 1] for each element in the dataset. It may not always be true that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the truth- membership degree because there is some kind of hesitation degree. On the basis of fuzzy sets, Atanassov [4] added a non-membership in the definition of intuitionistic fuzzy sets (IFSs) and later Smarandache [2] introduced the neutrosophic sets (NSs) with the appearance of the truth-membership degree (T), the falsehood-membership degree (F), and the indeterminacy degree (I). Wang et al. [3] proposed various set theoretical operators and linked to single valued neutrosophic sets The concept of neutrosophic sets have been successfully applied to many fields [16].

Fuzzy graph has been studied extensively in the past years [5,8,9]. Later on, Smarandache [1] proposed neutrosophic graphs in some special types such as neutrosophic offgraph, neutrosophic bipolar/tripolar/ multipolar graph. Presently, works on neutrosophic vertex-edge graphs and neutrosophic edge graphs are progressing rapidly. Broumi et al.[13] introduced certain types of single valued neutrosophic graphs (in short SVNG) such as strong single valued neutrosophic graph, constant single valued neutrosophic graph, complete single valued neutrosophic graph with their properties and examples. Neighborhood degree of a vertex and closed neighborhood degree of vertex in single valued neutrosophic graph were introduced in [15]. The necessary and sufficient condition for a single valued neutrosophic graph to be an isolated single valued neutrosophic graph has been presented in [10]. Other extensions of the neutrosophic graph have been described in [11,12, 14].

Up to now, to the best of our knowledge, there has been no study on the uniform single valued neutrosophic graph. Thus, we propose in this paper a new concept named the uniform single valued neutrosophic graph. An illustrative example and some properties are examined. Next, we develop an algorithmic approach for computing the complement of the single valued neutrosophic graph.

The remainder of this paper is organized as follows. In Section 2, we present the basic definitions. In section 3, we introduce the concept of uniform single valued neutrosophic graph and investigate its properties. Section 4 introduces an algorithm for computing the complement of single valued neutrosophic graphs. A numerical example is presented in Section 5. Finally, Section 6 outlines the conclusion of this paper and suggests several directions for future research.

2 Preliminaries

In this section, we have present the basic definitions of fuzzy sets, neutrosophic sets, single valued neutrosophic sets, fuzzy graphs, uniform fuzzy graphs, complement of single valued neutrosophic graph which will be useful to our main work in the next sections.

Definition 1[1]. Let X be the universe of discourse and its elements denoted by x. In fuzzy theory, a fuzzy set

A of universe X is defined by the function $T_A(x)$, called the membership function of set A.

$$T_A: X \rightarrow [0, 1] \qquad (1)$$

For any element x of universe X, $T_A(x)$ equals the degree, between 0 and 1, to which x is an element of set A, This degree represents the membership value or degree of membership of element x in set A.

Definition 2[1]**.** Let X be a space of points and let x ∈ X. A **neutrosophic set** A in X is characterized by a truth membership function T, an indeterminacy membership function I, and a falsehood membership function F which are real standard or nonstandard subsets of]−0,1+[, and T, I, F: X→]−0,1+[. The neutrosophic set can be represented as,

$$A=\{(x, T_A(x), I_A(x), F_A(x)): x \in X\} \qquad (4)$$

There is no restriction on the sum of T, I, F, So

$$^{-}0 \leq T_A(x)+ I_A(x)+F_A(x) \leq 3^{+}. \qquad (5)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0,1^{+}[$. Thus it is necessary to take the interval [0,1] instead of $]^{-}0,1^{+}[$. For practical applications, it is difficult to apply $]^{-}0,1^{+}[$ in the real life applications such as engineering and scientific problems.

Definition 3[3]**.** Let X be a space of objects with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS) is characterized by truth-membership function $T_A(x)$, an indeterminate-membership function $I_A(x)$, and a falsehood-membership function $F_A(x)$. For each point x in X, $T_A(x), I_A(x), F_A(x) \in [0, 1]$. A SVNS A can be written as,

$$A=\{(x, T_A(x), I_A(x), F_A(x)): x \in X\} \quad (6)$$

Definition 4 [5]**.** ***A fuzzy graph*** is a pair of functions G = (σ, μ) where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ. i.e σ : V → [0,1] and μ: VxV→[0,1] such that μ(uv) ≤ σ(u) ⋀ σ(v) for all u, v ∈ V where uv denotes the edge between u and v and σ(u) ⋀ σ(v) denotes the minimum of σ(u) and σ(v). σ is called the fuzzy vertex set of V and μ is called the fuzzy edge set of E.

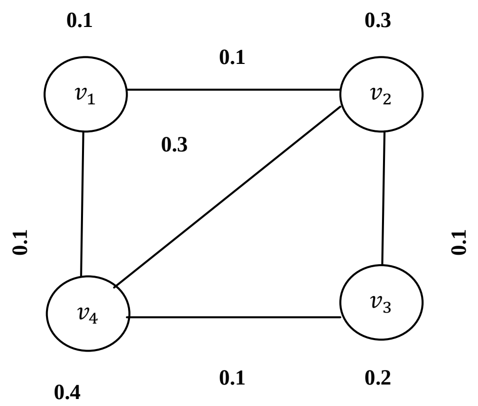


Fig.1. Fuzzy graph

Remark: The crisp graph G^* = (V, E) is a special case of the fuzzy graph G with each vertex and edge of (V, E) having degree of membership 1 (Fig. 1).

Definition5[6,8]**.** The complement of a fuzzy graph G = (σ, μ) is a fuzzy graph $\bar{G}$ = ($\bar{\sigma}$, $\bar{\mu}$) where $\bar{\sigma}$ =σ and $\bar{\mu}$(u,v) = σ(u)∧σ(v)-μ(u,v), ∀ u,v ∈ V.

Definition 6[6,8]. Let G = (σ, μ) be a fuzzy graph on a crisp graph G^* = (V, E). Let σ*={x∈ V | $\sigma(x)$> 0}.Then G is called a uniform fuzzy graph of level k if μ (x,y) = k, ∀ (x,y) ∈ (σ* × σ*) and $\sigma(x) = k$ where k isa positive real such that $0 < k_1 \leq 1$**.**

Definition 7[15].Let G = (V, E) be a single valued neutrosophic graph, then the **degree of a vertex** x_i is defined by $d_G(x_i)=d_G(x)=(d_T(x), d_I(x), d_F(x))$, $d_G(x_i) = (\sum_{x\neq y} T_B(x,y), \sum_{x\neq y} I_B(x,y), \sum_{x\neq y} I_B(x,y))$.

Definition 8[15].Let G = (V, E) be a single valued neutrosophic graph, then the **total degree of a vertex** x_i is defined by $td_G(x_i) = d_G(x) = (td_T(x), td_I(x), td_F(x))$, $td_G(x_i) = (\sum_{x\neq y} T_B(x,y) + T_A(x), \sum_{x\neq y} I_B(x,y) + I_A(x), \sum_{x\neq y} I_B(x,y) + F_A(x))$.

Definition 9[13]. Let G = (V, E) be a single valued neutrosophic graph, then the complement of single valued neutrosophic graph is defined as

1. $\bar{V}$ =V

2. $\overline{T_A}(x)=T_A(x)$, $\overline{I_A}(x) = I_A(x)$, $\overline{F_A}(x) =F_A(x)$ for all x∈V.

3. $\overline{T_B}(x,y)= \min [T_A(x), T(y)] - T_B(x,y)$

$\overline{I_B}(x,y)= \max [I_A(x), I_A(y)] - I_B(x,y)$ and

$\overline{F_B}(x,y)= \max [F_A(x), F_A(y)] - F_B(x,y)$, for all $(x,y) \in E$

Definition 10[13]. Let G = (V, E) be a single valued neutrosophic graph. If $d_G(x_i)= (k_1, k_2, k_3)$ for all $x_i \in$ V, then the single valued neutrosophic graph is called regular SVNG of degree (k_1, k_2, k_3)

Definition 11[13]. Let G = (V, E) be a single valued neutrosophic graph. If $td_G(x_i)$= (k_1, k_2, k_3) for all $x_i \in$ V, then the single valued neutrosophic graph is called Totally regular SVNG of degree (k_1, k_2, k_3)

III. Uniform Single Valued Neutrosophic Graph

In this section, we define the concept of uniform single valued neutrosophic graphs(in short USVNGs).

Definition 8. Let G = (A, B) be a single valued neutrosophic graph where A =(T_A, I_A, F_A) is a single valued neutrosophic vertex of G and B is a single valued neutrosophic edge set of G. Let A={x∈ V | $T_A(x)$> 0,$I_A(x)$> 0 and $F_A(x) > 0$}.Then G is called Uniform single valued neutrosophic graph of level (k_1, k_2, k_3) if T_B (x,y) = k_1,$I_A(x)$=$k_2$and F_B(x,y) = k_3∀ (x,y) ∈ (V× V) and $T_A(x)$ = k_1,$I_A(x) = k_2$and $F_A(x) = k_3$ where k_1, $k_2$and k_3 are some positive real such that 0 <k_1, k_2, k_3 ≤1.

Example 1. Consider an USVNG G= (A,B) on V={v_1,v_1,v_3,v_4} as shown in Fig.2.

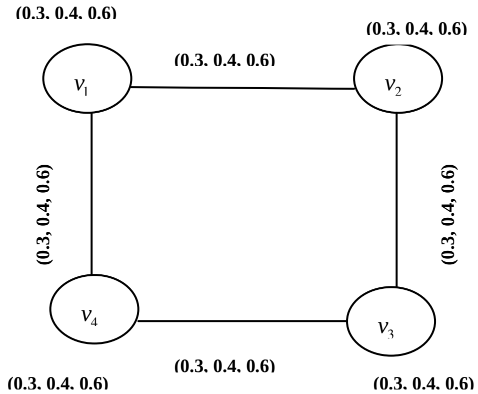


Fig. 2. USVNG.

Remark: The complement of an uniform single valued neutrosophic graph is always an empty graph.

Theorem1. If G = (A, B) is an uniform single valued neutrosophic graph of level (k_1, k_2, k_3) then G is a regular-USVNG.

Proof. Let A={x ∈ V | $T_A(x) >$ 0, $I_A(x) >$ 0 and $F_A(x) > 0$}. Suppose that G is a uniform single valued neutrosophic graph. Then T_B (x, y) = k_1,I_B (x, y) = k_2 and F_B (x,y) = k_3∀ (x,y) ∈ Eand $T_A(z)$ = k_1 , $I_A(z)$ = k_2 and $F_A(z) = k_3$∀ z ∈ V for some real k_1, $k_2$and k_3 where 0 <k_1, k_2, k_3 ≤1.

Let x ∈ V. Now $d_G(x)$=($d_T(x)$,$d_I(x)$, $d_F(x)$)

$$d_G(x) = (\sum_{x\neq y} T_B(\mathrm{x,y}), \sum_{x\neq y} I_B(\mathrm{x,y}), \sum_{x\neq y} F_B(\mathrm{x,y}))$$

=($\sum_{x\neq y} k_1$, $\sum_{x\neq y} k_2$, $\sum_{x\neq y} k_3$)

=((n-1) k_1, (n-1) k_2, (n-1) k_3)

$d_G(x)$=((n-1) k_1, (n-1) k_2, (n-1) k_3) ∀ x ∈ V

Therefore, G is regular uniform single valued neutrosophic graph.

Theorem 2. If G = (A, B) is a uniform single valued neutrosophic graph of level (k_1, k_2, k_3) then G is a totally regular- USVNG.

Proof. Let A={x ∈ V | $T_A(x) >$ 0, $I_A(x) >$ 0 and $F_A(x) > 0$}. Suppose that G is a uniform single valued neutrosophic graph. Then T_B (x, y) = k_1 ,I_B (x, y) = k_2 and F_B (x,y) = k_3∀ (x,y) ∈ Eand $T_A(z)$ = k_1 , $I_A(z)$ = k_2 and $F_A(z) = k_3$∀ z ∈ V for some real k_1, k_2 and k_3 where 0 <k_1, k_2, k_3 ≤1.Let x ∈ V. Now,

$td_G(x)$=($d_T(x) + T_A(x)$,$d_I(x) + I_A(x)$,$d_F(x) + F_A(x)$)

$$td_G(x) = (\sum_{x\neq y} T_B\,(\mathrm{x,y}) + T_A(x), \sum_{x\neq y} I_B(\mathrm{x,y}) + I_A(x), \sum_{x\neq y} F_B\,(\mathrm{x,y}) + F_A(x))$$

=(($\sum_{x\neq y} k_1$) + k_1, ($\sum_{x\neq y} k_2$) + k_2, ($\sum_{x\neq y} k_3$) + k_3)

=((n-1) k_1+k_1, (n-1) k_2+k_2, (n-1) k_3+k_3)

$td_G(x)$=(nk_1, n k_2, n k_3) ∀ x ∈ V.

Therefore, G is totally-regular uniform single valued neutrosophic graph.

Theorem 3. If G = (A, B) is a uniform single valued neutrosophic graph of level (k_1, k_2, k_3) on G^*=(V, E), then the order of G is O(G)= (nk_1, nk_2,nk_3).

Proof: Let A={x ∈ V | $T_A(x) >$ 0, $I_A(x) >$ 0 and $F_A(x) > 0$}. Suppose that G is a uniform single valued neutrosophic graph. Then T_B (x, y) = k_1 ,I_B (x, y) = k_2 and F_B (x,y) = k_3∀ (x,y) ∈ Eand $T_A(z)$ = k_1 , $I_A(z)$ = k_2 and $F_A(z) = k_3$∀ z ∈ V for some real k_1, k_2 and k_3 where 0 < k_1, k_2, k_3 ≤1.Let x ∈ V. Now

$O(G)$=($O_T(G)$,$O_I(G)$,$O_F(G)$)

$$O(G) = (\sum_{x\in V} T_A\,(\mathrm{x}), \sum_{x\in V} I_A(\mathrm{x}), \sum_{x\in V} f_A\,(\mathrm{x}))$$

=($\sum_{x\in V} k_1$, $\sum_{x\in V} k_2$, $\sum_{x\in V} k_3$)

Then,$O(G)$=(nk_1, n k_2, n k_3).

=($\sum_{\mathrm{x}\in\mathrm{V}} \mathrm{k}_1$, $\sum_{\mathrm{x}\in\mathrm{V}} \mathrm{k}_2$, $\sum_{\mathrm{x}\in\mathrm{V}} \mathrm{k}_3$)

Then,$O(G)$=(nk_1, n k_2, n k_3).

Theorem 4. The uniform single valued neutrosophic graph is a generalization of uniform fuzzy graph.

Proof: Straightforward.

IV. Computing Complement of Single Valued Neutrosophic Graph

In this section, we present in the last paper, a peudo-code of an algorithm computing the complement of single valued neutrosophic graph. This algorithm has the ability of computing the complement of fuzzy graphs, strong intuitionistic fuzzy graphs, uniform fuzzy graphs and also uniform single valued neutrosophic graphs.

The following flowchart demonstrates the algorithm to compute the complement operator is presented in Fig.3V.Numerical Example

In this section, we present an example to compute the complements of the uniform single valued neutrosophic graph. Consider a graph in Fig.4.

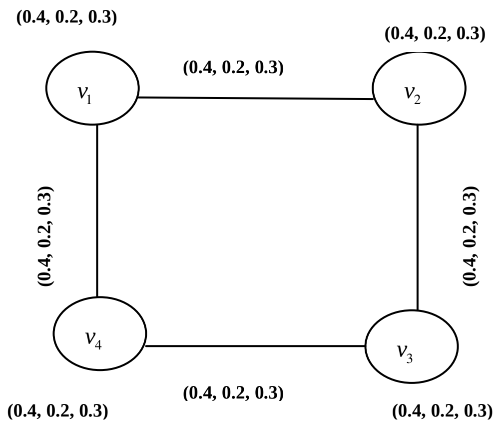
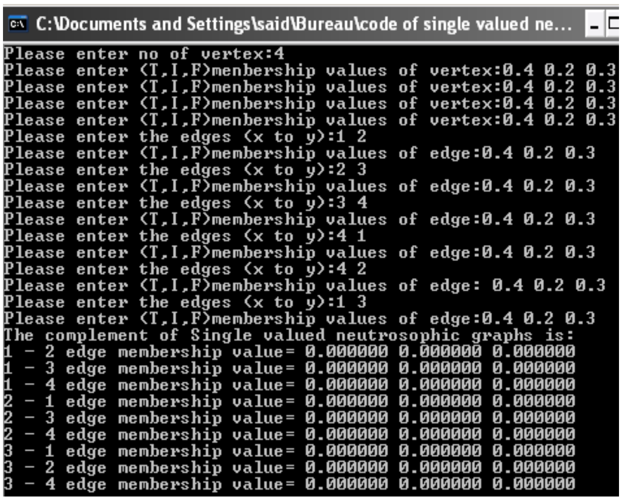


Fig. 4.A uniform single valued neutrosophic graph

Using the above pseudo code, the output result for the complement of a uniform single valued neutrosophic graph is in Fig. 5.



```
C:\Documents and Settings\said\Bureau\code of single valued ne...
Please enter no of vertex:4
Please enter (T,I,F)menbership values of vertex:0.4 0.2 0.3
Please enter (T,I,F)menbership values of vertex:0.4 0.2 0.3
Please enter (T,I,F)menbership values of vertex:0.4 0.2 0.3
Please enter (T,I,F)menbership values of vertex:0.4 0.2 0.3
Please enter the edges (x to y):1 2
Please enter (T,I,F)membership values of edge:0.4 0.2 0.3
Please enter the edges (x to y):2 3
Please enter (T,I,F)membership values of edge:0.4 0.2 0.3
Please enter the edges (x to y):3 4
Please enter (T,I,F)membership values of edge:0.4 0.2 0.3
Please enter the edges (x to y):4 1
Please enter (T,I,F)membership values of edge:0.4 0.2 0.3
Please enter the edges (x to y):4 2
Please enter (T,I,F)membership values of edge: 0.4 0.2 0.3
Please enter the edges (x to y):1 3
Please enter (T,I,F)membership values of edge:0.4 0.2 0.3
The complement of Single valued neutrosophic graphs is:
1 - 2 edge membership value= 0.000000 0.000000 0.000000
1 - 3 edge membership value= 0.000000 0.000000 0.000000
1 - 4 edge membership value= 0.000000 0.000000 0.000000
2 - 1 edge membership value= 0.000000 0.000000 0.000000
2 - 3 edge membership value= 0.000000 0.000000 0.000000
2 - 4 edge membership value= 0.000000 0.000000 0.000000
3 - 1 edge membership value= 0.000000 0.000000 0.000000
3 - 2 edge membership value= 0.000000 0.000000 0.000000
3 - 4 edge membership value= 0.000000 0.000000 0.000000
```

Fig. 5. The outputs

Example 2 Consider a fuzzy graph as shown in Fig.6

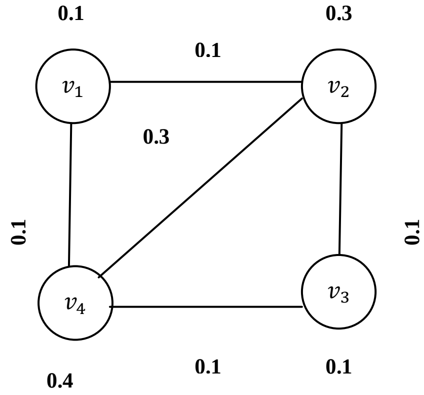
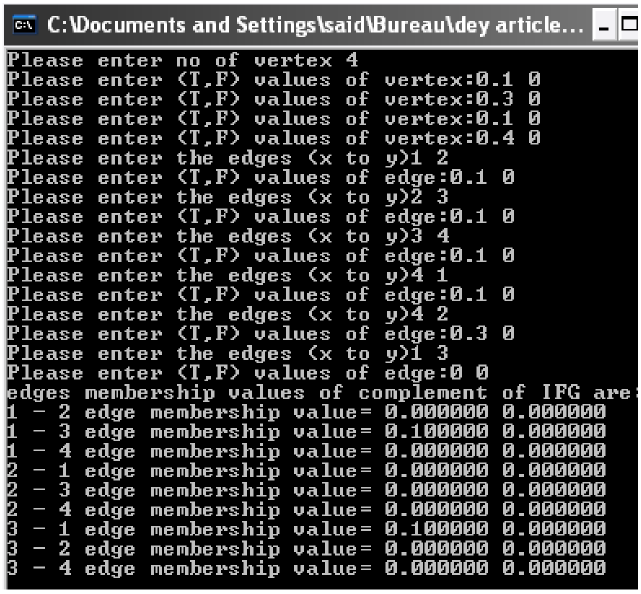


Fig. 6.Fuzzy graph

Using the above pseudo code, the output result for the complement of fuzzy graph is as follows:



```
C:\Documents and Settings\said\Bureau\dey article...
Please enter no of vertex 4
Please enter (T,F) values of vertex:0.1 0
Please enter (T,F) values of vertex:0.3 0
Please enter (T,F) values of vertex:0.1 0
Please enter (T,F) values of vertex:0.4 0
Please enter the edges (x to y)1 2
Please enter (T,F) values of edge:0.1 0
Please enter the edges (x to y)2 3
Please enter (T,F) values of edge:0.1 0
Please enter the edges (x to y)3 4
Please enter (T,F) values of edge:0.1 0
Please enter the edges (x to y)4 1
Please enter (T,F) values of edge:0.1 0
Please enter the edges (x to y)4 2
Please enter (T,F) values of edge:0.3 0
Please enter the edges (x to y)1 3
Please enter (T,F) values of edge:0 0
edges membership values of complement of IFG are:
1 - 2 edge membership value= 0.000000 0.000000
1 - 3 edge membership value= 0.100000 0.000000
1 - 4 edge membership value= 0.000000 0.000000
2 - 1 edge membership value= 0.000000 0.000000
2 - 3 edge membership value= 0.000000 0.000000
2 - 4 edge membership value= 0.000000 0.000000
3 - 1 edge membership value= 0.100000 0.000000
3 - 2 edge membership value= 0.000000 0.000000
3 - 4 edge membership value= 0.000000 0.000000
```

Example 3 Consider an uniform intuitionistic fuzzy graph as shown in Fig.7

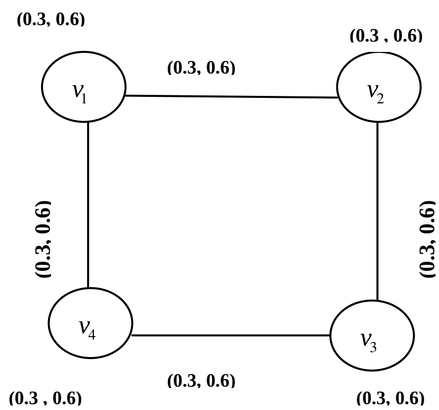
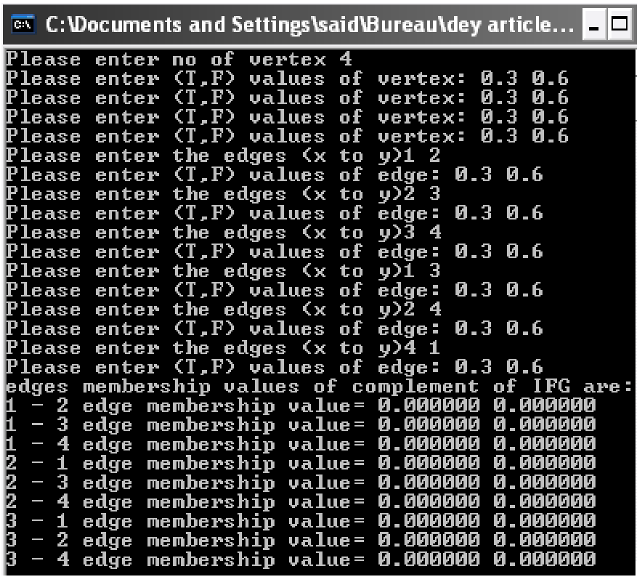


Fig.7. Uniform Intuitionistic fuzzy graph

Using the above pseudo code, the output result for the complement of uniform intuitionistic fuzzy graph is as follows


```
C:\Documents and Settings\said\Bureau\dey article...
Please enter no of vertex 4
Please enter (T,F) values of vertex: 0.3 0.6
Please enter (T,F) values of vertex: 0.3 0.6
Please enter (T,F) values of vertex: 0.3 0.6
Please enter (T,F) values of vertex: 0.3 0.6
Please enter the edges (x to y)1 2
Please enter (T,F) values of edge: 0.3 0.6
Please enter the edges (x to y)2 3
Please enter (T,F) values of edge: 0.3 0.6
Please enter the edges (x to y)3 4
Please enter (T,F) values of edge: 0.3 0.6
Please enter the edges (x to y)1 3
Please enter (T,F) values of edge: 0.3 0.6
Please enter the edges (x to y)2 4
Please enter (T,F) values of edge: 0.3 0.6
Please enter the edges (x to y)4 1
Please enter (T,F) values of edge: 0.3 0.6
edges membership values of complement of IFG are:
1 - 2 edge membership value= 0.000000 0.000000
1 - 3 edge membership value= 0.000000 0.000000
1 - 4 edge membership value= 0.000000 0.000000
2 - 1 edge membership value= 0.000000 0.000000
2 - 3 edge membership value= 0.000000 0.000000
2 - 4 edge membership value= 0.000000 0.000000
3 - 1 edge membership value= 0.000000 0.000000
3 - 2 edge membership value= 0.000000 0.000000
3 - 4 edge membership value= 0.000000 0.000000
```

VI. Conclusion

In this paper, we propose a new uniform single valued neutrosophic graph and an algorithm for computing its complement. Some theorems of the uniform single valued neutrosophic graph have been examined. The algorithm in this research also enables us to compute the complement of uniform single valued neutrosophic graph. In the future, we plan to extended this algorithm for computing the complement of others variants of single valued neutrosophic graphs.

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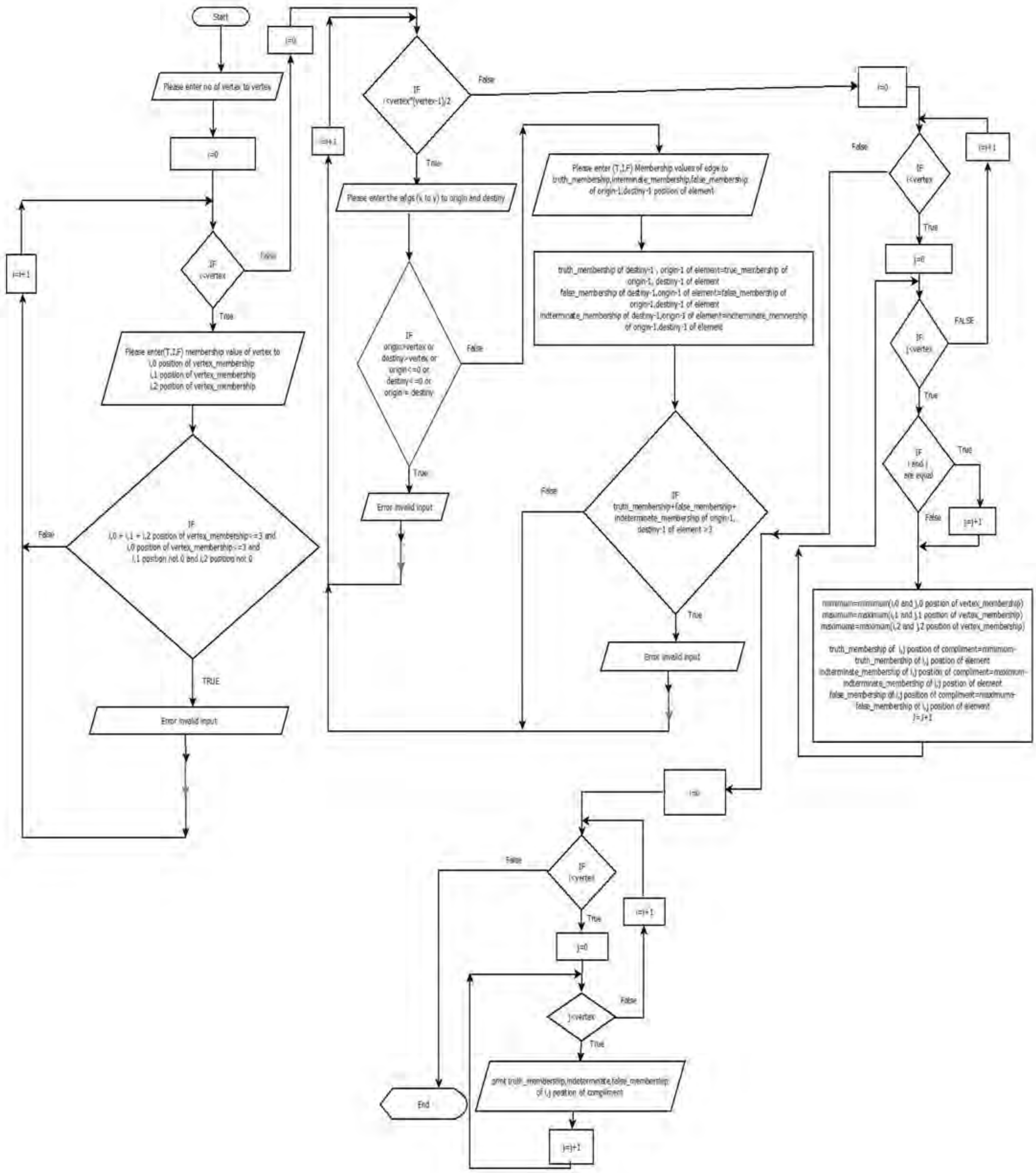
[16] More information on http://fs.gallup.unm.edu/NSS/.

Appendix

```
#include<stdio.h>
#include<conio.h>
#define max 100
typedef struct {
float
Truth_membership,Indterminate_membership,False_membership;
}fuzzy;
fuzzy
element[max][max],compliment[max][max];//element store the membership value of vertex.Compliment store the value of complimented graph.
int vertex;//store total number of vertex.
```



```
float vertex_membership[max][6];//store membership value of vertex.
void input()
{
int i,origin,destiny;//origin & destiny store the no. of vertex.And i for iteration.
        printf("Please enter no of vertex:");
        scanf("%d",&vertex);
        for(i=0;i<vertex;i++)
        {
        printf("Please enter (T,I,F)menbership values of vertex:");
        scanf("%f%f%f",&vertex_membership[i][0],&vertex_membership[i][1],&vertex_membership[i][2]);//store the membership value of vertex
        if(vertex_membership[i][0]+vertex_membership[i][1]+vertex_membership[i][2]>=3&&(vertex_membership[i][0]<=3&&vertex_membership[i][1]&&vertex_membership[i][2]))
        {
        printf("Error Invalid input\n");
        i--;
        }
        }
        for(i=0;i<vertex*(vertex-1)/2;i++)
        {
        printf("Please enter the edges (x to y):");
        scanf("%d%d",&origin,&destiny);

        if(origin>vertex||destiny>vertex||origin<=0||destiny<=0||destiny==origin)
        {
          printf("Error! Invalid input\n");
                i--;
        }
             else
        {
        printf("Please enter (T,I,F)membership values of edge:");
scanf("%f%f%f",&element[origin-1][destiny-1].Truth_membership,&element[origin-1][destiny-1].Indterminate_membership,&element[origin-1][destiny-1].False_membership);//store th membership value of edge.

element[destiny-1][origin
1].Truth_membership=element[origin-1][destiny-1].Truth_membership;//store the truth-membership value of edge.
element[destiny-1][origin-1].Indterminate_membership=element[origin-1][destiny-1].Indterminate_membership;//store the indterminate-membership value of edge.
element[destiny-1][origin-1].False_membership=element[origin-1][destiny-1].False_membership;//store the False-membership value of edge.
if(element[origin-1][destiny-1].Truth_membership+element[origin-1][destiny-1].Indterminate_membership+element[origin-1][destiny-1].False_membership>3)//store the membership value of edge.
{
printf("Error! Invalid input\n");
i--;
}
}
}


}
void output()
{
        int i,j;
        float maximum,minimum,maximuma;
        printf("The complement of Single valued neutrosophic graphs is:\n");
        for(i=0;i<vertex;i++)
{
        for(j=0;j<vertex;j++)
{
        if(i==j)
        j++;
if(vertex_membership[i][0]>vertex_membership[j][0])
minimum=vertex_membership[j][0];//find minimum value between two vertex.
else
minimum=vertex_membership[i][0];//find minimum value between two vertex.
if(vertex_membership[i][1]>vertex_membership[j][1])
maximum=vertex_membership[i][1];//find maximum value between two vertex.
else
maximum=vertex_membership[j][1];//find maximum value between two vertex.
```

```
if(vertex_membership[i][2]>vertex_membership[j][2])
maximuma=vertex_membership[i][2];//find maximum value between two vertex.
else
maximuma=vertex_membership[j][2];//find maximum value between two vertex.
compliment[i][j].Truth_membership=minimum-element[i][j].Truth_membership;//calculating compliment.
compliment[i][j].Indterminate_membership=maximum-element[i][j].Indterminate_membership;//calculating compliment.
compliment[i][j].False_membership=maximuma-element[i][j].False_membership;//calculating compliment.
}
}
        for(i=0;i<vertex-1;i++)
{
         for(j=0;j<vertex;j++)
{
         if(i==j)
 j++;
printf("%d - %d edge membership value= %f %f %f \n",i+1,j+1,compliment[i][j].Truth_membership,compliment[i][j].Indterminate_membership,compliment[i][j].False_membership);//printing complimented graph.
}
}
}
void main()
{
        input();
        output();
        getch();
           }
```

Extension of Crisp Functions on Neutrosophic Sets

Sabu Sebastian, Florentin Smarandache

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Abstract. In this paper, we generalize the definition of Neutrosophic sets and present a method for extending crisp functions on Neutrosophic sets and study some properties of such extended functions.

Keywords: Neutrosophic set, Multi-fuzzy set, Bridge function..

1 Introduction

L-fuzzy sets constitute a generalization of the notion of Zadeh's [26] fuzzy sets and were introduced by Goguen [8] in 1967, later Atanassov introduced the notion of the intuitionistic fuzzy sets [1] Gau and Buehrer [7] defined vague sets. Bustince and Burillo [2] showed that the notion of vague sets is the same as that of intuitionistic fuzzy sets. Deschrijver and Kerre [5] established the interrelationship between the theories of fuzzy sets, L-fuzzy sets, interval valued fuzzy sets, intuitionistic fuzzy sets, intuitionistic L-fuzzy sets, interval valued intuitionistic fuzzy sets, vague sets and gray sets [4].

2 Preliminaries

Definition 2.1. [26] Let X be a nonempty set. A fuzzy set A of X is a mapping $A : X \to [0, 1]$, that is,

$A = \{(x, \mu_A(x)) : \mu_A(x)$ is the grade of membership of x in A, $x \in X\}$. The set of all the fuzzy sets on X is denoted by $\mathcal{F}(X)$.

Definition 2.2. [8] Let X be a nonempty ordinary set, L a complete lattice. An L-fuzzy set on X is a mapping $A : X \to L$, that is the family of all the L-fuzzy sets on X is just L^X consisting of all the mappings from X to L.

Definition 2.3. [1] An Intuitionistic Fuzzy Set on X is a set

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where $\mu_A(x) \in [0, 1]$ denotes the membership degree and $\nu_A(x) \in [0, 1]$ denotes the non-membership degree of x in A and

$$\mu_A(x) + \nu_A(x) \leq 1, \forall x \in X.$$

The neutrosophic set (NS) was introduced by F. Smarandache [22] who introduced the degree of indeterminacy (i) as independent component in his manuscripts that was published in 1998.

Multi-fuzzy sets [12, 13, 16] was proposed in 2009 by Sabu Sebastian as an extension of fuzzy sets [8, 26] in terms of multi membership functions. In this paper we generalize the definition of neutrosophic sets and introduce extension of crisp functions on neutrosophic sets.

Definition 2.4. [22]A Neutrosophic Set on X is a set

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\},$$

where $T_A(x) \in [0, 1]$ denotes the truth membership degree, $I_A(x) \in [0, 1]$ denotes the indetermi-nancy membership degree and $F_A(x) \in [0, 1]$ denotes the falsity membership degree of x in A respectively and

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3, \forall x \in X.$$

For single valued neutrosophic logic (T, I, F), the sum of the components is: $0 \leq T + I + F \leq 3$ when all three components are independent; $0 \leq T + I + F \leq 2$ when two components are dependent, while the third one is independent from them; $0 \leq T + I + F \leq 1$ when all three components are dependent.

Definition 2.5. [12, 13, 16]Let X be a nonempty set, J be an indexing set and $\{L_j : j \in J\}$ a family of partially ordered sets. A **multi-fuzzy set A** in X is a set :

$$\mathbf{A} = \{\langle x, (\mu_j(x))_{j\in J}\rangle : x \in X,\ \mu_j \in L_j^X,\ j \in J\}.$$

The indexing set J may be uncountable. The function $\mu_{\mathbf{A}} = (\mu_j)_{j\in J}$ is called the membership function of the multi-fuzzy set **A** and $\prod_{j\in J} L_j$ is called the value domain.

If J = {1, 2, ..., n} or the set of all natural numbers, then the membership function $\mu_{\mathbf{A}} = \langle \mu_1, \mu_2, ...\rangle$ is a sequence.

In particular, if the sequence of the membership function having precisely n-terms and $L_j = [0, 1]$, for $J = \{1, 2, ..., n\}$, then n is called the dimension and $\mathbf{M^nFS}(X)$ denotes the set of all multi-fuzzy sets in X.

Properties of multi-fuzzy sets, relations on multi-fuzzy sets and multi-fuzzy extensions of crisp functions are depend on the order relations defined in the membership functions. Most of the results in the initial papers [12, 13, 15, 16, 18] are based on product order in the membership functions. The paper [21] discussed other order relations like dictionary order, reverse dictionary order on their membership functions.

Let $\{L_j : j \in J\}$ be a family of partially ordered sets, and $\mathbf{A} = \{\langle x, (\mu_j(x))_{j\in J}\rangle : x \in$ X, $\mu_j \in L_j^X$, $j \in J\}$ and $\mathbf{B} = \{\langle x, (\nu_j(x))_{j\in J}\rangle : x \in X$, $\nu_j \in L_j^X$, $j \in J\}$ be multi-fuzzy sets in a nonempty set X. Note that, if the order relation in their membership functions are either product order, dictionary order or reverse dictionary order[16, 21], then;

- $\mathbf{A} = \mathbf{B}$ if and only if $\mu_j(x) = \nu_j(x), \forall x \in X$ and for all $j \in J$
- $\mathbf{A} \sqcup \mathbf{B} = \{\langle x, (\mu_j(x) \vee_j \nu_j(x))_{j\in J}\rangle : x \in X\}$ and
- $\mathbf{A} \sqcap \mathbf{B} = \{\langle x, (\mu_j(x) \wedge_j \nu_j(x))_{j\in J}\rangle : x \in X\}$,

where $\vee_j$ and $\wedge_j$ are the supremum and infimum defined in L_j with partial order relation $\leq_j$. Set inclusion defined as follows:

- In product order, $\mathbf{A} \subset \mathbf{B}$ if and only if $\mu_j(x) < \nu_j(x), \forall x \in X$ and for all $j \in J$.
- In dictionary order, $A \subset B$ if and only if $\mu_1(x) < \nu_1(x)$ or if $\mu_1(x) = \nu_1(x)$ and $\mu_2(x) < \nu_2(x), \forall x \in X$.

Definition 2.6. *Let L be a lattice. A mapping $'$: $L \to L$ is called an order reversing involution [25], if for all $a, b \in L$:*

1. $a \leq b \Rightarrow b' \leq a'$;
2. $(a')' = a$.

Definition 2.7. [23] Let $'$: $M \to M$ and $'$: $L \to L$ be order reversing involutions. A mapping $h : M \to L$ is called an order homomorphism, if it satisfies the conditions:

1. $h(0_M) = 0_L$;
2. $h(\vee a_i) = \vee h(a_i)$;
3. $h^{-1}(b') = (h^{-1}(b))'$,

where $h^{-1} : L \to M$ is defined by, for every $b \in L$,
$h^{-1}(b) = \vee\{a \in M : h(a) \leq b\}$.

Generalized Zadeh extension of crisp functions [24] have prime importance in the study of fuzzy mappings. Sabu Sebastian [16, 13]generalized this concept as multi-fuzzy extension of crisp functions and it is useful to map a multi-fuzzy set into another multi-fuzzy set. In the case of a crisp function, there exists infinitely many multi-fuzzy extensions, even though the domain and range of multi-fuzzy extensions are same.

Definition 2.8. [16] Let $f : X \to Y$ and $h : \prod M_i \to \prod L_j$ be a functions. The multi-fuzzy extension of f and the inverse of the extension are $f : \prod M_i^X \to \prod L_j^Y$ and $f^{-1} : \prod L_j^Y \to \prod M_i^X$ defined by

$$f(A)(y) = \bigvee_{y=f(x)} h(A(x)),\ A \in \prod M_i^X,\ y \in Y$$

and

$$f^{-1}(B)(x) = h^{-1}(B(f(x))),\ B \in \prod L_j^Y,\ x \in X;$$

where h^{-1} is the upper adjoint [23] of h. The function $h : \prod M_i \to \prod L_j$ is called the **bridge function** of the multi-fuzzy extension of f.

Remark 2.9. In particular, the multi-fuzzy extension of a crisp function $f : X \to Y$ based on the bridge function $h : I^k \to I^n$ can be written as $f : \mathbf{M^kFS}(X) \to \mathbf{M^nFS}(Y)$ and $f^{-1} : \mathbf{M^nFS}(Y) \to \mathbf{M^kFS}(X)$, where

$$f(A)(y) = \sup_{y=f(x)} h(A(x)),\ A \in \mathbf{M^kFS}(X),\ y \in Y$$

and

$$f^{-1}(B)(x) = h^{-1}(B(f(x))), B \in \mathbf{M^nFS}(Y), x \in X.$$

In the following section $\prod M_i = \prod L_j = I^3$.

Remark 2.10. There exists infinitely many bridge functions. Lattice homomorphism, order homomorphism, lattice valued fuzzy lattices and strong L-fuzzy lattices are examples of bridge functions.

Definition 2.11. [10] A function $t : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm if $\forall a, b, c \in [0, 1]$: **(1)** $t(a, 1) = a$;

(2) $t(a, b) = t(b, a)$;

(3) $t(a, t(b, c)) = t(t(a, b), c)$;

(4) $b \le c$ implies $t(a, b) \le t(a, c)$.

Similarly, a t-conorm (s-norm) is a commutative, associative and non-decreasing mapping $s : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies the boundary condition:

$s(a, 0) = a$, for all $a \in [0, 1]$.

Definition 2.12. [9] A function $c : [0, 1] \to [0, 1]$ is called a complement (fuzzy) operation, if it satisfies the following conditions:

(1) $c(0) = 1$ and $c(1) = 0$,

(2) for all $a, b \in [0, 1]$, if $a \le b$, then $c(a) \ge c(b)$.

Definition 2.13. [9] A t-norm t and a t-conorm s are dual with respect to a fuzzy complement operation c if and only if

$c(t(a, b)) = s(c(a), c(b))$

and

$c(s(a, b)) = t(c(a), c(b))$,

for all $a, b \in [0, 1]$.

Definition 2.14. [9] Let n be an integer greater than or equal to 2. A function $m : [0, 1]^n \to [0, 1]$ is said to be an aggregation operation for fuzzy sets, if it satisfies the following conditions:

1. m is continuous;
2. m is monotonic increasing in all its arguments;
3. $m(0, 0, ..., 0) = 0$;
4. $m(1, 1, ..., 1) = 1$.

3 Neutrosophic Sets

In this section, we generalize the definition of neutrosophic sets on [0, 1]. Throughout the following sections $X$is the universe of discourse and $A \in \mathbf{M^3FS}(X)$ means A is a multi-fuzzy sets of dimension 3 with value domain I^3, where $I^3 = [0, 1] \times [0, 1] \times [0, 1]$. That is, $A \in (I^3)^X$.

Definition 3.1. Let X be a nonempty crisp set and $0 \le \alpha \le 3$. A multi-fuzzy set $A \in \mathbf{M^3FS}(X)$is called a neutrosophic set of order α, if

$\mathbf{A} = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X,$
$0 \le T_A(x) + I_A(x) + F_A(x) \le \alpha\}$.

Definition 3.2. Let A, B be neutrosophic sets in X of order 3 and let t, s, m, c be the t-norm, s-norm, aggregation operation and complement operation respectively. Then the union, intersection and complement are given by

1. $A \bigcup B = \{\langle x, s(T_A(x), T_B(x)), m(I_A(x), I_B(x)), t(F_A(x), F_B(x))\rangle : x \in X\}$;
2. $A \bigcap B = \{\langle x, t(T_A(x), T_B(x)), m(I_A(x), I_B(x)), s(F_A(x), F_B(x))\rangle : x \in X\}$;
3. $A^c = \{\langle x, c(T_A(x)), c(I_A(x)), c(F_A(x))\rangle : x \in X\}$.

4 Extension of crisp functions on neutrosophic set using order homomorphism as bridge function

Theorem 4.1. If an order homomorphism $h : I^3 \to I^3$ is the bridge function for the multi-fuzzy extension of a crisp function $f : X \to Y$, then for every $k \in K$ neutrosophic sets A_k in X and B_k in Y of order 3;

1. $A_1 \subseteq A_2$ implies $f(A_1) \subseteq f(A_2)$;
2. $f(\cup A_k) = \cup f(A_k)$;
3. $f(\cap A_k) \subseteq \cap f(A_k)$;
4. $B_1 \subseteq B_2$ implies $f^{-1}(B_1) \subseteq f^{-1}(B_2)$;
5. $f^{-1}(\cup B_k) = \cup f^{-1}(B_k)$;
6. $f^{-1}(\cap B_k) = \cap f^{-1}(B_k)$;
7. $(f^{-1}(B))' = f^{-1}(B')$;
8. $A \subseteq f^{-1}(f(A))$;
9. $f(f^{-1}(B)) \subseteq B$.

Proof.

1. $A_1 \subseteq A_2$ implies $A_1(x) \leq A_2(x), \forall x \in X$ and implies
$h(A_1(x)) \leq h(A_2(x)), \forall x \in X$.

Hence

$$\vee\{h(A_1(x)) : x \in X, y = f(x)\} \leq \vee\{h(A_2(x)) : x \in X, y = f(x)\}$$ and $f(A_1)(y) \leq f(A_2)(y)$, $\forall y \in Y$. That is, $f(A_1) \subseteq f(A_2)$.

2. For every $y \in Y$,

$$\begin{aligned} f(\cup A_k)(y) &= \vee\{h((\cup A_k)(x)) : x \in X, y = f(x)\} \\ &= \vee\{h(\vee A_k(x)) : x \in X,\ y = f(x)\} \\ &= \vee\{\vee_{k\in K} h(A_k(x)) : x \in X,\ y = f(x)\} \\ &= \vee_{k\in K} \vee \{h(A_k(x)) : x \in X,\ y = f(x)\} \\ &= \vee_{k\in K} f(A_k)(y), \end{aligned}$$

thus $f(\cup A_k) = \cup f(A_k)$.

3. For every $y \in Y$,

$$\begin{aligned} f(\cap A_k)(y) &= \vee\{h((\cap A_k)(x)) : x \in X, y = f(x)\} \\ &= \vee\{h(\wedge_{k\in K} A_k(x)) : x \in X,\ y = f(x)\} \\ &\leq \vee\{h(A_k(x)) : x \in X,\ y = f(x)\}, \end{aligned}$$

for each $k \in K$. Hence

$f(\cap A_k)(y) \leq \wedge_{k\in K} \vee \{h(A_k(x)) : x \in X, y = f(x)\} = \wedge_{k\in K} f(A_k)(y)$,

thus $f(\cap A_k) \subseteq \cap f(A_k)$.

4. $B_1 \subseteq B_2$ implies $B_1(y) \leq B_2(y), \forall y \in Y$. Hence

$$f^{-1}(B_1)(x) = h^{-1}(B_1(f(x))) \leq h^{-1}(B_2(f(x))) = f^{-1}(B_2)(x), \forall x \in X.$$

Therefore, $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

5. For every $x \in X$, we have

$$\begin{aligned} f^{-1}(\cup B_k)(x) &= h^{-1}((\cup B_k)(f(x))) = h^{-1}(\sup_{k\in K} B_k(f(x))) \\ &= \sup_{k\in K} h^{-1}(B_k(f(x))) = \sup_{k\in K} f^{-1}(B_k)(x) \\ &= (\cup f^{-1}(B_k))(x). \end{aligned}$$

Hence $f^{-1}(\cup B_k) = \cup f^{-1}(B_k)$.

6. For every $x \in X$, we have

$$\begin{aligned} f^{-1}(\cap B_k)(x) &= h^{-1}((\cap B_k)(f(x))) = h^{-1}(\inf_{k\in K} B_k(f(x))) \\ &= \inf_{k\in K} h^{-1}(B_k(f(x))) = \inf_{k\in K} f^{-1}(B_k)(x) \\ &= (\cap f^{-1}(B_k))(x). \end{aligned}$$

Hence $f^{-1}(\cap B_k) = \cap f^{-1}(B_k)$.

7. For every $x \in X$,

$$f^{-1}(B')(x) = h^{-1}(B'(f(x))) = h^{-1}(B(f(x)))' = (f^{-1}(B))'(x),$$ since $f^{-1}(B)(x) = h^{-1}(B(f(x)))$.

That is, $f^{-1}(B') = (f^{-1}(B))'$.

8. For every $x_0 \in X$,

$$\begin{aligned} A(x_0) &\leq \vee\{A(x) : x \in X,\ x \in f^{-1}(f(x_0)\} \\ &\leq h^{-1}(h(\vee\{A(x) : x \in X,\ x \in f^{-1}(f(x_0)\})) \\ &= h^{-1}(\vee\{h(A(x)) : x \in X,\ x \in f^{-1}(f(x_0))\}) \\ &= h^{-1}(f(A)(f(x_0))) \\ &= f^{-1}(f(A))(x_0). \end{aligned}$$

9. For every $y \in Y$

$$\begin{aligned} f(f^{-1}(B))(y) &= \sup_{y=f(x)} h(f^{-1}(B)(x)) \\ &= \sup_{y=f(x)} h(h^{-1}(B(f(x)))) \end{aligned}$$

$$= \quad h(h^{-1}(B(y)))$$
$$\leq \quad B(y).$$

Hence $f(f^{-1}(B)) \subseteq B$.

Proposition 4.2. If an order homomorphism $h : I^3 \to I^3$ is the bridge function for the extension of a crisp function $f : X \to Y$, then for any $k \in K$ neutrosophic sets A_k in X and B in Y:

1. $f(0_X) = 0_Y$;
2. $f(\cup A_k) = \cup f(A_k)$; and
3. $(f^{-1}(B))' = f^{-1}(B')$,

that is, the extension map f is an order homomorphism.

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Computing Operational Matrices in Neutrosophic Environments: A Matlab Toolbox

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Abstract. Neutrosophic set is a generalization of classical set, fuzzy set, and intuitionistic fuzzy set by employing a degree of truth (T), a degree of indeterminacy (I), and a degree of falsehood (F) associated with an element of the dataset. One of the most essential problems is studying set-theoretic operators in order to be applied to practical applications. Developing Matlab toolboxes for computing the operational matrices in neutrosophic environments is essential to gain more widely-used of neutrosophic algorithms. In this paper, we propose some computing procedures in Matlab for neutrosophic operational matrices, especially i) computing the single-valued neutrosophic matrix; ii) determining complement of a single-valued neutrosophic matrix; iii) computing max-min-min and min-max-max of two single-valued neutrosophic matrices; v) computing power of a single-valued neutrosophic matrix; vi) computing additional operation and subtraction of two single-valued neutrosophic matrices; and ix) computing transpose of a single-valued neutrosophic matrix. Numerical examples are given to illustrate their applicability.

Keywords: Matlab toolbox; Neutrosophic set; Single valued neutrosophic matrices; Set-theoretic operators

1 Introduction

There are many evidences in complex systems that an event or phenomenon cannot be modeled by a classical set [11,18]. For instance, the Schrödinger's Cat Theory says that the quantum state of a photoncan basically be in more than one place in the same time, which means that an element (quantum state) belongs and does not belong to a set (one place) inthe same time; or an element (quantum state) belongs to two different sets (two different places)in the same time [24]. Again, it is hard to judge the truth-value of a metaphor, or of an ambiguous statement, or of a social phenomenon which is positive from a standpoint and negative from another standpoint [24]. The classical mathematics does not practice any kind ofuncertainty in its tools, excluding possibly the case of probability, where it can handle a particular kind of uncertainty called randomness [11]. Therefore new techniques and modification of classical tools arerequired to model such uncertain phenomena [9]. Neutrosophic set (NS) [33] is a generalization of classical set, fuzzy set, and intuitionistic fuzzy set by employing a degree of truth (T), a degree of indeterminacy (I), and a degree of falsehood (F) associated with an element of the dataset proposed in 1998 by Smarandache. It has been successfully applied to many fields such as control theory[1], databases [4,5], medical diagnosis [7], decision making [23],topology [27]and graph theory [12-21].

NS has many advantages over other preceding sets. Specifically, triangular fuzzy numbers (TFNs) and neutrosophic numbers (NNs) are both generalizations of fuzzy numbers that are each characterized by three components [33]. TFNs and NNs have been widely used to represent uncertain and vague information in various areas such as engineering, medicine, communication science and decision science. However, NNs are far more accurate and convenient to be used to represent the uncertainty and hesitancy that exists in information, as compared to TFNs. NNs are characterized by three components, each of which clearly represents the degree of truth membership, indeterminacy membership and falsity membership of a NN with respect to an attribute. Therefore, we are able to tell the belongingness of the NN to the set of attributes that are being studied, by just looking at its structure. This provides a clear, concise and comprehensive method of representation of the different components of the membership of the number. This is in contrast to the structure of the TFN which only provides us with the maximum, minimum and initial values of the TFN, all of which can only tell us the path of the TFN, but does not tell us anything about the degree of non-belongingness of the TFN with respect to the set of attributes that are being studied. Furthermore, the structure of the TFN is not able to capture the hesitancy that naturally exists within the user in the process of assigning membership values. These reasons clearly show the advantages of NNs compared to TFNs.

One of the most essential problems in NS is studying set-theoretic operators (or operational matrices) in order to be applied to practical applications. Smarandache [33] and Wang et al.[41]proposed the concept of single-valued neutrosophic set and provided its set-theoretic operations and properties. Broumi and Smarandache [10] proposed some operations on interval neutrosophic sets (INSs) and studied their properties. Ye [43] defined the similarity measures between INSs on the basis of the hamming and Euclidean distances. Some set theoretic operations such as union, intersection and complement have also been proposed by Wang et al. [42].Broumi and Smarandache [8] also defined the correlation coefficient of interval neutrosophic set.Liu and Tang [26] presented some new operational laws for interval neutrosophic sets and studied their properties. More recent works on operational law and applications can be retrieved in [9, 24-26, 34, 44-45,47-50].

In practical point of view, developing Matlab toolboxes for computing the operational matrices in neutrosophic environments is essential to gain more widely-used of neutrosophic algorithms and methods. Zahariev [46] presented a new software package for fuzzy calculus in MATLAB environment whose main feature is solving fuzzy linear systems of equations and inequalities in fuzzy algebra. Peeva and Kyosev[30] developed a library for fuzzy relational calculus over the fuzzy algebra([0,1], max,min). The library includes various operations and compositions with fuzzy relation and intuitionistic fuzzy solving direct and inverse problem. Recently, Mumtaz et al. [3] implemented some functions in MATLAB for computing algebraic neutrosophic measures in medical diagnosis. Ashbacher [6] analyzed and developed some computing procedures for neutrosophic operations.Albeanu [2] described some neutrosophic computational models in order to identify a set of requirement for software implementation. Salama et al. [32] developed an Excel package for calculating neutrosophic data and analyzed them. Karunambigai and Kalaivani [22] developed a MATLAB program for computing power of an intuitionistic fuzzy matrix, strength of connectedness and index matrix of intuitionistic fuzzy graphs with suitable examples.

However, the existing Fuzzy Toolboxes in MATLAB does not propose options to evaluate the operations in neutrosophic environments. Thus, in this paper, we propose some computing procedures in Matlab for neutrosophic operational matrices, especially i) computing the single-valued neutrosophic matrix; ii) determining complement of a single-valued neutrosophic matrix; iii) computing max-min-min of two single-valued neutrosophic matrices; iv) computing min-max-max of two single-valued neutrosophic matrices; v) computing power of a single-valued neutrosophic matrix; vi) computing additional operation of two single-valued neutrosophic matrices; vii) computing subtraction of two single-valued neutrosophic matrices; and viii) computing transpose of a single-valued neutrosophic matrix. In order to illustrate their applicability, numerical examples are given and discussed.

The rest of this paper is organized as follows. Section 2 recalls some basic concepts of Neutrosophic Set. Section 3 presents the computing procedures in Matlab. Section 4 describes the numerical examples. Section 5 delineates conclusions and further studies of this research.

2 Fundamental and Basic Concepts

Definition 1[31]. ***Neutrosophic Set(NS)***

Let X be a space of points and let $x \angle X$. A neutrosophic set $\overline{S}$ in X is characterized by a truth membership function $T_{\overline{S}}$, an indeterminacy membership function $I_{\overline{S}}$, and a falsehood membership function $F_{\overline{S}}$. $T_{\overline{S}}$, $I_{\overline{S}}$ and $F_{\overline{S}}$ are real standard or non-standard subsets of $\rfloor 0^0, 1^{\cdot} \lfloor$. The neutrosophic set can be represented as

$$\overline{S} \cong \sim\!\!+ x, T_{\overline{S}} + x\,,, I_{\overline{S}} + x\,,, F_{\overline{S}} + x\,,,: x \angle X \Upsilon$$

The sum of $T_{\tilde{S}}(x)$, $I_{\tilde{S}}(x)$, and $F_{\tilde{S}}(x)$, is $0^0 \infty T_{\tilde{S}}(x) ,. I_{\tilde{S}}(x) ,. F_{\tilde{S}}(x) ,\infty 3$.

To use neutrosophic set in the real life applications such as engineering and scientific problems, it is necessary to consider the interval $[0,1]$ instead of $]0^0,1^{\cdot}[$ for technical applications.

Definition 2 [31]. Let $\tilde{A}_1=(T_1,I_1,F_1)$ and $\tilde{A}_2=(T_2,I_2,F_2)$ betwo single-valued neutrosophicnumbers. Then, the operations for NNs are defined as below:

(i) $\tilde{A}_1 \oplus \tilde{A}_2=(T_1+T_2\,0\,T_1T_2, I_1I_2, F_1F_2)$

(ii) $\tilde{A}_1 \otimes \tilde{A}_2=(T_1T_2, I_1 .\ I_2\,0\,I_1I_2, F_1 .\ F_2\,0\,F_1F_2)$

(iii) $o\tilde{A}\ =(1-(1\,0\,T_1)^o), I_1^o, F_1^o)$

(iv) $\tilde{A}_1^{o}=(T_1^{o}, 1\,0\,(1\,0\,I_1)^o, 1\,0\,(1\,0\,F_1)^o)$ where o A0

Definition3[31]. Let $\tilde{A}_1=(T_1,I_1,F_1)$ be a single-valued neutrosophic number. Then, the score function $s(\tilde{A}_1)$, the accuracy function $a(\tilde{A}_1)$ and the certainty function $c(\tilde{A}_1)$ of SVNN are defined as follows:

(i) $s(\tilde{A}_1) \div \dfrac{2 .\ T_1 0\ I_1\ 0\ F_1}{3}$

(ii) $a(\tilde{A}_1)=T_1-F_1$

(iii) $a(\tilde{A}_1)=T_1$

Definition 4[31]. Let $\tilde{A}_1=(T_1,I_1,F_1)$ and $\tilde{A}_2=(T_2,I_2,F_2)$ betwo single-valued neutrosophic numbers then

(i) $\tilde{A}_1 \prec \tilde{A}_2$ if $s(\tilde{A}_1)\prec s(\tilde{A}_2)$

(ii) $\tilde{A}_1 \succ \tilde{A}_2$ if $s(\tilde{A}_1)\succ s(\tilde{A}_2)$

(iii) $\tilde{A}_1 = \tilde{A}_2$ if $s(\tilde{A}_1)= s(\tilde{A}_2)$

Definition 5 [31]. The unit 0_n is defined by one of the four types:

(0_1) Type 1. $0_n \cong \{?\ x,(0,0,1)\ A\ x \angle X\}$

(0_2) Type 2. $0_n \cong \{?\ x,(0,1,1)\ A\ x \angle X\}$

(0_3) Type 3. $0_n \cong \{?\ x,(0,1,0)\ A\ x \angle X\}$

(0_4) Type 4. $0_n \cong \{?\ x,(0,0,0)\ A\ x \angle X\}$

Definition 6 [31]. The unit 1_n is defined by one of the four types:

(1_1) Type 1. $1_n \cong \{?\ x,(1,0,0)\ A\ x \angle X\}$

(1_2) Type 2. $1_n \cong \{?\ x,(1,0,1)\ A\ x \angle X\}$

(1_3) Type 3. $1_n \cong \{?\ x,(1,1,0)\ A\ x \angle X\}$

(1_4) **Type 4.** $1_n \cong \{?\ x,(1,1,1)\ A\ x \angle X\}$

III. Computing procedures for set-theoretic operations

For the sake of brevity, we use the following notations to denote the following types of matrices:

≠ a.m: Membership matrix.

≠ a.i: Indeterminacy membership matrix.

≠ a.n: Non-membership matrix.

3.1.Computing the single-valued neutrosophic matrix

The procedure is described as follows.

```
Function nm_out=nm(varargin); %single valued neutrosophic matrix class constructor.

%A = nm(Am,Ai,An) creates a single valued neutrosophic matrix

% with membership degrees from matrix Am

% indeterminate membership degrees from matrix Ai

%   and non-membership degrees from Matrix An.

% If the new matrix is not neutrosophic i.e. Am(i,j)+Ai(i,j+An(i,j)>3

% appears warning message, but the new object will be constructed.

If

length(varargin)==3

Am = varargin{1}; % Cell array indexing

Ai = varargin{2};

An = varargin{3};

end

nm_.m=Am;

nm_.i=Ai;

nm_.n=An;

nm_out=class(nm_,'im');

if ~checknm(nm_out)
```

```
disp('Warning! The created new object is NOT a Single valued neutrosophic matrix')

end
```

3.2. Determining complement of a single-valued neutrosophic matrix

In the literature, there are two definitions of complement of neutrosophic sets. They are implemented in this extended software package. To obtain the complement of a type 1 and type 2 of a single-valued neutrosophic matrix, simple call of the function named "complement1.m" or "complement2.m".

```
Function At=complement1(A);

% complement of type1 single valued neutrosophic matrix A

% "A" have to be single valued neutrosophic  matrix - "nm" object:

a.m=A.n;

a.i=A.i;

a.n=A.m;

At=nm(a.m,a.i,a.n);
```

```
Function At=complement2(A);

% complement of type2 single valued neutrosophic matrix A

% "A" have to be single valued neutrosophic  matrix - "nm" object:

a.m=1-A.m;

a.i=1-A.i;

a.n=1-A.n;

At=nm(a.m,a.i,a.n);
```

3.3. Computing max-min-min of two single-valued neutrosophic matrices

To obtain the max-min min of two single-valued neutrosophic matrices, simple call of the following function named "maxminmin.m" is needed:

```
Function At=maxminmin(A,B);

% maxminmin of two single valued neutrosophic matrix A and B

% "A" have to be single valued neutrosophic  matrix - "nm" object:

%"B" have to be single valued neutrosophic  matrix - "nm" object:

a.m=max(A.m,B.m);

a.i=min(A.i,B.i);

a.n=min(A.n,B.n);

At=nm(a.m,a.i,a.n);
```

3.4. Computing min-max-max of two single-valued neutrosophic matrices

To obtain the min-max max of two single-valued neutrosophic matrices, simple call of the following function named "minmaxmax.m" is needed:

```
Function At=minmaxmax(A,B);

% minmaxmax of two single valued neutrosophic matrix A and B

% "A" have to be single valued neutrosophic  matrix - "nm" object:

%"B" have to be single valued neutrosophic  matrix - "nm" object:

a.m=min(A.m,B.m);

a.i=max(A.i,B.i);

a.n=max(A.n,B.n);

At=nm(a.m,a.i,a.n);
```

3.5. Computing power of a single-valued neutrosophic matrix

To obtain the power of single-valued neutrosophic matrix, simple call of the following function named "power.m" is needed:

```
Function At=power(A,k);
```

```
%power of single valued neutrosophic
matrix A

% "A" have to be single valued neutro-
sophic  matrix - "nm" object:

for i =2 :k

a.m=(A.m).^k;

a.i=(A.i).^k;

a.m=(A.m).^k;

At=nm(a.m,a.i,a.m);

end
```

3.6. Computing additional operation of two single-valued neutrosophic matrices

To obtain the additional operation of two single-valued neutrosophic soft matrices, simple call of the following function named "softadd.m" is needed:

```
Function At=softadd(A,B);

% addition operations of two single
valued neutrosophic soft  matrix A and
B

% "A" have to be single valued neutro-
sophic  matrix - "nm" object:

a.m=max(A.m,B.m);

a.i=(A.i+B.i)/2;

a.n=min(A.n,B.n);

At=nm(a.m,a.i,a.n);
```

3.7. Computing subtraction of two single-valued neutrosophic matrices

To obtain the subtraction operation of two single-valued neutrosophic soft matrices, simple call of the following function named "softsub.m" is needed:

```
Function At=softsub(A,B);

% function st=disp_intui(A);
```

```
% substraction operations of two single
valued neutrosophic soft  matrix A and
B
```

```
% "A" have to be single valued neutro-
sophic  matrix - "nm" object:

a.m=min(A.m,B.n);

a.i=(A.i+B.i)/2;

a.n=max(A.n,B.m);

At=nm(a.m,a.i,a.n);
```

3.8. Computing transpose of a single-valued neutrosophic matrix

To obtain the power of single-valued neutrosophic matrix, simple call of the following function named "transpose.m" is needed:

```
Function At=transpose(A);

% transpose Single valued neutrosophic
matrix A

% "A" have to be single valued neutro-
sophic matrix - "nm" object:

a.m=(A.m)';

a.i=(A.i)';

a.n=(A.n)';

At=nm(a.m,a.i,a.n);
```

VI. Numerical Examples

In this section, we give several examples to illustrate solving some operations of the single-valued neutrosophic matrices.

Example 1. Input a neutrosophic matrix by a given structure in the toolbox.

```
%Enter the degree of membership of A in the variable a.m

>>a.m = [0 .5 .5 ;.3 0 .1 ;.3 .1 0 ; .1 .2 .1];
```

```
%Enter the degree of indterminate-membership of A in
the variable a.i

>>a.i = [1 .3 .2;.3 1 .4 ; .1 .5 1;.1 .5 .7 ];

%Enter the degree of non-membership of A in the variable
a.n
```

```
>>a.n = [0 .2 .3 ;.4 0 .5 ;.6 .1 0 ; .3 .5 .5];

%Enter the degree of membership of Bin the variable b.m

>>b.m = [0 .4 .2 ;.4 0 .1 ;.3 .2 0 ; .3 .3 .1];

%Enter the degree of indterminate-membership of Bin the variable b.i

>>b.i = [0 .5 .4 ;.3 0 .5 ;.8 .1 0 ; .3 .2 .4];

%Enter the degree of non-membership of Bin the variable b.n

>>b.n = [0 .5 .4 ;.3 0 .5 ;.8 .1 0 ; .3 .2 .4];
```

>>**A=nm(a.m,a.i,a.n)**

```
%This command returns a matrix A with degree of membership a.m,degree of indeterminate-membership a.i and degree of non-membership a.n%

A =

<0.00, 1.00, 0.00><0.50, 0.30, 0.20><0.50, 0.20, 0.30>

<0.30, 0.30, 0.40><0.00, 1.00, 0.00><0.10, 0.40, 0.50>

<0.30, 0.10, 0.60><0.10, 0.50, 0.10><0.00, 1.00, 0.00>

<0.10, 0.10, 0.30><0.20, 0.50, 0.50><0.10, 0.70, 0.50>
```

>>**B=nm(b.m,b.i,b.n)**

```
%This command returns a N matrix B with degree of membership b.m, degree of indeterminate-membership b.i and degree of non- membership b.n %

B =

<0.00, 0.00, 0.10><0.40, 0.50, 0.40><0.20, 0.40, 0.30>

<0.40, 0.30, 0.30><0.00, 0.00, 1.00><0.10, 0.50, 0.40>

<0.30, 0.80, 0.10><0.20, 0.10, 0.60><0.00, 0.00, 1.00>

<0.30, 0.30, 0.10><0.30, 0.20, 0.30><0.10, 0.40, 0.60>
```

Example 2. Evaluate the complement type 1 of the following matrix:

A=

$$\begin{pmatrix} <0.00,1.00,0.00> & <0.20,0.30,0.50> & <0.30,0.20,0.50> \\ <0.40,0.30,0.30> & <0.00,1.00,0.00> & <0.50,0.40,0.10> \\ <0.60,0.10,0.30> & <0.10,0.50,0.10> & <0.00,1.00,0.00> \\ <0.30,0.10,0.10> & <0.50,0.50,0.20> & <0.50,0.70,0.10> \end{pmatrix}$$

>>**complement1(A)**

```
% This command returns the complement1of N matrices A .

ans =

<0.00, 1.00, 0.00><0.20, 0.30, 0.50><0.30, 0.20, 0.50>

<0.40, 0.30, 0.30><0.00, 1.00, 0.00><0.50, 0.40, 0.10>

<0.60, 0.10, 0.30><0.10, 0.50, 0.10><0.00, 1.00, 0.00>

<0.30, 0.10, 0.10><0.50, 0.50, 0.20><0.50, 0.70, 0.10>
```

Example 3. Evaluate the complement type 2 of matrix above

>>**complement2(A)**

```
% This command returns the complement2

ans =

<1.00, 0.00, 1.00><0.50, 0.70, 0.80><0.50, 0.80, 0.70>

<0.70, 0.70, 0.60><1.00, 0.00, 1.00><0.90, 0.60, 0.50>

<0.70, 0.90, 0.40><0.90, 0.50, 0.90><1.00, 0.00, 1.00>

<0.90, 0.90, 0.70><0.80, 0.50, 0.50><0.90, 0.30, 0.50>
```

Example 4. Evaluate the min-max-max and max-min-min of these matrices:

A=

$$\begin{pmatrix} <0.00,1.00,0.00> & <0.20,0.30,0.50> & <0.30,0.20,0.50> \\ <0.40,0.30,0.30> & <0.00,1.00,0.00> & <0.50,0.40,0.10> \\ <0.60,0.10,0.30> & <0.10,0.50,0.10> & <0.00,1.00,0.00> \\ <0.30,0.10,0.10> & <0.50,0.50,0.20> & <0.50,0.70,0.10> \end{pmatrix}$$

B=

$$\begin{pmatrix} <0.00,0.00,0.10> & <0.40,0.50,0.40> & <0.20,0.40,0.30> \\ <0.40,0.30,0.30> & <0.00,0.00,1.00> & <0.10,0.50,0.40> \\ <0.30,0.80,0.10> & <0.20,0.10,0.60> & <0.00,0.00,1.00> \\ <0.30,0.30,0.10> & <0.30,0.20,0.30> & <0.10,0.40,0.60> \end{pmatrix}$$

>>**minmaxmax(A,B)**

```
% This command returns the min-max-max

ans =
```

```
<0.00, 1.00, 0.10><0.40, 0.50, 0.40><0.20, 0.40, 0.30>

<0.30, 0.30, 0.40><0.00, 1.00, 1.00><0.10, 0.50, 0.50>

<0.30, 0.80, 0.60><0.10, 0.50, 0.60><0.00, 1.00, 1.00>

<0.10, 0.30, 0.30><0.20, 0.50, 0.50><0.10, 0.70, 0.60>


>>maxminmin(A,B)

 % This command returns the max-min-min

ans =

<0.00, 0.00, 0.00><0.50, 0.30, 0.20><0.50, 0.20, 0.30>

<0.40, 0.30, 0.30><0.00, 0.00, 0.00><0.10, 0.40, 0.40>

<0.30, 0.10, 0.10><0.20, 0.10, 0.10><0.00, 0.00, 0.00>

<0.30, 0.10, 0.10><0.30, 0.20, 0.30><0.10, 0.40, 0.50>
```

Example 5. Evaluate the additional and subtraction operations of the matrices in Example

```
>>softadd(A,B)

% This command returns the addition of two neutrosophic matrices A and B

ans =

<0.00, 0.50, 0.00><0.50, 0.40, 0.20><0.50, 0.30, 0.30>

<0.40, 0.30, 0.30><0.00, 0.50, 0.00><0.10, 0.45, 0.40>

<0.30, 0.45, 0.10><0.20, 0.30, 0.10><0.00, 0.50, 0.00>

<0.30, 0.20, 0.10><0.30, 0.35, 0.30><0.10, 0.55, 0.50>
```

```
>>softsub(A,B)

 % This command returns the substraction of two neutrosophic matrices A and B

ans =

<0.00, 0.50, 0.00><0.40, 0.40, 0.40><0.30, 0.30, 0.30>

<0.30, 0.30, 0.40><0.00, 0.50, 0.00><0.10, 0.45, 0.50>

<0.10, 0.45, 0.60><0.10, 0.30, 0.20><0.00, 0.50, 0.00>

<0.10, 0.20, 0.30><0.20, 0.35, 0.50><0.10, 0.55, 0.50>
```

Example 6. Return the transpose of the matrix below:

A=

$$\begin{pmatrix} <0.00,1.00,0.00> & <0.20,0.30,0.50> & <0.30,0.20,0.50> \\ <0.40,0.30,0.30> & <0.00,1.00,0.00> & <0.50,0.40,0.10> \\ <0.60,0.10,0.30> & <0.10,0.50,0.10> & <0.00,1.00,0.00> \\ <0.30,0.10,0.10> & <0.50,0.50,0.20> & <0.50,0.70,0.10> \end{pmatrix}$$

```
>>transpose(A)

% This command returns the power of matrix A .

ans =

<0.00, 1.00, 0.00><0.30, 0.30, 0.40><0.30, 0.10, 0.60><0.10, 0.10, 0.30>

<0.50, 0.30, 0.20><0.00, 1.00, 0.00><0.10, 0.50, 0.10><0.20, 0.50, 0.50>

<0.50, 0.20, 0.30><0.10, 0.40, 0.50><0.00, 1.00, 0.00><0.10, 0.70, 0.50
```

Note: The functions described above enables us to compute the operations on fuzzy matrices and intuitionistic fuzzy matrices

Fuzzy matrix:

$$A_{FS}=\begin{pmatrix} <0.5,0,0> & <0.2,0,0> & <0.4,0,0> \\ <0.3,0,0> & <0.3,0,0> & <0.8,0,0> \\ <0.4,0,0> & <0.6,0,0> & <1,0,0> \\ <0.6,0,0> & <0.5,0,0> & <0.2,0,0> \end{pmatrix}$$

Intuitionisticfuzzy matrix:

$$A_{IFS}=$$

$$\begin{pmatrix} <0.5,0,0.2> & <0.2,0,0.1> & <0.4,0,0.4> \\ <0.3,0,0.2> & <0.3,0,0.4> & <0.8,0,0.3> \\ <0.4,0,0.3> & <0.6,0,0.8> & <0.3,0,0.5> \\ <0.6,0,0.5> & <0.5,0,0.9> & <0.2,0,0.2> \end{pmatrix}$$

CONCLUSION

This paper aimed to propose some new computing procedures in Matlab forset-theoretic operations in the neutrosophic set. The toolbox consists of 8 operations including forming the single-valued neutrosophic matrix, computing complement, power and transpose of a single-valued neutrosophic matrix, calculating the max-min-min, min-max-max, additional and subtraction operations of two single-valued neutrosophic matrices.The neutrosophic software package gives the ability for easy calculation of operations in associated problems.The high level of userfriendliness of the programs and functions also makes it very convenient to be used, and gives it a higher level of computational efficiency compared to the existing software packages for fuzzy calculus. We hope that they will support researches who are working in the field of neutrosophic decision making and neutrosophic graph theory.

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The Basic Notions for (over, off, under) Neutrosophic Geometric Programming Problems

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Huda E. Khalid, Florentin Smarandache, Ahmed K. Essa (2018). The Basic Notions for (over, off, under) Neutrosophic Geometric Programming Problems. *Neutrosophic Sets and Systems* 22, 50-62

Abstract. Neutrosophic (over, off, under) set and logic were defined for the first time in 1995 by Florentin Smarandache, and presented during 1995-2018 to various national and international conferences and seminars. The (over, off, under) neutrosophic geometric programming was put forward by Huda et al. in (2016) [8], in an attempt to define a new type of geometric programming using (over, off, under) neutrosophic less than or equal to. This paper completes the basic notions of (over, off, under) neutrosophic geometric programming illustrating its convexity condition, and its decomposition
theorems. The definitions of $(\alpha, \beta, \gamma) - cut$ and strong $(\alpha, \beta, \gamma) - cut$ are introduced, and some of their important properties are proved.

Keyword: Neutrosophic Set (NS), Neutrosophic Geometric Programming (NGP), (Over, Off, Under) Neutrosophic Convex Set, (sleeves, neut-sleeves, anti-sleeves) of Neutrosophic Sets, Ideal Sleeves, $(\alpha, \beta, \gamma) - \text{cut}$, Strong $(\alpha, \beta, \gamma) - \text{cut}$, Excluded Middle Law, Decomposition Theorems;

Introduction

B. Y. Cao set up the mathematical fundamentals of fuzzy geometric programming (FGP) [1], and introduced it at the second IFSA conference, in 1987, in (Tokyo). The formulation and uniqueness of the maximum solution of fuzzy neutrosophic geometric programming in the type of relational equations were firstly introduced by H.E. Khalid [14], later there was a novel method for finding the minimum solution in the same fuzzy neutrosophic relational equations on geometric programming presented on 2016 [15]. The most important paper which related with the basic role of this paper which regarded as the first attempt to present the notion of (over, off, under) neutrosophic less than or equal in geometric programming was established by Florentin S. and Huda E. [8]. The NGP method has been admitted by specialists and created a new branch of neutrosophic mathematics. Inspired by Smarandache's neutrosophic sets theory and (over, off, under) neutrosophic set theory [2, 5, 6], neutrosophic geometric programming emerges from the combination of neutrosophic sets with geometric programming. The present paper intends to discuss the (over, off, under) convexity in neutrosophic sets, introducing a new definition for convexity, and graphing the geometrical representations for (over, off, under) convexity property. Neutrosophic sleeves, neutrosophic neut-sleeves and neutrosophic anti-sleeves are also introduced in this research. Because each neutrosophic set can uniquely be represented by the family of all its $(\alpha, \beta, \gamma) - cut$, it is useful to enunciate the definition of $(\alpha, \beta, \gamma) - cut$ and prove some of its properties, similarly talking for strong$(\alpha, \beta, \gamma) - cut$. Any neutrosophic mathematical programming cannot be generated from the womb of fuzzy mathematical programming without the passage through intuitionistic fuzzy mathematical programming [17, 18], so we should be familiar with all aspects of intuitionistic mathematical programming fundamentals from the point of view of K. T. Atanassov [13, 16].

1 (Over, Off, Under) Convexity Property in Neutrosophic Sets

In this section, a new convexity behavior of the neutrosophic set will be given. Let X be an ordinary set whose generic elements are denoted by x. N(X) is the set of all neutrosophic sets included in X.

1.1 Definition [19]

A neutrosophic set $A \in N(x)$ is defined as $A = \{<\mu_A(x), \sigma_A(x), \nu_A(x)>: x \in X\}$ where $\mu_A(x), \sigma_A(x), \nu_A(x)$ represent the membership function, the indeterminacy function, the non-membership function respectively.

1.2 Definition [4]

A mapping $A: X \to [0,1]$, $x \to \mu_A(x)$, $x \to \sigma_A(x)$, $x \to \nu_A(x)$ is called a collection of neutrosophic elements, where μ_A a membership x corresponding to a neutrosophic set A, $\sigma_A(x)$ an indeterminacy membership x corresponding to a neutrosophic set A, $\nu_A(x)$ a non-membership x corresponding to a neutrosophic set A.

1.3 Definition

Suppose $A \in N(x)$. If $\forall x_1, x_2 \in X$, we call that A is an (over, off, under) convex neutrosophic set, iff the following conditions hold together:

1- $\mu_A(\lambda x_1 + (1-\lambda)x_2) > \min(\mu_A(x_1), \mu_A(x_2)\ \forall x_1, x_2 \in X$.

2- Let $\sigma_A(x) = \mu_A(x) \cap \nu_A(x)$ and

a- $\sigma_A(x)$ satisfies the convex condition,
i.e. $\sigma_A(\lambda x_1 + (1-\lambda)x_2) > \min(\sigma_A(x_1), \sigma_A(x_2)$ for some $x_1, x_2 \in X$.

b- $\sigma_A(x)$ satisfies the concave condition,
i.e.$\sigma_A(\lambda x_1 + (1-\lambda)x_2) < \max(\sigma_A(x_1), \sigma_A(x_2))$ for some $x_1, x_2 \in X$.

c- $\sigma_A(x)$ is neither convex nor concave at $t_1 \in X$, where $t_1 = \lambda x_2 + (1-\lambda)x_1$, and $\lambda = 0.5$, (i.e. $\sigma_A(x_1) = \sigma_A(x_2) = \sigma_A(t_1) = 0$) .

3- $\nu_A(\lambda x_1 + (1-\lambda)x_2) < \max(\nu_A(x_1), \nu_A(x_2))\ \forall x_1, x_2 \in X$.

For more details, see Figures 1, 2, and 3.

2 Geometrical Representation

This section illustrates the geometrical representation of the (over, off, under) convexity behavior in neutrosophic sets. Figures 1,2 and 3 illustrate the given notion as follow:

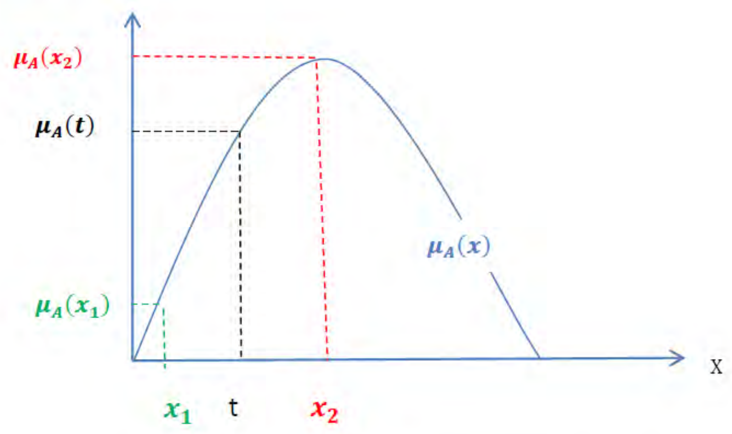

Figure 1: The convex condition of the truth membership function $\mu_A(x)$.

Here, $\mu_A(t) > \min(\mu_A(x_1), \mu_A(x_2)$, where $t = (\lambda x_1 + (1-\lambda)x_2)$; $\lambda \in [0,1]$, satisfying the condition for all $x_1, x_2 \in X$.

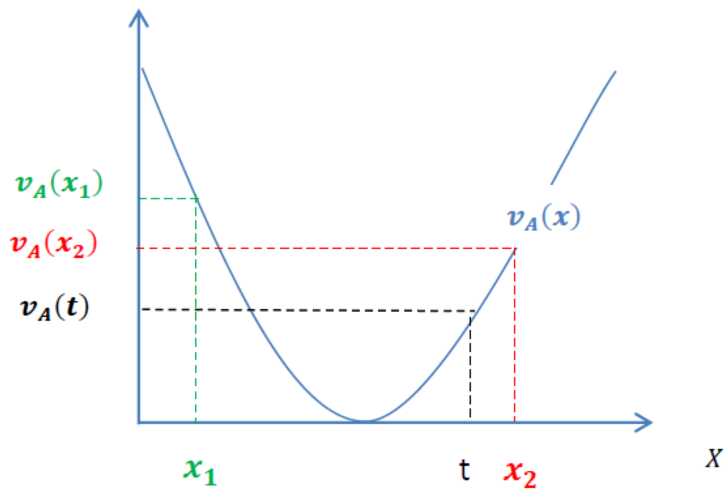

Figure 2: The concave condition of the falsehood membership function $v_A(x)$

Here, $v_A(t) < \max\ (v_A(x_1), v_A(x_2))$, where $t = (\lambda x_1 + (1-\lambda)x_2)$; $\lambda \in [0,1]$ the condition is happening for all $x_1, x_2 \in X$.

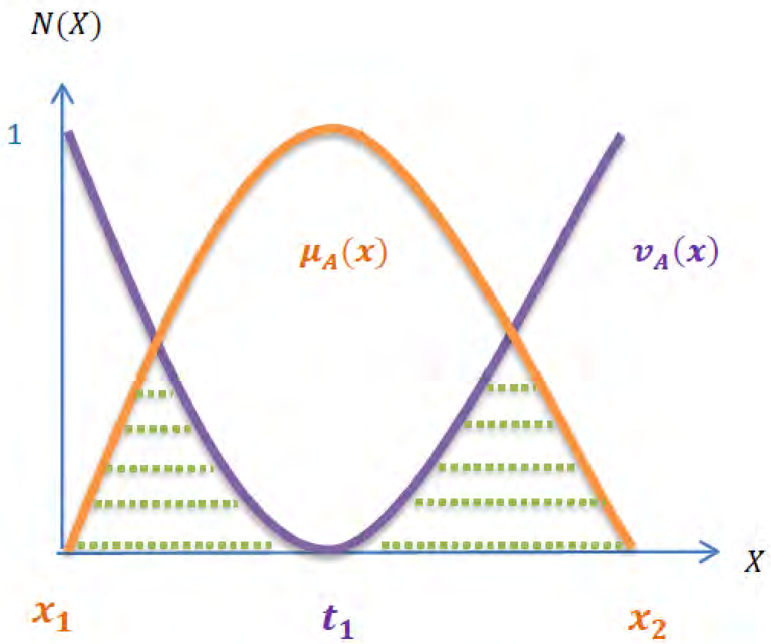

Figure 3: Here the indeterminate function is constructed from the intersection between the truth and falsehood membership functions; i.e. $\sigma_A(x) = \mu_A(x) \cap v_A(x)$. In this figure, the dashed point lines (i.e. shaded with green points represent the indeterminate region, here $\sigma_A(x)$ is neither convex nor concave at $\sigma_A(x_1) = \sigma_A(x_2) = \sigma_A(t_1) = 0$, where $t_1 = \lambda x_2 + (1-\lambda)x_1$, and $\lambda = 0.5$.

3 Neutrosophic Sleeves, Neutrosophic Anti-sleeves, Neutrosophic Neut-sleeves

This section introduces for the first time the notion of neutrosophic sleeves, its contradiction and its neutrality. Together with the definitions of the neutrosophic sleeve, neutrosophic anti-sleeve, neutrosophic unit-sleeve, we provided graphs; however, the graphs are imprecise, offering an illustration of the meaning of composite sleeves.

3.1 Definition

If a set-valued mapping $H: [0,1] \to N(x)$ satisfies $\forall\, \alpha_1, \alpha_2 \in [0,1]$, $\alpha_1 < \alpha_2 \Rightarrow H(\alpha_1) \supseteq H(\alpha_2)$, then H is called a collection neutrosophic sleeve on X. A set composed of all the collections of neutrosophic sleeves on X is written as (X). The ideal sleeve occurs when $H(\alpha_1) = H(\alpha_2)$.

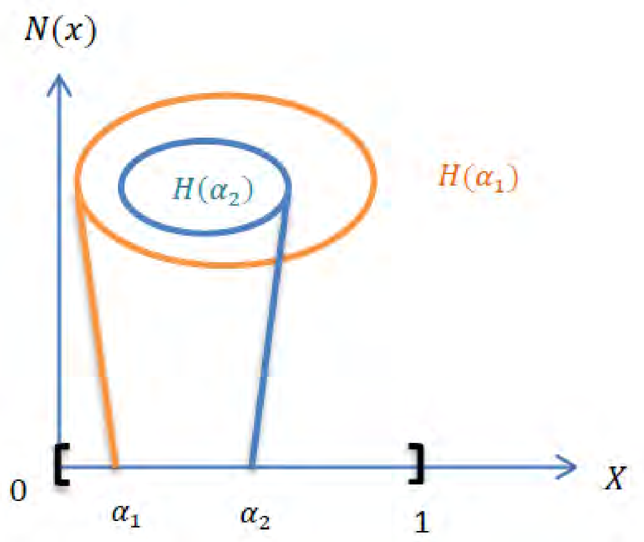

Figure 4: Neutrosophic sleeve

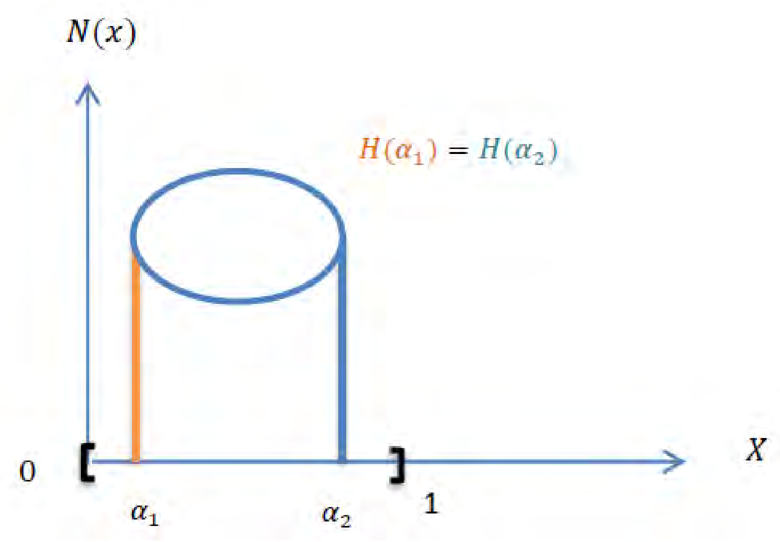

Figure 5: Neutrosophic ideal sleeve

3.2 Definition

If a set-valued mapping $H: [0,1] \to N(x)$ satisfies $\forall\, \alpha_1, \alpha_2, 2\alpha_2 - \alpha_1 \in [0,1]$, $\alpha_2 < 2\alpha_2 - \alpha_1 \Rightarrow H(\alpha_2) \supseteq H(2\alpha_2 - \alpha_1)$, then H is called a collection of neutrosophic anti-sleeves on X. A set composed of all the collection of neutrosophic anti-sleeves on X is written as $anti\ U(X)$. The ideal neutrosophic anti-sleeve on X occures when $H(\alpha_2) = H(2\alpha_2 - \alpha_1)$.

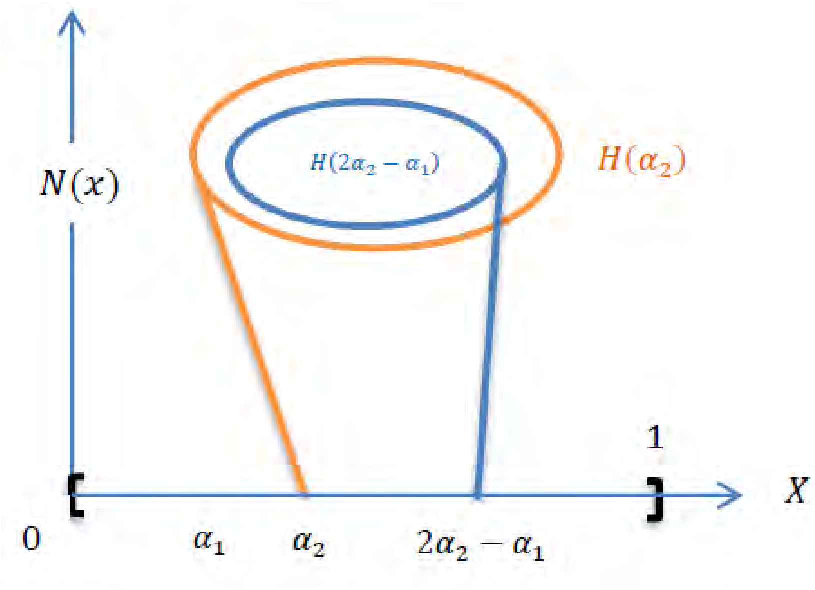

Figure 6: Neutrosophic anti-sleeve

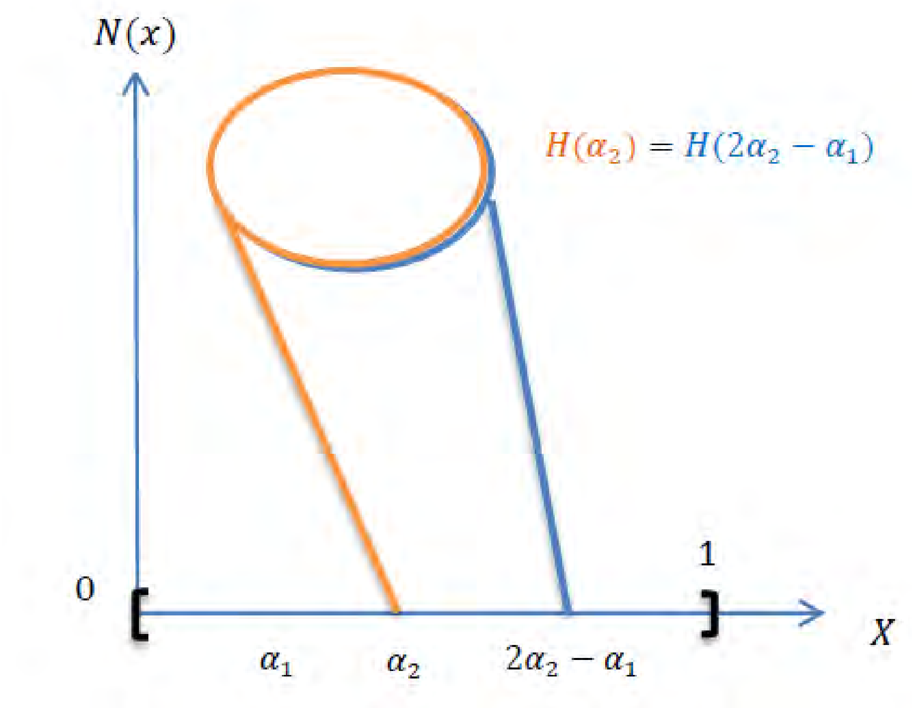

Figure 7: Neutrosophic ideal anti-sleeve

3.3 Definition

If a set-valued mapping $H: [0,1] \to N(x)$ satisfies $\forall\ \alpha_1, \alpha_2, 2\alpha_2 - \alpha_1 \in [0,1]$, $\alpha_1 < \alpha_2 < 2\alpha_2 - \alpha_1 \Rightarrow H(\alpha_1) \wedge H(2\alpha_2 - \alpha_1) = \min\{ H(\alpha_1), H(2\alpha_2 - \alpha_1)\} = H(\alpha_2)$, then H is called a collection of neutrosophic neut-sleeves on X . A set composed of all the collection of neutrosophic neut-sleeves on X is written as $neut\ U(X)$. The ideal neutrosophic neut-sleeve on X occurs in the case of $0 < \alpha_1 < \alpha_2 < 2\alpha_2 - \alpha_1 < 1 \Rightarrow H(\alpha_1) = H(\alpha_2) = H(2\alpha_2 - \alpha_1)$.

Note that:

The ideal case of neutrosophic neut-sleeve is composed from the ideal case of the neutrosophic sleeve combined with the ideal case of the neutrosophic anti-sleeve.

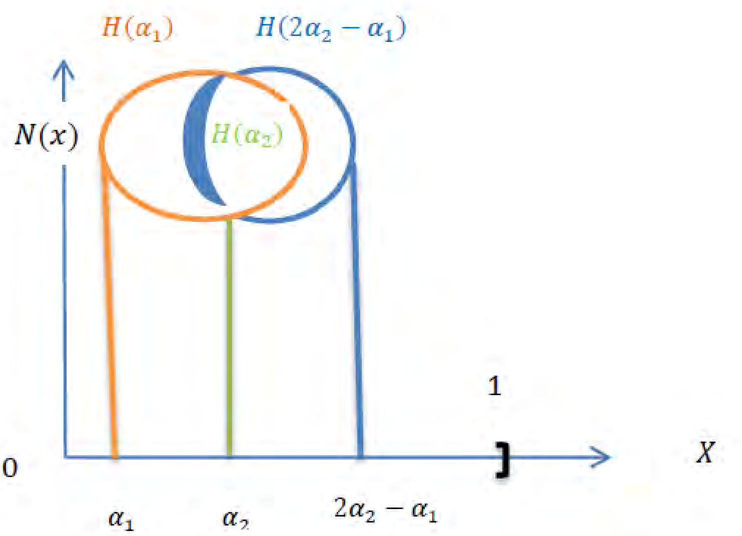

Figure 8: Neutrosophic neut-sleeve

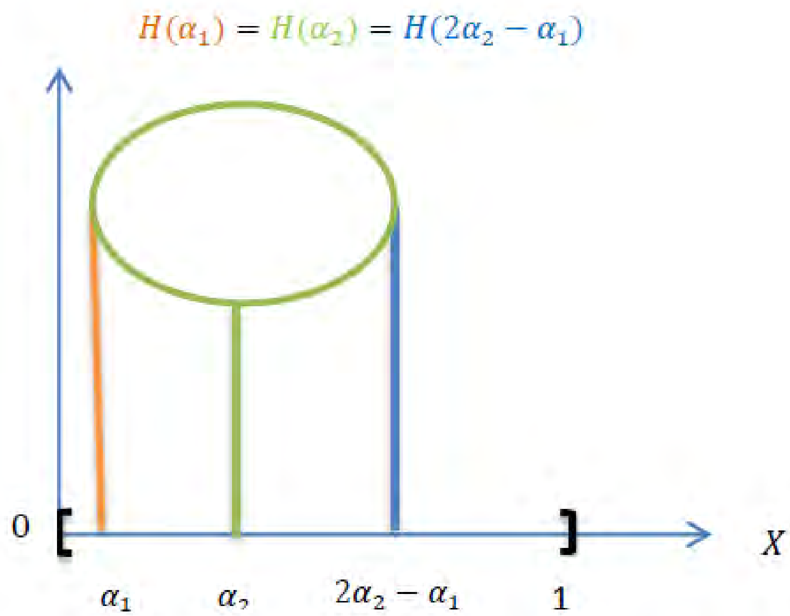

Figure 9: Neutrosophic ideal neut-sleeve

Note that:

All figures from 4 to 9 are just indicative graphs employed to understand the meaning of neutrosophic sleeves, neutrosophic anti-sleeves and neutrosophic neut-sleeves, but are not necessary accurate.

4 A new insight of the excluded-middle law in neutrosophic theory
4.1 The excluded-middle law in classical and fuzzy logics

In classical dialectics, the excluded middle law is the third of the three classic laws of thought. It states that for any proposal, either that proposal is true, or its contradictory is true. The earliest known formulation was in Aristotle's discussion of the principle of non-contradiction, where he said that of two contradictory suggestions, one of them must be true, and the other is false. Aristotle 384 BC, said that it is necessary for every claim there are two opposite parts, either confirm or deny and it is unattainable that there should be anything between the two parts of an opposition. We point out that fuzzy logic, intuitionistic fuzzy logic and neutrosophic logic no longer satisfy the excluded-middle law [3].

Let X be an ordinary fuzzy set, whose generic elements are denoted by x, a mapping $A: X \to [0,1]$, $x \to \mu_A(x)$ called a fuzzy set A, and let the complement of A be A^c with its membership function meaning $\mu_{A^c}(x) = 1 - \mu_A(x)$, then it is obvious that the excluded-middle law is not satisfied. $\mu_A(x) \cup \mu_{A^c}(x) \neq X$ (1)

$$\mu_A(x) \cap \mu_{A^c}(x) \neq \emptyset \tag{2}$$

Example:

Let $X = [0,1], \mu_A(x) = x$, then $\mu_{A^c}(x) = 1 - x$, while

$$(\mu_A \cup \mu_{A^c})(x) = \begin{cases} 1 - x & x \leq \frac{1}{2} \\ x & x > \frac{1}{2} \end{cases} \tag{3}$$

$$(\mu_A \cap \mu_{A^c})(x) = \begin{cases} x & x \leq \frac{1}{2} \\ 1 - x & x > \frac{1}{2} \end{cases} \tag{4}$$

Hence

$\mu_A \cup \mu_{A^c} \neq X$ & $\mu_A \cap \mu_{A^c} \neq \emptyset$

Especially,

$$(\mu_A \cup \mu_{A^c})\left(\frac{1}{2}\right) = (\mu_A \cap \mu_{A^c})\left(\frac{1}{2}\right) = \frac{1}{2} \tag{5}$$

A fuzzy set operation does not satisfy the excluded-middle law, which complicates the study of fuzzy sets. The fuzzy sets can provide more objective properties than the classical sets [1].

4.2 The excluded middle law with the perspective of (over, off, under) neutrosophic geometric programming

In the two-valued logic, all the designated values as types of truth and all the anti-designated values as types of untruth with gaps between truth-value (or falsehood-value). In the neutrosophic theory, one specifies the non-designated values as types of indeterminacy and thus, each neutrosophic consequences have degrees of designated, non-designated, and anti-designated values. However, the excluded middle law in the neutrosophic system does no longer work [7].

Even more, Smarandache (2014) [3] generalized the Law of Included Middle to the *Law of Included Multiple-Middles*, showing that in refined neutrosophic logic (2013), between truth (T) and falsehood (F) there are multiple types of sub-indeterminacies (I_1, I_2, ...) [10,11,12].

In upcoming definitions, the authors affirm that $\mu_A(x) \cap \mu_{A^c}(x) \neq \emptyset$ in neutrosophic environment; for example but not limited to, the nonlinear neutrosophic programming (i.e. example of neutrosophic geometric programming (NGP)).

4.2.1 Definition

Let $N(\mathrm{X})$ be the set of all neutrosophic variable vectors $x_i, i = 1,2, \ldots, m$, i.e. $N(X) = \{(x_1, x_2, \ldots, x_m)^T \mid x_i \in X\}$. The function g(x): N(X) $\rightarrow$ R $\cup$ I is said to be neutrosophic GP function of x, where $g(x) = \sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}$, $c_k \geq 0$ is a constant, γ_{kl} being an arbitrary real number.

4.2.2 Definition

Let $g(x)$ be a neutrosophic geometric function in any neutrosophic geometric programming, and let A_0 be the neutrosophic set for all functions $g(x)$ that are neutrosophically less than or equal to one.

$$A_0 = \{x_i \in \mathrm{X} : g(x) < \text{₦}\, 1\} = \{x_i \in \mathrm{X} : g(x) < 1, anti(\, g(x)) > 1, neut(\, g(x)) = 1\} \tag{6}$$

4.2.3 Definition

Let $g(x)$ be any neutrosophic geometric function written as a constraint in any neutrosophic geometric programming (NGP), where $x_i \in X = [0,1] \cup [0, nI]$ and $x = (x_1, x_2, \ldots, x_m)^T$ is an m-dimensional neutrosophic variable vector.

Call the inequality

$$g(x) < \text{₦}\, 1 \tag{7}$$

where " $<$ ₦" denotes the neutrosophied version of " $\leq$ " with the linguistic interpretation being "less than (the original claimed), greater than (the anti-claim of the original less than), or equal (neither the original claim nor the anti-claim)"

The constraint (7) can be redefined into three constraints as follow:-

$$\left.\begin{aligned} g(x) &< 1 \\ anti\ (g(x)) &> 1 \\ neut(\, g(x)) &= 1 \end{aligned}\right\} \tag{8}$$

4.2.4 Definition

Let A_0 be the set of all neutrosophic geometric functions that neutrosophically less than or equal to one,

i.e. $A_0 = \{x_i \in \mathrm{X}, g(x) < \text{₦}\, 1\} \Leftrightarrow A_0 = \{x_i \in \mathrm{X}: g(x) < 1, anti(\, g(x)) > 1, neut(\, g(x)) = 1\}$

It is significant to define the following membership functions:

$$\mu_{A_o}(\, g(x)) = \begin{cases} 1 & , \quad 0 \leq g(x) \leq 1 \\ \left(e^{\frac{-1}{d_o}(g(x)-1)} + e^{\frac{-1}{d_o}(anti(\, g(x))-1)} - 1\right), & \quad 1 < g(x) \leq 1 - d_o \ln 0.5 \end{cases} \tag{9}$$

$$\mu_{A_o}(anti(\, g(x))) = \begin{cases} 0 & , \quad 0 \leq g(x) \leq 1 \\ \left(1 - e^{\frac{-1}{d_o}(anti(\, g(x))-1)} - e^{\frac{-1}{d_o}(g(x)-1)}\right), & \quad 1 - d_o \ln 0.5 \leq g(x) \leq 1 + d_o \end{cases} \tag{10}$$

It is clear that $\mu_{A_o}(neut(\, g(x)))$ consists from the intersection of the following functions:

$$e^{\frac{-1}{d_o}(g(x)-1)} \quad , \quad 1 - e^{\frac{-1}{d_o}(anti(\, g(x))-1)}$$

i.e.

$$\mu_{A_o}(neut(\, g(x))) = \begin{cases} 1 - e^{\frac{-1}{d_o}(anti(\, g(x))-1)} & , \quad 1 \leq g(x) \leq 1 - d_o \ln 0.5 \\ e^{\frac{-1}{d_o}(g(x)-1)} & , 1 - d_o \ln 0.5 < g(x) \leq 1 + d_o \end{cases} \tag{11}$$

Note that $d_o > 0$ is a constant expressing a limit of the admissible violation of the neutrosophic geometric function $g(x)$.

Consequently,

$\mu_{A_o}(\, g(x)) \cap \mu_{A_o}\left(anti(\, g(x))\right) \neq \emptyset$,

Here $\mu_{A_o}(\, g(x)) \cap \mu_{A_o}\left(anti(\, g(x))\right) = \mu_{A_o}(neut(\, g(x)))$.

5 $(\alpha, \beta, \gamma) - \mathbf{cut}$ and strong $(\alpha, \beta, \gamma) - \mathbf{cut}$ of Neutrosophic sets

We put the following definitions as an initial step to prepare to prove the properties of $(\alpha, \beta, \gamma) - cut$ and strong $(\alpha, \beta, \gamma) - cut$ of neutrosophic sets.

5.1 Definition

Let $A \in N(x), \forall\ (\alpha, \beta, \gamma) \in [0,1]$, written $A_{(\alpha,\beta,\gamma)} = \{x: \mu_A(x) \geq \alpha, \sigma_A(x) \geq \beta, v_A(x) \leq \gamma\}$, $A_{(\alpha,\beta,\gamma)}$ is said to be an $(\alpha, \beta, \gamma) - cut$ set of a neutrosophic set A . Again, we write $A_{(\alpha,\beta,\gamma)^+} = \{x: \mu_A(x) > \alpha, \sigma_A(x) > \beta, v_A(x) < \gamma\}$, $A_{(\alpha,\beta,\gamma)^+}$ is said to be a strong $(\alpha, \beta, \gamma) - cut$ set of a neutrosophic set A, (α, β, γ) are confidence levels and $\alpha + \beta + \gamma \leq 3$.

5.2 Definition

Let $A \in N(x)$, written $A_{(0,0,1)^+} = \{x: \mu_A(x) > 0, \sigma_A(x) > 0, v_A(x) < 1\} = supp\ A$, $A_{(0,0,1)^+}$ is called a support of a neutrosophic set A. Again, $ker\ A = \{x: \mu_A(x) = 1, \sigma_A(x) = 0, v_A(x) = 0\}$ is called a kernel of neutrosophic set A, and A is a normal neutrosophic set for $\ker A\ \neq \emptyset$.

5.3 Definition

Let $A \in N(x)$, written $A \cup B = \{\left\langle x, \max\left(\mu_A(x), \mu_B(x)\right),\right.$ $\left.\max\left(\sigma_A(x), \sigma_B(x)\right),\ \min\left(v_A(x), v_B(x)\right)\right\rangle: x \in X\}$, the union of $A\&B$

$A \cap B = \{\left\langle x, \min\left(\mu_A(x), \mu_B(x)\right), \min\left(\sigma_A(x), \sigma_B(x)\right),\ \max\left(v_A(x), v_B(x)\right)\right\rangle: x \in X\}$, the intersection of $A\&B$.

5.4 Theorem

We have the following properties for $(\alpha, \beta, \gamma) - \mathrm{cut}$ and strong $(\alpha, \beta, \gamma) - \mathrm{cut}$ neutrosophic sets:

1- $A \subseteq B\ \Rightarrow\ A_{(\alpha,\beta,\gamma)} \subseteq B_{(\alpha,\beta,\gamma)}$

2- $(A \cup B)_{(\alpha,\beta,\gamma)} \supseteq A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$ equality holds if $\alpha + \beta + \gamma = 3$.
$(A \cap B)_{(\alpha,\beta,\gamma)} = A_{(\alpha,\beta,\gamma)} \cap B_{(\alpha,\beta,\gamma)}$.

3- $(A \cup B)_{(\alpha,\beta,\gamma)^+} \supseteq A_{(\alpha,\beta,\gamma)^+} \cup B_{(\alpha,\beta,\gamma)^+}$, equality holds if $\alpha + \beta + \gamma = 3$.
$(A \cap B)_{(\alpha,\beta,\gamma)^+} = A_{(\alpha,\beta,\gamma)^+} \cap B_{(\alpha,\beta,\gamma)^+}$.

Proof

1- Let $x \in A_{(\alpha,\beta,\gamma)} \Rightarrow \mu_A(x) \geq \alpha, \sigma_A(x) \geq \beta, v_A(x) \leq \gamma$
But B ⊇ A $\Rightarrow \mu_B(x) \geq \mu_A(x) \geq \alpha, \sigma_B(x) \geq \sigma_A(x) \geq \beta, v_B(x) \leq v_A(x) \leq \gamma$
$\Rightarrow \mu_B(x) \geq \alpha, \sigma_B(x) \geq \beta, v_B(x) \leq \gamma$
$\Rightarrow x \in B_{(\alpha,\beta,\gamma)}$, therefore $A_{(\alpha,\beta,\gamma)} \subseteq B_{(\alpha,\beta,\gamma)}$

2- $(A \cup B)_{(\alpha,\beta,\gamma)} \supseteq A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$
Since $A \subseteq (A \cup B)$, $B \subseteq (A \cup B)$ and from 1 above, we have:
$A_{(\alpha,\beta,\gamma)} \subseteq (A \cup B)_{(\alpha,\beta,\gamma)}$ (12)
$B_{(\alpha,\beta,\gamma)} \subseteq (A \cup B)_{(\alpha,\beta,\gamma)}$ (13)
Combine (12) with (13). The proof of property 2 is complete, i.e.
$(A \cup B)_{(\alpha,\beta,\gamma)} \supseteq A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$ (14)
If $\alpha + \beta + \gamma = 3$, we show that $(A \cup B)_{(\alpha,\beta,\gamma)} = A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$
Let $x \in (A \cup B)_{(\alpha,\beta,\gamma)} \Rightarrow \mu_A(x) \cup \mu_B(x) \geq \alpha, \sigma_A(x) \cup \sigma_B(x) \geq \beta, v_A(x) \cup v_B(x) \leq \gamma$
if $\mu_A(x) \geq \alpha$ *and* $\sigma_A(x) \geq \beta$ then $v_A(x) \leq 3 - \alpha - \beta = \gamma \Rightarrow x \in A_{(\alpha,\beta,\gamma)} \subseteq A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$
also if $\mu_B(x) \geq \alpha$ *and* $\sigma_B(x) \geq \beta$ then $v_B(x) \leq 3 - \alpha - \beta = \gamma \Rightarrow x \in B_{(\alpha,\beta,\gamma)} \subseteq A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$
$\Rightarrow x \in A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$
and so $(A \cup B)_{(\alpha,\beta,\gamma)} \subseteq A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$ (15)
From (14) and (15), we get
$(A \cup B)_{(\alpha,\beta,\gamma)} = A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$
We still need to prove that $(A \cap B)_{(\alpha,\beta,\gamma)} = A_{(\alpha,\beta,\gamma)} \cap B_{(\alpha,\beta,\gamma)}$
Proof
Since $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$
$\Rightarrow (A \cap B)_{(\alpha,\beta,\gamma)} \subseteq A_{(\alpha,\beta,\gamma)}$ & $(A \cap B)_{(\alpha,\beta,\gamma)} \subseteq B_{(\alpha,\beta,\gamma)}$ (16)
Let $x \in A_{(\alpha,\beta,\gamma)} \cap B_{(\alpha,\beta,\gamma)}$
$\Rightarrow x \in A_{(\alpha,\beta,\gamma)}$ & $x \in B_{(\alpha,\beta,\gamma)}$
$\Rightarrow \mu_A(x) \geq \alpha, \sigma_A(x) \geq \beta, v_A(x) \leq \gamma$ and $\mu_B(x) \geq \alpha, \sigma_B(x) \geq \beta, v_B(x) \leq \gamma$
$\Rightarrow \mu_A(x) \cap \mu_B(x) \geq \alpha, \sigma_A(x) \cap \sigma_B(x) \geq \beta, v_A(x) \cup v_B(x) \leq \gamma$
$\Rightarrow x \in (A \cap B)_{(\alpha,\beta,\gamma)}$
$\Rightarrow A_{(\alpha,\beta,\gamma)} \cap B_{(\alpha,\beta,\gamma)} \subseteq (A \cap B)_{(\alpha,\beta,\gamma)}$ (17)
From (16) and (17), we have
$(A \cap B)_{(\alpha,\beta,\gamma)} = A_{(\alpha,\beta,\gamma)} \cap B_{(\alpha,\beta,\gamma)}$
Note that:
The same technique that used for proving 2 will be employed to prove the properties of strong (α,β,γ)-cut in 3.

6 Representations of neutrosophic sets

The decomposition theorems of neutrosophic sets is a bridge between neutrosophic sets and ordinary ones. The principal feature of $(\alpha, \beta, \gamma) -$ cut and strong $(\alpha, \beta, \gamma) -$ cut sets in neutrosophic set theory is the capability to represent neutrosophic sets. We show in this section that each neutrosophic set can uniquely be represented by either the family of all its $(\alpha, \beta, \gamma) -$ cuts or the family of all its strong $(\alpha, \beta, \gamma) -$ cuts.
We can convert each of $(\alpha, \beta, \gamma) -$ cut and strong $(\alpha, \beta, \gamma) -$ cut to special neutrosophic sets denoted by ${}_{(\alpha,\beta,\gamma)}A$ *and* ${}_{(\alpha,\beta,\gamma)}^{+}A$, as follows:

for $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma \leq 3$, we have:

$$ {}_{(\alpha,\beta,\gamma)}A = \begin{cases} (\alpha,\beta,\gamma) & if\ x \in A_{(\alpha,\beta,\gamma)} \\ (0,0,1) & if\ x \notin A_{(\alpha,\beta,\gamma)} \end{cases} \tag{18} $$

$$ {}_{(\alpha,\beta,\gamma)}^{+}A = \begin{cases} (\alpha,\beta,\gamma) & if\ x \in A_{(\alpha,\beta,\gamma)^{+}} \\ (0,0,1) & if\ x \notin A_{(\alpha,\beta,\gamma)^{+}} \end{cases} \tag{19} $$

The representation of an arbitrary neutrosophic set A in terms of the special neutrosophic sets ${}_{(\alpha,\beta,\gamma)}A$, which are defined in terms of the $(\alpha,\beta,\gamma)-$ cuts of A by (18), is usually referred to as decomposition of A. In the following, we formulate and prove two basic decomposition theorems of neutrosophic sets.

6.1 First Decomposition theorem of neutrosophic set (NS)

For every $A \in N(x)$, $A = \bigcup_{\alpha,\beta,\gamma \in [0,1]} {}_{(\alpha,\beta,\gamma)}A$, where ${}_{(\alpha,\beta,\gamma)}A$ is defined by (18) and ∪ denotes the standard neutrosophic union.

Proof

For each particular $x \in X$, let
$\mu_A(x) = \mathrm{a}, \sigma_A(x) = \mathrm{b}, v_A(x) = c$. Then,

$$ \left(\bigcup_{\alpha,\beta,\gamma \in [0,1]} {}_{(\alpha,\beta,\gamma)}A\right)(x) = \left(\sup_{\alpha \in [0,1]} \mu_A(x), \quad \sup_{\beta \in [0,1]} \sigma_A(x), \quad \inf_{\gamma \in [0,1]} v_A(x)\right) $$

$$ = max\left[\left(\sup_{\alpha \in [0,a]} \mu_A(x), \quad \sup_{\beta \in [0,1]} \sigma_A(x), \quad \inf_{\gamma \in [0,1]} v_A(x)\right), \quad \left(\sup_{\alpha \in (a,1]} \mu_A(x), \quad \sup_{\beta \in [0,1]} \sigma_A(x), \quad \inf_{\gamma \in [0,1]} v_A(x)\right)\right] $$

For each $\alpha \in (a, 1]$, we have $\mu_A(x) = a < \alpha$ and, therefore, ${}_{(\alpha,\beta,\gamma)}A = (0,0,1)$. On the other hand, for each $\alpha \in [0, a]$, we have $\mu_A(x) = a \geq \alpha$, therefore, ${}_{(\alpha,\beta,\gamma)}A = (\alpha,\beta,\gamma)$.

The second step of the prove is to complete the maximum value for the second component $\left(i.e.\ \sup_{\beta \in [0,1]} \sigma_A(x)\right)$ as follow:

$$ \left(\bigcup_{\alpha,\beta,\gamma \in [0,1]} {}_{(\alpha,\beta,\gamma)}A\right)(x) $$

$$ = max\left[\left(\sup_{\alpha \in [0,a]} \mu_A(x), \quad \sup_{\beta \in [0,b]} \sigma_A(x), \quad \inf_{\gamma \in [0,1]} v_A(x)\right), \left(\sup_{\alpha \in [0,a]} \mu_A(x), \quad \sup_{\beta \in (b,1]} \sigma_A(x), \quad \inf_{\gamma \in [0,1]} v_A(x)\right)\right] $$

For each $\beta \in (b, 1]$, we have $\sigma_A(x) = b < \beta$ and, therefore, ${}_{(\alpha,\beta,\gamma)}A = (0,0,1)$. On the other hand, for each $\beta \in [0, b]$, we have $\sigma_A(x) = b \geq \beta$, therefore, ${}_{(\alpha,\beta,\gamma)}A = (\alpha,\beta,\gamma)$.

$$ = \left(\sup_{\alpha \in [0,a]} \mu_A(x), \quad \sup_{\beta \in [0,b]} \sigma_A(x), \quad \inf_{\gamma \in [0,1]} v_A(x)\right) $$

The final step of the proof is to complete the maximum value for the third component $\left(i.e.\ \inf_{\gamma \in [0,1]} v_A(x)\right)$,

$$ \left(\bigcup_{\alpha,\beta,\gamma \in [0,1]} {}_{(\alpha,\beta,\gamma)}A\right)(x) $$
$$ = max\left[\left(\sup_{\alpha \in [0,a]} \mu_A(x), \sup_{\beta \in [0,b]} \sigma_A(x), \quad \inf_{\gamma \in [0,c)} v_A(x)\right), \quad \left(\sup_{\alpha \in [0,a]} \mu_A(x), \sup_{\beta \in [0,b]} \sigma_A(x), \inf_{\gamma \in [c,1]} v_A(x)\right)\right] $$

For each $\gamma \in [c, 1]$, we have $v_A(x) = c \leq \gamma$, therefore, ${}_{(\alpha,\beta,\gamma)}A = (\alpha,\beta,\gamma)$. On the other hand, for each $\gamma \in [0, c)$, we have $v_A(x) = c > \gamma$, therefore, ${}_{(\alpha,\beta,\gamma)}A = (0,0,1)$.

Consequently,

$$ \left(\bigcup_{\alpha,\beta,\gamma \in [0,1]} {}_{(\alpha,\beta,\gamma)}A\right)(x) = \left(\sup_{\alpha \in [0,a]} \mu_A(x), \sup_{\beta \in [0,b]} \sigma_A(x), \quad \inf_{\gamma \in [c,1]} v_A(x)\right) = (\alpha,\beta,\gamma) = A(x) $$

Since the same argument is valid for each $x \in X$, the theorem is proved.

6.2 Second Decomposition Theorem of Neutrosophic Set (NS)

Let X be any non-empty set. For a neutrosophic subset $A \in N(X)$,
$A = \bigcup_{\alpha,\beta,\gamma \in [0,1]} {}_{(\alpha,\beta,\gamma)}\overset{+}{A}$, where ${}_{(\alpha,\beta,\gamma)}\overset{+}{A}$ is defined by (19) ,and $\cup$ denotes the standard neutrosophic union.

Proof:

For each particular $x \in X$, let $\mu_A(x) = \mathrm{a}, \sigma_A(x) = \mathrm{b}, v_A(x) = c$. Then

$$\left(\bigcup_{\alpha,\beta,\gamma \in [0,1]} {}_{(\alpha,\beta,\gamma)}\overset{+}{A}\right)(x) = \left(\sup_{\alpha \in [0,1]} \mu_A(x),\ \sup_{\beta \in [0,1]} \sigma_A(x),\ \inf_{\gamma \in [0,1]} v_A(x)\right)$$

$$= max\left[\left(\sup_{\alpha \in [0,a)} \mu_A(x), \sup_{\beta \in [0,1]} \sigma_A(x), \inf_{\gamma \in [0,1]} v_A(x)\right), \quad \left(\sup_{\alpha \in [a,1]} \mu_A(x), \sup_{\beta \in [0,1]} \sigma_A(x),\ \inf_{\gamma \in [0,1]} v_A(x)\right)\right]$$

For each $\alpha \in [a, 1]$, we have $\mu_A(x) = a \le \alpha$ and, therefore, ${}_{(\alpha,\beta,\gamma)}\overset{+}{A} = (0,0,1)$. On the other hand, for each $\alpha \in [0, a)$, we have $\mu_A(x) = a > \alpha$, therefore, ${}_{(\alpha,\beta,\gamma)}\overset{+}{A} = (\alpha, \beta, \gamma)$ the second step of the proof is to complete the maximum value for the second component $\left(i.e.\ \sup_{\beta \in [0,1]} \sigma_A(x)\right)$.

Again,

$$\left(\bigcup_{\alpha,\beta,\gamma \in [0,1]} {}_{(\alpha,\beta,\gamma)}\overset{+}{A}\right)(x)$$
$$= max\left[\left(\sup_{\alpha \in [0,a)} \mu_A(x),\ \sup_{\beta \in [0,b)} \sigma_A(x), \inf_{\gamma \in [0,1]} v_A(x)\right), \quad \left(\sup_{\alpha \in [a,1]} \mu_A(x),\ \sup_{\beta \in [b,1]} \sigma_A(x),\ \inf_{\gamma \in [0,1]} v_A(x)\right)\right]$$

For each $\beta \in [b, 1]$, we have $\sigma_A(x) = b \le \beta$ and, therefore, ${}_{(\alpha,\beta,\gamma)}\overset{+}{A} = (0,0,1)$. On the other hand, for each $\beta \in [0, b)$, we have $\sigma_A(x) = b > \beta$, therefore, ${}_{(\alpha,\beta,\gamma)}\overset{+}{A} = (\alpha, \beta, \gamma)$.

$$\left(\bigcup_{\alpha,\beta,\gamma \in [0,1]} {}_{(\alpha,\beta,\gamma)}\overset{+}{A}\right)(x) = \left(\sup_{\alpha \in [0,a)} \mu_A(x),\ \sup_{\beta \in [0,b)} \sigma_A(x),\ \inf_{\gamma \in [0,1]} v_A(x)\right)$$

The final step of the proof is to complete the maximum value for the third component $\left(i.e.\ \sup_{\beta \in [0,1]} \sigma_A(x)\right)$.

Finally,

$$\left(\bigcup_{\alpha,\beta,\gamma \in [0,1]} {}_{(\alpha,\beta,\gamma)}\overset{+}{A}\right)(x)$$
$$= max\left[\left(\sup_{\alpha \in [0,a)} \mu_A(x),\ \sup_{\beta \in [0,b)} \sigma_A(x),\ \inf_{\gamma \in [0,c]} v_A(x)\right), \quad \left(\sup_{\alpha \in [0,a)} \mu_A(x),\ \sup_{\beta \in [0,b)} \sigma_A(x),\ \inf_{\gamma \in (c,1]} v_A(x)\right)\right]$$

For each $\gamma \in (c, 1]$, we have $v_A(x) = c < \gamma$, therefore, ${}_{(\alpha,\beta,\gamma)}\overset{+}{A} = (\alpha, \beta, \gamma)$. On the other hand, for each $\gamma \in [0, c]$, we have $v_A(x) = c \ge \beta$, therefore, ${}_{(\alpha,\beta,\gamma)}\overset{+}{A} = (0,0,1)$.

Consequently,

$$\left(\bigcup_{\alpha,\beta,\gamma \in [0,1]} {}_{(\alpha,\beta,\gamma)}\overset{+}{A}\right)(x) = \left(\sup_{\alpha \in [0,a)} \mu_A(x),\ \sup_{\beta \in [0,b)} \sigma_A(x), \inf_{\gamma \in (c,1]} v_A(x)\right) = (\alpha, \beta, \gamma) = A(x)$$

Since the same argument is valid for each $x \in X$, therefore the theorem is proved.

■

Conclusion

Neutrosophic geometric programming (NGP) can find many application areas, such as power engineering, postal services, look for exemplars for eliminating waste-water in a power plant, or determining the power equipping radius in the electrical transformers. All the above-mentioned applications require building a strong neutrosophic theory for neutrosophic geometric programming (NGP), these aims lead the authors to present the (over, off, under) convexity condition in neutrosophic geometric functions. The needed of establishing the aspects of sleeves, neut sleeves and anti-sleeves were necessary. Furthermore, the basic concept of (α,β,γ)-cut and strong (α,β,γ)-cut of neutrosophic sets have been given. By strong definitions and given example, the authors proved that the excluded middle law has no longer satisfied in neutrosophic theory, this proof has been made by neutrosophic geometrical programming.

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Energy and Spectrum Analysis of Interval Valued Neutrosophic Graph using MATLAB

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Abstract. In recent time graphical analytics of uncertainty and indeterminacy has become major concern for data analytics researchers. In this direction, the mathematical algebra of neutrosophic graph is extended to interval-valued neutrosophic graph. However, building the interval-valued neutrosophic graphs, its spectrum and energy computation is addressed as another issues by research community of neutrosophic environment. To resolve this issue the current paper proposed some related mathematical notations to compute the spectrum and energy of interval-valued neutrosophic graph using the MATAB.

Keywords: Interval valued neutrosophic graphs. Adjacency matrix. Spectrum of IVNG. Energy of IVNG. Complete-IVNG.

1 Introduction

The handling uncertainty in the given data set is considered as one of the major issues for the research communities. To deal with this issue the mathematical algebra of neutrosophic set is introduced [1]. The calculus of neutrosophic sets (NSs)[1, 2] given a way to represent the uncertainty based on acceptation, rejection and uncertain part, independently. It is nothing but just an extension of fuzzy set [3], intuitionistic fuzzy set [4-6], and interval valued fuzzy sets [7] beyond the unipolar fuzzy space. It characterizes the uncertainty based on a truth-membership function (T), an indeterminate-membership function (I) and a falsity-membership function(F) independently of a defined neutrosophic set via real a standard or non-standard unit interval]$^{-}0$, 1^{+}[. One of the best suitable example is for the neutrosophic logic is win/loss and draw of a match, opinion of people towards an event is based on its acceptance, rejection and uncertain values. These properties of neutrosophic set differentiate it from any of the available approaches in fuzzy set theory while measuring the indeterminacy. Due to which mathematics of single valued neutrosophic sets (abbr. SVNS) [8] as well as interval valued neutrosophic sets (abbr.IVNS) [9-10] is introduced for precise analysis of indeterminacy in the given interval. The IVNS represents the acceptance, rejection and uncertain membership functions in the unit interval [0, 1] which helped a lot for knowledge processing tasks using different classifier [11], similarity method [12-14] as well as multi-decision making process [15-17] at user defined weighted method [18-24]. In this process a problem is addressed while drawing the interval-valued neutrosophic graph, its spectrum and energy analysis. To achieve this goal, the current paper tried to focus on introducing these related properties and its analysis using MATLAB.

2 Literature Review

There are several applications of graph theory which is a mathematical tool provides a way to visualize the given data sets for its precise analysis. It is utilized for solving several mathematical problems. In this process, a problem is addressed while representing the uncertainty and vagueness exists in any given attributes (i.e. vertices) and their corresponding relationship i.e edges. To deal with this problem, the properties of fuzzy graph [25-26] theory is extended to intuitionistic fuzzy graph [28-30], interval valued fuzzy graphs [31] is studied with applications [32—33]. In this case a problem is addressed while measuring with indeterminacy and its situation. Hence, the neutrosophic graphs and its properties is introduced by Smaranadache [34-37] to characterizes them using their truth, falsity, and indeterminacy membership-values (T, I, F) with its applications [38-40]. Broumi et al. [41] introduced neutrosophic graph theory considering (T, I, F) for vertices and edges in the graph specially termed as “Single valued neutrosophic graph theory (abbr. SVNG)” with its other properties [42-44]. Afterwards several researchers studied the neutrosophic graphs and its applications [65, 68]. Broumi et al. [50] utilized the

SVNGs to find the shortest path in the given network subsequently other researchers used it in different fields [51-53, 59-60, 65]. To measure the partial ignorance, Broumi et al. [45] introduced interval valued-neutrosophic graphs and its related operations [46-48] with its application in decision making process in various extensions[49, 54, 57 61, 62, 64,73-84].

Some other researchers introduced antipodal single valued neutrosophic graphs [63, 65], single valued neutrosophic digraph [68] for solving multi-criteria decision making. Naz et al.[69] discussed the concept of energy and laplacian energy of SVNGs. This given a major thrust to introduce it into interval-valued neutrosophic graph and its matrix. The matrix is a very useful tool in representing the graphs to computers, matrix representation of SVNG, some researchers study adjacency matrix and incident matrix of SVNG. Varol et al. [70] introduced single valued neutrosophic matrix as a generalization of fuzzy matrix, intuitionistic fuzzy matrix and investigated some of its algebraic operations including subtraction, addition, product, transposition. Uma et al. [66] proposed a determinant theory for fuzzy neutrosophic soft matrices. Hamidiand Saeid [72] proposed the concept of accessible single-valued neutrosophic graphs.

It is observed that, few literature have shown the study on energy of IVNG. Hence this paper, introduces some basic concept related to the interval valued neutrosophic graphs are developed with an interesting properties and its illustration for its various applications in several research field.

3 Preliminaries

This section consists some of the elementary concepts related to the neutrosophic sets, single valued neutrosophic sets, interval-valued neutrosophic sets, single valued neutrosophic graphs and adjacency matrix for establishing the new mathematical properties of interval-valued neutrosophic graphs. Readers can refer to following references for more detail about basics of these sets and their mathematical representations [1, 8, 41].

Definition 3.1:[1] Suppose ξ be a nonempty set. A neutrosophic set (abbr.NS) N in$\xi$is an object taking the form N_{NS}= {<x: $T_N(k)$, $I_N(k)$, $F_N(k)$>, k∈ ξ} (1)

Where $T_N(k)$:ξ →]$^-$0,1$^+$[, $I_N(k)$:ξ →]$^-$0,1$^+$[,$F_N(k)$:ξ →]$^-$0,1$^+$[are known as truth-membership function, indeterminate –membership function and false-membership unction, respectively. The neutrosophic sets is subject to the following condition:

$$^-0 \leq T_N(k)+I_N(k)+F_N(k) \leq 3^+ \qquad (2)$$

Definition 3.2:[8]Suppose ξ be a nonempty set. A single valued neutrosophic sets N (abbr. SVNs) inξ is an object taking the form:

$$N_{SVNs}=\{<k:T_N(k), I_N(k), F_N(k)>, k\in \xi\} \quad (3)$$

where $T_N(k)$, $I_N(k)$, $F_N(k)$ ∈ [0, 1] are mappings. $T_N(k)$denote the truth-membership function of an element x ∈ ξ, $I_N(k)$denote the indeterminate –membership function of an element k ∈ ξ.$F_N(k)$denote the false–membership function of an element k ∈ ξ. The SVNs subject to condition

$$0 \leq T_N(k)+I_N(k)+F_N(k) \leq 3 \qquad (4)$$

Example 3.3: Let us consider following example to understand the indeterminacy and neutrosophic logic:

In a given mobile phone suppose 100 calls came at end of the day.

1. 60 calls were received truly among them 50 numbers are saved and 10 were unsaved in mobile. In this case these 60 calls will be considered as truth membership i.e. 0.6.

2. 30 calls were not-received by mobile holder. Among them 20 calls which are saved in mobile contacts were not received due to driving, meeting, or phone left in home, car or bag and 10 were not received due to uncertain numbers. In this case all 30 not received numbers by any cause (i.e. driving, meeting or phone left at home) will be considered as Indeterminacy membership i.e. 0.3.

3. 10 calls were those number which was rejected calls intentionally by mobile holder due to behavior of those saved numbers, not useful calls, marketing numbers or other cases for that he/she do not want to pick or may be blocked numbers. In all cases these calls can be considered as false i.e. 0.1 membership value.

The above situation can be represented as (0.6, 0.3, 0.1) as neutrosophic set.

Definition 3.4: [10] Suppose ξ be a nonempty set. An interval valued neutrosophic sets N (abbr.IVNs) in ξ is an object taking the form:

$$N_{IVNs}=\{<k:\tilde{T}_N(k), \tilde{I}_N(k), \tilde{F}_N(k)>, k\in \xi>\} \quad (5)$$

Where $\tilde{T}_N(k)$, $\tilde{I}_N(k)$, $\tilde{F}_N(k) \subseteq int[0,1]$ are mappings. $\tilde{T}_N(k)=[T_N^L(k), T_N^U(k)]$ denote the interval truth-membership function of an element k$\in \xi$.$\tilde{I}_N(k)=[I_N^L(x), I_N^U(k)]$ denote the interval indeterminate-membership function of an element k$\in \xi$.$\tilde{F}_N(k)=[F_N^L(k), F_N^U(k)]$ denote the false-membership function of an element k$\in \xi$.

Definition 3.4: [10] For every two interval valued-neutrosophic sets A and B in ξ, we define

$$(N \cup M)(k)= ([T_C^L(k), T_C^U(k)], [I_C^L(k), I_C^U(k)], [F_C^L(k), F_C^U(k)]) \text{ for all } k \in \xi \quad (6)$$

Where

$T_C^L(k)= T_N^L(k) \vee T_M^L(k)$, $T_C^U(k)= T_N^U(k) \vee T_M^U(k)$

$I_C^L(k)= I_N^L(k) \wedge I_M^L(k)$, $I_C^U(k)= I_N^U(k) \wedge I_M^U(k)$

$F_C^L(k)= F_N^L(k) \wedge F_M^L(k)$, $F_C^U(k)= F_N^U(k) \wedge F_M^U(k)$

Definition 3.5: [41] A pair G=(V,E) is known as single valued neutrosophic graph (abbr.SVNG) if the following holds:

1. V= $\{k_i$:i=1,..,n$\}$ such as T_1:V→ [0,1] is the truth-membership degree, I_1:V→[0,1] is the indeterminate –membership degree and F_1:V→[0,1]is the false membership degree of $k_i \in$ V subject to condition

$$0 \leq T_1(k_i)+I_1(k_i)+F_1(k_i) \leq 3 \quad (7)$$

2. E=$\{(k_i, k_j)$: $(k_i, k_j) \in V \times V\}$ such as $T_2:V \times V \to [0,1]$ is the truth-memebership degree, $I_2:V \times V \to [0,1]$ is the indeterminate –membership degree and $F_2:V \times V \to [0,1]$ is the false-memebership degree of $(k_i,k_j) \in$ E defined as

$$T_2(k_i,k_j) \leq T_1(k_i) \wedge T_1(k_j) \quad (8)$$

$$I_2(k_i, k_j) \geq I_1(k_i) \vee I_1(k_j) \quad (9)$$

$$F_2(k_i, k_j) \geq F_1(k_i) \vee F_1(k_j) \quad (10)$$

Subject to condition $\quad 0 \leq T_2(k_ik_2)+I_2(k_1k_2)+F_2(k_ik_j) \leq 3 \ \forall \ (k_i, k_j) \in$ E. (11)

The Fig. 1 shows an illustration of SVNG.

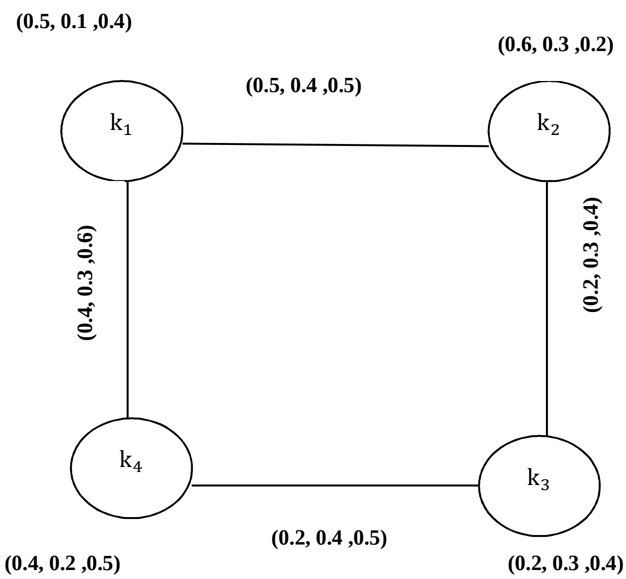

Fig. 1. An illustration of single valued neutrosophic graph

Definition 3.6[41]. A single valued neutrosophic graph G=(N, M) of G^*= (V, E) is termed strong single valued neutrosophic graph if the following holds:

$T_M(k_ik_j)= T_N(k_i) \wedge T_N(k_j)$ (12)

$I_M(k_ik_j) = I_N(k_i) \vee I_N(k_i)$ (13)

$F_M(k_ik_j)= F_N(k_i) \vee F_N(k_j)$ (14)

$\forall$ (k_i , k_j) ∈ E.

Where the operator ∧denote minimum and the operator ∨denote the maximum

Definition 3.8[41]. A single valued neutrosophic graph G=(N, M) of G^*= (V, E) is termed complete single valued neutrosophic graph if the following holds:

$T_M(k_ik_j)= T_N(k_i) \wedge T_N(k_j)$ (15)

$I_M(k_ik_j) = I_N(k_i) \vee I_N(k_i)$ (16)

$F_M(k_ik_j)= F_N(k_i) \vee F_N(k_j)$ (17)

$\forall k_i$, k_j ∈V.

Definition 3.9:[70] The Eigen value of a graph G are the Eigen values of its adjacency matrix.

Definition 3.10:[70]The spectrum of a graph is the set of all Eigen values of its adjacency matrix

$$\lambda_1 \geq \lambda_2 \dots \geq \lambda_n \qquad (18)$$

Definition 3.11:[70]The energy of the graph G is defined as the sum of the absolute values of its eigenvalues and denoted it by E(G):

$$E(G)=\sum_{i=1}^{n}|\lambda_i| \qquad (19)$$

4.Some Basic Concepts of Interval Valued Neutrosophic Graphs

Throughout this paper, we abbreviate G^*=(V, E) as a crisp graph, and G=(N, M) an interval valued neutrosophic graph.In this section we have defined some basic concepts of interval valued neutrosophic graphs and discuses some of their properties.

Definition 4.1:[45] A pair G=(V,E) is called an interval valued neutrosophic graph (abbr.IVNG) if the following holds:

1. V= {k_i:i=1,..,n} such as T_1^L:V→ [0,1] is the lower truth-membership degree,T_1^U:V→ [0,1] is the upper truth-membership degree,I_1^L:V→ [0,1] is the lower indeterminate-membership degree,I_1^U:V→ [0,1] is the upper indterminate-membership degree, and F_1^L:V→ [0,1] is the lower false-membership degree,F_1^U:V→ [0,1] is the upper false-membership degree,of v_i ∈ V subject to condition

$$0 \leq T_1^U(k_i)+I_1^U(k_i)+F_1^U(k_i) \leq 3 \qquad (20)$$

2. E={(k_i, k_j): $(k_i, k_j) \in V \times V$} such as $T_2^L: V \times V \rightarrow [0,1]$ is the lower truth-memebership degree, as $T_2^U: V \times V \rightarrow [0,1]$ is the upper truth-memebership degree, $I_2^L: V \times V \rightarrow [0,1]$ is the lower indeterminate-memebership degree, $I_2^U: V \times V \rightarrow [0,1]$ is the upper indeterminate-memebership degree and $F_2^L: V \times V \rightarrow [0,1]$ is the lower false-memebership degree, $F_2^U: V \times V \rightarrow [0,1]$ is the upper false-memebership degree of (k_i,k_j) ∈ E defined as

$$T_2^L(k_i,k_j)\leq T_1^L(k_i) \wedge T_1^L(k_j) ,T_2^U(k_i,k_j)\leq T_1^U(k_i) \wedge T_1^U(k_j) \qquad (21)$$

$$I_2^L(k_i,k_j)\geq I_1^L(k_i) \vee I_1^L(k_j) ,I_2^U(k_i,k_j)\geq I_1^U(k_i) \vee I_1^U(k_j) \qquad (22)$$

$$F_2^L(k_i,k_j)\geq F_1^L(k_i) \vee F_1^L(k_j) ,F_2^U(k_i,k_j)\geq F_1^U(k_i) \vee F_1^U(k_j) \qquad (23)$$

Subject to condition $0 \leq T_2^U(k_ik_2)+I_2^U(k_1k_2)+F_2^U(k_ik_j) \leq 3 \ \forall \ (k_i, k_j) \in$ E. (24)

Example 4.2. Consider a crisp graph G^* such that V= $\{k_1, k_2, k_3\}$, E=$\{k_1k_2, k_2k_3, k_3k_4\}$. Suppose N be an interval valued neutrosophic subset of V and suppose M an interval valued neutrosophic subset of E denoted by:

	k_1	k_2	k_3
T_N^L	0.3	0.2	0.1
T_N^U	0.5	0.3	0.3
I_N^L	0.2	0.2	0.2
I_N^U	0.3	0.3	0.4
F_N^L	0.3	0.1	0.3
F_N^U	0.4	0.4	0.5

	k_1k_2	k_2k_3	k_3k_1
T_M^L	0.1	0.1	0.1
T_M^U	0.2	0.3	0.2
I_M^L	0.3	0.4	0.3
I_M^U	0.4	0.5	0.5
F_M^L	0.4	0.4	0.4
F_M^U	0.5	0.5	0.6

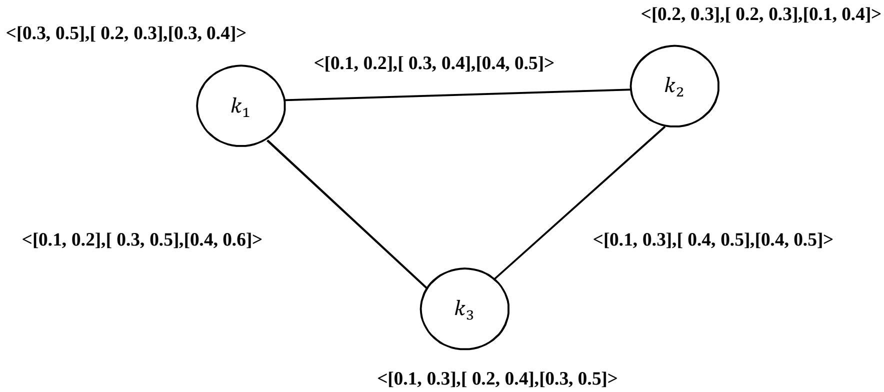

Fig. 2.Example of an interval valued neutrosophic graph

Definition 4.3A graph G=(N , M) is termed simple interval valued neutrosophic graph if it has neither self lops nor parallel edges in an interval valued neutrosophic graph.

Definition 4.4 The degree d(k) of any vertex k of an interval valued neutrosophic graph G=(N, M) is defined as follow:

$$d(v)= [\, d_T^L(k), d_T^U(k)],[d_I^L(k), d_I^U(k)],[d_F^L(k), d_F^U(k)] \quad (25)$$

Where

$d_T^L(k)= \sum_{k_i \neq k_j} T_M^L\,(k_ik_j)$ known as the degree of lower truth-membership vertex

$d_T^U(k)= \sum_{k_i \neq k_j} T_M^U\,(k_ik_j)$ known as the degree of upper truth-membership vertex

$d_I^L(k)= \sum_{k_i \neq k_j} I_M^L\,(k_ik_j)$ known as the degree of lower indterminate-membership vertex

$d_I^U(k)= \sum_{k_i \neq k_j} I_M^U\,(k_ik_j)$ known as the degree of upperindeterminate-membership vertex

$d_F^L(k)= \sum_{k_i \neq k_j} F_M^L\,(k_ik_j)$ known as the degree of lower false-membership vertex

$d_F^U(k)= \sum_{k_i \neq k_j} F_M^U\,(k_ik_j)$ known as the degree of upperfalse-membership vertex

Example 4.5 Consider an IVNG G=(N, M) presented in Fig. 4 with vertices set V=$\{k_i: i = 1,..,4\}$ and edges set E =$\{k_1k_4\ ,k_4k_3,\ k_3k_2, k_2k_1\}$.

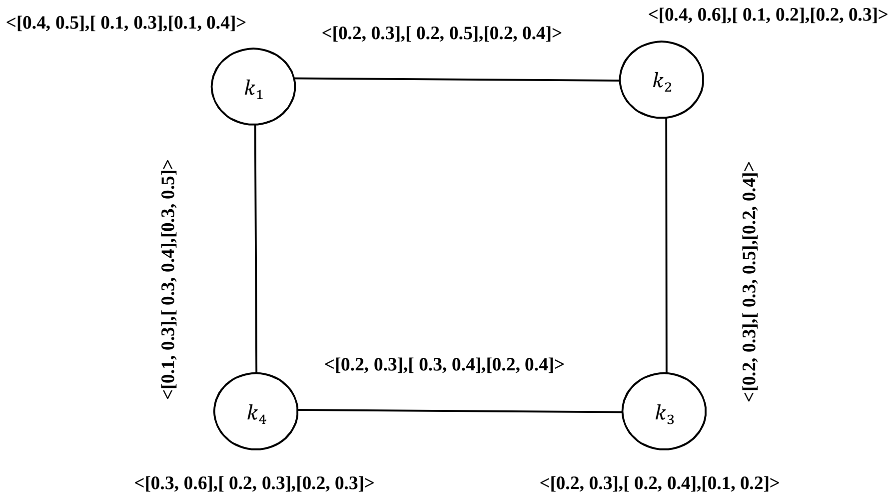

Fig. 4.Illutstrationof an interval valued neutrosophic graph

The degree of each vertex $k_i$is given as follows:

$d(\mathrm{k}_1)$= ([0.3, 0.6], [0.5, 0.9], [0.5, 0.9]),

$d(\mathrm{k}_2)$= ([0.4, 0.6], [0.5, 1.0], [0.4, 0.8]),

$d(\mathrm{k}_3)$= ([0.4, 0.6], [0.6, 0.9], [0.4, 0.8]),

$d(\mathrm{k}_4)$= ([0.3, 0.6], [0.6, 0.8], [0.5, 0.9]).

Definition 4.6. A graph G=(N, M) is termed regular interval valued neutrosophic graph if d(k)=r=([$\mathrm{r}_{1\mathrm{L}}$, $\mathrm{r}_{1\mathrm{U}}$], [$\mathrm{r}_{2\mathrm{L}}$, $\mathrm{r}_{2\mathrm{U}}$], [$\mathrm{r}_{3\mathrm{L}}$, $\mathrm{r}_{3\mathrm{U}}$]), ∀ k ∈ V.

(i.e.) if each vertex has same degree r, then G is said to be a regular interval valued neutrosophic graph of degree r.

Definition 4.7. A graph G=(N,M) is termed irregular interval valued neutrosophic graph if the degree of some vertices are different than other.

Example 4.8 Let us Suppose, G is a regular interval-valued neutrosophic graph as portrayed in Fig. 5 having vertex set V={k_1, k_2, k_3, k_4} and edge sets E={$\mathrm{k}_1\mathrm{k}_2$,$\mathrm{k}_2\mathrm{k}_3$, $\mathrm{k}_3\mathrm{k}_4$,$\mathrm{k}_4\mathrm{k}_1$} as follows.

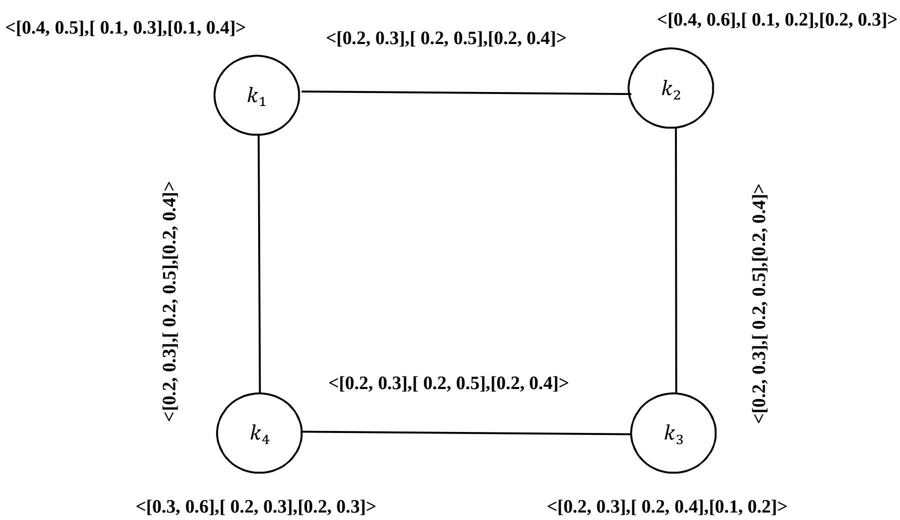

Fig.5 .Regular IVN-graph.

In the Fig. 5. All adjacent vertices k_1k_4 , k_4k_3 , k_3k_2 , k_2k_1 have the same degree equal <[0.4,0.6],[0.4,1],[0.4,0.8]>. Hence, the graph G is a regular interval valued neutrosophic graph.

Definition 4.9 A graph G= (N, M) on $G^*$is termed strong interval valued neutrosophic graph if the following holds:

$$T_M^L(k_i, k_j) = T_N^L(k_i) \wedge T_N^L(k_j)$$
$$T_M^U(k_i, k_j) = T_N^U(k_i) \wedge T_N^U(k_j)$$
$$I_M^L(k_i, k_j) = I_N^L(k_i) \vee I_N^L(k_j)$$
$$I_M^U(k_i, k_j) = I_N^U(k_i) \vee I_N^U(k_j)$$
$$F_M^L(k_i, k_j) = F_N^L(k_i) \vee F_N^L(k)$$
$$F_M^U(k_i, k_j) = F_N^U(k_i) \vee F_N^U(k_j) \ \forall (k_i, k_j) \in E \qquad (26)$$

Example 4.10. Consider the strong interval valued neutrosophic graph G=(N, M) in Fig. 6 with vertex set N ={k_1, k_2, k_3, k_4}and edge set M={k_1k_2, k_2k_3, k_3k_4, k_4k_1} as follows:

	k_1	k_2	k_3
T_N^L	0.3	0.2	0.1
T_N^U	0.5	0.3	0.3
I_N^L	0.2	0.2	0.2
I_N^U	0.3	0.3	0.4
F_N^L	0.3	0.1	0.3
F_N^U	0.4	0.4	0.5

	k_1k_2	k_2k_3	k_3k_1
T_M^L	0.2	0.1	0.1
T_M^U	0.3	0.3	0.3
I_M^L	0.2	0.2	0.2
I_M^U	0.3	0.4	0.4
F_M^L	0.3	0.3	0.3
F_M^U	0.4	0.4	0.5

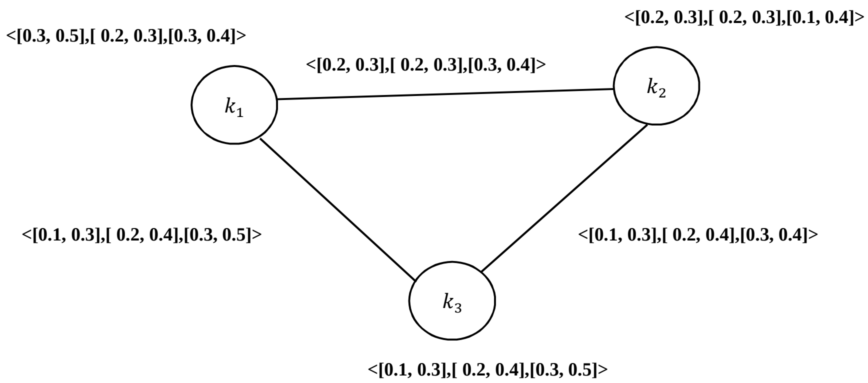

Fig.6.Illustration of strong IVNG

Proposition 4.11 For every $k_i, k_j \in V$, we have

$$T_M^L(k_i, k_j) = T_M^L(k_j, k_i) \text{ and } \quad T_M^U(k_i, k_j) = T_M^U(k_j, k_i)$$
$$I_M^L(k_i, k_j) = I_M^L(k_j, k_i) \text{ and } \quad I_M^U(k_i, k_j) = I_M^U(k_j, k_i)$$
$$F_M^L(k_i, k_j) = F_M^L(k_j, k_i) \text{ and } \quad F_M^U(k_i, k_j) = F_M^U(k_j, k_i) \qquad (27)$$

Proof. Suppose G =(N, M) be an interval valued neutrosophic graph, suppose $\boldsymbol{k_i}$ is a neigbourhood of $\boldsymbol{k_j}$ in G.Then , we have

$\boldsymbol{T^L_M(k_i, k_j)}$ =min [$\boldsymbol{T^L_N(k_i), T^L_N(k_j)}$] and $\boldsymbol{T^U_M(k_i, k_j)}$ =min [$\boldsymbol{T^U_N(k_i), T^U_N(k_j)}$]

$\boldsymbol{I^L_M(k_i, k_j)}$ =max [$\boldsymbol{I^L_N(k_i), I^L_N(k_j)}$] and $\boldsymbol{I^U_M(k_i, k_j)}$ =max[$\boldsymbol{I^U_N(k_i), I^U_N(k_j)}$]

$\boldsymbol{F^L_M(k_i, k_j)}$ =max [$\boldsymbol{F^L_N(k_i), F^L_N(k_j)}$] and $\boldsymbol{F^U_M(k_i, k_j)}$ =max [$\boldsymbol{F^U_N(k_i), F^U_N(k_j)}$]

Similarly we have also for

$\boldsymbol{T^L_M(k_j, k_i)}$ =min [$\boldsymbol{T^L_N(k_j), T^L_N(k_i)}$] and $\boldsymbol{T^U_M(k_j, k_i)}$ =min [$\boldsymbol{T^U_N(k_j), T^U_N(k_i)}$]

$\boldsymbol{I^L_M(k_j, k_i)}$ =max [$\boldsymbol{I^L_N(k_j), I^L_N(k_i)}$] and $\boldsymbol{I^U_M(k_j, k_i)}$ =max[$\boldsymbol{I^U_N(k_j), I^U_N(k_i)}$]

$\boldsymbol{F^L_M(k_j, k_i)}$ =max [$\boldsymbol{F^L_N(k_j), F^L_N(k_i)}$] and $\boldsymbol{F^U_M(k_j, k_i)}$ =max [$\boldsymbol{F^U_N(k_j), F^U_N(k_i)}$]

Thus

$\boldsymbol{T^L_M(k_i, k_j)} = \boldsymbol{T^L_M(k_j, k_i)}and\boldsymbol{T^U_M(k_i, k_j)} = \boldsymbol{T^U_M(k_j, k_i)}$
$\boldsymbol{I^L_M(k_i, k_j)} = \boldsymbol{I^L_M(k_j, k_i)}$***and***$\boldsymbol{I^U_M(k_i, k_j)} = \boldsymbol{I^U_M(k_j, k_i)}$
$\boldsymbol{F^L_M(k_i, k_j)} = \boldsymbol{F^L_M(k_j, k_i)}and\boldsymbol{F^U_M(k_i, k)} = \boldsymbol{F^U_M(k_j, k_i)}$

Definition 4.12 The graph G= (N, M) is termed an interval valued neutrosophic graph if the following holds

$\boldsymbol{T^L_M(k_i, k_j)}$ =min [$\boldsymbol{T^L_N(k_i), T^U_N(k_j)}$] and $\boldsymbol{T^U_M(k_i, k_j)}$ =min [$\boldsymbol{T^U_N(k_i), T^U_N(k_j)}$]

$\boldsymbol{I^L_M(k_i, k_j)}$ =max [$\boldsymbol{I^L_N(k_i), I^L_N(k_j)}$] and $\boldsymbol{I^U_M(k_i, k_j)}$ =max[$\boldsymbol{I^U_N(k_i), I^U_N(k_j)}$]

$\boldsymbol{F^L_M(k_i, k_j)}$ =max [$\boldsymbol{F^L_N(k_i), F^L_N(k_j)}$] and $\boldsymbol{F^U_M(k_i, k_j)}$ =max [$\boldsymbol{F^U_N(k_i), F^U_N(k_j)}$] $\forall \boldsymbol{k_i}, \boldsymbol{k_j} \in$ V (28)

Example 4.13. Consider the complete interval valued neutrosophic graph G=(N, M) portrayed in Fig. 7 with vertex set A ={k_1, k_2, k_3, k_4}and edge set E={k_1k_2,k_1k_3 ,k_2k_3, k_1k_4, k_3k_4 ,k_2k_4}as follows

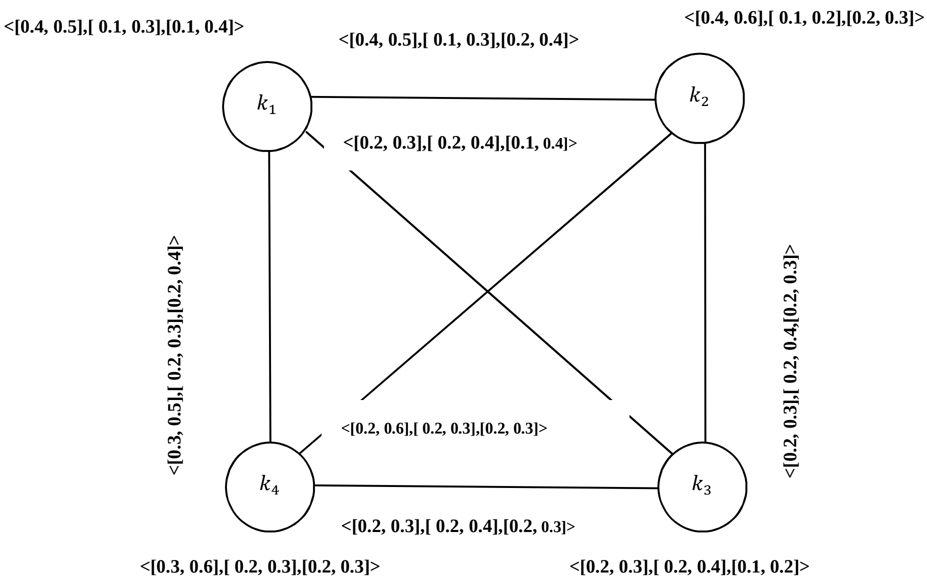

Fig.7 .Illustration of complete IVN-graph

In the following based on the extension of the adjacency matrix of SVNG [69], we defined the concept of adjacency matrix of IVNG as follow:

Definition 4.14: The adjacency matrix M(G) of IVNG G= (N, M) is defined as a square matrix M(G)=$[a_{ij}]$, with a_{ij}=<$\tilde{T}_M(k_i, k_j)$,$\tilde{I}_M^L(k_i, k_j)$,$\tilde{F}_M^L(k_i, k_j)$>, where

$\tilde{T}_M(k_i, k_j)$= [$T_M^L(k_i, k_j)$,$T^U(k_i, k_j)$] denote the strength of relationship

$\tilde{I}_M(k_i, k_j)$= [$I_M^L(k_i, k_j)$,$I_M^U(k_i, k_j)$] denote the strength of undecided relationship

$\tilde{F}_M(k_i, k_j)$=[$F_M^L(k_i, k_j)$,$F_M^U(k_i, k_j)$] denote the strength of non-relationship between k_i and k_j (29)

The adjacency matrix of an IVNG can be expressed as sixth matrices, first matrix contain the entries as lower truth-membership values, second contain upper truth-membership values, third contain lower indeterminacy-membership values, forth contain upper indeterminacy-membership, fifth contains lower non-membership values and the sixth contain the upper non-membership values, i.e.,

$$M(G)=<[T_M^L(k_i, k_j),T_M^U(k_i, k_j)]\,,\,[I_M^L(k_i, k_j),I_M^U(k_i, k_j)]\,,\,[F_M^L(k_i, k_j),F_M^U(k_i, k_j)]>, \quad (30)$$

From the Fig. 1, the adjacency matrix of IVNG is defined as:

$$\boldsymbol{M_G} = \begin{bmatrix} \mathbf{0} & <[\mathbf{0.1}, \mathbf{0.2}], [\mathbf{0.3}, \mathbf{0.4}], [\mathbf{0.4}, \mathbf{0.5}]> & <[\mathbf{0.1}, \mathbf{0.2}], [\mathbf{0.3}, \mathbf{0.5}], [\mathbf{0.4}, \mathbf{0.6}]> \\ <[\mathbf{0.1}, \mathbf{0.2}], [\mathbf{0.3}, \mathbf{0.4}], [\mathbf{0.4}, \mathbf{0.5}]> & \mathbf{0} & <[\mathbf{0.1}, \mathbf{0.3}], [\mathbf{0.4}, \mathbf{0.5}], [\mathbf{0.4}, \mathbf{0.5}]> \\ <[\mathbf{0.1}, \mathbf{0.2}], [\mathbf{0.3}, \mathbf{0.5}], [\mathbf{0.4}, \mathbf{0.6}]> & <[\mathbf{0.1}, \mathbf{0.3}], [\mathbf{0.4}, \mathbf{0.5}], [\mathbf{0.4}, \mathbf{0.5}]> & \mathbf{0} \end{bmatrix}$$

In the literature, there is no Matlab toolbox deals with neutrosophic matrix such as adjacency matrix and so on. Recently Broumi et al [58] developed a Matlab toolbox for computing operations on interval valued neutrosophicmatrices.So, we can inputted the adjacency matrix of IVNG in the workspace Matlab as portrayed in Fig. 8.

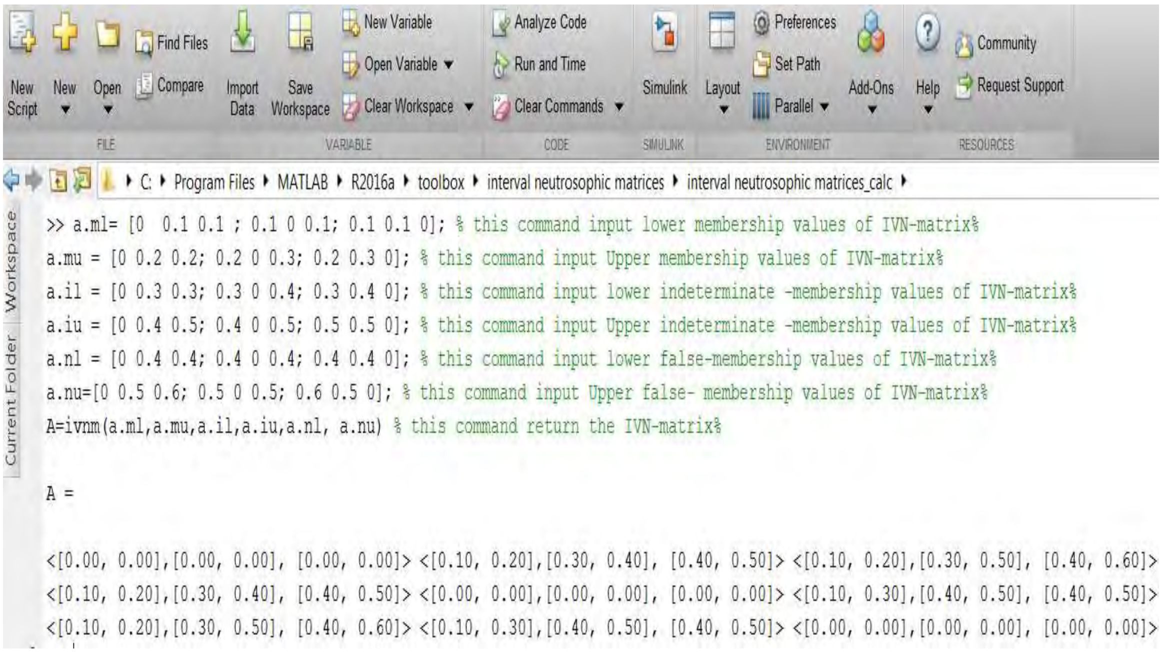

Fig. .8 Screen shot of Workspace MATLAB

Definition 4.15: The spectrum of adjacency matrix of an IVNG M(G) is defined as

$$<\tilde{R}, \tilde{S}, \tilde{Q}> = <[\tilde{R}^L, \tilde{R}^U], [\tilde{S}^L, \tilde{S}^U], <[\tilde{Q}^L, \tilde{Q}^U]> \quad (31)$$

Where $\tilde{R}^L$ is the set of eigenvalues of $M(T_M^L(k_i,k_j))$, $\tilde{R}^U$ is the set of eigenvalues of $M(T_B^U(k_i,k_j))$, $\tilde{S}^L$ is the set of eigenvalues of $M(I_M^L(k_i,k_j))$, $\tilde{S}^U$ is the set of eigenvalues of $M(I_M^U(k_i,k_j))$, $\tilde{Q}$ is the set of eigenvalues of $M(F_M^L(k_i,k_j))$ and $\tilde{Q}^U$ is the set of eigenvalue of $M(F_B^U(k_i,k_j))$ respectively.

Definition 4.16: The energy of an IVNG G= (N,M) is defined as

$$\mathbf{E(G)} = <\mathbf{E}(\tilde{\boldsymbol{T}}_{\boldsymbol{M}}(\boldsymbol{k_i},\boldsymbol{k_j})), \boldsymbol{E}(\tilde{\boldsymbol{I}}_{\boldsymbol{M}}^{\boldsymbol{L}}(\boldsymbol{k_i},\boldsymbol{k_j})), \boldsymbol{E}(\tilde{\boldsymbol{F}}_{\boldsymbol{M}}^{\boldsymbol{L}}(\boldsymbol{k_i},\boldsymbol{k_j}))> \quad (32)$$

Where

$$E(\tilde{T}_M(k_i,k_j)) = [E(T_M^L(k_ik_j)), E(T_M^U(k_ik_j))] = [\sum_{\substack{i=1\\ \lambda_i^L \in \tilde{R}^L}}^{n} |\lambda_i^L|, \sum_{\substack{i=1\\ \lambda_i^U \in \tilde{R}^U}}^{n} |\lambda_i^U|]$$

$$E(\tilde{I}_M(k_i,k_j)) = [E(I_M^L(k_ik_j)), E(I_M^U(k_ik_j))] = [\sum_{\substack{i=1\\ \delta_i^L \in \tilde{S}^L}}^{n} |\delta_i|, \sum_{\substack{i=1\\ \delta_i^U \in \tilde{S}^U}}^{n} |\delta_i^U|]$$

$$E(\tilde{F}_M(k_i,k_j)) = [E(F_M^L(k_ik_j)), E(F_M^U(k_ik_j))] = [\sum_{\substack{i=1\\ \zeta_i^L \in \tilde{Q}^L}}^{n} |\zeta_i|, \sum_{\substack{i=1\\ \zeta_i^U \in \tilde{Q}^U}}^{n} |\zeta_i^U|]$$

Definition 4.17: Two interval valued neutrosophic graphs G_1 and G_2 are termed equienergetic, if they have the same number of vertices and the same energy.

Proposition4.18: If an interval valued neutrosophic G is both regular and totally regular, then the eigen values are balanced on the energy.

$$\sum_{i=1}^{n} \pm\lambda_i^L = 0,\ \sum_{i=1}^{n} \pm\lambda_i^U = 0,\ \sum_{i=1}^{n} \pm\delta_i = 0,\ \sum_{i=1}^{n} \pm\delta_i^U = 0,\ \sum_{i=1}^{n} \pm\zeta_i = 0 \text{ and } \sum_{i=1}^{n} \pm\zeta_i^U = 0. \quad (33)$$

4.19. MATLAB program for findingspectrum of an interval valued neutrosophic graph

To generate the MATLAB program for finding the spectrum of interval valued neutrosophic graph. The program termed "Spec.m" is written as follow:

```
Function SG=Spec(A);
% Spectrum of an interval valued neutrosophic matrix A
% "A" have to be an interval valued neutrosophic  matrix - "ivnm" object:
a.ml=eig(A.ml);          % eigenvalues of lower membership of ivnm%
a.mu=eig(A.mu);          % eigenvalues of upper membership of ivnm%
a.il=eig(A.il);          % eigenvalues of lower rindeterminate-membership of ivnm%
a.iu=eig(A.iu);          % eigenvalues of upper indterminate- membership of ivnm%
a.nl=eig(A.nl);          % eigenvalues of lower false-membership of ivnm%
a.nu=eig(A.nu);          % eigenvalues of upper false-membership of ivnm%
SG=ivnm(a.ml,a.mu,a.il,a.iu,a.nl,a.nu);
```

4.20. MATLAB program for finding energy of an interval valued neutrosophic graph

To generate the MATLAB program for finding the energy of interval valued neutrosophic graph. The program termed "ENG.m"iswritten as follow:

```
function EG=ENG(A);
% energy of an interval valued neutrosophic matrix A
% "A" have to be an interval valued neutrosophic  matrix - "ivnm" object:
a.ml=sum(abs(eig(A.ml)));
a.mu=sum(abs(eig(A.mu)));
a.il=sum(abs(eig(A.il)));
a.iu=sum(abs(eig(A.iu)));
a.nl=sum(abs(eig(A.nl)));
a.nu=sum(abs(eig(A.nu)));
EG=ivnm(a.ml,a.mu,a.il,a.iu,a.nl,a.nu);
```

Example4.21: The spectrum and the energy of an IVNG, illustrated in Fig. 6, are given below:

Spec($T_M^L(k_ik_j)$)={ -0.10, -0.10,0.20}, Spec($T_M^U(k_ik_j)$)={-0.30,-0.17,0.47}
Spec($I_M^L(k_ik_j)$)={-0.40,-0.27,0.67}, Spec($I_M^U(k_ik_j)$)={-0.53,-0.40,0.93]}
Spec($F_M^L(k_ik_j)$)={-0.40,-0.40,0.80}, Spec($F_M^U(k_ik_j)$)={ -0.60,-0.47,1.07}

Hence,
Spec(G)={<[-0.10, -0.30], [-0.40, -0.53],[-0.40, -0.60]>, <[-0.10, -0.17], [-0.27, -0.40],[-0.40, -0.47]>, <[0.20, 0.47], [0.67, 0.93],[0.80, 1.07]>}
Now ,
E($T_M^L(k_ik_j)$)=0.40, E($T_M^U(k_ik_j)$)=0.94

E($I_M^L(k_ik_j)$)=1.34,E($I_M^U(k_ik_j)$)=1.87

E($F_M^L(k_ik_j)$)=1.60,E($F_M^U(k_ik_j)$)=2.14

Therefore
E(G)= <[0.40, 0.94],[1.34, 1.87], [1.60, 2.14]>
Based on toolbox MATLAB developed in [58], the readers can run the program termed "Spec.m", for computing the spectrum of graph, by writing in command window "Spec (A)" as described below:

```
>> Spec(A) % this command return the spectrum of IVN-matrix%
Warning! The created new object is NOT an interval valued neutrosophic matrix

ans =

<[-0.10, -0.30],[-0.40, -0.53], [-0.40, -0.60]>
<[-0.10, -0.17],[-0.27, -0.40], [-0.40, -0.47]>
<[0.20, 0.47],[0.67, 0.93], [0.80, 1.07]>
```

Similarly, the readers can also run the program termed "ENG.m", for computing the energy of graph, by writing in command window "ENG (A) as described below:

```
>> ENG(A) % this command return the Energy of IVN-matrix%
Warning! The created new object is NOT an interval valued neutrosophic matrix

ans =

<[0.40, 0.94],[1.34, 1.87], [1.60, 2.14]>
```

In term of the number of vertices and the sum of interval truth-membership, interval indeterminate-membership and interval false-membership, we define the upper and lower bounds on energy of an IVNG.

Proposition 4.22. Suppose G= (N, M) be an IVNG on n vertices and the adjacency matrix of G.then

$$\sqrt{2\sum_{1\le ij\le n}\left(T_M^L(k_ik_j)\right)^2+n(n-1)|T^L|^{2/N}}\le E(T_M^L(k_ik_j))\le\sqrt{2n\sum_{1\le ij\le n}\left(T_M^L(k_ik_j)\right)^2} \quad (34)$$

$$\sqrt{2\sum_{1\le ij\le n}\left(T_M^U(k_ik_j)\right)^2+n(n-1)|T^U|^{2/N}}\le E(T_M^U(k_ik_j))\le\sqrt{2n\sum_{1\le ij\le n}\left(T_M^U(k_ik_j)\right)^2} \quad (35)$$

$$\sqrt{2\sum_{1\le ij\le n}\left(I_M^L(k_ik_j)\right)^2+n(n-1)|I^L|^{2/N}}\le E(I_M^L(k_ik_j))\le\sqrt{2n\sum_{1\le ij\le n}\left(I_M^L(k_ik_j)\right)^2} \quad (36)$$

$$\sqrt{2\sum_{1\le ij\le n}\left(I_M^U(k_ik_j)\right)^2+n(n-1)|I^U|^{2/N}}\le E(I_M^U(k_ik_j))\le\sqrt{2n\sum_{1\le ij\le n}\left(I_M^U(k_ik_j)\right)^2} \quad (37)$$

$$\sqrt{2\sum_{1\le ij\le n}\left(F_M^L(k_ik_j)\right)^2+n(n-1)|F^L|^{2/N}}\le E(F_M^L(k_ik_j))\le\sqrt{2n\sum_{1\le ij\le n}\left(F_M^L(k_ik_j)\right)^2} \quad (38)$$

$$\sqrt{2\sum_{1\le ij\le n}\left(F_M^U(k_ik_j)\right)^2+n(n-1)|F^U|^{2/N}}\le E(F_M^U(k_ik_j))\le\sqrt{2n\sum_{1\le ij\le n}\left(F_M^U(k_ik_j)\right)^2} \quad (39)$$

Where $|\boldsymbol{T^L}|$,$|\boldsymbol{T^U}|$,$|\boldsymbol{I^L}|$,$|\boldsymbol{I^U}|$,$|\boldsymbol{F^L}|$and $|\boldsymbol{F^U}|$ are the determinant of $\boldsymbol{M(T_M^L(k_i,k_j))}$, $\boldsymbol{M(T_M^U(k_i,k_j))}$, $\boldsymbol{M(I_M^L(k_i,k_j))}$, $\boldsymbol{M(I_M^U(k_i,k_j))}$, $\boldsymbol{M(F_M^L(k_i,k_j))}$ and$\boldsymbol{M(F_M^U(k_i,k_j))}$,respectively.

Proof: proof is similar as in Theorem 3.2 [69]

Conclusion

This paper introduces some basic operations on interval-valued neutrosophic set to increase its utility in various fields for multi-decision process. To achieve this goal, a new mathematical algebra of interval-valued neutrosophic graphs, its energy as well as spectral computation is discussed with mathematical proof using MATLAB. In the near future, we plan to extend our research to interval valued neutrosophic digraphs and developed the concept of domination in interval valued-neutrosophic graphs. Same time the author will focus on handling its necessity for knowledge representation and processing tasks [85-87].

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Pestel analysis based on neutrosophic cognitive maps and neutrosophic numbers for the sinos river basin management

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Abstract. The Sinos River watershed is one of the most polluted water basins in Brazil with great efforts for its recovery through integral management. PESTEL is an analysis for the study of the external variables with influence in the efficiency of the organization or project. This paper presents a model to address problems encountered in the measurement and evaluation process of PESTEL analysis taking into account interdependencies among sub-factors and modeling uncertainty and indeterminacy in Sinos river basin. A Neutrosophic Cognitive Maps was used for modeling the integrated structure of PESTEL sub-factors. A quantitative analysis was developed based on static analysis and neutrosophic numbers. To demonstrate the applicability of the proposal in the Sinos river external factor analysis a case study is developed. Interdependencies among sub-factors were includes and uncertainty and indeterminacy were modeled in a practical way. Sub-factor was ranked and reduced, with Ecological, Technological and Social are the top three factors. The paper ends with a conclusion and future work recommendations.

Keywords: Sinos River Basin; PESTEL; Neutrosophy; Neutrosophic Cognitive Maps; Static Analysis

1 Introduction

PEST is an analysis for the study of the external variables with influence in the efficiency of the organization or project. These variables involved in the business environment are grouped in Political, Economical, Social, and Technological aspects [1].

The conceptual structure and nature of PEST require an integrated approach for considering importance and interrelation. The standard technical framework of the PEST approach mainly provides a general idea about macro conditions and the situation of an organization, so it is inadequate. Therefore, PEST analysis lacks a quantitative approach to the measurement of the interrelation between its factors. When the environment and legal factors are included, it is named PESTEL (Political, Economic, Socio-cultural, Technological, Environment, and Legal) analysis [2]. Political variables refer to the regulatory aspects that directly affect the enterprise. Here enter the taxes rules or business incentives in specific sectors, regulations on employment, the promotion of foreign trade, government stability, the system of government, international treaties or the existence of internal conflicts or with other current or future countries — also the way in which the different local, regional and national administrations are organized [3]. Economic variables relate to macroeconomic data, Gross domestic product (GDP) evolution, interest rates, inflation, unemployment rate, income level, exchange rates, access to resources, level of development, economic cycles. Current and future economic scenarios and economic policies should also be investigated.

Social variables take into account are demographic evolution, social mobility and changes in lifestyle — also the educational level and other cultural patterns, religion, beliefs, gender roles, tastes, fashions and consumption habits of society. In short, the social trends that may affect the enterprise business [3]. Technological variables are somewhat more complicated to analyze due to the high speed of the changes in this area. It is necessary to know the public investment in research and the promotion of technological development, the technology diffusion, the degree of obsolescence, the level of coverage, the digital device, the funds destined to R & D + I, as well as the trends in the use of new technologies. Ecological variables are the main factors to be analyzed aware of the conservation of the environment, environmental legislation, climate change, and temperature variations, natural risks, recycling levels, energy regulation and possible regulatory changes in this area[4]. Legal variables refer to legislation that is directly associated with the organization functions, information on licenses, labor legislation, intellectual property, health laws, and regulated sectors[5].

PESTEL analysis has deficiencies for a quantitative approach to the measurement of interrelation among factors are generally ignored [6]. Fuzzy cognitive maps (FCM) is a tool for modeling and analyzing interrelations [7]. Connections in FCMs are just numeric ones: the relationship of two events should be linear.

The Neutrosophy can operate with indeterminate and inconsistent information, while fuzzy sets and intuitionistic fuzzy sets do not describe them appropriately [4]. Neutrosophic cognitive maps (NCM) is an extension of FCM where was included the concept indeterminacy [8]. The concept of fuzzy cognitive maps fails to deal with the indeterminate relation [1].

In this paper, a PESTEL analysis based on neutrosophic cognitive maps is presented proposal methodological support and make possible of dealing with interdependence, feedback, and indeterminacy. Additionally, the new approach makes conceivable to category and to reduce factors.

This paper continues as follows: Section 2 reviews some essential concepts about the PESTEL analysis framework, NCM, and fuzzy numbers. In Section 3, a framework for the PESTEL shows a static analysis based on NCM. Section 4, displays a case study of the proposed model applied to social-environmental management of a river basin. The paper finishes with conclusions and additional work recommendations.

2. Case Study

The Sinos River Basin is one of the most contaminated water basins in Brazil [9] which leads to great efforts for its recovery through integral management. Due to the complex nature of the interrelations between the different factors involved in environmental quality management becomes intricate and therefore requires the use of tools that facilitate decision making[10]. Through a participatory exercise with stakeholder members of the COMITESINOS, external variables were identified and a diffuse cognitive map was constructed representing the relationships among the variables. This process of identifying PESTEL variables was carried out with the members of the committee, for which work sessions were held in the coordination meetings. To elaborate on the NCM, Mental Modeler tool of the website http://www.mentalmodeler.org/ was used.

Initially, factors and sub-factors were identified for Sinos river basin management as follows:

I. Relevant Political-Legal Aspects

In the political dimension, the following variables were identified:

1. Influence of the federal government in the watersheds management (N1)
2. Importance of the state government in the management of the basin (N2)
3. Control of the municipal government in the watershed management (N3)
4. Impact of bureaucracy on management (N4)
5. Corruption impact (N5)

II. Relevant economic and socio-economic aspects

In the socioeconomic dimension, the following variables were identified:

1. Poverty (N6)
2. Per capita income(N7)
3. Quality of solid waste collecting services (N8)
4. Quality liquid waste service (N9)
5. Water supply service (N10)
6. The quality of public health (N11)
7. Quality of sewage and sewage services (N12)

III. Relevant social aspects

In the social dimension, the variables identified were:

1. Public education (N13)
2. Population access to food (N14)
3. Access to the housing (N15)

IV. Relevant sociocultural aspects

In the sociocultural dimension, the variables identified were:
1. Perception of the environmental relevance in the local culture (N16)
2. Knowledge of environmental risk (N17)
3. Understanding of environmental awareness (N18)

V. Relevant technological aspects
The variables identified in the technical dimension were:
1. Innovation(N19)
2. Cleaner production(N20)
3. Eco-efficiency (N21)

For the ecological dimension was possible to identify the following variables:
1. Water quality index (WQI)(N22)
2. Air Quality index (AQI) (N23)
3. Landscape change and urban planning(N24)
4. Variations in the biodiversity index of ecosystems value (N25)
5. Climate Change (N31)
6. Soil Quality index (N32)

Legal dimension includes the following factors
1. Environmental Laws (N26)
2. Education regulation (27)
3. Health regulations (28
4. Environmental law (29)
5. Employment Laws (N30)
6. Consumer Law (33)

Interdependencies are identified and modeled using an NCM (Figure 1), with whose weighs represented in Table 1.

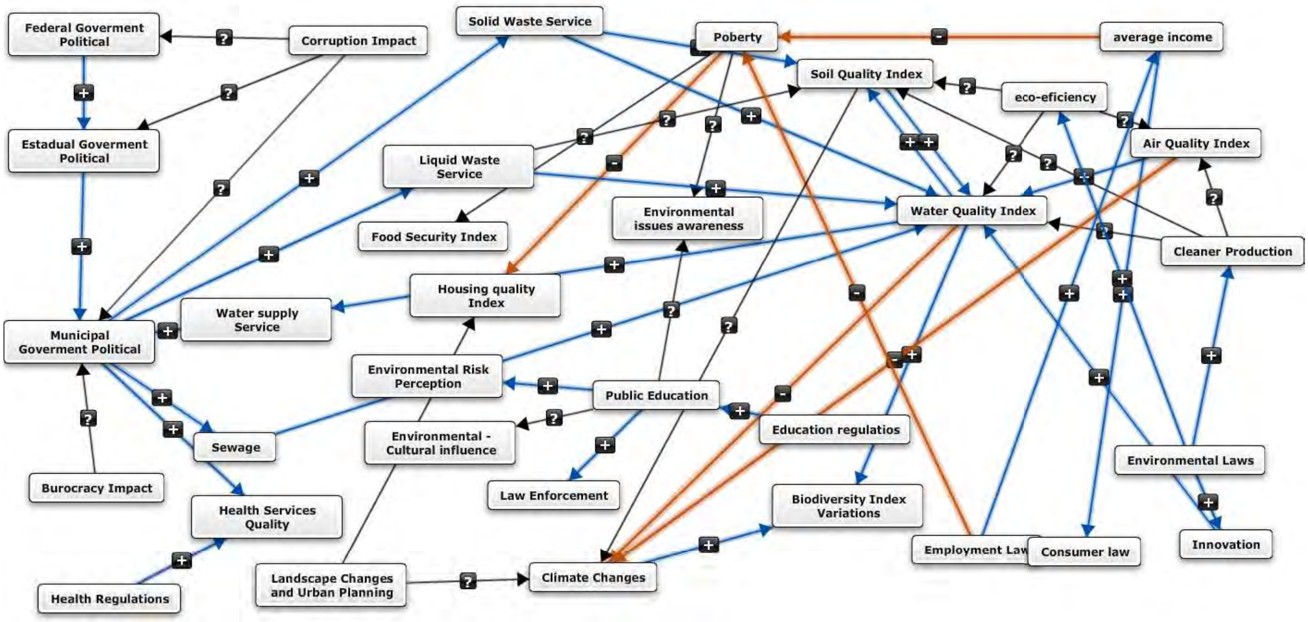

Figure 1: Fuzzy Neutrosophic Cognitive Maps of PESTEL factors.

2 Materials and Methods

2.1 Preliminaries

This article offers a first brief review by PESTEL analysis and the factors' interdependency. The following is a review of the basic concepts of NCM.

2.1.1 PESTEL Analysis

The PESTEL method is a prerequisite analysis with a network function to identify the characteristics of the environment in which an organization or project operates, provides data and information so that the organization can make predictions about new situations and circumstances and act accordingly. [12, 13]. The variables analyzed in PESTEL are identified and evaluated independently. [2] not taking into account interdependency. In [14] this approach based on fuzzy decision maps is presented taking into account the ambiguity, the uncertainty in their interrelationships.

This study presents a model to address the problems encountered in the PEST measurement and evaluation process, taking into account the interdependencies between the subfactors. NCM modeled the integrated structure of the PESTEL subfactor, and the quantitative analysis is developed from a static analysis that allows to classify and reduce the factors in line with the proposals presented in [15].

2.1.2 Neutrosophic Cognitive Maps.

The Neutrosophic Logic (NL) like a generalization of the fuzzy logic was introduced in 1995 [16]. According to this theory, a logical proposition P is characterized by three components:

NL (P) =(T,I,F) (1)

Where the neutrosophic component T is the degree of truth, F the degree of falsehood, and I is the degree of indeterminacy [7]. Neutrosophic set (NS) was introduced by F. Smarandache who introduced the degree of indeterminacy (i) as an independent component[11] .

A neutrosophic matrix content where the elements are $a = (a_{ij})$ have been replaced by elements in ⟨R∪I⟩. A neutrosophic graphic has at least one edge is a neutrosophic edge . If the indetermination is found in the cognitive map, it is called the neutrosophic cognitive map (NCM) [20]. NCM is based on neutrosophic logic to represent uncertainty and indeterminacy in cognitive maps [12]. An NCM is a directed graph in which at least one edge is an indeterminate border and is indicated by dashed lines [2] (Figure 2).

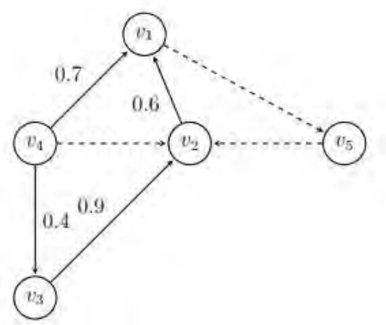

Figure 2: Fuzzy Neutrosophic Cognitive Maps example.

In [9] a static analysis of an NCM is presented.

2.1.3 Neutrosophic numbers

The result of the static analysis is in the form of neutrosophic numbers (a+bI, where I = indeterminacy) A de-neutrosification process as proposed by Salmeron and Smarandache could be applied giving final ranking value [13].

A neutrosophic number is a number as follows [14] :

N=d+I (2)

Where d is the determinacy part, and i is the indeterminate part. For example s: a=5 +I si i∈ [5,5.4] is equivalent to a∈[5,5.4].

Let $N_1=a_1+b_1$ I and $N_2=a_2+b_2$ I be two neutrosophic numbers then the following operational relation of neutrosophic numbers are defined as follows [8]:

$N_1+ N_2=a_1+a1+(b_1+b_2)$ I;

$N_1- N_2=a_1-a_1+(b_1-b_2)$ I

2.2 Proposed Framework

The aim was to develop and further detail a framework based on PESTEL and NCM [25]. The model was made in five steps (graphically, figure 3).

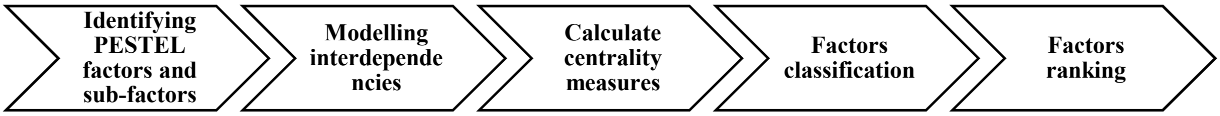

Figure 3: The proposed framework for PESTEL analysis [25]

2.2.1 Factors and sub-factors identification in the PESTEL method

In this step, the significant PESTEL factors and sub-factors were recognized. Identify factors and subfactors to form a hierarchical structure of the PESTEL model. Sub-factors are categorized according to the literature [2].

2.2.2 Modeling interdependencies

In this step causal interdependencies between PESTEL sub-factors are modeled, consists in the construction of NCM subfactors following the point views of an expert or expert team.

When a selection of experts (k) participates, the adjacency matrix of the collective MCD is calculated as follows:

$E=\mu\ (E_1,E_2,\ldots,E_k\ )$ (3)

The operator is usually the arithmetic mean [13].

2.2.3 Calculate centrality measures

Centrality measures are calculated [7] with absolute values of the NCM adjacency matrix [15]:

1. Outdegree $od(v_i)$ is the summation of the row of absolute values of a variable in the neutrosophic adjacency matrix, and It shows the cumulative strengths of connections (c_{ij}) exiting the variable.

$$od(v_i) = \sum_{i=1}^{N} c_{ij} \quad (4)$$

2. Indegree id(v_i) is the summation of the column of absolute values of a variable, and it shows the cumulative strength of variables come in the variable.

$$id(v_i) = \sum_{i=1}^{N} c_{ji} \quad (5)$$

3. The centrality degree (total degree $td(v_i)$), of a variable is the sum of its indegree and outdegree

$$td(v_i) = od(v_i) + id(v_i) \quad (6)$$

2.2.4 Factors classification and ranking

The factors were categorized according to the next rules:

The variables are a Transmitter (T) when having a positive or indeterminacy outdegree, $od(v_i)$ and zero indegree, $id(v_i)$.

The variables give a Receiver (R) when having a positive indegree or indeterminacy, $id(v_i)$., and zero outdegree, $od(v_i)$.

Variables receive the Ordinary (O) name when they have a non-zero degree, and these Ordinary variables can be considered more or less as receiving variables or transmitting variables, depending on the relation of their indegrees and outdegrees.

The de-neutrosophication process provides a range of numbers for centrality using as a ground the maximum & minimum values of I. A neutrosophic value is switched in an interval with these two values. ∈[0,1].

The contribution of a variable in an NCM can be known by calculating its degree of centrality, which shows how the variable is connected to other variables and what is the accumulated force of these connections. The median of the extreme values as proposed by Merigo [29] is used to give a centrality value :

$$\lambda([a_1, a_2]) = \frac{a_1 + a_2}{2} \quad (7)$$

Then

$$A > B \Leftrightarrow \frac{a_1 + a_2}{2} > \frac{b_1 + b_2}{2} \quad (8)$$

Finally, a ranking of variables could be given.

2.2.5 Factor prioritization

The numerical value obtained in the previous step is used for sub-factor prioritization and/or reduction. Threshold values may be set to 1.5 % of the total sum of total degree measures for subfactor reduction. Additionally, sub-factor could be grouped by parent factor and to extend the analysis to political, economic, social and technological general factor.

3 Results and Discussion

Case of study result.

0	0.28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0.28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0.39	0.33	0.25	0.28	0.25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	i	i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	i	-0.67	0	0	i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	-0.58	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.36
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.31	0	0	0	0	0	0	0	0	0	0.33	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.39	0	0	0	0	0	0	0	0	0	i	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.25	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.28	i	i	0	0	0	0	0	0	0	0.42	0	0	0	i	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.33	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	i	i	0	0	0	0	0	0	0	0	i	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	i	i	0	0	0	0	0	0	0	0	i	0
0	0	0	0	0	0	0	0	0	0.25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.36	0	0	0	0	0	-0.53	0.42	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.28	0	0	0	0	0	0	0	0	-0.5	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	i	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0.47	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0.36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.42	0.36	0.47	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	-0.58	0.42	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.28	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.31	0	0	0	0	0	0	0	0	i	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Tabla 1: Neutrosophic Adjacency Matrix

Nodes are initially classified (Table 2)

N1	T	N10	R	N19	O	N28	T
N2	O	N11	R	N20	O	N29	T
N3	O	N12	O	N21	O	N30	T
N4	T	N13	O	N22	O	N31	O
N5	T	N14	R	N23	O	N32	O
N6	O	N15	R	N24	O	N33	R
N7	O	N16	R	N25	R		
N8	O	N17	R	N26	R		
N9	O	N18	R	N27	T		

Tabla 2: Nodes classification

Total degree (Eq. 5) was calculated. Results are shown in Table 3.

N1	0,28	N9	0.72+i	N17	i	N25	0.64	N33	0.36
N2	0.56+i	N10	0.5	N18	2i	N26	0.42		
N3	1.78+2i	N11	0.64	N19	0.75	N27	0.47		
N4	I	N12	0.5	N20	0.36+3i	N28	0.36		
N5	2i	N13	1.17+3i	N21	0.47 + 3i	N29	1.25		
N6	1.83+2i	N14	i	N22	2.37+2i	N30	1+i		
N7	1.36	N15	0.67+i	N23	0.78+2i	N31	1.31+2i		
N8	1.03	N16	0.28	N24	2i	N32	1.06+4i		

Tabla 3: Total degree

"The next step is the de-neutrosophication process as proposes by Salmeron and Smarandache. I ∈[0,1] is replaced by both maximum and minimum values" [33]. In Table 4 are presented as interval values.

N1	0,28	N9	[0.72, 1.72]	N17	[0,1]	N25	0.64	N33	0.36
N2	[0.56, 1.56]	N10	0.5	N18	[0,2]	N26	0.42		
N3	[1.78, 2.78]	N11	0.64	N19	0.75	N27	0.47		
N4	[0,1]	N12	0.5	N20	[0.36, 3,36]	N28	0.36		
N5	[0, 2]	N13	[1.17, 4.17]	N21	[0.47, 3.47]	N29	1.25		
N6	[1.83,3.83]	N14	[0,1]	N22	[2.37, 4.37]	N30	[1, 2]		
N7	1.36	N15	[0.67, 1.67]	N23	[0.78, 2.78]	N31	[1.31, 3.31]		
N8	1.03	N16	0.28	N24	[0, 2]	N32	[1.06, 5.06]		

Tabla 4: De-neutrosophication, total degree values

Finally, we work with the median of the extreme values (Table 5) [29].

N1	0.28	N9	1.22	N17	0.5	N25	0.64	N33	0.36
N2	1.06	N10	0.5	N18	1	N26	0.42		
N3	2.28	N11	0.64	N19	0.75	N27	0.47		
N4	0.5	N12	0.5	N20	1.86	N28	0.36		
N5	1	N13	2.67	N21	1.97	N29	1.25		
N6	2.83	N14	0.5	N22	3.37	N30	1.5		
N7	1.36	N15	1.17	N23	1.75	N31	2.31		
N8	1.03	N16	0.28	N24	1	N32	3.06		

Tabla 5: Total degree using the median of the extreme values

Top 6 nodes according to centrality are represented in table 6.

N22	3,37
N32	3,06
N6	2,83
N13	2,67
N31	2,31
N3	2,28

Tabla 6: Top 6 nodes

Water quality index, Soil Quality index and Poverty are the top three factors. Centrality measures of subfactor were grouped according to their parent factors (Figure 4).

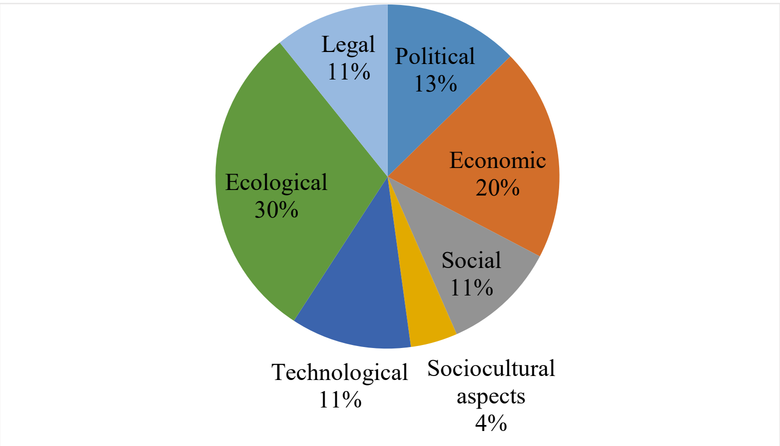

Figure 4: Aggregated total centrality values by factors

When the average is used as aggregation´s operator, the result is represented in Figure 5. Ecological, Technological and Social are the top three factors.

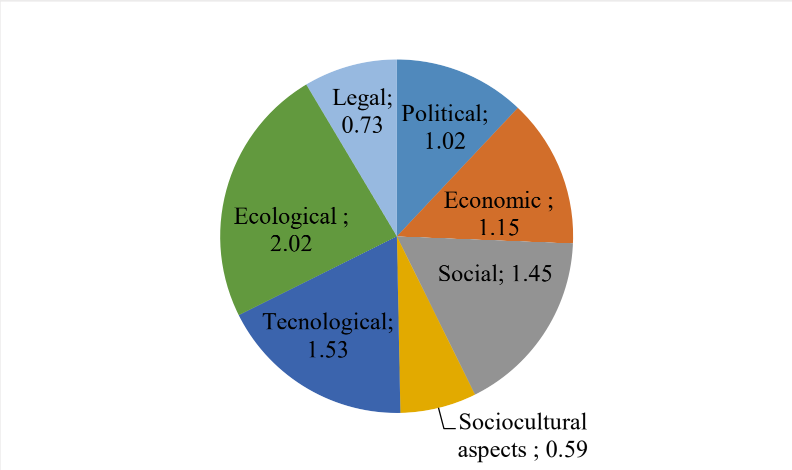

Figure 5: Average of total centrality values by factors

Factors with a little incidence of less than 1.5 % (0.606) are reduced for further analysis. In this case, we found nodes like N1, N4, N10, N14, N16, N17, N26, N27, N28 and N33.

After the application, in this case, study the model was found practical to use. The NCM gives high flexibility and takes into account interdependencies PESTEL analysis.

Conclusion

This study presents a model to address problems encountered in the measurement evaluation process of PESTEL analysis taking into account interdependencies among sub-factors and modeling uncertainty and indeterminacy in Sinos river basin. NCM modeled the integrated structure of PESTEL sub-factors, and quantitative analysis was developed based on static analysis and neutrosophic numbers.

To demonstrate the applicability of the proposal in the Sinos river external factor analysis a case study is developed. Sub-factor was ranked and reduced with Ecological, Technological, Social are the top three factors.

NCM modeled the integrated structure of PESTEL of factors and sub-factors. Our approach has many applications in complex decision problem that include interdependencies among criteria, and such as complex strategic decision support in river basin management.

Further works will concentrate on extending the model for dealing scenario analysis in conjunction with a multicriteria environment. Another area of future work is the development of a software tool.

Conflicts of Interest

No conflict of interest are declared by the authors.

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Data Availability Statement

The data used to support the findings of this study are included within the article.

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Neutrosophic General Finite Automata

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Abstract: The constructions of finite switchboard state automata is known to be an extension of finite automata in the view of commutative and switching automata. In this research, the idea of a neutrosophic is incorporated in the general fuzzy finite automata and general fuzzy finite switchboard automata to introduce neutrosophic general finite automata and neutrosophic general finite switchboard automata. Moreover, we define the notion of the neutrosophic subsystem and strong neutrosophic subsystem for both structures. We also establish the relationship between the neutrosophic subsystem and neutrosophic strong subsystem.

Keywords: Neutrosophic set, General fuzzy automata; switchboard; subsystems.

1 Introduction

It is well-known that the simplest and most important type of automata is finite automata. After the introduction of fuzzy set theory by [47] Zadeh in 1965, the first mathematical formulation of fuzzy automata was proposed by[46] Wee in 1967, considered as a generalization of fuzzy automata theory. Consequently, numerous works have been contributed towards the generalization of finite automata by many authors such as Cao and Ezawac [9], Jin et al [18], Jun [20], Li and Qiu [27], Qiu [34], Sato and Kuroki [36], Srivastava and Tiwari [41], Santos [35], Jun and Kavikumar [21], Kavikumar et al, [22, 23, 24] especially the simplest one by Mordeson and Malik [29]. In 2005, the theory of general fuzzy automata was firstly proposed by Doostfatemeh and Kermer [11] which is used to resolve the problem of assigning membership values to active states of the fuzzy automaton and its multi-membership. Subsequently, as a generalization, the concept of intuitionistic general fuzzy automata has been introduced and studied by Shamsizadeh and Zahedi [37], while Abolpour and Zahedi [6] proposed general fuzzy automata theory based on the complete residuated lattice-valued. As a further extension, Kavikumar et al [25] studied the notions of general fuzzy switchboard automata. For more details see the recent literature as [5, 12, 13, 14, 15, 16, 17].

The notions of neutrosophic sets was proposed by Smarandache [38, 39], generalizing the existing ordinary fuzzy sets, intuitionistic fuzzy sets and interval-valued fuzzy set in which each element of the universe has the degrees of truth, indeterminacy and falsity and the membership values are lies in $]0^-, 1^+[$, the nonstandard unit interval [40] it is an extension from standard interval [0,1]. It has been shown that fuzzy sets provides limited platform for computational complexity but neutrosophic sets is suitable for it. The neutrosophic sets is an appropriate mechanism for interpreting real-life philosophical problems but not for scientific problems since it is difficult to consolidate. In neutrosophic sets, the degree of indeterminacy can be defined independently since it is quantified explicitly which led to different from intuitionistic fuzzy sets. Single-valued neutrosophic set and interval neutrosophic set are the subclasses of the neutrosophic sets which was introduced by Wang et al. [44, 45] in order to examine kind of real-life and scientific problems. The applications of fuzzy sets have been found very useful in the domain of mathematics and elsewhere. A number of authors have been applied the concept of the neutrosophic set to many other structures especially in algebra [19, 28], decision-making [1, 2, 10, 30], medical [3, 4, 8], water quality management [33] and traffic control management [31, 32].

1.1 Motivation

In view of exploiting neutrosophic sets, Tahir et al. [43] introduced and studied the concept of single valued Neutrosophic finite state machine and switchboard state machine. Moreover, the fuzzy finite switchboard state machine is introduced into the context of the interval neutrosophic set in [42]. However, the realm of general structure of fuzzy automata in the neutrosophic environment has not been studied yet in the literature so far. Hence, it is still open to many possibilities for innovative research work especially in the context of neutrosophic general automata and its switchboard automata. The fundamental advantage of incorporating neutrosophic sets into general fuzzy automata is the ability to bring indeterminacy membership and nonmembership in each transitions and active states which help us to overcome the uncertain situation at the time of predicting next active state. Motivated by the work of [11], [36] and [38] the concept of neutrosophic general automata and neutrosophic general switchboard automata are introduced in this paper.

1.2 Main Contribution

The purpose of this paper is to introduce the primary algebraic structure of neutrosophic general finite automata and neutrosophic switchboard finite automata. The subsystem and strong subsystem of neutrosophic general finite automata and neutrosophic general finite switchboard f automata are exhibited. The relationship between these subsystems have been discussed and the characterizations of switching and commutative are discussed in the neutrosophic backdrop. We prove that the implication of a strong subsystem is a subsystem of neutrosophic general finite automata. The remainder of this paper is organised as follows. Section 2 provides the results and definitions concerning the general fuzzy automata. Section 3 describes the algebraic properties of the neutrosophic general finite automata. Finally, in section 4, the notion of the neutrosophic general finite switchboard automata is introduced. The paper concludes with Section 5.

2 Preliminaries

"For a nonempty set X, $\tilde{P}(X)$ denotes the set of all fuzzy sets on X.

Definition 2.1. [11] A general fuzzy automaton (GFA) is an eight-tuple machine $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ where

(a) Q is a finite set of states, $Q = \{q_1, q_2, \cdots, q_n\}$,

(b) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \cdots, a_m\}$,

(c) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subseteq \tilde{P}(Q)$,

(d) Z is a finite set of output symbols, $Z = \{b_1, b_2, \cdots, b_k\}$,

(e) $\omega : Q \to Z$ is the non-fuzzy output function,

(f) $F_1 : [0,1] \times [0,1] \to [0,1]$ is the membership assignment function,

(g) $\tilde{\delta} : (Q \times [0,1]) \times \Sigma \times Q \overset{F_1(\mu,\delta)}{\longrightarrow} [0,1]$ is the augmented transition function,

(h) $F_2 : [0,1]^* \to [0,1]$ is a multi-membership resolution function.

Noted that the function $F_1(\mu, \delta)$ has two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

This means that the membership value of the state q_j at time $t+1$ is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition. The usual options for the function $F(\mu, \delta)$ are $\max\{\mu, \delta\}, \min\{\mu, \delta\}$ and $(\mu + \delta)/2$. The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Let $Q_{act}(t_i)$ be the set of all active states at time $t_i, \forall i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$,

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists\, a \in \Sigma, \delta(q', a, q) \in \Delta\}, \forall i \geq 1.$$

Since $Q_{act}(t_i)$ is a fuzzy set, in order to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in Domain(Q_{act}(t_i))$ and $T \subset Domain(Q_{act}(t_i))$. Hereafter, we simply denote them as: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$. The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j leads to the multi-membership resolution algorithm.

Algorithm 2.2. [11] (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t+1$, the following algorithm will assign a unified membership value to it:

1. Each transition weight $\tilde{\delta}(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this v_i,

$$v_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

2. These membership values are not necessarily equal. Hence, they need to be processed by the multi-membership resolution function F_2.

3. The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j,

$$\mu^{t+1}(q_j) = F_2{}_{i=1}^{n}[v_i] = F_2{}_{i=1}^{n}[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))],$$

where

- n is the number of simultaneous transitions to the active state q_j at time $t+1$.
- $\delta(q_i, a_k, q_j)$ is the weight of a transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$ is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$ is the final membership value of q_j at time $t+1$.

Definition 2.3. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton, which is defined in Definition 2.1. The max-min general fuzzy automata is defined of the form:

$$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2),$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), \cdots\}$ and for every i, $i \geq 0$:

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p \\ 0, & \text{otherwise} \end{cases}$$

and for every i, $i \geq 1$: $\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p)$,

$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q') \wedge \tilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p))$$

and recursively

$$\tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \cdots u_n, p) = \bigvee \{\tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \cdots \wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \cdots, p_{n-1} \in Q_{act}(t_{n-1})\},$$

in which $u_i \in \Sigma, \forall 1 \leq i \leq n$ and assuming that the entered input at time t_i be $u_i, \forall 1 \leq i \leq n-1$.

Definition 2.4. [13] Let $\tilde{F}^*$ be a max-min GFA, $p \in Q, q \in Q_{act}(t_i), i \geq 0$ and $0 \leq \alpha < 1$. Then p is called a successor of q with threshold α if there exists $x \in \Sigma^*$ such that $\tilde{\delta}^*((q, \mu^{t_j}(q)), x, p) > \alpha$.

Definition 2.5. [13] Let $\tilde{F}^*$ be a max-min GFA, $q \in Q_{act}(t_i), i \geq 0$ and $0 \leq \alpha < 1$. Also let $S_\alpha(q)$ denote the set of all successors of q with threshold α. If $T \subseteq Q$, then $S_\alpha(T)$ the set of all successors of T with threshold α is defined by $S_\alpha(T) = \bigcup\{S_\alpha(q) : q \in T\}$.

Definition 2.6. [38] Let X be an universe of discourse. The neutrosophic set is an object having the form $A = \{\langle x, \mu_1(x), \mu_2(x), \mu_3(x)\rangle | \forall x \in X\}$ where the functions can be defined by $\mu_1, \mu_2, \mu_3 : X \rightarrow]0, 1[$ and μ_1 is the degree of membership or truth, μ_2 is the degree of indeterminacy and μ_3 is the degree of non-membership or false of the element $x \in X$ to the set A with the condition $0 \leq \mu_1(x) + \mu_2(x) + \mu_3(x) \leq 3$."

3 Neutrosophic General Finite Automata

Definition 3.1. An eight-tuple machine $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ is called neutrosophic general finite automata (*NGFA* for short), where

1. Q is a finite set of states, $Q = \{q_1, q_2, \cdots, q_n\}$,

2. Σ is a finite set of input symbols, $\Sigma = \{u_1, u_2, \cdots, u_m\}$,

3. $\tilde{R} = \{(q, \mu_1^{t_0}(q), \mu_2^{t_0}(q), \mu_3^{t_0}(q)) | q \in R\}$ is the set of fuzzy start states, $R \subseteq \tilde{P}(Q)$,

4. Z is a finite set of output symbols, $Z = \{b_1, b_2, \cdots, b_k\}$,

5. $\tilde{\delta} : (Q \times [0,1] \times [0,1] \times [0,1])) \times \Sigma \times Q \stackrel{F_1(\mu,\delta)}{\longrightarrow} [0,1] \times [0,1] \times [0,1]$ is the neutrosophic augmented transition function,

6. $\omega : (Q \times [0,1] \times [0,1] \times [0,1]) \to Z$ is the non-fuzzy output function,

7. $F_1 = (F_1^{\wedge}, F_1^{\wedge\vee}, F_1^{\vee})$, where $F_1^{\wedge} : [0,1] \times [0,1] \to [0,1]$, $F_2^{\wedge\vee} : [0,1] \times [0,1] \to [0,1]$ and $F_3^{\vee} : [0,1] \times [0,1] \to [0,1]$ are the truth, indeterminacy and false membership assignment functions, respectively. $F_1^{\wedge}(\mu_1, \tilde{\delta}_1)$, $F_2^{\wedge\vee}(\mu_2, \tilde{\delta}_2)$ and $F_3^{\vee}(\mu_3, \tilde{\delta}_3)$ are motivated by two parameters μ_1, μ_2, μ_3 and $\tilde{\delta}_1$, $\tilde{\delta}_2$, $\tilde{\delta}_3$ where μ_1, μ_2 and μ_3 are the truth, indeterminacy and false membership value of a predecessor and $\tilde{\delta}_1$, $\tilde{\delta}_2$ and $\tilde{\delta}_3$ are the truth, indeterminacy and false membership value of a transition,

8. $F_2 = (F_2^{\wedge}, F_2^{\wedge\vee}, F_2^{\vee})$, where $F_2^{\wedge} : [0,1]^* \to [0,1]$, $F_2^{\wedge\vee} : [0,1]^* \to [0,1]$ and $F_2^{\vee} : [0,1]^* \to [0,1]$ are the truth, indeterminacy and false multi-membership resolution function.

Remark 3.2. In Definition 3.1, the process that takes place upon the transition from the state q_i to q_j on an input u_k is represented by

$$\mu_1^{t_{k+1}}(q_j) = \tilde{\delta}_1((q_i, \mu_1^{t_k}(q_i)), u_k, q_j) = F_1^{\wedge}(\mu_1^{t_k}(q_i), \delta_1(q_i, u_k, q_j)) = \bigwedge(\mu_1^{t_k}(q_i), \delta_1(q_i, u_k, q_j)),$$

$$\mu_2^{t_{k+1}}(q_j) = \tilde{\delta}_2((q_i, \mu_2^{t_k}(q_i)), u_k, q_j) = F_1^{\wedge\vee}(\mu_2^{t_k}(q_i), \delta_2(q_i, u_k, q_j)) = \begin{cases} \bigvee(\mu_2^{t_k}(q_i), \delta_2(q_i, u_k, q_j)) & \text{if } t_k < t_{k+1} \\ \bigwedge(\mu_2^{t_k}(q_i), \delta_2(q_i, u_k, q_j)) & \text{if } t_k \geq t_{k+1} \end{cases},$$

$$\mu_3^{t_{k+1}}(q_j) = \tilde{\delta}_3((q_i, \mu_3^{t_k}(q_i)), u_k, q_j) = F_1^{\vee}(\mu_3^{t_k}(q_i), \delta_3(q_i, u_k, q_j)) = \bigvee(\mu_3^{t_k}(q_i), \delta_3(q_i, u_k, q_j)),$$

where

$$\tilde{\delta}((q_i.\mu^t(q_i)), u_k, q_j) = (\tilde{\delta}_1((q_i, \mu_1^t(q_i)), u_k, q_j), \tilde{\delta}_2((q_i, \mu_2^t(q_i)), u_k, q_j), \tilde{\delta}_3((q_i, \mu_3^t(q_i)), u_k, q_j)) \text{ and}$$

$$\delta(q_i, u_k, q_j) = (\delta_1(q_i, u_k, q_j), \delta_2(q_i, u_k, q_j), \delta_3(q_i, u_k, q_j)).$$

Remark 3.3. The algorithm for truth, indeterminacy and false multi-membership resolution for transition function is same as Algorithm 2.2 but the computation depends (see Remark 3.2) on the truth, indeterminacy and false membership assignment function.

Definition 3.4. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a NGFA. We define the max-min neutrosophic general fuzzy automaton $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}^*, \omega, F_1, F_2)$, where $\tilde{\delta}^* : (Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma^* \times Q \to [0,1] \times [0,1] \times [0,1]$ and define a neutrosophic set $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0,1] \times [0,1] \times [01]) \times \Sigma^* \times Q$ and for every i, $i \geq 0$:

$$\tilde{\delta}_1^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p \\ 0, & q \neq p \end{cases},$$
$$\tilde{\delta}_2^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 0, & q = p \\ 1, & q \neq p \end{cases},$$
$$\tilde{\delta}_3^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 0, & q = p \\ 1, & q \neq p \end{cases},$$

and for every i, $i \geq 1$:

$$\tilde{\delta}_1^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}_1((q, \mu^{t_{i-1}}(q)), u_i, p), \tilde{\delta}_2^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}_2((q, \mu^{t_{i-1}}(q)), u_i, p)$$
$$\tilde{\delta}_3^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}_3((q, \mu^{t_{i-1}}(q)), u_i, p)$$

and recursively,

$$\tilde{\delta}_1^*((q, \mu^{t_0}(q)), u_1 u_2 \cdots u_n, p) = \bigvee \{\tilde{\delta}_1((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}_1((p_1, \mu^{t_1}(p_1)), u_2, p_2) \wedge \cdots \wedge$$
$$\tilde{\delta}_1((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \cdots, p_{n-1} \in Q_{act}(t_{n-1})\},$$
$$\tilde{\delta}_2^*((q, \mu^{t_0}(q)), u_1 u_2 \cdots u_n, p) = \bigwedge \{\tilde{\delta}_2((q, \mu^{t_0}(q)), u_1, p_1) \vee \tilde{\delta}_2((p_1, \mu^{t_1}(p_1)), u_2, p_2) \vee \cdots \vee$$
$$\tilde{\delta}_2((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \cdots, p_{n-1} \in Q_{act}(t_{n-1})\},$$
$$\tilde{\delta}_3^*((q, \mu^{t_0}(q)), u_1 u_2 \cdots u_n, p) = \bigwedge \{\tilde{\delta}_3((q, \mu^{t_0}(q)), u_1, p_1) \vee \tilde{\delta}_3((p_1, \mu^{t_1}(p_1)), u_2, p_2) \vee \cdots \vee$$
$$\tilde{\delta}_3((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \cdots, p_{n-1} \in Q_{act}(t_{n-1})\},$$

in which $u_i \in \Sigma, \forall 1 \leq i \leq n$ and assuming that the entered input at time t_i be $u_i, \forall 1 \leq i \leq n-1$.

Example 3.5. Consider the NGFA in Figure 1 with several transition overlaps. Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where

- $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}$ be a set of states,
- $\Sigma = \{a, b\}$ be a set of input symbols,
- $\tilde{R} = \{(q_0, 0.7, 0.5, 0.2), (q_4, 0.6, 0.2, 0.45)\}$, set of initial states,
- the operation of $F_1^{\wedge}, F_1^{\wedge\vee}$ and $F_1^{\vee}$ are according to Remark 3.2,
- $Z = \emptyset$ and ω are not applicable (output mapping is not of our interest in this paper),
- $\tilde{\delta} : (Q \times [0,1] \times [0,1] \times [0,1])) \times \Sigma \times Q \xrightarrow{F_1(\mu,\delta)} [0,1] \times [0,1] \times [0,1]$, the neutrosophic augmented transition function.

Assuming that $\tilde{F}$ starts operating at time t_0 and the next three inputs are a, b, b respectively (one at a time), active states and their membership values at each time step are as follows:

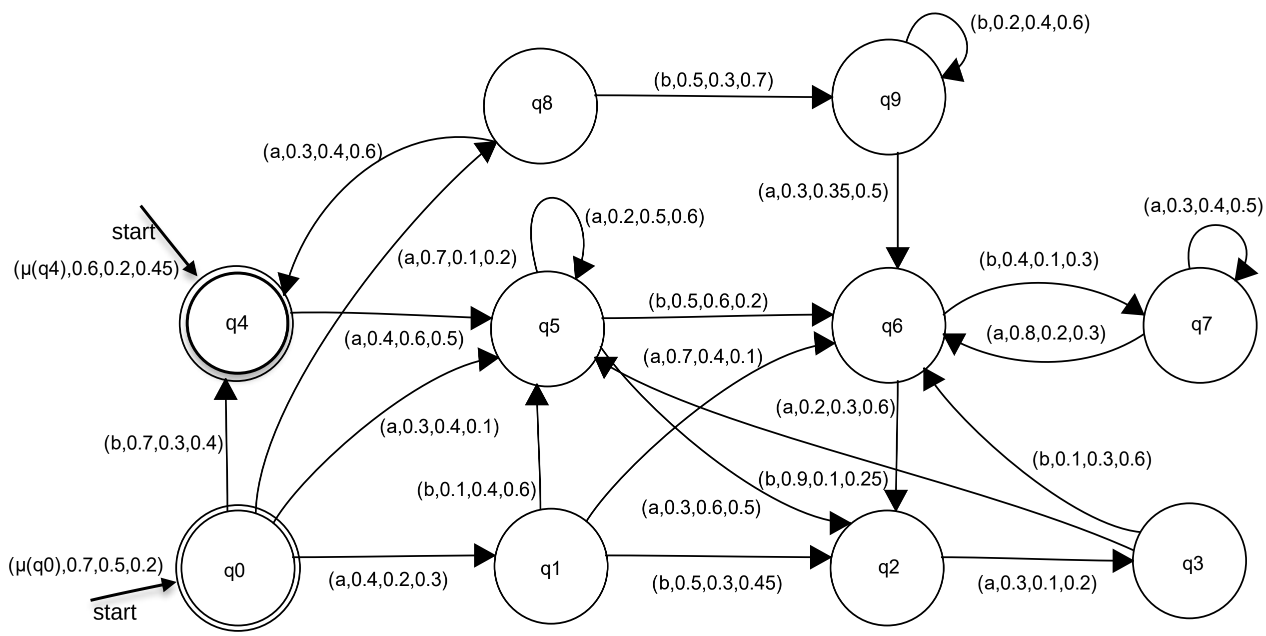

Figure 1: The NGFA of Example 3.5

- At time t_0: $Q_{act}(t_0) = \tilde{R} = \{(q_0, 0.7, 0.5, 0.2), (q_4, 0.6, 0.2, 0.45)\}$

- At time t_1, input is a. Thus q_1, q_5 and q_8 get activated. Then:

$$\begin{aligned}
\mu^{t_1}(q_1) &= \tilde{\delta}((q_0, \mu_1^{t_0}(q_0), \mu_2^{t_0}(q_0), \mu_3^{t_0}(q_0)), a, q_1)\\
&= \left[F_1^{\wedge}(\mu_1^{t_0}(q_0), \delta_1(q_0, a, q_1)), F_1^{\wedge\vee}(\mu_2^{t_0}(q_0), \delta_2(q_0, a, q_1)), F_1^{\vee}(\mu_3^{t_0}(q_0), \delta_3(q_0, a, q_1))\right]\\
&= [F_1^{\wedge}(0.7, 0.4), F_1^{\wedge\vee}(0.5, 0.2), F_1^{\vee}(0.2, 0.3)] = (0.4, 0.2, 0.3),
\end{aligned}$$

$$\begin{aligned}
\mu^{t_1}(q_8) &= \tilde{\delta}((q_0, \mu_1^{t_0}(q_0), \mu_2^{t_0}(q_0), \mu_3^{t_0}(q_0)), a, q_8)\\
&= \left[F_1^{\wedge}(\mu_1^{t_0}(q_0), \delta_1(q_0, a, q_8)), F_1^{\wedge\vee}(\mu_2^{t_0}(q_0), \delta_2(q_0, a, q_8)), F_1^{\vee}(\mu_3^{t_0}(q_0), \delta_3(q_0, a, q_8))\right]\\
&= [F_1^{\wedge}(0.7, 0.7), F_1^{\wedge\vee}(0.5, 0.1), F_1^{\vee}(0.2, 0.2)] = (0.7, 0.1, 0.2),
\end{aligned}$$

but q_5 is multi-membership at t_1. Then

$$\begin{aligned}
\mu^{t_1}(q_5) &= \underset{i=0\&4}{F_2}\left[F_1[\mu^{t_0}(q_i), \delta(q_i, a, q_5)]\right]\\
&= F_2\left[F_1[\mu^{t_0}(q_0), \delta(q_0, a, q_5)], F_1[\mu^{t_0}(q_0), \delta(q_4, a, q_5)]\right]\\
&= F_2\left[F_1[(0.7, 0.5, 0.2), (0.3, 0.4, 0.1)], F_1[(0.6, 0.2, 0.45), (0.4, 0.6, 0.5)]\right]\\
&= (F_2^{\wedge}[F_1^{\wedge}(0.7, 0.3), F_1^{\wedge}(0.6, 0.4)], F_2^{\wedge\vee}[F_1^{\wedge\vee}(0.5, 0.4), F_1^{\wedge\vee}(0.2, 0.6)],\\
&\qquad F_2^{\vee}[F_1^{\vee}(0.2, 0.1), F_1^{\vee}(0.45, 0.5)])\\
&= (F_2^{\wedge}(0.3, 0.4), F_2^{\wedge\vee}(0.4, 0.2), F_2^{\vee}(0.2, 0.5)) = (0.3, 0.2, 0.5).
\end{aligned}$$

Then we have:

$$\begin{aligned}Q_{act}(t_1) &= \{(q_1, \mu^{t_1}(q_1)), (q_5, \mu^{t_1}(q_5)), (q_8, \mu^{t_1}(q_8))\}\\ &= \{(q_1, 0.4, 0.2, 0.3), (q_5, 0.3, 0.2, 0.5), (q_8, 0.7, 0.1, 0.2)\}.\end{aligned}$$

- At t_2 input is b. q_2, q_5, q_6 and q_9 get activated. Then

$$\begin{aligned}\mu^{t_2}(q_5) &= \tilde{\delta}((q_1, \mu_1^{t_1}(q_1), \mu_2^{t_1}(q_1), \mu_3^{t_1}(q_1)), b, q_5)\\ &= \left[F_1^{\wedge}(\mu_1^{t_1}(q_1), \delta_1(q_1, b, q_5)), F_1^{\wedge\vee}(\mu_2^{t_1}(q_1), \delta_2(q_1, b, q_5)), F_1^{\vee}(\mu_3^{t_1}(q_1), \delta_3(q_1, b, q_5))\right]\\ &= [F_1^{\wedge}(0.4, 0.1), F_1^{\wedge\vee}(0.2, 0.4), F_1^{\vee}(0.3, 0.6)] = (0.1, 0.2, 0.6),\end{aligned}$$

$$\begin{aligned}\mu^{t_2}(q_6) &= \tilde{\delta}((q_5, \mu_1^{t_1}(q_5), \mu_2^{t_1}(q_5), \mu_3^{t_1}(q_5)), b, q_6)\\ &= \left[F_1^{\wedge}(\mu_1^{t_1}(q_5), \delta_1(q_5, b, q_6)), F_1^{\wedge\vee}(\mu_2^{t_1}(q_5), \delta_2(q_5, b, q_6)), F_1^{\vee}(\mu_3^{t_1}(q_5), \delta_3(q_5, b, q_6))\right]\\ &= [F_1^{\wedge}(0.3, 0.5), F_1^{\wedge\vee}(0.2, 0.6), F_1^{\vee}(0.5, 0.2)] = (0.3, 0.2, 0.5),\end{aligned}$$

$$\begin{aligned}\mu^{t_2}(q_9) &= \tilde{\delta}((q_8, \mu_1^{t_1}(q_8), \mu_2^{t_1}(q_8), \mu_3^{t_1}(q_8)), b, q_9)\\ &= \left[F_1^{\wedge}(\mu_1^{t_1}(q_8), \delta_1(q_8, b, q_9)), F_1^{\wedge\vee}(\mu_2^{t_1}(q_8), \delta_2(q_8, b, q_9)), F_1^{\vee}(\mu_3^{t_1}(q_8), \delta_3(q_8, b, q_9))\right]\\ &= [F_1^{\wedge}(0.7, 0.5), F_1^{\wedge\vee}(0.1, 0.3), F_1^{\vee}(0.2, 0.7)] = (0.5, 0.1, 0.7),\end{aligned}$$

but q_2 is multi-membership at t_2. Then:

$$\begin{aligned}\mu^{t_2}(q_2) &= \underset{i=1\&5}{F_2}\left[F_1[\mu^{t_1}(q_i), \delta(q_i, b, q_2)]\right]\\ &= F_2\left[F_1[\mu^{t_1}(q_1), \delta(q_1, b, q_2)], F_1[\mu^{t_1}(q_5), \delta(q_5, b, q_2)]\right]\\ &= F_2\left[F_1[(0.4, 0.2, 0.3), (0.5, 0.3, 0.45)], F_1[(0.3, 0.2, 0.5), (0.1, 0.4, 0.6)]\right]\\ &= (F_2^{\wedge}[F_1^{\wedge}(0.4, 0.5), F_1^{\wedge}(0.3, 0.1)], F_2^{\wedge\vee}[F_1^{\wedge\vee}(0.2, 0.3), F_1^{\wedge\vee}(0.2, 0.4)],\\ &\qquad F_2^{\vee}[F_1^{\vee}(0.3, 0.45), F_1^{\vee}(0.5, 0.6)])\\ &= (F_2^{\wedge}(0.4, 0.1), F_2^{\wedge\vee}(0.2, 0.2), F_2^{\vee}(0.3, 0.5)) = (0.1, 0.2, 0.5).\end{aligned}$$

Then we have:

$$\begin{aligned}Q_{act}(t_2) &= \{(q_2, \mu^{t_2}(q_2)), (q_5, \mu^{t_2}(q_5)), (q_6, \mu^{t_2}(q_6)), (q_9, \mu^{t_2}(q_9))\}\\ &= \{(q_2, 0.1, 0.2, 0.5), (q_5, 0.1, 0.2, 0.6), (q_6, 0.3, 0.2, 0.5), (q_9, 0.5, 0.1, 0.7)\}.\end{aligned}$$

- At t_3 input is b. q_2, q_6, q_7 and q_9 get activated and none of them is multi-membership. It is easy to verify that:

$$\begin{aligned}Q_{act}(t_3) &= \{(q_2, \mu^{t_3}(q_2)), (q_6, \mu^{t_3}(q_6)), (q_7, \mu^{t_3}(q_7)), (q_9, \mu^{t_3}(q_9))\}\\ &= \{(q_2, 0.1, 0.1, 0.6), (q_6, 0.1, 0.2, 0.6), (q_7, 0.3, 0.1, 0.5), (q_9, 0.3, 0.1, 0.5)\}.\end{aligned}$$

Proposition 3.6. *Let $\tilde{F}$ be a NGFA, if $\tilde{F}^*$ is a max-min NGFA, then for every $i \geq 1$,*

$$\tilde{\delta}_1^*((q,\mu^{t_{i-1}}(q)),xy,p) = \bigvee_{r\in Q_{act}(t_i)} \left[\tilde{\delta}_1^*((p,\mu^{t_{i-1}}(p)),x,r) \wedge \tilde{\delta}_1^*((r,\mu^{t_{i-1}}(r)),y,q)\right],$$
$$\tilde{\delta}_2^*((q,\mu^{t_{i-1}}(q)),xy,p) = \bigwedge_{r\in Q_{act}(t_i)} \left[\tilde{\delta}_2^*((p,\mu^{t_{i-1}}(p)),x,r) \vee \tilde{\delta}_2^*((r,\mu^{t_{i-1}}(r)),y,q)\right],$$
$$\tilde{\delta}_3^*((q,\mu^{t_{i-1}}(q)),xy,p) = \bigwedge_{r\in Q_{act}(t_i)} \left[\tilde{\delta}_3^*((p,\mu^{t_{i-1}}(p)),x,r) \vee \tilde{\delta}_3^*((r,\mu^{t_{i-1}}(r)),y,q)\right],$$

for all $p,q \in Q$ and $x,y \in \Sigma^*$.

Proof. Since $p,q \in Q$ and $x,y \in \Sigma^*$, we prove the result by induction on $|y| = n$. First, we assume that $n = 0$, then $y = \Lambda$ and so $xy = x\Lambda = x$. Thus, for all $r \in Q_{act}(t_i)$

$$\begin{aligned}
\bigvee \left[\tilde{\delta}_1^*((p,\mu^{t_{i-1}}(p)),x,r) \wedge \tilde{\delta}_1^*((r,\mu^{t_{i-1}}(r)),y,q)\right] &= \bigvee \left[\tilde{\delta}_1^*((p,\mu^{t_{i-1}}(p)),x,r) \wedge \tilde{\delta}_1^*((r,\mu^{t_{i-1}}(r)),\Lambda,q)\right] \\
&= \tilde{\delta}_1^*((p,\mu^{t_{i-1}}(p)),x,r) = \tilde{\delta}_1^*((q,\mu^{t_{i-1}}(q)),xy,p), \\
\bigwedge \left[\tilde{\delta}_2^*((p,\mu^{t_{i-1}}(p)),x,r) \vee \tilde{\delta}_2^*((r,\mu^{t_{i-1}}(r)),y,q)\right] &= \bigwedge \left[\tilde{\delta}_2^*((p,\mu^{t_{i-1}}(p)),x,r) \vee \tilde{\delta}_2^*((r,\mu^{t_{i-1}}(r)),\Lambda,q)\right] \\
&= \tilde{\delta}_2^*((p,\mu^{t_{i-1}}(p)),x,r) = \tilde{\delta}_2^*((q,\mu^{t_{i-1}}(q)),xy,p), \\
\bigwedge \left[\tilde{\delta}_3^*((p,\mu^{t_{i-1}}(p)),x,r) \vee \tilde{\delta}_3^*((r,\mu^{t_{i-1}}(r)),y,q)\right] &= \bigwedge \left[\tilde{\delta}_3^*((p,\mu^{t_{i-1}}(p)),x,r) \vee \tilde{\delta}_3^*((r,\mu^{t_{i-1}}(r)),\Lambda,q)\right] \\
&= \tilde{\delta}_3^*((p,\mu^{t_{i-1}}(p)),x,r) = \tilde{\delta}_3^*((q,\mu^{t_{i-1}}(q)),xy,p).
\end{aligned}$$

The result holds for $n = 0$. Now, continue the result is true for all $u \in \Sigma^*$ with $|u| = n-1$, where $n > 0$. Let $y = ua$, where $a \in \Sigma$ and $u \in \Sigma^*$. Then

$$\begin{aligned}
&\tilde{\delta}_1^*((q,\mu^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_1^*((q,\mu^{t_{i-1}}(q)),xua,p) = \bigvee_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q,\mu^{t_{i-1}}(q)),xu,r) \wedge \tilde{\delta}_1((r,\mu^{t_i}(r)),a,p)\Big) \\
&= \bigvee_{r\in Q_{act}(t_i)} (\bigvee_{s\in Q_{act}(t_i)} (\tilde{\delta}_1((q,\mu^{t_{i-1}}(q)),x,s) \wedge \tilde{\delta}_1((s,\mu^{t_{i-1}}(s)),u,r)) \wedge \tilde{\delta}_1((r,\mu^{t_i}(r)),a,p)) \\
&= \bigvee_{r,s\in Q_{act}(t_i)} (\tilde{\delta}_1((q,\mu^{t_{i-1}}(q)),x,s) \wedge \tilde{\delta}_1((s,\mu^{t_{i-1}}(s)),u,r) \wedge \tilde{\delta}_1((r,\mu^{t_i}(r)),a,p)) \\
&= \bigvee_{s\in Q_{act}(t_i)} (\tilde{\delta}_1((q,\mu^{t_{i-1}}(q)),x,s) \wedge (\bigvee_{r\in Q_{act}(t_i)} (\tilde{\delta}_1((s,\mu^{t_{i-1}}(s)),u,r) \wedge \tilde{\delta}_1((r,\mu^{t_i}(r)),a,p)))) \\
&= \bigvee_{s\in Q_{act}(t_i)} (\tilde{\delta}_1((q,\mu^{t_{i-1}}(q)),x,s) \wedge \tilde{\delta}_1((s,\mu^{t_i}(r)),ua,p))) = \bigvee_{s\in Q_{act}(t_i)} (\tilde{\delta}_1((q,\mu^{t_{i-1}}(q)),x,s) \wedge \tilde{\delta}_1((s,\mu^{t_i}(r)),y,p))),
\end{aligned}$$

$$\begin{aligned}
&\tilde{\delta}_2^*((q,\mu^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_2^*((q,\mu^{t_{i-1}}(q)),xua,p) = \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q,\mu^{t_{i-1}}(q)),xu,r) \vee \tilde{\delta}_2((r,\mu^{t_i}(r)),a,p)\Big) \\
&= \bigwedge_{r\in Q_{act}(t_i)} (\bigwedge_{s\in Q_{act}(t_i)} (\tilde{\delta}_2((q,\mu^{t_{i-1}}(q)),x,s) \vee \tilde{\delta}_2((s,\mu^{t_{i-1}}(s)),u,r)) \vee \tilde{\delta}_2((r,\mu^{t_i}(r)),a,p)) \\
&= \bigwedge_{r,s\in Q_{act}(t_i)} (\tilde{\delta}_2((q,\mu^{t_{i-1}}(q)),x,s) \vee \tilde{\delta}_2((s,\mu^{t_{i-1}}(s)),u,r) \vee \tilde{\delta}_2((r,\mu^{t_i}(r)),a,p)) \\
&= \bigwedge_{s\in Q_{act}(t_i)} (\tilde{\delta}_2((q,\mu^{t_{i-1}}(q)),x,s) \vee (\bigwedge_{r\in Q_{act}(t_i)} (\tilde{\delta}_2((s,\mu^{t_{i-1}}(s)),u,r) \vee \tilde{\delta}_2((r,\mu^{t_i}(r)),a,p)))) \\
&= \bigwedge_{s\in Q_{act}(t_i)} (\tilde{\delta}_2((q,\mu^{t_{i-1}}(q)),x,s) \vee \tilde{\delta}_2((s,\mu^{t_i}(r)),ua,p))) = \bigwedge_{s\in Q_{act}(t_i)} (\tilde{\delta}_2((q,\mu^{t_{i-1}}(q)),x,s) \vee \tilde{\delta}_2((s,\mu^{t_i}(r)),y,p))),
\end{aligned}$$

$$\tilde{\delta}_3^*((q,\mu^{t_{i-1}}(q)),xy,p)=\tilde{\delta}_3^*((q,\mu^{t_{i-1}}(q)),xua,p)=\bigwedge_{r\in Q_{act}(t_i)}\Big(\tilde{\delta}_3((q,\mu^{t_{i-1}}(q)),xu,r)\vee\tilde{\delta}_3((r,\mu^{t_i}(r)),a,p)\Big)$$
$$=\bigwedge_{r\in Q_{act}(t_i)}(\bigwedge_{s\in Q_{act}(t_i)}(\tilde{\delta}_3((q,\mu^{t_{i-1}}(q)),x,s)\vee\tilde{\delta}_3((s,\mu^{t_{i-1}}(s)),u,r))\vee\tilde{\delta}_3((r,\mu^{t_i}(r)),a,p))$$
$$=\bigwedge_{r,s\in Q_{act}(t_i)}(\tilde{\delta}_3((q,\mu^{t_{i-1}}(q)),x,s)\vee\tilde{\delta}_3((s,\mu^{t_{i-1}}(s)),u,r)\vee\tilde{\delta}_3((r,\mu^{t_i}(r)),a,p))$$
$$=\bigwedge_{s\in Q_{act}(t_i)}(\tilde{\delta}_3((q,\mu^{t_{i-1}}(q)),x,s)\vee(\bigwedge_{r\in Q_{act}(t_i)}(\tilde{\delta}_3((s,\mu^{t_{i-1}}(s)),u,r)\vee\tilde{\delta}_3((r,\mu^{t_i}(r)),a,p))))$$
$$=\bigwedge_{s\in Q_{act}(t_i)}(\tilde{\delta}_3((q,\mu^{t_{i-1}}(q)),x,s)\vee\tilde{\delta}_3((s,\mu^{t_i}(r)),ua,p)))=\bigwedge_{s\in Q_{act}(t_i)}(\tilde{\delta}_3((q,\mu^{t_{i-1}}(q)),x,s)\vee\tilde{\delta}_3((s,\mu^{t_i}(r)),y,p))).$$

Hence the result is valid for $|y|=n$. This completes the proof. □

Definition 3.7. Let $\tilde{F}^*$ be a max-min NGFA, $p\in Q, q\in Q_{act}(t_i), i\geq 0$ and $0\leq\alpha<1$. Then p is called a successor of q with threshold α if there exists $x\in\Sigma^*$ such that $\tilde{\delta}_1^*((q,\mu_1^{t_j}(q)),x,p)>\alpha$, $\tilde{\delta}_2^*((q,\mu_2^{t_j}(q)),x,p)<\alpha$ and $\tilde{\delta}_3^*((q,\mu_3^{t_j}(q)),x,p)<\alpha$.

Definition 3.8. Let $\tilde{F}^*$ be a max-min NGFA, $q\in Q_{act}(t_i), i\geq 0$ and $0\leq\alpha<1$. Also let $S_\alpha(q)$ denote the set of all successors of q with threshold α. If $T\subseteq Q$, then $S_\alpha(T)$ the set of all successors of T with threshold α is defined by $S_\alpha(T)=\bigcup\{S_\alpha(q):q\in T\}$.

Definition 3.9. Let $\tilde{F}^*$ be a max-min NGFA. Let $\mu=\langle\mu_1,\mu_2,\mu_3\rangle$ and $\tilde{\delta}^*=\langle\tilde{\delta}_1^*,\tilde{\delta}_2^*,\tilde{\delta}_3^*\rangle$ in $(Q\times[0,1]\times[0,1]\times[0,1])\times\Sigma^*\times Q$ be a neutrosophic set in Q. Then μ is a neutrosophic subsystem of $\tilde{F}^*$, say $\mu\subseteq\tilde{F}^*$ if for every j, $1\leq j\leq k$ such that $\mu_1^{t_j}(p)\geq\tilde{\delta}_1^*((q,\mu_1^{t_j}(q)),x,p)$, $\mu_2^{t_j}(p)\leq\tilde{\delta}_2^*((q,\mu_2^{t_j}(q)),x,p)$, $\mu_3^{t_j}(p)\leq\tilde{\delta}_3^*((q,\mu_3^{t_j}(q)),x,p)$. $\forall q,p\in Q$ and $x\in\Sigma^*$.

Example 3.10. Let $Q=\{p,q\}$, $\Sigma=\{a\}$. Let $\mu=\langle\mu_1,\mu_2,\mu_3\rangle$ and $\tilde{\delta}^*=\langle\tilde{\delta}_1^*,\tilde{\delta}_2^*,\tilde{\delta}_3^*\rangle$ in $(Q\times[0,1]\times[0,1]\times[0,1])\times\Sigma^*\times Q$ be a neutrosophic set in Q such that $\mu_1^{t_j}(p)=0.8$, $\mu_2^{t_j}(p)=0.7$, $\mu_3^{t_j}(p)=0.5$, $\mu_1^{t_j}(q)=0.5$, $\mu_2^{t_j}(q)=0.6$, $\mu_3^{t_j}(q)=0.8$, $\delta_1(q,x,p)=0.7$, $\delta_2(q,x,p)=0.9$ and $\delta_3(q,x,p)=0.7$. Then

$$\tilde{\delta}_1^*((q,\mu_1^{t_j}(q)),x,p)=F_1^{\wedge}(\mu_1^{t_j}(q),\delta_1(q,x,p))=\min\{0.5,0.7\}=0.5\leq\mu_1^{t_j}(p),$$
$$\tilde{\delta}_2^*((q,\mu_2^{t_j}(q)),x,p)=F_2^{\wedge\vee}(\mu_2^{t_j}(q),\delta_2(q,x,p))=\max\{0.6,0.9\}=0.9\geq\mu_2^{t_j}(p),\ \ (\text{since } t<t_j)$$
$$\tilde{\delta}_3^*((q,\mu_3^{t_j}(q)),x,p)=F_3^{\vee}(\mu_3^{t_j}(q),\delta_3(q,x,p))=\max\{0.8,0.7\}=0.8\geq\mu_3^{t_j}(p).$$

Hence μ is a neutrosophic subsystem of $\tilde{F}^*$.

Theorem 3.11. *Let $\tilde{F}^*$ be a NGFA and let $\mu=\langle\mu_1,\mu_2,\mu_3\rangle$ and $\tilde{\delta}^*=\langle\tilde{\delta}_1^*,\tilde{\delta}_2^*,\tilde{\delta}_3^*\rangle$ in $(Q\times[0,1]\times[0,1]\times[0,1])\times\Sigma^*\times Q$ be a neutrosophic set in Q. Then μ is a neutrosophic subsystem of $\tilde{F}^*$ if and only if $\mu_1^{t_j}(p)\geq\tilde{\delta}_1^*((q,\mu_1^{t_j}(q)),x,p)$, $\mu_2^{t_j}(p)\leq\tilde{\delta}_2^*((q,\mu_2^{t_j}(q)),x,p)$, $\mu_3^{t_j}(p)\leq\tilde{\delta}_3^*((q,\mu_3^{t_j}(q)),x,p)$, for all $q\in Q_{(act)}(t_j)$, $p\in Q$ and $x\in\Sigma^*$.*

Proof. Suppose that μ is a neutrosophic subsystem of $\tilde{F}^*$. Let $q\in Q_{(act)}(t_j)$, $p\in Q$ and $x\in\Sigma^*$. The proof is by induction on $|x|=n$. If $n=0$, then $x=\Lambda$. Now if $q=p$, then $\tilde{\delta}_1^*((p,\mu_1^{t_i}(p)),\Lambda,p)=F_1^{\wedge}(\mu_1^{t_i}(p),\tilde{\delta}_1(p,\Lambda,p))=\mu_1^{t_i}(p)$, $\tilde{\delta}_2^*((p,\mu_2^{t_i}(p)),\Lambda,p)=F_1^{\wedge\vee}(\mu_2^{t_i}(p),\tilde{\delta}_2(p,\Lambda,p))=\mu_2^{t_i}(p)$, $\tilde{\delta}_3^*((p,\mu_3^{t_i}(p)),\Lambda,p)=F_1^{\vee}(\mu_3^{t_i}(p),\tilde{\delta}_3(p,\Lambda,p))=\mu_3^{t_i}(p)$.

If $q\neq p$, then $\tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),\Lambda,p)=F_1^{\wedge}(\mu_1^{t_i}(q),\tilde{\delta}_1(q,\Lambda,p))=0\leq\mu_1^{t_j}(p)$, $\tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),\Lambda,p)=F_1^{\wedge\vee}(\mu_2^{t_i}(q),\tilde{\delta}_2(q,\Lambda,p))=1\geq\mu_2^{t_j}(p)$, $\tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),\Lambda,p)=F_1^{\vee}(\mu_3^{t_i}(q),\tilde{\delta}_3(q,\Lambda,p))=1\geq\mu_3^{t_j}(p)$.

Hence the result is true for $n = 0$. For now, we assume that the result is valid for all $y \in \Sigma^*$ with $|y| = n-1$, $n > 0$. For the y above, let $x = u_1 \cdots u_n$ where $u_i \in \Sigma, i = 1, 2, \cdots n$. Then

$$\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1 \cdots u_n, p) = \bigvee \left(\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1, r_1) \wedge \cdots \wedge \tilde{\delta}_1^*((r_{n-1}, \mu_1^{t_{i+n}}(r_{n-1})), u_n, p) \right)$$
$$\leq \bigvee \left(\tilde{\delta}_1^*((r_{n-1}, \mu_1^{t_{i+n}}(r_{n-1})), u_n, p) | r_{n-1} \in Q_{(act)}(t_{i+n}) \right) \leq \bigvee \mu_1^{t_j}(p) = \mu_1^{t_j}(p),$$

$$\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1 \cdots u_n, p) = \bigwedge \left(\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1, r_1) \vee \cdots \vee \tilde{\delta}_2^*((r_{n-1}, \mu_2^{t_{i+n}}(r_{n-1})), u_n, p) \right)$$
$$\leq \bigwedge \left(\tilde{\delta}_2^*((r_{n-1}, \mu_2^{t_{i+n}}(r_{n-1})), u_n, p) | r_{n-1} \in Q_{(act)}(t_{i+n}) \right) \leq \bigwedge \mu_2^{t_j}(p) = \mu_2^{t_j}(p),$$

$$\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1 \cdots u_n, p) = \bigwedge \left(\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1, r_1) \vee \cdots \vee \tilde{\delta}_3^*((r_{n-1}, \mu_3^{t_{i+n}}(r_{n-1})), u_n, p) \right)$$
$$\leq \bigwedge \left(\tilde{\delta}_3^*((r_{n-1}, \mu_3^{t_{i+n}}(r_{n-1})), u_n, p) | r_{n-1} \in Q_{(act)}(t_{i+n}) \right) \leq \bigwedge \mu_3^{t_j}(p) = \mu_3^{t_j}(p),$$

where $r_1 \in Q_{(act)}(t_{i+1}) \cdots r_{n-1} \in Q_{(act)}(t_{i+n})$. Hence $\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p)$, $\mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p)$, $\mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p)$. The converse is trivial. This proof is completed. □

Definition 3.12. Let $\tilde{F}^*$ be a NGFA. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q. Then μ is a neutrosophic strong subsystem of $\tilde{F}^*$, say $\mu \subseteq \tilde{F}^*$, if for every i, $1 \leq i \leq k$ such that $p \in S_\alpha(q)$, then for $q, p \in Q$ and $x \in \Sigma$, $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$, for every $1 \leq j \leq k$.

Theorem 3.13. *Let $\tilde{F}^*$ be a NGFA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q. Then μ is a strong neutrosophic subsystem of $\tilde{F}^*$ if and only if there exists $x \in \Sigma^*$ such that $p \in S_\alpha(q)$, then $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$, for all $q \in Q_{(act)}(t_j)$, $p \in Q$.*

Proof. Suppose that μ is a strong neutrosophic subsystem of $\tilde{F}^*$. Let $q \in Q_{(act)}(t_j)$, $p \in Q$ and $x \in \Sigma^*$. The proof is by induction on $|x| = n$. If $n = 0$, then $x = \Lambda$. Now if $q = p$, then $\delta_1^*((p, \mu_1^{t_i}(p)), \Lambda, p) = 1$, $\delta_2^*((p, \mu_2^{t_i}(p)), \Lambda, p) = 0$, $\delta_3^*((p, \mu_3^{t_i}(p)), \Lambda, p) = 0$ and $\mu_1^{t_j}(p) = \mu_1^{t_j}(p)$, $\mu_2^{t_j}(p) = \mu_2^{t_j}(p)$, $\mu_3^{t_j}(p) = \mu_3^{t_j}(p)$. If $q \neq p$, then $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), \Lambda, p) = F_1^\wedge(\mu_1^{t_i}(q), \tilde{\delta}_1(q, \Lambda, p)) = c \leq \mu_1^{t_j}(p)$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), \Lambda, p) = F_1^{\wedge\vee}(\mu_2^{t_i}(q), \tilde{\delta}_2(q, \Lambda, p)) = d \geq \mu_2^{t_j}(p)$, $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), \Lambda, p) = F_1^\vee(\mu_3^{t_i}(q), \tilde{\delta}_3(q, \Lambda, p)) = e \geq \mu_3^{t_j}(p)$. Hence the result is true for $n = 0$. For now, we assume that the result is valid for all $u \in \Sigma^*$ with $|u| = n-1$, $n > 0$. For the u above, let $x = u_1 \cdots u_n$ where $u_i \in \Sigma^*, i = 1, 2, \cdots n$. Suppose that $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) > c$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) < d$, $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) < e$. Then

$$\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1 \cdots u_n, p) = \bigvee \left\{ \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), u_1, p_1) \wedge \cdots \wedge \tilde{\delta}_1^*((p_{n-1}, \mu_1^{t_{i+n}}(p_{n-1})), u_n, p) \right\} > c,$$
$$\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1 \cdots u_n, p) = \bigwedge \left\{ \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), u_1, p_1) \vee \cdots \vee \tilde{\delta}_2^*((p_{n-1}, \mu_2^{t_{i+n}}(p_{n-1})), u_n, p) \right\} < d,$$
$$\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1 \cdots u_n, p) = \bigwedge \left\{ \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), u_1, p_1) \vee \cdots \vee \tilde{\delta}_3^*((p_{n-1}, \mu_3^{t_{i+n}}(p_{n-1})), u_n, p) \right\} < e,$$

where $p_1 \in Q_{(act)}(t_i), \cdots, p_{n-1} \in Q_{(act)}(t_{i+n})$.

This implies that $\tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),u_1,p_1) > c, \cdots, \tilde{\delta}_1^*((p_{n-1},\mu_1^{t_{i+n}}(p_{n-1})),u_n,p) > c$, $\tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),u_1,p_1) < d, \cdots, \tilde{\delta}_2^*((p_{n-1},\mu_2^{t_{i+n}}(p_{n-1})),u_n,p) < d$, $\tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),u_1,p_1) < e, \cdots, \tilde{\delta}_3^*((p_{n-1},\mu_3^{t_{i+n}}(p_{n-1})),u_n,p) < e$. Hence $\mu_1^{t_j}(p) \geq \mu_1^{t_{i+n}}(p_{n-1}), \mu_1^{t_{i+n}}(p) \geq \mu_1^{t_{i+n-1}}(p_{n-2}), \cdots, \mu_1^{t_i}(p_1) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_{i+n}}(p_{n-1}), \mu_2^{t_{i+n}}(p) \leq \mu_2^{t_{i+n-1}}(p_{n-2}), \cdots, \mu_2^{t_i}(p_1) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_{i+n}}(p_{n-1}), \mu_3^{t_{i+n}}(p) \leq \mu_3^{t_{i+n-1}}(p_{n-2}), \cdots, \mu_3^{t_i}(p_1) \leq \mu_3^{t_j}(q)$. Thus $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$. The converse is trivial. The proof is completed. □

4 Neutrosophic General Finite Switchboard Automata

Definition 4.1. Let $\tilde{F}^*$ be a max-min NGFA. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ be a neutrosophic set in $(Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma \times Q$ in Q. Then

1. $\tilde{F}^*$ is switching, if it satisfies $\forall p, q \in Q$, $a \in \Sigma$ and for every i, $i \geq 0$, $\tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),a,p) = \tilde{\delta}_1^*((p,\mu_1^{t_i}(p)),a,q)$, $\tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),a,p) = \tilde{\delta}_2^*((p,\mu_2^{t_i}(p)),a,q)$, $\tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),a,p) = \tilde{\delta}_3^*((p,\mu_3^{t_i}(p)),a,q)$.

2. $\tilde{F}^*$ is commutative, if it satisfies $\forall p, q \in Q$, $a, b \in \Sigma$ and for every i, $i \geq 1$, $\tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),ab,p) = \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),ba,p)$, $\tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),ab,p) = \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),ba,p)$, $\tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),ab,p) = \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),ba,p)$.

3. $\tilde{F}^*$ is Neutrosophic General Finite Switchboard Automata (NGFSA, for short), if $\tilde{F}^*$ satisfies both switching and commutative.

Proposition 4.2. *Let $\tilde{F}$ be a NGFA, if $\tilde{F}^*$ is a commutative NGFSA, then for every $i \geq 1$,*

$$\begin{aligned}
\tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),xa,p) &= \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),ax,p),\\
\tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),xa,p) &= \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),ax,p),\\
\tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),xa,p) &= \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),ax,p),
\end{aligned}$$

for all $q \in Q_{act}(t_{i-1}), p \in S_c(q)$, $a \in \Sigma$ and $x \in \Sigma^*$.

Proof. Since $p \in S_c(q)$ then $q \in Q_{act}(t_{i-1})$ and $|x| = n$. If $n = 0$, then $x = \Lambda$. Thus

$$\begin{aligned}
\tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),xa,p) = \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),\Lambda a,p) = \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),a,p) &= \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),a\Lambda,p)\\
&= \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),ax,p),\\
\tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),xa,p) = \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),\Lambda a,p) = \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),a,p) &= \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),a\Lambda,p)\\
&= \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),ax,p),\\
\tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),xa,p) = \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),\Lambda a,p) = \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),a,p) &= \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),a\Lambda,p)\\
&= \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),ax,p).
\end{aligned}$$

Suppose the result is true for all $u \in \Sigma^*$ with $|u| = n-1$, where $n > 0$. Let $x = ub$, where $b \in \Sigma$. Then

$$\tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), xa, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), uba, p) = \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), u, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), ba, p)\Big)$$
$$= \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), u, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), ab, p)\Big) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), uab, p)$$
$$= \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), ua, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), b, p)\Big)$$
$$= \bigvee_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q, \mu_1^{t_{i-1}}(q)), au, r) \wedge \tilde{\delta}_1((r, \mu_1^{t_i}(r)), b, p)\Big) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), aub, p) = \tilde{\delta}_1^*((q, \mu_1^{t_{i-1}}(q)), ax, p),$$

$$\tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), xa, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), uba, p) = \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), u, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), ba, p)\Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), u, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), ab, p)\Big) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), uab, p)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), ua, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), b, p)\Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q, \mu_2^{t_{i-1}}(q)), au, r) \vee \tilde{\delta}_2((r, \mu_2^{t_i}(r)), b, p)\Big) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), aub, p) = \tilde{\delta}_2^*((q, \mu_2^{t_{i-1}}(q)), ax, p),$$

$$\tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), xa, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), uba, p) = \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), u, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), ba, p)\Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), u, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), ab, p)\Big) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), uab, p)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), ua, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), b, p)\Big)$$
$$= \bigwedge_{r \in Q_{act}(t_i)} \Big(\tilde{\delta}_3((q, \mu_3^{t_{i-1}}(q)), au, r) \vee \tilde{\delta}_3((r, \mu_3^{t_i}(r)), b, p)\Big) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), aub, p) = \tilde{\delta}_3^*((q, \mu_3^{t_{i-1}}(q)), ax, p).$$

This completes the proof. □

Proposition 4.3. *Let $\tilde{F}$ be a NGFA, if $\tilde{F}^*$ is a switching NGFSA, then for every $i \geq 0$, $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q)$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q)$, $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q)$, for all $p, q \in Q_{act}(t_i)$ and $x \in \Sigma^*$.*

Proof. Since $p, q \in Q_{act}(t_i)$ and $x \in \Sigma^*$, we prove the result by induction on $|x| = n$. First, we assume that $x = \Lambda$, whenever $n = 0$. Then we have $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), \Lambda, p) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), \Lambda, q) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q)$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), \Lambda, p) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), \Lambda, q) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q)$ $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), \Lambda, p) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), \Lambda, q) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q)$. Thus, the theorem holds for $x = \Lambda$. Now, we assume that the results holds for all $u \in \Sigma^*$ such that $|u| = n-1$ and $n > 0$. Let

$a \in \Sigma$ and $x \in \Sigma^*$ be such that $x = ua$. Then

$$\tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),x,p) = \tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),ua,p) = \bigvee_{r\in Q_{act}(t_{i+1})} \Big(\tilde{\delta}_1((q,\mu_1^{t_i}(q)),u,r) \wedge \tilde{\delta}_1((r,\mu_1^{t_{i+1}}(r)),a,p)\Big)$$
$$= \bigvee_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_1((r,\mu_1^{t_i}(r)),u,q) \wedge \tilde{\delta}_1(p,\mu_1^{t_{i+1}}(p)),a,r)\Big) = \bigvee_{r\in Q_{act}(t_{i+1})} \Big(\tilde{\delta}_1((p,\mu_1^{t_i}(p)),a,r) \wedge \tilde{\delta}_1((r,\mu_1^{t_{i+1}}(r)),u,q)\Big)$$
$$= \tilde{\delta}_1^*((p,\mu_1^{t_i}(p)),au,q) = \tilde{\delta}_1^*((p,\mu_1^{t_i}(p)),ua,q) = \tilde{\delta}_1^*((p,\mu_1^{t_i}(p)),x,q),$$

$$\tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),x,p) = \tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),ua,p) = \bigwedge_{r\in Q_{act}(t_{i+1})} \Big(\tilde{\delta}_2((q,\mu_2^{t_i}(q)),u,r) \vee \tilde{\delta}_2((r,\mu_2^{t_{i+1}}(r)),a,p)\Big)$$
$$= \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_2((r,\mu_2^{t_i}(r)),u,q) \vee \tilde{\delta}_2(p,\mu_2^{t_{i+1}}(p)),a,r)\Big) = \bigwedge_{r\in Q_{act}(t_{i+1})} \Big(\tilde{\delta}_2((p,\mu_2^{t_i}(p)),a,r) \vee \tilde{\delta}_2((r,\mu_2^{t_{i+1}}(r)),u,q)\Big)$$
$$= \tilde{\delta}_2^*((p,\mu_2^{t_i}(p)),au,q) = \tilde{\delta}_2^*((p,\mu_2^{t_i}(p)),ua,q) = \tilde{\delta}_2^*((p,\mu_2^{t_i}(p)),x,q),$$

$$\tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),x,p) = \tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),ua,p) = \bigwedge_{r\in Q_{act}(t_{i+1})} \Big(\tilde{\delta}_3((q,\mu_3^{t_i}(q)),u,r) \vee \tilde{\delta}_3((r,\mu_3^{t_{i+1}}(r)),a,p)\Big)$$
$$= \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_3((r,\mu_3^{t_i}(r)),u,q) \vee \tilde{\delta}_3(p,\mu_3^{t_{i+1}}(p)),a,r)\Big) = \bigwedge_{r\in Q_{act}(t_{i+1})} \Big(\tilde{\delta}_3((p,\mu_3^{t_i}(p)),a,r) \vee \tilde{\delta}_3((r,\mu_3^{t_{i+1}}(r)),u,q)\Big)$$
$$= \tilde{\delta}_3^*((p,\mu_3^{t_i}(p)),au,q) = \tilde{\delta}_3^*((p,\mu_3^{t_i}(p)),ua,q) = \tilde{\delta}_3^*((p,\mu_3^{t_i}(p)),x,q).$$

Hence, the result is true for $|u| = n$. This completes the proof. □

Proposition 4.4. *Let $\tilde{F}$ be a NGFA, if $\tilde{F}^*$ is a NGFSA, then for every $i \geq 1$, $\tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_1^*((p,\mu_1^{t_{i-1}}(p)),yx,q)$, $\tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_2^*((p,\mu_2^{t_{i-1}}(p)),yx,q)$, $\tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_3^*((p,\mu_3^{t_{i-1}}(p)),yx,q)$ for all $p,q \in Q$ and $x,y \in \Sigma^*$.*

Proof. Since $p,q \in Q$ and $x,y \in \Sigma^*$, we prove the result by induction on $|x| = n$. First, we assume that $n = 0$, then $x = \Lambda$. Thus
$\tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),x\Lambda,p) = \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),\Lambda x,p) = \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),yx,p)$,
$\tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),x\Lambda,p) = \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),\Lambda x,p) = \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),yx,p)$,
$\tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),x\Lambda,p) = \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),\Lambda x,p) = \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),yx,p)$.

Suppose that
$\tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),xu,p) = \tilde{\delta}_1^*((p,\mu_1^{t_{i-1}}(p)),ux,q)$, $\tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),xu,p) = \tilde{\delta}_2^*((p,\mu_2^{t_{i-1}}(p)),ux,q)$,
$\tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),xu,p) = \tilde{\delta}_3^*((p,\mu_3^{t_{i-1}}(p)),ux,q)$, for every $u \in \Sigma^*$.

Now, continue the result is true for all $u \in \Sigma^*$ with $|u| = n-1$, where $n > 0$. Let $y = ua$, where $a \in \Sigma$

and $u \in \Sigma^*$. Then

$$\tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),xua,p) = \bigvee_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q,\mu_1^{t_{i-1}}(q)),xu,r) \wedge \tilde{\delta}_1((r,\mu_1^{t_i}(r)),a,p)\Big)$$
$$= \bigvee_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_1((q,\mu_1^{t_{i-1}}(q)),ux,r) \wedge \tilde{\delta}_1((r,\mu_1^{t_i}(r)),a,p)\Big)$$
$$= \bigvee_{r\in Q_{act}(t_{i-1})} \Big(\tilde{\delta}_1((r,\mu_1^{t_{i-1}}(r)),ux,q) \wedge \tilde{\delta}_1((p,\mu_1^{t_i}(p)),a,r)\Big)$$
$$= \bigvee_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_1((p,\mu_1^{t_{i-1}}(p)),a,r) \wedge \tilde{\delta}_1((r,\mu_1^{t_i}(r)),ux,q)\Big)$$
$$= \tilde{\delta}_1^*((p,\mu_1^{t_{i-1}}(p)),aux,q) = \bigvee_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_1((p,\mu_1^{t_{i-1}}(p)),au,r) \wedge \tilde{\delta}_1((r,\mu_1^{t_i}(r)),x,q)\Big)$$
$$= \bigvee_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_1((p,\mu_1^{t_{i-1}}(p)^),ua,r) \wedge \tilde{\delta}_1((r,\mu_1^{t_i}(r)),x,q)\Big)$$
$$= \tilde{\delta}_1^*((p,\mu_1^{t_{i-1}}(p)),uax,q) = \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),uax,p) = \tilde{\delta}_1^*((q,\mu_1^{t_{i-1}}(q)),yx,p),$$

$$\tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),xua,p) = \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q,\mu_2^{t_{i-1}}(q)),xu,r) \vee \tilde{\delta}_2((r,\mu_2^{t_i}(r)),a,p)\Big)$$
$$= \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_2((q,\mu_2^{t_{i-1}}(q)),ux,r) \vee \tilde{\delta}_2((r,\mu_2^{t_i}(r)),a,p)\Big)$$
$$= \bigwedge_{r\in Q_{act}(t_{i-1})} \Big(\tilde{\delta}_2((r,\mu_2^{t_{i-1}}(r)),ux,q) \vee \tilde{\delta}_2((p,\mu_2^{t_i}(p)),a,r)\Big)$$
$$= \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_2((p,\mu_2^{t_{i-1}}(p)),a,r) \vee \tilde{\delta}_2((r,\mu_2^{t_i}(r)),ux,q)\Big)$$
$$= \tilde{\delta}_2^*((p,\mu_2^{t_{i-1}}(p)),aux,q) = \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_2((p,\mu_2^{t_{i-1}}(p)),au,r) \wedge \tilde{\delta}_2((r,\mu_2^{t_i}(r)),x,q)\Big)$$
$$= \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_2((p,\mu_2^{t_{i-1}}(p)^),ua,r) \vee \tilde{\delta}_2((r,\mu_2^{t_i}(r)),x,q)\Big)$$
$$= \tilde{\delta}_2^*((p,\mu_2^{t_{i-1}}(p)),uax,q) = \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),uax,p) = \tilde{\delta}_2^*((q,\mu_2^{t_{i-1}}(q)),yx,p),$$

$$\tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),xy,p) = \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),xua,p) = \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_3((q,\mu_3^{t_{i-1}}(q)),xu,r) \vee \tilde{\delta}_3((r,\mu_3^{t_i}(r)),a,p)\Big)$$
$$= \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_3((q,\mu_3^{t_{i-1}}(q)),ux,r) \vee \tilde{\delta}_3((r,\mu_3^{t_i}(r)),a,p)\Big)$$
$$= \bigwedge_{r\in Q_{act}(t_{i-1})} \Big(\tilde{\delta}_3((r,\mu_3^{t_{i-1}}(r)),ux,q) \vee \tilde{\delta}_3((p,\mu_3^{t_i}(p)),a,r)\Big)$$
$$= \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_3((p,\mu_3^{t_{i-1}}(p)),a,r) \vee \tilde{\delta}_3((r,\mu_3^{t_i}(r)),ux,q)\Big)$$
$$= \tilde{\delta}_3^*((p,\mu_3^{t_{i-1}}(p)),aux,q) = \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_3((p,\mu_3^{t_{i-1}}(p)),au,r) \wedge \tilde{\delta}_3((r,\mu_3^{t_i}(r)),x,q)\Big)$$
$$= \bigwedge_{r\in Q_{act}(t_i)} \Big(\tilde{\delta}_3((p,\mu_3^{t_{i-1}}(p)^),ua,r) \vee \tilde{\delta}_3((r,\mu_3^{t_i}(r)),x,q)\Big)$$
$$= \tilde{\delta}_3^*((p,\mu_3^{t_{i-1}}(p)),uax,q) = \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),uax,p) = \tilde{\delta}_3^*((q,\mu_3^{t_{i-1}}(q)),yx,p).$$

This completes the proof. □

Definition 4.5. Let $\tilde{F}^*$ be a GNFSA. Let $\mu = \langle \mu_1, \mu_2, \mu_3\rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^*\rangle$ in $(Q \times [0,1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q. Then μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$, say $\mu \subseteq \tilde{F}^*$, if for every j, $1 \leq j \leq k$ such that $\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p)$, $\mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p)$, $\mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p)$. $\forall q, p \in Q$ and $x \in \Sigma$.

Theorem 4.6. *Let $\tilde{F}^*$ be a NGFSA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q. Then μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$ if and only if $\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_j}(q)), x, p)$, $\mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_j}(q)), x, p)$, $\mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_j}(q)), x, p)$, for all $q \in Q_{(act)}(t_j)$, $p \in Q$ and $x \in \Sigma^*$.*

Proof. The proof of the theorem is similar to Theorem 3.11 and it is clear that μ satisfies switching and commutative, since $\tilde{F}^*$ is NGFSA. This proof is completed. □

Definition 4.7. Let $\tilde{F}^*$ be a NGFSA. Let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q. Then μ is a neutrosophic strong switchboard subsystem of $\tilde{F}^*$, say $\mu \subseteq \tilde{F}^*$, if for every i, $1 \leq i \leq k$ such that $p \in S_\alpha(q)$, then for $q, p \in Q$ and $x \in \Sigma$, $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$, for every $1 \leq j \leq k$.

Theorem 4.8. *Let $\tilde{F}^*$ be a NGFA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ and $\tilde{\delta}^* = \langle \tilde{\delta}_1^*, \tilde{\delta}_2^*, \tilde{\delta}_3^* \rangle$ in $(Q \times [0,1] \times [0,1] \times [0,1]) \times \Sigma^* \times Q$ be a neutrosophic set in Q. Then μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$ if and only if there exists $x \in \Sigma^*$ such that $p \in S_\alpha(q)$, then $\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q)$, $\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q)$, $\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q)$, for all $q \in Q_{(act)}(t_j)$, $p \in Q$.*

Proof. The proof of the theorem is similar to Theorem 3.13 and it is clear that μ satisfies switching and commutative, since $\tilde{F}^*$ is NGFSA. The proof is completed. □

Theorem 4.9. *Let $\tilde{F}^*$ be a NGFSA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ be a neutrosophic subset of Q. If μ is a neutrosohic switchboard subsystem of $\tilde{F}^*$, then μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$.*

Proof. Assume that $\tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) > 0$, $\tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) < 1$ and $\tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) < 1$, for all $x \in \Sigma$. Since μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$, we have

$$\mu_1^{t_j}(p) \geq \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p), \quad \mu_2^{t_j}(p) \leq \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p), \quad \mu_3^{t_j}(p) \leq \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p).$$

for all $q \in Q_{(act)}(t_j)$, $p \in Q$ and $x \in \Sigma$. As μ is switching, then we have

$$\begin{aligned}
\mu_1^{t_j}(p) &\geq \tilde{\delta}_1^*((q, \mu_1^{t_i}(q)), x, p) = \tilde{\delta}_1^*((p, \mu_1^{t_i}(p)), x, q) = \mu_1^{t_j}(q), \\
\mu_2^{t_j}(p) &\leq \tilde{\delta}_2^*((q, \mu_2^{t_i}(q)), x, p) = \tilde{\delta}_2^*((p, \mu_2^{t_i}(p)), x, q) = \mu_2^{t_j}(q), \\
\mu_3^{t_j}(p) &\leq \tilde{\delta}_3^*((q, \mu_3^{t_i}(q)), x, p) = \tilde{\delta}_3^*((p, \mu_3^{t_i}(p)), x, q) = \mu_3^{t_j}(q).
\end{aligned}$$

As μ is commutative, then $x = uv$, we have

$$\begin{aligned}
\mu_1^{t_j}(p) &\geq \tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),x,p) = \tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),uv,p)\\
&= \bigvee \left\{\tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),u,r) \wedge \tilde{\delta}_1^*((r,\mu_1^{t_{i+1}}(r)),v,p) | r \in Q_{1(act)}(t_{i+1})\right\}\\
&= \bigvee \left\{\tilde{\delta}_1^*((r,\mu_1^{t_i}(r)),u,q) \wedge \tilde{\delta}_1^*((p,\mu_1^{t_{i+1}}(p)),v,r) | r \in Q_{1(act)}(t_{i+1})\right\}\\
&= \bigvee \left\{\tilde{\delta}_1^*((p,\mu_1^{t_{i+1}}(p)),v,r) \wedge \tilde{\delta}_1^*((r,\mu_1^{t_i}(r)),u,q) | r \in Q_{1(act)}(t_{i+1})\right\}\\
&= \tilde{\delta}_1^*((p,\mu_1^{t_{i+1}}(p)),vu,q) = \tilde{\delta}_1^*((p,\mu_1^{t_{i+1}}(p)),uv,q) = \tilde{\delta}_1^*((p,\mu_1^{t_{i+1}}(p)),x,q) \geq \mu_1^{t_j}(q),\\
\mu_2^{t_j}(p) &\leq \tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),x,p) = \tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),uv,p)\\
&= \bigwedge \left\{\tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),u,r) \vee \tilde{\delta}_2^*((r,\mu_2^{t_{i+1}}(r)),v,p) | r \in Q_{1(act)}(t_{i+1})\right\}\\
&= \bigwedge \left\{\tilde{\delta}_2^*((r,\mu_2^{t_i}(r)),u,q) \vee \tilde{\delta}_2^*((p,\mu_2^{t_{i+1}}(p)),v,r) | r \in Q_{1(act)}(t_{i+1})\right\}\\
&= \bigwedge \left\{\tilde{\delta}_2^*((p,\mu_2^{t_{i+1}}(p)),v,r) \vee \tilde{\delta}_2^*((r,\mu_2^{t_i}(r)),u,q) | r \in Q_{1(act)}(t_{i+1})\right\}\\
&= \tilde{\delta}_2^*((p,\mu_2^{t_{i+1}}(p)),vu,q) = \tilde{\delta}_2^*((p,\mu_2^{t_{i+1}}(p)),uv,q) = \tilde{\delta}_2^*((p,\mu_2^{t_{i+1}}(p)),x,q) \leq \mu_2^{t_j}(q),\\
\mu_3^{t_j}(p) &\leq \tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),x,p) = \tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),uv,p)\\
&= \bigwedge \left\{\tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),u,r) \vee \tilde{\delta}_3^*((r,\mu_3^{t_{i+1}}(r)),v,p) | r \in Q_{1(act)}(t_{i+1})\right\}\\
&= \bigwedge \left\{\tilde{\delta}_3^*((r,\mu_3^{t_i}(r)),u,q) \vee \tilde{\delta}_3^*((p,\mu_3^{t_{i+1}}(p)),v,r) | r \in Q_{1(act)}(t_{i+1})\right\}\\
&= \bigwedge \left\{\tilde{\delta}_3^*((p,\mu_3^{t_{i+1}}(p)),v,r) \vee \tilde{\delta}_3^*((r,\mu_3^{t_i}(r)),u,q) | r \in Q_{1(act)}(t_{i+1})\right\}\\
&= \tilde{\delta}_3^*((p,\mu_3^{t_{i+1}}(p)),vu,q) = \tilde{\delta}_3^*((p,\mu_3^{t_{i+1}}(p)),uv,q) = \tilde{\delta}_3^*((p,\mu_3^{t_{i+1}}(p)),x,q) \leq \mu_3^{t_j}(q).
\end{aligned}$$

Hence μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$. □

Theorem 4.10. *Let $\tilde{F}^*$ be a NGFSA and let $\mu = \langle \mu_1, \mu_2, \mu_3 \rangle$ be a neutrosophic subset of Q. If μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$, then μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$.*

Proof. Let $q, p \in Q$. Since μ is a strong neutrosophic switchboard subsystem of $\tilde{F}^*$ and μ is switching, we have for all $x \in \Sigma$, since $\tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),x,p) > 0$, $\tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),x,p) < 1$ and $\tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),x,p) < 1$, $\forall x \in \Sigma$,
$\mu_1^{t_j}(p) \geq \mu_1^{t_j}(q) \geq \tilde{\delta}_1^*((p,\mu_1^{t_i}(p)),x,q) \geq \tilde{\delta}_1^*((q,\mu_1^{t_i}(q)),x,p)$,
$\mu_2^{t_j}(p) \leq \mu_2^{t_j}(q) \leq \tilde{\delta}_2^*((p,\mu_2^{t_i}(p)),x,q) \leq \tilde{\delta}_2^*((q,\mu_2^{t_i}(q)),x,p)$,
$\mu_3^{t_j}(p) \leq \mu_3^{t_j}(q) \leq \tilde{\delta}_3^*((p,\mu_3^{t_i}(p)),x,q) \leq \tilde{\delta}_3^*((q,\mu_3^{t_i}(q)),x,p)$.

It is clear that μ is commutative. Thus μ is a neutrosophic switchboard subsystem of $\tilde{F}^*$. □

5 Conclusions

This paper attempt to develop and present a new general definition for neutrosophic finite automata. The general definition for (strong) subsystem also examined and discussed their properties. A comprehensive analysis and an appropriate methodology to manage the essential issues of output mapping in general fuzzy automata were studied by Doostfatemen and Kremer [11]. Their approach is consistent with the output which is either associated with the states (Moore model) or with the transitions (Mealy model). Interval-valued fuzzy subsets have many applications in several areas. The concept of interval-valued fuzzy sets have been studied in various algebraic structures, see [7, 26]. On the basis [11] and [7], the future work will focus on general interval-valued neutrosophic finite automata with output respond to input strings.

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Implementation of Neutrosophic Function Memberships Using MATLAB Program

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Abstract. Membership function (MF) plays a key role for getting an output of a system and hence it influences system's performance directly. Therefore choosing a MF is an essential task in fuzzy logic and neutrosophic logic as well. Uncertainty is usually represented by MFs. In this paper, a novel Matlab code is derived for trapezoidal neutrosophic function and the validity of the proposed code is proved with illustrative graphical representation

Keywords: Membership function, Matlab code, Trapezoidal neutrosophic function, Graphical representation

1 Introduction

The membership function (MF) designs a structure of practical relationship to relational structure numerically where the elements lies between 0 and 1. By determining the MFs one can model the relationship between the cognitive and stimuli portrayal in fuzzy set theory [1]. The computed MF will provide a solution to the problem and the complete process can be observed as a training and acceptable approximation to the function from the behavior of the objects [2]. This kind of MFs can be utilized for the fuzzy implication appeared in the given rules to examine more examples [3].

The MFs of fuzzy logic is nothing but a stochastic representation and are used to determine a probability space and its value may be explained as probabilities. The stochastic representation will to know the reasoning and capability of fuzzy control [4]. MFs which are characterized in a single domain where the functions are in terms of single variable are playing a vital role in fuzzy logic system. FMFs determine the degree of membership (M/S) which is a crisp value. Generally MFs are considered as either triangular or trapezoidal as they are adequate, can be design easily and flexible [5].

MFs can be carried out using hardware [6]. MFs are taking part in most of the works done under fuzzy environment without checking their existence for sure and also in the connection between a studied characteristic for sure and its reference set won't be problematic as it is a direct measurement [7]. It is adorable to have continuously differentiable MFs with less parameters [8]. MFs plays an important role in fuzzy classifier (FC). In traditional FC, the domain of every input variable is separated into various intervals. All these intervals is assumed to be a FS and a correlated MF is determined. Hence the input space is separated again into various sub regions which are all parallel in to input axes and a fuzzy rule is defined for all these sub regions if the input belongs to the sub region then it is also belongs to the associated class with the sub region.

Further the degrees of M/S of an unidentified input for all the FSs are evaluated and the input is restricted into the class with maximum degree of M/S. Thus the MFs are directly control the performance of the fuzzy classifier [10]. If the position of the MF is changed then the direct methods maximize the understanding rate of the training data by calculating the total increase directly [11]. Estimation of the MF is usually based on the level of information gained with the experiment transferred by the numerical data [12]. Due to the important role of MFs, concepts of fuzzy logic have been applied in many of the control systems for controlling the robot, nuclear reactor, climate, speed of the car, power systems, memory device under fuzzy logic, aircraft flight, mobile robots and focus of a camcorder.

There has been a habit of restrain the MFs into a well-known formats like triangular, trapezoidal and standard Gaussian or sigmoid types [13]. In information systems the incomplete information can be designed by rough sets [20]. Neutrosophy has established the base for the entire family of novel mathematical theories which generalizes the counterparts of the conventional and fuzzy sets [21]. The success of an approach depends on the MFs and hence designing MFs is an important task for the process and the system. Theory of FSs contributes the way of handling impreciseness, uncertainty and vagueness in the software metrics. The uncertainty of the problem can be solved b considering MFs in an expert system under fuzzy setting. Triangular and trapezoidal MFs are flexible representation of domain expert knowledge and where the computational complexity is less. Hence the derivation of the MF is need to be clarified.

The MFs are continuous and maps from any closed interval to [0,1]. Also which are all either monotonically decreasing or increasing or both [22]. A connectively flexible aggregation of crisp and imprecise knowledge is possible with the horizontal MFs which are capable of introducing uncertainty directly [23]. There are effective methods for calculating MFs of FSs connected with few multi criteria decision making problem [25]. Due to the possibility of having some degree of hesitation, one could not define the non-membership degree by subtracting membership degree from 1 [26]. The degree of the fuzzy sets will be determined by FMFs. [30] Crisp value is converted into fuzzy during fuzzification process. If uncertainty exists on the variable then becomes fuzzy and could be characterized by MFs. The degree of MF is determined by fuzzification.

In the real world problems satisfaction of the decision maker is not possible at most of the time due to impreciseness and incompleteness of the information of the data. Fuzziness exist in the FS is identified by the MF [27]. he uncertainty measure is the possible MF of the FS and is interpreted individually. This is the advantage of MFs especially one needs to aggregate the data and human expert knowledge. Designing MFs vary according to the ambition of their use. Membership functions influence a quality of inference [31].

Neutrosophy is the connecting idea with its opposite idea also with non-committal idea to get the common parts with unknown things [36]. Artificial network, fuzzy clustering, genetic algorithm are some methods to determine the MFs and all these consume time with complexities. The MFs plays a vital role in getting the output. The methods are uncertain due to noisy data and difference of opinion of the people. The most suitable shape and widely used MFs in fuzzy systems are triangular and trapezoidal [37]. Properties and relations of multi FSs and its extension are depending on the order relations of the MFs [38]. FS is the class of elements with a continuum of grades of M/S [39].

The logic of neutrosophic concept is an explicit frame trying to calculate the truth, IIndetrminacy and falsity. Smarandache observes the dissimilarity of intuitionistic fuzzy logic (IFL) and neutrosophic logic (NL). NL could differentiate absolute truth (AT) and relative truth (RT) by assigning 1^+ for AT and 1 for RT and is also applied in the field of philosophy. Hence the standard interval [0,1] used in IFS is extended to non-standard $]^-0,1^+[$ in NL. There is not condition on truth, indeterminacy and falsity which are all the subsets of non-standard unitary interval. This is the reason of considering $^-0\ ^-0 \le \inf T \le \inf I \le \inf F \le \sup T \le \sup I \le \sup F \le 3^+$ and which is useful to characterize para consistent and incomplete information [40]. The generalized form of trapezoidal FNs, trapezoidal IFNs, triangular FN and TIFNs are the trapezoidal and triangular neutrosophic fuzzy number [48].

2 Review of Literature

The authors of, [Zysno 1] presented a methodology to determine the MFs analytically. [Sebag and Schoenauer 2] Established algorithms to determine functions from examples. [Bergadano and Cutello 3] proposed an effective technique to learn MFs for fuzzy predicates. [Hansson 4] introduce a stochastic perception of the MFs based on fuzzy logic. [Kelly and Painter 5] proposed a methodology to define N-dimensional fuzzy MFs (FMFs) which is a generalized form of one dimensional MF generally used in fuzzy systems. [Peterson et al. 6] presented a hardware implementation of MF. [Royo and Verdegay 7] examined about the characterization of the different cases where the endurance of the MF is assured.

[Grauel and L. A. Ludwig 8] proposed a class of MFs for symmetrically and asymmetrically in exponential order and constructed a more adaptive MFs. [Straszecka 9] presented preliminaries and methodology to define the MFs of FSs and discussed about application of FS with its universe, certainty of MFs and format. [Abe 10] examined the influence of the MFs in fuzzy classifier. [Abe 11] proved that by adjusting the slopes and positions the performance of the fuzzy rule classification can be improved [Pedrycz and G. Vukovich 12] imposed on an influential issue of determining MF. [J. M. Garibaldi and R. I. John 13] focused more MFs which considered as the alternatives in fuzzy systems [T. J. Ross 14] established the methodology of MFs.

[Brennan, E. Martin 15] proposed MFs for dimensional proximity. [Hachani et al. 16] Proposed a new incremental method to represent the MFs for linguistic terms. [Gasparovica et al. 17] examined about the suitable MF for data analysis in bioinformatics. [Zade and Ismayilova 18] investigated a class of MFs which conclude the familiar types of MFs for FSs. [Bilgic 19] proposed a method of measuring MFs. [Broumi et al. 20] established rough neutrosophic sets and their properties. [Salama et al. 21] proposed a technique for constructing. [Yadava and Yadav 22] proposed an approach for constructing the MFs of software metrics. [Piegat and M. Landowski 23] proposed horizontal MFs to determine the FS instead of usual vertical MFs. [Mani 24] reviewed the relation between different meta theoretical concepts of probability and rough MFs critically.

[Sularia 25] showed their interest of multi-criteria decision analysis under fuzzy environment. [Ali and F. Smarandache 26] Introduced complex NS. [Goyal et al. 27] proposed a circuit model for Gaussian MF. [Can and Ozguven 28] proposed fuzzy logic controller with neutrosophic MFs. [Ali et al. 29] introduced δ-equalities and their properties of NSs. [Radhika and Parvathi 30] introduced different fuzzification methods for intuitionistic fuzzy environment. [Porebski and Straszecka 31] examined diagnosing rules for driving data which can be described by human experts. [Hong et al. 32] accumulated the concepts of fuzzy MFs using fuzzy c-means clustering method.

[Kundu 33] proposed an improved method of approximation of piecewise linear MFs with the support of approximation of cut function obtained by sigmoid function. [Wang 34] proposed the operational laws of fuzzy ellipsoid numbers and straight connection between the MFs which are located on the junctions and edges. [Mani 35] studied the contemplation of theory of probability over rough MFs. [Christianto and Smarandache 36] offered a new perception at Liquid church and neutrosophic MF. [Asanka and A. S. Perera 37] introduced a new approach of using box plot to determine fuzzy Function with some conditions. [Sebastian and F. Smarandache 38] generalized the concepts of NSs and its extension method. [Reddy 39] proposed a FS with two MFs such as Belief and Disbelief. [Lupianeza 40] determined NSs and Topology.

[Zhang et al. 41] derived FMFs analytically. [Wang 42] framed a framework theoretically to construct MFs in a hierarchical order. [Germashev et al. 43] proposed convergence of series of FNs along with Unimodal membership. [Marlen and Dorzhigulov 44] implemented FMF with Memristor. [Ahmad et al. 45] introduced MFs and fuzzy rules for Harumanis examinations [Buhentala et al. 46] explained about the procedure and process of the Takagi-Sugeno fuzzy model. [Broumi et al. 47-55] proposed few concepts of NSs, triangular and trapezoidal NNs.

From this literature study, to the best our knowledge there is no contribution of work on deriving membership function using Matlab under neutrosophic environment and hence it's a motivation of the present work.

3 Preliminaries

Definition:A trapezoidal neutrosophic number $a=\langle (a,b,c,d);w_a,u_a,y_a\rangle$ is a special neutrosophic set on the real number set R, whose truth-membership, indeterminacy– membership and falsity-membership functions are defined as follows:

$$\mu_a(x)=\begin{cases}\dfrac{(x-a)}{(b-a)}w_a & , \quad a\le x\le b\\ w_a & , \quad b\le x\le c\\ \dfrac{(d-x)}{(d-c)}w_a & , \quad c\le x\le d\\ 0 & , \quad otherwise\end{cases}
\qquad
v_a(x)=\begin{cases}\dfrac{(b-x)+u_a(x-a)}{(b-a)} & , \quad a\le x\le b\\ u_a & , \quad b\le x\le c\\ \dfrac{(x-c)+u_a(d-x)}{(d-c)} & , \quad c\le x\le d\\ 1 & , \quad otherwise\end{cases}$$

$$\lambda_a(x)=\begin{cases}\dfrac{(b-x)+y_a(x-a)}{(b-a)} & , \quad a\le x\le b\\ y_a & , \quad b\le x\le c\\ \dfrac{(x-c)+y_a(d-x)}{(d-c)} & , \quad c\le x\le d\\ 1 & , \quad otherwise\end{cases}$$

4. Proposed Matlab code to find Trapezoidal Neutrosophic Function

In this section, trapezoidal neutrosophic function has been proposed using Matlab program and for the differennt membership values, pictorical representation is given and the Matlab code is designed as follows.

```
Trapezoidal neutrosophic Function (trin)
%x=45:70;
%[y,z]=trin(x,50,55,60,65, 0.6, 0.4,0.6)%

U truth membership
V indterminacy membership
W :falsemembership

function [y,z,t]=trin(x,a,b,c,d,u,v,w)
y=zeros(1,length(x));
z=zeros(1,length(x));
t=zeros(1,length(x));
for j=1:length(x)
if(x(j)<=a)
   y(j)=0;
   z(j)=1;
   t(j)=1;
elseif(x(j)>=a)&&(x(j)<=b)
y(j)=u*(((x(j)-a)/(b-a)));
z(j)=(((b-x(j))+v*(x(j)-a))/(b-a));
t(j)=(((b-x(j))+w*(x(j)-a))/(b-a));
elseif(x(j)>=b)&&(x(j)<=c)
y(j)=u;
   z(j)=v;
   t(j)=w;
elseif(x(j)>=c)&&(x(j)<=d)
    y(j)=u*(((d-x(j))/(d-c)));
    z(j)=(((x(j)-c)+v*(d-x(j)))/(d-c));
    t(j)=(((x(j)-c)+w*(d-x(j)))/(d-c));
elseif(x(j)>=d)
   y(j)=0;
z(j)=1;
t(j)=1;
end
end
plot(x,y,x,z,x,t)
legend('Membership function','indeterminate function','Non-membership function')
end
```

4.1 Example

The figure 1 portrayed the pictorical representation of the trapezoidal neutrosophic function $a=\langle(0.3,0.5,0.6,0.7);0.4,0.2,0.3\rangle$

The line command to show this function in Matlab is written below:

```
x=0:0.01:1;
[y,z,t]=trin(x,0.3,0.5,0.6,0.7, 0.4, 0.2,0.3)
```

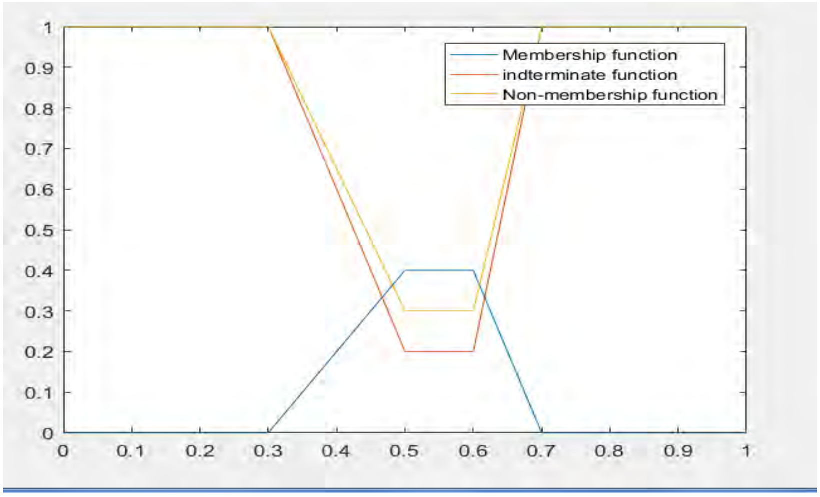

Figure 1: Trapezoidal neutrosophic function for example 4.1

4.2 Example

The figure 2 portrayed the trapezoidal neutrosophic function of $a = \langle (50,55,60,65);0.6,0.4,0.3 \rangle$

The line command to show this function in Matlab is written below:

```
>> x=45:70;
[y,z]=trin(x,50,55,60,65, 0.6, 0.4,0.3)
```

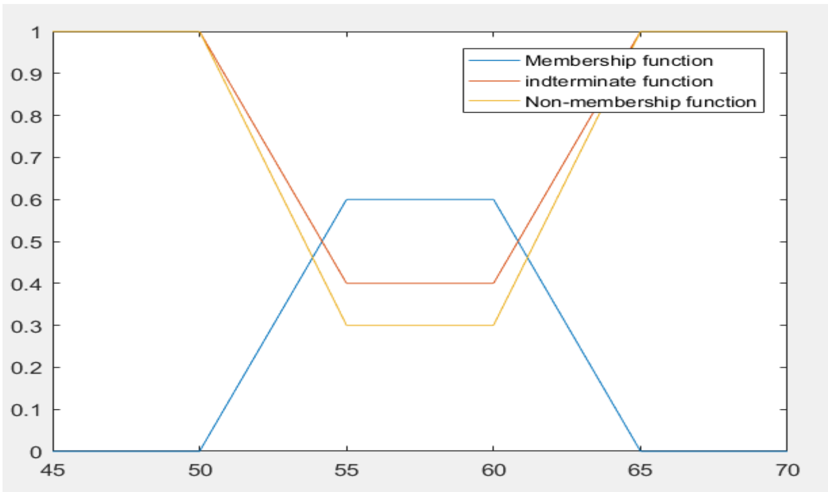

Figure 2: Trapezoidal neutrosophic function for example 4.2

4.3 Example

The figure 3 portrayed the triangular neutrosophic function of $a = \langle (0.3,0.5,0.5,0.7);0.4,0.2,0.3 \rangle$

The line command to show this function in Matlab is written below:

```
x= 0:0.01:1;
[y,z,t]=trin(x,0.3, 0.5,0.5,0.7, 0.4, 0.2,0.3)
```

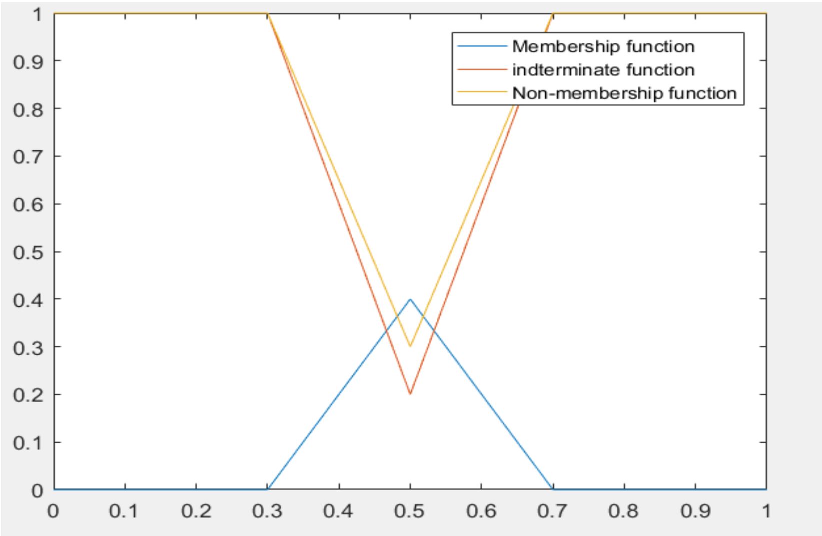

Figure 3: Triangular neutrosophic function for example 3

Remark: if b= c, the trapezoidal neutrosophic function degenerate to triangular neutrosophic function as protrayed in figure 3.

5. Qualitative analysis of different types of graphs

The following analysis helps to know the importance of the neutrosophic graph where the limitations are possible as mentioned in the table for fuzzy and intuitionistic fuzzy graphs.

Types of graphs	Advantages	Limitations
Graphs	• Models of relations • describing information involving relationship between objects • Objects are represented by vertices and relations by edges • Vertex and edge sets are crisp	• Unable to handle fuzzy relation (FR)
Fuzzy graphs (FGs)	• Symmetric binary fuzzy relation on a fuzzy subset • Uncertainty exist in the description of the objects or in the relationships or in both • Able to handle FR with membership value • FGs models are more useful and practical in nature	• Not able to deal interval data
Interval valued FGs	• Edge set of a graphs is a collection of intervals	• Unable to deal the case of non membership
Intuitionistic fuzzy graphs (IntFGs)	• Gives more certainty into the problems • Minimize the cost of operation and enhance efficiency • Contributes a adjustable model to define uncertainty and vagueness exists in decision making • Able to deal non membership of a relation	• Unable to handle interval data

Interval valued IntFGs	• Capable of dealing interval data	• Unable to deal indeterminacy

6. Conclusion

Choosing a MF is an essential task of all the fuzzy and neutrosophic system (Control system or decision making process). Due to the simplicity (less computational complexity) and flexibility triangular and trapezoidal membership functions are widely used in many real world applications. In this paper, trapezoidal neutrosophic membership function is derived using Matlab with illustrative example. In future, this work may be extended to interval valued trapezoidal and triangular neutrosophic membership functions.

Notes

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Application of Bipolar Neutrosophic Sets to Incidence Graphs

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Abstract: In this research paper, we apply the idea of bipolar neutrosophic sets to incidence graphs. We present some notions, including bipolar neutrosophic incidence graphs, bipolar neutrosophic incidence cycle and bipolar neutrosophic incidence tree. We define strong path, strength and incidence strength of strongest path in bipolar neutrosophic incidence graphs. We investigate some properties of bipolar neutrosophic incidence graphs. We also describe an application of bipolar neutrosophic incidence graphs.

Keywords: Bipolar neutrosophic incidence graphs; Bipolar neutrosophic incidence cycle; Bipolar neutrosophic incidence tree.

1 Introduction

Graph theory is a mathematical structure which is used to represent a relationship between objects. It has been very successful in engineering and natural sciences. Sometimes, in many cases, graph theoretical concepts may be imprecise. To handle such cases, in 1975, Rosenfeld [1] gave the idea of fuzzy graphs. He considered fuzzy relations and proposed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Bhutani and Rosenfeld [2] studied the strong edges in fuzzy graphs. By applying bipolar fuzzy sets [3] to graphs, Akram [4] introduced the notion of bipolar fuzzy graphs. He described the different methods to construct the bipolar fuzzy graphs and discussed the some of their properties. Broumi et al [5] introduced the single-valued neutrosophic graphs by applying the concept of single-valued neutrosophic sets to graphs. Later on, Akram and Sarwar [6] studied the novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. They developed the independent and dominating sets of bipolar

neutrosophic graphs. Ishfaq et al [13, 14] introduced the rough neutrosophic digraphs and their applications. Later Akram et al [15] introduced the decision making approach based on neutrsophic rough information.
Dinesh [7, 8] studied the graph structures and introduced the fuzzy incidence graphs. Fuzzy incidence graphs not only give the limitation of the relation between elements contained in a set, but also give the influence or impact of an element to its relation pair. Fuzzy incidence graphs play an important role to interconnect the networks. Incidence relations have significant parts in human and natural made networks, including pipe, road, power and interconnection networks. Later Mathew and Mordeson [9] introduced the connectivity concepts in fuzzy incidence graphs and also introduced fuzzy influence graphs [10]. In this paper, we apply the idea of bipolar neutrosophic sets to incidence graphs and introduce a new concept, namely bipolar neutrosophic incidence graphs.
Some of essential preliminaries from [7] and [11] are given below:
Let V^* be a non-empty set. Then $G^* = (V^*, E^*, I^*)$ is an incidence graph, where E^* is a subset of $V^* \times V^*$ and I^* is a subset of $V^* \times E^*$.
A fuzzy incidence graph on an incidence graph $G^* = (V^*, E^*, I^*)$ is an ordered triplet $G' = (\mu', \lambda', \psi')$, where μ' is a fuzzy set on V^*, λ' is a fuzzy relation on V^* and ψ' is a fuzzy set on $V^* \times E^*$ such that

$$\psi'(y, yz) \leq \mu'(y) \wedge \lambda'(yz), \ \ \forall y, z \in V^*.$$

A bipolar neutrosophic set on a non-empty set V^* is an object having the form

$$B = \{(b, T_Y^+(b), I_Y^+(b), F_Y^+(b), T_Y^-(b), I_Y^-(b), F_Y^-(b)) : b \in V^*\},$$

where, $T_b^+, I_b^+, F_b^+ : V^* \longrightarrow [0, 1]$ and $T_b^-, I_b^-, F_b^- : V^* \longrightarrow [-1, 0]$.
For other notations and applications, readers are referred to [15-21].

2 Bipolar Neutrosophic Incidence Graphs

Definition 2.1. A *bipolar neutrosophic incidence graphs* (BNIG) on an incidence graph $G^* = (V^*, E^*, I^*)$ is an ordered triplet $G = (X, Y, Z)$, where

(1) X is a bipolar neutrosophic set on V^*.

(2) Y is a bipolar neutrosophic relation on V^*.

(3) Z is a bipolar neutrosophic set on $V^* \times E^*$ such that

$$\begin{aligned}
T_Z^+(x, xy) &\leq \min\{T_X^+(x), T_Y^+(xy)\}, \quad & T_Z^-(x, xy) &\geq \max\{T_X^-(x), T_Y^-(xy)\},\\
I_Z^+(x, xy) &\leq \min\{I_X^+(x), I_Y^+(xy)\}, \quad & I_Z^-(x, xy) &\geq \max\{I_X^-(x), I_Y^-(xy)\},\\
F_Z^+(x, xy) &\geq \max\{F_X^+(x), F_Y^+(xy)\}, \quad & F_Z^-(x, xy) &\leq \min\{F_X^-(x), F_Y^-(xy)\}, \forall x, y \in V^*.
\end{aligned}$$

Example 2.2. Let $G^* = (V^*, E^*, I^*)$ be an incidence graph, as shown in Fig. 1, where $V^* = \{w, x, y, z\}$, $E^* = \{wx, xy, yz, zw\}$ and $I^* = \{(w, wx), (x, wx), (x, xy), (y, xy), (y, yz), (z, yz), (z, zw), (w, zw)\}$. Let X be a bipolar neutrosophic set on V^* given as

$$\begin{aligned}
X = \{&(w,\ 0.2,\ 0.4,\ 0.7, -0.1, -0.2, -0.4), (x,\ 0.3,\ 0.5,\ 0.9, -0.1, -0.6, -0.7),\\
&(y,\ 0.4,\ 0.6,\ 0.9, -0.1, -0.2, -0.8), (z,\ 0.5,\ 0.6,\ 0.8, -0.2, -0.8, -0.6)\}.
\end{aligned}$$

Let Y be a bipolar neutrosophic relation on V^* given as

$$Y = \{(wx,\ 0.1,\ 0.2,\ 0.8, -0.1, -0.2, -0.9), (xy,\ 0.2,\ 0.4,\ 0.7, -0.2, -0.3, -0.9), \\ (yz,\ 0.1,\ 0.2,\ 0.8, -0.1, -0.2, -0.9), (zw,\ 0.2,\ 0.3,\ 0.6, -0.1, -0.2, -0.7)\}.$$

Let Z be a bipolar neutrosophic set on $V^* \times E^*$ given as

$$Z = \{((w, wx), 0.1,\ 0.1,\ 0.8, -0.2, -0.2, -0.9), ((x, wx), 0.1,\ 0.2,\ 0.8, -0.2, -0.3,\ -0.9), \\ ((x,\ xy), 0.2,\ 0.3,\ 0.8, -0.2, -0.4, -0.9), ((y,\ xy), 0.1,\ 0.1,\ 0.8, -0.2, -0.2,\ -0.9), \\ ((y,\ yz), 0.1,\ 0.2,\ 0.7, -0.2, -0.3, -0.9), ((z,\ yz), 0.1,\ 0.2,\ 0.7, -0.2, -0.3,\ -0.7), \\ ((z, zw), 0.1,\ 0.1,\ 0.8, -0.2, -0.2, -0.9), ((w, zw), 0.2,\ 0.3,\ 0.5, -0.3, -0.3, -0.8)\}.$$

Then $G = (X, Y, Z)$ is a BNIG as shown in Fig. 2.

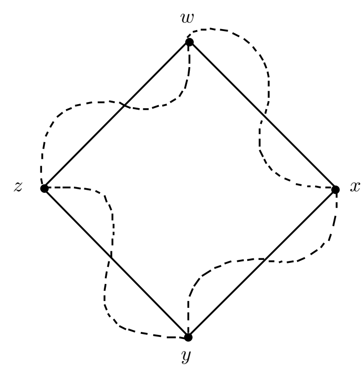

Figure 1: $G^* = (V^*, E^*, I^*)$

Definition 2.3. Let $G = (X, Y, Z)$ be a BNIG of G^*. Then *support* of $G = (X, Y, Z)$ is denoted by supp(G)=(supp(X),supp(Y),supp(Z)) such that

$$\text{supp}(X) = \{x \in X | T_X^+(x) > 0,\ I_X^+(x) > 0,\ F_X^+(x) > 0, \\ T_X^-(x) < 0,\ I_X^-(x) < 0,\ F_X^-(x) < 0\},$$

$$\text{supp}(Y) = \{xy \in Y | T_Y^+(xy) > 0,\ I_Y^+(xy) > 0,\ F_Y^+(xy) > 0, \\ T_Y^-(xy) < 0,\ I_Y^-(xy) < 0,\ F_Y^-(xy) < 0\},$$

$$\text{supp}(Z) = \{(x, xy) \in Z | T_Z^+(x, xy) > 0,\ I_Z^+(x, xy) > 0,\ F_Z^+(x, xy) > 0, \\ T_Z^-(x, xy) < 0,\ I_Z^-(x, xy) < 0,\ F_Z^-(x, xy) < 0\}.$$

Definition 2.4. A sequence
$x_0, (x_0, x_0x_1), x_0x_1, (x_1, x_0x_1), x_1, \cdot.., x_{n-1}, (x_{n-1}, x_{n-1}x_n), x_{n-1}x_n, (x_n, x_{n-1}x_n), x_n$
of vertices, edges and pairs in BNIG G is called *walk*.

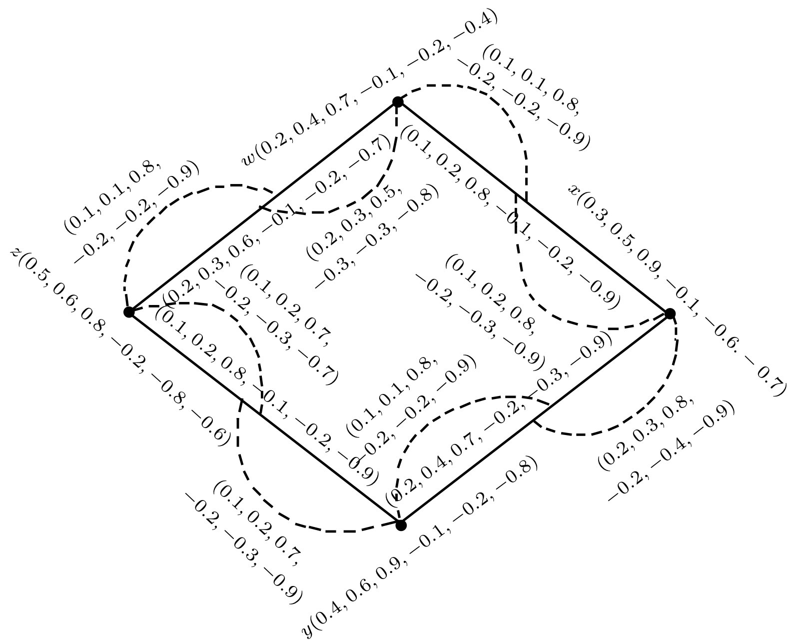

Figure 2: BNIG $G = (X, Y, Z)$

If $x_0 = x_n$, it is a *close walk*.
If edges are distinct, it is a *trail*.
If pairs are distinct, it is an *incidence trail*.
If vertices are distinct, it is a *path*.
If pairs are distinct, it is an *incidence path*.

Example 2.5. In a BNIG $G = (X, Y, Z)$ as shown in Fig.2,
$w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz, (z, yz), z, (z, zw), zw, (w, zw), w, (w, wx), wx, (x, wx), x$ ia a walk. It is not a path, trail and an incidence trail.

$$w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz, (z, yz), z$$

is a path, trail and an incidence trail.

Definition 2.6. The BNIG $G = (X, Y, Z)$ is a *cycle* if and only if supp(G)=(supp(X),supp(Y),supp(Z)) is a cycle.

Example 2.7. In a BNIG $G = (X, Y, Z)$ as shown in Fig.2, consider a walk

$$w, (w, wx), wx, (x, wx), x, (x, xy), xy, (y, xy), y, (y, yz), yz, (z, yz), z, (z, zw), zw, (w, zw), w.$$

which is a cycle. So $G = (X, Y, Z)$ is a cycle.

Definition 2.8. The BNIG $G = (X, Y, Z)$ is a *bipolar neutrosophic cycle* if and only if

$$\text{supp}(G) = (\text{supp}(X), \text{supp}(Y), \text{supp}(Z))$$

is a cycle and there exist at least two $xy \in$ supp(Y) such that

$$\begin{aligned}
T_Y^+(xy) &= \min\{T_Y^+(uv)\ |uv \in \text{supp}(Y)\},\\
I_Y^+(xy) &= \min\{I_Y^+(uv)\ \ |uv \in \text{supp}(Y)\},\\
F_Y^+(xy) &= \max\{F_Y^+(uv)|uv \in \text{supp}(Y)\},\\
T_Y^-(xy) &= \max\{T_Y^-(uv)|uv \in \text{supp}(Y)\},\\
I_Y^-(xy) &= \max\{I_Y^-(uv)\ |uv \in \text{supp}(Y)\},\\
F_Y^-(xy) &= \min\{F_Y^-(uv)\ |uv \in \text{supp}(Y)\}.
\end{aligned}$$

Example 2.9. In a BNIG $G = (X, Y, Z)$ as shown in Fig.2,
we have

$$\begin{aligned}
T_Y^+(wx) &= \ \ \ 0.1 = \min\{\ T_Y^+(wx),\ T_Y^+(xy),\ T_Y^+(yz),\ T_Y^+(zw)\},\\
I_Y^+(wx) &= \ \ \ 0.2 = \min\{\ I_Y^+(wx),\ I_Y^+(xy),\ \ \ I_Y^+(yz),\ I_Y^+(zw)\},\\
F_Y^+(wx) &= \ \ \ 0.8 = \max\{F_Y^+(wx),\ F_Y^+(xy),\ F_Y^+(yz),\ F_Y^+(zw)\},\\
T_Y^-(wx) &= -0.1 = \max\{T_Y^-(wx),\ T_Y^-(xy),\ T_Y^-(yz),\ T_Y^-(zw)\},\\
I_Y^-(wx) &= -0.2 = \max\{\ I_Y^-(wx),\ I_Y^-(xy),\ \ \ I_Y^-(yz),\ I_Y^-(zw)\},\\
F_Y^-(wx) &= -0.9 = \min\{\ F_Y^-(wx),\ F_Y^-(xy),\ F_Y^-(yz),\ F_Y^-(zw)\}.
\end{aligned}$$

Also

$$\begin{aligned}
T_Y^+(yz) &= \ \ \ 0.1 = \min\{\ T_Y^+(wx),\ T_Y^+(xy),\ T_Y^+(yz),\ T_Y^+(zw)\},\\
I_Y^+(yz) &= \ \ \ 0.2 = \min\{\ I_Y^+(wx),\ I_Y^+(xy),\ \ \ I_Y^+(yz),\ I_Y^+(zw)\},\\
F_Y^+(yz) &= \ \ \ 0.8 = \max\{F_Y^+(wx),\ F_Y^+(xy),\ F_Y^+(yz),\ F_Y^+(zw)\},\\
T_Y^-(yz) &= -0.1 = \max\{T_Y^-(wx),\ T_Y^-(xy),\ T_Y^-(yz),\ T_Y^-(zw)\},\\
I_Y^-(yz) &= -0.2 = \max\{\ I_Y^-(wx),\ I_Y^-(xy),\ \ \ I_Y^-(yz),\ I_Y^-(zw)\},\\
F_Y^-(yz) &= -0.9 = \min\{\ F_Y^-(wx),\ F_Y^-(xy),\ F_Y^-(yz),\ F_Y^-(zw)\}.
\end{aligned}$$

So $G = (X, Y, Z)$ is a bipolar neutrosophic cycle.

Definition 2.10. The BNIG $G = (X, Y, Z)$ is a *bipolar neutrosophic incidence cycle* if and only if it is a bipolar neutrosophic cycle and there exist at least two $(x, xy) \in$ supp(Z) such that

$$\begin{aligned}
T_Z^+(x, xy) &= \min\{T_Z^+(u, vw)\ |(u, vw) \in \text{supp}(Z)\},\\
I_Z^+(x, xy) &= \min\{I_Z^+(u, vw)\ \ |(u, vw) \in \text{supp}(Z)\},\\
F_Z^+(x, xy) &= \max\{F_Z^+(u, vw)|(u, vw) \in \text{supp}(Z)\},\\
T_Z^-(x, xy) &= \max\{T_Z^-(u, vw)|(u, vw) \in \text{supp}(Z)\},\\
I_Z^-(x, xy) &= \max\{I_Z^-(u, vw)\ |(u, vw) \in \text{supp}(Z)\},\\
F_Z^-(x, xy) &= \min\{F_Z^-(u, vw)\ |(u, vw) \in \text{supp}(Z)\}.
\end{aligned}$$

Example 2.11. In a BNIG $G = (X, Y, Z)$ as shown in Fig.2, we have

$$\begin{aligned}
T_Z^+(w,wx) &= \quad 0.1 = \min\{T_Z^+(u,vw)\ |(u,vw) \in \text{supp}(Z)\},\\
I_Z^+(w,wx) &= \quad 0.1 = \min\{I_Z^+(u,vw)\ \ |(u,vw) \in \text{supp}(Z)\},\\
F_Z^+(w,wx) &= \quad 0.8 = \max\{F_Z^+(u,vw)|(u,vw) \in \text{supp}(Z)\},\\
T_Z^-(w,wx) &= -0.2 = \max\{T_Z^-(u,vw)|(u,vw) \in \text{supp}(Z)\},\\
I_Z^-(w,wx) &= -0.2 = \max\{I_Z^-(u,vw)\ |(u,vw) \in \text{supp}(Z)\},\\
F_Z^-(w,wx) &= -0.9 = \min\{F_Z^-(u,vw)\ |(u,vw) \in \text{supp}(Z)\}.
\end{aligned}$$

and

$$\begin{aligned}
T_Z^+(y,xy) &= \quad 0.1 = \min\{T_Z^+(u,vw)\ |(u,vw) \in \text{supp}(Z)\},\\
I_Z^+(y,xy) &= \quad 0.1 = \min\{I_Z^+(u,vw)\ \ |(u,vw) \in \text{supp}(Z)\},\\
F_Z^+(y,xy) &= \quad 0.8 = \max\{F_Z^+(u,vw)|(u,vw) \in \text{supp}(Z)\},\\
T_Z^-(y,xy) &= -0.2 = \max\{T_Z^-(u,vw)|(u,vw) \in \text{supp}(Z)\},\\
I_Z^-(y,xy) &= -0.2 = \max\{I_Z^-(u,vw)\ |(u,vw) \in \text{supp}(Z)\},\\
F_Z^-(y,xy) &= -0.9 = \min\{F_Z^-(u,vw)\ |(u,vw) \in \text{supp}(Z)\}.
\end{aligned}$$

So $G = (X, Y, Z)$ is a bipolar neutrosophic incidence cycle.

Definition 2.12. If $G = (X, Y, Z)$ is a BNIG, then $H = (X^*, Y^*, Z^*)$ is a *bipolar neutrosophic incidence subgraph* of G if

$$X^* \subseteq X,\ Y^* \subseteq Y,\ Z^* \subseteq Z.$$

$H = (X^*, Y^*, Z^*)$ is a *spanning subgraph* if $X = X^*$.

Definition 2.13. *Strength of the strongest path* from x to y in BNIG $G = (X, Y, Z)$ is defined as

$$\begin{aligned}
T^+_{\rho^\infty}(x,y) &= \bigvee_{i=1}^{k} T^+_{\rho_i}(x,y),\quad I^+_{\rho^\infty}(x,y) = \bigvee_{i=1}^{k} I^+_{\rho_i}(x,y),\quad F^+_{\rho^\infty}(x,y) = \bigwedge_{i=1}^{k} F^+_{\rho_i}(x,y),\\
T^-_{\rho^\infty}(x,y) &= \bigwedge_{i=1}^{k} T^-_{\rho_i}(x,y),\quad I^-_{\rho^\infty}(x,y) = \bigwedge_{i=1}^{k} I^-_{\rho_i}(x,y),\quad F^-_{\rho^\infty}(x,y) = \bigvee_{i=1}^{k} F^-_{\rho_i}(x,y).
\end{aligned}$$

where $\rho(x, y)$ is the strength of path from x to y such that

$$\begin{aligned}
T_\rho^+(x,y) &= \wedge\{T_Y^+(xy)|xy \in \text{supp}(Y)\},\\
I_\rho^+(x,y) &= \wedge\{I_Y^+(xy)\ |xy \in \text{supp}(Y)\},\\
F_\rho^+(x,y) &= \vee\{F_Y^+(xy)|xy \in \text{supp}(Y)\},\\
T_\rho^-(x,y) &= \vee\{T_Y^-(xy)|xy \in \text{supp}(Y)\},\\
I_\rho^-(x,y) &= \vee\{I_Y^-(xy)\ |xy \in \text{supp}(Y)\},\\
F_\rho^-(x,y) &= \wedge\{F_Y^-(xy)|xy \in \text{supp}(Y)\}.
\end{aligned}$$

Definition 2.14. *Incidence strength of the strongest path* from x to wy in BNIG $G = (X, Y, Z)$ is defined as

$$T^+_{\psi^\infty}(x, wy) = \bigvee_{i=1}^{k} T^+_{\psi_i}(x, wy), \quad T^-_{\psi^\infty}(x, wy) = \bigwedge_{i=1}^{k} T^-_{\psi_i}(x, wy),$$
$$I^+_{\psi^\infty}(x, wy) = \bigvee_{i=1}^{k} I^+_{\psi_i}(x, wy), \quad I^-_{\psi^\infty}(x, wy) = \bigwedge_{i=1}^{k} I^-_{\psi_i}(x, wy),$$
$$F^+_{\psi^\infty}(x, wy) = \bigwedge_{i=1}^{k} F^+_{\psi_i}(x, wy), \quad F^-_{\psi^\infty}(x, wy) = \bigvee_{i=1}^{k} F^-_{\psi_i}(x, wy).$$

where $\psi(x, wy)$ is the incidence strength of path from x to wy such that

$$\begin{aligned}
T^+_{\psi}(x, wy) &= \wedge \{T^+_Z(x, wy) | (x, wy) \in \text{supp}(Z)\},\\
I^+_{\psi}(x, wy) &= \wedge \{I^+_Z(x, wy) | (x, wy) \in \text{supp}(Z)\},\\
F^+_{\psi}(x, wy) &= \vee \{F^+_Z(x, wy) | (x, wy) \in \text{supp}(Z)\},\\
T^-_{\psi}(x, wy) &= \vee \{T^-_Z(x, wy) | (x, wy) \in \text{supp}(Z)\},\\
I^-_{\psi}(x, wy) &= \vee \{I^-_Z(x, wy) | (x, wy) \in \text{supp}(Z)\},\\
F^-_{\psi}(x, wy) &= \wedge \{F^-_Z(x, wy) | (x, wy) \in \text{supp}\}(Z).
\end{aligned}$$

Example 2.15. In a BNIG $G = (X, Y, Z)$ as shown in Fig.3
the strength of path $w, (w, wy), wy, (y, wy), y, (y, yz), yz, (z, yz), z$ is

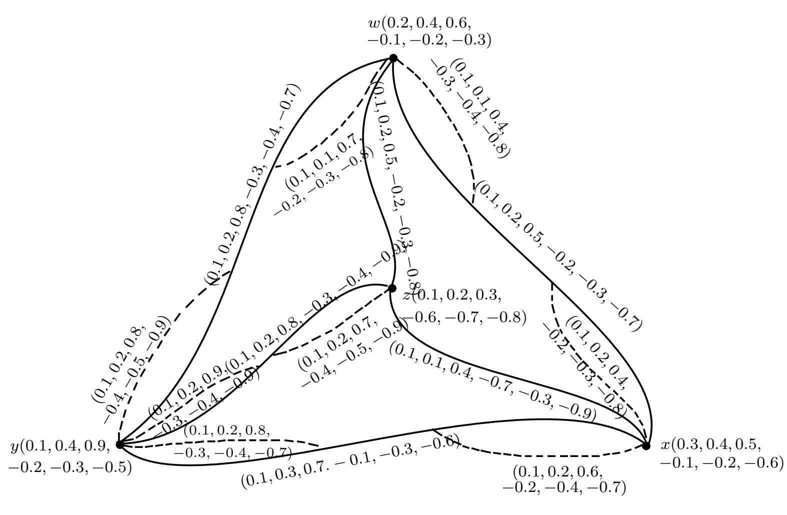

Figure 3: BNIG $G = (X, Y, Z)$

$$(0.1, 0.1, 0.8, -0.3, -0.4, -0.9),$$

the strength of path $w,(w,wx),wx,(x,wx),x,(x,xy),xy,(y,xy),y,(y,yz),yz,(z,yz),z$ is

$$(0.1,0.2,0.8,-0.1,-0.3,-0.9),$$

the strength of the strongest path from w to z is

$$(0.1,0.2,0.8,-0.3,-0.4,-0.9).$$

In a BNIG $G=(X,Y,Z)$ as shown in Fig.3
the incidence strength of the path $w,(w,wy),wy,(y,wy),y,(y,yz),yz$ is

$$(0.1,0.1,0.9,-0.2,-0.3,-0.9),$$

the incidence strength of the path $w,(w,wx),wx,(x,wx),x,(x,xy),xy,(y,xy),y,(y,yz),yz$ is

$$(0.1,0.1,0.8,-0.2,-0.3,-0.9),$$

the incidence strength of strongest path from w to yz is

$$(0.1,0.1,0.8,-0.2,-0.3,-0.9).$$

Definition 2.16. BNIG $G=(X,Y,Z)$ is called a *tree* if and only if supp(G)=(supp(X),supp(Y),supp(Z)) is a tree.

Definition 2.17. $G=(X,Y,Z)$ is a *bipolar single-valued neutrosophic tree* if and only if bipolar neutrosophic incidence spanning subgraph $H=(X,Y^*,Z^*)$ of $G=(X,Y,Z)$ is a tree such that

$$T^+_Y(xy)<T^+_{\phi^\infty}(x,y),\ \ I^+_Y(xy)< I^+_{\phi^\infty}(x,y),\ \ F^+_Y(xy)>F^+_{\phi^\infty}(x,y),$$
$$T^-_Y(xy)>T^-_{\phi^\infty}(x,y),\ \ I^-_Y(xy)> I^-_{\phi^\infty}(x,y),\ \ F^-_Y(xy)<F^-_{\phi^\infty}(x,y),\ \ \forall xy\in \text{supp}(Y)\backslash\text{supp}(Y^*).$$

where $\phi^\infty(x,y)$ is the strength of strongest path from x to y in $H=(X,Y^*,Z^*)$.

Definition 2.18. $G=(X,Y,Z)$ is a *bipolar neutrosophic incidence tree* if and only if bipolar neutrosophic incidence spanning subgraph $H=(X,Y^*,Z^*)$ of $G=(X,Y,Z)$ is a tree such that

$$T^+_Z(x,xy)<T^+_{\tau^\infty}(x,xy),\ \ I^+_Z(x,xy)< I^+_{\tau^\infty}(x,xy),\ \ F^+_Z(x,xy)>F^+_{\tau^\infty}(x,xy),$$
$$T^-_Z(x,xy)>T^-_{\tau^\infty}(x,xy),\ \ I^-_Z(x,xy)> I^-_{\tau^\infty}(x,xy),\ \ F^-_Z(x,xy)<F^-_{\tau^\infty}(x,xy),\ \ \forall (x,xy)\in \text{supp}(Z)\backslash\text{supp}(Z^*).$$

where $\tau^\infty(x,xy)$ is the strength of strongest path from x to xy in $H=(X,Y^*,Z^*)$.

Example 2.19. $G=(X,Y,Z)$ is a bipolar neutrosophic tree as shown in Fig.4 because a bipolar neutrosophic incidence spanning subgraph $H = (X, Y^*, Z^*)$ of $G = (X, Y, Z)$ as shown in Fig.5 is a tree and

$$\begin{aligned}
T_Y^+(wx) &= \quad 0.1 < \ \ 0.2 = T_{\phi^\infty}^+(w, x),\\
I_Y^+(wx) &= \quad 0.1 < \ \ 0.2 = I_{\phi^\infty}^+(w, x),\\
F_Y^+(wx) &= \quad 0.9 > \ \ 0.7 = F_{\phi^\infty}^+(w, x),\\
T_Y^-(wx) &= -0.1 > -0.2 = T_{\phi^\infty}^-(w, x),\\
I_Y^-(wx) &= -0.2 > -0.3 = I_{\phi^\infty}^-(w, x),\\
F_Y^-(wx) &= -0.9 < -0.8 = F_{\phi^\infty}^-(w, x).
\end{aligned}$$

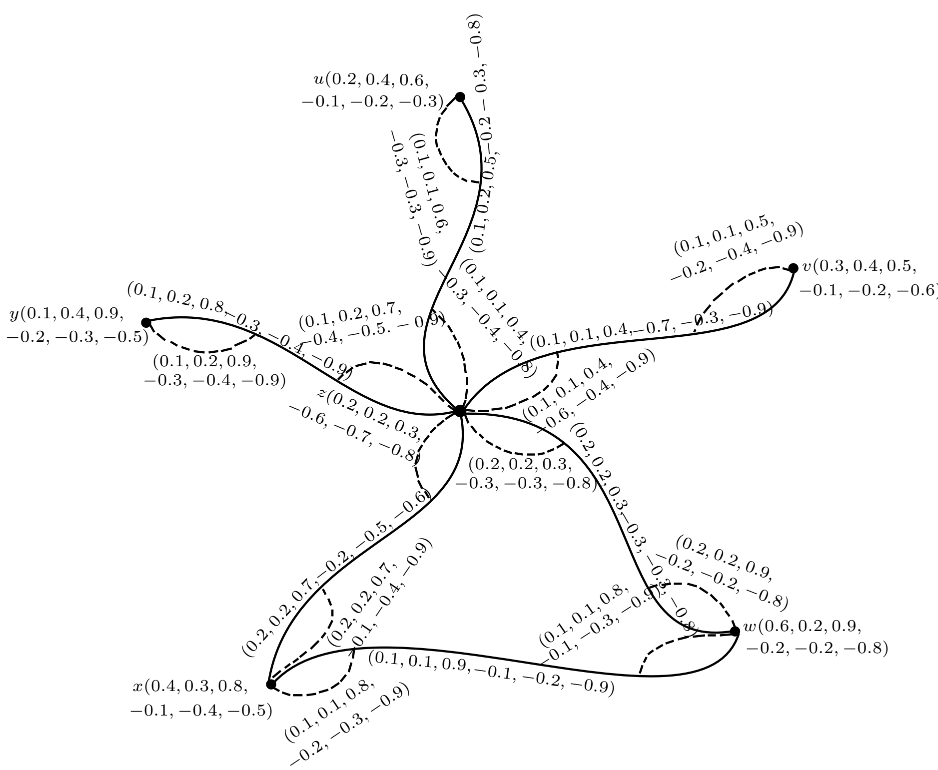

Figure 4: BNIG $G = (X, Y, Z)$

Theorem 2.20. *Let $G = (X, Y, Z)$ be a cycle. Then $G = (X, Y, Z)$ is a bipolar neutrosophic cycle if and only if $G = (X, Y, Z)$ is not a bipolar neutrosophic tree.*

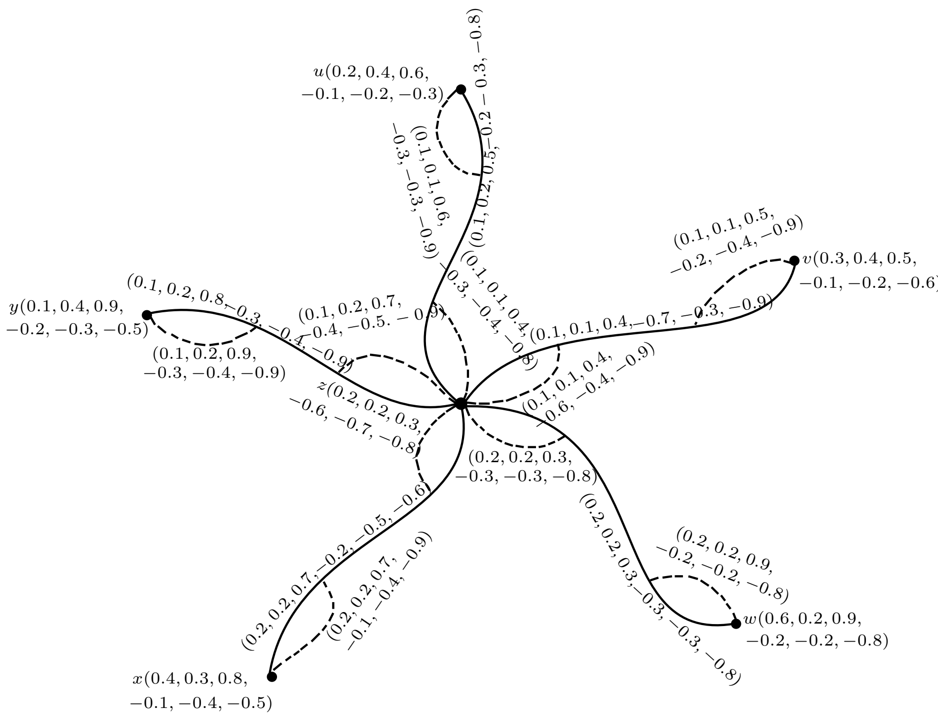

Figure 5: $H=(X,Y^*,Z^*)$

Proof. Let $G=(X,Y,Z)$ be a bipolar neutrosophic cycle. So there exists $uv, xy \in \text{supp}(Y)$ such that

$$
\begin{aligned}
T_Y^+(uv) &= T_Y^+(xy) = \wedge \{T_Y^+(yz) | yz \in \text{supp}(Y)\},\\
I_Y^+(uv) &= I_Y^+(xy) = \wedge \{I_Y^+(yz) | yz \in \text{supp}(Y)\},\\
F_Y^+(uv) &= F_Y^+(xy) = \vee \{F_Y^+(yz) | yz \in \text{supp}(Y)\},\\
T_Y^-(uv) &= T_Y^-(xy) = \vee \{T_Y^-(yz) | yz \in \text{supp}(Y)\},\\
I_Y^-(uv) &= I_Y^-(xy) = \vee \{I_Y^-(yz) | yz \in \text{supp}(Y)\},\\
F_Y^-(uv) &= F_Y^-(xy) = \wedge \{F_Y^-(yz) | yz \in \text{supp}(Y)\}.
\end{aligned}
$$

If $H=(X,Y^*,Z^*)$ is a spanning bipolar neutrosophic incidence tree of $G=(X,Y,Z)$, then $\text{supp}(Y)\backslash\text{supp}(Y^*)=\{yz\}$ for some $y,z \in V$ because $G=(X,Y,Z)$ is a cycle.
Hence there exists no path between y and z in $H=(X,Y^*,Z^*)$ such that

$$
\begin{aligned}
&T_Y^+(yz) < T_{\phi\infty}^+(y,z), \quad I_Y^+(yz) < I_{\phi\infty}^+(y,z), \quad F_Y^+(yz) > F_{\phi\infty}^+(y,z),\\
&T_Y^-(yz) > T_{\phi\infty}^-(y,z), \quad I_Y^-(yz) > I_{\phi\infty}^-(y,z), \quad F_Y^-(yz) < F_{\phi\infty}^-(y,z).
\end{aligned}
$$

Thus, $G=(X,Y,Z)$ is not a bipolar neutrosophic tree.
Conversely, let $G=(X,Y,Z)$ be not a bipolar neutrosophic tree. Because $G=(X,Y,Z)$ is a cycle, so for all

$yz \in$ supp(Y), $H = (X, Y^*, Z^*)$ is spanning bipolar neutrosophic incidence tree in $G = (X, Y, Z)$ such that

$$T_Y^+(yz) \geq T_{\phi^\infty}^+(y, z), \quad I_Y^+(yz) \geq I_{\phi^\infty}^+(y, z), \quad F_Y^+(yz) \leq F_{\phi^\infty}^+(y, z),$$
$$T_Y^-(yz) \leq T_{\phi^\infty}^-(y, z), \quad I_Y^-(yz) \leq I_{\phi^\infty}^-(y, z), \quad F_Y^-(yz) \geq F_{\phi^\infty}^-(y, z).$$

where

$$T_{Y^*}^+(yz) = 0, \ I_{Y^*}^+(yz) = 0, \ F_{Y^*}^+(yz) = 0,$$
$$T_{Y^*}^-(yz) = 0, \ I_{Y^*}^-(yz) = 0, \ F_{Y^*}^-(yz) = 0.$$

and

$$T_{Y^*}^+(uv) = T_Y^+(uv), \ I_{Y^*}^+(uv) = I_Y^+(uv), \ F_{Y^*}^+(uv) = F_Y^+(uv),$$
$$T_{Y^*}^-(uv) = T_Y^-(uv), \ I_{Y^*}^-(uv) = I_Y^-(uv), \ F_{Y^*}^-(uv) = F_Y^-(uv), \forall \ uv \in \text{supp}(Y) \setminus \{yz\}.$$

Hence, there exists more than one edge such that

$$T_Y^+(yz) = \wedge \{T_Y^+(xy) | xy \in \text{supp}(Y)\},$$
$$I_Y^+(yz) = \wedge \{ I_Y^+(xy) | xy \in \text{supp}(Y)\},$$
$$F_Y^+(yz) = \vee \{F_Y^+(xy) | xy \in \text{supp}(Y)\},$$
$$T_Y^-(yz) = \vee \{T_Y^-(xy) | xy \in \text{supp}(Y)\},$$
$$I_Y^-(yz) = \vee \{ I_Y^-(xy) | xy \in \text{supp}(Y)\},$$
$$F_Y^-(yz) = \wedge \{F_Y^-(xy) | xy \in \text{supp}(Y)\}.$$

Thus, $G = (X, Y, Z)$ is a bipolar neutrosophic cycle. □

Theorem 2.21. *If $G = (X, Y, Z)$ is a bipolar neutrosophic tree and supp$(G) = ($supp(X), supp(Y), supp$(Z))$ is not a tree, then there exists at least one edge $xy \in$ supp(Y) such that*

$$T_Y^+(xy) < T_{\mu^\infty}^+(x, y), \quad I_Y^+(xy) < I_{\mu^\infty}^+(x, y), \quad F_Y^+(xy) > F_{\mu^\infty}^+(x, y),$$
$$T_Y^-(xy) > T_{\mu^\infty}^-(x, y), \quad I_Y^-(xy) > I_{\mu^\infty}^-(x, y), \quad F_Y^-(xy) < F_{\mu^\infty}^-(x, y).$$

where $\mu^\infty(x, y)$ is the strength of strongest path between u and v in $G = (X, Y, Z)$.

Proof. Let $G = (X, Y, Z)$ be a bipolar neutrosophic tree, then there exists a bipolar neutrosophic spanning subgraph $H = (X, Y^*, Z^*)$ that is tree and

$$T_Y^+(xy) < T_{\rho^\infty}^+(x, y), \quad I_Y^+(xy) < I_{\rho^\infty}^+(x, y), \quad F_Y^+(xy) > F_{\rho^\infty}^+(x, y),$$
$$T_Y^-(xy) > T_{\rho^\infty}^-(x, y), \quad I_Y^-(xy) > I_{\rho^\infty}^-(x, y), \quad F_Y^-(xy) < F_{\rho^\infty}^-(x, y), \forall uv \in \text{supp}(Y) \setminus \text{supp}(Y^*).$$

Also

$$T_{\rho^\infty}^+(x, y) \leq T_{\mu^\infty}^+(x, y), \ I_{\rho^\infty}^+(x, y) \leq I_{\mu^\infty}^+(x, y), \ F_{\rho^\infty}^+(x, y) \geq F_{\mu^\infty}^+(x, y),$$
$$T_{\rho^\infty}^-(x, y) \geq T_{\mu^\infty}^-(x, y), \ I_{\rho^\infty}^-(x, y) \geq I_{\mu^\infty}^-(x, y), \ F_{\rho^\infty}^-(x, y) \leq F_{\mu^\infty}^-(x, y).$$

Thus,

$$T_Y^+(xy)<T_{\mu^\infty}^+(x,y),\ \ I_Y^+(xy)<I_{\mu^\infty}^+(x,y),\ \ F_Y^+(xy)>F_{\mu^\infty}^+(x,y),$$
$$T_Y^-(xy)>T_{\mu^\infty}^-(x,y),\ \ I_Y^-(xy)>I_{\mu^\infty}^-(x,y),\ \ F_Y^-(xy)<F_{\mu^\infty}^-(x,y), \forall uv \in \text{supp}(Y)\backslash\text{supp}(Y^*)$$

and by hypothesis there exists at least one edge $xy \in$ supp(Y). □

Theorem 2.22. *Let $G=(X,Y,Z)$ be a cycle. Then $G=(X,Y,Z)$ is a bipolar neutrosophic incidence cycle if and only if $G=(X,Y,Z)$ is not a bipolar neutrosophic incidence tree.*

Proof. Let $G=(X,Y,Z)$ be a bipolar neutrosophic incidence cycle. Then there exist at least two $(x,wy)\in$ supp(Z) such that

$$T_Z^+(x,yz)=\min\{T_Z^+(u,vw)\ |(u,vw)\in \text{supp}(Z)\},$$
$$I_Z^+(x,yz)=\min\{I_Z^+(u,vw)\ |(u,vw)\in \text{supp}(Z)\},$$
$$F_Z^+(x,yz)=\max\{F_Z^+(u,vw)|(u,vw)\in \text{supp}(Z)\},$$
$$T_Z^-(x,yz)=\max\{T_Z^-(u,vw)|(u,vw)\in \text{supp}(Z)\},$$
$$I_Z^-(x,yz)=\max\{I_Z^-(u,vw)\ |(u,vw)\in \text{supp}(Z)\},$$
$$F_Z^-(x,yz)=\min\{F_Z^-(u,vw)\ |(u,vw)\in \text{supp}(Z)\}.$$

If $H=(X,Y^*,Z^*)$ is a spanning bipolar neutrosophic incidence tree of $G=(X,Y,Z)$, then supp$(Z)\backslash$supp$(Z^*)=\{(x,yz)\}$ for some $x\in V$ $yz\in$ supp(Y).
Hence there exists no path between x and yz in $H=(X,Y^*,Z^*)$ such that

$$T_Z^+(x,yz)<T_{\tau^\infty}^+(x,yz),\ \ I_Z^+(x,yz)<I_{\tau^\infty}^+(x,yz),\ \ F_Z^+(x,yz)>F_{\tau^\infty}^+(x,yz),$$
$$T_Z^-(x,yz)>T_{\tau^\infty}^-(x,yz),\ \ I_Z^-(x,yz)>I_{\tau^\infty}^-(x,yz),\ \ F_Z^-(x,yz)<F_{\tau^\infty}^-(x,yz).$$

Thus, $G=(X,Y,Z)$ is not a bipolar neutrosophic incidence tree.
Conversely, let $G=(X,Y,Z)$ be not a bipolar neutrosophic incidence tree. Then for all $(x,yz)\in$ supp(Z), $H=(X,Y^*,Z^*)$ is spanning bipolar neutrosophic incidence tree in $G=(X,Y,Z)$ such that

$$T_Z^+(x,yz)\geq T_{\tau^\infty}^+(x,yz),\ \ I_Z^+(x,yz)\geq I_{\tau^\infty}^+(x,yz),\ \ F_Z^+(x,yz)\leq F_{\tau^\infty}^+(x,yz),$$
$$T_Z^-(x,yz)\leq T_{\tau^\infty}^-(x,yz),\ \ I_Z^-(x,yz)\leq I_{\tau^\infty}^-(x,yz),\ \ F_Z^-(x,yz)\geq F_{\tau^\infty}^-(x,yz).$$

where

$$T_{Z^*}^+(x,yz)=0,\ I_{Z^*}^+(x,yz)=0,\ F_{Z^*}^+(x,yz)=0,$$
$$T_{Z^*}^-(x,yz)=0,\ I_{Z^*}^-(x,yz)=0,\ F_{Z^*}^-(x,yz)=0.$$

and

$$T_{Z^*}^+(u,vw)=T_Z^+(u,vw),\ \ I_{z^*}^+(u,vw)=I_Z^+(u,vw),\ F_{Z^*}^+(u,vw)=F_Z^+(u,vw),$$
$$T_{Z^*}^-(u,vw)=T_Z^-(u,vw),\ I_{Z^*}^-(u,vw)=I_Z^-(u,vw),\ F_{Z^*}^-(u,vw)=F_Z^-(u,vw), \forall\ (u,vw)\in \text{supp}(Z)\backslash\{(x,yz)\}.$$

Hence, there exists more than one pair such that

$$\begin{aligned}
T_Z^+(u,vw) &= \wedge\{T_Z^+(x,yz)|(x,yz)\in \text{supp}(Z)\},\\
I_Z^+(u,vw) &= \wedge\{I_Z^+(x,yz)|(x,yz)\in \text{supp}(Z)\},\\
F_Z^+(u,vw) &= \vee\{F_Z^+(x,yz)|(x,yz)\in \text{supp}(Z)\},\\
T_Z^-(u,vw) &= \vee\{T_Z^-(x,yz)|(x,yz)\in \text{supp}(Z)\},\\
I_Z^-(u,vw) &= \vee\{I_Z^-(x,yz)|(x,yz)\in \text{supp}(Z)\},\\
F_Z^-(u,vw) &= \wedge\{F_Z^-(x,yz)|(x,yz)\in \text{supp}(Z)\}.
\end{aligned}$$

Thus, $G=(X,Y,Z)$ is a bipolar neutrosophic incidence cycle. □

Definition 2.23. Let $G=(X,Y,Z)$ be a BNIG. An edge xy is called a *strong edge* if

$$\begin{aligned}
T_Y^+(xy) &\geq T_{\epsilon^\infty}^+(x,y), & T_Y^-(xy) &\leq T_{\epsilon^\infty}^-(x,y),\\
I_Y^+(xy) &\geq I_{\epsilon^\infty}^+(x,y), & I_Y^-(xy) &\leq I_{\epsilon^\infty}^-(x,y),\\
F_Y^+(xy) &\leq F_{\epsilon^\infty}^+(x,y), & F_Y^-(xy) &\geq F_{\epsilon^\infty}^-(x,y).
\end{aligned}$$

An edge xy is called α-*strong* if

$$\begin{aligned}
T_Y^+(xy) &> T_{\epsilon^\infty}^+(x,y), & T_Y^-(xy) &< T_{\epsilon^\infty}^-(x,y),\\
I_Y^+(xy) &> I_{\epsilon^\infty}^+(x,y), & I_Y^-(xy) &< I_{\epsilon^\infty}^-(x,y),\\
F_Y^+(xy) &< F_{\epsilon^\infty}^+(x,y), & F_Y^-(xy) &> F_{\epsilon^\infty}^-(x,y).
\end{aligned}$$

An edge xy is called β-*strong* if

$$\begin{aligned}
T_Y^+(xy) &= T_{\epsilon^\infty}^+(x,y), & T_Y^-(xy) &= T_{\epsilon^\infty}^-(x,y),\\
I_Y^+(xy) &= I_{\epsilon^\infty}^+(x,y), & I_Y^-(xy) &= I_{\epsilon^\infty}^-(x,y),\\
F_Y^+(xy) &= F_{\epsilon^\infty}^+(x,y), & F_Y^-(xy) &= F_{\epsilon^\infty}^-(x,y).
\end{aligned}$$

where $\epsilon^\infty(x,y)$ is the strength of strongest path between x and y.

Definition 2.24. Let $G=(X,Y,Z)$ be a BNIG. An edge xy is called a δ-*edge* if

$$\begin{aligned}
T_Y^+(xy) &< T_{\epsilon^\infty}^+(x,y), & T_Y^-(x,y) &> T_{\epsilon^\infty}^-(x,y),\\
I_Y^+(xy) &< I_{\epsilon^\infty}^+(x,y), & I_Y^-(x,y) &> I_{\epsilon^\infty}^-(x,y),\\
F_Y^+(xy) &> F_{\epsilon^\infty}^+(x,y), & F_Y^-(x,y) &< F_{\epsilon^\infty}^-(x,y).
\end{aligned}$$

Definition 2.25. Let $G=(X,Y,Z)$ be a BNIG. A pair (w,xy) is called a *strong pair* if

$$\begin{aligned}
T_Z^+(w,xy) &\geq T_{\eta^\infty}^+(w,xy), & T_Z^-(w,xy) &\leq T_{\eta^\infty}^-(w,xy),\\
I_Z^+(w,xy) &\geq I_{\eta^\infty}^+(w,xy), & I_Z^-(w,xy) &\leq I_{\eta^\infty}^-(w,xy),\\
F_Z^+(w,xy) &\leq F_{\eta^\infty}^+(w,xy), & F_Z^-(w,xy) &\geq F_{\eta^\infty}^-(w,xy).
\end{aligned}$$

A pair (w, xy) is called α-*strong* if

$$\begin{aligned}
&T_Z^+(w,xy) > T_{\eta^\infty}^+(w,xy),\ \ T_Z^-(w,xy) < T_{\eta^\infty}^-(w,xy),\\
&I_Z^+(w,xy) > I_{\eta^\infty}^+(w,xy),\ \ \ I_Z^-(w,xy) < I_{\eta^\infty}^-(w,xy),\\
&F_Z^+(w,xy) < F_{\eta^\infty}^+(w,xy),\ \ F_Z^-(w,xy) > F_{\eta^\infty}^-(w,xy).
\end{aligned}$$

A pair (w, xy) is called β-*strong* if

$$\begin{aligned}
&T_Z^+(w,xy) = T_{\eta^\infty}^+(w,xy),\ \ T_Z^-(w,xy) = T_{\eta^\infty}^-(w,xy),\\
&I_Z^+(w,xy) = I_{\eta^\infty}^+(w,xy),\ \ \ I_Z^-(w,xy) = I_{\eta^\infty}^-(w,xy),\\
&F_Z^+(w,xy) = F_{\eta^\infty}^+(w,xy),\ \ F_Z^-(w,xy) = F_{\eta^\infty}^-(w,xy).
\end{aligned}$$

where $\eta^\infty(w, xy)$ is incidence strength of strongest path between w and xy.

Definition 2.26. Let $G = (X, Y, Z)$ be a BNIG. A pair (w, xy) is called a δ-*pair* if

$$\begin{aligned}
&T_Z^+(w,xy) < T_{\eta^\infty}^+(w,xy),\ \ T_Z^-(w,xy) > T_{\eta^\infty}^-(w,xy),\\
&I_Z^+(w,xy) < I_{\eta^\infty}^+(w,xy),\ \ \ I_Z^-(w,xy) > I_{\eta^\infty}^-(w,xy),\\
&F_Z^+(w,xy) > F_{\eta^\infty}^+(w,xy),\ \ F_Z^-(w,xy) < F_{\eta^\infty}^-(w,xy).
\end{aligned}$$

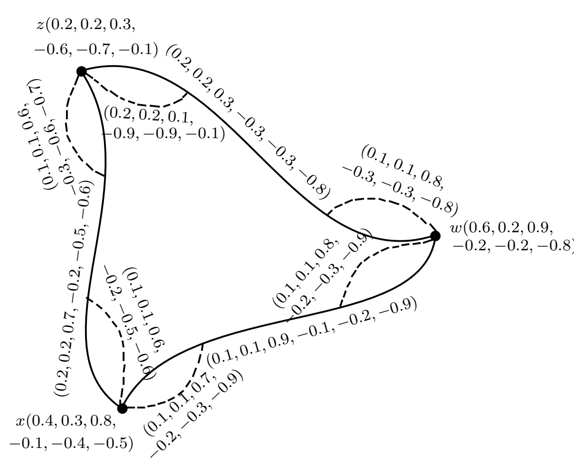

Figure 6: BNIG $G = (X, Y, Z)$

Example 2.27. In Fig.6 all edges except xw are strong. Indeed, wz and xz are $\alpha-$strong edges. whereas, a pair (z, wz) is an α-strong pair and (w, xw) is a β-strong pair.

Definition 2.28. A path P in $G = (X, Y, Z)$ is called a *strong path* if all edges and pairs of P are strong. If strong path is closed, then it is called a *strong cycle*.

Example 2.29. In Fig.7 a path $x, (x, xu), xu, (u, xu), u, (u, uw), uw, (w, uw), w$ is strong path.

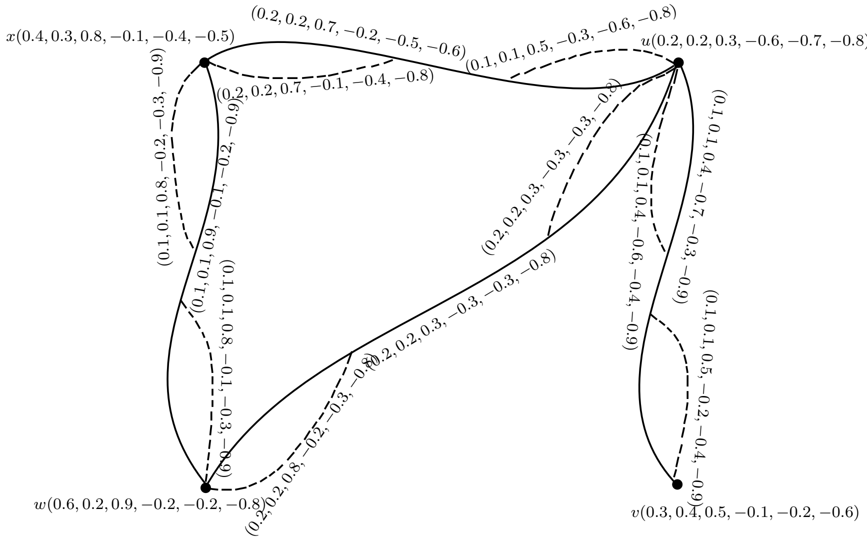

Figure 7: BNIG $G = (X, Y, Z)$

Theorem 2.30. *Let $G = (X, Y, Z)$ be a BNIG. A pair (w, xy) is strong if*

$$\begin{aligned}
T_Z^+(w, xy) &= \vee\{T_Z^+(u, vw)|(u, vw) \in \textit{supp}(Z)\},\\
I_Z^+(w, xy) &= \vee\{I_Z^+(u, vw)\ |(u, vw) \in \textit{supp}(Z)\},\\
F_Z^+(w, xy) &= \wedge\{F_Z^+(u, vw)|(u, vw) \in \textit{supp}(Z)\},\\
T_Z^-(w, xy) &= \wedge\{T_Z^-(u, vw)|(u, vw) \in \textit{supp}(Z)\},\\
I_Z^-(w, xy) &= \wedge\{I_Z^-(u, vw)\ |(u, vw) \in \textit{supp}(Z)\},\\
F_Z^-(w, xy) &= \vee\{F_Z^-(u, vw)|(u, vw) \in \textit{supp}(Z)\}.
\end{aligned}$$

Proof. Let $\psi^\infty(w, xy)$ be an incidence strength of strongest path between w and xy in $G = (X, Y, Z)$, then

$$\begin{aligned}
T_{\psi^\infty}^+(w, xy) &\leq T_Z^+(w, xy),\ \ T_{\psi^\infty}^-(w, xy) \geq T_Z^-(w, xy),\\
I_{\psi^\infty}^+(w, xy) &\leq I_Z^+(w, xy),\ \ \ I_{\psi^\infty}^-(w, xy) \geq I_Z^-(w, xy),\\
F_{\psi^\infty}^+(w, xy) &\geq F_Z^+(w, xy),\ \ F_{\psi^\infty}^-(w, xy) \leq F_Z^-(w, xy).
\end{aligned}$$

If (w, xy) is only one pair such that

$$\begin{aligned}
T_Z^+(w, xy) &= \vee\{T_Z^+(u, vw)|(u, vw) \in \text{supp}(Z)\},\\
I_Z^+(w, xy) &= \vee\{I_Z^+(u, vw)\ |(u, vw) \in \text{supp}(Z)\},\\
F_Z^+(w, xy) &= \wedge\{F_Z^+(u, vw)|(u, vw) \in \text{supp}(Z)\},\\
T_Z^-(w, xy) &= \wedge\{T_Z^-(u, vw)|(u, vw) \in \text{supp}(Z)\},\\
I_Z^-(w, xy) &= \wedge\{I_Z^-(u, vw)\ |(u, vw) \in \text{supp}(Z)\},\\
F_Z^-(w, xy) &= \vee\{F_Z^-(u, vw)|(u, vw) \in \text{supp}(Z)\}.
\end{aligned}$$

then for every path between u and vw, we have

$$\begin{aligned}
T^+_{\psi^\infty}(u,vw)<T^+_Z(w,xy),\ \ T^-_{\psi^\infty}(u,vw) >T^-_Z(w,xy),\\
I^+_{\psi^\infty}(u,vw) <I^+_Z(w,xy),\ \ \ I^-_{\psi^\infty}(u,vw) > I^-_Z(w,xy),\\
F^+_{\psi^\infty}(u,vw)>F^+_Z(w,xy),\ \ F^-_{\psi^\infty}(u,vw) <F^-_Z(w,xy).
\end{aligned}$$

hence

$$\begin{aligned}
T^+_{\psi^\infty}(w,xy)<T^+_Z(w,xy),\ \ T^-_{\psi^\infty}(w,xy) >T^+_Z(w,xy),\\
I^+_{\psi^\infty}(w,xy) <I^+_Z(w,xy),\ \ \ I^-_{\psi^\infty}(w,xy) > I^+_Z(w,xy),\\
F^+_{\psi^\infty}(w,xy)>F^+_Z(w,xy),\ \ F^-_{\psi^\infty}(w,xy) <F^+_Z(w,xy).
\end{aligned}$$

Thus, (w,xy) is an α-strong pair. If (w,xy) is not unique, then

$$\begin{aligned}
T^+_Z(w,xy)=T^+_{\psi^\infty}(w,xy),\ \ T^-_Z(w,xy) =T^-_{\psi^\infty}(w,xy),\\
I^+_Z(w,xy) =I^+_{\psi^\infty}(w,xy),\ \ \ I^-_Z(w,xy) = I^-_{\psi^\infty}(w,xy),\\
F^+_Z(w,xy)=F^+_{\psi^\infty}(w,xy),\ \ F^-_Z(w,xy) =F^-_{\psi^\infty}(w,xy).
\end{aligned}$$

Hence (w,xy) is β-strong pair. □

Theorem 2.31. *If $G=(X,Y,Z)$ is a bipolar neutrosophic incidence tree and P is a strong path between any two vertices x and y. Then P have maximum strength between x and y.*

Proof. Let P be only one strong path between x and y. Because P is strong, all edges and pairs of P are in the spanning bipolar neutrosophic incidence tree H of G. We prove that P is a path between x and y having maximum strength.
Suppose, on contrary that P is not a path having maximum strength from x to y and P' is such a path. Then P and P' are not equal, hence P and and reversal of P' form a cycle. Since H^* is tree, so there exist no cycle in H, . Hence any edge $x'y'$ of P' must not exist in H.
By definition of G, we have

$$\begin{aligned}
T^+_Y(x'y') < T^+_{\phi^\infty}(x',y'),\ \ I^+_Y(x'y')< I^+_{\phi^\infty}(x',y'),\ \ F^+_Y(x'y') > F^+_{\phi^\infty}(x',y'),\\
T^-_Y(x'y') > T^-_{\phi^\infty}(x',y'),\ \ I^-_Y(x'y')> I^-_{\phi^\infty}(x',y'),\ \ F^-_Y(x'y') < F^-_{\phi^\infty}(x',y').
\end{aligned}$$

It means there exist a path between x' and y' in H and we can replace all edges $x'y'$ of P' which not exist in H by a path P^* from x to y in H. Hence P^* is at least as strong as P'. Hence P^* and P cannot be equal. So, P and reversal of P^* form a cycle in H, which is a contradiction to the fact that H^* is tree.
Hence our assumption P is not a path having maximum strength from x to y is wrong. □

3 Application to Illegal Migration

Suppose Mr.Kamran wants to travel from Bangladesh to India illegally. For this he use all borders line between Bangladesh and India. He have three ways, first one is a direct way, i.e. Bangladesh to India, second one is Bangladesh to Pakistan and Pakistan to India and the third one is Bangladesh to Bhutan, Bhutan to

Pakistan, Pakistan to Nepal and Nepal to India. Let V={Bangladesh(BGD), Bhutan(BTN), Pakistan(PAK), Nepal(NPL), India(IND)} be the set of countries and $E = \{(BGD, BTN), (BTN, PAK), (PAK, NPL), (NPL, IND), (BGD, PAK), (PAK, IND), (BGD, IND)\}$ a subset of $V \times V$.
Let X be the bipolar neutrosophic set on V, which is given as

$$\begin{aligned} X = \{&(BGD, 0.3, 0.2, 0.6, -0.1, -0.2, -0.5), (BTN, 0.3, 0.6, 0.9, -0.2, -0.4, -0.6),\\ &(PAK, 0.4, 0.5, 0.6, -0.1, -0.3, -0.4), (NPL, 0.9, 0.7, 0.8, -0.4, -0.3, -0.4),\\ &(IND, 0.6, 0.9, 0.1, -0.1, -0.2, -0.3)\}. \end{aligned}$$

Let Y be the bipolar neutrosophic relation on V, which is given as

$$\begin{aligned} Y = \{&((BGD, BTN), 0.1, 0.2, 0.8, -0.2, -0.3, -0.7), ((BTN, PAK), 0.2, 0.5, 0.9, -0.3, -0.3, -0.7),\\ &((PAK, NPL), 0.3, 0.4, 0.7, -0.2, -0.4, -0.5), ((NPL, IND), 0.5, 0.6, 0.7, -0.2, -0.3, -0.5),\\ &((BGD, PAK), 0.3, 0.1, 0.6, -0.2, -0.2, -0.6), ((PAK, IND), 0.4, 0.4, 0.5, -0.1, -0.3, -0.5),\\ &((BGD, IND), 0.2, 0.1, 0.5, -0.1, -0.3, -0.6)\}. \end{aligned}$$

Let Z be the bipolar neutrosophic set on $V \times E$, which is given as

$$\begin{aligned} Z = \{&((BGD, (BGD, BTN)), 0.1, 0.1, 0.7, -0.1, -0.3, -0.8),\\ &((BTN, (BGD, BTN)), 0.1, 0.2, 0.8, -0.3, -0.3, -0.8),\\ &((BTN, (BTN, PAK)), 0.2, 0.4, 0.8, -0.2, -0.3, -0.8),\\ &((PAK, (BTN, PAK)), 0.2, 0.4, 0.8, -0.2, -0.4, -0.7),\\ &((PAK, (PAK, NPL)), 0.3, 0.3, 0.5, -0.1, -0.4, -0.5),\\ &((NPL, (PAK, NPL)), 0.2, 0.3, 0.8, -0.2, -0.3, -0.6),\\ &((NPL, (NPL, IND)), 0.4, 0.5, 0.7, -0.3, -0.3, -0.6),\\ &((IND, (NPL, IND)), 0.4, 0.5, 0.5, -0.1, -0.2, -0.7),\\ &((BGD, (BGD, PAK)), 0.1, 0.1, 0.5, -0.2, -0.3, -0.7),\\ &((PAK, (BGD, PAK)), 0.1, 0.1, 0.5, -0.2, -0.2, -0.6),\\ &((PAK, (PAK, IND)), 0.3, 0.3, 0.5, -0.1, -0.3, -0.6),\\ &((IND, (PAK, IND)), 0.4, 0.3, 0.4, -0.1, -0.3, -0.6),\\ &((BGD, (BGD, IND)), 0.1, 0.1, 0.4, -0.2, -0.2, -0.7),\\ &((IND, (BGD, IND)), 0.1, 0.1, 0.5, -0.1, -0.3, -0.8)\}. \end{aligned}$$

Thus, $G = (X, Y, Z)$ is a BNIG as shown in Fig.8.

Let $T^{+}_{\rho}(u, v)$ represent the degree of protection for an illegal immigrant to use u as origin and come to a destination v. There are three paths from BGD to IND

$$P_1 : BGD,(BGD, (BGD, IND)),(BGD, IND), (IND, (BGD, IND)),IND.$$

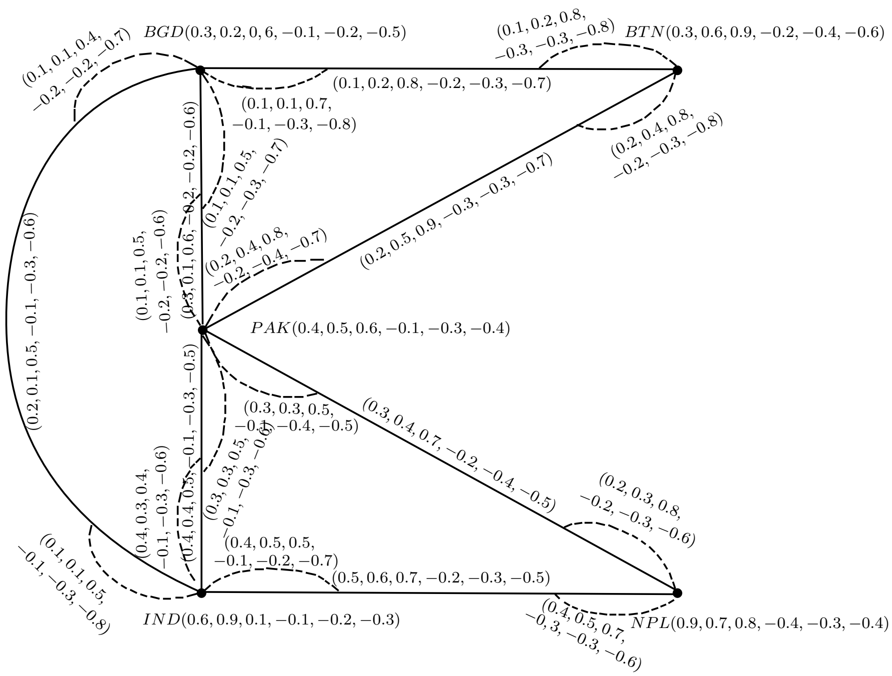

Figure 8: BNIG $G = (X, Y, Z)$

$$\begin{aligned} P_2 : BGD, & (BGD,(BGD,PAK)),(BGD,PAK),(PAK,(BGD,PAK)),PAK, \\ & (PAK,(PAK,IND)),(PAK,IND),(IND,(PAK,IND)),IND. \end{aligned}$$

$$\begin{aligned} P_3 : BGD, & (BGD,(BGD,BTN)),(BGD,BTN),(BTN,(BGD,BTN)),BTN, \\ & (BTN,(BTN,PAK)),(BTN,PAK),(PAK,(BTN,PAK)),PAK, \\ & (PAK,(PAK,NPL)),(PAK,NPL),(NPL,(PAK,NPL)),NPL, \\ & (NPL,(NPL,IND)),(NPL,IND),(IND,(NPL,IND)),IND. \end{aligned}$$

$\rho^{\infty}(BGD, IND)$ is the strength of strongest path between BGD and IND. This is the safest path between BGD and IND. To calculate the value of $\rho^{\infty}(BGD, IND)$, we need the strength of paths P_1, P_2 and P_3, which is denoted by $\rho_{P_1}(BGD, IND)$, $\rho_{P_2}(BGD, IND)$ and $\rho_{P_3}(BGD, IND)$, respectively.
By calculation, we have

$$\rho_{P_1}(BGD, IND) = (0.2, 0.1, 0.5, -0.1, -0.3, -0.6),$$

$$\rho_{P_2}(BGD, IND) = (0.3, 0.1, 0.6, -0.1, -0.2, -0.6),$$

$$\rho_{P_3}(BGD, IND) = (0.1, 0.2, 0.9, -0.2, -0.3, -0.7).$$

$\rho^{\infty}(BGD, IND)$ =$(0.3, 0.2, 0.5, -0.2, -0.3, -0.6)$.
We see that

$$T^{+}_{\rho^{\infty}}(BGD, IND) = T^{+}_{\rho_{P_2}}(BGD, IND).$$

Hence P_2 is safest path for an illegal immigrant.
We present proposed method in the following Algorithm 3.1.

3.1 Algorithm

1. Input the vertex set V^*.
2. Input the edge set $E^* \subseteq V^* \times V^*$.
3. Input the bipolar neutrosophic set X on V^*.
4. Input the bipolar neutrosophic relation Y on V^*.
5. Input the bipolar neutrosophic set Z on $V^* \times E^*$.
6. Calculate the strength of path $\rho(x, y)$ from x to y such that

$$\begin{aligned}
T_\rho^+(x,y) &= \wedge \{T_Y^+(xy) | xy \in \text{supp}(Y)\},\\
I_\rho^+(x,y) &= \wedge \{I_Y^+(xy) \, | xy \in \text{supp}(Y)\},\\
F_\rho^+(x,y) &= \vee \{F_Y^+(xy) | xy \in \text{supp}(Y)\},\\
T_\rho^-(x,y) &= \vee \{T_Y^-(xy) | xy \in \text{supp}(Y)\},\\
I_\rho^-(x,y) &= \vee \{I_Y^-(xy) \, | xy \in \text{supp}(Y)\},\\
F_\rho^-(x,y) &= \wedge \{F_Y^-(xy) | xy \in \text{supp}(Y)\}.
\end{aligned}$$

7. Calculate the incidence strength $\rho^\infty(x, y)$ of strongest path from x to y such that

$$T^+_{\rho^\infty}(x,y) = \bigvee_{i=1}^{k} T^+_{\rho_i}(x,y), \quad I^+_{\rho^\infty}(x,y) = \bigvee_{i=1}^{k} I^+_{\rho_i}(x,y), \quad F^+_{\rho^\infty}(x,y) = \bigwedge_{i=1}^{k} F^+_{\rho_i}(x,y),$$
$$T^-_{\rho^\infty}(x,y) = \bigwedge_{i=1}^{k} T^-_{\rho_i}(x,y), \quad I^-_{\rho^\infty}(x,y) = \bigwedge_{i=1}^{k} I^-_{\rho_i}(x,y), \quad F^-_{\rho^\infty}(x,y) = \bigvee_{i=1}^{k} F^-_{\rho_i}(x,y).$$

8. The safest path is $S(v_k) = \bigvee\limits_{i=1}^{k} T^+_{\rho_i}(x,y)$.
9. If v_k has more than one value then any path can be chosen.

4 Conclusion

Graph theory has become a branch of applied mathematics. Graph theory is considered as a mathematical tool for modeling and analyzing different mathematical structure, but it does not give the relationship between element and its relation pair. We have introduced BNIG which not only give the limitation of the relation between elements contained in a set, but also give the influence or impact of an element to its relation pair. We have defined the bipolar neutrosophic incidence cycle and tree. An application to illegal migration is presented using strength of strongest path in BNIG.

Conflict of interest: The authors declare that they have no conflict of interest.

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Single Valued Neutrosophic Coloring

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Abstract: Neutrosophic set was introduced by Smarandache in 1998. Due to some real time situation, decision makers deal with uncertainty and inconsistency to identify the best result. Neutrosophic concept helps to investigate the vague or indeterminacy values. Graph structures used to reduce the complications in solving the system of equations for finding the decision of some real-life problems. In this research study, we introduced the single-valued neutrosophic coloring concept. We introduce various notions, single valued neutrosophic vertex coloring, single valued neutrosophic edge coloring, and single valued neutrosophic total coloring and support those definitions with some examples.

Keywords: single-valued neutrosophic graphs; single-valued neutrosophic vertex coloring; single-valued neutrosophic edge coloring; single-valued neutrosophic total coloring.

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1. Introduction

Graph theory plays a vital role in real time problems Graph represents the connection among the points by lines and is the useful tool to solve the network problems. It is applicable in many fields such as computer science, physical science, electrical communication engineering, economics and Operation Research etc. In 1852, Francis Guthrie's four-color conjecture gave the sparkle for the new branch, graph coloring in graph theory. Graph coloring is assigning the color to the vertices or edges or both vertices and edges of the graph based on some conditions. After three decades got the solution to Guthrie's conjecture. Graph coloring technique used in many areas like telecommunication, scheduling, computer networks etc. Sometime in real-life have to deal with imprecise data and uncertain relation between points, in that case fuzzy technique where came. In 1965, Fuzzy set theory was introduced by Zadeh [39] and further work on fuzzy graph theory developed by A. Rosenfeld [33] in 1975. The fuzzy chromatic number was introduced by Munoz et al. [36] in 2004 and extended by C.Eslahchi and B.N.Onagh [23] in 2006. In 2009, S.Lavanya and R.Sattanathan [30] introduced the concept fuzzy total coloring. In 2014, Anjaly Kishore, M.S.Sunitha [7] discussed the strong chromatic number of fuzzy graphs in their research paper.

Intuitionistic fuzzy sets are dealing membership and non-membership data. Kassimir T.Atanassov [13] introduced the concept of intuitionistic fuzzy sets in 1986 and intuitionistic fuzzy graph in 1999. Ismail and Rifayathali [28] discussed the coloring of intuitionaistic fuzzy graphs using (α, β) cuts in 2015, Rifayathali et al. [32] discussed intuitionistic fuzzy coloring and strong intuitionistic fuzzy coloring in 2017 and 2018.

Vague set concept introduced by Gau and Buehrer [26] in 1993 and in 2014, Akram et al. [11] discussed vague graphs and further work extended by Borzooei et al. [14, 15], Vertex and Edge coloring of Vague graphs were introduced by Arindam Dey et al [12] in 2018.

In all real-time cases, the membership and non-membership values are not enough to find the result. Sometimes the vague or indeterminacy qualities need to be considered for the decision making, in that case intuitionistic fuzzy logic insufficient to give the solution. This situation reasoned for to move the new concept, F.Smarandache came with a solution"Neutrosophic logic". Neutrosophic logic play a vital role in several of the real valued problems like law, medicine, industry, finance, engineering, IT, etc.

Neutrosophic set was introduced by F.Smarandache [35] in 1998, Neutrosophic set a generalisation of the intuitionistic fuzzy set. It consists truth value, indeterminacy value and false values.Wang et al. [38] worked on Single valued neutrosophic sets in 2010. Strong Neutrosophic graph and its properties were introduced and discussed by Dhavaseelan et al. [25] in 2015 and Single valued neutrosophic concept introduced in 2016 by Akram and Shahzadi [8, 9, 10]. Broumi et al. [16, 17, 18, 19, 20, 21, 22] extended their works in Single valued neutrosophic graphs, Isolated single valued graphs, Uniform single valued graphs, Interval valued neutrosophic graphs (IVNG) and Bipolar neutrosophic graphs. Dhavaseelan et al. [24] in 2018, discussed Single valued co-neutrosophic graphs in their paper. Sinha et al. [34] extended the single valued work for signed digraphs in 2018 and Vasile [37] proposed five penta-valued refined neutrosophic indexes representation in his work. In 2019, jan et al. in their paper [29] have reviewed the following definitions: Interval-Valued Fuzzy Graphs (IVFG), Interval-Valued Intuitionistic Fuzzy Graphs (IVIFG), Complement of IVFG, SVNG, IVNG and the complement of SVNG and IVNG. They have modified those definitions, supported with some examples. Neutrosophic graphs happen to play a vital role in the building of neutrosophic models. Also, these graphs can be used in networking, Computer technology, Communication, Genetics, Economics, Sociology, Linguistics, etc., when the concept of indeterminacy is present.

Abdel-Basset et al. used Neutrosophic concept in their papers [1, 2, 3, 4, 5, 6, 31] to find the decisions for some real-life operation research and IoT-based enterprises in 2019. The above papers given the idea to interlink the graph coloring concept in SVNG when deal with vague or indeterminacy qualities.

In this research paper, we introduced the concept of single valued neutrosophic vertex coloring, single valued neutrosophic edge coloring and single valued neutrosophic total coloring of single valued neutrosophic graph and also Strong and Complete Single valued neutrosophic graph coloring are discussed with examples.

Definition 1.1. [35]

Let X be a space of points(objects). A neutrosophic set A in X is characterized by truth-membership function $t_A(x)$, an indeterminacy-membership function $i_A(x)$ and a falsity-membership function $f_A(x)$. The functions $t_A(x)$, $i_A(x)$, and $f_A(x)$, are real standard or non-standard subsets of $]0^-, 1^+[$. That is, $t_A(x): X \to]0^-, 1^+[$, $i_A(x): X \to]0^-, 1^+[$ and $f_A(x): X \to]0^-, 1^+[$ and $0^- \leq t_A(x) + i_A(x) + f_A(x) \leq 3^+$.

Definition 1.2. [9]

A single-valued neutrosophic graphs (SVNG) G = (X, Y) is a pair where X: N → [0,1] is a single-valued neutrosophic set on N and Y: N × N → [0,1] is a single-valued neutrosophic relation on N such that

$$t_Y(xy) \leq \min\{t_X(x), t_X(y)\},$$

$$i_Y(xy) \leq \min\{i_X(x), i_X(y)\},$$

$$f_Y(xy) \leq \max\{f_X(x), f_X(y)\},$$

for all x,y ∈ N. X and Y are called the single-valued neutrosophic vertex set of G and the single-valued neutrosophic edge set of G, respectively. A single-valued neutrosophic relation Y is said to be symmetric if $t_Y(xy) = t_Y(yx)$, $i_Y(xy) = i_Y(yx)$ and $f_Y(xy) = f_Y(yx)$, for all x,y ∈ N. Single-valued neutrosophic be abbreviated here as SVN.

Definition 1.3. [10]

The complement of a SVNG G = (X, Y) is a SVNG $\bar{G} = (\bar{X}, \bar{Y})$, where

1. $\bar{X} = X$
2. $\bar{t_X}(x) = t_X(x), \bar{i_X}(x) = i_X(x), \bar{f_X}(x) = f_X(x)$ $for\ all\ x \in X$
3. $\bar{t_X}(xy) = \begin{cases} \min\{t_X(x), t_X(y)\}\ if\ \ t_Y(xy) = 0 \\ \min\{t_X(x), t_X(y)\} - t_Y(xy)\ if\ \ t_Y(xy) > 0 \end{cases}$

$$\bar{i_X}(xy) = \begin{cases} \min\{i_X(x), i_X(y)\}\ if\ \ i_Y(xy) = 0 \\ \min\{i_X(x), i_X(y)\} - i_Y(xy)\ if\ \ i_Y(xy) > 0 \end{cases}$$

$$\bar{f_X}(xy) = \begin{cases} \max\{f_X(x), f_X(y)\}\ if\ \ f_Y(xy) = 0 \\ \max\{f_X(x), f_X(y)\} - f_Y(xy)\ if\ \ f_Y(xy) > 0 \end{cases}$$

for all $x, y \in X$.

2. Single-Valued Neutrosophic Vertex Coloring (SVNVC)

In this section, we have developed SVNVC and this coloring has verified through some examples of SVNG, CSVNG and SSVNG. Also discussed some theorems.

Definition 2.1.

A family $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ of SVN fuzzy set is called a k-SVNVC of a SVNG G = (X, Y) if

1. $\vee \gamma_i(x) = X,\ \forall x \in X$
2. $\gamma_i \wedge \gamma_j = 0$
3. For every incident vertices of edge xy of G, $\min\{\gamma_i(m_1(x)), \gamma_i(m_1(y))\} = 0$, $\min\{\gamma_i(i_1(x)), \gamma_i(i_1(y))\} = 0\ and\ \max\{\gamma_i(n_1(x)), \gamma_i(n_1(y))\} = 1, (1 \leq i \leq k)$.

This k-SVNVC of G is denoted by $\chi_v(G)$, is called the SVN chromatic number of the SVNG G.

Example 2.2.

Consider the SVNG G = (X,E) with SVN vertex set $X = \{x_1, x_2, x_3, x_4, x_5\}$ and SVN edge set $E = \{X_i X_j | ij = 12,14,15,23,24,25,34,35,45\}$ the membership functions defined as,

$$\left(m_1(x_i), i_1(x_i), n_1(x_i)\right) = \begin{cases} (0.3,0.2,0.6) \; for \; i = 1,2 \\ (0.7,0.1,0.2) \; for \; i = 3 \\ (0.2,0.1,0.7) \; for \; i = 4 \\ (0.5,0.1,0.7) \; for \; i = 5 \end{cases}$$

$$\left(m_2(x_i x_j), i_2(x_i x_j), n_2(x_i x_j)\right) = \begin{cases} (0.3,0.2,0.6) \; for \; ij = 12 \\ (0.2,0.1,0.7) \; for \; ij = 14,24,34,45 \\ (0.3,0.1,0.6) \; for \; ij = 15,23,25 \\ (0.5,0.1,0.7) \; for \; ij = 35 \end{cases}$$

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be a family of SVN fuzzy sets defined on X as follows:

$$\gamma_1(x_i) = \begin{cases} (0.3,0.2,0.6) \; for \; i = 1,3 \\ (0,0,1) \; for \; others \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} (0.7,0.1,0.2) \; for \; i = 2 \\ (0,0,1) \; for \; others \end{cases}$$

$$\gamma_3(x_i) = \begin{cases} (0.5,0.1,0.7) \; for \; i = 4 \\ (0,0,1) \; for \; others \end{cases}$$

$$\gamma_4(x_i) = \begin{cases} (0.2,0.1,0.7) \; for \; i = 5 \\ (0,0,1) \; for \; others \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ fulfilled the conditions of SVNVC of the graph G. Any families below four points could not satisfy our definition. Hence the SVN chromatic number $\chi_v(G)$ of the above example is 4.

Definition 2.3.

A SVNG G = (X, Y) is called complete single-valued neutrosophic graph (CSVNG) if the following conditions are satisfied:

$$t_Y(xy) = \min\{t_X(x), t_X(y)\},$$

$$i_Y(xy) = \min\{i_X(x), i_X(y)\},$$

$$f_Y(xy) = \max\{f_X(x), f_X(y)\},$$

for all x,y ∈ X.

Definition 2.4.

A SVNG G = (X, Y) is called strong single-valued neutrosophic graph (SSVNG) if the following conditions are satisfied:

$$t_Y(xy) = \min\{t_X(x), t_X(y)\},$$

$$i_Y(xy) = \min\{i_X(x), i_X(y)\},$$

$$f_Y(xy) = \max\{f_X(x), f_X(y)\},$$

for all (x,y) ∈ Y .

Example 2.5.

Consider the SSVNG G = (X,Y) with SVN vertex set $X = \{x_1, x_2, x_3, x_4, x_5\}$ and SVN edge set $Y = \{x_i x_j | ij = 12,15,23,34,45\}$ the membership functions defined as,

$$\left(m_1(x_i), i_1(x_i), n_1(x_i)\right) = \begin{cases} (0.1,0.2,0.9)\ for\ i = 1 \\ (0.6,0.7,0.4)\ for\ i = 2 \\ (0.3,0.3,0.7)\ for\ i = 3 \\ (0.7,0.8,0.2)\ for\ i = 4 \\ (0.5,0.5,0.6)\ for\ i = 5 \end{cases}$$

$$\left(m_2(x_i x_j), i_2(x_i x_j), n_2(x_i x_j)\right) = \begin{cases} (0.1,0.2,0.9)\ for\ ij = 12,15 \\ (0.3,0.3,0.7)\ for\ ij = 23,34 \\ (0.5,0.5,0.6)\ for\ ij = 45 \end{cases}$$

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ be a family of SVN fuzzy sets defined on X as follows:

$$\gamma_1(x_i) = \begin{cases} (0.1,0.2,0.9)\ for\ i = 1 \\ (0.3,0.3,0.7)\ for\ i = 3 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} (0.6,0.7,0.4)\ for\ i = 2 \\ (0.7,0.8,0.2)\ for\ i = 4 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_3(x_i) = \begin{cases} (0.5,0.5,0.6)\ for\ i = 5 \\ (0,0,1)\ for\ others \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ fulfilled the conditions of Strong SVNVC of the graph G. Any families below three points could not satisfy our definition. Hence the SSVN chromatic number $\chi_v(G)$ of the above example is 3.

Example 2.6.

Consider the CSVNG G = (X,Y) with SVN vertex set $X = \{x_1, x_2, x_3, x_4, x_5\}$ and SVN edge set $Y = \{x_i x_j | ij = 12,13,14,23,24,34\}$ the membership functions defined as,

$$\left(m_1(x_i), i_1(x_i), n_1(x_i)\right) = \begin{cases} (0.7,0.7,0.1)\ for\ i = 1 \\ (0.6,0.7,0.3)\ for\ i = 2 \\ (0.3,0.3,0.7)\ for\ i = 3 \\ (0.1,0.1,0.8)\ for\ i = 4 \end{cases}$$

$$\left(m_2(x_i x_j), i_2(x_i x_j), n_2(x_i x_j)\right) = \begin{cases} (0.6,0.7,0.3)\ for\ ij = 12 \\ (0.3,0.3,0.7)\ for\ ij = 13,23 \\ (0.1,0.1,0.8)\ for\ ij = 14,24,34 \end{cases}$$

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be a family of SVN fuzzy sets defined on X as follows:

$$\gamma_1(x_i) = \begin{cases} (0.7,0.7,0.1)\ for\ i = 1 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} (0.6,0.7,0.3)\ for\ i = 2 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_3(x_i) = \begin{cases} (0.3,0.3,0.7)\ for\ i = 3 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_4(x_i) = \begin{cases} (0.1,0.1,0.8)\ for\ i = 4 \\ (0,0,1)\ for\ others \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ fulfilled the conditions of complete SVNVC of the graph G. Any families below four points could not satisfy our definition. Hence the SVN chromatic number $\chi_v(G)$ of the above example is 4.

Theorem 2.7.

For any graph CSVNG with n vertices, $\chi_v(G) = n$.

Proof:

By the definition of CSVNG, all the vertices are adjacent to each other. Each color class contains exactly one vertex with the value (t_X(x), t_X(x),t_X(x))> 0, thus remaining vertices are with the value (t_X(x), t_X(x),t_X(x)) = 0. Hence $\chi_v(G) = n$.

Theorem 2.8.

For any SSVNG G, then $\overline{\overline{\chi_v}}(G) = \chi_v(G)$.

Proof. It is obvious.

3. Single-Valued Neutrosophic Edge Coloring (SVNEC)

In this section, we introduced and discussed SVNEC with an example and theorems.

Definition 3.1.

A family $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ of SVN fuzzy set is called a k-SVNEC of a SVNG G = (X,Y) if

1. $\vee\, \gamma_i(xy) = Y,\ \forall xy \in Y$
2. $\gamma_i \wedge \gamma_j = 0$
3. For every strong edge xy of G, $\min\{\gamma_i(m_2(xy))\} = 0$, $\min\{\gamma_i(i_2(xy))\} = 0\ and\ \max\{\gamma_i(n_2(xy))\} = 1, (1 \le i \le k)$.

This k-SVNEC of G is denoted by$\chi_e(G)$, is called the SVN chromatic number of the SVNG G.
Example 3.2.
Consider the SVNG G = (X,Y) with SVN vertex set $X = \{x_1, x_2, x_3, x_4\}$ and SVN edge set $Y = \{x_i x_j | ij = 12,13,14,23,24,34\}$ the membership functions defined as,

$$\left(m_1(x_i), i_1(x_i), n_1(x_i)\right) = \begin{cases} (0.3,0.1,0.6)\ for\ i = 1 \\ (0.2,0.1,0.4)\ for\ i = 2 \\ (0.5,0.2,0.4)\ for\ i = 3 \\ (0.4,0.1,0.4)\ for\ i = 4 \end{cases}$$

$$\left(m_2(x_ix_j), i_2(x_ix_j), n_2(x_ix_j)\right) = \begin{cases} (0.2,0.1,0.4)\ for\ ij = 12,23,24 \\ (0.3,0.1,0.6)\ for\ ij = 13,14 \\ (0.4,0.1,0.4)\ for\ ij = 24 \end{cases}$$

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ be a family of SVN fuzzy sets defined on Y as follows:

$$\gamma_1(x_ix_j) = \begin{cases} (0.2,0.1,0.4)\ for\ i = 12,34 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_2(x_ix_j) = \begin{cases} ((0.3,0.1,0.6))\ for\ i = 14,23 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_3(x_ix_j) = \begin{cases} (0.4,0.1,0.4)\ for\ i = 13,24 \\ (0,0,1)\ for\ others \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ fulfills the conditions of SVNEC of SVNG. Any families below three members could not satisfy our definition. Hence, the SVN chromatic number $\chi_e(G)$ of the above example is 3.

4. Single-Valued Neutrosophic Total Coloring (SVNTC)

In this section, we defined SVNTC supported by an example.

Definition 4.1.

A family $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ of SVN fuzzy sets on the SVN vertex set X is called a k-SVNTC of SVNG G = (X, Y) if

1. $\vee\, \gamma_i(x) = X,\ \forall x \in X$ and $\vee\, \gamma_i(xy) = Y,\ \forall xy \in Y$
2. $\gamma_i \wedge \gamma_j = 0$
3. For every incident vertices of edge xy of G, $\min\{\gamma_i(m_1(x)), \gamma_i(m_1(y))\} = 0$, $\min\{\gamma_i(i_1(x)), \gamma_i(i_1(y))\} = 0\ and\ \max\{\gamma_i(n_1(x)), \gamma_i(n_1(y))\} = 1, (1 \le i \le k)$. For every strong edge xy of G, $\min\{\gamma_i(m_2(xy))\} = 0$, $\min\{\gamma_i(i_2(xy))\} = 0\ and\ \max\{\gamma_i(n_2(xy))\} = 1, (1 \le i \le k)$.

This k-SVNTC of G is denoted by $\chi_t(G)$, is called the SVN chromatic number of the SVNG G.

Example 4.2.

Consider the SVNG G = (X,Y) with SVN vertex set $X = \{x_1, x_2, x_3, x_4, x_5\}$ and SVN edge set $Y = \{x_ix_j | ij = 12,13,14,15,23,24,25,34,35,45\}$ the membership functions defined as,

$$\left(m_1(x_i), i_1(x_i), n_1(x_i)\right) = \begin{cases} (0.3,0.1,0.7)\ for\ i = 1 \\ (0.5,0.3,0.5)\ for\ i = 2 \\ (0.4,0.2,0.6)\ for\ i = 3 \\ (0.8,0.6,0.2)\ for\ i = 4 \\ (0.7,0.5,0.3)\ for\ i = 5 \end{cases}$$

$$\left(m_2(x_ix_j), i_2(x_ix_j), n_2(x_ix_j)\right) = \begin{cases} (0.3,0.1,0.7)\ for\ ij = 12,13,14,15 \\ (0.8,0.6,0.2)\ for\ ij = 45 \\ (0.4,0.2,0.6)\ for\ ij = 23,24,25 \\ (0.5,0.3,0.5)\ for\ ij = 34,35 \end{cases}$$

Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ be a family of SVN fuzzy sets defined on Y as follows:

$$\gamma_1(x_i) = \begin{cases} (0.3,0.1,0.7)\ for\ i = 1 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} (0.5,0.3,0.5)\ for\ i = 2 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_3(x_i) = \begin{cases} (0.4,0.2,0.6)\ for\ i = 3 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_4(x_i) = \begin{cases} (0.8,0.6,0.2)\ for\ i = 4 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_5(x_i) = \begin{cases} (0.7,0.5,0.3)\ for\ i = 5 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_1(x_i x_j) = \begin{cases} (0.3,0.1,0.7)\ for\ i = 12 \\ (0.5,0.3,0.5)\ for\ i = 35 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_2(x_i x_j) = \begin{cases} (0.3,0.1,0.7)\ for\ i = 13 \\ (0.4,0.2,0.6)\ for\ i = 24 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_3(x_i x_j) = \begin{cases} (0.3,0.1,0.7)\ for\ i = 14 \\ (0.4,0.2,0.6)\ for\ i = 25 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_4(x_i x_j) = \begin{cases} (0.8,0.6,0.2)\ for\ i = 45 \\ (0.4,0.2,0.6)\ for\ i = 23 \\ (0,0,1)\ for\ others \end{cases}$$

$$\gamma_5(x_i x_j) = \begin{cases} (0.3,0.1,0.7)\ for\ i = 15 \\ (0.5,0.3,0.5)\ for\ i = 34 \\ (0,0,1)\ for\ others \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ fulfills the conditions of SVNTC of SVNG. Any families below five members could not satisfy our definition. Hence the SVN chromatic number $\chi_t(G)$ of the above example is 5.

5. **Conclusions**

Single Valued Neutrosophic Coloring concept introduced in this paper. Single valued neutrosophic vertex coloring, single valued neutrosophic edge coloring and single valued neutrosophic total coloring are defined. All thus definitions are developed and supported by some of the examples. In future, it will be extended to examine the theory of SVNC with the irregular colorings of graphs.

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Machine learning in Neutrosophic Environment: A Survey

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Abstract: Veracity in big data analytics is recognized as a complex issue in data preparation process, involving imperfection, imprecision and inconsistency. Single-valued Neutrosophic numbers (SVNs), have prodded a strong capacity to model such complex information. Many Data mining and big data techniques have been proposed to deal with these kind of dirty data in preprocessing stage. However, only few studies treat the imprecise and inconsistent information inherent in the modeling stage. However, this paper summarizes all works done about mapping machine learning algorithms from crisp number space to Neutrosophic environment. We discuss also contributions and hybridization of machine learning algorithms with Single-valued Neutrosophic numbers (SVNs) in modeling imperfect information, and then their impacts on resolving reel world prob-lems. In addition, we identify new trends for future research, then we introduce, for the first time, a taxonomy of Neutrosophic learning algorithms, clarifying what algorithms are already processed or not, which makes it easier for domain researchers.

Keywords: Neutrosophic; Machine Learning; Single-valued Neutrosophic numbers; Neutrosophic simple linear regression; Neutrosophic-k-NN; Neutrosophic-SVM; Neutrosophic C-means; Neutrosophic Hierarchical Clustering.

1. Introduction

Although Machine learning algorithms have caught extensive attention in last decade, seen their abilities to solve a wide problems remained obscure for years. Most of these techniques work under the some hypotheses that data should be pure, perfect and complete information. As a result, formally if the learning problems are formulated under a set of indeterminate or inconsistent information, the machine learning system becomes unable to work and the data must treated in preparation phase, which is make data science process very long, and impracticable.

However, real learning problems are often involves imperfect information such as uncertainty, inconsistency, inaccuracy and incompleteness. If we can modeling the learning problem as it in real form, exploiting the information's imperfections, we can reduce the data science process which is in many times come back from modeling that is the last step to preparation step that is the first step in the process of data science.

Single-valued neutrosophic set (SVNs) aims to provide a framework to model imperfect information. In contrast to classical machine learning methods, single-valued neutrosophic learning algorithm manipulate information with imperfections to deal with learning problems modeling complex information. To improve the performance of existing learning algorithms and handle the imperfect information in real-world, many machine learning techniques has recently been mapped into Neutrosophic Sets (NSs) environment.

Hence, the main notions and concepts of Neutrosophic are defined, also some achievements and its extensions on the NSs are undertaken. Thus, to manipulate indeterminacy, uncertainty, or inconsistency in information, that often characterizes real situations, Smarandache [1 - 3], introduced Neutrosophic set (NS), which consists of three elements, truth-membership, an indeterminacy membership, and a falsity-membership degrees independently.

Every element of the NS's features has not only a certain degree of truth(T), but also a falsity degree (F) and indeterminacy degree(I). This concept is generated from many others such as crisp set, intuitionistic fuzzy set, fuzzy set, interval-valued fuzzy set, interval-valued intuitionistic fuzzy set, etc.

Nonetheless, the NS as a philosophical concept is hard to apply in real applications. In order to overcome this situation, Smarandache and al. [4] concretize this concept introducing single-valued neutrosophic set (SVNS). SVNS can be applied quite well in real scientific and engineering fields to handle the uncertainty, imprecise, incomplete, and inconsistent information. Broumi and Smarandache [5, 6] studied basic properties of similarity and distances applied in Neutrosophic environment using single valued neutrosophic set (SVN).

Hybridization between Neutrosophic and machine learning algorithms, have also been studied, several papers [7- 11] on Neutrosophic Machine Learning (NML) have been published in the last few years.

However, there is no survey papers summarize those new learning techniques and approaches, removing the barrier for researchers currently working in the area of Neutrosophic Machine Learning. This has the twofold advantage of making such techniques more readily reachable by researchers and, conversely, avoid wasting time for to have idea which Machine learning approaches to be mapped to Neutrosophic.

The rest of this paper is organized as follows. We discuss the origins of the connection between Neutrosophic and machine learning in Section 2. Next, in Section 3, we summarize a wide variety of hybrid Neutrosophic Machine Learning techniques. Research trends and outstanding issues are discussed in Section 4.1. Then, in section 4.2, we introduce, for the first time, a taxonomy of Neutrosophic learning algorithms, clarifying what algorithms are already processed or not, which makes it easier for domain researchers.

2. Origins of connection between Neutrosophic and Machine learning

We cannot understand this connection without understanding how the Neutrosophic community works. In recent years there has been an augmenting passion from this community of neutrosophic in working, in different directions, the use of Neutrosophic to treat imperfections information in many methods and domaines. This has led to the development of a new mathematic domaine called Neutrosophic, then the connections with many others areas, such as machine learning and artificial intelligence. In the early 1999s, the pioneer of the field Florentin Smarandache generalized the intuitionistic fuzzy set (IFS), paraconsistent set, and intuitionistic set to the neutrosophic set (NS), and he underlined the distinctions between NS and IFS by reel examples. With his biggest passion and faith, Florentin Smarandache, in a quiet small town in south U.S. called Gallup, start defend his theory of Neutrality and why the three elements truth-membership (T), indeterminacy (I), and falsehood-nonmembership (F) are over 1, reproducing the history of science by story as many concepts and theory that considered primitives, and then changed by new ones.

In addition to several papers of the Neutrosophic science international association (NSIA) members, gathered in Encyclopedia Neutrosophic Researchers [12], much advances has been done. Today there are several fields of Neutrosophic to tackle a variety of problems, including Neutrosophic Computing and Machine Learning. These efforts are valued by launching a science international journal of Neutrosophic Computing and Machine Learning [13], which issued its 7th volume in 2019. In which, all published papers have wrote by NSIA's researchers.

The international journal of Neutrosophic Computing and Machine Learning with its all volumes can be seen as broad overview of the field of machine learning in Neutrosophic provided by NSIA's researchers.

The main contributions of this paper: (1) summarizes research achievements on Neutrosophic Computing and Machine Learning from the point of view of non NSIA's researchers. In a different way, try to collect the different articles on Neutrosophic machine learning papers published on several journals around the world other than those published in Neutrosophic Computing and Machine Learning journal, among it, each volume is can be considered a state of art. In order to present to researchers, the global state of art of advances research on Neutrosophic Machine Learning approaches. (2) Try to taxonomy, cluster and identify differences Neutrosophic Machines learning approaches.

3. Literature review

There are several Machine learning in Neutrosophic algorithms and approaches surveyed in this article. Then, a natural questions arise: how we can categorize all hybrid methods?

Our view of the general relationship between the fields of machine learning and Neutrosophic is the re-searchers try to map the basic operations from crisp number to Neutrosophic environment, however they rewrite machine learning algorithm instead of using simple mathematical formulas, and they use Neutrosophic formulas. But the main question should the researchers in this hybrid field (Machine learning and Neutrosophic) respond is, does this hybridization make sense to tackle the real world issues or just a theoretical formulation?

Before trying to respond this question, we synthesis all hybrid methods according to commonly used categories, summary all surveyed papers in a table 1. There are four categories of machine learning algorithms, supervised learning with two subcategories classification and prediction, semi-supervised learning, unsupervised learning and reinforcement learning.

3.1. Neutrosophic supervised learning

3.1.1. Neutrosophic Classification

Neutrosophic-k-NN Classifier [14]: K-Nearest Neighbor (K-NN) method isn't a learning method, but based on saving the training examples (all training examples), at prediction time, it find the k training examples $(x_1, y_1), \cdots, (x_k, y_k)$ that are closest to the test example x, and then affect to the most frequent class among those y_i's. This initial version of K-NN suffers from slowness because to classify x, one need to loop over all training examples. Actually, some tricks to speed are introduced such as classes represented by medoid (Representative point), or centroid (central value), etc. The Neutrosophic K-NN method we present here is the mapping of method based on Centroid, in which we consider c_j the center of cluster or class j, a constant m, regularization parameter δ, and (T_{ij}, I_{ij}, F_{ij}), where T_{ij} denote truth, I_{ij} indeterminacy and N_{ij} falsity membership values of point i for class j.

$$T_{ij} = \frac{(x_i - c_j)^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C}(x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - c_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \tag{1}$$

$$F_{ij} = \frac{\delta^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C}(x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - c_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \tag{2}$$

$$I_{ij} = \frac{(x_i - c_{imax})^{-(\frac{2}{m-1}}}{\sum_{j=1}^{C}(x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - c_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \tag{3}$$

At the time of prediction, the membership value of unknown point x_u to class j is defined by as follow:

$$x_{uj} = \frac{\sum_{i=1}^{k} d_i(T_{ij} + F_{ij} - I_{ij})}{\sum_{i=1}^{k} d_i}, \tag{4}$$

With $d_i = \frac{1}{(x_u - x_i)^{\frac{2}{q-1}}}$

Then unknown point x_u get the label of class maximizing $max\{x_{uj}; j = 1,2 \cdots, C\}$.

The authors didn't show the usefulness of the proposed method but they proposed an interesting idea to apply it on imbalanced data-set problems.

Neutrosophic SVM (N-SVM) [15] : Let's assume that (x_i, y_i) a set of training data, in which eve

with $i = 1,2 \cdots, N$

$$t_i = 1 - \frac{\|(x_j - C_+)\|}{max_{x_k \in P}\|(x_j - C_+)\|}, \tag{5}$$

ry x_i belonging to class y_i with a triple t_i, f_i, and i_i as its Neutrosophic components.

$$i_i = 1 - \frac{\|(x_j - C_{all})\|}{max_{x_k \in P}\|(x_j - C_{all})\|}, \tag{6}$$

$$f_i = 1 - \frac{\|(x_j - C_-)\|}{max_{x_k \in P}\|(x_j - C_-)\|}, \tag{7}$$

Where P and N represent the positive and negative samples subsets respectively, $y_i = +1$ for all $x_i \in P$ and $y_i = -1$ for $x_i \in N$.

$$t_i = 1 - \frac{\|(x_j - C_-)\|}{max_{x_k \in N}\|(x_j - C_-)\|}, \tag{8}$$

$$i_i = 1 - \frac{\|(x_j - C_{all})\|}{max_{x_k \in N}\|(x_j - C_{all})\|}, \tag{9}$$

$$f_i = 1 - \frac{\|(x_j - C_+)\|}{max_{x_k \in N}\|(x_j - C_+)\|}, \tag{10}$$

with $C_+ = \frac{1}{n_+}\sum_{k=1}^{n_+} x_k$, $C_- = \frac{1}{n_-}\sum_{k=1}^{n_-} x_k$, and $C_{all} = \frac{1}{2}(C_+ + C_-)$

We define g_j as weighting function:

$$g_j = t_i + i_i - f_i, \tag{11}$$

The optimal hyper-plane problem in the reformulated SVM is the solution to:

$$\text{minimize } g_j = \frac{1}{2}\omega \cdot \omega \sum_{j=1}^{k} g_j \zeta_j, \tag{12}$$

Subject to

$$\boldsymbol{y_j(\omega_j + b) > 1 - \zeta_j \quad i = 1, 2 \cdots, n} \tag{13}$$

N-SVM (Neutrosophic-Support Vector Machine) improves performance over standard SVM and reduces the effects of outliers in learning samples.

3.1.2. Neutrosophic Regression

Neutrosophic simple linear Regression: Salama and al. [16] studied and introduced Neutrosophic simple linear regression model with its possible utility to predict value of a dependent variable y according to predictor variable x. Below a pseudo code of Neutrosophic Linear Regression algorithm.

Algorithm 1 Neutrosophic Simple Linear Regression

Require: Training data $(x_i, y_j), i, j = 1,2, \cdots, N$

A model define the relationship between input x and y, y = ax + b, where (a and b) represent estimated Neutrosophic (intercept and slope) coefficients, y estimated Neutrosophic output

Define degree of membership, non-membership, and indeterminacy :
$((\mu_A(x_1), \lambda_A(x_1), \nu_A(x_1)), (\mu_B(x_1), \lambda_B(x_1), \nu_B(x_1)), i, j = 1,2 \cdots, N$
Define cost function $J(a, b) = \sum (ax_i + b - y_i)^2$
Repeat

Calculate the gradients of J

Update the weights a

Repeat until the cost $J(a, b)$ stops reducing, or some other predefined termination criteria is met

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3.2. Neutrosophic unsupervised learning

3.2.1. Neutrosophic Clustering

Neutrosophic C-means: In this method, authors [10] have given a meaning to the three basic Neutrosophic components T_{ij} as membership values belonging to the determinate clusters I_i as boundary regions, and N_i noisy data set.

$$\overline{c}_{imax} = \frac{c_{pi} + c_{qi}}{2}, \tag{14}$$

We define p_i and q_i are the cluster numbers with the biggest and second biggest value of T respectively, and m is a constant.

$$p_i = \lambda \cdot argmax_{j=1,2\cdots,C}(T_{ij}), \tag{15}$$

$$q_i = argmax_{j \neq p_i \cap 1,2 \cdots, C}(T_{ij}), \tag{16}$$

Membership Neutrosophic values are defined by follow formulas:

$$T_{ij} = \frac{\varpi_2 \varpi_3 (x_i - c_j)^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C} (x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - c_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \tag{17}$$

$$F_{ij} = \frac{\varpi_1 \varpi_3 \delta^{-(\frac{2}{m-1})}}{\sum_{j=1}^{C} (x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - c_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \tag{18}$$

$$I_{ij} = \frac{\varpi_1 \varpi_2 (x_i - c_{imax})^{-(\frac{2}{m-1}}}{\sum_{j=1}^{C} (x_i - c_j)^{-(\frac{2}{m-1})} + (x_i - c_{imax})^{-(\frac{2}{m-1})} + \delta^{-(\frac{2}{m-1})}}, \tag{19}$$

with $\boldsymbol{i = 1, 2 \cdots, N}$

$$c_j = \frac{\sum_{i=1}^{N} (\varpi_1 T_{ij})^m x_i}{\sum_{i=1}^{N} (\varpi_1 T_{ij})^m}, \tag{20}$$

$$J_{NCM}(T, F, I, c) = \sum_{i=1}^{N} (\varpi_1 T_{ij})^m (x_i - c_j)^2 + \sum_{i=1}^{N} (\varpi_2 F_i)^m (x_i - \overline{c}_{imax})^2 + \delta^2 \sum_{i=1}^{N} (\varpi_3 I_i)^m, \tag{21}$$

The separation between classes is performed by iteration optimizing objective function, that is based on updating the Neutrosophic membership values (T_{ij},F_i, I_i), the centers c_j , and $\overline{c}_{imax}$ according to the equations defined above. The loop stop when $\| T_{ij}^{(k+1)} - T_{ij}^{(k)} \| < \epsilon$ with ϵ is condition check and k is step.

For nonlinear clustering problem an extended Method have been proposed called Kernel NCMA in which we use a function kernel K, $K(x_i, z_j)$ instead of$(x_i - z_j)$, such as $K(x_i, \overline{c}_{imax})$ in place of $x_i - \overline{c}_{imax}$. The NCMA can be summarized as follow :

Algorithm 2 KNCM algorithm

Assign each data into the class with the largest TM
Choose kernel function and its parameters
Initialize $T^{(0)}$, $F^{(0)}$, $I^{(0)}$, $C, m, \delta, \epsilon, \varpi_1, \varpi_2, \varpi_3$ parameters
While $\| T_{ij}^{(k+1)} - T_{ij}^{(k)} \| < \epsilon$ **do**
Calculate the centers vectors $c^{(k)}$ at k ste
Compute the $\overline{c}_{imax}$ using the clusters centers with the largest and second largest value of T_{ij}
Update $T_{ij}(k)$ to $T_{ij}(k+1)$, $F_{ij}(k)$ to $T_{ij}(k+1)$, and $I_{ij}(k)$ to $I_{ij}(k+1)$
End while

NCM and KNCM as mentioned by authors may handle veracity in data such as outliers and noise using their new objective function. And then possibility to deal with raw data in modeling phase instead while data cleaning phase.

3.2.2. Neutrosophic Hierarchical Clustering

Agglomerative Hierarchical Clustering Algorithm [17]: First, every SVNSs A_k with $(k = 1, \cdots, n)$ considered as single cluster. In a loop, until we get a single cluster of size n, the SVNSs A_k the SVNS are then compared to each other and are merged into a single group based on the closest pair of groups (with the smallest distance), based on a weighted distance (Hamming distance or Euclidean distance). At each stage, only two clusters can be merged and they cannot be separated once merged. The center of each cluster is recalculated using the arithmetic mean of the SVNS offered to the cluster. The distance between the centers of each group is considered as the distance between two groups.

Algorithm 3 Agglomerative Hierarchical Clustering algorithm

Let us consider a collection of n SVNSs $A_k(k = 1, \cdots, n)$
Assign each of the n SVNSs $A_k(k = 1, \cdots, n)$ to a single cluster
While All A_k clustered into a single cluster of size n **do**
SVNSs $A_k(k = 1, \cdots, n)$ are then compared among themselves and are merged them into a single
Cluster according to the closest (with smaller distance) pair of clusters, based on a weighted distance
(Hamming distance or Euclidean distance)
End while

Table 1. List of major contributions on machine learning algorithms in Neutrosophic environment.

Authors	Title	Reference	Publisher
Salama, A. A., Eisa, M., ELhafeez, S. A., Lotfy, M. M. (2015)	Review of recommender systems algorithms utilized in social networks based e-Learning systems neutrosophic system	[18]	Neutrosophic Sets and Systems 8 : 32-40
Ansari, A. Q., Biswas, R., Aggarwal, S. (2013)	Neutrosophic classifier: An extension of fuzzy classifer	[19]	Applied Soft Computing, 13(1), 563-573
Zhang, M., Zhang, L., Cheng, H. D. (2010)	A neutrosophic approach to image segmentation based on watershed method	[20]	Signal Processing, 90(5), 1510-1517
Zhang, X., Bo, C., Smarandache, F., Dai, J. (2018)	New inclusion relation of neutrosophic sets with applications and related lattice structure	[21]	International Journal of Machine Learning and Cybernetics, 9, 1753-1763
Mondal, K. A. L. Y. A. N., Pramanik, S. U. R. A. P. A. T. I., Giri, B. C. (2016)	Role of neutrosophic logic in data mining. New Trends in Neutrosophic Theory and Application	[22]	Pons Editions, Brussels, 15-23.
Sengur, A., Guo, Y. (2011)	Color texture image segmentation based on neutrosophic set and wavelet transformation	[23]	Computer Vision and Image Understanding,115(8), 1134-1144
Akbulut, Y., engr, A., Guo, Y., Smarandache, F. (2017)	A novel neutrosophic weighted extreme learning machine for imbalanced data set	[24]	Symmetry, 9(8), 142
Kraipeerapun, P., Fung, C. C., Wong, K. W. (2007 August)	Ensemble neural networks using interval neutrosophic sets and bagging	[25]	In Third International Conference on Natural Computation (ICNC 2007) (Vol. 1, pp. 386-390). IEEE
Kavitha, B., Karthikeyan, S., Maybell, P. S(2012)	An ensemble design of intrusion detection system for handling uncertainty using Neutrosophic Logic Classifier	[26]	Knowledge-Based Systems, 28, 88-96

Ye, J. (2014).	Single-valued neutrosophic minimum spanning tree and its clustering method	[27]	Journal of intelligent Systems, 23(3), 311-324
Thanh, N. D., Ali, M. (2017, July)	Neutrosophic recommender system for medical diagnosis based on algebraic similarity measure and clustering	[28]	In 2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE) (pp. 1-6). IEEE
Akbulut, Y., engr, A., Guo, Y., Polat, K. (2017)	KNCM: Kernel neutrosophic c-means clustering	[10]	Applied Soft Computing, 52, 714-724
Kraipeerapun, P., Fung, C. C., Wong, K. W. (2006)	Multiclass classification using neural networks and interval neutrosophic sets	[29]	World Scientific and Engineering Academy and Society (WSEAS)
Ali, M., Khan, M., Tung, N. T. (2018)	Segmentation of dental X-ray images in medical imaging using neutrosophic orthogonal matrices	[30]	Expert Systems with Applications, 91, 434-441
Long, H. V., Ali, M., Khan, M., Tu, D. N. (2019)	A novel approach for fuzzy clustering based on neutrosophic association matrix	[31]	Computers and Industrial Engineering, 127, 687-697
Kraipeerapun, P., Fung, C. C. (2008, February)	Comparing performance of interval neutrosophic sets and neural networks with support vector machines for binary classification problems	[32]	In 2008 2nd IEEE International Conference on Digital Ecosystems and Technologies (pp. 34-37). IEEE
Thanh, N. D., Ali, M. (2017)	A novel clustering algorithm in a neutrosophic recommender system for medical diagnosis	[33]	Cognitive Computation, 9(4), 526-544
Gaber, T., Ismail, G., Anter, A., Soliman, M., Ali, M., Semary, N., Snasel, V. (2015, August)	Thermogram breast cancer prediction approach based on Neutrosophic sets and fuzzy c-means algorithm	[34]	In 2015 37th Annual International Conference of the IEEE Engineering in Medicine and Biology Society (EMBC) (pp. 4254-4257). IEEE
Ye, J. (2017)	Single-valued neutrosophic clustering algorithms based on similarity measures	[35]	Journal of Classification, 34(1), 148-162
Tuan, T. M., Chuan, P. M., Ali, M., Ngan, T. T., Mittal, M. (2018)	Fuzzy and neutrosophic modeling for link prediction in social networks	[36]	Evolving Systems, 1-6
Ju, W., Cheng, H. D. (2008, December)	Discrimination of outer membrane proteins using reformulated support vector machine based on neutrosophic set	[37]	In 11th Joint International Conference on Information Sciences. Atlantis Press
Shan, J., Cheng, H. D., Wang, Y. (2012)	A novel segmentation method for breast ultrasound images based on neutrosophic lmeans clustering	[38]	Medical physics, 39(9), 5669-5682
Basha, S. H., Abdalla, A. S., Hassanien, A. E. (2016, December)	GNRCS: hybrid classification system based on neutrosophic logic and genetic algorithm	[39]	In 2016 12th International Computer Engineering Conference (ICENCO) (pp. 53-58). IEEE
Kraipeerapun, P., Fung, C. C., Wong, K. W. (2007)	Uncertainty assessment using neural networks and interval neutrosophic sets for multiclass classification problems	[40]	WSEAS Transactions on Computers, 6(3)
Dhingra, G., Kumar, V., Joshi, H. D. (2019)	A novel computer vision based neutrosophic approach for leaf disease identification and classification	[41]	Measurement, 135, 782-794
Rashno, E., Akbari, A., Nasersharif, B. (2019)	A Convolutional Neural Network model based on Neutrosophy for Noisy Speech Recognition	[42]	arXiv preprint arXiv:1901.10629

4. Discussions

4.1. Research trends and open issues

Hybridization between Neutrosophic and machine learning algorithms, have also been studied. In supervision learning, Akbulut and al. [14] introduced intuitive supervised learning method called Neutrosophic-k-NN Classifier K-Nearest Neighbor (K-NN). Due to its results as a powerful machine learning methods, several tries to map SVM in Neutrosophic, Ju and al. [15] proposed Neutrosophic-support vector machines (N-SVM). In [32], authors Compared performance of interval neutrosophic sets and neural networks with support vector machines for binary classification problems. Ju and al [37] reformulated SVM, based on neutrosophic set, to discriminate outer membrane proteins using reformulated support vector machine based on neutrosophic set. In recent years, Artificial neural networks (ANN) has recognized huge advances, which explain many attempts of hybridization between ANN and Neutrosophic, Kraipeerapun and al. [40] demonstrated how to assess uncertainty using neural networks and interval neutrosophic sets for multi-class classification problems, then its application on multi-class classification problems [29], afterward, for more robustness ensemble neural networks using interval neutrosophic sets and bagging [25].

Likewise, in unsupervised learning, Alsmadi and al. [7] introduced a hybrid Fuzzy C-Means and Neutrosophic for jaw lesions segmentation. Inspired from fuzzy c-means and the neutrosophic set framework, Guo and al. [9] proposed a new clustering algorithm, neutrosophic c-means (NCM), for uncertain data cluster-ing. Akbulu and al. [10] developed KNCM: Kernel Neutrosophic c-Means Clustering, neutrosophic c-means (NCM), in order to alleviate the limitations of the popular fuzzy c-means (FCM) clustering algorithm by introducing a new objective function which contains two types of rejection. To deal with indeterminacy, Qureshi and al. [11] improved the Method for Image Segmentation Using K-Means Clustering with Neutrosophic Logic. Ye and al. [35] proposed Single-valued neutrosophic clustering algorithms based on similarity measures. Akhtar and al. [8] applied K-mean algorithm in Neutrosophics environment for Image Segmentation, Gaber and al. [34] to predict thermogram breast cancer, and Shan and al. [38] use neutrosophic l-means clustering to breast ultrasound images based.

Conversely, in reinforcement learning, we haven't find any resources about mixture between the both approaches, because this type of algorithms of reinforcement is under development, to be subject of hybridization.

4.2. Taxonomy of Neutrosophic Machine learning

The trends also involve the question of where machine learning areas to apply Neutrosophic, whether to it is more appropriate to employ instead of crisp number the SVN numbers. Hence, we have classified different Neutrosophic machine learning algorithms. Below a summarizing of all Neutrosophic Learning Methods and algorithms, according to standard taxonomy of machine learning.

- Supervised (inductive) learning (training data includes desired outputs)
    - Prediction : (Regression) to predict continuous values
        - Neutrosophic simple linear regression
    - Classification (discrete labels) : predict categorical values
        - Neutrosophic-k-NN [14]
        - Neutrosophic-Support Vector Machines (N-SVM)[15], [32],[37]
        - Neutrosophy-Artificial neural networks (N-ANN)[40], [29]
        - Neutrosophy-Ensemble neural networks, Bagging [25]
- Unsupervised learning (training data does not include desired outputs)
    - Clustering
        - Neutrosophic C-Means (NCM) [7], [9], [11], [35], [8], [38], [34]
        - Kernel Neutrosophic c-Means(KNCM) [10]
    - Neutrosophic Hierarchical Clustering
        - Neutrosophic Agglomerative Hierarchical Clustering [17]
        - Neutrosophic Divisive Hierarchical Clustering
    - Finding association (in features)
    - Dimension reduction
- Neutrosophic semi-supervised learning : Neutrosophic Semi-supervised learning (training data includes a few desired outputs
- Neutrosophic Reinforcement learning : Learning from sequential data
    - Q-Learning
    - State-Action-Reward-State-Action (SARSA)
    - Deep Q Network (DQN)
    - Deep Deterministic Policy Gradient (DDPG)

5. Conclusions

In this paper, we have explored how Neutrosophic contributes to enhance machine learning algorithms generally and how to modeling and exploit information's imperfection such as uncertainty as a source of information, not a kind of noises. We tried to cover hybrid approaches. However, it is still several machine learning algorithms to map to Neutrosophic environment, demonstrate the utility of Neutrosophic with machine learning to tackle real world challenges.

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Neutrosophic Bipolar Vague Set and its Application to Neutrosophic Bipolar Vague Graphs

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Abstract: A bipolar model is a significant model wherein positive data revels the liked object, while negative data speaks the disliked object. The principle reason for analysing the vague graphs is to demonstrate the stability of few properties in a graph, characterized or to be characterized in using vagueness. In this present research article, the new concept of neutrosophic bipolar vague sets are initiated. Further, its application to neutrosophic bipolar vague graphs are introduced. Moreover, some remarkable properties of strong neutrosophic bipolar vague graphs, complete neutrosophic bipolar vague graphs and complement neutrosophic bipolar vague graphs are explored and the proposed ideas are outlined with an appropriate example

Keywords: Neutrosophic bipolar vague set, Neutrosophic bipolar vague graphs, Complete neutrosophic bipolar vague graph, Strong neutrosophic bipolar vague graph.

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1. Introduction

Fuzzy set theory richly contains progressive frameworks comprising of data with various degrees of accuracy. Vague sets are first investigated by Gau and Buehrer [30] which is an extension of fuzzy set theory. Various issues in real-life problems have fluctuations, one has to handle these vulnerabilities, vague set is introduced. Vague sets are regarded as a special case of context dependent fuzzy sets and it is applicable in real-time systems consisting of information with multiple levels of precision. So as to deal with the uncertain and conflicting data, the neutrosophic set is presented by the creator Smarandache and studied widely about it [13, 21, 28, 31, 41, 42, 4, 5, 43, 44, 22, 23, 45]. Neutrosophic sets are the more generalized sets, one can manage with uncertain informations in a more successful way with a progressive manner when appeared differently in relation to fuzzy sets. It have the greater adaptability, accuracy and similarity to the framework when contrasted with past existing fuzzy models. The neutrosophic set has three completely independent parts, which are truth-membership degree, indeterminacy-membership degree and falsity-membership degree with the sum of these values lies between **0** and **3**; therefore, it is applied to many different areas, such as algebra [32, 33] and decision-making problems (see [46] and references therein).

Bipolar fuzzy sets are extension of fuzzy sets whose membership degree ranges from $[-\mathbf{1},\mathbf{1}]$. Themembership degree $(\mathbf{0},\mathbf{1}]$ represents that an object satisfies a certain property whereas the membership degree $[-\mathbf{1},\mathbf{0})$ represents that the element satisfies the implicit counter-property. The positive information indicates that the consideration to be possible and negative information indicates that the consideration is granted to be impossible. Notable that bipolar fuzzy sets and vague sets appear to be comparative, but they are completely different sets. Even though both sets handle with incomplete data, they will not adapt the indeterminate or inconsistent information which appears in many domains like decision support systems. Many researchers pay attention to the development of neutrosophic and bipolar neutrosophic graphs [39, 40]. For example, in [17], the authors studied neutrosophic soft topological K-algebras. In [48], complex neutrosophic graphs are developed. Bipolar single valued neutrosophic graphs are established in [25]. Bipolar neutrosophic sets and its application to incidence graphs are discussed in [15]. In [16], bipolar neutrosophic graphs are established.

Recently, a variety of decision making problems are based on two-sided bipolar judgements on a positive side and a negative side. Nowadays bipolar fuzzy sets are playing a substantial role in chemistry, economics, computer science, engineering, medicine and decision making problems (for more details see [27, 28, 31, 34, 38, 46] and references therein). Akram [8] introduced bipolar fuzzy graphs and discuss its various properties and several new concepts on bipolar neutrosophic graphs and bipolar neutrosophic hypergraphs have been studied in [7] and references therein. In [4], he established the certain notions including strong neutrosophic soft graphs and complete neutrosophic soft graphs. The author Shawkat Alkhazaleh introduces the concept of neutrosophic vague set theory [6]. The authors [3] introduces the concept of neutrosophic vague soft expert set which is a combination of neutrosophic vague set and soft expert set to improve the reasonability of decision making in reality. It is remarkable that the Definition 2.6 in [37] has a flaw and it not defined in a proper manner. We focussed on to redefine that definition in a proper way and explained with an example and also we applied to neutrosophic bipolar vague graphs. Motivation of the mentioned works as earlier [10], we mainly contribute the definition of neutrosophic bipolar vague set is redefined. In addition, it is applied to neutrosophic bipolar vague graphs and strong neutrosophic bipolar vague graphs. The developed results will find an application in NBVGs and also in decision making. The objectives in this work as follows:

- Newly defined the neutrosophic bipolar vague set
- Introduce the operations like union and intersection with example in section 2.
- In section 3, neutrosophic bipolar vague graphs are developed with an example. Further, the concepts of neutrosophic bipolar vague subgraph, adjacency, path, connectedness and degree of neutrosophic bipolar vague graph are evolved.
- Further we presented some remarkable properties of strong neutrosophic bipolar vague graphs in section 5, followed by a remark by comparing other types of bipolar graphs. The obtained results will improve the existing result [37].

2. Preliminaries

Definition 2.1 *[18] A vague set A on a non empty set X is a pair (T_A, F_A), where $T_A\colon X \to [0,1]$ and $F_A\colon X \to [0,1]$are true membership and false membership functions, respectively, such that*

$$0 \le T_A(x) + F_A(y) \le 1 \text{ for any } x \in X.$$

Let X and Y be two non-empty sets. A vague relation R of X to Y is a vague set R on $X \times Y$ that is $R = (T_R, F_R)$, where $T_R: X \times Y \to [0,1], F_R: X \times Y \to [0,1]$ which satisfies the condition:

$$0 \le T_R(x,y) + F_R(x,y) \le 1 \text{ for any } x \in X.$$

Let $G = (V, E)$ be a graph. A pair $G = (J, K)$ is called a vague graph on G^* or a vague graph where $J = (T_J, F_J)$ is a vague set on V and $K = (T_K, F_K)$ is a vague set on $E \subseteq V \times V$ such that for each $xy \in E$,

$$T_K(xy) \le (T_J(x) \wedge T_J(y)) \text{ and } F_K(xy) \ge (T_J(x) \vee F_J(y)).$$

Definition 2.2 *[4] A Neutrosophic set A is contained in another neutrosophic set B, (i.e) $A \subseteq B$ if $\forall x \in X, T_A(x) \le T_B(x), I_A(x) \ge I_B(x)$ and $F_A(x) \ge F_B(x)$.*

Definition 2.3 *[27, 30] Let X be a space of points (objects), with a generic elements in X denoted by x. A single valued neutrosophic set (SVNS) A in X is characterized by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$ and falsity-membership-function $F_A(x)$.*

For each point x in X, $T_A(x), F_A(x), I_A(x) \in [0,1], A = \{\langle x, T_A(x), F_A(x), I_A(x)\rangle\}$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 2.4 *[9] A neutrosophic graph is defined as a pair $G^* = (V, E)$ where*

(i) $V = \{v_1, v_2, .., v_n\}$ such that $T_1 = V \to [0,1]$, $I_1 = V \to [0,1]$ and $F_1 = V \to [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$$

(ii) $E \subseteq V \times V$ where $T_2 = E \to [0,1]$, $I_2 = E \to [0,1]$ and $F_2 = E \to [0,1]$ are such that

$$T_2(uv) \le \{T_1(u) \wedge T_1(v)\},$$
$$I_2(uv) \le \{I_1(u) \wedge I_1(v)\},$$
$$F_2(uv) \le \{F_1(u) \vee F_1(v)\},$$
$$\text{and } 0 \le T_2(uv) + I_2(uv) + F_2(uv) \le 3, \forall uv \in E.$$

Definition 2.5 *[46] A bipolar neutrosophic set A in X is defined as an object of the form*

$A = \{< x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x) >: x \in X\}$, where $T^P, I^P, F^P: X \to [0,1]$ and $T^N, I^N, F^N: X \to [-1,0]$ The Positive membership degree $T^P(x), I^P(x), F^P(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ corresponding to a bipolar neutrosophic set A and the negative membership degree $T^N(x), I^N(x), F^N(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ to some implicit counter-property corresponding to a bipolar neutrosophic set A.

Definition 2.6 *[46] Let X be a non-empty set. Then we call* $A = \{\langle x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x)\rangle, x \in X\}$a bipolar single valued neutrosophic relation on X such that $T_A^P(x,y) \in [0,1], I_A^P(x,y) \in [0,1], F_A^P(x,y) \in [0,1]$ and $T_A^N(x,y) \in [-1,0], I_A^N(x,y) \in [-1,0], F_A^N(x,y) \in [-1,0]$.

Definition 2.7 *[46] Let $A = (T_P^A, I_P^A, F_P^A, T_N^A, I_N^A, F_N^A)$ and $B = (T_P^B, I_P^B, F_P^B, T_N^B, I_N^B, F_N^B)$ be bipolar single valued neutrosophic set on X. If $B = (T_P^B, I_P^B, F_P^B, T_N^B, I_N^B, F_N^B)$ is a bipolar single valued neutrosophic relation on $A = (T_P^A, I_P^A, F_P^A, T_N^A, I_N^A, F_N^A)$ then*

$$T_B^P(xy) \le (T_A^P(x) \wedge T_A^P(y)), T_B^N(xy) \ge (T_A^N(x) \vee T_A^N(y))$$

$$I_B^P(xy) \geq (I_A^P(x) \vee I_A^P(y)), I_B^N(xy) \leq (I_A^N(x) \wedge I_A^N(y))$$
$$F_B^P(xy) \geq (F_A^P(x) \vee F_A^P(y)), F_B^N(xy) \leq (F_A^N(x) \wedge F_A^N(y))$$

A bipolar single valued neutrosophic relation B on X is called symmetric if $T_B^P(xy) = T_B^P(yx), I_B^P(xy) = I_B^P(yx), F_B^P(xy) = F_B^P(yx)$ and $T_B^N(xy) = T_B^N(yx), I_B^N(xy) = I_B^N(yx), F_B^N(xy) = F_B^N(yx)$ for all $xy \in X$.

Definition 2.8 *[6] A neutrosophic vague set A_{NV} (NVS in short) on the universe of discourse X written as* $A_{NV} = \{\langle x, \hat{T}_{A_{NV}}(x), \hat{I}_{A_{NV}}(x), \hat{F}_{A_{NV}}(x)\rangle, x \in X\}$ whose truth-membership, indeterminacy membership and falsity-membership function is defined as $\hat{T}_{A_{NV}}(x) = [\hat{T}^-(x), \hat{T}^+(x)], [\hat{I}^-(x), \hat{I}^+(x)], [\hat{F}^-(x), \hat{F}^+(x)]$, where $T^+(x) = 1 - F^-(x), F^+(x) = 1 - T^-(x)$, and $0 \leq T^-(x) + I^-(x) + F^-(x) \leq 2$.

Definition 2.9 *[20] The complement of NVS A_{NV} is denoted by A_{NV}^c and it is defined by*

$$\hat{T}_{A_{NV}}^c(x) = [1 - T^+(x), 1 - T^-(x)],$$
$$\hat{I}_{A_{NV}}^c(x) = [1 - I^+(x), 1 - I^-(x)],$$
$$\hat{F}_{A_{NV}}^c(x) = [1 - F^+(x), 1 - F^-(x)],$$

Definition 2.10 *[6] Let A_{NV} and B_{NV} be two NVSs of the universe U. If for all $u_i \in U, \hat{T}_{A_{NV}}(u_i) =$* $\hat{T}_{B_{NV}}(u_i), \hat{I}_{A_{NV}}(u_i) = \hat{I}_{B_{NV}}(u_i), \hat{F}_{A_{NV}}(u_i) = \hat{F}_{B_{NV}}(u_i)$ then the NVS A_{NV} are included by B_{NV}, denoted by $A_{NV} \subseteq B_{NV}$ where $1 \leq i \leq n$.

Definition 2.11 *[6] The union of two NVSs A_{NV} and B_{NV} is a NVSs, C_{NV}, written as $C_{NV} = A_{NV} \cup B_{NV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by*

$$\hat{T}_{C_{NV}}(x) = [(\hat{T}_{A_{NV}}^-(x) \vee \hat{T}_{B_{NV}}^-(x)), (\hat{T}_{A_{NV}}^+(x) \vee \hat{T}_{B_{NV}}^+(x))]$$
$$\hat{I}_{C_{NV}}(x) = [(\hat{I}_{A_{NV}}^-(x) \wedge \hat{I}_{B_{NV}}^-(x)), (\hat{I}_{A_{NV}}^+(x) \wedge \hat{I}_{B_{NV}}^+(x))]$$
$$\hat{F}_{C_{NV}}(x) = [(\hat{F}_{A_{NV}}^-(x) \wedge \hat{F}_{B_{NV}}^-(x)), (\hat{F}_{A_{NV}}^+(x) \wedge \hat{F}_{B_{NV}}^+(x))]$$

Definition 2.12 *[6] The intersection of two NVSs A_{NV} and B_{NV} is a NVSs C_{NV}, written as $C_{NV} = A_{NV} \cap B_{NV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by*

$$\hat{T}_{C_{NV}}(x) = [(\hat{T}_{A_{NV}}^-(x) \wedge \hat{T}_{B_{NV}}^-(x)), (\hat{T}_{A_{NV}}^+(x) \wedge \hat{T}_{B_{NV}}^+(x))]$$
$$\hat{I}_{C_{NV}}(x) = [(\hat{I}_{A_{NV}}^-(x) \vee \hat{I}_{B_{NV}}^-(x)), (\hat{I}_{A_{NV}}^+(x) \vee \hat{I}_{B_{NV}}^+(x))]$$
$$\hat{F}_{C_{NV}}(x) = [(\hat{F}_{A_{NV}}^-(x) \vee \hat{F}_{B_{NV}}^-(x)), (\hat{F}_{A_{NV}}^+(x) \vee \hat{F}_{B_{NV}}^+(x))]$$

Definition 2.13 *[39] Let $G^* = (V, E)$ be a graph. A pair $G = (J, K)$ is called a neutrosophic vague graph (NVG) on G^* or a neutrosophic graph where $J = (\hat{T}_J, \hat{I}_J, \hat{F}_J)$ is a neutrosophic vague set on V and $K = (\hat{T}_K, \hat{I}_K, \hat{F}_K)$ is a neutrosophic vague set $E \subseteq V \times V$ where*

(1)$V = \{v_1, v_2, \ldots, v_n\}$ such that $T_J^-: V \to [0,1], I_J^-: V \to [0,1], F_J^-: V \to [0,1]$ which satisfies the condition $F_J^- = [1 - T_J^+]$

$T_J^+: V \to [0,1], I_J^+: V \to [0,1], F_J^+: V \to [0,1]$ which satisfies the condition $F_J^+ = [1 - T_1^-]$

denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i \in V$, and

$$0 \leq T_J^-(v_i) + I_J^-(v_i) + F_J^-(v_i) \leq 2.$$
$$0 \leq T_J^+(v_i) + I_J^+(v_i) + F_J^+(v_i) \leq 2.$$

(2) $E \subseteq V \times V$ where

$$T_K^-: V \times V \to [0,1], I_K^-: V \times V \to [0,1], F_K^-: V \times V \to [0,1]$$
$$T_K^+: V \times V \to [0,1], I_K^+: V \times V \to [0,1], F_K^+: V \times V \to [0,1]$$

denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i, v_j \in E$ respectively and such that

$$0 \le T_K^-(v_i) + I_K^-(v_i) + F_K^-(v_i) \le 2.$$
$$0 \le T_K^+(v_i) + I_K^+(v_i) + F_K^+(v_i) \le 2.$$

such that

$$T_K^-(xy) \le \{T_J^-(x) \wedge T_J^-(y)\}$$
$$I_K^-(xy) \le \{I_J^-(x) \wedge I_J^-(y)\}$$
$$F_K^-(xy) \le \{F_J^-(x) \vee F_J^-(y)\},$$

similarly

$$T_K^+(xy) \le \{T_J^+(x) \wedge T_J^+(y)\}$$
$$I_K^+(xy) \le \{I_J^+(x) \wedge I_J^+(y)\}$$
$$F_K^+(xy) \le \{F_J^+(x) \vee F_J^+(y)\}.$$

Example 2.14 *Consider a neutrosophic vague graph* $G = (J, K)$ *such that* $J = \{a, b, c\}$ *and* $K = \{ab, bc, ca\}$ *defined by*

$$\hat{a} = T[0.5,0.6], I[0.4,0.3], F[0.4,0.5], \hat{b} = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6],$$
$$\hat{c} = T[0.4,0.4], I[0.5,0.3], F[0.6,0.6]$$
$$a^- = (0.5,0.4,0.4), b^- = (0.4,0.7,0.4), c^- = (0.4,0.5,0.6)$$
$$a^+ = (0.6,0.3,0.5), b^+ = (0.6,0.3,0.6), c^+ = (0.4,0.3,0.6)$$

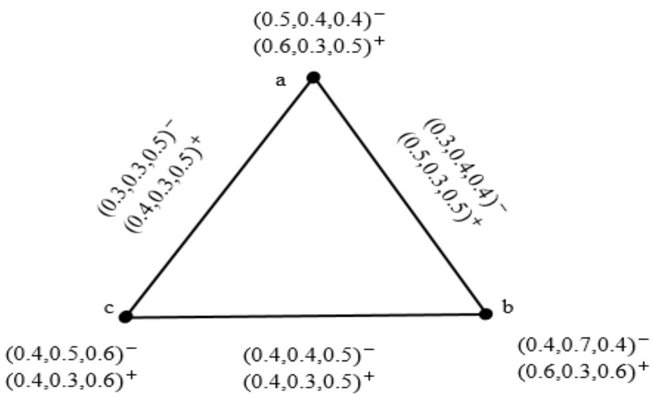


Figure 1neutrosophic vague graph

3. Neutrosophic Bipolar Vague Set

In this section, the definition of NBVS, complement of NBVS, operations like union, intersection are elaborated with an example.

Definition 3.1 *In a universe of discourse* X, *the neutrosophic bipolar vague set (NBVS), denoted as* A_{NBVS} *represented as,*

$$A_{NBV} = \{\langle x, \hat{T}^P_{A_{NBV}}(x), \hat{I}^P_{A_{NBV}}(x), \hat{F}^P_{A_{NBV}}(x), \hat{T}^N_{A_{NBV}}(x), \hat{I}^N_{A_{NBV}}(x), \hat{F}^N_{A_{NBV}}(x)\rangle, x \in X\}$$

whose truth-membership, indeterminacy membership and falssity-membership function is expanded as

$$\hat{T}^{P}_{A_{NBV}}(x) = [(T^{-})^{P}(x), (T^{+})^{P}(x)], \hat{I}^{P}_{A_{NBV}}(x) = [(I^{-})^{P}(x), (I^{+})^{P}(x)], \hat{F}^{P}_{A_{NBV}}(x) = [(F^{-})^{P}(x), (F^{+})^{P}(x)],$$

where $(T^{+})^{P}(x) = 1 - (F^{-})^{P}(x), (F^{+})^{P}(x) = 1 - (T^{-})^{P}(x)$, and provided that,

$$0 \leq (T^{-})^{P}(x) + (I^{-})^{P}(x) + (F^{-})^{P}(x) \leq 2.$$

Also

$$\hat{T}^{N}_{A_{NBV}}(x) = [(T^{-})^{N}(x), (T^{+})^{N}(x)], \hat{I}^{N}_{A_{NBV}}(x) = [(I^{-})^{N}(x), (I^{+})^{N}(x)], \hat{F}^{N}_{A_{NBV}}(x) = [(F^{-})^{N}(x), (F^{+})^{N}(x)],$$

where $(T^{+})^{N}(x) = -1 - (F^{-})^{N}(x), (F^{+})^{N}(x) = -1 - (T^{-})^{N}(x)$,

and provided that,

$$0 \geq (T^{-})^{N}(x) + (I^{-})^{N}(x) + (F^{-})^{N}(x) \geq -2.$$

Example 3.2 *Let* $U = \{x_1, x_2, x_3\}$ *be a set of universe we define the NBV set* A_{NBV} *as follows*

$$A_{NBV} = \{\frac{x_1}{[0.3,0.6]^P, [0.5,0.5]^P, [0.4,0.7]^P, [-0.3,-0.5]^N, [-0.4,-0.4]^N, [-0.5,-0.7]^N},$$

$$\frac{x_2}{[0.4,0.6]^P, [0.4,0.6]^P, [0.4,0.6]^P, [-0.4,-0.4]^N, [-0.5,-0.5]^N, [-0.6,-0.6]^N},$$

$$\frac{x_3}{[0.3,0.7]^P, [0.6,0.4]^P, [0.3,0.7]^P, [-0.4,-0.6]^N, [-0.5,-0.6]^N, [-0.4,-0.6]^N}\}$$

Definition 3.3 *IN NBVS, the complement of* A^{c}_{NBV} *be expanded as,*

$$(\hat{T}^{c}_{A_{NBV}}(x))^{P} = \{(1 - T^{+}(x))^{P}, (1 - T^{-}(x))^{P}\}, (\hat{T}^{c}_{A_{NBV}}(x))^{N} = \{(-1 - T^{+}(x))^{N}, (-1 - T^{-}(x))^{N}\}$$

$$(\hat{I}^{c}_{A_{NBV}}(x))^{P} = \{(1 - I^{+}(x))^{P}, (1 - I^{-}(x))^{P}\}, (\hat{I}^{c}_{A_{NBV}}(x))^{N} = \{(-1 - I^{+}(x))^{N}, (-1 - I^{-}(x))^{N}\}$$

$$(\hat{F}^{c}_{A_{NBV}}(x))^{P} = \{(1 - F^{+}(x))^{P}, (1 - F^{-}(x))^{P}\}, (\hat{F}^{c}_{A_{NBV}}(x))^{N} = \{(-1 - F^{+}(x))^{N}, (-1 - F^{-}(x))^{N}\}$$

Example 3.4 *Considering above example we have*

$$A_{NBV} = \{\frac{x_1}{[0.7,0.4]^P, [0.5,0.5]^P, [0.6,0.3]^P, [-0.7,-0.5]^N, [-0.6,-0.6]^N, [-0.5,-0.3]^N},$$

$$\frac{x_2}{[0.6,0.4]^P, [0.6,0.4]^P, [0.6,0.4]^P, [-0.6,-0.6]^N, [-0.5,-0.5]^N, [-0.4,-0.4]^N},$$

$$\frac{x_3}{[0.7,0.3]^P, [0.4,0.6]^P, [0.7,0.3]^P, [-0.6,-0.4]^N, [-0.5,-0.4]^N, [-0.6,-0.4]^N}\}$$

Definition 3.5 *Two NBVSs* A_{NBV} *and* B_{NBV} *of the universe* U *are said to be equal, if for all* $u_i \in U$,

$$(\hat{T}_{A_{NBV}})^{P}(u_i) = (\hat{T}_{B_{NBV}})^{P}(u_i), (\hat{I}_{A_{NBV}})^{P}(u_i) = (\hat{I}_{B_{NBV}})^{P}(u_i), (\hat{F}_{A_{NBV}})^{P}(u_i) = (\hat{F}_{B_{NBV}})^{P}(u_i)$$

and

$$(\hat{T}_{A_{NBV}})^{N}(u_i) = (\hat{T}_{B_{NBV}})^{N}(u_i), (\hat{I}_{A_{NBV}})^{N}(u_i) = (\hat{I}_{B_{NBV}})^{N}(u_i), (\hat{F}_{A_{NBV}})^{N}(u_i) = (\hat{F}_{B_{NBV}})^{N}(u_i).$$

Definition 3.6 *In the Universe* U*, two NBVSs,* A_{NBV}*,* B_{NBV} *be given as,*

$$(\hat{T}_{A_{NBV}})^{P}(u_i) \leq (\hat{T}_{B_{NBV}})^{P}(u_i), (\hat{I}_{A_{NBV}})^{P}(u_i) \geq (\hat{I}_{B_{NBV}})^{P}(u_i), (\hat{F}_{A_{NBV}})^{P}(u_i) \geq (\hat{F}_{B_{NBV}})^{P}(u_i)$$

and

$$(\hat{T}_{A_{NBV}})^{N}(u_i) \geq (\hat{T}_{B_{NBV}})^{N}(u_i), (\hat{I}_{A_{NBV}})^{N}(u_i) \leq (\hat{I}_{B_{NBV}})^{N}(u_i), (\hat{F}_{A_{NBV}})^{N}(u_i) \leq (\hat{F}_{B_{NBV}})^{N}(u_i)$$

then the NBVS $(A_{NBV})^P$ are included by $(B_{NBV})^P$, denoted by $(A_{NBV})^P \subseteq (B_{NBV})^P$ where $1 \leq i \leq n$ and $(A_{NBV})^N$ are included by $(B_{NBV})^N$, denoted by $(A_{NBV})^N \subseteq (B_{NBV})^N$ where $1 \leq i \leq n$.

Definition 3.7 *The union of two NVSs A_{NBV} and B_{NBV} is a NBVSs, C_{NBV}, written as $C_{NBV} = A_{NBV} \cup B_{NBV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NBV} and B_{NBV} by*

$$(\hat{T}_{C_{NBV}})^P(x) = [((T^-_{A_{NBV}})^P(x) \vee (T^-_{B_{NBV}})^P(x)), ((T^+_{A_{NBV}})^P(x) \vee (T^+_{B_{NBV}})^P(x))]$$
$$(\hat{I}_{C_{NBV}})^P(x) = [((I^-_{A_{NBV}})^P(x) \wedge (I^-_{B_{NBV}})^P(x)), ((I^+_{A_{NBV}})^P(x) \wedge (I^+_{B_{NBV}})^P(x))]$$
$$(\hat{F}_{C_{NBV}})^P(x) = [((F^-_{A_{NBV}})^P(x) \wedge (F^-_{B_{NBV}})^P(x)), ((F^+_{A_{NBV}})^P(x) \wedge (F^+_{B_{NBV}})^P(x))], \text{and}$$
$$(\hat{T}_{C_{NBV}})^N(x) = [((T^-_{A_{NBV}})^N(x) \wedge (T^-_{B_{NBV}})^N(x)), ((T^+_{A_{NBV}})^N(x) \wedge (T^+_{B_{NBV}})^N(x))]$$
$$(\hat{I}_{C_{NBV}})^N(x) = [((I^-_{A_{NBV}})^N(x) \vee (I^-_{B_{NBV}})^N(x)), ((I^+_{A_{NBV}})^N(x) \vee (I^+_{B_{NBV}})^N(x))]$$
$$(\hat{F}_{C_{NBV}})^N(x) = [((F^-_{A_{NBV}})^N(x) \vee (F^-_{B_{NBV}})^N(x)), ((F^+_{A_{NBV}})^N(x) \vee (F^+_{B_{NBV}})^N(x))]$$

Definition 3.8 *The intersection of two NVSs A_{NBV} and B_{NBV} is a NBVSs C_{NBV}, written as $C_{NBV} = A_{NBV} \cap B_{NBV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NBV} and B_{NBV} by*

$$(\hat{T}_{C_{NBV}})^P(x) = [((T^-_{A_{NBV}})^P(x) \wedge (T^-_{B_{NBV}})^P(x)), ((T^+_{A_{NBV}})^P(x) \wedge (T^+_{B_{NBV}})^P(x))]$$
$$(\hat{I}_{C_{NBV}})^P(x) = [((I^-_{A_{NBV}})^P(x) \vee (I^-_{B_{NBV}})^P(x)), ((I^+_{A_{NBV}})^P(x) \vee (I^+_{B_{NBV}})^P(x))]$$
$$(\hat{F}_{C_{NBV}})^P(x) = [((F^-_{A_{NBV}})^P(x) \vee (F^-_{B_{NBV}})^P(x)), ((F^+_{A_{NBV}})^P(x) \vee (F^+_{B_{NBV}})^P(x))], \text{and}$$
$$(\hat{T}_{C_{NBV}})^N(x) = [((T^-_{A_{NBV}})^N(x) \vee (T^-_{B_{NBV}})^N(x)), ((T^+_{A_{NBV}})^N(x) \vee (T^+_{B_{NBV}})^N(x))]$$
$$(\hat{I}_{C_{NBV}})^N(x) = [((I^-_{A_{NBV}})^N(x) \wedge (I^-_{B_{NBV}})^N(x)), ((I^+_{A_{NBV}})^N(x) \wedge (I^+_{B_{NBV}})^N(x))]$$
$$(\hat{F}_{C_{NBV}})^N(x) = [((F^-_{A_{NBV}})^N(x) \wedge (F^-_{B_{NBV}})^N(x)), ((F^+_{A_{NBV}})^N(x) \wedge (F^+_{B_{NBV}})^N(x))]$$

Definition 3.9 *Let U be a set of universe and let A_{NBV} and B_{NBV} be NBVSs, then the union $A_{NBV} \cap B_{NBV}$ is defined as follows:*

$$A_{NBV} = \{\frac{x_1}{[0.3,0.6]^P, [0.6,0.6]^P, [0.4,0.7]^P, [-0.4,-0.7]^N, [-0.6,-0.6]^N, [-0.3,-0.6]^N},$$
$$\frac{x_2}{[0.4,0.6]^P, [0.6,0.4]^P, [0.4,0.6]^P, [-0.5,-0.5]^N, [-0.7,-0.3]^N, [-0.5,-0.5]^N},$$
$$\frac{x_3}{[0.7,0.8]^P, [0.6,0.6]^P, [0.2,0.3]^P, [-0.5,-0.4]^N, [-0.5,-0.5]^N, [-0.6,-0.5]^N}\}$$

$$B_{NBV} = \{\frac{x_1}{[0.2,0.8]^P, [0.5,0.4]^P, [0.2,0.8]^P, [-0.5,-0.7]^N, [-0.7,-0.7]^N, [-0.3,-0.5]^N},$$
$$\frac{x_2}{[0.3,0.8]^P, [0.6,0.5]^P, [0.2,0.7]^P, [-0.5,-0.6]^N, [-0.4,-0.3]^N, [-0.4,-0.5]^N},$$
$$\frac{x_3}{[0.2,0.5]^P, [0.5,0.2]^P, [0.5,0.8]^P, [-0.5,-0.5]^N, [-0.4,-0.3]^N, [-0.5,-0.5]^N}\}$$

$$A_{NBV} \cap B_{NBV} = H_{NBV}$$
$$= \{\frac{x_1}{[0.2,0.6]^P, [0.6,0.6]^P, [0.4,0.8]^P, [-0.4,-0.7]^N, [-0.7,-0.7]^N, [-0.3,-0.6]^N},$$
$$\frac{x_2}{[0.3,0.6]^P, [0.6,0.5]^P, [0.4,0.7]^P, [-0.5,-0.5]^N, [-0.7,-0.3]^N, [-0.5,-0.5]^N},$$

$$\frac{x_3}{[0.2,0.5]^P, [0.6,0.6]^P, [0.5,0.8]^P, [-0.5,-0.4]^N, [-0.5,-0.5]^N, [-0.6,-0.5]^N}\}$$

4 Neutrosophic Bipolar Vague graphs

In this section, neutrosophic bipolar vague graphs are defined. The concepts of neutrosophic bipolar vague subgraph, adjacency, path, connectedness and degree of neutrosophic bipolar vague graph are discussed.

Definition 4.1 *In a crisp graph $G^* = (V, E)$. A pair $G = (J, K)$ is called a neutrosophic bipolar vague graph (NBVG) on G^* or a neutrosophic bipolar vague graph where J is a neutrosophic bipolar vague set and K is a neutrosophic bipolar vague relation in G^* such that $J^P = ((\hat{T}_J)^P, (\hat{I}_J)^P, (\hat{F}_J)^P), J^N = ((\hat{T}_J)^N, (\hat{I}_J)^N, (\hat{F}_J)^N)$ is a neutrosophic bipolar vague set on V and $K^P = ((\hat{T}_K)^P, (\hat{I}_K)^P, (\hat{F}_K)^P), K^N = ((\hat{T}_K)^N, (\hat{I}_K)^N, (\hat{F}_K^N))$ is a neutrosophic Bipolar vague set $E \subseteq V \times V$ where*

(1) $V = \{v_1, v_2, \dots, v_n\}$ such that

$$(T_J^-)^P: V \to [0,1], (I_J^-)^P: V \to [0,1], (F_J^-)^P: V \to [0,1]$$

which satisfies the condition $(F_J^-)^P = [1 - (T_J^+)^P]$

$$(T_J^+)^P: V \to [0,1], (I_J^+)^P: V \to [0,1], (F_J^+)^P: V \to [0,1]$$

which satisfies the condition $(F_J^+)^P = [1 - (T_J^-)^P]$, and

$$(T_J^-)^N: V \to [-1,0], (I_J^-)^N: V \to [-1,0], (F_J^-)^N: V \to [-1,0]$$

which satisfies the condition $(F_J^-)^N = [-1 - (T_J^+)^N]$

$(T_J^+)^N: V \to [-1,0], (I_J^+)^N: V \to [-1,0], (F_J^+)^N: V \to [-1,0]$ which satisfies the condition $(F_J^+)^N = [-1 - (T_J^-)^N]$ denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i \in V$, and

$$0 \le (T_J^-)^P(v_i) + (I_J^-)^P(v_i) + (F_J^-)^P(v_i) \le 2$$
$$0 \le (T_J^+)^P(v_i) + (I_J^+)^P(v_i) + (F_J^+)^P(v_i) \le 2$$
$$0 \ge (T_J^-)^N(v_i) + (I_J^-)^N(v_i) + (F_J^-)^N(v_i) \ge -2$$
$$0 \le (T_J^+)^N(v_i) + (I_J^+)^N(v_i) + (F_J^+)^N(v_i) \ge -2.$$

(2) $E \subseteq V \times V$ where

$$(T_K^-)^P: V \times V \to [0,1], (I_K^-)^P: V \times V \to [0,1], (F_K^-)^P: V \times V \to [0,1]$$
$$(T_K^+)^P: V \times V \to [0,1], (I_K^+)^P: V \times V \to [0,1], (F_K^+)P: V \times V \to [0,1]\text{and}$$
$$(T_K^-)^N: V \times V \to [-1,0], (I_K^-)^N: V \times V \to [-1,0], (F_K^-)^N: V \times V \to [-1,0]$$
$$(T_K^+)^N: V \times V \to [-1,0], (I_K^+)^N: V \times V \to [-1,0], (F_K^+)^N: V \times V \to [-1,0]$$

denotes the degree of truth membership function, indeterminacy membership and falsity membership of the element $v_i, v_j \in E$ respectively and such that

$$0 \le (T_K^-)^P(v_i, v_j) + (I_K^-)^P(v_i, v_j) + (F_K^-)^P(v_i, v_j) \le 2$$
$$0 \le (T_K^+)^P(v_i, v_j) + (I_K^+)^P(v_i, v_j) + (F_K^+)^P(v_i, v_j) \le 2$$
$$0 \ge (T_K^-)^N(v_i, v_j) + (I_K^-)^N(v_i, v_j) + (F_K^-)^N(v_i, v_j) \ge -2$$
$$0 \ge (T_K^+)^N(v_i, v_j) + (I_K^+)^N(v_i, v_j) + (F_K^+)^N(v_i, v_j) \ge -2,$$

such that

$$(T_K^-)^P(xy) \le \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\}$$
$$(I_K^-)^P(xy) \le \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\}$$

$$(F_K^-)^P(xy) \leq \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\}$$
$$(T_K^+)^P(xy) \leq \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\}$$
$$(I_K^+)^P(xy) \leq \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\}$$
$$(F_K^+)^P(xy) \leq \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\},$$

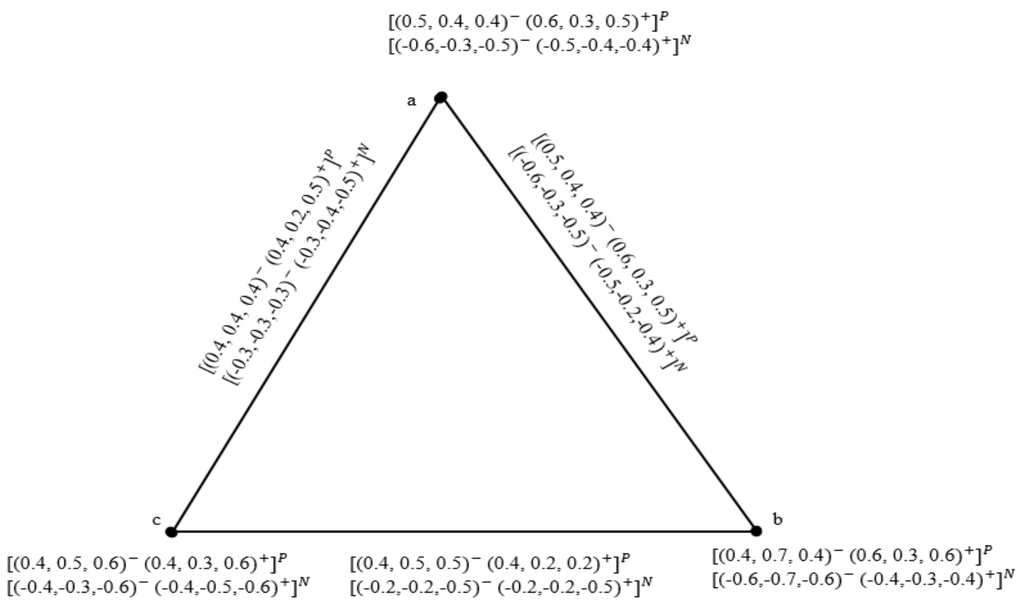
and

$$(T_K^-)^N(xy) \geq \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\}$$
$$(I_K^-)^N(xy) \geq \{(I_J^-)^N(x) \vee (I_K^-)^N(y)\}$$
$$(F_K^-)^N(xy) \geq \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\},$$
$$(T_K^+)^N(xy) \geq \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\}$$
$$(I_K^+)^N(xy) \geq \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\}$$
$$(F_K^+)^N(xy) \geq \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\}.$$

Example 4.2 *Consider a neutrosophic bipolar vague graph* $G = (J, K)$ *such that* $J = \{a, b, c\}$ *and* $K = \{ab, bc, ca\}$ *defined by*

$$(\hat{a})^P = T[0.5,0.6], I[0.4,0.3], F[0.4,0.5],$$
$$(\hat{b})^P = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6],$$
$$(\hat{c})^P = T[0.4,0.4], I[0.5,0.3], F[0.6,0.6]$$
$$(a^-)^P = (0.5,0.4,0.4), (b^-)^P = (0.4,0.7,0.4), (c^-)^P = (0.4,0.5,0.6)$$

$$(a^+)^P = (0.6,0.3,0.5), (b^+)^P = (0.6,0.3,0.6), (c^+)^P = (0.4,0.3,0.6)$$
$$(\hat{a})^N = T[-0.6,-0.5], I[-0.3,-0.4], F[-0.5,-0.4],$$
$$(\hat{b})^N = T[-0.6,-0.4], I[-0.7,-0.3], F[-0.6,-0.4],$$
$$(\hat{c})^N = T[-0.4,-0.4], I[-0.3,-0.5], F[-0.6,-0.6]$$
$$(a^-)^N = (-0.6,-0.3,-0.5), (b^-)^N = (-0.6,-0.7,-0.6), (c^-)^N = (-0.4,-0.3,-0.6)$$
$$(a^+)^N = (-0.5,-0.4,-0.4), (b^+)^P = (-0.4,-0.3,-0.4), (c^+)^P = (-0.4,-0.5,-0.6)$$



$Figure$ 2 NEUTROSOPHIC BIPOLAR VAGUE GRAPH

Definition 4.3 *A neutrosophic bipolar vague graph* $H = (J'(x), K'(x))$ *is said to be a neutrosophic bipolar vague subgraph of the NVG* $G = (J, K)$ *if* $J'(x) \subseteq J(x)$ *and* $K'(xy) \subseteq K'(xy)$, *in other words, if*

$$(\hat{T}'_J)^P(x) \le (\hat{T}_J)^P(x)$$
$$(\hat{I}'_J)^P(x) \le (\hat{I}_J)^P(x)$$
$$(\hat{F}'_J)^P(x) \le (\hat{F}_J)^P(x)\ \forall x \in V$$
$$(\hat{T}'_K)^P(xy) \le (\hat{T}_K)^P(xy)$$
$$(\hat{I}'_K)^P(xy) \le (\hat{I}_K)^P(xy)$$
$$(\hat{F}'_K)^P(xy) \le (\hat{F}_K)^P(xy), \forall xy \in E.$$

Also,

$$(\hat{T}'_J)^N(x) \ge (\hat{T}_J)^N(x)$$
$$(\hat{I}'_J)^N(x) \ge (\hat{I}_J)^N(x)$$
$$(\hat{F}'_J)^N(x) \ge (\hat{F}_J)^N(x), \forall x \in V$$

and

$$(\hat{T}'_K)^N(xy) \ge (\hat{T}_K)^N(xy)$$
$$(\hat{I}'_K)^N(xy) \ge (\hat{I}_K)^N(xy)$$
$$(\hat{F}'_K)^N(xy) \ge (\hat{F}_K)^N(xy), \forall xy \in E.$$

Definition 4.4 *The two vertices are said to be adjacent in a neutrosophic bipolar vague graph* $G = (J, K)$ *if*

$$(T_K^-)^P(xy) = \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\}$$
$$(I_K^-)^P(xy) = \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\}$$
$$(F_K^-)^P(xy) = \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\},$$
$$(T_K^+)^P(xy) = \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\}$$
$$(I_K^+)^P(xy) = \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\}$$
$$(F_K^+)^P(xy) = \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\},$$
$$(T_K^-)^N(xy) = \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\}$$
$$(I_K^-)^N(xy) = \{(I_J^-)^N(x) \vee (I_K^-)^N(y)\}$$
$$(F_K^-)^N(xy) = \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\},$$
$$(T_K^+)^N(xy) = \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\}$$
$$(I_K^+)^N(xy) = \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\}$$
$$(F_K^+)^N(xy) = \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\},$$

Here, x is the neighbour of y and vice versa, also (xy) is incident at x and y.

Definition 4.5 *In a neutrosophic bipolar vague graph* $G = (J, K)$, *a path* ρ *is meant to be a sequence of different points* $x_0, x_1, \dots, x_n$ *such an extent that*

$$(T_K^-)^P(x_{i-1}, x_1) > 0, (I_K^-)^P(x_{i-1}, x_1) > 0, (F_K^-)^P(x_{i-1}, x_1) > 0,$$
$$(T_K^+)^P(x_{i-1}, x_1) > 0, (I_K^+)^P(x_{i-1}, x_1) > 0, (F_K^+)^P(x_{i-1}, x_1) > 0,$$

and

$$(T_K^-)^N(x_{i-1}, x_1) < 0, (I_K^-)^N(x_{i-1}, x_1) < 0, (F_K^-)^N(x_{i-1}, x_1) < 0,$$
$$(T_K^+)^N(x_{i-1}, x_1) < 0, (I_K^+)^N(x_{i-1}, x_1) < 0, (F_K^+)^N(x_{i-1}, x_1) < 0,$$

for every i lies between 0 and 1. $n \le 1$ is known as the path length.. A single vertex x_i can represent as a path.

Definition 4.6 *A neutrosophic bipolar vague graph $G = (J, K)$, if every pair of vertices has at least one neutrosophic bipolar vague path between them is known as connected, otherwise it is disconnected.*

Definition 4.7 *A vertex $x_i \in V$ of neutrosophic bipolar vague graph $G = (J, K)$ is said to be isolatedvertex if there is no effective edge incident at x_i.*

Definition 4.8 *A vertex in a neutrosophic bipolar vague graph $G = (J, K)$ having exactly one neighbours is called a pendent vertex. Otherwise, it is called non-pendent vertex. An edge in a neutrosophic bipolar vague graph incident with a pendent vertex is called a pendent edge other words it is called non-pendent edge. A vertex in a neutrosophic bipolar vague graph adjacent to the pendent vertex is called an support of the pendent edge.*

Definition 4.9 *A neutrosophic bipolar vague graph $G = (J, K)$ that has neither self loops nor parallel edge is called simple neutrosophic bipolar vague graph.*

Definition 4.10 *Let $G = (J, K)$ be a neutrosophic bipolar vague graph. Then the degree of a vertex $x \in G$ is a sum of degree truth membership, sum of indeterminacy membership and sum of falsity membership of all those edges which are incident on vertex x denoted by*

$$(d(x))^P = ([(d_{T_J}^-)P(x), (d_{T_J}^+)^P(x)], [(d_{I_J}^-)^P(x), (d_{I_J}^+)^P(x)], [(d_{F_J}^-)^P(x), (d_{F_J}^+)^P(x)])$$
$$(d(x))^N = ([(d_{T_J}^-)N(x), (d_{T_J}^+)^N(x)], [(d_{I_J}^-)^N(x), (d_{I_J}^+)^N(x)], [(d_{F_J}^-)^N(x), (d_{F_J}^+)^N(x)])$$

where $(d_{T_J}^-)^P(x) = \sum_{x \neq y} (T_K^-)^P(xy)$, $(d_{T_J}^+)^P(x) = \sum_{x \neq y} (T_K^+)^P(xy)$ denotes the positive degree of truth membership vertex, $(d_{I_J}^-)^P(x) = \sum_{x \neq y} (I_K^-)^P(xy)$, $(d_{I_J}^+)^P(x) = \sum_{x \neq y} (I_K^+)^P(xy)$ denotes the positive degree of indeterminacy membership vertex, $(d_{F_J}^-)^P(x) = \sum_{x \neq y} (F_K^-)^P(xy)$, $(d_{F_J}^+)^P(x) = \sum_{x \neq y} (F_K^+)^P(xy)$ denotes the positive degree of falsity membership vertex for all $x, y \in J$.

Similarly, $(d_{T_J}^-)^N(x) = \sum_{x \neq y} (T_K^-)^N(xy)$, $(d_{T_J}^+)^N(x) = \sum_{x \neq y} (T_K^+)^N(xy)$ denotes the negative degree of truth membership vertex, $(d_{I_J}^-)^N(x) = \sum_{x \neq y} (I_K^-)^N(xy)$, $(d_{I_J}^+)^N(x) = \sum_{x \neq y} (I_K^+)^N(xy)$ denotes the negative degree of indeterminacy membership vertex, $(d_{F_J}^-)^N(x) = \sum_{x \neq y} (F_K^-)^N(xy)$, $(d_{F_J}^+)^N(x) = \sum_{x \neq y} (F_K^+)^N(xy)$ denotes the negative degree of falsity membership vertex for all $x, y \in J$.

Definition 4.11 *A neutrosophic bipolar vague graph $G = (J, K)$ is called constant if degree of each vertex is $A = (A_1, A_2, A_3)$ that is $d(x) = (A_1, A_2, A_3)$ for all $x \in V$.*

5 Strong Neutrosophic Bipolar Vague Graphs

In this section, we presented some remarkable properties of strong neutrosophic bipolar vague graphs and a remark is provided by comparing other types of bipolar graphs. Finally conclusion is given.

Definition 5.1 *A neutrosophic bipolar vague graph $G = (J, K)$ of $G^* = (V, E)$ is called strong neutrosophic bipolar vague graph if*

$$(T_K^-)^P(xy) = \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\}$$
$$(I_K^-)^P(xy) = \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\}$$
$$(F_K^-)^P(xy) = \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\},$$
$$(T_K^+)^P(xy) = \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\}$$
$$(I_K^+)^P(xy) = \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\}$$
$$(F_K^+)^P(xy) = \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\},$$

$$(T_K^-)^N(xy) = \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\}$$
$$(I_K^-)^N(xy) = \{(I_J^-)^N(x) \vee (I_K^-)^N(y)\}$$
$$(F_K^-)^N(xy) = \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\},$$
$$(T_K^+)^N(xy) = \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\}$$
$$(I_K^+)^N(xy) = \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\}$$
$$(F_K^+)^N(xy) = \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\}, \forall((xy) \in K)$$

Definition 5.2 *The complement of neutrosophic bipolar vague graph $G = (J, K)$ on G^* is a neutrosophic bipolar vague graph G^c where*

- $(J^c)^P(x) = (J)^P(x)$
- $(T_J^{-c})^P(x) = (T_J^-)^P(x), (I_J^{-c})^P(x) = (I_J^-)^P(x), (F_J^{-c})^P(x) = (F_J^-)^P(x)$ for all $x \in V$.
- $(T_J^{+c})^P(x) = (T_J^+)^P(x), (I_J^{+c})^P(x) = (I_J^+)^P(x), (F_J^{+c})^P(x) = (F_J^+)^P(x)$ for all $x \in V$.
- $(T_K^{-c})^P(xy) = \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\} - (T_K^-)^P(xy)$, $(I_K^{-c})^P(xy) = \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\} - (I_K^-)^P(xy)$

$$(F_K^{-c})^P(xy) = \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\} - (F_K^-)^P(xy) \text{ for all } (xy) \in E$$

- $(T_K^{+c})^P(xy) = \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\} - (T_K^+)^P(xy)$, $(I_K^{+c})^P(xy) = \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\} - (I_K^+)^P(xy)$

$$(F_K^{+c})^P(xy) = \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\} - (F_K^+)^P(xy) \text{ for all } (xy) \in E$$

- $(J^c)^N(x) = (J)^N(x)$
- $(T_J^{-c})^N(x) = (T_J^-)^N(x), (I_J^{-c})^N(x) = (I_J^-)^N(x), (F_J^{-c})^N(x) = (F_J^-)^N(x)$ for all $x \in V$.
- $(T_J^{+c})^N(x) = (T_J^+)^N(x), (I_J^{+c})^N(x) = (I_J^+)^N(x), (F_J^{+c})^N(x) = (F_J^+)^N(x)$ for all $x \in V$.
- $(T_K^{-c})^N(xy) = \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\} - (T_K^-)^N(xy)$

$$(I_K^{-c})^N(xy) = \{(I_J^-)^N(x) \vee (I_J^-)^N(y)\} - (I_K^-)^N(xy)$$
$$(F_K^{-c})^N(xy) = \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\} - (F_K^-)^N(xy) \text{ for all } (xy) \in E$$

- $(T_K^{+c})^N(xy) = \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\} - (T_K^+)^N(xy)$

$$(I_K^{+c})^N(xy) = \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\} - (I_K^+)^N(xy)$$
$$(F_K^{+c})^N(xy) = \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\} - (F_K^+)^N(xy) \text{ for all } (xy) \in E$$

Remark 5.3 *If $G = (J, K)$ is a neutrosophic bipolar vague graph on G^* then from above definition, it follows that G^{c^c} is given by the neutrosophic bipolar vague graph $G^{c^c} = (J^{c^c}, K^{c^c})$ on G^* where*

- $((J^c)^c)^P(x) = (J(x))^P$
- $((T_J^{-c})^c)^P(x) = (T_J^-)^P(x), ((I_J^{-c})^c)^P(x) = (I_J^-)^P(x), ((F_J^{-c})^c)^P(x) = (F_J^-)^P(x)$ for all $x \in V$.
- $((T_J^{+c})^c)^P(x) = (T_J^+)^P(x), ((I_J^{+c})^c)^P(x) = (I_J^+)^P(x), ((F_J^{+c})^c)^P(x) = (F_J^+)^P(x)$ for all $x \in V$.
- $((T_K^{-c})^c)^P(xy) = \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\} - (T_K^-)^P(xy)$

$$((I_K^{-c})^c)^P(xy) = \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\} - (I_K^-)^P(xy)$$
$$((F_K^{-c})^c)^P(xy) = \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\} - (F_K^-)^P(xy) \text{ for all } (xy) \in E$$

- $((T_K^{+c})^c)^P(xy) = \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\} - (T_K^+)^P(xy)$

$$((I_K^{+c})^c)^P(xy) = \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\} - (I_K^+)^P(xy)$$
$$((F_K^{+c})^c)^P(xy) = \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\} - (F_K^+)^P(xy) \text{ for all } (xy) \in E$$

- $((J^c)^c)^N(x) = (J(x))^N$

- $((T_J^{-c})^c)^N(x) = (T_J^-)^N(x), ((I_J^{-c})^c)^N(x) = (I_J^-)^N(x), ((F_J^{-c})^c)^N(x) = (F_J^-)^N(x)$ for all $x \in V$.
- $((T_J^{+c})^c)^N(x) = (T_J^+)^N(x), ((I_J^{+c})^c)^N(x) = (I_J^+)^N(x), ((F_J^{+c})^c)^N(x) = (F_J^+)^N(x)$ for all $x \in V$.
- $((T_K^{-c})^c)^N(xy) = \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\} - (T_K^-)^N(xy)$

$$((I_K^{-c})^c)^N(xy) = \{(I_J^-)^N(x) \vee (I_J^-)^N(y)\} - (I_K^-)^N(xy)$$

$$((F_K^{-c})^c)^N(xy) = \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\} - (F_K^-)^N(xy) \text{ for all } (xy) \in E$$

- $((T_K^{+c})^c)^N(xy) = \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\} - (T_K^+)^N(xy)$

$$((I_K^{+c})^N)^N(xy) = \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\} - (I_K^+)^N(xy)$$

$$((F_K^{+c})^c)^N(xy) = \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\} - (F_K^+)^N(xy) \text{ for all } (xy) \in E.$$

for any neutrosophic bipolar vague graph G, G^c is strong neutrosophic bipolar vague graph and $G \subseteq G^c$.

Definition 5.4 *Suppose G^c is the complement of neutrosophic bipolar vague graph G. In a strong neutrosophic bipolar vague graph G, $G \cong G^c$ then it is called self-complementary.*

Proposition 5.5 *Let $G = (J, K)$ be a strong neutrosophic bipolar vague graph if*

$$(T_K^-)^P(xy) = \{(T_J^-)^P(x) \wedge (T_J^-)^P(y)\}$$
$$(I_K^-)^P(xy) = \{(I_J^-)^P(x) \wedge (I_J^-)^P(y)\}$$
$$(F_K^-)^P(xy) = \{(F_J^-)^P(x) \vee (F_J^-)^P(y)\},$$
$$(T_K^+)^P(xy) = \{(T_J^+)^P(x) \wedge (T_J^+)^P(y)\}$$
$$(I_K^+)^P(xy) = \{(I_J^+)^P(x) \wedge (I_J^+)^P(y)\}$$
$$(F_K^+)^P(xy) = \{(F_J^+)^P(x) \vee (F_J^+)^P(y)\},$$
$$(T_K^-)^N(xy) = \{(T_J^-)^N(x) \vee (T_J^-)^N(y)\}$$
$$(I_K^-)^N(xy) = \{(I_J^-)^N(x) \vee (I_K^-)^N(y)\}$$
$$(F_K^-)^N(xy) = \{(F_J^-)^N(x) \wedge (F_J^-)^N(y)\},$$
$$(T_K^+)^N(xy) = \{(T_J^+)^N(x) \vee (T_J^+)^N(y)\}$$
$$(I_K^+)^N(xy) = \{(I_J^+)^N(x) \vee (I_J^+)^N(y)\}$$
$$(F_K^+)^N(xy) = \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\}, \forall((xy) \in K)$$

Then G is self complementary.

Proof. Let $G = (J, K)$ be a strong neutrosophic bipolar vague graph such that

$$(\hat{T}_K)^P(xy) = \frac{1}{2}[(\hat{T}_J)^P(x) \wedge (\hat{T}_J)^P(y)]$$

$$(\hat{I}_K)^P(xy) = \frac{1}{2}[(\hat{I}_J)^P(x) \wedge (\hat{I}_J)^P(y)]$$

$$(\hat{F}_K)^P(xy) = \frac{1}{2}[(\hat{F}_J)^P(x) \vee (\hat{F}_J)^P(y)],$$

and

$$(\hat{T}_K)^N(xy) = \frac{1}{2}[(\hat{T}_J)^N(x) \vee (\hat{T}_J)^N(y)]$$

$$(\hat{I}_K)^N(xy) = \frac{1}{2}[(\hat{I}_J)^N(x) \vee (\hat{I}_J)^N(y)]$$

$$(\hat{F}_K)^N(xy) = \frac{1}{2}[(\hat{F}_J)^N(x) \wedge (\hat{F}_J)^N(y)]$$

for all $xy \in J$ then $G \approx G^{c^c}$, implies G is self complementary. Hence proved

Proposition 5.6 *Assume that, G is a self complementary neutrosophic bipolar vague graph then*

$$\sum_{x\neq y} (\hat{T}_K)^P(xy) = \frac{1}{2}\sum_{x\neq y} \{(\hat{T}_J)^P(x) \wedge (\hat{T}_J)^P(y)\}$$

$$\sum_{x\neq y} (\hat{I}_K)^P(xy) = \frac{1}{2}\sum_{x\neq y} \{(\hat{I}_J)^P(x) \wedge (\hat{I}_J)^P(y)\}$$

$$\sum_{x\neq y} (\hat{F}_K)^P(xy) = \frac{1}{2}\sum_{x\neq y} \{(\hat{F}_J)^P(x) \vee (\hat{F}_J)^P(y)\}$$

$$\sum_{x\neq y} (\hat{T}_K)^N(xy) = \frac{1}{2}\sum_{x\neq y} \{(\hat{T}_J)^N(x) \vee (\hat{T}_J)^N(y)\}$$

$$\sum_{x\neq y} (\hat{I}_K)^N(xy) = \frac{1}{2}\sum_{x\neq y} \{(\hat{I}_J)^N(x) \vee (\hat{I}_J)^N(y)\}$$

$$\sum_{x\neq y} (\hat{F}_K)^N(xy) = \frac{1}{2}\sum_{x\neq y} \{(\hat{F}_J)^N(x) \wedge (\hat{F}_J)^N(y)\}$$

Proof. Suppose that G be an self complementary neutrosophic bipolar vague graph, by its definition, we have isomorphism $f: J_1 \to J_2$ satisfy

$$(\hat{T}_{J_1}^c)^P(f(x)) = (\hat{T}_{J_1})^P(f(x)) = (\hat{T}_{J_1})^P(x)$$
$$(\hat{I}_{J_1}^c)^P(f(x)) = (\hat{I}_{J_1})^P(f(x)) = (\hat{I}_{J_1})^P(x)$$
$$(\hat{F}_{J_1}^c)^P(f(x)) = (\hat{F}_{J_1})^P(f(x)) = (\hat{F}_{J_1})^P(x)$$

and

$$(\hat{T}_{K_1}^c)^P(f(x), f(y)) = (\hat{T}_{K_1})^P(f(x), f(y)) = (\hat{T}_{K_1})^P(xy)$$
$$(\hat{I}_{K_1}^c)^P(f(x), f(y)) = (\hat{I}_{K_1})^P(f(x), f(y)) = (\hat{I}_{K_1})^P(xy)$$
$$(\hat{F}_{K_1}^c)^P(f(x), f(y)) = (\hat{F}_{K_1})^P(f(x), f(y)) = (\hat{F}_{K_1})^P(xy)$$

we have $(\hat{T}_{K_1}^c)^P(f(x), f(y)) = ((\hat{T}_{J_1}^c)^P(x) \wedge (\hat{T}_{J_1}^c)^P(y)) - (\hat{T}_{K_1})^P(f(x), f(y))$.

i.e,$(\hat{T}_{K_1})^P(xy) = ((\hat{T}_{J_1}^c)^P(x) \wedge (\hat{T}_{J_1}^c)^P(y)) - (\hat{T}_{K_1})^P(f(x), f(y))$.

$(\hat{T}_{K_1})^P(xy) = ((\hat{T}_{J_1}^c)^P(x) \wedge (\hat{T}_{J_1}^c)^P(y)) - (\hat{T}_{K_1})^P(xy)$, hence

$\sum_{x\neq y} (\hat{T}_{K_1})^P(xy) + \sum_{x\neq y} (\hat{T}_{K_1})^P(xy) = \sum_{x\neq y} ((\hat{T}_{J_1})^P(x) \wedge (\hat{T}_{J_1})^P(y))$.

Similarly, $\sum_{x\neq y} (\hat{I}_{K_1})^P(xy) + \sum_{x\neq y} (\hat{I}_{K_1})^P(xy) = \sum_{x\neq y} ((\hat{I}_{J_1})^P(x) \wedge (\hat{I}_{J_1})^P(y))$

$$\sum_{x\neq y} (\hat{F}_{K_1})^P(xy) + \sum_{x\neq y} (\hat{F}_{K_1})^P(xy) = \sum_{x\neq y} ((\hat{F}_{J_1})^P(x) \vee (\hat{F}_{J_1})^P(y))$$

$$2\sum_{x\neq y} (\hat{T}_{K_1})^P(xy) = \sum_{x\neq y} ((\hat{T}_{J_1})^P(x) \wedge (\hat{T}_{J_1})^P(y))$$

$$2\sum_{x\neq y} (\hat{I}_{K_1})^P(xy) = \sum_{x\neq y} ((\hat{I}_{J_1})^P(x) \wedge (\hat{I}_{J_1})^P(y))$$

$$2\sum_{x\neq y} (\hat{F}_{K_1})^P(xy) = \sum_{x\neq y} ((\hat{F}_{J_1})^P(x) \vee (\hat{F}_{J_1})^P(y))$$

Similarly one can prove for the negative condition, from the equation of the proposition (5.5) holds.

Proposition 5.7 *Suppose G_1 and G_2 is neutrosophic bipolar vague graph which is strong, $\overline{G_1} \approx \overline{G_2}$(isomorphism)*

Proof. Assume that G_1 and G_2 are isomorphic there exist a bijective map $f: J_1 \to J_2$ satisfying,

$$(\hat{T}_{J_1})^P(x) = (\hat{T}_{J_2})^P(f(x)),$$
$$(\hat{I}_{J_1})^P(x) = (\hat{I}_{J_2})^P(f(x)),$$
$$(\hat{F}_{J_1})^P(x) = (\hat{F}_{J_2})^P(f(x)), \text{for all} x \in J_1$$
$$(\hat{T}_{J_1})^N(x) = (\hat{T}_{J_2})^N(f(x)),$$
$$(\hat{I}_{J_1})^N(x) = (\hat{I}_{J_2})^N(f(x)),$$
$$(\hat{F}_{J_1})^N(x) = (\hat{F}_{J_2})^N(f(x)), \text{for all} x \in J_1$$

and

$$(\hat{T}_{K_1})^P(xy) = (\hat{T}_{K_2})^P(f(x), f(y))$$
$$(\hat{I}_{K_1})^P(xy) = (\hat{I}_{K_2})^P(f(x), f(y))$$
$$(\hat{F}_{K_1})^P(xy) = (\hat{F}_{K_2})^P(f(x), f(y)) \forall xy \in K_1$$
$$(\hat{T}_{K_1})^N(xy) = (\hat{T}_{K_2})^N(f(x), f(y))$$
$$(\hat{I}_{K_1})^N(xy) = (\hat{I}_{K_2})^N(f(x), f(y))$$
$$(\hat{F}_{K_1})^N(xy) = (\hat{F}_{K_2})^N(f(x), f(y)) \forall xy \in K_1$$

by definition (5.2) we have

$$(T^c_{K_1})^P(xy) = ((T_{J_1})^P(x) \wedge (T_{J_1})^P(y)) - (T_{K_1})^P(xy)$$
$$= ((T_{J_2})^P f(x) \wedge (T_{J_2})^P f(y)) - (T_{K_2})^P(f(x)f(y))$$
$$= (T^c_{K_2})^P(f(x)f(y))$$
$$(I^c_{K_1})^P(xy) = ((I_{J_1})^P(x) \wedge (I_{J_1})^P(y)) - (I_{K_1})^P(xy)$$
$$= ((I_{J_2})^P f(x) \wedge (I_{J_2})^P f(y)) - (I_{K_2})^P(f(x)f(y))$$
$$= (I^c_{K_2})^P(f(x)f(y))$$
$$(F^c_{K_1})^P(xy) = ((F_{J_1})^P(x) \vee (F_{J_1})^P(y)) - (F_{K_1})^P(xy)$$
$$= ((F_{J_2})^P f(x) \vee (F_{J_2})^P f(y)) - (F_{K_2})^P(f(x)f(y))$$
$$= (F^c_{K_2})^P(f(x)f(y))$$

Hence $G^c_1 \approx G^c_2$ for all $(xy) \in K_1$

Definition 5.8 *A neutrosophic bipolar vague graph $G = (J, K)$ is complete if*

$$(T^-_K)^P(xy) = \{(T^-_J)^P(x) \wedge (T^-_J)^P(y)\}$$
$$(I^-_K)^P(xy) = \{(I^-_J)^P(x) \wedge (I^-_J)^P(y)\}$$
$$(F^-_K)^P(xy) = \{(F^-_J)^P(x) \vee (F^-_J)^P(y)\},$$
$$(T^+_K)^P(xy) = \{(T^+_J)^P(x) \wedge (T^+_J)^P(y)\}$$
$$(I^+_K)^P(xy) = \{(I^+_J)^P(x) \wedge (I^+_J)^P(y)\}$$
$$(F^+_K)^P(xy) = \{(F^+_J)^P(x) \vee (F^+_J)^P(y)\},$$
$$(T^-_K)^N(xy) = \{(T^-_J)^N(x) \vee (T^-_J)^N(y)\}$$
$$(I^-_K)^N(xy) = \{(I^-_J)^N(x) \vee (I^-_K)^N(y)\}$$
$$(F^-_K)^N(xy) = \{(F^-_J)^N(x) \wedge (F^-_J)^N(y)\},$$
$$(T^+_K)^N(xy) = \{(T^+_J)^N(x) \vee (T^+_J)^N(y)\}$$
$$(I^+_K)^N(xy) = \{(I^+_J)^N(x) \vee (I^+_J)^N(y)\}$$

$$(F_K^+)^N(xy) = \{(F_J^+)^N(x) \wedge (F_J^+)^N(y)\}, \forall((xy) \in J)$$

Remark 5.9 The complement of NBVGs are NBVGs provided the graph is strong. According to [9], the complement of Single-Valued Neutrosophic Graph (SVNG) is not a SVNG. By the same idea, we implement the definition for NBVGs to obtain the proposed concepts. For other type of bipolar graphs, the complement of Bipolar Fuzzy Graph (BFG) is BFG [6]. The complement of Bipolar Fuzzy Soft Graph (BFSG) and Bipolar Neutrosophic Graph (BNG) are BFSG and BNG, [14, 16] respectively, provided if the graph is strong. The complement of complete bipolar SVNG is bipolar SVFG [25].

Conclusion

This present work characterised the new concept of neutrosophic bipolar vague sets and its application to NBVGs are introduced. Moreover, some remarkable properties of strong NBVGs, complete NBVGs and complement NBVGs have been investigated and the proposed concepts are illustrated with the examples. The obtained results are extended to interval neutrosophic bipolar vague sets. Further we can extend to investigate the domination number, regular and isomorphic properties of the proposed graph.

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A Python Tool for Implementations on Bipolar Neutrosophic Matrices

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Abstract: Bipolar neutrosophic matrices (BNM) are obtained by bipolar neutrosophic sets. Each bipolar neutrosophic number represents an element of the matrix. The matrices are representable multi-dimensional arrays (3D arrays). The arrays have nested list data type. Some operations, especially the composition is a challenging algorithm in terms of coding because there are so many nested lists to manipulate. This paper presents a Python tool for bipolar neutrosophic matrices. The advantage of this work, is that the proposed Python tool can be used also for fuzzy matrices, bipolar fuzzy matrices, intuitionistic fuzzy matrices, bipolar intuitionistic fuzzy matrices and single valued neutrosophic matrices.

Keywords: Python; Neutrosophic sets; bipolar neutrosophic sets; matrix; composition operation

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1. Introduction

Smarandache [1] gave the concept of neutrosophic set (NS) by considering the triplets independent components whose values belong to real standard or nonstandard unit interval] - 0, 1+[. Later on, Smarandache [1] gave single valued neutrosophic set (SVNS) to apply into the various engineering applications. The various properties of SVNS is being studied by Wang et al. [2]. Further, Zhang et al. [3] presented a concept of interval-valued NS (IVNS) where the different membership degrees are represented by interval. In [4] Deli et al. introduced the concept of bipolar neutrosophic sets and their applications based on multicriteria decision making problems. The same author [5] proposed the bipolar neutrosophic refined sets and their applications in medical diagnosis for more details about the applications and its sets, we refer to [6]. Since the existence of NS, various scholars have presented the approaches related to SVNS and bipolar neutrosophic sets into the different fields. For instance, Mumtaz et al. [7] developed the concept of bipolar neutrosophic soft sets that combines soft sets and bipolar neutrosophic sets. In [8, 9] Broumi et al. introduced the notion of bipolar single valued neutrosophic graph theory and its shortest path problem. Dey et al. [10] considered TOPSIS method for solving the decision making problem under bipolar neutrosophic environment. Akram et al. [11] described bipolar neutrosophic TOPSIS method and bipolar neutrosophic ELECTRE-I method. Akram and Sarwar [12] studied the novel multiple criteria decision making methods based on bipolar neutrosophic sets and bipolar neutrosophic graphs. Akram and Sitara [13] introduced the concept of bipolar single-valued neutrosophic graph structures and discussed certain notions of bipolar single-valued neutrosophic graph structures with examples. Singh [14] introduced bipolar neutrosophic graph representation of concept lattice and it's processing using granular computing. Mullai and Broumi [15] presented shortest path problem by minimal spanning tree algorithm using bipolar neutrosophic numbers. Uluçay et al. [16] defined similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. Based on literal neutrosophic numbers, Mamouni et al. [17] defined the addition and multiplication of two neutrosophic fuzzy matrices. in the light of Fuzzy Neutrosophic soft sets, Arockiarani [18] present a new technique for handling decision making problems and proposed some new notions on matrix representation. Karaaslan and Hayat [19] introduced some novel operations on neutrosophic matrices. Uma et al. [20] introduced two types of fuzzy neutrosophic soft Matrices. The same authors in [21] decomposed fuzzy neutrosophic soft matrix by means of its section of fuzzy neutrosophic soft matrix of Type-I. Hassan et al. [22] defined some special types of bipolar single valued neutrosophic graphs. Akram and Siddique [23] discussed certain types of edge irregular bipolar neutrosophic graphs. Pramanik [24] developed cross entropy measures of bipolar neutrosophic sets and interval bipolar neutrosophic sets. Wang *et al*. [25] defined Frank operations of bipolar neutrosophic numbers (BNNs) and proposed Frank bipolar neutrosophic Choquet Bonferroni mean operators by combining Choquet integral operators and Bonferroni mean operators based on Frank operations of BNNs. In the same study, Akram and Nasir [26] introduced the concept of p-competition bipolar neutrosophic graphs. then they defined generalization of bipolar neutrosophic competition graphs called m-step bipolar neutrosophic competition graphs. AKRAM and SHUM [27] defined Bipolar Neutrosophic Planar Graphs. Hashim et al. [28] provide an application of neutrosophic bipolar fuzzy sets in daily life's problem related with HOPE foundation that is planning to build a children hospital. Akram, and Luqman [29] generalized the concept of bipolar neutrosophic sets to hypergraphs. Das et al. [30] proposes an algorithmic approach for group decision making (GDM) problems using neutrosophic soft matrix (NSM) and relative weights of experts.

Broumi et al. [31-34] applied the concept of IVNS on graph theory and studied some interesting results. Broumi et al. [35] developed a Matlab toolbox for computing operational matrices under the SVNS environments. Pramanik et al [36] developed a hybrid structure termed "rough bipolar neutrosophic set". In [37] Pramanik et al. presented Bipolar neutrosophic projection based models for solving multi-attribute decision making problems. Broumi et al [38] developed the concept of bipolar complex neutrosophic sets and its application in decision making problem. Akram, et al.[39] applied the concept of bipolar neutrosophic sets to incidence graphs and studied some properties. For more details on the application of neutrosophic set theory, we refer the readers to [46-52].

Among all the above, matrices play a vital job in the expansion region of science and engineering. However, the classical matrix theory neglects the role of uncertainties during the analysis. Therefore, the decision process may contain a lot of uncertainties. Thus, the role of the fuzzy matrices and their extension including triangular fuzzy matrices, type-2 triangular fuzzy matrices, interval valued fuzzy matrices, intuitionistic fuzzy matrices, interval valued intuitionistic fuzzy matrices are studied deeply by several scholars. In [40] Zahariev, developed a Matlab software package to the fuzzy algebras. In

[41], authors solved intuitionistic fuzzy relational rational calculus problems using a fuzzy toolbox. Later on, in [42] Karunambigai and Kalaivani proposed some computing procedures in Matlab for intuitionistic fuzzy operational matrices with suitable examples. Uma et al. [43] studied determinant theory for fuzzy neutrosophic soft square matrices. Also, in [44] Uma et al. introduced the determinant and adjoint of a square Fuzzy Neutrosophic Soft Matrices (FNSMs) a defined the circular FNSM and study some relations on square FNSM such as reflexivity, transitivity and circularity.

Recently few researchers [45] developed a Python programs for computing operations on neutrosophic numbers, but all these programs cannot deal with neutrosophic matrices, to do best of our knowledge, there is no work conducted on developing python codes to compute the operations on single valued neutrosophic matrices and bipolar neutrosophic matrices. Thus, there is a need to develop the work in that direction. For it, the presented paper discusses various operations of bipolar neutrosophic sets and their corresponding Python code for different metrics. To achieve it, rest of the manuscript is summarized as. In section 2, some concepts related to SVNS, BNS are presented. Section 3 deals with the generations of Python programs for bipolar neutrosophic matrices with a numerical example and lastly, conclusion is summarized in section 4.

2.BACKGROUND AND BIPOLAR NEUTROSOPHIC SETS

In this section, some basic concepts on SVNS, BNS are briefly presented over the universal set ξ [1, 2, 4].

Definition 2.1 [1] A set A is said to be A neutrosophic set 'A' consists of three components namely truth, indeterminate and falsity denoted by T_A, I_A(x) and F_A(x) such that

$T_A(x), I_A(x),\ F_A(x) \in]\ ^-0, 1^+[$ and $-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3+$ (1)

Definition 2.2 [2] A SVNS 'A' on X is given as

A = $\{< x: T_A(x), I_A(x),\ F_A(x) > x \in \xi\}$ (2)

where the functions T_A(x), I_A(x), F_A(x) $\in$ [0. 1] are named "degree of truth, indeterminacy and falsity membership of x in A", such that

$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ (3)

Definition 2.3[4]. A bipolar neutrosophic set A in ξ is defined as an object of the form

A={<x, $(T_A^P(x), I_A^P(x), F_A^P(x), T_A^N(x), I_A^N(x), F_A^N(x))$>: x $\in$ ξ }, where $T_A^P(x), I_A^P(x), F_A^P(x)$: $\xi \to$ [1, 0] and $T_A^N(x), I_A^N(x), F_A^N(x)$: $\xi \to$ [-1, 0]. The positive membership degree $T_A^P(x), I_A^P(x), F_A^P(x)$enotes the truth membership, indeterminate membership and false membership of an element $\in \xi$ corresponding to a bipolar neutrosophic set whereas the negative membership degree $T_A^N(x), I_A^N(x), F_A^N(x)$denotes the truth membership, indeterminate membership and false membership of an element $x \in \xi$ to some implicit counter-property corresponding to a bipolar neutrosophic set A. For convenience a bipolar neutrosophic number is represented by

A= <$(T_A^P, I_A^P, F_A^P, T_A^N, I_A^-, F_A^-$> (4)

Definition 2.4 [4]. In order to make a comparison between two BNN. The score function is applied to compare the grades of BNS. This function shows that greater is the value, the greater is the bipolar neutrosophic sets and by using this concept paths can be ranked. Suppose

$\tilde{A} =< T^P, I^P, F^P, T^N, I^N, F^N >$ be a bipolar neutrosophic number. Then, the score function $s(\tilde{A})$, accuracy function $a(\tilde{A})$ and certainty function $c(\tilde{A})$ of a BNN are defined as follows:

(i) $s(\tilde{A}) = \left(\frac{1}{6}\right) \times \left[T^P + 1 - I^P + 1 - F^P + 1 + T^N - I^N - F^N\right]$ (5)

(ii) $a(\tilde{A}) = T^P - F^P + T^N - F^N$ (6)

(iii) $c(\tilde{A}) = T^P - F^N$ (7)

Comparison of bipolar neutrosophic numbers

Let $\tilde{A}_1 =< T_1^p, I_1^p, F_1^p, T_1^n, I_1^n, F_1^n >$ and $\tilde{A}_2 =< T_2^p, I_2^p, F_2^p, T_2^n, I_2^n, F_2^n >$ be two bipolar neutrosophic numbers then

i. If $s(\tilde{A}_1) \succ s(\tilde{A}_2)$, then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$.

ii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, and $a(\tilde{A}_1) \succ a(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$.

iii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, $a(\tilde{A}_1) = a(\tilde{A}_2)$, and $c(\tilde{A}_1) \succ c(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$.

iv. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, $a(\tilde{A}_1) = a(\tilde{A}_2)$, and $c(\tilde{A}_1) = c(\tilde{A}_2)$ then $\tilde{A}_1$ is equal to $\tilde{A}_2$, that is, $\tilde{A}_1$ is indifferent to $\tilde{A}_2$, denoted by $\tilde{A}_1 = \tilde{A}_2$.

Definition 2.5 [4]: A bipolar neutrosophic matrix (BNM) of order m× n is defined as
$A_{\mathrm{BNM}}=[< a_{ij}, a_{ij_T}^P, a_{ij_I}^P, a_{ij_F}^P, a_{ij_T}^N, a_{ij_I}^N, a_{ij_F}^N >]_{\mathrm{m\times n}}$ where
$a_{ij_T}^P$ is the positive membership value of element a_{ij} in A.
$a_{ij_T}^N$ is the negative membership value of element a_{ij} in A.
$a_{ij_T}^P$ is the positive indeterminate-membership value of element a_{ij} in A.
$a_{ij_T}^N$ is the negative indeterminate-membership value of element a_{ij} in A.
$a_{ij_T}^P$ is the positive non- membership value of element a_{ij} in A.
$a_{ij_T}^N$ is the negative non-membership value of element a_{ij} in A.

For simplicity, we write A as $A_{\mathrm{BNM}}= [< a_{ij_T}^P, a_{ij_I}^P, a_{ij_F}^P, a_{ij_T}^N, a_{ij_I}^N, a_{ij_F}^N >]_{\mathrm{m\times n}}$.

3.COMPUTING THE BIPOLAR NEUTROSOPHIC MATRIX OPERATIONS USING PYTHON LANGUAGE

To generate the Python program for inputting the single valued neutrosophic matrices. The procedure is described as follows:

3.1 Checking the matrix is BNM or not

To generate the Python program for deciding for a given the matrix is bipolar neutrosophic matrix or, simple call of the function **BNMChecking ( )** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for
BNM Checking
#A1.shape and A2.shape returns (3, 3, 6) the dimension of A. (row, column, numbers of element
(Bipolar Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 3 columns
# A.shape[2] = Each bipolar neutrosophic number has 6 tuple as usual
#One can use any matrices having arbitrary dimension
import numpy as np
#A1 is a BNM
A1= np.array([    [[0.000, 0.001, 0.002, -0.003, -0.004, -0.005],   [0.010, 0.011, 0.012, -0.013, -0.014, -
0.015] , [0.020, 0.021, 0.022, -0.023, -0.024, -0.025]    ],
[[0.100,0.101,0.102,-0.103,-0.104, -0.105], [0.110,0.111,0.112,-0.113,-0.114,-0.115], [0.120,0.121,0.122,-
0.123,-0.124,-0.125]     ],
  [[0.200,0.201,0.202,-0.203,-0.204,-0.205], [0.210, 0.211,0.212,-0.213,-0.214,-0.215], [0.220,0.221,0.222,-
0.223,-0.224,-0.225]      ] ])
#A2 is not BNM
A2= np.array([    [[0.000, 0.001, 0.002, -0.003, -0.004, -0.005],   [0.010, 0.011, 0.012, -0.013, -0.014, -
0.015] , [0.020, 0.021, 0.022, -0.023, -0.024, -0.025]    ],
[[0.100,0.101,0.102,-0.103,-0.104, -0.105],       [0.110,0.111,0.112,-0.113,-0.114,-0.115],
[0.120,0.121,0.122,-0.123,-0.124,-0.125]      ],
[[0.200,0.201,0.202,-0.203, 0.204,-0.205],        [0.210, 0.211,0.212,-0.213,-0.214,-0.215],
[0.220,0.221,0.222,-0.223,-0.224,-0.225]      ] ])
def BNMChecking (A):
    dimA=A.shape
    control=0
    counter = 0
    for i in range (0,dimA[0]):
        if counter == 1:
            break
        for j in range (0,dimA[0]):
            if counter == 1:
                break
            for   d in range (0, dimA[2]):
                if   counter ==0:
```

```
                    if (d==0 or d==1 or d==2) :
                        if   not (0 <=   A[i][j][d] <= 1):
                            counter=1
                            print (A[i][j], ' is not a bipolar neutrosophic number, so the matrix
is not a BNM')
                            control=1
                            break
                    if   (d==3 or d==4 or d==5) :
                        if not (-1 <=   A[i][j][d] <= 0) :
                            counter=1
                            print (A[i][j], ' is not a bipolar neutrosophic number, so the matrix
is not a BNM')
                            control=1
                            break
                else:
                    print (A[i][j], ' is not a bipolar neutrosophic number, so the matrix is not a
BNM')
                    break
    if control==0:
        print ('The matrix is a BNM')
```

Example 1. In this example we evaluate the checking the matrix C is BNM or not of order 4X4:

C=

$$\begin{pmatrix} <.5,.7,.2,-.7,-.3,-.6> & <.4,.4,.5,-.7,-.8,-.4> & <.7,.7,.5,-.8,-.7,-.6> & <.1,.5,.7,-.5,-.2,-.8> \\ <.9,.7,.5,-.7,-.7,-.1> & <.7,.6,.8,-.7,-.5,-.1> & <.9,.4,.6,-.1,-.7,-.5> & <.5,.2,.7,-.5,-.1,-.9> \\ <.9,.4,.2,-.6,-.3,-.7> & <.2,.2,.2,-.4,-.7,-.4> & <.9,.5,.5,-.6,-.5,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.7,.2,-.8,-.6,-.1> & <.3,.5,.2,-.5,-.5,-.2> & <.5,.4,.5,-.1,-.7,-.2> & <.2,.4,.8,-.5,-.5,-.6> \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python environment like this:

```
Shell  AST
>>> %Run 'bnm checking.py'
>>> C= np.array([ [ [0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-0.8]],
                [[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]],
                 [[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]],
                 [[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]] ])

>>> BNMChecking (C)
 The matrix is a BNM
...
```

3.2. Determining complement of bipolar neutrosophic matrix

For a given BNM A= $[< T_{ij}^P, I_{ij}^P, F_{ij}^P, T_{ij}^N, I_{ij}^N, F_{ij}^N >]_{m\times n}$, the complement of A is defined as follow:

$$A^c= [< \{1\} - T_{ij}^P, \{1\} - I_{ij}^P, \{-1\} - F_{ij}^P, \{1\} - T_{ij}^N, \{-1\} - I_{ij}^N, \{-1\} - F_{ij}^N >]_{m\times n} \quad (8)$$

$$A^c= [< F_{ij}^P, \{1\} - I_{ij}^P, T_{ij}^P, F_{ij}^N, \{-1\} - I_{ij}^N, T_{ij}^N >]_{m\times n} \quad (9)$$

To generate the Python program for finding complement of bipolar neutrosophic matrix, simple call of the function **BNMCompelementOf()** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for
(8)
import numpy as np
A= np.array([   [ [0.3,0.6,1,-0.2,-0.54,-0.4],   [0.1,0.2,0.8,-0.5,-0.34,-0.7]],
                [ [0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22]],
                [ [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32] ]
                ])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar
Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = each bipolar neutrosophic number with 6 tuple as usual
def BNMCompelementOf( A ):
    global Ac
    dimA=A.shape                          # Dimension of the matrix
    Ac= []     # Empty matrix with dimension of A to create complement of A
        for i in range (0,dimA[0]):       # for rows, here 3
        H=[]
        for j in range (0,dimA[1]):   # for columns, here 2
            H.extend([ [ 1-A[i][j][0], 1-A[i][j][1], 1-A[i][j][2], -1-(-A[i][j][3]), -1-(-A[i][j][4]), -1-(-
A[i][j][5]) ] ])
            Ac.append(H)
    print   ('A= ', A)
    print ('*******************************************************************')
    print('Ac= ', np.array(Ac))
```

The function BNMCompelementOf (A) the below returns the complement matrix of a given bipolar neutrosophic matrix A for (9).

```
# BNM is representable by 3D Numpy Array ====> row, column and bipolar neutrosophic
numbers having 6 tuples for (9)
import numpy as np
A= np.array([   [ [0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7]   ],
                [ [0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22] ],
                [ [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32]    ]])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar
Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = Each bipolar neutrosophic number with 6 tuple as usual
def BNMCompelementOf( A ):
    global Ac
    dimA=A.shape                          # Dimension of the matrix
    Ac= []
```

```
    for i in range (0,dimA[0]):          # for rows, here 3
        H=[]
        for j in range (0,dimA[1]):    # for columns, here 2
            H.extend([[ A[i][j][2], 1-A[i][j][1], A[i][j][0], A[i][j][5], -1-(-A[i][j][4]), A[i][j][3] ] ])
        Ac.append(H)
  print   ('A= ', A)
  print ('********************************************************************')
  print ('********************************************************************')
  print('Ac= ', np.array(Ac))
```

The bipolar neutrosophic matrix A is a simple example, one can create his/her BNM and try it into the function **BNMCompelementOf** ( ):

3.3. Determining the score, accuracy and certainty matrices of bipolar neutrosophic matrix

To generate the python program for obtaining the score matrix, accuracy of bipolar neutrosophic matrix, simple call of the functions **ScoreMatrix( )**, **AccuracyMatrix( )** and **CertaintyMatrix( )** are defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for (5,
6 and 7)
  import numpy as np
  A= np.array([      [          [0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7]                    ],
                     [          [0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22]          ],
                     [          [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32]     ]])
  def ScoreMatrix( A ):
      score=[]
      dimA=A.shape                                    # Dimension of the matrix
      for i in range (0,dimA[0]):              # for rows, here 3
          H=[]
          for j in range (0,dimA[1]):           # for columns, here 2
              H.extend([ [ ( A[i][j][0] + 1 - A[i][j][1] + 1 - A[i][j][2] + 1 + A[i][j][3] - A[i][j][4] -
A[i][j][5] )/6 ] ])
          score.append(H)
      print('Score Matrix= ', np.array(score))
  def AccuracyMatrix ( A ):
      accuracy=[]
      dimA=A.shape                                    # Dimension of the matrix
      for i in range (0,dimA[0]):              # for rows, here 3
          H=[]
          for j in range (0,dimA[1]):           # for columns, here 2
              H.extend([ [   A[i][j][0] - A[i][j][2] + A[i][j][3] - A[i][j][5]    ] ])
          accuracy.append(H)
      print('Accuracy Matrix= ', np.array(accuracy))
```

```
def CertaintyMatrix ( A ):
    certainty = []
    dimA=A.shape                          # Dimension of the matrix
    for i in range (0,dimA[0]):           # for rows, here 3
        H=[]
        for j in range (0,dimA[1]):       # for columns, here 2
            H.extend([ [   A[i][j][0] - A[i][j][5]   ] ])
        certainty.append(H)
    print('Certainty Matrix= ', np.array(certainty))
```

3.4. Computing union of two bipolar neutrosophic matrices

The union of two bipolar neutrosophic matrices A and B is defined as follow:

$$A \cup B = C = \left[< c^P_{ij_T}, c^P_{ij_I}, c^P_{ij_F}, c^N_{ij_T}, c^N_{ij_I}, c^N_{ij_F} >\right]_{\mathrm{m}\times\mathrm{n}} \tag{10}$$

where

$c^P_{ij_T} = a^P_{ij_T} \vee b^P_{ij_T}$, $\quad c^N_{ij_T} = a^N_{ij_T} \wedge b^N_{ij_T}$

$c^P_{ij_I} = a^P_{ij_I} \wedge b^P_{ij_I}$, $\quad c^N_{ij_I} = a^N_{ij_I} \vee b^N_{ij_I}$

$c^P_{ij_F} = a^P_{ij_F} \wedge b^P_{ij_F}$, $\quad c^N_{ij_F} = a^N_{ij_F} \vee b^N_{ij_F}$

To generate the python program for finding the union of two bipolar neutrosophic matrices, simple call of the following function **Union(A, B)** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for
(10)
import numpy as np
A= np.array([   [       [0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7]   ],
                [         [0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22] ],
                [   [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32]] ])
B= np.array([   [         [0.32,0.4,0.1,-0.25,-0.54,-0.4], [0.13,0.2,0.11,-0.55,-0.35,-0.72]      ],
                [[0.17,0.19,0.66,-0.87,-0.64,-0.92], [0.25,0.36,0.88,-0.33,-0.54,-0.22] ],
                [ [0.15,0.28,0.67,-0.39,-0.27,-0.55],[0.24,0.73,0.28,-0.26,-0.53,-0.52] ]
                ])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar
Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = each bipolar neutrosophic number with 6 tuple as usual
union=[]
def Union( A, B ):
    if A.shape == B.shape:
        dimA=A.shape
        for i in range (0,dimA[0]):         # for rows, here 3
            H=[]
            for j in range (0,dimA[1]):     # for columns, here 2
```

```
                    H.extend([[ max(A[i][j][0],B[i][j][0]) , min(A[i][j][1], B[i][j][1]), min(A[i][j][2], B[i][j][2]), max(A[i][j][3],B[i][j][3]), min(A[i][j][4], B[i][j][4]), min(A[i][j][5], B[i][j][5]) ] ])
                    union.append(H)
      print('union= ', np.array(union)
```

Example 2. In this example we Evaluate the union of the two bipolar neutrosophic matrices C and D of order 4X4:

C=

$$\begin{pmatrix} <.5,.7,.2,-.7,-.3,-.6> & <.4,.4,.5,-.7,-.8,-.4> & <.7,.7,.5,-.8,-.7,-.6> & <.1,.5,.7,-.5,-.2,-.8> \\ <.9,.7,.5,-.7,-.7,-.1> & <.7,.6,.8,-.7,-.5,-.1> & <.9,.4,.6,-.1,-.7,-.5> & <.5,.2,.7,-.5,-.1,-.9> \\ <.9,.4,.2,-.6,-.3,-.7> & <.2,.2,.2,-.4,-.7,-.4> & <.9,.5,.5,-.6,-.5,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.7,.2,-.8,-.6,-.1> & <.3,.5,.2,-.5,-.5,-.2> & <.5,.4,.5,-.1,-.7,-.2> & <.2,.4,.8,-.5,-.5,-.6> \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python code like this:

C= np.array([[[0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-0.8]],[[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]], [[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]], [[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]]])

D=

$$\begin{pmatrix} <.3,.4,.3,-.5,-.4,-.2> & <.1,.2,.7,-.5,-.2,-.3> & <.3,.2,.6,-.4,-.8,-.7> & <.2,.1,.3,-.2,-.4,-.4> \\ <.2,.2,.7,-.3,-.3,-.5> & <.3,.5,.6,-.6,-.7,-.4> & <.6,.5,.4,-.3,-.6,-.8> & <.3,.4,.4,-.3,-.5,-.3> \\ <.5,.3,.1,-.4,-.2,-.4> & <.5,.4,.3,-.3,-.8,-.2> & <.5,.8,.6,-.2,-.2,-.4> & <.4,.6,.5,-.1,-.6,-.5> \\ <.6,.1,.7,-.7,-.4,-.8> & <.4,.6,.4,-.4,-.2,-.5> & <.4,.9,.3,-.5,-.5,-.3> & <.4,.5,.4,-.3,-.7,-.4> \end{pmatrix}$$

The bipolar neutrosophic matrix D can be inputted in Python code like this:

D= np.array([[[0.3,0.4, 0.3,-0.5,-0.4,-0.2], [0.1,0.2,0.7,-0.5,-0.2,-0.3], [0.3,0.2,0.6,-0.4,-0.8,-0.7], [0.2,0.1,0.3,-0.2,-0.4,-0.4]], [[0.2,0.2,0.7,-0.3,-0.3,-0.5], [0.3,0.5,0.6,-0.6,-0.7,-0.4], [0.6,0.5,0.4,-0.3,-0.6,-0.8], [0.3,0.4,0.4,-0.3,-0.5,-0.3]], [[0.5,0.3,0.1,-0.4,-0.2,-0.4], [0.5,0.4,0.3,-0.3,-0.8,-0.2], [0.5,0.8,0.6,-0.2,-0.2,-0.4], [0.4,0.6,0.5,-0.1,-0.6,-0.5]], [[0.6,0.1,0.7,-0.7,-0.4,-0.8], [0.4,0.6,0.4,-0.4,-0.2,-0.5], [0.4,0.9,0.3,-0.5,-0.5,-0.3], [0.4,0.5,0.4,-0.3,-0.7,-0.4]]])

So, the union matrix of two bipolar neutrosophic matrices is portrayed as follow

$$C_{BNS} \cup D_{BNS} = \begin{pmatrix} <.5,.4,.2,-.7,-.3,-.2> & <.4,.2,.5,-.7,-.2,-.3> & <.7,.2,.5,-.8,-.7,-.6> & <.2,.1,.3,-.5,-.2,-.4> \\ <.9,.2,.5,-.7,-.3,-.1> & <.7,.5,.6,-.7,-.5,-.1> & <.9,.4,.4,-.3,-.6,-.5> & <.5,.2,.4,-.5,-.1,-.3> \\ <.9,.3,.1,-.6,-.2,-.4> & <.5,.2,.2,-.4,-.7,-.2> & <.9,.5,.5,-.6,-.2,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.1,.2,-.8,-.4,-.1> & <.4,.5,.2,-.5,-.2,-.2> & <.5,.4,.3,-.5,-.5,-.2> & <.4,.4,.4,-.5,-.5,-.4> \end{pmatrix}$$

The result of union matrix of two bipolar neutrosophic matrices C and D can be obtained by the call of the command Union (C, D):

```
>>> Union(C, D)
Union =
[[[ 0.5   0.4   0.2 -0.7 -0.3 -0.2]   [ 0.4   0.2   0.5 -0.7 -0.2 -0.3] [ 0.7   0.2   0.5 -0.8 -0.7 -0.6]     [ 0.2   0.1   0.3 -0.5 -0.2 -0.4]]
 [[ 0.9   0.2   0.5 -0.7 -0.3 -0.1]   [ 0.7   0.5   0.6 -0.7 -0.5 -0.1]     [ 0.9   0.4   0.4 -0.3 -0.6 -0.5]     [ 0.5   0.2   0.4 -0.5 -0.1 -0.3]]
 [[ 0.9   0.3   0.1 -0.6 -0.2 -0.4]     [ 0.5   0.2   0.2 -0.4 -0.7 -0.2]     [ 0.9   0.5   0.5 -0.6 -0.2 -0.2]     [ 0.7   0.5   0.3 -0.4 -0.2 -0.2]]
 [[ 0.9   0.1   0.2 -0.8 -0.4 -0.1]     [ 0.4   0.5   0.2 -0.5 -0.2 -0.2]     [ 0.5   0.4   0.3 -0.5 -0.5 -0.2]     [ 0.4   0.4   0.4 -0.5 -0.5 -0.4]]]
```

3.5. Computing intersection of two bipolar neutrosophic matrices

The union of two bipolar neutrosophic matrices A and B is defined as follow:

$$A \cap B = D = \left[< d^P_{ij_T}, d^P_{ij_I}, d^P_{ij_F}, d^N_{ij_T}, d^N_{ij_I}, d^N_{ij_F} >\right]_{m\times n} \quad (11)$$

Where

$d^P_{ij_T} = a^P_{ij_T} \wedge b^P_{ij_T}, \quad d^N_{ij_T} = a^N_{ij_T} \vee b^N_{ij_T}$
$d^P_{ij_I} = a^P_{ij_I} \vee b^P_{ij_I}, \quad d^N_{ij_I} = a^N_{ij_I} \wedge b^N_{ij_I}$
$d^P_{ij_F} = a^P_{ij_F} \vee b^P_{ij_F}, \quad d^N_{ij_F} = a^N_{ij_F} \wedge b^N_{ij_F}$

To generate the python program for finding the intersection of two bipolar neutrosophic matrices, simple call of the function Intersection (A, B) is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for
(11)
import numpy as np
A= np.array([   [         [0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7]            ],
                [         [0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22]     ],
                [         [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32]  ]
                ])
B= np.array([   [         [0.32,0.4,0.1,-0.25,-0.54,-0.4], [0.13,0.2,0.11,-0.55,-0.35,-0.72]    ],
                [         [0.17,0.19,0.66,-0.87,-0.64,-0.92], [0.25,0.36,0.88,-0.33,-0.54,-0.22] ],
                [         [0.15,0.28,0.67,-0.39,-0.27,-0.55],[0.24,0.73,0.28,-0.26,-0.53,-0.52] ]])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar
Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = each bipolar neutrosophic number with 6 tuple as usual
intersection=[]
def Intersection( A, B ):
    if A.shape == B.shape:
        dimA=A.shape
        for i in range (0,dimA[0]):         # for rows, here 3
            H=[]
            for j in range (0,dimA[1]):   # for columns, here 2
                H.extend([[ min(A[i][j][0],B[i][j][0]) , max(A[i][j][1], B[i][j][1]), max(A[i][j][2],
B[i][j][2]), min(A[i][j][3],B[i][j][3]), max(A[i][j][4], B[i][j][4]), max(A[i][j][5], B[i][j][5]) ] ])
                intersection.append(H)
    print('Intersection= ', np.array(intersection))
```

Example 3. In this example we evaluate the intersection of the two bipolar neutrosophic matrices C and D of order 4X4:

C=

$$\begin{pmatrix} <.5,.7,.2,-.7,-.3,-.6> & <.4,.4,.5,-.7,-.8,-.4> & <.7,.7,.5,-.8,-.7,-.6> & <.1,.5,.7,-.5,-.2,-.8> \\ <.9,.7,.5,-.7,-.7,-.1> & <.7,.6,.8,-.7,-.5,-.1> & <.9,.4,.6,-.1,-.7,-.5> & <.5,.2,.7,-.5,-.1,-.9> \\ <.9,.4,.2,-.6,-.3,-.7> & <.2,.2,.2,-.4,-.7,-.4> & <.9,.5,.5,-.6,-.5,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.7,.2,-.8,-.6,-.1> & <.3,.5,.2,-.5,-.5,-.2> & <.5,.4,.5,-.1,-.7,-.2> & <.2,.4,.8,-.5,-.5,-.6> \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python code like this:

C= np.array([[[0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-0.8]], [[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]], [[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]], [[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]])

D=

$$\begin{pmatrix} <.3,.4,.3,-.5,-.4,-.2> & <.1,.2,.7,-.5,-.2,-.3> & <.3,.2,.6,-.4,-.8,-.7> & <.2,.1,.3,-.2,-.4,-.4> \\ <.2,.2,.7,-.3,-.3,-.5> & <.3,.5,.6,-.6,-.7,-.4> & <.6,.5,.4,-.3,-.6,-.8> & <.3,.4,.4,-.3,-.5,-.3> \\ <.5,.3,.1,-.4,-.2,-.4> & <.5,.4,.3,-.3,-.8,-.2> & <.5,.8,.6,-.2,-.2,-.4> & <.4,.6,.5,-.1,-.6,-.5> \\ <.6,.1,.7,-.7,-.4,-.8> & <.4,.6,.4,-.4,-.2,-.5> & <.4,.9,.3,-.5,-.5,-.3> & <.4,.5,.4,-.3,-.7,-.4> \end{pmatrix}$$

The bipolar neutrosophic matrix D can be inputted in Python code like this:

D= np.array([[[0.3, 0.4, 0.3,-0.5,-0.4,-0.2], [0.1,0.2,0.7,-0.5,-0.2,-0.3], [0.3,0.2,0.6,-0.4,-0.8,-0.7], [0.2,0.1,0.3,-0.2,-0.4,-0.4]], [[0.2,0.2,0.7,-0.3,-0.3,-0.5], [0.3,0.5,0.6,-0.6,-0.7,-0.4], [0.6,0.5,0.4,-0.3,-0.6,-0.8], [0.3,0.4,0.4,-0.3,-0.5,-0.3]],

[[0.5,0.3,0.1,-0.4,-0.2,-0.4], [0.5,0.4,0.3,-0.3,-0.8,-0.2], [0.5,0.8,0.6,-0.2,-0.2,-0.4], [0.4,0.6,0.5,-0.1,-0.6,-0.5]],

[[0.6,0.1,0.7,-0.7,-0.4,-0.8], [0.4,0.6,0.4,-0.4,-0.2,-0.5], [0.4,0.9,0.3,-0.5,-0.5,-0.3], [0.4,0.5,0.4,-0.3,-0.7,-0.4]]])

So, the intersection matrix of two bipolar neutrosophic matrices is portrayed as follow

$$C_{BNS} \cap D_{BNS} = \begin{pmatrix} <.3,.7,.3,-.5,-.4,-.6> & <.1,.4,.7,-.5,-.8,-.4> & <.3,.7,.6,-.4,-.8,-.7> & <.1,.5,.7,-.2,-.4,-.8> \\ <.2,.7,.7,-.3,-.7,-.5> & <.3,.6,.8,-.6,-.7,-.4> & <.6,.5,.6,-.1,-.7,-.8> & <.3,.4,.7,-.3,-.5,-.9> \\ <.5,.4,.2,-.4,-.3,-.7> & <.2,.4,.3,-.3,-.7,-.4> & <.5,.8,.6,-.2,-.5,-.4> & <.4,.6,.5,-.1,-.6,-.5> \\ <.6,.7,.7,-.7,-.6,-.8> & <.3,.6,.4,-.4,-.5,-.5> & <.4,.9,.5,-.1,-.7,-.3> & <.2,.5,.8,-.3,-.7,-.6> \end{pmatrix}$$

The result of intersection matrix of two bipolar neutrosophic matrices C and D can be obtained by the call of the command Intersection (C, D):

```
>>> Intersection (C, D)
Intersection =
[[[ 0.3   0.7   0.3 -0.5 -0.4 -0.6]   [ 0.1   0.4   0.7 -0.5 -0.8 -0.4]   [ 0.3   0.7   0.6 -0.4 -0.8 -0.7]   [ 0.1   0.5   0.7 -0.2 -0.4 -0.8]]
 [[ 0.2   0.7   0.7 -0.3 -0.7 -0.5]   [ 0.3   0.6   0.8 -0.6 -0.7 -0.4]   [ 0.6   0.5   0.6 -0.1 -0.7 -0.8]   [ 0.3   0.4   0.7 -0.3 -0.5 -0.9]]
 [[ 0.5   0.4   0.2 -0.4 -0.3 -0.7]   [ 0.2   0.4   0.3 -0.3 -0.8 -0.4]   [ 0.5   0.8   0.6 -0.2 -0.5 -0.4]   [ 0.4   0.6   0.5 -0.1 -0.6 -0.5]]
 [[ 0.6   0.7   0.7 -0.7 -0.6 -0.8]   [ 0.3   0.6   0.4 -0.4 -0.5 -0.5]   [ 0.4   0.9   0.5 -0.1 -0.7 -0.3]   [ 0.2   0.5   0.8 -0.3 -0.7 -0.6]]]
```

3.6. Computing addition operation of two bipolar neutrosophic matrices.

The addition of two bipolar neutrosophic matrices A and B is defined as follow:

$$A \oplus B = S = [< s^P_{ij_T}, s^P_{ij_I}, s^P_{ij_F}, s^N_{ij_T}, s^N_{ij_I}, s^N_{ij_F} >]_{m\times n} \quad (12)$$

Where

$s^P_{ij_T} = a^P_{ij_T} + b^P_{ij_T} - a^P_{ij_T}.b^P_{ij_T}$, $\quad s^N_{ij_T} = -(a^N_{ij_T}.b^N_{ij_T})$

$s^P_{ij_I} = a^P_{ij_I}.b^P_{ij_I}$, $\quad s^N_{ij_I} = -(-a^N_{ij_I} - b^N_{ij_I} - a^N_{ij_I}.b^N_{ij_I})$

$s^P_{ij_F} = a^P_{ij_F}.b^P_{ij_F}$, $\quad s^N_{ij_F} = -(-a^N_{ij_F} - b^N_{ij_F} - a^N_{ij_F}.b^N_{ij_F})$

To generate the python program for obtaining the addition of two bipolar neutrosophic matrices, simple call of the function **Addition (A, B)** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for (12)
```

```
import numpy as np
A= np.array([    [[0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7] ],
                 [[0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22]],
                 [ [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32] ]])
B= np.array([    [[0.32,0.4,0.1,-0.25,-0.54,-0.4], [0.13,0.2,0.11,-0.55,-0.35,-0.72] ],
                 [[0.17,0.19,0.66,-0.87,-0.64,-0.92], [0.25,0.36,0.88,-0.33,-0.54,-0.22] ],
                 [[0.15,0.28,0.67,-0.39,-0.27,-0.55],[0.24,0.73,0.28,-0.26,-0.53,-0.52]     ]])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar
Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = each bipolar neutrosophic number with 6 tuples as usual
addition=[]
def Addition( A, B ):
    if A.shape == B.shape:
        dimA=A.shape
        for i in range (0,dimA[0]):          # for rows, here 3
            H=[]
            for j in range (0,dimA[1]):    # for columns, here 2
                H.extend([[A[i][j][0]+B[i][j][0]-A[i][j][0]*B[i][j][0],     A[i][j][1]* B[i][j][1],
A[i][j][2]* B[i][j][2] -(-A[i][j][3]*B[i][j][3]), -(-A[i][j][4]-B[i][j][4] -A[i][j][4]*B[i][j][4] ), -(-A[i][j][5]-
B[i][j][5]-A[i][j][5]*B[i][j][5]) ]])
            addition.append(H)
    print('Addition= ', np.array(addition))
```

Example 4. In this example we evaluate the addition of the two bipolar neutrosophic matrices C and D of order 4X4:

C=

$$\begin{pmatrix} <.5,.7,.2,-.7,-.3,-.6> & <.4,.4,.5,-.7,-.8,-.4> & <.7,.7,.5,-.8,-.7,-.6> & <.1,.5,.7,-.5,-.2,-.8> \\ <.9,.7,.5,-.7,-.7,-.1> & <.7,.6,.8,-.7,-.5,-.1> & <.9,.4,.6,-.1,-.7,-.5> & <.5,.2,.7,-.5,-.1,-.9> \\ <.9,.4,.2,-.6,-.3,-.7> & <.2,.2,.2,-.4,-.7,-.4> & <.9,.5,.5,-.6,-.5,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.7,.2,-.8,-.6,-.1> & <.3,.5,.2,-.5,-.5,-.2> & <.5,.4,.5,-.1,-.7,-.2> & <.2,.4,.8,-.5,-.5,-.6> \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python code like this:

```
C= np.array([ [ [0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-
0.8]],[[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]],

[[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]],

[[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]]])
```

D=

$$\begin{pmatrix} <.3,.4,.3,-.5,-.4,-.2> & <.1,.2,.7,-.5,-.2,-.3> & <.3,.2,.6,-.4,-.8,-.7> & <.2,.1,.3,-.2,-.4,-.4> \\ <.2,.2,.7,-.3,-.3,-.5> & <.3,.5,.6,-.6,-.7,-.4> & <.6,.5,.4,-.3,-.6,-.8> & <.3,.4,.4,-.3,-.5,-.3> \\ <.5,.3,.1,-.4,-.2,-.4> & <.5,.4,.3,-.3,-.8,-.2> & <.5,.8,.6,-.2,-.2,-.4> & <.4,.6,.5,-.1,-.6,-.5> \\ <.6,.1,.7,-.7,-.4,-.8> & <.4,.6,.4,-.4,-.2,-.5> & <.4,.9,.3,-.5,-.5,-.3> & <.4,.5,.4,-.3,-.7,-.4> \end{pmatrix}$$

The bipolar neutrosophic matrix D can be inputted in Python code like this:

D= np.array([[[0.3,0.4,0.3,-0.5,-0.4,-0.2], [0.1,0.2,0.7,-0.5,-0.2,-0.3], [0.3,0.2,0.6,-0.4,-0.8,-0.7], [0.2,0.1,0.3,-0.2,-0.4,-0.4]], [[0.2,0.2,0.7,-0.3,-0.3,-0.5], [0.3,0.5,0.6,-0.6,-0.7,-0.4], [0.6,0.5,0.4,-0.3,-0.6,-0.8], [0.3,0.4,0.4,-0.3,-0.5,-0.3]],

[[0.5,0.3,0.1,-0.4,-0.2,-0.4], [0.5,0.4,0.3,-0.3,-0.8,-0.2], [0.5,0.8,0.6,-0.2,-0.2,-0.4], [0.4,0.6,0.5,-0.1,-0.6,-0.5]], [[0.6,0.1,0.7,-0.7,-0.4,-0.8], [0.4,0.6,0.4,-0.4,-0.2,-0.5], [0.4,0.9,0.3,-0.5,-0.5,-0.3], [0.4,0.5,0.4,-0.3,-0.7,-0.4]]])

So, the addition matrix of two bipolar neutrosophic matrices is portrayed as follow

$$C_{BNS} \oplus D_{BNS} = \begin{pmatrix} <.65,.28,.06,-.35,-.58,-.68> & <.46,.08,.35,-.35,-.84,-.58> & <.79,.14,.30,-.32,-.94,-.88> & <.28,.05,.21,-.10,-.52,-.88> \\ <.92,.14,.35,-.21,-.79,-.55> & <.79,.30,.48,-.42,-.85,-.46> & <.96,.20,.24,-.03,-.88,-.90> & <.65,.08,.28,-.15,-.55,-.93> \\ <.65,.12,.02,-.24,-.44,-.82> & <.60,.08,.06,-.12,-.94,-.52> & <.95,.40,.30,-.12,-.60,-.52> & <.82,.30,.15,-.04,-.68,-.60> \\ <.96,.07,.14,-.56,-.76,-.82> & <.58,.30,.08,-.20,-.60,-.60> & <.70,.36,.15,-.05,-.85,-.44> & <.52,.20,.32,-.15,-.85,-.76> \end{pmatrix}$$

The result of addition matrix of two bipolar neutrosophic matrices C and D can be obtained by the call of the command addition (C, D):

```
>>> Addition(C, D)

Addition=

[[[ 0.65   0.28   0.06   0.35 -0.58 -0.68] [ 0.46   0.08   0.35   0.35 -0.84 -0.58] [ 0.79   0.14   0.3    0.32 -0.94 -0.88] [ 0.28   0.05   0.21   0.1    -0.52 -0.88]]

[[ 0.92   0.14   0.35   0.21 -0.79 -0.55]    [ 0.79   0.3    0.48   0.42 -0.85 -0.46]    [ 0.96   0.2    0.24   0.03 -0.88 -0.9 ]    [ 0.65   0.08   0.28   0.15 -0.55 -0.93]]

[[ 0.95   0.12   0.02   0.24 -0.44 -0.82]    [ 0.6    0.08   0.06   0.12 -0.94 -0.52]    [ 0.95   0.4    0.3    0.12 -0.6   -0.52]    [ 0.82   0.3    0.15   0.04 -0.68 -0.6 ]]

[[ 0.96   0.07   0.14   0.56 -0.76 -0.82]    [ 0.58   0.3    0.08   0.2    -0.6   -0.6 ]    [ 0.7    0.36   0.15   0.05 -0.85 -0.44]    [ 0.52   0.2    0.32   0.15 -0.85 -0.76]]]
```

3.7. Computing product of two bipolar neutrosophic matrices

The product of two bipolar neutrosophic matrices A and B is defined as follow:

$$A \odot B = R = \left[< r^P_{ij_T}, r^P_{ij_I}, r^P_{ij_F}, r^N_{ij_T}, r^N_{ij_I}, r^N_{ij_F} >\right]_{\mathrm{m} \times \mathrm{n}} \tag{13}$$

Where

$$r^P_{ij_T} = a^P_{ij_T}.\, b^P_{ij_T}, \quad r^N_{ij_T} = -\left(-a^N_{ij_T} - b^N_{ij_T} - a^N_{ij_T}.\, b^N_{ij_T}\right)$$
$$r^P_{ij_I} = a^P_{ij_I} + b^P_{ij_I} - a^P_{ij_I}.\, b^P_{ij_I}, \quad r^N_{ij_I} = -\left(a^N_{ij_I}.\, b^N_{ij_I}\right)$$
$$r^P_{ij_F} = a^P_{ij_F} + b^P_{ij_F} - a^P_{ij_F}.\, b^P_{ij_F}, \quad r^N_{ij_F} = -\left(a^N_{ij_F}.\, b^N_{ij_F}\right)$$

To generate the python program for finding the product operation of two bipolar neutrosophic matrices, simple call of the function **Product (A, B)** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for (13)
import numpy as np
A= np.array([   [          [0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7]                    ],
                 [          [0.1,0.12,0,-0.27,-0.44,-0.92], [0.5,0.33,0.58,-0.33,-0.24,-0.22]         ],
                 [          [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32]     ]])
B= np.array([   [          [0.32,0.4,0.1,-0.25,-0.54,-0.4], [0.13,0.2,0.11,-0.55,-0.35,-0.72]                 ],
```

```
                    [           [0.17,0.19,0.66,-0.87,-0.64,-0.92], [0.25,0.36,0.88,-0.33,-0.54,-0.22]            ],
                    [            [0.15,0.28,0.67,-0.39,-0.27,-0.55],[0.24,0.73,0.28,-0.26,-0.53,-0.52]      ]])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar
Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = each bipolar neutrosophic number with 6 tuple as usual
product=[]
def Product( A, B ):
    if A.shape == B.shape:
        dimA=A.shape
        for i in range (0,dimA[0]):          # for rows, here 3
            H=[]
            for j in range (0,dimA[1]):    # for columns, here 2
                H.extend([[ A[i][j][0]*B[i][j][0]) , A[i][j][1]+ B[i][j][1]- (A[i][j][1]*B[i][j][1]),
A[i][j][2]+ B[i][j][2]- (A[i][j][2]*B[i][j][2]), -(-A[i][j][3]-B[i][j][3]-A[i][j][3]*B[i][j][3]), -(A[i][j][4]*
B[i][j][4]), -(A[i][j][5]* B[i][j][5]) ] ])
            product.append(H)
    print(' Product = ', np.array(product))
```

Example 5. In this example we evaluate the product of the two bipolar neutrosophic matrices C and D of order 4X4:

C=

$$\begin{pmatrix} \langle .5,.7,.2,-.7,-.3,-.6 \rangle & \langle .4,.4,.5,-.7,-.8,-.4 \rangle & \langle .7,.7,.5,-.8,-.7,-.6 \rangle & \langle .1,.5,.7,-.5,-.2,-.8 \rangle \\ \langle .9,.7,.5,-.7,-.7,-.1 \rangle & \langle .7,.6,.8,-.7,-.5,-.1 \rangle & \langle .9,.4,.6,-.1,-.7,-.5 \rangle & \langle .5,.2,.7,-.5,-.1,-.9 \rangle \\ \langle .9,.4,.2,-.6,-.3,-.7 \rangle & \langle .2,.2,.2,-.4,-.7,-.4 \rangle & \langle .9,.5,.5,-.6,-.5,-.2 \rangle & \langle .7,.5,.3,-.4,-.2,-.2 \rangle \\ \langle .9,.7,.2,-.8,-.6,-.1 \rangle & \langle .3,.5,.2,-.5,-.5,-.2 \rangle & \langle .5,.4,.5,-.1,-.7,-.2 \rangle & \langle .2,.4,.8,-.5,-.5,-.6 \rangle \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python code like this:

```
C= np.array([ [ [0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-
0.8]], [[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]],
[[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]],
[[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]] ])
```

$$\mathbf{D}=\begin{pmatrix} \langle .3,.4,.3,-.5,-.4,-.2 \rangle & \langle .1,.2,.7,-.5,-.2,-.3 \rangle & \langle .3,.2,.6,-.4,-.8,-.7 \rangle & \langle .2,.1,.3,-.2,-.4,-.4 \rangle \\ \langle .2,.2,.7,-.3,-.3,-.5 \rangle & \langle .3,.5,.6,-.6,-.7,-.4 \rangle & \langle .6,.5,.4,-.3,-.6,-.8 \rangle & \langle .3,.4,.4,-.3,-.5,-.3 \rangle \\ \langle .5,.3,.1,-.4,-.2,-.4 \rangle & \langle .5,.4,.3,-.3,-.8,-.2 \rangle & \langle .5,.8,.6,-.2,-.2,-.4 \rangle & \langle .4,.6,.5,-.1,-.6,-.5 \rangle \\ \langle .6,.1,.7,-.7,-.4,-.8 \rangle & \langle .4,.6,.4,-.4,-.2,-.5 \rangle & \langle .4,.9,.3,-.5,-.5,-.3 \rangle & \langle .4,.5,.4,-.3,-.7,-.4 \rangle \end{pmatrix}$$

The bipolar neutrosophic matrix D can be inputted in Python code like this:

```
D= np.array([[[0.3,0.4,0.3,-0.5,-0.4,-0.2], [0.1,0.2,0.7,-0.5,-0.2,-0.3], [0.3,0.2,0.6,-0.4,-0.8,-0.7], [0.2,0.1,0.3,-0.2,-0.4,-
0.4]], [[0.2,0.2,0.7,-0.3,-0.3,-0.5], [0.3,0.5,0.6,-0.6,-0.7,-0.4], [0.6,0.5,0.4,-0.3,-0.6,-0.8], [0.3,0.4,0.4,-0.3,-0.5,-0.3]],

 [[0.5,0.3,0.1,-0.4,-0.2,-0.4], [0.5,0.4,0.3,-0.3,-0.8,-0.2], [0.5,0.8,0.6,-0.2,-0.2,-0.4], [0.4,0.6,0.5,-0.1,-0.6,-0.5]],

 [[0.6,0.1,0.7,-0.7,-0.4,-0.8], [0.4,0.6,0.4,-0.4,-0.2,-0.5], [0.4,0.9,0.3,-0.5,-0.5,-0.3], [0.4,0.5,0.4,-0.3,-0.7,-0.4]]])
```

So, the product matrix of two bipolar neutrosophic matrices is portrayed as follow

C_{BNS}⊙D_{BNS}=

$$\begin{pmatrix} <.15,.82,.44,-.85,-.12,-.12> & <.04,.52,.85,-.85,-1.16,-.12> & <.21,.76,.80,-.88,-.56,-.42> & <.02,.55,.79,-.60,-.008,-.32> \\ <.18,.76,.85,-.79,-.21,-.05> & <.21,.80,.92,-.88,-.35,-.04> & <.54,.70,.76,-.37,-.42,-0.40> & <.15,.52,.82,-.65,-.05,-.27> \\ <.45,.58,.28,-.76,-.06,-.28> & <.10,.52,.44,-.58,-.56,-.08> & <.45,.90,.80,-.68,-.10,-.08> & <.28,.80,.65,-.46,-.12,-.10> \\ <.54,.73,.76,-.94,-.24,-.08> & <.12,.80,.52,-.70,-.10,-.10> & <.20,.94,.65,-.55,-.35,-.06> & <.08,.70,.88,-.65,-.35,-.24> \end{pmatrix}$$

The result of product matrix of two bipolar neutrosophic matrices C and D can be obtained by the call of the command Product (C, D):

```
>>> Product(C, D)
Product=
[[[ 0.15   0.82   0.44 -0.85 -0.12 -0.12]    [ 0.04   0.52   0.85 -0.85 -0.16 -0.12]    [ 0.21   0.76   0.8   -0.88 -
0.56 -0.42] [ 0.02   0.55   0.79 -0.6   -0.08 -0.32]]
 [[ 0.18   0.76   0.85 -0.79 -0.21 -0.05]    [ 0.21   0.8    0.92 -0.88 -0.35 -0.04]    [ 0.54   0.7    0.76 -0.37 -
0.42 -0.4 ]    [ 0.15   0.52   0.82 -0.65 -0.05 -0.27]]
 [[ 0.45   0.58   0.28 -0.76 -0.06 -0.28]    [ 0.1    0.52   0.44 -0.58 -0.56 -0.08]    [ 0.45   0.9    0.8   -0.68 -
0.1   -0.08]    [ 0.28   0.8    0.65 -0.46 -0.12 -0.1 ]]
 [[ 0.54   0.73   0.76 -0.94 -0.24 -0.08] [ 0.12   0.8    0.52 -0.7   -0.1   -0.1 ] [ 0.2    0.94   0.65 -0.55 -0.35 -
0.06] [ 0.08   0.7    0.88 -0.65 -0.35 -0.24]]]
```

3.8. Computing transpose of bipolar neutrosophic matrix

To generate the python program for finding the transpose of bipolar neutrosophic matrix, simple call of the function **Transpose (A)** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for
transpose
import numpy as np
A=np.array([[ [0.3,0.6,1,-0.2,-0.54,-0.4], [0.1,0.2,0.8,-0.5,-0.34,-0.7]    ],
                 [ [0.1,0.12,0,-0.27,-0.44,-0.92],[0.5,0.33,0.58,-0.33,-0.24,-0.22]],
                [ [0.11,0.22,0.6,-0.29,-0.24,-0.52],[0.22,0.63,0.88,-0.28,-0.54,-0.32]] ])
#A.shape gives (3, 2, 6) the dimension of A. (row, column, numbers of element (Bipolar
Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 2 columns
# A.shape[2] = each bipolar neutrosophic number with 6 tuple as usual
def Transpose( A ):
    DimA= A. shape
    print (' the matrix ', DimA[0],' x ', DimA[1], ' dimension')
    trA = A.transpose()
    DimtrA= trA. shape
    print ('\n')
    print (' its transpose ', DimtrA[1],' x ', DimtrA[2], ' dimension')
    print ('\n' )
    print(' Transpose = ', trA)
```

Example 6. In this example we evaluate the transpose of the bipolar neutrosophic matrix C of order 4X4:

C=

$$\begin{pmatrix} <.5,.7,.2,-.7,-.3,-.6> & <.4,.4,.5,-.7,-.8,-.4> & <.7,.7,.5,-.8,-.7,-.6> & <.1,.5,.7,-.5,-.2,-.8> \\ <.9,.7,.5,-.7,-.7,-.1> & <.7,.6,.8,-.7,-.5,-.1> & <.9,.4,.6,-.1,-.7,-.5> & <.5,.2,.7,-.5,-.1,-.9> \\ <.9,.4,.2,-.6,-.3,-.7> & <.2,.2,.2,-.4,-.7,-.4> & <.9,.5,.5,-.6,-.5,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.7,.2,-.8,-.6,-.1> & <.3,.5,.2,-.5,-.5,-.2> & <.5,.4,.5,-.1,-.7,-.2> & <.2,.4,.8,-.5,-.5,-.6> \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python code like this:

```
C= np.array([ [ [0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-0.8]], [[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]],

[[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]],

[[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]] ])
```

So, the transpose matrix of bipolar neutrosophic matrices is portrayed as follow

<0.50, 0.70, 0.20,-0.70, -0.30, -0.60>	**<0.90, 0.70, 0.50,-0.70, -0.70, -0.10>**	**<0.30, 0.40, 0.20,-0.60, -0.30, -0.70>**	**<0.90, 0.70, 0.20,-0.80, -0.60, -0.10>**
<0.40, 0.40, 0.50,-0.70, -0.80, -0.40>	**<0.70, 0.60, 0.80,-0.70, -0.50, -0.10>**	**<0.20, 0.20, 0.20,-0.40, -0.70, -0.40>**	**<0.30, 0.50, 0.20,-0.50, -0.50, -0.20>**
<0.70, 0.70, 0.50,-0.80, -0.70, -0.60>	**<0.90, 0.40, 0.60,-0.10, -0.70, -0.50>**	**<0.90, 0.50, 0.50,-0.60, -0.50, -0.20>**	**<0.50, 0.40, 0.50,-0.10, -0.70, -0.20>**
<0.10, 0.50, 0.70,-0.50, -0.20, -0.80>	**<0.50, 0.20, 0.70,-0.50, -0.10, -0.90>**	**<0.70, 0.50, 0.30,-0.40, -0.20, -0.20>**	**<0.20, 0.40, 0.80,-0.50, -0.50, -0.60>**

```
>>> Transpose(C)
  The matrix 4 x4 dimension
  Its transpose 4 x 4 dimension
Transpose =
[[[ 0.5   0.9   0.9   0.9] [ 0.4   0.7   0.2   0.3]   [ 0.7   0.9   0.9   0.5] [ 0.1   0.5   0.7   0.2]]
[[ 0.7   0.7   0.4   0.7] [ 0.4   0.6   0.2   0.5] [ 0.7   0.4   0.5   0.4] [ 0.5   0.2   0.5   0.4]]
[[ 0.2   0.5   0.2   0.2] [ 0.5   0.8   0.2   0.2] [ 0.5   0.6   0.5   0.5] [ 0.7   0.7   0.3   0.8]]
[[-0.7 -0.7 -0.6 -0.8]     [-0.7 -0.7 -0.4 -0.5]      [-0.8 -0.1 -0.6 -0.1]      [-0.5 -0.5 -0.4 -0.5]]
[[-0.3 -0.7 -0.3 -0.6]      [-0.8 -0.5 -0.7 -0.5]      [-0.7 -0.7 -0.5 -0.7]      [-0.2 -0.1 -0.2 -0.5]]
[[-0.6 -0.1 -0.7 -0.1]     [-0.4 -0.1 -0.4 -0.2]     [-0.6 -0.5 -0.2 -0.2]     [-0.8 -0.9 -0.2 -0.6]]]
```

3.9 Computing composition of two bipolar neutrosophic matrices

To generate the python program for finding the composition of two bipolar neutrosophic matrices, simple call of the function **Composition ()** is defined as follow:

```
# BNM is represented by 3D Numpy Array => row, column and bipolar number with 6 tuples for Composition
#A.shape and B.shape returns (3, 3, 6) the dimension of A. (row, column, numbers of element (Bipolar Neutrosophic Number, 6 elements) )
# A.shape[0] = 3 rows
# A.shape[1] = 3 columns
# A.shape[2] = Each bipolar neutrosophic number has 6 tuple as usual
#One can use matrices with any dimensions but dimensions of two matrices must be the same and nxn
```

```
import math
import numpy as np
A= np.array( [ [ [0.3, 0.6, 1, -0.2, -0.54, -0.4], [0.1, 0.2, 0.8, -0.5, -0.34, -0.7], [0.020,0.021,0.022,-0.023,-
0.024,-0.025]    ],
[ [0.17,0.19,0.66,-0.87,-0.64,-0.92], [0.25,0.36,0.88,-0.33,-0.54,-0.22], [0.120,0.121,0.122,-0.123,-0.124,-
0.125]    ],
[ [0.15,0.28,0.67,-0.39,-0.27,-0.55],[0.24,0.73,0.28,-0.26,-0.53,-0.52], [0.220,0.221,0.222,-0.223,-0.224,-
0.225]    ]    ] )
B=np.array([ [0.11,0.22,0.6,-0.29,-0.24,-0.52], [0.32,0.4,0.1,-0.25,-0.54,-0.4], [0.13,0.2,0.11,-0.55,-0.35,-
0.72] ],
[ [0.100,0.101,0.102,-0.103,-0.104,-0.105], [1,0.111,0.112,-0.113,-0.114,-0.115], [0.720,0.821,0.152,-
0.143,-0.194,-0.1]    ],
[ [0,0.73,0.202,-0.203,-0.204,-0.205],    [0.22,0.63,0.88,-0.28,-0.54,-0.32], [0.3,0,0.47,-0.223,-0.254,-0.295]
] ] )
def Composition( A, B ):
    global composition
    composition=[]
    dimA = A.shape
    H=[ ]
    if A.shape == B.shape and dimA[0] == dimA[1]:
        for i in range (0,dimA[0]):
            for j in range (0,dimA[0]):
                counter0=0
                for d in range (0, dimA[0]):
                    if counter0 ==0:
                        maxtt    =        [ A[i][d][0],B[d][j][0] ]
                        maxT = min(maxtt)
                        minii =    [A[i][d][1],B[d][j][1] ]
                        minI =    max(minii)
                        minff = [ A[i][d][2],B[d][j][2]]
                        minF = max( minff)
                        minntt= [      A[i][d][3],B[d][j][3] ]
                        minNT = max (minntt)
                        maxnii = [ A[i][d][4],B[d][j][4]    ]
                        maxNI =    min( maxnii    )
                        maxnff= [      A[i][d][5],B[d][j][5]    ]
                        maxNF = min (maxnff)
                        counter0    = 1
                    else:
                        maxT1               = [    A[i][d][0],B[d][j][0]    ]
                        maxT11            = min(maxT1)
                        maxT112        = [    maxT11 ,    maxT    ]
```

```
                    maxT          =   max(maxT112)
                    minI1      =   [ A[i][d][1],B[d][j][1]   ]
                    minI11    =   max(minI1)
                    minI112   = [ minI11, minI     ]
                    minI          = min(   minI112)
                    minF1          =      [ A[i][d][2],B[d][j][2]   ]
                    minF11       =   max(minF1)
                    minF112     = [ minF11, minF]
                    minF             = min(minF112 )
                    minNT1       =    [   A[i][d][3],B[d][j][3]        ]
                    minNT11     =   max( minNT1     )
                    minNT112   =     [ minNT11, minNT ]
                    minNT          =   min( minNT112   )
                    maxNI1         = [ A[i][d][4],B[d][j][4] ]
                    maxNI11      = min(   maxNI1     )
                    maxNI112    = [   maxNI11, maxNI   ]
                    maxNI            = max(maxNI112)
                    maxNF1       =   [ A[i][d][5],B[d][j][5]   ]
                    maxNF11     =   min ( maxNF1 )
                    maxNF112 =   [ maxNF11, maxNF ]
                    maxNF          =   max (   maxNF112   )
                H.append( [maxT,   minI, minF, minNT, maxNI, maxNF] )
        composition.extend(H)
    global nested
    nested = [   ]
    for k in range( int(math.sqrt(len(composition))) ):
                nested.append(composition[k:k+int(math.sqrt(len(composition))) ] )
    print('Composition= ', np.array(nested))
```

Example 7. In this example we evaluate the composition of the two bipolar neutrosophic matrices C and D of order 4X4:

C=

$$\begin{pmatrix} <.5,.7,.2,-.7,-.3,-.6> & <.4,.4,.5,-.7,-.8,-.4> & <.7,.7,.5,-.8,-.7,-.6> & <.1,.5,.7,-.5,-.2,-.8> \\ <.9,.7,.5,-.7,-.7,-.1> & <.7,.6,.8,-.7,-.5,-.1> & <.9,.4,.6,-.1,-.7,-.5> & <.5,.2,.7,-.5,-.1,-.9> \\ <.9,.4,.2,-.6,-.3,-.7> & <.2,.2,.2,-.4,-.7,-.4> & <.9,.5,.5,-.6,-.5,-.2> & <.7,.5,.3,-.4,-.2,-.2> \\ <.9,.7,.2,-.8,-.6,-.1> & <.3,.5,.2,-.5,-.5,-.2> & <.5,.4,.5,-.1,-.7,-.2> & <.2,.4,.8,-.5,-.5,-.6> \end{pmatrix}$$

The bipolar neutrosophic matrix C can be inputted in Python code like this:

```
C= np.array([ [ [0.5,0.7,0.2,-0.7,-0.3,-0.6], [0.4,0.4,0.5,-0.7,-0.8,-0.4], [0.7,0.7,0.5,-0.8,-0.7,-0.6], [0.1,0.5,0.7,-0.5,-0.2,-0.8]], [[0.9,0.7,0.5,-0.7,-0.7,-0.1], [0.7,0.6,0.8,-0.7,-0.5,-0.1], [0.9,0.4,0.6,-0.1,-0.7,-0.5], [0.5,0.2,0.7,-0.5,-0.1,-0.9]], [[0.9,0.4,0.2,-0.6,-0.3,-0.7], [0.2,0.2,0.2,-0.4,-0.7,-0.4], [0.9,0.5,0.5,-0.6,-0.5,-0.2], [0.7,0.5,0.3,-0.4,-0.2,-0.2]], [[0.9,0.7,0.2,-0.8,-0.6,-0.1], [0.3,0.5,0.2,-0.5,-0.5,-0.2], [0.5,0.4,0.5,-0.1,-0.7,-0.2], [0.2,0.4,0.8,-0.5,-0.5,-0.6]] ])
```

D=

$$\begin{pmatrix} <.3,.4,.3,-.5,-.4,-.2> & <.1,.2,.7,-.5,-.2,-.3> & <.3,.2,.6,-.4,-.8,-.7> & <.2,.1,.3,-.2,-.4,-.4> \\ <.2,.2,.7,-.3,-.3,-.5> & <.3,.5,.6,-.6,-.7,-.4> & <.6,.5,.4,-.3,-.6,-.8> & <.3,.4,.4,-.3,-.5,-.3> \\ <.5,.3,.1,-.4,-.2,-.4> & <.5,.4,.3,-.3,-.8,-.2> & <.5,.8,.6,-.2,-.2,-.4> & <.4,.6,.5,-.1,-.6,-.5> \\ <.6,.1,.7,-.7,-.4,-.8> & <.4,.6,.4,-.4,-.2,-.5> & <.4,.9,.3,-.5,-.5,-.3> & <.4,.5,.4,-.3,-.7,-.4> \end{pmatrix}$$

The bipolar neutrosophic matrix D can be inputted in Python code like this:

D= np.array([[[0.3,0.4,0.3,-0.5,-0.4,-0.2], [0.1,0.2,0.7,-0.5,-0.2,-0.3], [0.3,0.2,0.6,-0.4,-0.8,-0.7], [0.2,0.1,0.3,-0.2,-0.4,-0.4]], [[0.2,0.2,0.7,-0.3,-0.3,-0.5], [0.3,0.5,0.6,-0.6,-0.7,-0.4], [0.6,0.5,0.4,-0.3,-0.6,-0.8], [0.3,0.4,0.4,-0.3,-0.5,-0.3]], [[0.5,0.3,0.1,-0.4,-0.2,-0.4], [0.5,0.4,0.3,-0.3,-0.8,-0.2], [0.5,0.8,0.6,-0.2,-0.2,-0.4], [0.4,0.6,0.5,-0.1,-0.6,-0.5]], [[0.6,0.1,0.7,-0.7,-0.4,-0.8], [0.4,0.6,0.4,-0.4,-0.2,-0.5], [0.4,0.9,0.3,-0.5,-0.5,-0.3], [0.4,0.5,0.4,-0.3,-0.7,-0.4]]])

So, the composition matrix of two bipolar neutrosophic matrices is portrayed as follow

$C_{BNS} \odot D_{BNS}$=

$$\begin{pmatrix} <.5,.4,.3,-.5,-.4,-.5> & <.5,.5,.5,-.6,-.2,-.4> & <.5,.5,.5,-.5,-.5,-.6> & <.4,.4,.3,-.3,-.4,-.4> \\ <.5,.5,.5,-.6,-.2,-.4> & <.5,.5\ .5,-.5,-.5,-.6> & <.4,.4,.3,-.3,-.4,-.4> & <.5,.2,.5,-.5,-.4,-.2> \\ <.5\ .5,.5,-.5,-.5,-.6> & <.4,.4,.4,-.3,-.4,-.4> & <.5,.2,.5,-.5,-.4,-.2> & <.5,.4,.6,-.6,-.2,-.3> \\ <.4,.4,.3,-.3,-.4,-.1> & <.5,.2,.5,-.5,-.4\ -.2> & <.5,.4,.6,-.6,-.2,-.3> & <.6,.6,.6,-.5,-.5,-.5> \end{pmatrix}$$

The result of composition t matrix of two bipolar neutrosophic matrices C and D can be obtained by the call of the command Composition (C, D):

```
>>> Composition(C, D)

Composition=

[[[ 0.5   0.4   0.3 -0.5 -0.4 -0.5] [ 0.5   0.5   0.5 -0.6 -0.2 -0.4] [ 0.5   0.5   0.5 -0.5 -0.5 -0.6] [ 0.4   0.4   0.3 -0.3 -0.4 -0.4]]

[[ 0.5   0.5   0.5 -0.6 -0.2 -0.4] [ 0.5   0.5   0.5 -0.5 -0.5 -0.6] [ 0.4   0.4   0.3 -0.3 -0.4 -0.4] [ 0.5   0.2   0.5 -0.5 -0.4 -0.2]]

[[ 0.5   0.5   0.5 -0.5 -0.5 -0.6] [ 0.4   0.4   0.3 -0.3 -0.4 -0.4] [ 0.5   0.2   0.5 -0.5 -0.4 -0.2] [ 0.5   0.4   0.6 -0.6 -0.2 -0.3]]

[[ 0.4   0.4   0.3 -0.3 -0.4 -0.4] [ 0.5   0.2   0.5 -0.5 -0.4 -0.2] [ 0.5   0.4   0.6 -0.6 -0.2 -0.3] [ 0.6   0.6   0.6 -0.5 -0.5 -0.5]]]
```

4. Conclusion

In this paper, we have presented a useful Python tool for the calculations of matrices obtained by bipolar neutrosophic sets. The matrices have nested list data type, in other words, multi-dimensional arrays in the Python Programming Language. The importance of this work, is that the proposed Python tool can be used also for fuzzy matrices, bipolar fuzzy matrices, intuitionistic fuzzy matrices, bipolar intuitionistic fuzzy matrices and single valued neutrosophic matrices. This work will be extending with the implementation of Bipolar Complex Neutrosophic Matrices in the future. We have used Python Numpy module in order to provide convenience for possible users. We hope that the tool might be useful in data science, physics, scientific computing, decision making, engineering studies and other fields.

Author Contributions

S.T. implemented codes of the bipolar neutrosophic matrices and their operations and created the scripts on Python 3.7 by using Numpy module. S.B. offered the project paper and reviewed the implementations. Conceptualization, S.B. and S.T.; Methodology, S.T.; Validation, S.B., S.T., A.B., M.T and F.S.; Investigation, S.B. and S.T.; Resources, S.B., S.T., A.B., M.T and F.S; Writing-Original Draft

Preparation, S.B..; Writing—Review and Editing, S.B., S.T., A.B., M.T and F.S.; Supervision, S.B. and F.S.

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On NGSR Closed Sets in Neutrosophic Topological Spaces

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Anitha S, Mohana K, Florentin Smarandache (2019). On NGSR Closed Sets in Neutrosophic Topological Spaces. *Neutrosophic Sets and Systems* 28, 171-178

Abstract: The intention of this paper is to introduce the concept of GSR-closed sets in terms of neutrosophic topological spaces. Some of the properties of NGSR-closed sets are obtained. In addition, we inspect NGSR-continuity and NGSR-contra continuity in neutrosophic topological spaces.

Keywords: neutrosophic topology, NGSR-closed set, NGSR-continuous, NGSR-contra continuous mappings.

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1. Introduction

In 1965, fuzzy concept was proposed by Zadeh [43] and he studied membership function. Chang [14] developed the theory of fuzzy topology in 1967. The notions of inclusion, union, intersection, complement, relation, convexity, and so forth, are expanded to such sets and several properties of these notions are established by various authors.

Atanassov [10, 11, 12] generalized the idea of fuzzy set to intuitionistic fuzzy set by adding the degree of non-membership. The intuitionistic fuzzy topology was advanced by Coker [16] using the notion of intuitionistic fuzzy sets. Intuitionistic fuzzy point was given by Coker et.al [15]. These approaches gave a wide field for exploration in the area of intuitionistic fuzzy topology and its application. Burillo et al.[13]studied the intuitionistic fuzzy relation and their properties. Thakur et.al [44] introduced generalized closed set in intuitionistic fuzzy topology. Various researchers [8, 24, 26, 33, 37, 38] extended the results of generalization of various Intuitionistic fuzzy closed sets in many directions.

The concepts of neutrosophy was introduced by Florentin Smarandache [18, 19, 20] in which he developed the degree of indeterminacy. In comparing with more uncertain ideology, the neutrosophic set can accord with indeterminacy situation. Salama et.al [34,35,36] transformed the idea of neutrosophic crisp set into neutrosophic topological spaces and introduced generalized neutrosophic set and generalized neutrosophic topological Spaces. Ishwarya et.al [22] studied Neutrosophic semi open sets in Neutrosophic topological spaces. Abdel-Basset et.al [1,2,3,4,5,6] gave a novel neutrosophic approach. Many researchers [28, 30, 31, 41, 42] added and studied semi open sets, α open sets, pre-open sets, semi alpha open sets etc., and developed several interesting properties and applications in Neutrosophic Topology. Several authors [7, 25, 27, 32, 39, 44] have contributed in topological spaces.

Mohana K et.al [29] introduced gsr -closed sets in soft topology in 2017. In this article we tend to provide the idea of NGSR-closed sets and NGSR-open sets. Also, we presented NGSR continuous and NGSR-contra continuous mappings.

2 Preliminaries

Definition 2.1. [20] Let X be a non-empty fixed set. A neutrosophic set (NS) A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ where $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ represent the degree of membership, degree of indeterminacy and the degree of nonmembership respectively of each element $x \in X$ to the set A.

A Neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ can be identified as an ordered triple $\langle \mu_A(x), \sigma_A(x), \nu_A(x)\rangle$ in $]^-0, 1^+[$ on X.

Definition 2.2. [20] Let $A = \langle \mu_A(x), \sigma_A(x), \nu_A(x)\rangle$ be a NS on X, then the complement C(A) may be defined as

1. $C(A) = \{\langle x, 1 - \mu_A(x), 1 - \nu_A(x)\rangle : x \in X\}$
2. $C(A) = \{\langle x, \nu_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$
3. $C(A) = \{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)\rangle : x \in X\}$

Note that for any two neutrosophic sets A and B,

4. $C(A \cup B) = C(A) \cap C(B)$
5. $C(A \cap B) = C(A) \cup C(B)$.

Definition 2.3. [20] For any two neutrosophic sets $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \sigma_B(x), \nu_B(x)\rangle : x \in X\}$ we may have

1. $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x), \sigma_A(x) \le \sigma_B(x)$ and $\nu_A(x) \ge \nu_B(x) \forall\, x \in X$
2. $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x), \sigma_A(x) \ge \sigma_B(x)$ and $\nu_A(x) \ge \nu_B(x) \forall\, x \in X$
3. $A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x)$ and $\nu_A(x) \vee \nu_B(x)\rangle$
4. $A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x)$ and $\nu_A(x) \vee \nu_B(x)\rangle$
5. $A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x)$ and $\nu_A(x) \wedge \nu_B(x)\rangle$
6. $A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x)$ and $\nu_A(x) \wedge \nu_B(x)\rangle$

Definition 2.4. [34] A neutrosophic topology (NT) on a non-empty set X is a family τ of neutrosophic subsets in X satisfies the following axioms:

(NT_1) $0_N, 1_N \in \tau$
(NT_1) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$
(NT_1) $\cup G_i \in \tau\ \forall \{G_i : i \in J\} \subseteq \tau$

Definition 2.5. [34] Let A be an NS in NTS X. Then

Nint(A) = ∪ {G : G is an NOS in X and G ⊆ A} is called a neutrosophic interior of A

Ncl(A) = ∩ {K : K is an NCS in X and A ⊆ K} is called a neutrosophic closure of A

Definition 2.6. [18] A NS A of a NTS X is said to be

(1) a neutrosophic pre-open set (NPOS) if $A \subseteq NInt(NCl(A))$ and a neutrosophic pre-closed(NPCS) if $NCl(NInt(A)) \subseteq A$.

(2) a neutrosophic semi-open set (NSOS) if A ⊆ NCl(NInt(A)) and a neutrosophic semi-closed set (NSCS) if $\mathrm{NInt}(\mathrm{NCl}(A)) \subseteq A$.

(3) a neutrosophic α-open set (NαOS) if $A \subseteq \mathrm{NInt}(\mathrm{NCl}(\mathrm{NInt}(A)))$ and a neutrosophic α-closed set (NαCS) if $\mathrm{NCl}(\mathrm{NInt}(\mathrm{NCl}(A))) \subseteq A$.

(4) a neutrosophic regular open set (NROS) if $A = \mathrm{Nint}(\mathrm{Ncl}(A))$ and a neutrosophic regular closed set (NRCS) if $\mathrm{Ncl}(\mathrm{Nint}(A)) = A$.

Definition 2.7. [22] Consider a NS A in a NTS (X, τ).Then the neutrosophic semi interior and the neutrosophic
semi closure are defined as

Nsint(A) = ∪{G: G is a N Semi open set in X and G ⊆ A}

Nscl(A) = ∩ {K: K is a N Semi closed set in X and A ⊆ K }

Definition 2.8. [38] A subset A of a neutrosophic topological space (X, τ) is called a neutrosophic α generalized closed (Nαg-closed) set if Nαcl(A) ⊆ U whenever A ⊆ U and U is neutrosophic α-open in (X, τ).

3. NGSR closed sets

Definition 3.1. A NS A in a NTS X is stated to be a neutrosophic gsr closed set (NGSR-Closed set) if Nscl (A) ⊆ U for every A ⊆ U and U is a NROS (Neutrosophic Regular Open set) in X.

The complement C(A) of a NGSR-closed set A is a NGSR-open set in X.

Example 3.2. Let X = {a, b} and $\tau = \{0_1, G, 1_N\}$ be NT in which G_1 ⟨x, (0.4, 0.1), (0.3,0.2), (0.5, 0.5)⟩ and G_2 = ⟨x, (0.4,0,4), (0.4, 0.3), (0.5,0.4)⟩. Here A = ⟨x, (0.4,0.4), (0.3,0.2), (0.4,0.5)⟩ is an NGSR-closed set.

Theorem 3.3. Each NCS is a NGSR-closed set in X.

Proof. Let A ⊆ U wherein U is a NROS in X. Let A be an NCS in X.

We got Nscl(A) ⊆ Ncl(A) ⊆ U. Consequently A is a NGSR-closed set in X.

Example 3.4. Let X = {a, b} and $\tau = \{0_1, G, 1_N\}$ be an NT having G_1 = ⟨x, (0.4, 0.1), (0.3, 0.2), (0.5, 0.5)⟩ and G_2 = ⟨x, (0.4, 0.4), (0.4; 0.3), (0.5, 0.4)⟩ .Here A = ⟨x, (0.4,0.4), (0.3,0.2), (0.4,0.5)⟩ is an NGSR-closed set, however not NCS.

Theorem 3.5. Each Nα − closed set is a NGSR-closed set in X.

Proof. Let A ⊆ U inwhich U is a NROS in X. Let A be an Nα − closed set in X.

Now Nscl(A) ⊆ N ⊆ cl(A) ⊆ U . Consequently A is a NGSR-closed set in X.

Example 3.6. Let X ={a, b} and $\tau = \{0_1, G, 1_N\}$ be an NT in which

G_1 = ⟨x, (0.6,0.2), (0.1,0.5), (0.5, 0.4)⟩ and G_2 = ⟨ x, (0.5, 03), (0.3, 0.2), (0.6,0.4) ⟩

Here A = ⟨x, (0.6,0.3), (0.1,0.6), (0.5,0.4) ⟩ is an NGSR-closed set, but not Nα-closed set as

$$\mathrm{Ncl}(\mathrm{Nint}(\mathrm{Ncl}(A))) = C(A) \nsubseteq A.$$

Theorem 3.7. Each Nsemi-closed set is a NGSR-closed set in X.

Proof. Suppose A is an Nsemi-closed set and $A \subseteq U$ wherein U is a NROS in X. Now $(A) = A \cup Nint(Ncl(A)) \subseteq A \cup A = A$. Therefore A is a NGSR-closed set in X.

Example 3.8. Let X = {a, b} and $\tau = \{0_1, G, 1_N\}$ be an NT in which

$G_1 = \langle x, (0.4,0.5), (0.3,0.2), (0.5,0.5)\rangle$ and $G_2 = \langle x, (0.4, 0.4), (0.4,0.3), (0.5, 0.4)\rangle$

Then $A = \langle x, (0.4,0.4), (0.3,0.2), (0.4,0.5)\rangle$ is an NGSR-closed set, however not Nsemi-closed set as $Nint(Ncl(A)) = G_1 \nsubseteq A$.

Theorem 3.9. Each $N\alpha G - closed$ set is a NGSR-closed set in X.

Proof. Let $A \subseteq U$ where U is a NROS in X. Let A be an $N\alpha G - closed$ set in X. Now $Nscl(A) \subseteq N\alpha cl(A) \subseteq U$. Therefore A is a NGSR-closed set in X.

Example 3.10. Let X = {a, b} and $\tau = \{0_1, G, 1_N\}$ be an NT where

$G_1 = \langle x, (0.6, 0.2), (0.1,0.5), (0.5, 0.4) \rangle$ and $G_2 = \langle x, (0.5, 0.3), (0.3,0.2), (0.6,0.4) \rangle$

Then $A = \langle x, (0.6,0.3), (0.1,0.6), (0.5,0.4) \rangle$ is an NGSR-closed set but not NαG-closed set.

Remark 3.11. The counter examples shows that NGSR-closed set is independent of NPCS.

Example 3.12. Let X = {a, b} and $\tau = \{0_1, G, 1_N\}$ be an NT where

$G_1 = \langle x, (0.6, 0.2), (01, 0.5), (0.5,0.4) \rangle$ and $G_2 = \langle x, (0.5,03.), (0.3, 0.2), (0.6,0.4) \rangle$

Here $A = \langle x, (0.6,0.3), (0.1,0.6), (0.5,0.4) \rangle$ be an NGSR-closed set, but not NPCS as Ncl(Nint(A)) = C(B) $\nsubseteq$ A .

Example 3.13. Let X ={a, b} and $\tau = \{0_1, G, 1_N\}$ be an NT where

$G_1 = \langle x, (0{:}5;\ 0{:}4), (0{:}3;\ 0{:}2), (0{:}5;\ 0{:}6) \rangle$,$G_2 = \langle x, (0{:}8;\ 0{:}7), (0{:}4;\ 0{:}3), (0{:}2;\ 0{:}3) \rangle$ and

$G_3 = \langle x, (0{:}2;\ 0{:}1), (0{:}3;\ 0{:}2), (0{:}8;\ 0{:}9) \rangle$

Then $A = \langle x, (0.5,0.3), (0.3,0.2), (0.5,0.7) \rangle$ is an NPCS, but not NGSR-closed set.

Theorem 3.14. Consider a NTS (X, τ). Then for each A ∈ NGSR-closed set and for each B ∈ NS in X, $A \subseteq B \subseteq Nscl(A)$ implies B ∈ NGSR-closed in (X, τ) .

Proof. Assume that $B \subseteq U$ and U is a NROS in (X, τ) which shows that $A \subseteq B, A \subseteq U$. Via speculation, B ⊆ Nscl(A). Consequently $Nscl(B) \subseteq Nscl(Nscl(A)) = Nscl(A) \subseteq U$, given that A is an NGSR-closed set in (X, τ). As a result B ∈ NGSR-closed in (X, τ).

Theorem 3.15. Consider a NROS A and a NGSR-closed set in (X, τ), then A is a NSemi-closed set in (X, τ).

Proof. Due to the fact A ⊆ A and A is a NROS in (X, τ),Via speculation, Nscl(A) ⊆ A.

However A ⊆ Nscl(A). Therefore Nscl(A) = A. Consequently A is a Nsemi-closed set in (X, τ).

Theorem 3.16. Let (X, τ) be a NTS. Then for each A ∈ NGSR-open X and for every B ∈ NS(X), Nsint (A) ⊆B⊆A implies B ∈ NGSR-open set in X.

Proof. Let A be any NGSR-open set of X and B be any NS of X. By means of speculation Nsint ⊆ B ⊆A. Then C(A) is a NGSR-closed in X and C(A) ⊆ C(B) ⊆Nscl(C(A)). By using Theorem 3.5, C(B) is a NGSRclosed in (X, τ). Thus B is a NGSR-Open in (X, τ). Hence B ∈ NGSR-open in X.

Theorem 3.17. A NS A is a NGSR-open in (X, τ) if and only if F _ Nsint(A) everytime F is a NRCS in (X, τ) and F ⊆ A.

Proof. **Necessity:** Assume that A is a NGSR-open in (X, τ) and F is a NRCS in (X, τ) such that F ⊆ A. Then C(F) is a NROS and C(A) ⊆ C(F). Via speculation C(A) is a NGSR-closed set in (X, τ), we've Nscl(C(A)) ⊆ C(F). Therefore F ⊆ Nsint(A).

Sufficiency: Let U be a NROS in (X, τ) such that C(A) ⊆ U. By hypothesis, C(U) ⊆ Nsint(A). Consequently Nscl(C(A)) ⊆ U and C(A) is an NGSR-closed set in (X, τ). Thus A is a NGSR-open set in (X, τ).

Theorem 3.18. A is Nsemi-closed if it is both Nsemi-open and NGSR-closed.
Proof. Considering A is each Nsemi-open and NGSR-closed set in X, then Nscl(A) ⊆ A. We additionally have A ⊆Nscl(A). Accordingly, Nscl(A) = A. Therefore, A is an Nsemi-closed set in X.

4 On NGSR-Continuity and NGSR-Contra Continuity

Definition 4.1. Let f be a mapping from a neutrosophic topological space (X, τ) to a neutrosophic topological space (Y, σ). Then f is referred to as a neutrosophic gsr-continuous(NGSR-continuous) mapping if $f^{-1}(B)$ is a NGSR-open set in X, for each neutrosophic-open set B in Y .

Theorem 4.2. Consider a mapping $f : (X,\tau) \to (Y,\sigma)$. Then (1) and (2) are equal.
(1) f is NGSR-continuous
(2) The inverse image of each N-closed set B in Y is NGSR-closed set in X.
Proof. This can be proved with the aid of using the complement and Definition 4.1.

Theorem 4.3. Consider an NGSR-continuous mapping $f : (X,\tau) \to (Y,\sigma)$ then the subsequent assertions hold:
(1) for all neutrosophic sets A in X, $f(NGSRNcl(A)) \subseteq Ncl(f(A))$
(2) for all neutrosophic sets B in Y, $NGSRNcl\,(f^{-1}(B)) \subseteq f^{-1}(Ncl(B))$.
Proof. (1) Let Ncl(f(A)) be a neutrosophic closed set in Y and f be NGSR-continuous, then it follows that f^{-1} (Ncl(f(A))) is NGSR-closed in X. In view that $A \subseteq f^{-1}\,(Ncl(f(A))), NGSRcl(A) \subseteq f^{-1}\,(Ncl(f(A)))$. Hence, $f(NGSRNcl(A)) \subseteq Ncl(f(A))$.
(2) We get $f(NGSRcl(f^{-1}\,(B))) \subseteq Nclf(f^{-1}\,(B))) \subseteq Ncl(B)$.
Hence, $NGSRcl(f^{-1}\,(B)) \subseteq f^{-1}\,(Ncl(B))$ by way of changing A with B in (1).

Definition 4.4. Let f be a mapping from a neutrosophic topological space (X, τ) to a neutrosophic topological space (Y, σ). Then f is known as neutrosophic gsr-contra continuous(NGSR-contra continuous) mapping if f^{-1} (B) is a NGSR-closed set in X for each neutrosophic-open set B in Y .

Theorem 4.5. Consider a mapping $f : (X,\tau) \to (Y,\sigma)$. Then the subsequent assertions are equivalent:
(1) f is a NGSR-contra continuous mapping
(2) f^{-1} (B) is an NGSR-closed set in X, for each NOS B in Y.
Proof. (1) ⟹ (2) Assume that f is NGSR-contra continuous mapping and B is NOS in Y. Then B_c is an NCS in Y. It follows that, $f^{-1}(B^c)$ is an NGSR-open set in X. For this reason, f^{-1} (B) is an NGSR-closed set in X.
(2) ⟹ (1) The converse is similar.

Theorem 4.6. Consider a bijective mapping $f : (X,\tau) \to (Y,\sigma)$. from an

NTS(X, τ)into an NTS(Y, σ).If $Ncl(f(A)) \subseteq f(NGSRint(A))$, for each NS B in X, then the mapping f is NGSR-contra continuous.

Proof. Consider a NCS B in Y. Then $Ncl(B) = B$ and f is onto, by way of assumption, $f(NGSRint(f^{-1}(B))) \subseteq Ncl(f(f^{-1}(B))) = Ncl(B) = B$. Consequently, $f^{-1}(f(NGSRint(f^{-1}(B)))) \subseteq f^{-1}(B)$. Additionally due to the fact that f is an into mapping, we have $NGSRint(f^{-1}(B)) = f^{-1}(f(NGSRint(f^{-1}(B)))) \subseteq f^{-1}(B)$.Consequently, $NGSRint(f^{-1}(B)) = f^{-1}(B)$, so f1(B) is an NGSR-open set in X. Hence, f is a NGSR-contra continuous mapping.

5. Conclusion and Future work

Neutrosophic topological space concept is used to deal with vagueness. This paper introduced NGSR closed set and some of its properties were discussed and derived some contradicting examples. This idea can be developed and extended in the real life applications such as in medical field and so on.

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Neutrosophic Weibull distribution and Neutrosophic Family Weibull Distribution

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Abstract: Many problems in life are filled with ambiguity, uncertainty, impreciseness etc., therefore we need to interpret these phenomena. In this paper, we will focus on studying neutrosophic Weibull distribution and its family, through explaining its special cases , and the functions' relationship with neutrosophic Weibull such as Neutrosophic Inverse Weibull, Neutrosophic Rayleigh, Neutrosophic three parameter Weibull, Neutrosophic Beta Weibull, Neutrosophic five Weibull, Neutrosophic six Weibull distributions (various parameters).This study will enable us to deal with indeterminate or inaccurate problems in a flexible manner. These problems will follow this family of distributions. In addition, these distributions are applied in various domains, such as reliability, electrical engineering, Quality Control etc. Some properties and examples for these distributions are discussed.

Keywords: Weibull distribution, Neutrosophic logic, Neutrosophic number, Neutrosophic Weibull, Neutrosophic inverse Weibull, Neutrosophic Rayleigh, Neutrosophic Weibull with (three, four, five, six) parameters.

1. Introduction

The real world is overstuffed with vague, unclear, fuzzy (problems, situations, ideas). The classical probability ignores extreme, aberrant, unclear values, and therefore a new adequate tool had to emerge. Neutrosophic logic was introduced by Smarandache in 1995, as a generalization for the fuzzy logic and intuitionistic fuzzy logic [5, 6]. Smarandache [3, 7, 8] and Salamaa.et.al [3, 4] were presented the fundamental concepts of neutrosophic set. Smarandache extended the fuzzy set to the neutrosophic set [1, 3], introducing the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where]-0, 1+[is the non-standard unit interval. Smarandache presented the neutrosophic statistics, which the data can be enigmatic, vague, imprecise, incomplete, even unknown.

The extension of classical distributions according to the neutrosophic logic means that the parameters of classical distribution take undetermined values[1,2,3,10], which allows dealing with all the situations that one may encounter while working with statistical data and especially when working with vague and inaccurate statistical data, such as the sample size may not be exactly known. The sample size could be between 50 and 70; the statistician is not sure about 20 sample persons if they belong or not to the population of interest; or because the 20 sample persons only partially belong to the population of interest, while partially they don't belong. This mean, in classical statistics all data are determined, while in neutrosophic statistic the data or a part of it are indeterminate in some degree. The neutrosophic researchers presented studies in objects different in neutrosophic statistic, such as Salama, Rafief [29], Abdel-Basset and others, see [20-28]. For more than a decade, Weibull distribution has been applied extensively in many areas and particularly used in the analysis of lifetime data for reliability engineering or biology (Rinne, 2008). However, the Weibull distribution has a weakness for modeling phenomenon with non-monotone failure rate. In this paper, we will define and study the Neutrosophic Weibull distribution, Neutrosophic family Weibull distribution for varies cases as Neutrosophic Weibull, Neutrosophic beta Weibull, Neutrosophic inverse Weibull, Neutrosophic Rayleigh, Neutrosophic with (three, four, five, six) parameters, and discuss some properties of these distributions, illustrated through examples and graphs.

2. Terminologies

In this section, we present some basic axioms of neutrosophic logic, and in particular, the work of Smarandache in [3, 7, 8] and Salama et al. [3, 4]. Smarandache introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where] 0-,1+[is nonstandard unit interval.

2.1 Some definitions

Definition 1 [1, 2, 3] "Neutrosophy is a new branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their".

Definition 2 [1, 2, 3] Let T, I,F be real standard or nonstandard subsets of] 0-,1+[, with
Sup_T=t_sup, inf_T=t_inf
Sup_I=i_sup, inf_I=i_inf
Sup_F=f_sup, inf_F=f_inf
n-sup=t_sup+i_sup+f_sup
n-inf=t_inf+i_inf+f_inf,
T, I, F are called neutrosophic components.

Definition 3 [4, 5] Let X be a non-empty fixed set. A neutrosophic set (NS for short) A is an object having the form {x, $(\mu_A(x), \delta_A(x), \gamma_A(x)): x \in X\}$, where $\mu_A(x)$, $\delta_A(x)$ *and* $\gamma_A(x)$ which represent the degree of member ship function , the degree of indeterminacy , and the degree of non-member ship , respectively of each element x $\in$ X to the set A .

Definition 4 [4, 5] The NSS 0_N and 1_N in X as follows:

0_N may be defined as:

$$0_1 = \{x \;\; 0,0,1: x \in X\}$$
$$0_2 = \{x \;\; 0,1,1: x \in X\}$$
$$0_3 = \{x \;\; 0,1,0: x \in X\}$$
$$0_4 = \{x \;\; 0,0,0: x \in X\}$$

1_N may be defined as:

$$1_1 = \{x \;\; 1,0,0: x \in X\}$$
$$1_2 = \{x \;\; 1,0,1: x \in X\}$$
$$1_3 = \{x \;\; 1,0,0: x \in X\}$$
$$1_4 = \{x \;\; 1,1,1: x \in X\}$$

2.2 *Neutrosophic probability*

Neutrosophic probability is a generalization of the classical probability in which the chance that event $A = \{X, A_1, A_2, A_3\}$ occurs is P(A1) true, P(A2) indeterminate , P(A3) false on a space X, then $NP(A) = \{X, P(A_1), P(A_2), P(A_3)\}$.

Definition 5 [3,4]

Let A and B be a neutrosophic events on a space X, then $NP(A) = \{X, P(A_1), P(A_2), P(A_3)\}$

And $NP(B) = \{X, P(B_1), P(B_2), P(B_3)\}$ their neutrosophic probabilities.

Definition 6 [3,4]

Let A and B be a neutrosophic events on a space X, and$NP(A) = \{X, P(A_1), P(A_2), P(A_3)\}$, and $NP(B) = \{X, P(B_1), P(B_2), P(B_3)\}$ are neutrosophic probabilities. Then we define

$NP(A \cap B) = \{X, P(A_1 \cap B_1), P(A_2 \cap B_2), P(A_3 \cap B_3)\}$

$NP(A \cup B) = \{X, P(A_1 \cup B_1), P(A_2 \cup B_2), P(A_3 \cup B_3)\}$

$NP(A^c) = \{X, P(A_1{}^c), P(A_2{}^c), P(A_3{}^c)\}$

3 Weibull Distribution

Weibull distribution is one of most important distributions because it is widely used in reliability analysis, industrial and electrical engineering, in distribution of life time, in extreme value theory, … etc.; this distribution has various cases dependent on number of parameters such as two or three or five parameters α is the scale parameter, β is the shape parameter and γ is the location parameter. Also, it can be used to model a state where the failure function increases, decreases or remains constant with time.

4 Neutrosophic Weibull Distribution

A neutrosophic Weibull distribution (Neut-Weibull) of a continuous variable X is a classical Weibull distribution of x, but such that its mean α or β or γ are unclear or imprecise.

For example, α or β or γ can be an interval (open or closed or half open or half close) or can be set(s) with two or more elements. Then, the probability density function (p.d.f.) is:

$f_N(X) = \frac{\beta_N}{\alpha_N^{\beta_N}} X^{\beta_N - 1} e^{-(X/\alpha_N)^{\beta_N}}$, $X > 0$, Where β_N: is the shape parameter of distribution Net-Weibull, α_N: is the scale parameter of distribution Net- Weibull, such that N is a neutrosophic number.

4.1 Properties of Neutrosophic Weibull Distribution

- The distribution function (c.d.f.) is:

  $F_N(X) = 1 - e^{-(X/\alpha_N)^{\beta_N}}$,

  $E_N(X) = \alpha_N \Gamma\left(\frac{\beta_N + 1}{\beta_N}\right)$,

  $V_N(X) = \alpha^2{}_N \left[\Gamma\left(\frac{\beta_N + 2}{\beta_N}\right)\right] - \left[\Gamma\left(\frac{\beta_N + 1}{\beta_N}\right)\right]^2$.

- The hazard function is:

  $h_N = \beta_N X^{\beta_N - 1} X^{(\beta_N - 1/\alpha_N)^{\beta_N}}$.

- The moment rth about mean is:

  $\alpha_N{}^r \Gamma\left(\beta_N + \frac{r}{\beta_N}\right)$

- So, the reliability or survival function is:

$$\overline{F_N}(X) = e^{-(X/\alpha_N)^{\beta_N}}.$$

Now, we put β_N=1 in the formula (1), and we get the neutrosophic exponential distribution [13].

4.2 Example of Neutrosophic Weibull distribution

Let the product be an electric generator produced with high capacity of trademark that has a Weibull distribution with parameter α=1, β=[1.5,2]. Compute the probability of electric generator fails before the expiration of a five years warranty.

Solution :

In this example, we note that the shape parameter is indeterminate.

The electric generator can work through to one year:

$$f_N(X) = \frac{[1.5,2]}{\alpha_N^{[1.5,2]}} X^{[1.5,2]-1} e^{-(X/\alpha_N)^{[1.5,2]}}$$

If we take $\beta = 1.5$, and $\alpha = 1$

$$f_N(X = 1) = 0.5518$$

the probability of electric generator fails before the expiration of a five years warranty:

$$P(X \leq 5) = 1 - e^{-(5/1)^{1.5}}, = 0.999986$$

If we take $\beta = 2$, and $\alpha = 1$

$$f_N(X = 1) = 0.7357$$

$$P(X \leq 5) = 1 - e^{-(5/1)^{5}}, = 0.999999$$

Thus, the probability that the electric machine fails has the range between [0.5518, 0.7357].

Now, suppose $\beta = 2$ and $= [1,2]$, i.e the scale parameter α is indeterminate.

We take $\alpha = 1$ and $\beta = 2$

$$f_N(X = 1) = \frac{2}{e^1} = 0.7357$$

We take $\alpha = 2$ and $\beta = 2$

$$f_N(X = 1) = \frac{1}{2e^{1/4}} = 0.3894$$

In this case, the probability that the electric machine fails has the range between [0.7357, 0.3894]. Also, we can take more values of X, showed in Figure (1).

Now, we can compute

$F_N(X) = 1 - 1/e = 0.6321\,, \quad if\ \alpha = 1$

$F_N(X) = 1 - e^{1}/_{1.2840} = 0.2212\,, \quad if\ \alpha = 2.$

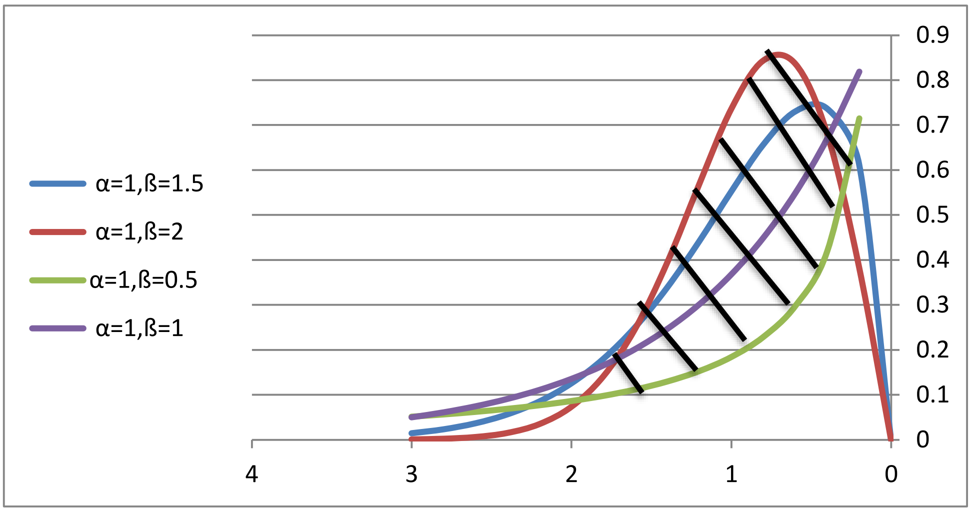

Figure 1: Neutrosophic Weibull distribution.

4.3 Comparison between Neutrosophic Weibull distribution and Weibull distribution

1- In classical Weibull, we noted that if the $\beta = 3.6$ *or more*, the probability distribution function (p.d.f) takes value error because it is greater than one, and this contradicts with law of probability,consedered Extreme values, while in neutrosophic Weibull this is applicable. See Figure (2).

2- In classical Weibull distribution, when X is increasing, the p.d.f. is decreasing, while in Neutrosophic Weibull distribution the p.d.f is unpredictable because of the aberrant values.

3- Many values that are larger than one are neglected in Weibull distribution, meanwhile in Neutrosophic Weibull these values are considered.

4- When $\alpha = \beta = 1$, the p.d.f. will equal zero when X=701,while in neutrosophic Weibull X can be of other values such as X={701,100} or [701,100] in this case p.d.f can be of different values.

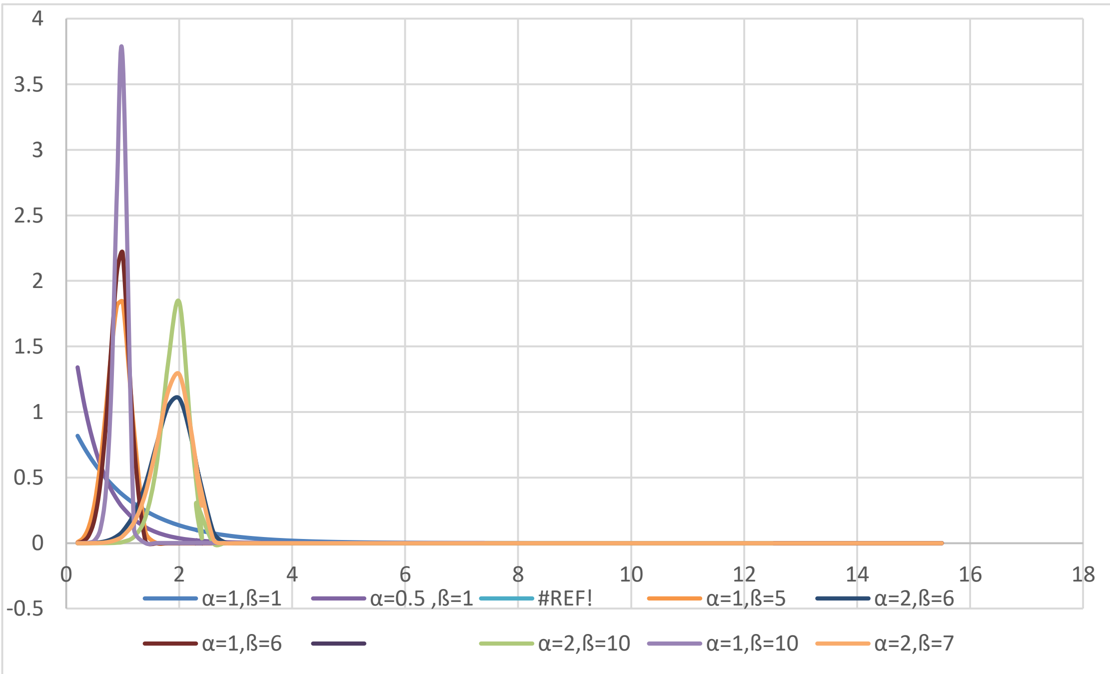

Figure 2: p.d.f of neutrosophic Weibull more than one.

5 The Family of Neutrosophic Weibull

In this section, we study the various types of Net-Weibull, such as neutrosophic Rayleigh distribution, neutrosophic inverse Weibull distribution, neutrosophic Beta-Weibull distribution and (three, four, five, six)-parameters Weibull distributions.

5.1 Neutrosophy Rayleigh Distribution

A Rayleigh distribution is often observed when the total size of a vector is linked to its directional components. Considering this distribution is important in the error analysis of various systems or individuals. It is also considered as a model for testing life failure/expiration. Rayleigh distribution is used in the study of the event of sea wave rise in the oceans and the study of wind speed, as well as in the information of the strength of signals and radiation at peak time of communications. The distribution is widely applied:

- In communications theory, to model multiple paths of dense scattered signals getting to a receiver;
- In the physics, to model wind speed, wave heights and sound/light radiation;
- In engineering, to measure the lifetime of an object, since the lifetime depends on the object's age (resistors, transformers, and capacitors in aircraft radar sets);
- In medical imaging examination, to study noise variance in magnetic resonance imaging.

Now, we define the probability density function of neutrosophic Rayleigh distribution as follows:

$R_N(X) = \frac{X}{{\delta_N}^2} e^{-X^2 \ /2{\delta_N}^2}$, $X > 0,\ \delta_N$ is the scale parameter.

this parameter δ_N can take the values of an interval or a set:
cumulative distribution is $F_N(X) = 1 - e^{-X^2 \ /2{\delta_N}^2}$,
the mean of Neutrosophic Rayleigh distribution is

$$E(X) = \delta_N \sqrt{\frac{\pi}{2}},$$

the variance: var(x) = 2-π/2 ${\delta_N}^2$.

5.2 Neutrosophic Weibull with 3 Parameters

We can obtain the neutrosophic Weibull with 3-parameters by relaying on Weibull with 2-parameters and adding the third parameter, namely the location parameter (γ), this is in classical probability . Now, we define the neutrosophic Weibull with three parameters (an indeterminacy may exist in one parameter or in all parameters). Neutrosophic Weibull with 3-parameters is defined as follows:

$$f_N(X) = [\beta_N \frac{(X-\gamma_N)^{\beta_N - 1}}{\alpha_N^{\beta_N}}] e^{-((X-\gamma_N)/\alpha_N)^{\beta_N}} \quad , \ \gamma_N \le X$$

- The distribution function is:

  $F_N(X) = 1 - e^{-((X-\gamma_N)/\alpha_N)^{\beta_N}} \quad , \ \gamma_N \le X$
- The hazard function is:

  $h_N(X) = \beta_N \quad (X - \gamma_N)^{\beta_N} (1 \ /\alpha_N)^{\beta_N} \quad , \qquad \gamma_N \le X$
- The survival function is

  $\overline{F_N}(X) = e^{-((X-\gamma_N)/\alpha_N)^{\beta_N}}$
- The variance

$$V_N(X) = \alpha^2{}_N\left[\Gamma\left(\frac{\beta_N+2}{\beta_N}\right)\right] - \left[\Gamma\left(\frac{\beta_N+1}{\beta_N}\right)\right]^2,$$

- The expected value $E_N(X) = \gamma_N + \alpha_N\Gamma\left(\frac{\beta_N+1}{\beta_N}\right)$.

5.3 Four-Parameter Neutrosophic-Beta-Weibull

The Beta-Weibull was first proposed by Famoye et al. (2005) [11,12, 15]. We now define the new density function of neutrosophic-Beta-Weibull distribution (NBW) in neutrosophic logic with indeterminacy points for random variable or parameters as follows:

$$f(X) = \frac{\Gamma(c_N+\gamma_N)}{\Gamma(c_N)\Gamma(\gamma_N)}\frac{\alpha_N}{\beta_N}\left(\frac{X}{\beta_N}\right)^{\alpha_N-1}[1-e^{-(X/\beta_N)^{\alpha_N}}]^{c_N-1}e^{-\gamma_N(X/\beta_N)^{\alpha_N}}$$

$$X > 0,\ \gamma_N, \beta_N, \alpha_N > 0$$

where these parameters $\gamma_N, \beta_N, \alpha_N$ can be set(s) or interval (closed or open or half):

Because $\lim_{x\to 0} f(X) = \lim_{x\to 0} \frac{\Gamma(c_N+\gamma_N)}{\Gamma(c_N)\Gamma(\gamma_N)}\frac{\alpha_N}{\beta_N}\left(\frac{X}{\beta_N}\right)^{\alpha_N-1}\left[1-e^{-(X/\beta_N)^{\alpha_N}}\right]^{c_N-1}e^{-\gamma_N(X/\beta_N)^{\alpha_N}}$

$$= \frac{\Gamma(c_N+\gamma_N)}{\Gamma(c_N)\Gamma(\gamma_N)}\frac{\alpha_N}{\beta_N}\left(\frac{X}{\beta_N}\right)^{\alpha_N-1} e^{-\gamma_N(X/\beta_N)^{\alpha_N}}\left[1-e^{-(X/\beta_N)^{\alpha_N}}\right]^{c_N-1}$$

$$= \frac{\Gamma(c_N+\gamma_N)}{\Gamma(c_N)\Gamma(\gamma_N)}\frac{\alpha_N}{\beta_N}\left(\frac{X}{\beta_N}\right)^{\alpha_N-1} e^{-\gamma_N(X/\beta_N)^{\alpha_N}}\left[1-\frac{1}{2!}\left(\frac{X}{\beta_N}\right)^{\alpha_N}+\frac{1}{3!}\left(\frac{X}{\beta_N}\right)^{2\alpha_N}-\frac{1}{4!}\left(\frac{X}{\beta_N}\right)^{3\alpha_N}+\cdots\right]^{c_N-1}$$

Then the probability of density function is equal to

$$= \lim_{x\to 0}\frac{\alpha_N}{\beta_N}\frac{\Gamma(c_N+\gamma_N)}{\Gamma(c_N)\Gamma(\gamma_N)}\left(\frac{X}{\beta_N}\right)^{\alpha_N-1} = \begin{cases} \infty & \alpha_N c_N < 1 \\ \frac{\alpha_N}{\beta_N}\frac{\Gamma(c_N+\gamma_N)}{\Gamma(c_N)\Gamma(\gamma_N)} & \alpha_N c_N = 1 \\ 0 & \alpha_N c_N > 1 \end{cases}$$

where $\beta_N, c_N, \gamma_N, \alpha_N,$ are Neutrosophy numbers.

- When $c_N = \gamma_N = 1$, then the (NBW) is reduced to neutrosophic Weibull distribution.
- When $\beta_N = \alpha_N = 1, c_N = 2, \gamma_N = \delta\sqrt{2}$, the NBW is reduced to neutrosophy Rayleigh.
- In (1958) Kies defined the survival function to Weibull with four parameters in classical distribution.

Here we define Neutrosophic survival function in Neutrosophic distribution as follows:

$$\overline{F_N}(X) = e^{\left\{-\gamma_N\left(\frac{x-\alpha_N}{\beta_N-x}\right)^{k_N}\right\}},\ \gamma_N > 0, k_N > 0,\ 0 < \alpha_N < X < \beta_N < \infty.$$

5.4 Neutrosophic Weibull Distribution with 5 Parameters

Phani in (1987) [14] suggested model with survival function has five parameters. We define the neutrosophic Weibull with 5-parameters:

$$\overline{F_N}(X) = e^{\frac{-\gamma_N[X-\alpha_N]^{b_1}}{[\beta_N - X]^{b_2}}},\quad \gamma_N, b_1, b_2 > 0,\quad 0 < \alpha_N < X < \beta_N < \infty.$$

5.5 Neutrosophic Weibull Distribution with 6 Parameters

T, W, and Uraiwan in (2014) [15] proposed a mixed distribution is Beta exponential Weibull Poisson distribution. We define the neutrosophic Beta exponential Weibull Poisson distribution as follows:

Let x be the neutrosophic random variable with parameters $\gamma_N, k_N, \alpha_N, \beta_N, b_1, b_2$;

$$f(x) = \frac{\beta_N \alpha_N \gamma_N \; {k_N}^{\beta_N} \; X^{\beta_N - 1} u (1-u)^{\alpha_N - 1} e^{\gamma_N (1-u)^{\alpha_N}}}{\mathrm{B}(b_1, b_2)(e^{\gamma_N} - 1)} \cdot \left[\frac{e^{\gamma_N (1-u)^{\alpha_N}} - 1}{(e^{\gamma_N} - 1)}\right]^{\alpha_N - 1} \left[1 - \frac{e^{\gamma_N (1-u)^{\alpha_N}} - 1}{(e^{\gamma_N} - 1)}\right]^{b1_N - 1}$$

where $u = e^{-(X k_N)^{\beta_N}}$.

5.6 Neutrosophic Inverse Weibull Distribution

Keller et al. (1985) used the inverse Weibull distribution for reliability analysis of commercial vehicle engines. Here, we define Neutrosophic inverse Weibull distribution as follows:

$f_N(t) = \beta_N \alpha_N^{\beta_N} t^{-\beta_N - 1} e^{-(\alpha_N / t)^{\beta_N}}$, $t > 0$, So the Hazard function is $h_N(t) = \frac{\beta_N \alpha_N^{\beta_N} t^{-\beta_N - 1} e^{-(\alpha_N / t)^{\beta_N}}}{1 - e^{-(\alpha_N / t)^{\beta_N}}}$.

6 Applications

Many applications of Weibull families distributions are suitable for modeling and analysis of floods, rainfall, sea, electronic, manufacturing products, navigation and transportation control. The theories and tools of reliability engineering are applied into widespread fields, such as electronic and manufacturing products, aerospace equipment, earthquake and volcano forecasting, communication, navigation and transportation control, medical processor to the survival analysis of human being or biological species, and so on [14]. So the neutrosophic has the multi-applied in Decision-making, introduced by Abdel-Basset and others.

7 Conclusions

The study of neutrosophic probability distributions gives us a more comprehensive space in the applied field, as it takes into account more than the value of the distribution parameters and not only one value as in the classical distributions, and thus we will be able to solve and explain many of the issues that have been hindering us or we tended to ignore in classical logic. In this paper, we defined th new neutrosophic clasical distribution, the neutrosophic Weibull distribution and neutrosophic family Weibull (neutrosophic inverse Weibull, Neutrosophic Rayleigh distribution, Neutrosophic Weibull distribution with (3, 4, 5, 6)-parameters, and give clear examples. Because the weibull distribution has many applications in different fields.such as control system, relability and others. We also study some properties of these distributions (mean, variance, failure function and reliability function). In the future, we will apply these distributions to many problems and we will examine other distributions in neutrosophic logic.

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On Neutrosophic Vague Graphs

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S. Satham Hussain, R. Jahir Hussain, Florentin Smarandache (2019). On Neutrosophic Vague Graphs. *Neutrosophic Sets and Systems* 28, 245-258

Abstract: In this work, the new concept of neutrosophic vague graphs are introduced form the ideas of neutrosophic vague sets. Moreover, some remarkable properties of strong neutrosophic vague graphs, complete neutrosophic vague graphs and self-complementary neutrosophic vague graphs are investigated and the proposed concepts are described with suitable examples.

Keywords: Neutrosophic vague graphs, Complete neutrosophic vague graph, Strong neutrosophic vague graph.

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1. Introduction

Initially, vague set theory was first investigated by Gau and Buehrer [30] which is an extension of fuzzy set theory. Vague sets are regarded as a special case of context-dependent fuzzy sets. In order to handle the indeterminate and inconsistent information, the neutrosophic set is introduced by the author Smarandache and studied extensively about neutrosophic set [14] - [37] and it receives applications in many fields. In neutrosophic set, the indeterminacy value is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are defined completely independent, if the sum of these values lies between 0 and 3. The new developments of neutrosophic theory are extensively studied in [1] - [6]. Molodtsov [28] firstly introduced the soft set theory as a general mathematical tool to with uncertainty and vagueness. Akram [9] established the certain notions including strong neutrosophic soft graphs and complete neutrosophic soft graphs. The authors [7] first introduce the concept of neutrosophic vague soft expert set which is a combination of neutrosophic vague set and soft expert set to improve the reasonability of decision making in reality. Neutrosophic vague set theory are introduced in [8]. The operations on single valued neutrosophic graphs are studied in [11]. Further, intuitionistic neutrosophic soft set and graphs are developed in [13]. Now, the domination in vague graphs and its is application are discussed in [16]. Intuitionistic neutrosophic soft set are studied in [18]. Interval valued neutrosophic graphs are developed by the author Broumi [22,23,25]. Interval neutrosophic vague sets are intiated in [31]. Motivation of the aforementioned works, we introduced the concept of neutrosophic vague graphs and strong neutrosophic vague graphs. This is a new developed theory which is the combination of neutrsophic graphs and vague graphs. Here the sum of Truth, Intermediate and False membership value lies between 0 and 2 since the truth and false membership are dependent variables. Here the complement of neutrosophic vague graphs is again neutrosophic vague graphs. This development theory will be applied in Operation Research, Social network problems. Particularly, fake profile is one of the big problems of social networks. Now, it has become easier to create a fake profile. People often use fake profile to insult, harass someone, involve in unsocial activities, etc. This model can be reformulated in the abstract form to be applied in neutrosophic vague graphs. The major contribution of this work as follows:

- Newly introduced neutrosophic vague graphs, neutrosophic vague subgraphs, constant neutrosophic vague graphs with examples.
- Further we presented some remarkable properties of strong neutrosophic vague graphs with suitable examples.

2 Preliminaries

Definition 2.1 [10] A vague set A on a non empty set X is a pair (T_A, F_A), where $T_A: X \to [0,1]$ and $F_A: X \to [0,1]$are true membership and false membership functions, respectively, such that

$$0 \leq T_A(r) + F_A(r) \leq 1 \text{ for any } r \in X.$$

Let X and Y be two non-empty sets. A vague relation R of X to Y is a vague set R on $X \times Y$ that is $R = (T_R, F_R)$, where $T_R: X \times Y \to [0,1], F_R: X \times Y \to [0,1]$ which satisfies the condition:

$$0 \leq T_R(r,s) + F_R(r,s) \leq 1 \text{ for any } r \in X.$$

Let $G = (R, S)$ be a graph. A pair $G = (J, K)$ is named as a vague graph on G^* or a vague graph where $J = (T_J, F_J)$ is a vague set on R and $K = (T_K, F_K)$ is a vague set on $S \subseteq R \times R$ such that for each $rs \in S$,

$$T_K(rs) \leq (T_J(r) \wedge T_J(s)) \& F_K(rs) \geq (T_J(r) \vee F_J(s)).$$

Definition 2.2 [9] A Neutrosophic set $A \subset B$, (i.e) $A \subseteq C$ if $\forall r \in X, T_A(r) \leq T_B(r), I_A(r) \geq I_B(r)$and $F_A(r) \geq F_B(r)$.

Definition 2.3 [12, 26, 30] Let X be a space of points (objects), with a generic elements in X known by r. A single valued neutrosophic set (SVNS) A in X is characterized by truth-membership function $T_A(r)$, indeterminacy-membership function $I_A(r)$ and falsity-membership-function $F_A(r)$. For each point r in X, $T_A(r), F_A(r), I_A(r) \in [0,1]$.

$$A = \{r, T_A(r), F_A(r), I_A(r)\} \text{ and } 0 \leq T_A(r) + I_A(r) + F_A(r) \leq 3$$

Definition 2.4 [19, 20] A neutrosophic graph is represented as a pair $G^* = (V, E)$ where

(i) $R = \{r_1, r_2, .., r_n\}$ such that $T_1 = R \to [0,1]$, $I_1 = R \to [0,1]$ and $F_1 = R \to [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and

$$0 \leq T_A(r) + I_A(r) + F_A(r) \leq 3$$

(ii) $S \subseteq R \times R$ where $T_2 = S \to [0,1]$, $I_2 = S \to [0,1]$ and $F_2 = S \to [0,1]$ are such that

$$T_2(rs) \leq \{T_1(r) \wedge T_1(s)\},$$

$$I_2(rs) \geq \{I_1(r) \vee I_1(s)\},$$

$$F_2(rs) \geq \{F_1(r) \vee F_1(s)\},$$

$$\text{and } 0 \leq T_2(rs) + I_2(rs) + F_2(rs) \leq 3, \forall rs \in R$$

Definition 2.5 [8] A neutrosophic vague set A_{NV} (NVS in short) on the universe of discourse X written as

$$A_{NV} = \{\langle r, \hat{T}_{A_{NV}}(r), \hat{I}_{A_{NV}}(r), \hat{F}_{A_{NV}}(r)\rangle, r \in X\}$$

whose truth-membership, indeterminacy membership and falsity-membership function is defined as

$$\hat{T}_{A_{NV}}(r) = [\hat{T}^-(r), \hat{T}^+(r)], [\hat{I}^-(r), \hat{I}^+(r)], [\hat{F}^-(r), \hat{F}^+(r)],$$

where $T^+(r) = 1 - F^-(r), F^+(r) = 1 - T^-(r)$, and $0 \le T^-(r) + I^-(r) + F^-(r) \le 2$.

Definition 2.6 [8] The complement of NVS A_{NV} is denoted by A^c_{NV} and it is represented by

$$\hat{T}^c_{A_{NV}}(r) = [1 - T^+(r), 1 - T^-(r)],$$
$$\hat{I}^c_{A_{NV}}(r) = [1 - I^+(r), 1 - I^-(r)],$$
$$\hat{F}^c_{A_{NV}}(r) = [1 - F^+(r), 1 - F^-(r)],$$

Definition 2.7 [8] Let A_{NV} and B_{NV} be two NVSs of the universe U. If for all $r_i \in U$,

$$\hat{T}_{A_{NV}}(r_i) = \hat{T}_{B_{NV}}(r_i), \hat{I}_{A_{NV}}(r_i) = \hat{I}_{B_{NV}}(r_i), \hat{F}_{A_{NV}}(r_i) = \hat{F}_{B_{NV}}(r_i)$$

then the NVS A_{NV} are included by B_{NV}, denoted by $A_{NV} \subseteq B_{NV}$ where $1 \le i \le n$.

Definition 2.8 [7] The union of two NVSs A_{NV} and B_{NV} is a NVSs, C_{NV}, written as $C_{NV} = A_{NV} \cup B_{NV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by

$$\hat{T}_{C_{NV}}(x) = [\max(\hat{T}^-_{A_{NV}}(r)\hat{T}^-_{B_{NV}}(r)), \max(\hat{T}^+_{A_{NV}}(r)\hat{T}^+_{B_{NV}}(r))]$$
$$\hat{I}_{C_{NV}}(x) = [\min(\hat{I}^-_{A_{NV}}(r)\hat{I}^-_{B_{NV}}(r)), \min(\hat{I}^+_{A_{NV}}(r)\hat{I}^+_{B_{NV}}(r))]$$
$$\hat{F}_{C_{NV}}(x) = [\min(\hat{F}^-_{A_{NV}}(r)\hat{F}^-_{B_{NV}}(r)), \min(\hat{F}^+_{A_{NV}}(r)\hat{F}^+_{B_{NV}}(r))]$$

Definition 2.9 [7] The intersection of two NVSs A_{NV} and B_{NV} is a NVSs C_{NV}, written as $C_{NV} = A_{NV} \cap B_{NV}$, whose truth membership function, indeterminacy-membership function and false-membership function are related to those of A_{NV} and B_{NV} by

$$\hat{T}_{C_{NV}}(x) = [\min(\hat{T}^-_{A_{NV}}(r)\hat{T}^-_{B_{NV}}(r)), \min(\hat{T}^+_{A_{NV}}(r)\hat{T}^+_{B_{NV}}(r))]$$
$$\hat{I}_{C_{NV}}(x) = [\max(\hat{I}^-_{A_{NV}}(r)\hat{I}^-_{B_{NV}}(r)), \max(\hat{I}^+_{A_{NV}}(r)\hat{I}^+_{B_{NV}}(r))]$$
$$\hat{F}_{C_{NV}}(x) = [\max(\hat{F}^-_{A_{NV}}(r)\hat{F}^-_{B_{NV}}(r)), \max(\hat{F}^+_{A_{NV}}(r)\hat{F}^+_{B_{NV}}(r))]$$

3 NEUTROSOPHIC VAGUE GRAPH

Definition 3.1 Let $G^* = (R, S)$ be a graph. A pair $G = (J, K)$ is named as a neutrosophic vague graph (NVG) on G^* or a neutrosophic graph where $J = (\hat{T}_J, \hat{I}_J, \hat{F}_J)$ is a neutrosophic vague set on R and $K = (\hat{T}_K, \hat{I}_K, \hat{F}_K)$ is a neutrosophic vague set $S \subseteq R \times R$ where

(1) $R = \{r_1, r_2, \dots, r_n\}$ such that

$$T^-_J : R \to [0,1], I^-_J : R \to [0,1], F^-_J : R \to [0,1]$$

which satisfies the condition $F^-_J = [1 - T^+_J]$

$$T^+_J : R \to [0,1], I^+_J : R \to [0,1], F^+_J : R \to [0,1]$$

which satisfies the condition $F^+_J = [1 - T^-_1]$ indicates the degree of truth membership function, indeterminacy membership and falsity membership of the element $r_i \in R$., and

$$0 \le T^-_J(r_i) + I^-_J(r_i) + F^-_J(r_i) \le 2.$$
$$0 \le T^+_J(r_i) + I^+_J(r_i) + F^+_J(r_i) \le 2.$$

(2) $S \subseteq R \times R$ where

$$T^-_K : R \times R \to [0,1], I^-_K : R \times R \to [0,1], F^-_K : R \times R \to [0,1]$$
$$T^+_K : R \times R \to [0,1], I^+_K : R \times R \to [0,1], F^+_K : R \times R \to [0,1]$$

indicates the degree of truth membership function, indeterminacy membership and falsity membership of the element $r_i, r_j \in S$. respectively and such that

$$0 \leq T_K^-(r_i) + I_K^-(r_i) + F_K^-(r_i) \leq 2.$$
$$0 \leq T_K^+(r_i) + I_K^+(r_i) + F_K^+(r_i) \leq 2.$$

such that

$$T_K^-(rs) \leq \{T_J^-(r) \wedge T_J^-(s)\}$$
$$I_K^-(rs) \leq \{I_J^-(r) \wedge I_J^-(s)\}$$
$$F_K^-(rs) \leq \{F_J^-(r) \vee F_J^-(s)\},$$

similarly

$$T_K^+(rs) \leq \{T_J^+(r) \wedge T_J^+(s)\}$$
$$I_K^+(rs) \leq \{I_J^+(r) \wedge I_J^+(s)\}$$
$$F_K^+(rs) \leq \{F_J^+(r) \vee F_J^+(s)\}.$$

Example 3.2 A neutrosophic vague graph $G = (J, K)$ such that $J = \{a, b, c\}$ and $K = \{ab, bc, ca\}$ indicated by

$$\hat{a} = T[0.5,0.6], I[0.4,0.3], F[0.4,0.5], \hat{b} = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6],$$
$$\hat{c} = T[0.4,0.4], I[0.5,0.3], F[0.6,0.6]$$
$$a^- = (0.5,0.4,0.4), b^- = (0.4,0.7,0.4), c^- = (0.4,0.5,0.6)$$
$$a^+ = (0.6,0.3,0.5), b^+ = (0.6,0.3,0.6), c^+ = (0.4,0.3,0.6)$$

.

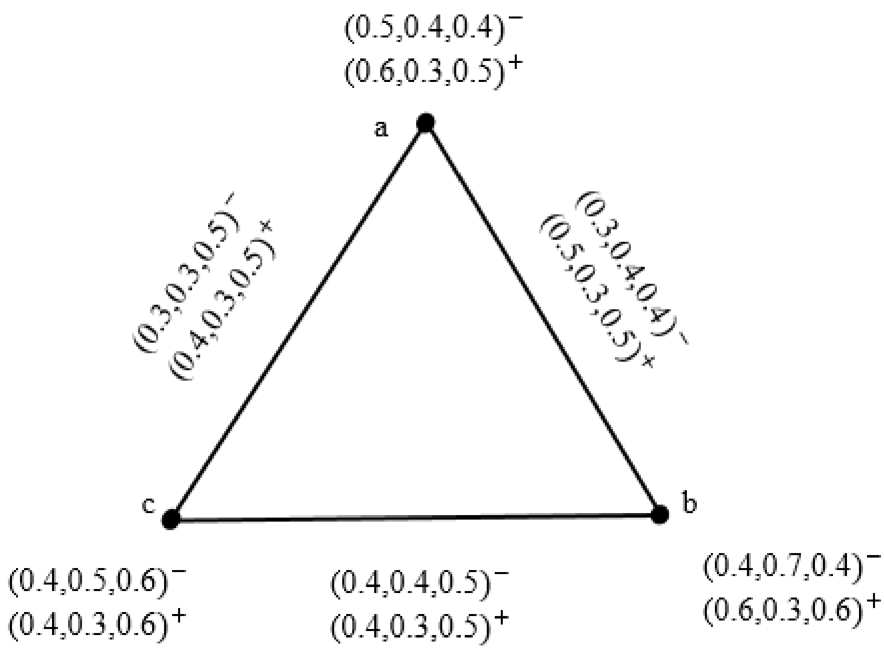


Figure 1
NEUTROSOPHIC VAGUE GRAPH

Definition 3.3 A neutrosophic vague graph $H = (J'(r), K'(r))$ is meant to be a neutrosophic vague subgraph of the NVG $G = (J, K)$ if $J'(r) \subseteq J(r)$ and $K'(rs) \subseteq K'(rs)$ in other words, if

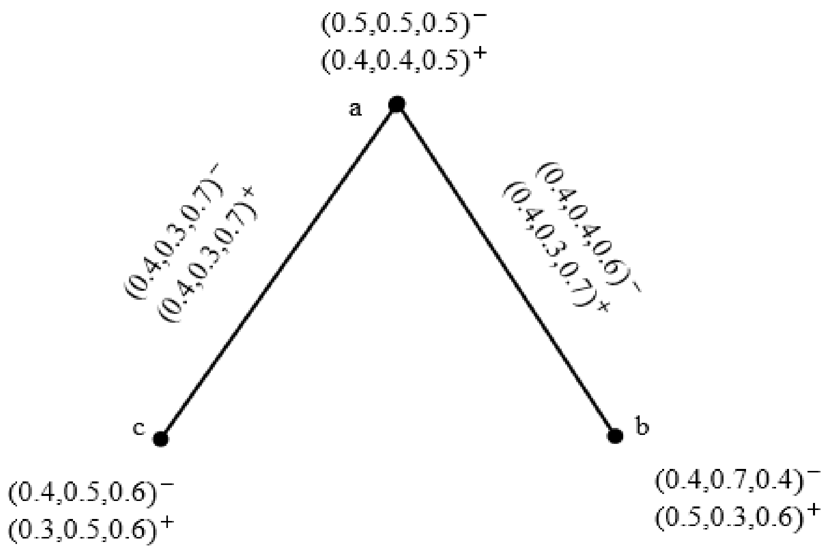
$$\hat{T}_J'^-(r) \leq \hat{T}_J^-(r)$$
$$\hat{I}_J'^-(r) \leq \hat{I}_J^-(r)$$

$$\hat{F}_J^{'-}(r) \geq \hat{F}_J^-(r) \forall r \in R$$

and

$$\hat{T}_K^{'+}(rs) \leq \hat{T}_K^-(rs)$$
$$\hat{I}_K^{'+}(rs) \leq \hat{I}_K^-(rs)$$
$$\hat{F}_K^{'+}(rs) \geq \hat{F}_K^-(rs) \forall (rs) \in S.$$

Example 3.4 A neutrosophic vague graph $G = (J, K)$ in Figure (1)



H_1 Figure 2

H_1 is a neutrosophic vague subgraph of G

Definition 3.5 The two vertices are said to be adjacent in a neutrosophic vague graph $G = (J, K)$ if

$$T_K^-(rs) = \{T_J^-(r) \wedge T_J^-(s)\}$$
$$I_K^-(rs) = \{I_J^-(r) \wedge I_J^-(s)\}$$
$$F_K^-(rs) = \{F_J^-(r) \vee F_J^-(s)\} \quad \text{and}$$
$$T_K^+(rs) = \{T_J^+(r) \wedge T_J^+(s)\}$$
$$I_K^+(rs) = \{I_J^+(r) \wedge I_J^+(s)\}$$
$$F_K^+(rs) = \{F_J^+(r) \vee F_J^+(s)\}$$

In this case, r and s are known to be neighbours and (rs) is incident at r and s also.

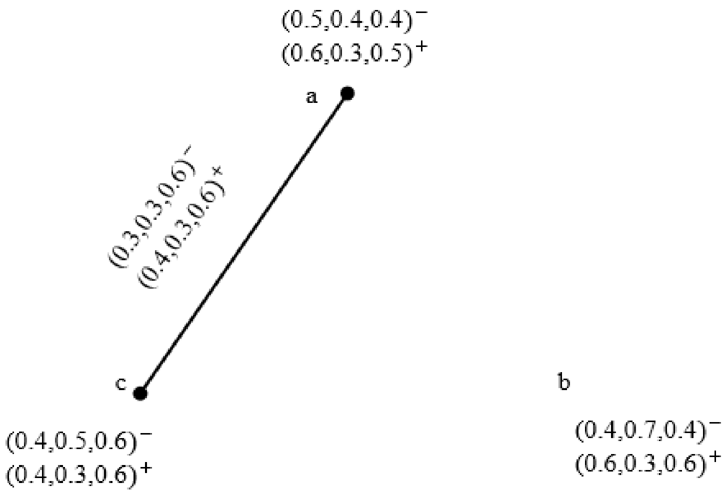
Definition 3.6 A path ρ in a NVG $G = (J, K)$ is a sequence of distinct vertices $r_0, r_1, \ldots, r_n$ such that $T_K^-(r_{i-1}, r_1) > 0$, $I_K^-(r_{i-1}, r_1) > 0$, $F_K^-(r_{i-1}, r_1) > 0$, $T_K^+(r_{i-1}, r_1) > 0$, $I_K^+(r_{i-1}, r_1) > 0$, $F_K^+(r_{i-1}, r_1) > 0$, for $0 \leq i \leq 1$, here $n \leq 1$ is called the length of the path ρ. A single node or single vertex r_i may all consider as a path.

Definition 3.7 A neutrosophic vague graph $G = (J, K)$ is said to be connected if every pair of vertices has at least on neutrosophic vague path between them otherwise it is disconnected.

Definition 3.8 A vertex $r_i \in R$ of neutrosophic vague graph $G = (J, K)$ called as a pendent vertex if there is no effective edge incident at x_i.

Definition 3.9 A vertex in a neutrosophic vague graph $G = (J, K)$ having exactly one neighbours is called a isolated vertex. otherwise,it is called non-isolated vertex. An edge in a neutrosophic vague graph incident with a isolated vertex is called a isolated edge other words it is called non-isolated edge. A vertex in a neutrosophic vague graph adjacent to the isolated vertex is called an support of the pendent edge.

Example 3.10 A neutrosophic vague graph $G = (J, K)$ in figure (3)



H_1 Figure 3

NEUTROSOPHIC VAGUE GRAPH

In figure (3), the neutrosophic vague vertex b is an pendent vertex.

Definition 3.11 Let $G = (J, K)$ be a neutrosophic vague graph. Then the degree of a vertex $r \in G$ is a sum of degree truth membership, sum of indeterminacy membership and sum of falsity membership of all those edges which are incident on vertex r represented by $d(r) = ([d^-_{T_J}(r), d^+_{T_J}(r)], [d^-_{I_J}(r), d^+_{I_J}(r)], [d^-_{F_J}(r), d^+_{F_J}(r)])$ where

$d^-_{T_J}(r) = \sum_{r \neq s} T^-_K(rs), d^+_{T_J}(r) = \sum_{r \neq s} T^+_K(rs)$ indicates the degree of truth membership vertex

$d^-_{I_J}(r) = \sum_{r \neq s} I^-_K(rs), d^+_{I_J}(r) = \sum_{r \neq s} I^+_K(rs)$ indicates the degree of indeterminacy membership vertex

$d^-_{F_J}(r) = \sum_{r \neq s} F^-_K(rs), d^+_{F_J}(r) = \sum_{r \neq s} F^+_K(rs)$ indicates the degree of falsity membership vertex for all $r, s \in J$.

Example 3.12 A neutrosophic vague graph $G = (J, K)$ in figure (1), we have the degree of each vertex as follows

$$d^-_T(a) = (0.6,0.7,0.9), d^-_F(b) = (0.7,0.8,1.3), d^-_F(c) = (0.7,0.7,1.0),$$

$$d^+_T(a) = (0.9,0.6,1.0), d^+_F(b) = (0.9,0.6,1.0), d^+_F(c) = (0.8,0.6,1.0)$$

Definition 3.13 A neutrosophic vague graph $G = (J, K)$ is called constant if degree of each vertex is $A = (A_1, A_2, A_3)$ that is $d(x) = (A_1, A_2, A_3)$ for all $x \in V$.

Example 3.14 Consider a neutrosophic vague graph $G = (J, K)$ in figure (4)defined by

$$\hat{a} = T[0.5,0.6], I[0.6,0.4], F[0.4,0.5], \hat{b} = T[0.4,0.4], I[0.4,0.6], F[0.6,0.6],$$
$$\hat{c} = T[0.4,0.6], I[0.7,0.3], F[0.4,0.6], \hat{d} = T[0.6,0.4], I[0.3,0.7], F[0.6,0.4]$$
$$a^- = (0.5,0.6,0.4), b^- = (0.4,0.4,0.6), c^- = (0.4,0.7,0.4), d^- = (0.6,0.3,0.6)$$
$$a^+ = (0.6,0.4,0.5), b^+ = (0.4,0.6,0.6), c^+ = (0.6,0.3,0.6), d^+ = (0.4,0.7,0.4).$$

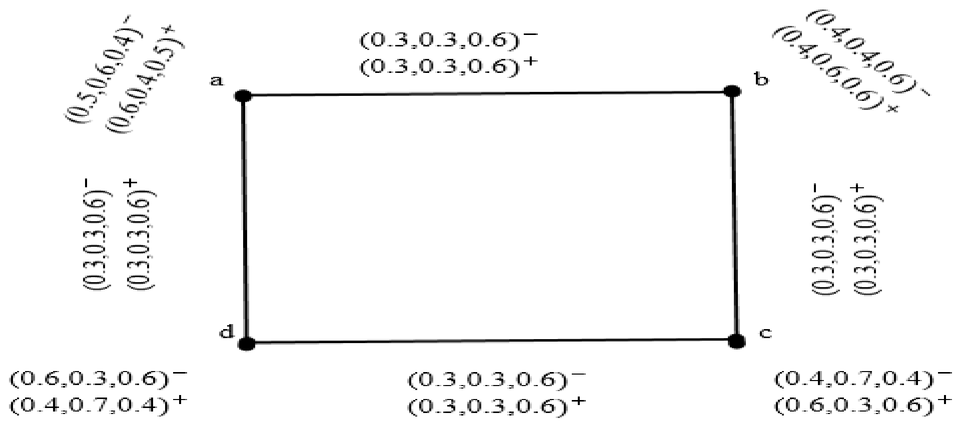


Figure 4

CONSTANT NEUTROSOPHIC VAGUE GRAPH

Clearly as it is seen in figure(4) G is constant neutrosophic vague graph since the degree of $(\hat{a}, \hat{b}, \hat{c}, \hat{d})$ and $\hat{d} = (0.6,0.6,1.2)$.

Definition 3.15 The complement of neutrosophic vague graph $G = (J, K)$ on G^* is a neutrosophic vague graph G^c on G^* where

- $J^c(r) = J(r)$
- $T_J^{-c}(r) = T_J^-(r), I_J^{-c}(r) = I_J^-(r), F_J^{-c}(r) = F_J^-(r)$ for all $r \in R$.
- $T_J^{+c}(r) = T_J^+(r), I_J^{+c}(r) = I_J^+(r), F_J^{+c}(r) = F_J^+(r)$ for all $r \in R$.
- $T_K^{-c}(rs) = \{T_J^-(r) \wedge T_J^-(s)\} - T_K^-(rs)$
$$I_K^{-c}(rs) = \{I_J^-(r) \wedge I_J^-(s)\} - I_K^-(rs)$$
$$F_K^{-c}(rs) = \{F_J^-(r) \vee F_J^-(s)\} - F_K^-(rs) \text{ for all } (rs) \in S$$
- $T_K^{+c}(rs) = \{T_J^+(r) \wedge T_J^+(s)\} - T_K^+(rs)$
$$I_K^{+c}(rs) = \{I_J^+(r) \wedge I_J^+(s)\} - I_K^+(rs)$$
$$F_K^{+c}(rs) = \{F_J^+(r) \vee F_J^+(s)\} - F_K^+(rs) \text{ for all } (rs) \in S$$

4 Strong Neutrosophic Vague Graphs

Definition 4.1 A neutrosophic vague graph $G = (J, K)$ of $G^* = (R, S)$ is named as a strong neutrosophic vague graph if

$$T_K^-(rs) = \{T_J^-(r) \wedge T_J^-(s)\}$$
$$I_K^-(rs) = \{I_J^-(r) \wedge I_J^-(s)\}$$
$$F_K^-(rs) = \{F_J^-(r) \vee F_J^-(s)\} \quad \text{and}$$
$$T_K^+(rs) = \{T_J^+(r) \wedge T_J^+(s)\}$$
$$I_K^+(rs) = \{I_J^+(r) \wedge I_J^+(s)\}$$
$$F_K^+(rs) = \{F_J^+(r) \vee F_J^+(s)\} \text{ for all } (rs \in S)$$

Example 4.2 A neutrosophic vague graph $G = (J, K)$ such that $J = \{a, b, c\}$ and $K = \{ab, bc, ca\}$ defined by $\hat{a} = T[0.3,0.4], I[0.4,0.6], F[0.6,0.7], \hat{b} = T[0.6,0.4], I[0.6,0.7], F[0.6,0.4]$,

$$\hat{c} = T[0.7,0.7], I[0.5,0.6], F[0.3,0.3]$$

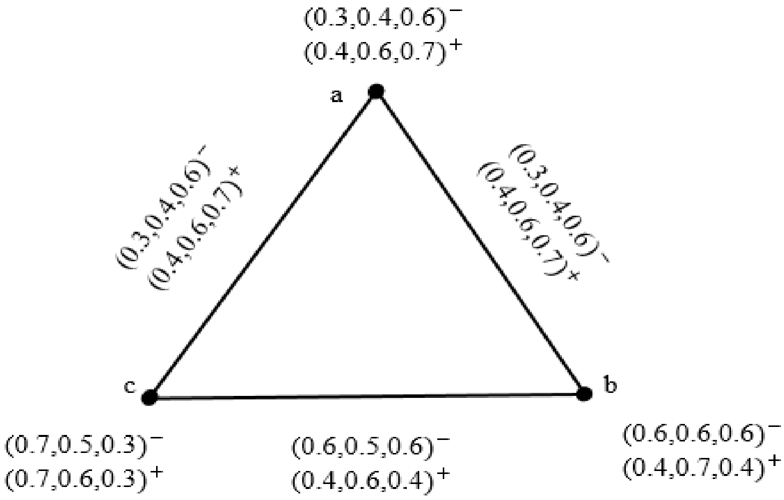


Figure 5

STRONG NEUTROSOPHIC VAGUE GRAPH

Remark 4.3 If $G = (J, K)$ is a neutrosophic vague graph on G^* then from above definition, it follow that G^{c^c} is given by the neutrosophic vague graph $G^{c^c} = J^{c^c}, K^{c^c}$ on G^* where

- $(J^c)^c(r) = J(r)$
- $(T_J^{-c})^c(r) = T_J^-(r), I_J^{-c}(r) = I_J^-(r), F_J^{-c}(r) = F_J^-(r)$ for all $r \in R$.
- $(T_J^{+c})^c(r) = T_J^+(r), I_J^{+c}(r) = I_J^+(r), F_J^{+c}(r) = F_J^+(r)$ for all $r \in R$.
- $(T_K^{-c})^c(rs) = \{T_J^-(r) \wedge T_J^-(s)\} - T_K^-(rs)$

$$(I_K^{-c})^c(rs) = \{I_J^-(r) \wedge I_J^-(s)\} - I_K^-(rs)$$

$$(F_K^{-c})^c(rs) = \{F_J^-(r) \vee F_J^-(s)\} - F_K^-(rs) \text{ for all } (rs) \in S$$

- $(T_K^{+c})^c(rs) = \{T_J^+(r) \wedge T_J^+(s)\} - T_K^+(rs)$

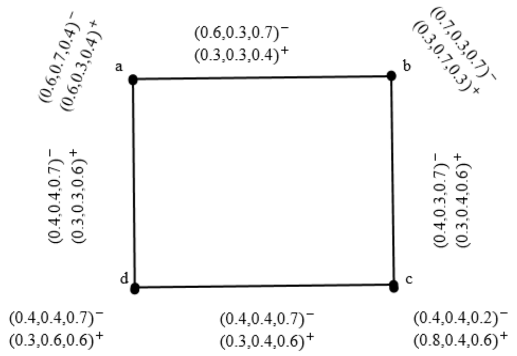
$$(I_K^{+c})^c(rs) = \{I_J^+(r) \wedge I_J^+(s)\} - I_K^+(rs)$$

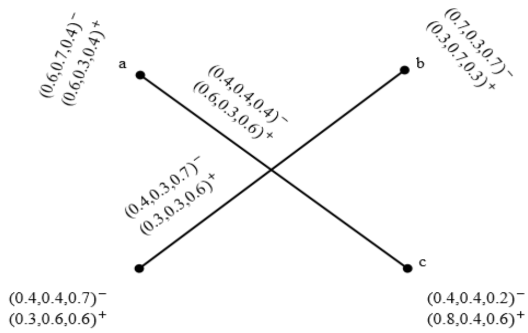
$$(F_K^{+c})^c(rs) = \{F_J^+(r) \vee F_J^+(s)\} - F_K^+(rs) \text{ for all } (rs) \in S$$

for any neutrosophic vague graph G, G^c is strong neutrosophic graph and $G \subseteq G^c$

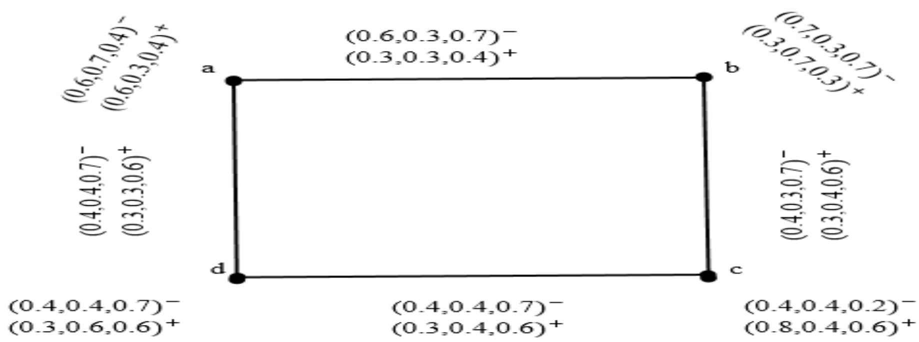
Definition 4.4 A strong neutrosophic graph G is called self-complementary if $G \cong G^c$ where G^c is the complement of neutrosophic vague graph G.

Example 4.5 A neutrosophic vague graph $G = (J, K)$ such that $J = \{a, b, c, d\}$ and $K = \{ab, bc, cd, da\}$ defined as follows: consider a neutrosophic vague graph G as in figure(6)





G STRONG NVG **G^c STRONG NVG**



G^{c^c} STRONG NVG

Clearly, as it is seen in figure (6) $G \cong G^{c^c}$.

Hence G is self complementary.

Proposition 4.6 Let $G = (J, K)$ be a strong neutrosophic vague graph if

$$T_K^-(rs) = \{T_J^-(r) \wedge T_J^-(s)\}$$

$$I_K^-(rs) = \{I_J^-(r) \wedge I_J^-(s)\}$$

$$F_K^-(rs) = \{F_J^-(r) \vee F_J^-(s)\}$$

$$T_K^+(rs) = \{T_J^+(r) \wedge T_J^+(s)\}$$

$$I_K^+(rs) = \{I_J^+(r) \wedge I_J^+(s)\}$$

$$F_K^+(rs) = \{F_J^+(r) \vee F_J^+(s)\} \text{ for all } rs \in K$$

Then G is self complementary.

Proof. Let $G = (J, K)$ be a strong neutrosophic vague graph such that

$$\hat{T}_K(rs) = \frac{1}{2}\min[\hat{T}_J(r), \hat{T}_J(s)]$$

$$\hat{I}_K(rs) = \frac{1}{2}\min[\hat{I}_J(r), \hat{I}_J(s)]$$

$$\hat{F}_K(rs) = \frac{1}{2}\max\hat{F}_J(r), \hat{F}_J(s)$$

for all $rs \in J$ then $G \approx G^{c^c}$ under the identity map $I: J \to J$. Hence G is self complementary

Proposition 4.7 Let G be a self complementary neutrosophic vague graph then

$$\sum_{r\neq s} \hat{T}_K(rs) = \frac{1}{2}\sum_{r\neq s} \min\{\hat{T}_J(r), \hat{T}_J(s)\}$$
$$\sum_{r\neq s} \hat{I}_K(rs) = \frac{1}{2}\sum_{r\neq s} \min\{\hat{I}_J(r), \hat{I}_J(s)\}$$
$$\sum_{r\neq s} \hat{F}_K(rs) = \frac{1}{2}\sum_{r\neq s} \max\{\hat{F}_J(r), \hat{F}_J(s)\}$$

Proof. If G be an self complementary neutrosophic vague graph then there exist an isomorphism $f: J_1 \to J_2$ satisfy

$$\hat{T}^c_{J_1}(f(r)) = \hat{T}_{J_1}(f(r)) = \hat{T}_{J_1}(r)$$
$$\hat{I}^c_{J_1}(f(r)) = \hat{I}_{J_1}(f(r)) = \hat{I}_{J_1}(r)$$
$$\hat{F}^c_{J_1}(f(r)) = \hat{F}_{J_1}(f(r)) = \hat{F}_{J_1}(r)$$

and

$$\hat{T}^c_{K_1}(f(r), f(s)) = \hat{T}_{K_1}(f(r), f(s)) = \hat{T}_{K_1}(rs)$$
$$\hat{I}^c_{K_1}(f(r), f(s)) = \hat{I}_{K_1}(f(r), f(s)) = \hat{I}_{K_1}(rs)$$
$$\hat{F}^c_{K_1}(f(r), f(s)) = \hat{F}_{K_1}(f(r), f(s)) = \hat{F}_{K_1}(rs)$$

we have $\hat{T}^c_{K_1}(f(r), f(s)) = \min(\hat{T}^c_{J_1}(r), \hat{T}^c_{J_1}(s)) - \hat{T}_{K_1}(f(r), f(s))$. i.e, $\hat{T}_{K_1}(rs) = \min(\hat{T}^c_{J_1}(r), \hat{T}^c_{J_1}(s)) - \hat{T}_{K_1}(f(r), f(s))$. $\hat{T}_{K_1}(rs) = \min(\hat{T}^c_{J_1}(r), \hat{T}^c_{J_1}(s)) - \hat{T}_{K_1}(rs)$. that is

$$\textstyle\sum_{r\neq s} \hat{T}_{K_1}(rs) + \sum_{r\neq s} \hat{T}_{K_1}(rs) = \sum_{r\neq s} \min(\hat{T}_{J_1}(r), \hat{T}_{J_1}(s)).$$

Similarly, $\sum_{r\neq s} \hat{I}_{K_1}(rs) + \sum_{r\neq s} \hat{I}_{K_1}(rs) = \sum_{r\neq s} \min(\hat{I}_{J_1}(r), \hat{I}_{J_1}(s))$

$$\sum_{r\neq s} \hat{F}_{K_1}(rs) + \sum_{r\neq s} \hat{F}_{K_1}(rs) = \sum_{r\neq s} \max(\hat{F}_{J_1}(r), \hat{F}_{J_1}(s))$$
$$2\sum_{r\neq s} \hat{T}_{K_1}(rs) = \sum_{r\neq s} \min(\hat{T}_{J_1}(r), \hat{T}_{J_1}(s))$$
$$2\sum_{r\neq s} \hat{I}_{K_1}(rs) = \sum_{r\neq s} \min(\hat{I}_{J_1}(r), \hat{I}_{J_1}(s))$$
$$2\sum_{r\neq s} \hat{F}_{K_1}(rs) = \sum_{r\neq s} \max(\hat{F}_{J_1}(r), \hat{F}_{J_1}(s))$$

from the equation of the proposition (4.8) holds.

Proposition 4.8 Let G_1 and G_2 be strong neutrosophic vague graph $\overline{G_1} \approx \overline{G_2}$(isomorphism)

Proof. Assume that G_1 and G_2 are isomorphic there exist a bijective map $f: J_1 \to J_2$ satisfying,

$$\hat{T}_{J_1}(r) = \hat{T}_{J_2}(f(r)), \hat{I}_{J_1}(r) = \hat{I}_{J_2}(f(r)), \hat{F}_{J_1}(r) = \hat{F}_{J_2}(f(r)), \text{forallr} \in J_1$$

and

$$\hat{T}_{K_1}(rs) = \hat{T}_{K_2}(f(r), f(s))$$
$$\hat{I}_{K_1}(rs) = \hat{I}_{K_2}(f(r), f(s))$$
$$\hat{F}_{K_1}(r) = \hat{F}_{K_2}(f(r), f(s)) \forall rs \in K_1$$

by definition (4.3) we have

$T^c_{K_1}(rs) = \min(T_{J_1}(r), T_{J_1}(s)) - T_{K_1}(rs)$
$= \min(T_{J_2}f(r), T_{J_2}f(s)) - T_{K_2}(f(r)f(s))$
$= T^c_{K_2}(f(r)f(s))$
$I^c_{K_1}(rs) = \min(I_{J_1}(r), I_{J_1}(s)) - I_{K_1}(rs)$
$= \min(I_{J_2}f(r), I_{J_2}f(s)) - I_{K_2}(f(r)f(s))$
$= I^c_{K_2}(f(r)f(s))$
$F^c_{K_1}(rs) = \max(F_{J_1}(r), F_{J_1}(s)) - F_{K_1}(rs)$
$= \max(F_{J_2}f(r), F_{J_2}f(s)) - F_{K_2}(f(r)f(s))$
$= F^c_{K_2}(f(r)f(s))$

Hence $G^c_1 \approx G^c_2$ for all $(rs) \in K_1$

Definition 4.9 A neutrosophic vague graph $G = (J, K)$ is complete if

$$T^-_K(rs) = \{T^-_J(r) \wedge T^-_J(s)\}$$

$$I^-_K(rs) = \{I^-_J(r), I^-_J(s)\}$$

$$F^-_K(rs) = \{F^-_J(r) \vee F^-_J(s)\},$$

similarly,

$$T^+_K(rs) = \{T^+_J(r) \wedge T^+_J(s)\}$$

$$I^+_K(rs) = \{I^+_J(r) \wedge I^+_J(s)\}$$

$$F^+_K(rs) = \{F^+_J(r) \vee F^+_J(s)\} \text{for} r, s \in J.$$

Example 4.10 Consider a neutrosophic vague graph $G = (J, K)$ such that $J = \{a, b, c, d\}$ and $K = \{ab, bc, cd, da\}$ defined by

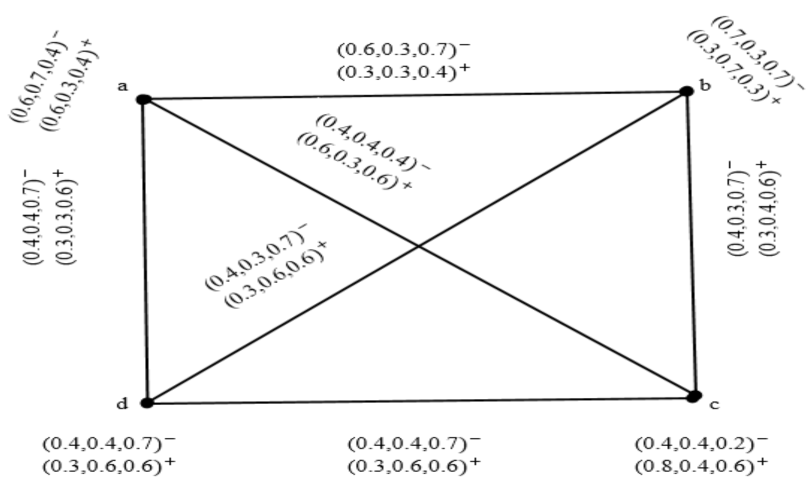


Figure 7

COMPLETE NEUTROSOPHIC VAGUE GRAPH

Definition 4.11 The complement of neutrosophic vague graph $G = (J, K)$ of $G^* = (V, E)$ is a neutrosophic vague complete graph $G = (J^c, K^c)$ on $G^* = (R, S^c)$ where

(1) $J^c(r_i) = J(r_i)$

(2) $\widehat{T}_J^c(r_i) = \widehat{T}_J(r_i), \hat{I}_J^c(r_i) = \hat{I}_J(r_i), \widehat{F}_J^c(r_i) = \widehat{F}_J(r_i)$ for all $r_i \in J$

(3)$\widehat{T}_K^c(r_i s_j) = (\widehat{T}_J(r_i) \wedge \widehat{T}_J(s_j)) - \widehat{T}_K(r_i s_j)$

$$\hat{I}_K^c(r_i s_j) = (\hat{I}_J(r_i) \wedge \hat{I}_J(s_j)) - \hat{I}_K(r_i, s_j)$$

$$\widehat{F}_K^c(r_i s_j) = (\widehat{F}_J(r_i) \vee \widehat{F}_J(s_j)) - \widehat{F}_K(r_i s_j) \quad \text{for all } (r_i s_j) \in K$$

Proposition 4.12 The complement of complete neutrosophic vague graph with no edge. or if G is complete then G^c the edge is empty.

Proof. Let $G = (J, K)$ be a complete neutrosophic vague graph so

$$\widehat{T}_K(r_i s_j) = (\widehat{T}_J(r_i) \wedge \widehat{T}_J(s_j))$$

$$\hat{I}_K(r_i s_j) = (\hat{I}_J(r_i) \wedge \widehat{T}_J(s_j))$$

$$\widehat{F}_K(r_i s_j) = (\widehat{F}_J(r_i) \vee \widehat{F}_J(s_j)) \forall (r_i, s_j) \in J$$

Hence in G^c. Now,

$$\widehat{T}_K^c(r_i s_j) = (\widehat{T}_J(r_i) \wedge \widehat{T}_J(s_j)) - \widehat{T}_K(r_i s_j)$$

$= (\widehat{T}_J(r_i) \wedge \widehat{T}_J(s_j)) - (\widehat{T}_J(r_i) \wedge \widehat{T}_J(s_j)) \ \forall\, i, j, \ldots, n$

$= 0 \ \forall, i, j, \ldots, n.$

and

$\hat{I}_K^c(r_i s_j) = (\hat{I}_J(r_i) \wedge \hat{I}_J(s_j)) - \hat{I}_K(r_i s_j)$

$= (\hat{I}_J(r_i) \wedge \hat{I}_J(s_j)) - (\hat{I}_J(r_i) \wedge \hat{I}_J(s_j)) \forall\, i, j, \ldots, n$

$= 0 \forall\, i, j, \ldots, n.$

Similarly $\widehat{F}_K^c(r_i, s_j) = 0.$ Thus,$(\widehat{T}_K(r_i, s_j), \hat{I}_K(r_i, s_j), \widehat{F}_K(r_i, s_j)) = (0,0,0)$

Hence, the edge set of G^c is empty if G is a complete neutrosophic vague graph.

Conclusion and futute directions:

This work dealt with the new concept of neutrosophic vague graphs. Moreover, some remarkable properties of strong neutrosophic vague graphs, complete neutrosophic vague graphs and self-complementary neutrosophic vague graphs have been investigated and the proposed concepts were described with suitable examples. Further we can extend to investigate the regular and isomorphic properties of the proposed graph. This can be applied to social network model and operations research.

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An Introduction to Neutrosophic Bipolar Vague Topological Spaces

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Abstract: The main objective of this paper is to make known to a new concept of generalised neutrosophic bipolar vague sets and also defined neutrosophic bipolar vague topology in topological spaces. Also, we introduce generalized neutrosophic bipolar vague closed sets and conferred its properties.

Keywords: Bipolar set, Vague set, Neutrosophic set, Neutrosophic Bipolar Vague set, Neutrosophic Bipolar Vague Topological Spaces.

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1. Introduction

Levine [24] studied the Generalized closed sets in general topology. Several investigations were conducted on the generalizations of the notion of the fuzzy set, after the introduction of the concept of fuzzy sets by Zadeh [34]. In the traditional fuzzy sets, the membership degree of component ranges over the interval [0, 1]. Few types of fuzzy set extensions in the fuzzy set theory are present, for example, intuitionistic fuzzy sets[12], interval-valued fuzzy sets[32], vague sets[30] etc. As a generalization of Zadeh's fuzzy set, the notion of vague set theory was first introduced by Gau W.L and Buehrer D.J [22]. In 1996, H.Bustince & P.Burillo indicated that vague sets are intuitionistic fuzzy sets [15].

Intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets can handle only unfinished information but not the indeterminate and unreliable information which happens normally in actual circumstances. Hence, the conception of a neutrosophic set is very common, and then it can overcome the aforesaid issues on the intuitionistic fuzzy set and the interval-valued intuitionistic fuzzy set. In 1995, the definition of Smarandache's neutrosophic set, neutrosophic sets and neutrosophic logic have been useful in many real applications to handle improbability. Neutrosophy is a branch of philosophy which studies the source, nature and scope of neutralities, as well as their interactions with different ideational scales [31]. The neutrosophic set uses one single value to indicate the truth-membership grade, indeterminacy-membership degree and falsity membership grade of an element in the universe X. The theory has been brought into extensive application in varieties of field [1-6, 8, 10, 11, 14, 17, 23, 27, 33, 35] for dealing with indeterminate and unreliable information in actual domain. The conception of Neutrosophic Topological space was introduced by A.A.Salama and S.A.Alblowi [29].

Bipolar-valued fuzzy sets, which was introduced by Lee [25, 26] is an extension of fuzzy sets whose membership degree range is extended from the interval [0, 1] to [-1, 1]. The membership degrees of the Bipolar valued fuzzy sets signify the degree of satisfaction to the property analogous to a fuzzy set and its counter-property in a bipolar valued fuzzy set, if the membership degree is 0 it means that the elements are unrelated to the corresponding property. Furthermore if the membership degree is on (0, 1] it indicates that the elements somewhat fulfil the property, and if the membership degree is on

[-1,0) it indicates that elements somewhat satisfy the entire counter property. After that, Deli et al. [21] announced the concept of bipolar neutrosophic sets, as an extension lead of neutrosophic sets. In the bipolar neutrosophic sets, the positive membership degree $T^{+}(x), I^{+}(x), F^{+}(x)$ signifies the truth membership, indeterminate membership and false membership of an element x ∈ X analogous to a bipolar neutrosophic set A and the negative membership degree $T^{-}(x),\ I^{-}(x), F^{-}(x)$ signifies the truth membership, indeterminate membership and false membership of an element x ∈ X to some implied counter-property analogous to a bipolar neutrosophic set A. There are quite a few extensions of Neutrosophic Bipolar sets such as Neutrosophic Bipolar Soft sets [7] and Rough Neutrosophic Bipolar sets [28].

Neutrosophic vague set is a combination of neutrosophic set and vague set which was well-defined by Shawkat Alkhazaleh [30]. Neutrosophic vague theory is a useful tool to practise incomplete, indeterminate and inconsistent information. In this paper, we introduced the perception of a neutrosophic bipolar vague set as a combination of neutrosophic set, Bipolar set and vague set and we also define the concept of generalised Neutrosophic Bipolar Vague set.

2. Preliminaries

Definition 2.1[16]: Let X be the universe. Then a bipolar valued fuzzy sets, A on X is defined by positive membership function μ_A^{+}: X→[0,1] and a negative membership function μ_A^{-}: X→[-1,0]. For sake of easiness, we shall practice the symbol A= {< x, μ_A^{+}(x), μ_A^{-}(x)>: x ∈ X}.

Definition 2.2[18]: Let A and B be two bipolar valued fuzzy sets then their union, intersection and complement are well-defined as follows:

(i)$\mu_{A\cup B}^{+}(x)$ = max { μ_A^{+}(x), μ_B^{+}(x)}.

(ii) $\mu_{A\cup B}^{-}(x)$ = min { μ_A^{-}(x), μ_B^{-}(x)}.

(iii) $\mu_{A\cap B}^{+}(x)$ = min { μ_A^{+}(x), μ_B^{+}(x)}.

(iv) $\mu_{A\cap B}^{-}(x)$ = max { μ_A^{-}(x), μ_B^{-}(x)}.

(v) $\mu_{\bar{A}}^{+}(x)$= 1-μ_A^{+}(x) and $\mu_{\bar{A}}^{-}(x)$= -1-μ_A^{+}(x) for all x∈ X.

Definition 2.3[15]: A vague set A in the universe of discourse U is a pair (t_A , f_A) where t_A : U→[0,1], f_A : U→[0,1] denote the mapping such that $t_A + f_A \leq 1$ for all u∈ U .The function t_A and f_A are called true membership function and false membership function respectively. The interval [t_A ,1-f_A] is called the vague value of u in A, and denoted by ν_A(u), i.e ν_A(u)=[t_A ,1-f_A].

Definition 2.4[15]: Let A be a non-empty set and the vague set A and B in the form A= {<x, t_A ,1-f_A >:x∈ X}, B={<x, t_B ,1-f_B >:x∈ X}.

Then

(i)A⊆ B if and only if $t_A(x) \leq t_A(x)$ and 1-f_B(x) ≤ 1-f_B(x).

(ii)A∪ B={<max(t_A(x),t_B(x)), max(1-f_A(x),1-f_B(x))>/x∈X}.

(iii) A∩ B={<min(t_A(x),t_B(x)), min(1-f_A(x),1-f_B(x))>/x∈X}.

(iv)$\bar{A}$={<x, f_A(x) ,1-t_A(x)>:x∈X}.

Definition 2.5[14]:Let X be a universe of discourse. Then a neutrosophic set is well-defined as: A = {⟨x, T_A(x), I_A(x), F_A(x)⟩:x ∈ X}, which is categorized by a truth-membership function T_A:X →]0−,1+[, an indeterminacy membership function I_A:X →]0−,1+[and a falsity-membership function F_A:X →]0−,1+[. There is no restriction to the sum of T_A(x), I_A(x) and F_A(x), so $0^{-} \leq \sup T_A(x) \leq \sup I_A(x) \leq \sup F_A(x) \leq 3^{+}$.

Definition 2.6[30]: A neutrosophic vague set A_{NBV}(NVS in short) on the universe of discourse X written as,

A_{NBV} = $\{<\ \hat{T}_{NBV}(x),\ \hat{I}_{NBV}(x),\ \hat{F}_{NBV}(x) >: x \in X\}$ whose truth-membership, indeterminacy-membership and falsity-membership functions is defined as,

$\hat{T}_{NBV}(x)$ = $[T^{-}, T^{+}]$, $\hat{I}_{NBV}(x)$ = $[I^{-}, I^{+}]$, $\hat{F}_{NBV}(x)$ = $[F^{-}, F^{+}]$ where

$T^{+} = 1 - F^{-}$, $F^{+} = 1 -$and T^{-}, $^{-}0 \leq T^{-}+I^{-} + F^{-} \leq 2^{+}$.

3. Bipolar Neutrosophic Vague Set:

Under this division, we present and well-defined the notion of neutrosophic bipolar vague set and its operations.

Definition 3.1: If A={< x, $[T_A^-,T_A^+]^+,[I_A^-,I_A^+]^+,[F_A^-,F_A^+]^+,[T_A^-,T_A^+]^-,[I_A^-,I_A^+]^-,[F_A^-,F_A^+]^-$ >} and B={< x, $[T_B^-,T_B^+]^+,[I_B^-,I_B^+]^+,[F_B^-,F_B^+]^+,[T_B^-,T_B^+]^-,[I_B^-,I_B^+]^-,[F_B^-,F_B^+]^-$>} where
$(T^+)^+=1-(F^-)^+$, $(F^+)^+=1-(T^-)^+$ and $(T^+)^-=-1-(F^-)^-$, $(F^+)^-=-1-(T^-)^-$ T^+, I^+, F^+: X→[0,1] and T^-,I^-,F^-:X→[-1,0] are two neutrosophic bipolar vague sets then their union, intersection and complement are well-defined as follows:

1.A∪B= { $\max[T_A^-,T_B^-]^+$, $\max[T_A^+,T_B^+]^+$,
$\min[I_A^-,I_B^-]^+$, $\min[I_A^+,I_B^+]^+$,
$\min[F_A^-,F_B^-]^+$, $\min[F_A^+,F_B^+]^+$,
$\min[T_A^-,T_B^-]^-$, $\min[T_A^+,T_B^+]^-$,
$\max[I_A^-,I_B^-]^-$, $\max[I_A^+,I_B^+]^-$,
$\max[F_A^-,F_B^-]^-$, $\max[F_A^+,F_B^+]^-$}.

2.A∩B={ $\min[T_A^-,T_B^-]^+$, $\min[T_A^+,T_B^+]^+$,
$\max[I_A^-,I_B^-]^+$, $\max[I_A^+,I_B^+]^+$,
$\max[F_A^-,F_B^-]^+$, $\max[F_A^+,F_B^+]^+$,
$\max[T_A^-,T_B^-]^-$, $\max[T_A^+,T_B^+]^-$,
$\min[I_A^-,I_B^-]^-$, $\min[I_A^+,I_B^+]^-$,
$\min[F_A^-,F_B^-]^-$, $\min[F_A^+,F_B^+]^-$}.

3. $\bar{A}$={<$[F_A^-,F_A^+]^+,[1-I_A^-,1-I_A^+]^+,[T_A^-,T_A^+]^+,[F_A^-,F_A^+]^-,[1-I_A^-,1-I_A^+]^-,[T_A^-,T_A^+]^-$>}.

Definition 3.2: Suppose A and B be two neutrosophic bipolar vague sets defined over a universe of disclosure X. We say that A ⊆ B if and only if $[T_A^-\le T_B^-]^+$, $[T_A^+\le T_B^+]^+$, $[I_A^-\ge I_B^-]^+$, $[I_A^+\ge I_B^+]^+$, $[F_A^-\ge F_B^-]^+$, $[F_A^+\ge F_B^+]^+$, $[T_A^-\ge T_B^-]^-$, $[T_A^+\ge T_B^+]^-$, $[I_A^-\le I_B^-]^-$, $[I_A^+\le I_B^+]^-$, $[F_A^-\le F_B^-]^-$, $[F_A^+\le F_B^+]^-$.

Definition 3.3: A bipolar vague topology NBVT on a nonempty set X is a family NBV_τ of Neutrosophic bipolar vague set in X sustaining the following axioms:

1. 0, 1 ∈ NBV_τ.
2. $G_1 \cap G_2 \in NBV_\tau$, for any $G_1, G_2 \in NBV_\tau$.
3. $\cup G_i \in NBV_\tau$ for any arbitrary family { G_i: $G_i \in NBV_\tau$, i∈I}.

Under such case the pair (X, NBV_τ) is known as the neutrosophic bipolar vague topological space and any NBVS in NBV_τ is known as bipolar vague open set in X . The complement $\bar{A}$ of a neutrosophic bipolar vague open set (NBVOS) A in a neutrosophic bipolar vague topological space (X, NBV_τ) is referred as a neutrosophic bipolar vague closed (NBVCS) in X.

Example 3.4: Assume X={u,v},
A_{NBV}= $\{\frac{u}{[0.5,0.7][0.5,0.5][0.3,0.5][-0.4,-0.1][-0.5,-0.6][-0.9,-0.6]},\frac{v}{[0.3,0.6][0.4,0.4][0.4,0.7][-0.2,-0.2][-0.6,-0.8][-0.8,-0.8]}\}$,
B_{NBV}= $\{\frac{u}{[0.5,0.9][0.3,0.3][0.1,0.5][-0.4,-0.3][-0.4,-0.4][-0.7,-0.6]},\frac{v}{[0.4,0.6][0.2,0.2][0.4,0.6][-0.5,-0.3][-0.5,-0.5][-0.7,-0.5]}\}$.
Then the family NBV_τ={0,1,A,B} of neutrosophic bipolar vague sets in X is a NBVT on X.

Definition 3.5: Suppose (X, NBV_τ) is a neutrosophic bipolar vague topological space and
A={< x, $[T_A^-,T_A^+]^+,[I_A^-,I_A^+]^+,[F_A^-,F_A^+]^+,[T_A^-,T_A^+]^-,[I_A^-,I_A^+]^-,[F_A^-,F_A^+]^-$ >} be a NBVS inX . Then the neutrosophic bipolar vague interior and neutrosophic bipolar vague closure of A are well-defined by,
NBVcl(A)=∩{K:K is a NBVCS in X and A⊆K},
NBVint(A)=∪{G:G is a NBVOS in X and G⊆A}.
Note that NBVcl(A) is a NBVCS and NBVint(A) is a NBVOS in X . Further,

1. A is a NBVCS in X iff NBVcl(A)=A
2. A is a NBVOS in X iff NBVint(A)=A.

Example 3.6: Assume that X={a, b},
A={x,$\frac{a}{[0.5,0.7][0.5,0.5][0.3,0.5][-0.4,-0.1][-0.5,-0.6][-0.9,-0.6]},\frac{b}{[0.3,0.6][0.4,0.4][0.4,0.7][-0.2,-0.2][-0.6,-0.8][-0.8,-0.8]}$}
B= $\{\frac{a}{[0.5,0.9][0.3,0.3][0.1,0.5][-0.4,-0.3][-0.4,-0.4][-0.7,-0.6]},\frac{b}{[0.4,0.6][0.2,0.2][0.4,0.6][-0.5,-0.3][-0.5,-0.5][-0.7,-0.5]}\}$.
Then the family NBV_τ={0,1,A,B} of a neutrosophic bipolar vague sets in X is NBVT on X. If,
F={<x,$\frac{a}{[0.5,0.4][0.5,0.5][0.6,0.5][-0.6,-0.4][-0.3,-0.3][-0.6,-0.4]},\frac{b}{[0.5,0.7][0.1,0.1][0.3,0.5][-0.3,-0.4][-0.2,-0.2][-0.6,-0.7]}$ >}

Then, NBVint(A)=∪{G:G is a NBVOS in X and G⊆F}=0 and NBVcl(A)=∩{K:K is a NBVCS in X and F⊆K}=1.

Proposition 3.7: For any NBVS A in (X, NBV_τ) we have,

1. NBVcl($\bar{A}$)=$\overline{\text{NBVint(A)}}$
2. NBVint($\bar{A}$)=$\overline{\text{NBVcl(A)}}$.

Proof: Let A={< x, $[T_A^-, T_A^+]^+, [I_A^-, I_A^+]^+, [F_A^-, F_A^+]^+, [T_A^-, T_A^+]^-, [I_A^-, I_A^+]^-, [F_A^-, F_A^+]^-$ >} and suppose that NBVOS's contained in A are indexed by the family
{<x, $[T_{G_i}^-, T_{G_i}^+]^+, [I_{G_i}^-, I_{G_i}^+]^+, [F_{G_i}^-, F_{G_i}^+]^+, [T_{G_i}^-, T_{G_i}^+]^-, [I_{G_i}^-, I_{G_i}^+]^-, [F_{G_i}^-, F_{G_i}^+]^-$ >: $i \in J$}. Then
NBVint(A)= <$x, \cup[T_{G_i}^-, T_{G_i}^+]^+, \cap[I_{G_i}^-, I_{G_i}^+]^+, \cap[F_{G_i}^-, F_{G_i}^+]^+, \cap[T_{G_i}^-, T_{G_i}^+]^-, \cup[I_{G_i}^-, I_{G_i}^+]^-, \cup[F_{G_i}^-, F_{G_i}^+]^-$ > and hence
$\overline{\text{NBVint(A)}}$ =< $x, \cap[F_{G_i}^-, F_{G_i}^+]^+, \cup[1-I_{G_i}^-, 1-I_{G_i}^+]^+, \cup[T_{G_i}^-, T_{G_i}^+]^+, \cup[F_{G_i}^-, F_{G_i}^+]^-, \cap[1-I_{G_i}^-, 1-I_{G_i}^+]^-, \cap[T_{G_i}^-, T_{G_i}^+]^-$ > --------- (1)
Since,
$\bar{A}$={<$[F_A^-, F_A^+]^+, [1-I_A^-, 1-I_A^+]^+, [T_A^-, T_A^+]^+, [F_A^-, F_A^+]^-, [1-I_A^-, 1-I_A^+]^-, [T_A^-, T_A^+]^-$>}. Where
$[T_{G_i}^-, T_{G_i}^+]^+ \le [T_A^-, T_A^+]^+$, $[I_{G_i}^-, I_{G_i}^+]^+ \ge [I_A^-, I_A^+]^+$, $[F_{G_i}^-, F_{G_i}^+]^+ \ge [F_A^-, F_A^+]^+$, $[T_{G_i}^-, T_{G_i}^+]^- \ge [T_A^-, T_A^+]^-$,
$[I_{G_i}^-, I_{G_i}^+]^- \le [I_A^-, I_A^+]^-$, $[F_{G_i}^-, F_{G_i}^+]^- \le [F_A^-, F_A^+]^-$ for every i∈J we obtain that
{< $x, [F_{G_i}^-, F_{G_i}^+]^+, [1-I_{G_i}^-, 1-I_{G_i}^+]^+, [T_{G_i}^-, T_{G_i}^+]^+, [F_{G_i}^-, F_{G_i}^+]^-, [1-I_{G_i}^-, 1-I_{G_i}^+]^-, [T_{G_i}^-, T_{G_i}^+]^-$ > i∈J}
Is the family of NBVS's containing$\bar{A}$, that is ,
NBVcl($\bar{A}$)= < $x, \cap[F_{G_i}^-, F_{G_i}^+]^+, \cup[1-I_{G_i}^-, 1-I_{G_i}^+]^+, \cup[T_{G_i}^-, T_{G_i}^+]^+, \cup[F_{G_i}^-, F_{G_i}^+]^-, \cap[1-I_{G_i}^-, 1-I_{G_i}^+]^-, \cap[T_{G_i}^-, T_{G_i}^+]^-$ > ------------ (2).
Hence from (1) and (2) we get NBVcl($\bar{A}$)=$\overline{\text{NBVint(A)}}$
(2) follows from (1).

Proposition 3.8: If (X, NBV_τ) is a NBVTS and A,B be are NBVS's in X. Then the following properties hold:

1. NBVint(A)⊆A
2. A⊆ NBvcl(A)
3. A⊆ B ⇒ NBVint(A) ⊆ NBVint(B)
4. A⊆ B ⇒ NBVcl(A) ⊆ NBVcl(B)
5. NBVint(NBVint(A))= NBVint(A)
6. NBVcl(NBVcl(A)= NBVcl(A)
7. NBVint($A \cap B$)= NBVint(A) ∩ NBVint(B)
8. NBVcl(A∪ B)= NBVcl(A)∪ NBVcl(B)
9. NBVint(1)=1
10. NBVcl(0)=0

Definition 3.9: Suppose (X, NBV_τ) and (Y, NBV_σ) be two neutrosophic bipolar vague topological spaces and $\psi: X \to Y$ be a function. Then ψ is referred to be a neutrosophic bipolar vague continuous iff the preimage of each neutrosophic bipolar vague open set in Y is a neutrosophic bipolar vague open set in X.

Proposition 3.10: Suppose A, {A_i: i∈J} be a neutrosophic bipolar vague set in X, and B, {B_j: j∈K} be a neutrosophic bipolar vague set in Y, and let $\psi: X \to Y$ be a function. Then,

(a) A_1⊆A_2⟺ ψ(A_1) ⊆ ψ(A_2)
(b) B_1⊆B_2⟺ ψ^{-1}(B_1) ⊆ ψ^{-1}(B_2)
(c) ψ^{-1}(∪B_i)= ∪ ψ^{-1}(B_i) and ψ^{-1}(∩B_i)= ∩ ψ^{-1}(B_i)

Proof: Obvious.

Proposition 3.11: The subsequent are equivalent to each other.

1. $\psi: X \to Y$ is neutrosophic bipolar vague continuous.
2. ψ^{-1}(NBVint(B)) ⊆ NBVint(ψ^{-1}(B)) for each NBVOS B in Y.
3. NBVcl(ψ^{-1}(B)) ⊆ ψ^{-1}(NBVcl(B)) for each NBVOS B in Y.

Proof: (1)⟹(2) Given $\psi: X \to Y$ is neutrosophic bipolar vague continuous.

Then we have to show that ψ^{-1}(NBVint(B)) $\subseteq$ NBVint(ψ^{-1}(B)) for each NBVOS B in Y.
Let B={< y, $[T_B^-, T_B^+]^+$, $[I_B^-, I_B^+]^+$, $[F_B^-, F_B^+]^+$, $[T_B^-, T_B^+]^-$, $[I_B^-, I_B^+]^-$, $[F_B^-, F_B^+]^-$>} be NBVOS in Y.
NBVint(B)=
$\{ < y, \cup[T_{H_i}^-, T_{H_i}^+]^+, \cap[I_{H_i}^-, I_{H_i}^+]^+, \cap[F_{H_i}^-, F_{H_i}^+]^+, \cap[T_{H_i}^-, T_{H_i}^+]^-, \cup[I_{H_i}^-, I_{H_i}^+]^-, \cup[F_{H_i}^-, F_{H_i}^+]^- >$:i∈I}
Where,
$[T_{H_i}^-, T_{H_i}^+]^+ \leq [T_B^-, T_B^+]^+$, $[I_{H_i}^-, I_{H_i}^+]^+ \geq [I_B^-, I_B^+]^+$, $[F_{H_i}^-, F_{H_i}^+]^+ \geq [F_B^-, F_B^+]^+$, $[T_{H_i}^-, T_{H_i}^+]^- \geq [T_B^-, T_B^+]^-$, $[I_{H_i}^-, I_{H_i}^+]^- \leq [I_B^-, I_B^+]^-$, $[F_{H_i}^-, F_{H_i}^+]^- \leq [F_B^-, F_B^+]^-$ for every i ∈ I. By the definition of continuity ψ^{-1}(NBVint(B)) is a neutrosophic bipolar vague open set in NBV_τ. Now,
ψ^{-1}(NBVint(B)) ={
$\psi^{-1}(< y, \cup[T_{H_i}^-, T_{H_i}^+]^+, \cap[I_{H_i}^-, I_{H_i}^+]^+, \cap[F_{H_i}^-, F_{H_i}^+]^+, \cap[T_{H_i}^-, T_{H_i}^+]^-, \cup[I_{H_i}^-, I_{H_i}^+]^-, \cup[F_{H_i}^-, F_{H_i}^+]^- >)\}$
= {(<x, $\psi^{-1}(\cup[T_{H_i}^-, T_{H_i}^+]^+)$, $\psi^{-1}(\cap[I_{H_i}^-, I_{H_i}^+]^+)$, $\psi^{-1}\left(\cap[F_{H_i}^-, F_{H_i}^+]^+\right)$, $\psi^{-1}(\cap[T_{H_i}^-, T_{H_i}^+]^-)$, $\psi^{-1}(\cup[I_{H_i}^-, I_{H_i}^+]^-)$, $\psi^{-1}(\cup[F_{H_i}^-, F_{H_i}^+]^-)$ >)}.
= {<x, $\cup[\psi^{-1}[[T_{H_i}^-, T_{H_i}^+]^+]$, $\cap[\psi^{-1}[I_{H_i}^-, I_{H_i}^+]^+]$, $\cap[\psi^{-1}[F_{H_i}^-, F_{H_i}^+]^+]$, $\cap[\psi^{-1}[T_{H_i}^-, T_{H_i}^+]^-]$, $\cup[\psi^{-1}[[I_{H_i}^-, I_{H_i}^+]^-]$, $\cup[\psi^{-1}[[F_{H_i}^-, F_{H_i}^+]^-]$>}.
$\subseteq$ NBVint(ψ^{-1}(B))
(2)⟹(1). Given ψ^{-1}(NBVint(B)) $\subseteq$ NBVint(ψ^{-1}(B)) for each NBVOS B in Y. Let
B={< y, $[T_B^-, T_B^+]^+$, $[I_B^-, I_B^+]^+$, $[F_B^-, F_B^+]^+$, $[T_B^-, T_B^+]^-$, $[I_B^-, I_B^+]^-$, $[F_B^-, F_B^+]^-$>} be NBVOS in Y. We know that B is a neutrosophic bipolar vague open in Y if and only if NBVint(B)=B and hence ψ^{-1}(NBVint(B))= ψ^{-1}(B). But according to our supposition ψ^{-1}(NBVint(B)) $\subseteq$ NBVint(ψ^{-1}(B)), therefore we get ψ^{-1}(B) $\subseteq$ NBVint(ψ^{-1}(B), i.e., ψ^{-1}(B) is a NBVS in X and thus ψ is a neutrosophic bipolar vague continuous.
(1)⟹ (3) Given $\psi: X \to Y$ is neutrosophic bipolar vague continuous.
Suppose B={< y, $[T_B^-, T_B^+]^+$, $[I_B^-, I_B^+]^+$, $[F_B^-, F_B^+]^+$, $[T_B^-, T_B^+]^-$, $[I_B^-, I_B^+]^-$, $[F_B^-, F_B^+]^-$>} be NBVOS in Y.
Also suppose NBVcl(B) =
{<y,$\cap[T_{K_i}^-, T_{K_i}^+]^+, \cup[I_{K_i}^-, I_{K_i}^+]^+, \cup[F_{K_i}^-, F_{K_i}^+]^+, \cup[T_{K_i}^-, T_{K_i}^+]^-, \cap[I_{K_i}^-, I_{K_i}^+]^-, \cap[F_{K_i}^-, F_{K_i}^+]^-$>:i∈ I},where
$[T_{K_i}^-, T_{K_i}^+]^+ \leq [T_B^-, T_B^+]^+$,$[I_{K_i}^-, I_{K_i}^+]^+ \geq [I_B^-, I_B^+]^+$,$[F_{K_i}^-, F_{K_i}^+]^+ \geq [F_B^-, F_B^+]^+$,
$[T_{K_i}^-, T_{K_i}^+]^- \geq [T_B^-, T_B^+]^-$,$[I_{K_i}^-, I_{K_i}^+]^- \leq [I_B^-, I_B^+]^-$, $[F_{K_i}^-, F_K^+]^- \leq [F_B^-, F_B^+]^-$ for every i ∈ I. Since ψ is a neutrosophic bipolar vague continuous iff the inverse image of each NBVCS in Y is a NBVCS in X, therefore ψ^{-1}(NBVcl(B)) is a NBVCS in X.
Now, ψ^{-1} (NBVcl(B))= $\{\psi^{-1}(< y, \cap[T_{K_i}^-, T_{K_i}^+]^+, \cup[I_{K_i}^-, I_{K_i}^+]^+, \cup[F_{K_i}^-, F_{K_i}^+]^+, \cup[T_{K_i}^-, T_{K_i}^+]^-, \cap[I_{K_i}^-, I_{K_i}^+]^-, \cap[F_{K_i}^-, F_{K_i}^+]^- >)\}$
={(<x, $\psi^{-1}(\cap[T_{K_i}^-, T_{K_i}^+]^+)$, $\psi^{-1}(\cup[I_{K_i}^-, I_{K_i}^+]^+)$, $\psi^{-1}\left(\cup[F_{K_i}^-, F_{K_i}^+]^+\right)$, $\psi^{-1}(\cup[T_{K_i}^-, T_{K_i}^+]^-)$, $\psi^{-1}(\cap[I_{K_i}^-, I_{K_i}^+]^-)$, $\psi^{-1}(\cap[F_{K_i}^-, F_{K_i}^+]^-)$ >)}.
= {<x, $\cap[\psi^{-1}[[T_{K_i}^-, T_{K_i}^+]^+]$, $\cup[\psi^{-1}[I_{K_i}^-, I_{K_i}^+]^+]$, $\cup[\psi^{-1}[F_{K_i}^-, F_{K_i}^+]^+]$, $\cup[\psi^{-1}[T_{K_i}^-, T_{K_i}^+]^-]$, $\cap[\psi^{-1}[[I_{K_i}^-, I_{K_i}^+]^-]$, $\cap[\psi^{-1}[[F_{K_i}^-, F_{K_i}^+]^-]$>}
$\supseteq$ NBVcl(ψ^{-1}(B))
(3)⟹(1)
Given NBVcl(ψ^{-1}(B))⊆ ψ^{-1}(NBVcl(B)), for each NBVOS B in Y. Let
B={< y, $[T_B^-, T_B^+]^+$, $[I_B^-, I_B^+]^+$, $[F_B^-, F_B^+]^+$, $[T_B^-, T_B^+]^-$, $[I_B^-, I_B^+]^-$, $[F_B^-, F_B^+]^-$>} be NBVCS in Y. Since NBVcl(B)=B. But it is given that NBVcl(ψ^{-1}(B))⊆ ψ^{-1}(NBVcl(B)), hence NBVcl(ψ^{-1}(B))⊆ ψ^{-1}(B). Hence ψ^{-1}(B)= NBVcl(ψ^{-1}(B)), i.e., ψ^{-1}(B) is a NBVCS in X and this proves that ψ is a neutrosophic bipolar vague continuous.

4. Generalized Neutrosophic Bipolar Vague Closed Sets:

Definition 4.1: Suppose if (X, NBV_τ) be a neutrosophic bipolar vague topological space. A neutrosophic bipolar vague set A in (X, NBV_τ) is referred to be a generalized neutrosophic bipolar vague closed set if NBVcl(A) $\subseteq G$ whenever A⊆G and G is a neutrosophic bipolar vague open. The complement of a generalized neutrosophic bipolar vague closed set is generalized neutrosophic bipolar vague open set.

Definition 4.2: Suppose let (X, NBV_τ) be a neutrosophic bipolar vague topological space and let A be a neutrosophic bipolar vague set in X. The generalized neutrosophic bipolar vague closure (GNBVcl for short) and the generalized neutrosophic bipolar vague interior (GNBVint for short) of A are well-defined by,

1) GNBVcl(A)=⋂{G:G is a generalized neutrosophic bipolar vague closed sets in X and A⊆ G},
2) GNBVint(A)=⋃{G:G is a generalized neutrosophic bipolar vague open sets in X and A⊇ G}.

Remark 4.3: Every NBVCS is generalized neutrosophic bipolar vague closed but not conversely.

Example 4.4: Assume that X= {u,v} and NBV_τ={0,1,F} is a NBVT on X where,

F=$<x, \frac{u}{[0.5,0.9][0.3,0.3][0.1,0.5][-0.4,-0.3][-0.4,-0.4][-0.7,-0.6]}, \frac{v}{[0.4,0.6][0.2,0.2][0.4,0.6][-0.5,-0.3][-0.5,-0.5][-0.7,-0.5]}>$

Then the neutrosophic bipolar vague set,

A=$< x, \frac{u}{[0.5,0.7][0.5,0.5][0.3,0.5][-0.4,-0.1][-0.5,-0.6][-0.9,-0.6]}, \frac{v}{[0.3,0.6][0.4,0.4][0.4,0.7][-0.2,-0.2][-0.6,-0.8][-0.8,-0.8]} >$ is a generalized neutrosophic bipolar vague closed but not NBVC in X.

Proposition 4.5: Suppose that (X, NBV_τ) be a neutrosophic bipolar vague topological space. If A is a generalized neutrosophic bipolar vague closed set and A⊆B⊆NBVcl(A), then B is a generalized neutrosophic bipolar vague closed set.

Proof: Suppose let G be a neutrosophic bipolar vague open set in (X, NBV_τ), such that B⊆G. Since A⊆B, A⊆ G. Now A is a generalized neutrosophic bipolar vague closed set and NBVcl(A) ⊆G. But NBVcl(B) ⊆ NBVcl(A). Since NBVcl(B) ⊆ NBVcl(A) ⊆G, NBVcl(B) ⊆G. Hence B is a generalized neutrosophic bipolar vague closed set.

Proposition 4.6: Suppose if A is a neutrosophic bipolar vague open set and generalized neutrosophic bipolar vague closed set in (X, NBV_τ), then A is said to be a neutrosophic bipolar vague closed set in X.

Proof: Assume that A is a neutrosophic bipolar vague open set in X. Since A⊆A, by hypothesis NBVcl(A) ⊆A. Then from definition A⊆ NBVcl(A). Therefore NBVcl(A)= A. Hence A is neutrosophic bipolar vague closed set in X.**Proposition 4.7**: Suppose that NBVint(A) ⊆B⊆A and assume A is a generalized neutrosophic bipolar vague open set then B is also a generalized neutrosophic bipolar vague open set.

Proof: Now, $\bar{A} \subseteq \bar{B} \subseteq \overline{\text{NBVint(A)}}$= NBVcl($\bar{A}$). As A is a generalized neutrosophic bipolar vague open, $\bar{A}$ is a generalized neutrosophic bipolar vague closed set. By proposition 4.5, $\bar{B}$ generalized neutrosophic bipolar vague closed set. That is, B is also a generalized neutrosophic bipolar vague open set.

Definition 4.8: Suppose (X, NBV_τ) and (Y, NBV_σ)be any two neutrosophic bipolar vague topological spaces.

1. A map ψ: (X, NBV_τ)→(Y, NBV_σ) is referred to be a generalized neutrosophic bipolar vague continuous if the inverse image of every neutrosophic bipolar vague open set in (Y, NBV_σ) is a generalized neutrosophic bipolar vague open set in (X, NBV_τ).
2. A map ψ: (X, NBV_τ)→(Y, NBV_σ) is called as a generalized neutrosophic bipolar vague irresolute if the inverse image of every generalized neutrosophic bipolar vague open set in (Y, NBV_σ) is a generalized neutrosophic bipolar vague open set in (X, NBV_τ).

Proposition 4.9: Suppose (X, NBV_τ) and (Y, NBV_σ)be any two neutrosophic bipolar vague topological spaces. A mapping ψ: (X, NBV_τ)→(Y, NBV_σ) is referred to be generalized neutrosophic bipolar vague continuous function mapping. Then for every neutrosophic bipolar vague set A in X, ψ(GNBVcl(A)) ⊆ NBVcl($\psi(A)$).

Proof: Assume A to be a neutrosophic bipolar vague set in (X, NBV_τ). Since NBVcl($\psi(A)$) is a neutrosophic bipolar vague closed set and since ψ is a generalized neutrosophic bipolar vague continuous mapping, the set ψ^{-1}(NBVcl($\psi(A)$)) is a generalized neutrosophic bipolar vague closed set and thus ψ^{-1}(NBVcl($\psi(A)$)) ⊇A.

Now, GNBVcl(A))⊆ $\psi^{-1}\left(\text{NBVcl}(\psi(A))\right)$. Therefore ψ(GNBVcl(A)) ⊆ NBVcl($\psi(A)$).

Proposition 4.10: If (X, NBV_τ) and (Y, NBV_σ) are two neutrosophic bipolar vague topological spaces. Let the mapping ψ : (X, NBV_τ) →(Y, NBV_σ) be a generalized neutrosophic bipolar vague continuous mapping. Then for every neutrosophic bipolar vague set A in Y, GNBVcl(ψ^{-1}(A)) ⊆ ψ^{-1}(NBVcl(A).

Proof: Assume A to be a neutrosophic bipolar vague set in (Y, NBV_σ). Let B= ψ^{-1}(A). Then, ψ(B)= ψ(ψ^{-1}(A)) ⊆A. By proposition 4.10, ψ(GNBVcl(ψ^{-1}(A))) ⊆NBV cl(ψ(ψ^{-1}(A))). Thus, GNBVcl(ψ^{-1}(A)) ⊆ ψ^{-1}(NBVcl(A).

Proposition 4.11: Suppose let (X, NBV_τ) and (Y, NBV_σ) be any two neutrosophic bipolar vague topological spaces. Let ψ : (X, NBV_τ) →(Y, NBV_σ) is referred to be a neutrosophic bipolar vague continuous mapping, then it is a generalized neutrosophic bipolar vague continuous mapping.

Proof: Suppose let A be a neutrosophic bipolar vague open set in (Y, NBV_σ). Since the mapping ψ is a neutrosophic bipolar vague continuous mapping, $\psi^{-1}(A)$ is a neutrosophic bipolar vague open set in (X, NBV_τ). Every neutrosophic bipolar vague open set is a generalized neutrosophic bipolar vague open set. Now, $\psi^{-1}(A)$ is a generalized neutrosophic bipolar vague open set in (X, NBV_τ). Hence ψ is thus a generalized neutrosophic bipolar vague continuous mapping.

The converse of the proposition need not be true as shown in example.

Example 4.12: Assume that X={a,b}, Y={u,v} and,

A=<x, $\frac{a}{[0.5,0.4][0.5,0.5][0.6,0.5][-0.6,-0.4][-0.3,-0.3][-0.6,-0.4]}$, $\frac{b}{[0.6,0.7][0.1,0.1][0.3,0.4][-0.3,-0.4][-0.2,-0.2][-0.6,-0.7]}$>,

B=<x, $\frac{a}{[0.5,0.3][0.5,0.5][0.7,0.5][-0.4,-0.2][-0.4,-0.4][-0.8,-0.6]}$, $\frac{b}{[0.5,0.4][0.2,0.2][0.6,0.5][-0.3,-0.4][-0.2,-0.2][-0.6,-0.7]}$>.

Then NBV_τ={0,1,A} and NBV_σ={0,1,B} are NBVT on X and Y respectively. Define a mapping ψ: (X, NBV_τ)→(Y, NBV_σ) by ψ(a)=u and ψ(b)=v. then ψ is a generalized neutrosophic bipolar vague continuous mapping but not bipolar vague continuous mapping.

Proposition 4.13: Suppose let (X, NBV_τ) and (Y, NBV_σ) be any two neutrosophic bipolar vague topological spaces. A mapping ψ: (X, NBV_τ)→(Y, NBV_σ) is said to be a generalized neutrosophic bipolar vague irresolute mapping, then it is a generalized neutrosophic bipolar vague continuous mapping.

Proof: Let A be a neutrosophic bipolar vague open set in (Y, NBV_σ). Since every neutrosophic bipolar vague open set is a generalized neutrosophic bipolar vague open set in (Y, NBV_σ), but ψ is a generalized neutrosophic bipolar vague irresolute mapping, $\psi^{-1}(A)$ is a generalized neutrosophic bipolar vague open set in (X, NBV_τ). Thus ψ is a generalized neutrosophic bipolar vague continuous mapping.

Proposition 4.14: Suppose let (X, NBV_τ), (Y, NBV_σ) and (Z, NBV_ρ) be any three bipolar vague topological spaces. Let ψ: (X, NBV_τ)→(Y, NBV_σ) be a generalized neutrosophic bipolar vague irresolute mapping and ψ_1: (Y, NBV_σ)→(Z, NBV_ρ) be a generalized neutrosophic bipolar vague continuous mapping. Then $\psi_1 \circ \psi$ is a generalized neutrosophic bipolar vague continuous mapping.

Proof: Let A be a neutrosophic bipolar vague open set in (Z, NBV_ρ). Since ψ_1 is a generalized neutrosophic bipolar vague continuous mapping, ψ_1^{-1} (A) is a generalized neutrosophic bipolar vague open set in (Y, NBV_σ) . Since ψ is a generalized neutrosophic bipolar vague irresolute mapping, ψ^{-1}(ψ_1^{-1} (A)) is a generalized neutrosophic bipolar vague open set in (X, NBV_τ). Thus $\psi_1 \circ \psi$ is a generalized neutrosophic bipolar vague continuous mapping.

Conclusion:

This paper presented the new concept of Neutrosophic Bipolar Vague sets and studied some basic operational relation of Neutrosophic Bipolar Vague set. Then a generalization of NBVSs in closed set is done. As a future work, we shall continue to work in the application of NBVS to other domains, such as medical diagnosis, pattern recognition and decision making.

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Data Envelopment Analysis for Simplified Neutrosophic Sets

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Abstract: In recent years, there has been a growing interest in neutrosophic theory, and there are several methods for solving various problems under neutrosophic environment. However, a few papers have discussed the Data envelopment analysis (DEA) with neutrosophic sets. So, in this paper, we propose an input-oriented DEA model with simplified neutrosophic numbers and present a new strategy to solve it. The proposed method is based on the weighted arithmetic average operator and has a simple structure. Finally, the new approach is illustrated with the help of a numerical example.

Keywords: Data envelopment analysis; Neutrosophic set; Simplified neutrosophic sets (SNSs); Aggregation operator.

1. Introduction

With the advent of technology and the complexity and volume of information, senior executives have required themselves to apply scientific methods to determine and increase the productivity of the organization under their jurisdiction. Data envelopment analysis (DEA) is a mathematical technique to evaluate the relative efficiency of a set of some homogeneous units called decision-making units (DMUs) that use multiple inputs to produce multiple outputs. DMUs are called homogeneous because they all employ the same inputs to produce the same outputs. DEA by constructing an efficiency frontier measures the relative efficiency of decision making units (DMUs). Charnes et al. [1] developed a DEA model (CCR) based on the seminal work of Farrell [2] under the assumption of constant returns to scale (CRS). Banker et al. [3] extended the pioneering work Charnes et al. [1] and proposed a model conventionally called BCC to measure the relative efficiency under the assumption of variable returns to scale (VRS). DEA technique has just been effectively connected in various cases such as broadcasting companies [4], banking institutions [5-8], R&D organizations [9-10], health care services [11-12], manufacturing [13-14], telecommunication [15], and supply chain management [16-19]. However, data in the standard models are certain, but there are numerous circumstances in real life where we have to face uncertain parameters. Zadeh [20] first proposed the theory of fuzzy sets (FSs) against certain logic where the membership degree is a real number between zero and one. After this work, many researchers studied on this topic; details of some researches can be observed in [21-30]. Several researchers also proposed some models of DEA under fuzzy environment [31-42]. However, Zadeh's fuzzy sets cannot deal with certain cases in which it is difficult to define the membership degree using one specific value. To overcome this lack of knowledge, Atanassov [43] introduced an extension of the FSs that called the intuitionistic fuzzy sets (IFSs). Although the theory of IFSs can handle incomplete information in various real-world issues, it cannot address all types of uncertainty such as indeterminate and inconsistent information.

Therefore, Smarandache [44-45], proposed the neutrosophic set (NS) as a strong general framework that generalizes the classical set concept, fuzzy set [20], interval-valued fuzzy set [46], intuitionistic fuzzy set [43], and interval-valued intuitionistic fuzzy set [47]. Neutrosophic set (NS) can deal with uncertain, indeterminate and incongruous information where the indeterminacy is quantified explicitly and truth membership, indeterminacy membership and falsity membership are completely independent. It can effectively describe uncertain, incomplete and inconsistent information and overcomes some limitations of the existing methods in depicting uncertain decision information. Moreover, some extensions of NSs, including interval neutrosophic set [48-51], bipolar neutrosophic set [52-54], single-valued neutrosophic set [55-59], simplified neutrosophic sets [60-64], multi-valued neutrosophic set [65-67], and neutrosophic linguistic set [68-70] have been presented and applied to solve various problems; see [71-80].

Although there are several approaches to solving various problems under neutrosophic environment, to the best of our knowledge, there are few investigations regarding DEA with neutrosophic sets. The first attempt has been proposed by Edalatpanah in [81] and further research has been presented in [82]. So, in this paper, we design a model of DEA with simplified neutrosophic numbers (SNNs) and establish a new strategy to solve it. The proposed method is based on the weighted arithmetic average operator and has a simple structure.

This paper organized as follows: some basic knowledge, concepts and arithmetic operations on SNNs are introduced in Section 2. In Section 3, we review some concepts of DEA and the input-oriented BCC model. In Section 4, we introduce the mentioned model of DEA under the simplified neutrosophic environment and propose a method to solve it. In Section 5, an example demonstrates the application of the proposed model. Finally, some conclusions and future research are offered in Section 6.

2. Simplified neutrosophic sets

Smarandache [44-45] has provided a variety of real-life examples for possible applications of his neutrosophic sets; however, it is difficult to apply neutrosophic sets to practical problems. Therefore, Ye [60] reduced neutrosophic sets of non-standard intervals into a kind of simplified neutrosophic sets (SNSs) of standard intervals that will preserve the operations of the neutrosophic sets. In this section, we will review the concept of SNSs, which are a subclass of neutrosophic sets briefly.

Definition 1 [60]. Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. If the functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are singleton subintervals/subsets in the real standard [0, 1], that is $T_A(x)$: $X \to [0,1]$, $I_A(x)$: $X \to [0,1]$, and $F_A(x)$: $X \to [0,1]$. Then, a simplification of the neutrosophic set A is denoted by $A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\}$, which is called a SNS. Also, SNS satisfies the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 2 [60]. For SNSs A and B, $A \subseteq B$ if and only if $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, and $F_A(x) \ge F_B(x)$ for every x in X.

Definition 3 [63]. Let A, B be two SNSs. Then the arithmetic relations are defined as:

$$(i)\, A \oplus B = < T_A(x) + T_B(x) - T_A(x) T_B(x), I_A(x) I_B(x), F_A(x) F_B(x) >, \tag{1}$$

$$(ii)\, A \otimes B = < T_A(x) T_B(x), I_A(x) + I_B(x) - I_A(x).I_B(x), F_A(x) + F_B(x) - F_A(x).F_B(x) >, \tag{2}$$

$$(iii)\, \lambda A = < 1-(1-T_A(x))^{\lambda}, (I_A(x))^{\lambda}, (F_A(x))^{\lambda} >, \lambda > 0. \tag{3}$$

$$(iv)\, A^{\lambda} = < T_A^{\lambda}(x), 1-(1-I_A(x))^{\lambda}, 1-(1-F_A(x))^{\lambda} >, \lambda > 0. \tag{4}$$

Definition 4 [60]. Let A_j (j = 1, 2, ... , n) be a SNS. The simplified neutrosophic weighted arithmetic average operator is defined as:

$$F_{\omega}(A_1, \dots, A_n) = \sum_{j=1}^{n} \omega_j A_j \tag{5}$$

where $W=(\omega_1,\omega_2,\ldots,\omega_n)$ is the weight vector of Aj, $\omega_j\in[0,1]$ and $\sum_{j=1}^{n}\omega_j=1$.

Theorem 1 [63]. For the simplified neutrosophic weighted arithmetic average operator, the aggregated result is as follows:

$$F_\omega(A_1,\ldots,A_n)=\left\langle 1-\prod_{j=1}^{n}(1-T_{A_j}(x))^{\omega_j},\prod_{j=1}^{n}(I_{A_j}(x))^{\omega_j},\prod_{j=1}^{n}(F_{A_j}(x))^{\omega_j}\right\rangle. \tag{6}$$

3. The input-oriented BCC model of DEA

Data envelopment analysis (DEA) is a linear programming method for assessing the efficiency and productivity of decision-making units (DMUs). In the traditional DEA literature, various well-known DEA approaches can be found such as CCR and BCC models [1, 3]. The efficiency of a DMU is established as the ratio of sum weighted output to sum weighted input, subjected to happen between one and zero. Let DMUO is under consideration, then input-oriented BCC model for the relative efficiency is as follows [3]:

$$\begin{aligned}
&Min \quad \theta_o\\
&s.t\\
&\quad \sum_{j=1}^{n}\lambda_j x_{ij}\le\theta_o x_{i_o} \qquad , \qquad i=1,2,\ldots,m\\
&\quad \sum_{j=1}^{n}\lambda_j y_{rj}\ge y_{ro} \qquad , \qquad r=1,2,\ldots,s\\
&\quad \sum_{j=1}^{n}\lambda_j=1\\
&\quad \lambda_j\ge 0 \qquad , \qquad j=1,2,\ldots,n
\end{aligned} \tag{7}$$

In this model, each DMU (suppose that we have n DMUs) uses m inputs x_{ij} $(i=1,2,\ldots,m)$, to obtains s outputs y_{rj} $(r=1,2,\ldots,s)$. Here $u_r(r=1,2,\ldots,s)$ and $v_i(i=1,2,\ldots,m)$, are the weights of the i th input and r th output. This model is calculated for every DMU to find out its best input and output weights. If $\theta_o^*=1$, we say that the DMU$_o$ is efficient otherwise it is inefficient.

4. Simplified Neutrosophic Data Envelopment Analysis

In this section, we establish DEA under simplified neutrosophic environment. Consider the input and output for the j th DMU as $x_{ij}^{\aleph}=(T_{x_{ij}},I_{x_{ij}},F_{x_{ij}})$, $y_{rj}^{\aleph}=(T_{y_{rj}},I_{y_{rj}},F_{y_{rj}})$ which are the simplified neutrosophic numbers (SNN). Then the simplified neutrosophic BCC model that called SNBCC is defined as follows:

$$\begin{aligned}
&Min \quad \theta_o\\
&s.t\\
&\quad \sum_{j=1}^{n}\lambda_j x_{ij}^{\aleph}\le\theta_o x_{i_o}^{\aleph}, \qquad i=1,2,\ldots,m\\
&\quad \sum_{j=1}^{n}\lambda_j y_{rj}^{\aleph}\ge y_{r_o}^{\aleph}, \qquad r=1,2,\ldots,s\\
&\quad \sum_{j=1}^{n}\lambda_j=1,\\
&\quad \lambda_j\ge 0, \qquad j=1,2,\ldots,n.
\end{aligned} \tag{8}$$

Next, to solve the model (8) we propose the following algorithm:

Algorithm 1.

Step 1. Consider the DEA model (8) that the inputs and outputs of each DMU are SNN.

Step 2. Using the Definition 3 and Theorem 1, the SNBCC model of Step 1 can be transformed into the following model:

$$
\begin{aligned}
& Min \qquad \theta_o \\
& s.t \\
& \left(1-\prod_{j=1}^{n}(1-T_{x_{ij}})^{\lambda_j}, \prod_{j=1}^{n}(I_{x_{ij}})^{\lambda_j}, \prod_{j=1}^{n}(F_{x_{ij}})^{\lambda_j}\right) \le \left(1-(1-T_{x_{io}})^{\theta_o}, (I_{x_{io}})^{\theta_o}, (F_{x_{io}})^{\theta_o}\right) \\
& \left(1-\prod_{j=1}^{n}(1-T_{y_{rj}})^{\lambda_j}, \prod_{j=1}^{n}(I_{y_{rj}})^{\lambda_j}, \prod_{j=1}^{n}(F_{y_{rj}})^{\lambda_j}\right) \ge \left(T_{y_{ro}}, I_{y_{ro}}, F_{y_{ro}}\right) \\
& \sum_{j=1}^{n}\lambda_j = 1, \\
& \lambda_j \ge 0, \qquad j = 1,2,...,n.
\end{aligned} \tag{9}
$$

Step 3. Using Definition 2, the SNBCC model of Step 2 can be transformed into the following model:

$$
\begin{aligned}
& Min \qquad \theta_o \\
& s.t \\
& \prod_{j=1}^{n}(1-T_{x_{ij}})^{\lambda_j} \ge (1-T_{x_{io}})^{\theta_o}, \qquad i = 1,2,\dots,m \\
& \prod_{j=1}^{n}(I_{x_{ij}})^{\lambda_j} \ge (I_{x_{io}})^{\theta_o}, \qquad i = 1,2,\dots,m \\
& \prod_{j=1}^{n}(F_{x_{ij}})^{\lambda_j} \ge (F_{x_{io}})^{\theta_o}, \qquad i = 1,2,\dots,m \\
& \prod_{j=1}^{n}(1-T_{y_{rj}})^{\lambda_j} \le (1-T_{y_{ro}}), \qquad r = 1,2,\dots,s \\
& \prod_{j=1}^{n}(I_{y_{rj}})^{\lambda_j} \le I_{y_{ro}}, \qquad r = 1,2,\dots,s \\
& \prod_{j=1}^{n}(F_{y_{rj}})^{\lambda_j} \le F_{y_{ro}}, \qquad r = 1,2,\dots,s \\
& \sum_{j=1}^{n}\lambda_j = 1, \\
& \lambda_j \ge 0, \qquad j = 1,2,\dots,n.
\end{aligned} \tag{10}
$$

Step 4. Using the natural logarithm, transform the nonlinear model of (10) into the following linear model:

$$
Min \qquad \theta_o \tag{11}
$$

$$s.t$$

$$
\sum_{j=1}^{n}\lambda_j \ln(1-T_{x_{ij}}) \ge \theta_o \ln(1-T_{x_{io}}), \qquad i = 1,2,...,m \tag{12}
$$

$$
\sum_{j=1}^{n}\lambda_j \ln(I_{x_{ij}}) \ge \theta_o \ln(I_{x_{io}}), \qquad i = 1,2,...,m \tag{13}
$$

$$
\sum_{j=1}^{n}\lambda_j \ln(F_{x_{ij}}) \ge \theta_o \ln(F_{x_{io}}), \qquad i = 1,2,...,m \tag{14}
$$

$$
\sum_{j=1}^{n}\lambda_j \ln(1-T_{y_{rj}}) \le \ln(1-T_{y_{ro}}), \qquad r = 1,2,...,s \tag{15}
$$

$$
\sum_{j=1}^{n}\lambda_j \ln(I_{y_{rj}}) \le \ln(I_{y_{ro}}), \qquad r = 1,2,...,s \tag{16}
$$

$$\sum_{j=1}^{n} \lambda_j \ln(F_{y_{rj}}) \le \ln(F_{y_{ro}}), \qquad r = 1,2,...,s \tag{17}$$

$$\sum_{j=1}^{n} \lambda_j = 1, \tag{18}$$

$$\lambda_j \ge 0, \qquad j = 1,2,...,n.$$

Step 5. Run model (11) and obtain the optimal solution.

5. Numerical example

In this section, an example of DEA problem under simplified neutrosophic environment is used to demonstrate the validity and effectiveness of the proposed model.

Example 5.1. Consider 10 DMUs with three inputs and outputs where all the input and output data are designed as SNN (see tables 1 and 2).

Table 1. DMUs with three SNN inputs

DMUS	Inputs 1	Inputs 2	Inputs 3
DMU1	<0.75, 0.1, 0.15>	<0.75,0.1, 0.15>	<0.8, 0.05, 0.1>
DMU2	<0.85, 0.2,0.15>	<0.6, 0.05,0.05>	<0.9, 0.1, 0.2>
DMU3	<0.9, 0.01, 0.05>	<0.95, 0.01, 0.01>	<0.98, 0.01, 0.01>
DMU4	<0.7,0.2, 0.1>	<0.65, 0.2, 0.15>	<0.8, 0.05, 0.2>
DMU5	<0.9, 0.05, 0.1>	<0.95, 0.05, 0.05>	<0.7, 0.2, 0.4>
DMU6	<0.85, 0.2, 0.1>	<0.7, 0.05, 0.1>	<0.6, 0.2, 0.3>
DMU7	<0.8, 0.3, 0.1>	<0.9, 0.5, 0.1>	<0.8, 0.1, 0.3>
DMU8	<0.55, 0.3, 0.35>	<0.65, 0.2, 0.25>	<0.5, 0.35, 0.4>
DMU9	<0.8, 0.05, 0.1>	<0.9, 0.01, 0.05>	<0.8, 0.05, 0.1>
DMU10	<0.6, 0.1, 0.3>	<0.8. 0.3. 0.1>	<0.65, 0.2, 0.1>

Table 2. DMUs with three SNN outputs.

DMUS	Outputs 1	Outputs 2	Outputs 3
DMU1	<0.7, 0.15, 0.2>	<0.7,0.15, 0.2>	<0.65, 0.2, 0.25>
DMU2	<0.15, 0.2,0.25>	<0.15, 0.2,0.25>	<0.25, 0.15, 0.05>
DMU3	<0.75, 0.1, 0.15>	<0.7, 0.15, 0.2>	<0.8, 0.05, 0.1>
DMU4	<0.5,0.35, 0.4>	<0.6, 0.25, 0.3>	<0.55, 0.3, 0.35>
DMU5	<0.6, 0.2, 0.25>	<0.6, 0.15, 0.4>	<0.3, 0.5, 0.5>
DMU6	<0.55, 0.3, 0.35>	<0.5, 0.5, 0.5>	<0.6, 0.25, 0.3>
DMU7	<0.8, 0.1, 0.2>	<0.3, 0.01, 0.05>	<0.9, 0.05, 0.05>
DMU8	<0.8, 0.1, 0.3>	<0.8, 0.25, 0.3>	<0.85, 0.2, 0.2>
DMU9	<0.65, 0.2, 0.25>	<0.7, 0.15, 0.2>	<0.75, 0.1, 0.15>
DMU10	<0.6, 0.1, 0.5>	<0.75. 0.1. 0.3>	<0.8, 0.3, 0.5>

Next, we use Algorithm.1 to solve the mentioned performance assessment problem. For example, The Algorithm.1 for DMU_1 can be used as follows:

Step 1. Obtain the SNBCC model (8):

$$Min \quad \theta_1$$
$$s.t$$

$$\begin{pmatrix} \lambda_1 <0.75,0.1,0.15> \oplus \lambda_2 <0.85,0.2,0.15> \oplus \lambda_3 <0.9,0.01,0.05> \oplus \\ \lambda_4 <0.7,0.2,0.1> \oplus \lambda_5 <0.9,0.05,0.1> \oplus \lambda_6 <0.85,0.2,0.1> \oplus \\ \lambda_7 <0.8,0.3,0.35> \oplus \lambda_8 <0.8,0.05,0.1> \oplus \lambda_9 <0.6,0.1,0.3> \oplus \\ \lambda_{10} <0.6,0.1,0.3> \end{pmatrix} \le \left(\theta_1 <0.75,0.1,0.15>\right),$$

$$\begin{pmatrix} \lambda_1 <0.7,0.1,0.2> \oplus \lambda_2 <0.6,0.05,0.05> \oplus \lambda_3 <0.95,0.01,0.01> \oplus \\ \lambda_4 <0.65,0.2,0.15> \oplus \lambda_5 <0.95,0.05,0.05> \oplus \lambda_6 <0.7,0.05,0.1> \oplus \\ \lambda_7 <0.9,0.5,0.1> \oplus \lambda_8 <0.65,0.2,0.25> \oplus \lambda_9 <0.9,0.01,0.05> \oplus \\ \lambda_{10} <0.8,0.3,0.1> \end{pmatrix} \le \left(\theta_1 <0.7,0.1,0.2>\right),$$

$$\begin{pmatrix} \lambda_1 <0.8,0.05,0.1> \oplus \lambda_2 <0.9,0.1,0.2> \oplus \lambda_3 <0.98,0.01,0.01> \oplus \\ \lambda_4 <0.8,0.05,0.2> \oplus \lambda_5 <0.7,0.2,0.4> \oplus \lambda_6 <0.6,0.2,0.3> \oplus \\ \lambda_7 <0.8,0.1,0.3> \oplus \lambda_8 <0.5,0.35,0.4> \oplus \lambda_9 <0.7,0.05,0.1> \oplus \\ \lambda_{10} <0.65,0.2,0.1> \end{pmatrix} \le \left(\theta_1 <0.8,0.05,0.1>\right),$$

$$\begin{pmatrix} \lambda_1 <0.7,0.15,0.2> \oplus \lambda_2 <0.15,0.2,0.25> \oplus \lambda_3 <0.75,0.1,0.15> \oplus \\ \lambda_4 <0.5,0.35,0.4> \oplus \lambda_5 <0.6,0.2,0.25> \oplus \lambda_6 <0.55,0.3,0.35> \oplus \\ \lambda_7 <0.8,0.1,0.2> \oplus \lambda_8 <0.8,0.1,0.3> \oplus \lambda_9 <0.65,0.2,0.25> \oplus \\ \lambda_{10} <0.6,0.1,0.5> \end{pmatrix} \ge \left(<0.7,0.15,0.2>\right),$$

$$\begin{pmatrix} \lambda_1 <0.6,0.1,0.3> \oplus \lambda_2 <0.2,0.1,0.3> \oplus \lambda_3 <0.7,0.15,0.2> \oplus \\ \lambda_4 <0.6,0.25,0.3> \oplus \lambda_5 <0.6,0.15,0.4> \oplus \lambda_6 <0.5,0.5,0.5> \oplus \\ \lambda_7 <0.3,0.01,0.05> \oplus \lambda_8 <0.8,0.25,0.3> \oplus \lambda_9 <0.7,0.15,0.2> \oplus \\ \lambda_{10} <0.75,0.1,0.3> \end{pmatrix} \ge \left(<0.6,0.1,0.3>\right),$$

$$\begin{pmatrix} \lambda_1 <0.65,0.2,0.25> \oplus \lambda_2 <0.25,0.15,0.05> \oplus \lambda_3 <0.8,0.05,0.1> \oplus \\ \lambda_4 <0.55,0.3,0.35> \oplus \lambda_5 <0.3,0.5,0.5> \oplus \lambda_6 <0.6,0.25,0.3> \oplus \\ \lambda_7 <0.9,0.05,0.05> \oplus \lambda_8 <0.85,0.2,0.2> \oplus \lambda_9 <0.75,0.1,0.15> \oplus \\ \lambda_{10} <0.8,0.3,0.5> \end{pmatrix} \ge \left(<0.65,0.2,0.25>\right),$$

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_4+\lambda_6+\lambda_7+\lambda_8+\lambda_9+\lambda_{10}=1,$$
$$\lambda_j \ge 0, \qquad j=1,2,...,10.$$

Step 2. Using the Step 4 of Algorithm 1, we have:

$$Min \quad \theta_1$$
$$s.t$$

(Using Eq. (12))

$$\lambda_1 \ln(0.25)+\lambda_2 \ln(0.15)+\lambda_3 \ln(0.1)+\lambda_4 \ln(0.3)+\lambda_5 \ln(0.1)+$$
$$\lambda_6 \ln(0.15)+\lambda_7 \ln(0.2)+\lambda_8 \ln(0.2)+\lambda_9 \ln(0.4)+\lambda_{10} \ln(0.4) \ge \theta_1 \ln(0.25),$$

$$\lambda_1 \ln(0.3) + \lambda_2 \ln(0.4) + \lambda_3 \ln(0.05) + \lambda_4 \ln(0.35) + \lambda_5 \ln(0.05) + \lambda_6 \ln(0.3) + \lambda_7 \ln(0.1) + \lambda_8 \ln(0.35) + \lambda_9 \ln(0.1) + \lambda_{10} \ln(0.2) \geq \theta_1 \ln(0.3)$$

$$\lambda_1 \ln(0.2) + \lambda_2 \ln(0.1) + \lambda_3 \ln(0.02) + \lambda_4 \ln(0.2) + \lambda_5 \ln(0.3) + \lambda_6 \ln(0.4) + \lambda_7 \ln(0.2) + \lambda_8 \ln(0.5) + \lambda_9 \ln(0.3) + \lambda_{10} \ln(0.35) \geq \theta_1 \ln(0.2)$$

(Using Eq. (13))

$$\lambda_1 \ln(0.1) + \lambda_2 \ln(0.2) + \lambda_3 \ln(0.01) + \lambda_4 \ln(0.2) + \lambda_5 \ln(0.05) + \lambda_6 \ln(0.2) + \lambda_7 \ln(0.3) + \lambda_8 \ln(0.05) + \lambda_9 \ln(0.1) + \lambda_{10} \ln(0.1) \geq \theta_1 \ln(0.1)$$

$$\lambda_1 \ln(0.1) + \lambda_2 \ln(0.05) + \lambda_3 \ln(0.01) + \lambda_4 \ln(0.2) + \lambda_5 \ln(0.05) + \lambda_6 \ln(0.05) + \lambda_7 \ln(0.5) + \lambda_8 \ln(0.2) + \lambda_9 \ln(0.01) + \lambda_{10} \ln(0.3) \geq \theta_1 \ln(0.1)$$

$$\lambda_1 \ln(0.05) + \lambda_2 \ln(0.05) + \lambda_3 \ln(0.01) + \lambda_4 \ln(0.05) + \lambda_5 \ln(0.2) + \lambda_6 \ln(0.2) + \lambda_7 \ln(0.1) + \lambda_8 \ln(0.35) + \lambda_9 \ln(0.05) + \lambda_{10} \ln(0.2) \geq \theta_1 \ln(0.05)$$

(Using Eq. (14))

$$\lambda_1 \ln(0.15) + \lambda_2 \ln(0.15) + \lambda_3 \ln(0.05) + \lambda_4 \ln(0.1) + \lambda_5 \ln(0.1) + \lambda_6 \ln(0.1) + \lambda_7 \ln(0.35) + \lambda_8 \ln(0.1) + \lambda_9 \ln(0.3) + \lambda_{10} \ln(0.3) \geq \theta_1 \ln(0.15)$$

$$\lambda_1 \ln(0.2) + \lambda_2 \ln(0.05) + \lambda_3 \ln(0.01) + \lambda_4 \ln(0.15) + \lambda_5 \ln(0.05) + \lambda_6 \ln(0.1) + \lambda_7 \ln(0.1) + \lambda_8 \ln(0.25) + \lambda_9 \ln(0.05) + \lambda_{10} \ln(0.1) \geq \theta_1 \ln(0.2)$$

$$\lambda_1 \ln(0.1) + \lambda_2 \ln(0.2) + \lambda_3 \ln(0.01) + \lambda_4 \ln(0.2) + \lambda_5 \ln(0.4) + \lambda_6 \ln(0.3) + \lambda_7 \ln(0.3) + \lambda_8 \ln(0.4) + \lambda_9 \ln(0.1) + \lambda_{10} \ln(0.1) \geq \theta_1 \ln(0.1)$$

(Using Eq. (15))

$$\lambda_1 \ln(0.3) + \lambda_2 \ln(0.85) + \lambda_3 \ln(0.25) + \lambda_4 \ln(0.5) + \lambda_5 \ln(0.4) + \lambda_6 \ln(0.45) + \lambda_7 \ln(0.2) + \lambda_8 \ln(0.2) + \lambda_9 \ln(0.35) + \lambda_{10} \ln(0.4) \leq \ln(0.3),$$

$$\lambda_1 \ln(0.4) + \lambda_2 \ln(0.8) + \lambda_3 \ln(0.3) + \lambda_4 \ln(0.4) + \lambda_5 \ln(0.4) + \lambda_6 \ln(0.5) + \lambda_7 \ln(0.7) + \lambda_8 \ln(0.2) + \lambda_9 \ln(0.3) + \lambda_{10} \ln(0.25) \leq \ln(0.4),$$

$$\lambda_1 \ln(0.35) + \lambda_2 \ln(0.75) + \lambda_3 \ln(0.2) + \lambda_4 \ln(0.45) + \lambda_5 \ln(0.7) + \lambda_6 \ln(0.4) + \lambda_7 \ln(0.1) + \lambda_8 \ln(0.15) + \lambda_9 \ln(0.25) + \lambda_{10} \ln(0.2) \leq \ln(0.35),$$

(Using Eq. (16))

$$\lambda_1 \ln(0.15) + \lambda_2 \ln(0.2) + \lambda_3 \ln(0.1) + \lambda_4 \ln(0.35) + \lambda_5 \ln(0.2) + \lambda_6 \ln(0.3) + \lambda_7 \ln(0.1) + \lambda_8 \ln(0.1) + \lambda_9 \ln(0.2) + \lambda_{10} \ln(0.1) \leq \ln(0.15),$$

$$\lambda_1 \ln(0.1) + \lambda_2 \ln(0.1) + \lambda_3 \ln(0.15) + \lambda_4 \ln(0.25) + \lambda_5 \ln(0.15) + \lambda_6 \ln(0.5) + \lambda_7 \ln(0.01) + \lambda_8 \ln(0.25) + \lambda_9 \ln(0.15) + \lambda_{10} \ln(0.1) \leq \ln(0.1),$$

$$\lambda_1 \ln(0.2)+\lambda_2 \ln(0.15)+\lambda_3 \ln(0.05)+\lambda_4 \ln(0.3)+\lambda_5 \ln(0.5)+$$
$$\lambda_6 \ln(0.25)+\lambda_7 \ln(0.05)+\lambda_8 \ln(0.2)+\lambda_9 \ln(0.1)+\lambda_{10} \ln(0.3)\le \ln(0.2),$$

(Using Eq. (17))

$$\lambda_1 \ln(0.2)+\lambda_2 \ln(0.25)+\lambda_3 \ln(0.15)+\lambda_4 \ln(0.4)+\lambda_5 \ln(0.25)+$$
$$\lambda_6 \ln(0.35)+\lambda_7 \ln(0.2)+\lambda_8 \ln(0.3)+\lambda_9 \ln(0.25)+\lambda_{10} \ln(0.5)\le \ln(0.2),$$

$$\lambda_1 \ln(0.3)+\lambda_2 \ln(0.3)+\lambda_3 \ln(0.2)+\lambda_4 \ln(0.3)+\lambda_5 \ln(0.4)+$$
$$\lambda_6 \ln(0.5)+\lambda_7 \ln(0.05)+\lambda_8 \ln(0.3)+\lambda_9 \ln(0.2)+\lambda_{10} \ln(0.3)\le \ln(0.3),$$

$$\lambda_1 \ln(0.25)+\lambda_2 \ln(0.05)+\lambda_3 \ln(0.1)+\lambda_4 \ln(0.35)+\lambda_5 \ln(0.5)+$$
$$\lambda_6 \ln(0.3)+\lambda_7 \ln(0.05)+\lambda_8 \ln(0.2)+\lambda_9 \ln(0.15)+\lambda_{10} \ln(0.5)\le \ln(0.25),$$

(Using Eq. (18))

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_4+\lambda_6+\lambda_7+\lambda_8+\lambda_9+\lambda_{10}=1,$$
$$\lambda_j \ge 0, \qquad j=1,2,\ldots,10.$$

Step 3. After computations with Lingo, we obtain $\theta_1^* = 0.9068$ for *DMU₁*.

Similarly, for the other DMUs, we report the results in Table 3.

Table 3. The efficiencies of the other DMUs

DMUs	1	2	3	4	5	6	7	8	9	10
θ^*	0.9068	0.9993	0.5153	0.9973	0.6382	0.6116	1	1	0.6325	1
Rank	4	2	8	3	5	7	1	1	6	1

By these results, we can see that DMUs 7, 8, and 10 are efficient and others are inefficient.

6. Conclusions and future work

There are several approaches to solving various problems under neutrosophic environment. However, to the best of our knowledge, the Data Envelopment Analysis (DEA) has not been discussed with neutrosophic sets until now. This paper, therefore, plans to fill this gap and a new method has been designed to solve an input-oriented DEA model with simplified neutrosophic numbers. A numerical example has been illustrated to show the efficiency of the proposed method. The proposed approach has produced promising results from computing efficiency and performance aspects. Moreover, although the model, arithmetic operations and results presented here demonstrate the effectiveness of our approach, it could also be considered in other DEA models and their applications to banks, police stations, hospitals, tax offices, prisons, schools and universities. As future researches, we intend to study these problems.

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Correlation Measure for Pythagorean Neutrosophic Sets with T and F as Dependent Neutrosophic Components

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Abstract: In this paper, we study the new concept of Pythagorean neutrosophic set with T and F as dependent neutrosophic components [PNS]. Pythagorean neutrosophic set with T and F as dependent neutrosophic components [PNS] is introduced as a generalization of neutrosophic set (In neutrosophic sets, there are three special cases, here we take one of the special cases. That is, membership and non-membership degrees are dependent components and indeterminacy is independent) and Pythagorean fuzzy set. In PNS sets, membership, non-membership and indeterminacy degrees are gratifying the condition $0 \leq (u_A(x))^2 + (\zeta_A(x))^2 + (v_A(x))^2 \leq 2$ instead of $u_A(x) + \zeta_A(x) + v_A(x) > 2$ as in neutrosophic sets. We investigate the basic operations of PNS sets. Also, the correlation measure of PNS set is proposed and proves some of their basic properties. The concept of this correlation measures of PNS set is the extension of correlation measures of Pythagorean fuzzy set and neutrosophic set. Then, using correlation of PNS set measure, the application of medical diagnosis is given.

Keywords: Pythagorean fuzzy set, Pythagorean Neutrosophic set with T and F as dependent neutrosophic components [PNS], Correlation measure and Medical diagnosis.

Introduction

Fuzzy sets were firstly initiated by L.A.Zadeh [36] in 1965. Zadeh's idea of fuzzy set evolved as a new tool having the ability to deal with uncertainties in real-life problems and discussed only membership function. After the extensions of fuzzy set theory Atanassov [7] generalized this concept and introduced a new set called intuitionistic fuzzy set (IFS) in 1986, which can be describe the non-membership grade of an imprecise event along with its membership grade under a restriction that the sum of both membership and non-membership grades does not exceed 1. IFS has its greatest use in practical multiple attribute decision making problems.In some practical problems.In some practical problems, the sum of membership and non-membership degree to which an alternative satisfying attribute provided by decision maker(DM) may be bigger than 1.

Yager [30] was decided to introduce the new concept known as Pythagorean fuzzy sets. Pythagorean fuzzy sets has limitation that their square sum is less than or equal to 1. IFS was failed to deal with indeterminate and inconsistent information which exist in beliefs system, therefore, Smarandache [22] in 1995 introduced new concept known as neutrosophic set(NS) which generalizes fuzzy sets and intuitionistic fuzzy sets and so on. A neutrosophic set includes truth membership, falsity membership and indeterminacy membership.

In 2006, F.Smarandache introduced, for the first time, the degree of dependence (and consequently the degree of independence) between the components of the fuzzy set, and also between the components of the neutrosophic set. In 2016, the refined neutrosophic set was generalized to the degree of dependence or independence of subcomponents [22]. In neutrosophic set [22], if truth membership and falsity membership are 100% dependent and indeterminacy is 100% independent, that is $0 \leq u_A(x) + \zeta_A(x) + v_A(x) \leq 2$. Sometimes in real life, we face many problems which cannot be handled by using neutrosophic for example when $u_A(x) + \zeta_A(x) + v_A(x) > 2$. In such condition, a neutrosophic set has no ability to obtain any satisfactory result. To state this condition, we give an example: the truth membership, falsity membership and indeterminacy values are $\frac{8}{10}, \frac{5}{10}$ *and* $\frac{9}{10}$ respectively. This satisfies the condition that their sums exceeds 2 and are not presented to neutrosophic set. So, In Pythagorean neutrosophic set with T and F are dependent neutrosophic components [PNS] of condition is as their square sum does not exceeds 2. Here, T and F are dependent neutrosophic components and we make $u_A(x), v_A(x)$ *as* Pythagorean, then $\left(u_A(x)\right)^2 + \left(v_A(x)\right)^2 \leq 1$ with $u_A(x), v_A(x)$ *in* [0,1]. If $\zeta_A(x)$ is an Independent from them, then $0 \leq \zeta_A(x) \leq 1$. Then $0 \leq \left(u_A(x)\right)^2 + \left(\zeta_A(x)\right)^2 + \left(v_A(x)\right)^2 \leq 2$, with $u_A(x), \zeta_A(x), v_A(x)$ *in* [0,1]. We consider in general the degree of dependence between $u_A(x), \zeta_A(x), v_A(x)$ *is* 1 , hence $u_A(x), \zeta_A(x), v_A(x) \leq 3 - 1 = 2$.

Correlation coefficients are beneficial tools used to determine the degree of similarity between objects. The importance of correlation coefficients in fuzzy environments lies in the fact that these types of tools can feasibly be applied to problems of pattern recognition, MADM, medical diagnosis and clustering, etc. In other research, Ye[33] proposed three vector similarity measure for SNSs, an instance of SVNS and INS, includingthe Jaccard, Dice, and cosine similarity measures for SVNS and INSs, and applied them to multi-criteria decision-making problems with simplified neutrosophic information. Hanafy et al. [16] proposed the correlation coefficients of neutrosophic sets and studied some of their basic properties. Based on centroid method, Hanafy et al. [17], introduced and studied the concepts of correlation and correlation coefficient of neutrosophic sets and studied some of their properties.

Recently Bromi and Smarandache defined the Haudroff distance between neutrosophic sets and some similarity measures based on the distance such as; set theoretic approach and matching function to calculate the similarity degree between neutrosophic sets. In the same year, Broumi and Smarandache [11] also proposed the correlation coefficient between interval neutrsphic sets.

In this paper, we have to study the concept of Pythagorean neutrosophic set with T and F are neutrosophic components and also define the correlation measure of Pythagorean neutrosophic set with T and F are dependent neutrosophic components [PNS] and prove some of its properties. Then, using correlation of Pythagorean neutrosophic fuzzy set with T and F are dependent neutrosophic components [PNS] measure, the application of medical diagnosis is given.

Preliminaries

Definition 2.1 [1] Let E be a universe. An intuitionistic fuzzy set A on E can be defined as follows:

$$A = \{< x, u_A(x), v_A(x) >: x \in E\}$$

Where $u_A: E \to [0,1]$ *and* $v_A: E \to [0,1]$ such that $0 \leq u_A(x) + v_A(x) \leq 1$ for any $x \in E$. Where, $u_A(x)$ and $v_A(x)$ is the degree of membership and degree of non-membership of the element x, respectively.

Definition 2.2 [18, 24]
Let X be a non-empty set and I the unit interval [0, 1]. A Pythagorean fuzzy set S is an object having the form $A = \{(x, u_A(x), v_A(x)): x \in X\}$ where the functions $u_A: X \to [0,1]$ and $v_A: X \to [0,1]$ denote respectively the degree of membership and degree of non-membership of each element $x \in X$ to the set P, and $0 \leq \left(u_A(x)\right)^2 + (v_A(x))^2 \leq 1$ for each $x \in X$.

Definition 2.3[15] Let X be a non-empty set (universe). A neutrosophic set A on X is an object of the form: $A = \left\{\left(x, u_A(x), \zeta_A(x), v_A(x)\right): x \in X\right\}$,

Where $u_A(x), \zeta_A(x), v_A(x) \in [0,1], 0 \leq u_A(x) + \zeta_A(x) + v_A(x) \leq 2, for\ all\ x\ in\ X$. $u_A(x)$ is the degree of membership, $\zeta_A(x)$ is the degree of inderminancy and $v_A(x)$ is the degree of non-membership. Here $u_A(x)$ and $v_A(x)$ are dependent components and $\zeta_A(x)$ is an independent components.

Definition 2.4 Let X be a nonempty set and I the unit interval [0,1]. A neutrosophic set A and B of the form $A = \{(x, u_A(x), \zeta_A(x), v_A(x)): x \in X\}$ and $B = \{(x, u_B(x), \zeta_B(x), v_B(x)): x \in X\}$. Then

1) $A^C = \left\{\left(x, v_A(x), \zeta_A(x), u_A(x)\right): x \in X\right\}$
2) $A \cup B = \left\{\left(x, \max\left(u_A(x), u_B(x)\right), \min\left(\zeta_A(x), \zeta_B(x)\right), \min(v_A(x), v_B(x))\right): x \in X\right\}$
3) $A \cap B = \left\{(x, \min\left(u_A(x), u_B(x)\right), \max\left(\zeta_A(x), \zeta_B(x)\right), \max(v_A(x), v_B(x)): x \in X\right\}$

3. Pythagorean Neutrosophic set with T and F are dependent neutrosophic components [PNS]:

Definition 3.1 Let X be a non-empty set (universe). A Pythagorean neutrosophic set with T and F are dependent neutrosophic components [PNS] A on X is an object of the form $A = \left\{\left(x, u_A(x), \zeta_A(x), v_A(x)\right): x \in X\right\}$,

Where $u_A(x), \zeta_A(x), v_A(x) \in [0,1], 0 \leq \left(u_A(x)\right)^2 + \left(\zeta_A(x)\right)^2 + \left(v_A(x)\right)^2 \leq 2, for\ all\ x\ in\ X$. $u_A(x)$ is the degree of membership, $\zeta_A(x)$ is the degree of inderminancy and $v_A(x)$ is the degree of non-membership .Here $u_A(x)$ and $v_A(x)$ are dependent components and $\zeta_A(x)$ is an independent components.

Definition 3.2 Let X be a nonempty set and I the unit interval [0, 1]. A Pythagorean neutrosophic set with T and F are dependent neutrosophic components [PNS] A and B of the form

$A = \{(x, u_A(x), \zeta_A(x), v_A(x)): x \in X\}$ and $B = \{(x, u_B(x), \zeta_B(x), v_B(x)): x \in X\}$. Then
1) $A^C = \left\{\left(x, v_A(x), \zeta_A(x), u_A(x)\right): x \in X\right\}$
2) $A \cup B = \left\{\left(x, \max\left(u_A(x), u_B(x)\right), \max\left(\zeta_A(x), \zeta_B(x)\right), \min(v_A(x), v_B(x))\right): x \in X\right\}$
3) $A \cap B = \left\{(x, \max\left(u_A(x), u_B(x)\right), \max\left(\zeta_A(x), \zeta_B(x)\right), \min(v_A(x), v_B(x)): x \in X\right\}$

Definition 3.3 Let X be a nonempty set and I the unit interval [0, 1]. A Pythagorean neutrosophic set with T and F are dependent neutrosophic components [PNS] A and B of the form
$A = \{(x, u_A(x), \zeta_A(x), v_A(x)): x \in X\}$ and $B = \{(x, u_B(x), \zeta_B(x), v_B(x)): x \in X\}$.
Then the correlation coefficient of A and B

$$\rho(A, B) = \frac{C(A, B)}{\sqrt{C(A, A). C(B, B)}} \qquad (1)$$

$$C(A,B) = \sum_{i=1}^{n} \left(\left(u_A(x_i)\right)^2 . \left(u_B(x_i)\right)^2 + \left(\zeta_A(x_i)\right)^2 . \left(\zeta_B(x_i)\right)^2 + \left(v_A(x_i)\right)^2 . \left(v_B(x_i)\right)^2 \right)$$

$$C(A,A) = \sum_{i=1}^{n} \left(\left(u_A(x_i)\right)^2 . \left(u_A(x_i)\right)^2 + \left(\zeta_A(x_i)\right)^2 . \left(\zeta_A(x_i)\right)^2 + \left(v_A(x_i)\right)^2 . \left(v_A(x_i)\right)^2 \right)$$

$$C(B,B) = \sum_{i=1}^{n} \left(\left(u_B(x_i)\right)^2 . \left(u_B(x_i)\right)^2 + \left(\zeta_B(x_i)\right)^2 . \left(\zeta_B(x_i)\right)^2 + \left(v_B(x_i)\right)^2 . \left(v_B(x_i)\right)^2 \right)$$

Preposition 3.4 The defined correlation measure between PNS A and PNS B satisfies the following properties

(i) $0 \leq \rho(A,B) \leq 1$

(ii) $\rho(A,B) = 1 \; if \; and \; only \; if \; A = B$

(iii) $\rho(A,B) = \rho(B,A)$.

Proof:

(i) $0 \leq \rho(A,B) \leq 1$

As the membership, inderminate and non-membership functions of the PNS lies between 0 and 1, $\rho(A,B)$ also lies between 0 and 1.

We will prove $C(A,B) = \sum_{i=1}^{n} \left(\left(u_A(x_i)\right)^2 . \left(u_B(x_i)\right)^2 + \left(\zeta_A(x_i)\right)^2 . \left(\zeta_B(x_i)\right)^2 + \left(v_A(x_i)\right)^2 . \left(v_B(x_i)\right)^2 \right)$

$$= \left(\left(u_A(x_1)\right)^2 . \left(u_B(x_1)\right)^2 + \left(\zeta_A(x_1)\right)^2 . \left(\zeta_B(x_1)\right)^2 + \left(v_A(x_1)\right)^2 . \left(v_B(x_1)\right)^2 \right) + \left(\left(u_A(x_2)\right)^2 . \left(u_B(x_2)\right)^2 + \left(\zeta_A(x_2)\right)^2 . \left(\zeta_B(x_2)\right)^2 + \left(v_A(x_2)\right)^2 . \left(v_B(x_2)\right)^2 \right) + \cdots + \left(\left(u_A(x_n)\right)^2 . \left(u_B(x_n)\right)^2 + \left(\zeta_A(x_n)\right)^2 . \left(\zeta_B(x_n)\right)^2 + \left(v_A(x_n)\right)^2 . \left(v_B(x_n)\right)^2 \right)$$

By Cauchy-Schwarz inequality, $(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2 \leq ({x_1}^2 + {x_2}^2 + \cdots + {x_n}^2).({y_1}^2 + {y_2}^2 + \cdots + {y_n}^2)$, where $(x_1 + x_2 + \cdots + x_n) \in R^n \; and \; (y_1 + y_2 + \cdots + y_n) \in R^n$, we get

$$\left(C(A,B)\right)^2 = \left(\left(u_A(x_1)\right)^4 + \left(\zeta_A(x_1)\right)^4 + \left(v_A(x_1)\right)^4 \right) + \left(\left(u_A(x_2)\right)^4 + \left(\zeta_A(x_2)\right)^4 + \left(v_A(x_2)\right)^4 \right) + \ldots + \left(\left(u_A(x_n)\right)^4 + \left(\zeta_A(x_n)\right)^4 + \left(v_A(x_n)\right)^4 \right)$$

$$\times \left(\left(u_B(x_1)\right)^4 + \left(\zeta_B(x_1)\right)^4 + \left(v_B(x_1)\right)^4 \right) + \left(\left(u_B(x_2)\right)^4 + \left(\zeta_B(x_2)\right)^4 + \left(v_B(x_2)\right)^4 \right) + \cdots + \left(\left(u_B(x_n)\right)^4 + \left(\zeta_B(x_n)\right)^4 + \left(v_B(x_n)\right)^4 \right)$$

$$= \left(\left(u_A(x_1)\right)^2 . \left(u_A(x_1)\right)^2 + \left(\zeta_A(x_1)\right)^2 . \left(\zeta_A(x_1)\right)^2 + \left(v_A(x_1)\right)^2 . \left(v_A(x_1)\right)^2 \right) + \left(\left(u_A(x_2)\right)^2 . \left(u_A(x_2)\right)^2 + \left(\zeta_A(x_2)\right)^2 . \left(\zeta_A(x_2)\right)^2 + \left(v_A(x_2)\right)^2 . \left(v_A(x_2)\right)^2 \right) + \cdots +$$

$$\left(\left(u_A\,(x_n)\right)^2.\left(u_A(x_n)\right)^2+\left(\zeta_A\,(x_n)\right)^2.\left(\zeta_A(x_n)\right)^2+\left(v_A\,(x_n)\right)^2.\left(v_A(x_n)\right)^2\right)\times$$
$$\left(\left(u_B(x_1)\right)^2\left(u_B(x_1)\right)^2+\left(\zeta_B(x_1)\right)^2\left(\zeta_B(x_1)\right)^2+\left(v_B(x_1)\right)^2\left(v_B(x_1)\right)^2\right)+$$
$$\left(\left(u_B(x_2)\right)^2\left(u_B(x_2)\right)^2+\left(\zeta_B(x_2)\right)^2\left(\zeta_B(x_2)\right)^2+\left(v_B(x_2)\right)^2\left(v_B(x_2)\right)^2\right)+\cdots+$$
$$\left(\left(u_B(x_n)\right)^2\left(u_B(x_n)\right)^2+\left(\zeta_B(x_n)\right)^2+\left(v_B(x_n)\right)^2\left(v_B(x_n)\right)^2\right)$$

$$=C(A,A)\times C(B,B).$$

Therefore, $\left(C(A,B)\right)^2\le C(A,A)\times C(B,B)$ and thus $\rho(A,B)\le 1$.

Hence we obtain the following propertity $0\le\ \rho(A,B)\le 1$

(ii) $\rho(A,B)=1\ if\ and\ only\ if\ A=B$

Let the two PNS A and B be equal (i.e A = B). Hence for any

$u_A(x_i)=u_B(x_i)$, $\zeta_A(x_i)=\zeta_B(x_i)$ and $v_A(x_i)=v_B(x_i)$,

Then $C(A,A)=C(B,B)=\sum_{i=1}^{n}\left(\left(u_A\,(x_i)\right)^2.\left(u_A(x_i)\right)^2+\left(\zeta_A\,(x_i)\right)^2.\left(\zeta_A(x_i)\right)^2+\left(v_A\,(x_i)\right)^2.\left(v_A(x_i)\right)^2\right)$

And $C(A,B)=\sum_{i=1}^{n}\left(\left(u_A\,(x_i)\right)^2.\left(u_B(x_i)\right)^2+\left(\zeta_A\,(x_i)\right)^2.\left(\zeta_B(x_i)\right)^2+\left(v_A\,(x_i)\right)^2.\left(v_B(x_i)\right)^2\right)$

$$=\sum_{i=1}^{n}\left(\left(u_A\,(x_i)\right)^2.\left(u_A(x_i)\right)^2+\left(\zeta_A\,(x_i)\right)^2.\left(\zeta_A(x_i)\right)^2+\left(v_A\,(x_i)\right)^2.\left(v_A(x_i)\right)^2\right)=C(A,A)$$

Hence

$$\rho(A,B)=\frac{C(A,B)}{\sqrt{C(A,A).C(B,B)}}$$

$$=\frac{C(A,A)}{\sqrt{C(A,A).C(A,A)}}=1$$

Let the $\rho(A,B)=1$.Then, the unite measure is possible only if

$$\frac{C(A,B)}{\sqrt{C(A,A).C(B,B)}}=1$$

This refer that $u_A(x_i)=u_B(x_i)$, $\zeta_A(x_i)=\zeta_B(x_i)$ and $v_A(x_i)=v_B(x_i)$,

for all i. Hence A = B.

(iii) If $\rho(A,B)=\rho(B,A)$, it obvious that

$$\frac{C(A,B)}{\sqrt{C(A,A).C_{NPFS}(B,B)}}=\frac{C(A,B)}{\sqrt{C(A,A).C(B,B)}}=\rho(B,A)$$

as

$$C(A,B) = \sum_{i=1}^{n}\left(\left(u_A(x_i)\right)^2.\left(u_B(x_i)\right)^2 + \left(\zeta_A(x_i)\right)^2.\left(\zeta_B(x_i)\right)^2 + \left(v_A(x_i)\right)^2.\left(v_B(x_i)\right)^2\right)$$
$$= \sum_{i=1}^{n}\left(\left(u_B(x_i)\right)^2.\left(u_A(x_i)\right)^2 + \left(\zeta_B(x_i)\right)^2.\left(\zeta_A(x_i)\right)^2 + \left(v_B(x_i)\right)^2.\left(v_A(x_i)\right)^2\right)$$
$$C(B,A)$$

Hence the proof.

Definition 3.5

Let A and B be two PNSs, then the correlation coefficient is defined as

$$\rho'(A,B) = \frac{C(A,B)}{max\{C(A,A).C(B,B)\}} \tag{2}$$

Theorem 3.6

The defined correlation measure between PNS A and PNS B satisfies the following properties

(i) $0 \le \rho'(A,B) \le 1$

(ii) $\rho'(A,B) = 1\ if\ and\ only\ if\ A = B$

(iii) $\rho'(A,B) = \rho'(B,A)$.

Proof: The property (i) and (ii) is straight forward, so omit here. Also $\rho'(A,B) \ge 0$ is evident. We now prove only $\rho'(A,B) \le 1$.

Since Theorem 3.4, we have $(C(A,B))^2 \le C(A,A).C(B,B)$. Therefore, $C(A,B) \le max\{C(A,A),C(B,B)\}$ and thus $\rho'(A,B) \le 1$.

However, in many practical situations, the different set may have taken different weights, and thus, weight ω_i of the element $x_i \in X$ $(i = 1,2,\dots,n)$ should be taken into account. In the following, we develop a weighted correlation coefficient between PNSs. Let $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ be the weight vector of the elements $x_i(i = 1,2,\dots.,n)$ with $\omega_i \ge 0\ and\ \sum_{i=1}^{n}\omega_i = 1$, then we have extended the above correlation coefficient $\rho(A,B)\ and\ \rho'(A,B)$ to weighted correlation coefficient as follows:

$$\rho'' = \frac{C_\omega(A,B)}{\sqrt{C_\omega(A,A).C_\omega(B,B)}} \tag{3}$$

$$C_\omega(A,B) = \sum_{i=1}^{n}\omega_i\left(\left(u_A(x_i)\right)^2.\left(u_B(x_i)\right)^2 + \left(\zeta_A(x_i)\right)^2.\left(\zeta_B(x_i)\right)^2 + \left(v_A(x_i)\right)^2.\left(v_B(x_i)\right)^2\right)$$

$$C_\omega(A,A) = \sum_{i=1}^{n}\omega_i\left(\left(u_A(x_i)\right)^2.\left(u_A(x_i)\right)^2 + \left(\zeta_A(x_i)\right)^2.\left(\zeta_A(x_i)\right)^2 + \left(v_A(x_i)\right)^2.\left(v_A(x_i)\right)^2\right)$$

$$C_\omega(B,B) = \sum_{i=1}^{n}\omega_i\left(\left(u_B(x_i)\right)^2.\left(u_B(x_i)\right)^2 + \left(\zeta_B(x_i)\right)^2.\left(\zeta_B(x_i)\right)^2 + \left(v_B(x_i)\right)^2.\left(v_B(x_i)\right)^2\right)$$

And

$$\rho''' = \frac{C_\omega(A,B)}{max\{C_\omega(A,A).C_\omega(B,B)\}} \tag{4}$$

$$= \frac{\sum_{i=1}^{n}\omega_i\left(\left(u_A(x_i)\right)^2.\left(u_B(x_i)\right)^2+\left(\zeta_A(x_i)\right)^2.\left(\zeta_B(x_i)\right)^2+\left(v_A(x_i)\right)^2.\left(v_B(x_i)\right)^2\right)}{max\left\{\begin{array}{l}\sum_{i=1}^{n}\omega_i\left(\left(u_A(x_i)\right)^2.\left(u_A(x_i)\right)^2+\left(\zeta_A(x_i)\right)^2.\left(\zeta_A(x_i)\right)^2+\left(v_A(x_i)\right)^2.\left(v_A(x_i)\right)^2\right),\\ \sum_{i=1}^{n}\omega_i\left(\left(u_B(x_i)\right)^2.\left(u_B(x_i)\right)^2+\left(\zeta_B(x_i)\right)^2.\left(\zeta_B(x_i)\right)^2+\left(v_B(x_i)\right)^2.\left(v_B(x_i)\right)^2\right)\end{array}\right\}}$$

It can be easy to verify that if $\omega = \left(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}\right)^T$, then Equation (3) and (4) reduce that (1) and (2), respectively.

Theorem 3.7

Let $\omega = (\omega_1,\omega_2,\dots,\omega_n)^T$ be the weight vector of $x_i (i = 1,2,\dots.,n)$ with $\omega_i \geq 0$ *and* $\sum_{i=1}^{n}\omega_i = 1$, then the weighted correlation coefficient between the PNSs A and B defined by Equation (3) satisfies:

(i) $0 \leq \rho''(A,B) \leq 1$

(ii) $\rho''(A,B) = 1$ *if and only if* $A = B$

(iii) $\rho''(A,B) = \rho''(B,A)$.

Proof:

The property (i) and (ii) are straight forward so omit here. Also $\rho''(A,B) \geq 0$ is evident so we need to show only $\rho''(A,B) \leq 1$.

Since,

$$C_\omega(A,B) = \sum_{i=1}^{n}\omega_i\left(\left(u_A(x_i)\right)^2.\left(u_B(x_i)\right)^2+\left(\zeta_A(x_i)\right)^2.\left(\zeta_B(x_i)\right)^2+\left(v_A(x_i)\right)^2.\left(v_B(x_i)\right)^2\right)$$

$$\begin{aligned}&= \omega_1\left(\left(u_A(x_1)\right)^2.\left(u_B(x_1)\right)^2+\left(\zeta_A(x_1)\right)^2.\left(\zeta_B(x_1)\right)^2+\left(v_A(x_1)\right)^2.\left(v_B(x_1)\right)^2\right)+\\ &\quad \omega_2\left(\left(u_A(x_2)\right)^2.\left(u_B(x_2)\right)^2+\left(\zeta_A(x_2)\right)^2.\left(\zeta_B(x_2)\right)^2+\left(v_A(x_2)\right)^2.\left(v_B(x_2)\right)^2\right)+\dots+\\ &\quad \omega_n\left(\left(u_A(x_n)\right)^2.\left(u_B(x_n)\right)^2+\left(\zeta_A(x_n)\right)^2.\left(\zeta_B(x_n)\right)^2+\left(v_A(x_n)\right)^2.\left(v_B(x_n)\right)^2\right)\end{aligned}$$

$$\begin{aligned}&= \left(\sqrt{\omega_1}\left(u_A(x_1)\right)^2.\sqrt{\omega_1}\left(u_B(x_1)\right)^2+\sqrt{\omega_1}\left(\zeta_A(x_1)\right)^2.\sqrt{\omega_1}\left(\zeta_B(x_1)\right)^2+\sqrt{\omega_1}\left(v_A(x_1)\right)^2.\sqrt{\omega_1}\left(v_B(x_1)\right)^2\right)\\ &\quad+\left(\sqrt{\omega_2}\left(u_A(x_2)\right)^2.\sqrt{\omega_2}\left(u_B(x_2)\right)^2+\sqrt{\omega_2}\left(\zeta_A(x_2)\right)^2.\sqrt{\omega_2}\left(\zeta_B(x_2)\right)^2\right.\\ &\quad\left.+\sqrt{\omega_2}\left(v_A(x_2)\right)^2.\sqrt{\omega_2}\left(v_B(x_2)\right)^2\right)+\dots+\\ &\quad\left(\sqrt{\omega_n}\left(u_A(x_n)\right)^2.\sqrt{\omega_n}\left(u_B(x_n)\right)^2+\sqrt{\omega_n}\left(\zeta_A(x_n)\right)^2.\sqrt{\omega_n}\left(\zeta_B(x_n)\right)^2+\right.\\ &\quad\left.\sqrt{\omega_n}\left(v_A(x_n)\right)^2.\sqrt{\omega_n}\left(v_B(x_n)\right)^2\right)\end{aligned}$$

By using Cauchy-Schwarz inequality, we get

$$\begin{aligned}(C_\omega(A,B))^2 \le\ & \left(\omega_1\left(u_A(x_1)\right)^2.\left(u_A(x_1)\right)^2+\left(\zeta_A(x_1)\right)^2.\left(\zeta_A(x_1)\right)^2+\left(v_A(x_1)\right)^2.\left(v_A(x_1)\right)^2\right)+\\ & \left(\omega_2\left(u_A(x_2)\right)^2.\left(u_A(x_2)\right)^2+\left(\zeta_A(x_2)\right)^2.\left(\zeta_A(x_2)\right)^2+\left(v_A(x_2)\right)^2.\left(v_A(x_2)\right)^2\right)+\\ & \dots+\left(\omega_n\left(u_A(x_n)\right)^2.\left(u_A(x_n)\right)^2+\left(\zeta_A(x_n)\right)^2.\left(\zeta_A(x_n)\right)^2+\left(v_A(x_n)\right)^2.\left(v_A(x_n)\right)^2\right)\times\\ & \left(\omega_1\left(u_B(x_1)\right)^2\left(u_B(x_1)\right)^2+\left(\zeta_B(x_1)\right)^2\left(\zeta_B(x_1)\right)^2+\left(v_B(x_1)\right)^2\left(v_B(x_1)\right)^2\right)+\\ & \left(\omega_2\left(u_B(x_2)\right)^2\left(u_B(x_2)\right)^2+\left(\zeta_B(x_2)\right)^2\left(\zeta_B(x_2)\right)^2+\left(v_B(x_2)\right)^2\left(v_B(x_2)\right)^2\right)\\ & +\dots+\left(\omega_n\left(u_B(x_n)\right)^2\left(u_B(x_n)\right)^2+\left(\zeta_B(x_n)\right)^2\left(\zeta_B(x_n)\right)^2+\left(v_B(x_n)\right)^2\left(v_B(x_n)\right)^2\right)\end{aligned}$$

$$\begin{aligned}&=\sum_{i=1}^{n}\omega_i\left(\left(u_A(x_i)\right)^2.\left(u_A(x_i)\right)^2+\left(\zeta_A(x_i)\right)^2.\left(\zeta_A(x_i)\right)^2+\left(v_A(x_i)\right)^2.\left(v_A(x_i)\right)^2\right)\times\\ &\quad \textstyle\sum_{i=1}^{n}\omega_i\left(\left(u_B(x_i)\right)^2.\left(u_B(x_i)\right)^2+\left(\zeta_B(x_i)\right)^2.\left(\zeta_B(x_i)\right)^2+\left(v_B(x_i)\right)^2.\left(v_B(x_i)\right)^2\right)\\ &= C_\omega(A,A)\times C_\omega(B,B)\end{aligned}$$

Therefore, $C_\omega(A,B) \le \sqrt{C_\omega(A,A)\times C_\omega(B,B)}$ and hence $0 \le \rho''(A,B) \le 1$.

Theorem 3.8

The correlation coefficient of two PNSs A and B as defined in Equation (4), that is, $\rho'''(A,B)$ satisfies the same properties as those in Theorem 3.7

Proof: The proof of this theorem is similar to that of Theorem 3.6.

5. Application

In this section, we give some application of PNS in medical diagnosis problem using correlation measure.

Medical Diagnosis Problem

As medical diagnosis contains lots of uncertainties and increased volume of information available to physicians from new medical technologies, the process of classifying different set of symptoms under a single name of disease becomes difficult.In some practical problems, there is the possibility of each element having different truth membership , inderminate and false membership functions.The proposed correlation measure among the patients Vs. symptoms and symptoms Vs. diseases gives the proper medical diagnosis. Now, an example of a medical diagnosis will be presented

Example

Let P= $\{P_1, P_2, P_3\}$ be a set of patients, D= $\{Viral\ Fever, Malaria, Typhoid, Dengu\}$ be a set of diseases and S= $\{Temperature$, Headache, Cough, Joint pain$\}$ be a set of symptoms.

Table 1: M (the relation between Patient and Symptoms)

M	$Temperature$	Headache	Cough	Joint pain
P_1	(0.8,0.7,0.6)	(0.5,0.3,0.8)	(0.6,0.9,0.4)	(0.3,0.5,0.2)
P_2	(0.2,0.7,0.9)	(0.5,0.9,0.8)	(0.4,0.6,0.3)	(0.1,0.2,0.9)
P_3	(0.3,0.1,0.5)	(0.8,0.5,0.6)	(0.4,0.8,0.9)	(0.5,0.7,0.2)

Table 2: N (the relation between Symptoms and Diseases)

N	$Viral\ Fever$	$Malaria$	$Typhoid$	$Dengu$
$Temperature$	(0.9,0.5,0.4)	(0.5,0.3,0.6)	(0.8,0.9,0.4)	(0.2,0.8,0.5)
Headache	(0.1,0.5,0.3)	(0.5,0.6,0.7)	(0.4,0.5,0.9)	(0.9,0.8,0.3)
Cough	(0.3,0.7,0.8)	(0.9,0.7,0.4)	(0.1,0.3,0.9)	(0.5,0.3,0.8)
Joint pain	(0.7,0.3,0.5)	(0.8,0.9,0.6)	(0.5,0.7,0.6)	(0.1,0.5,0.8)

Using Equations (1), we get the value of $\rho(A, B)$

Table 3: M and N (Correlation Measure)

M	$Viral\ Fever$	$Malaria$	$Typhoid$	$Dengu$
P_1	**0.7670**	0.5363	0.5965	0.5446
P_2	0.4638	**0.6253**	0.4873	0.5434
P_3	0.4596	0.6606	0.6072	**0.7401**

Using Equations (2), we get the value of $\rho'(A, B)$

Table 4: M and N (Correlation Measure)

M	$Viral\ Fever$	$Malaria$	$Typhoid$	$Dengu$
P_1	**0.6997**	0.5223	0.5786	0.5357
P_2	0.3670	**0.5292**	0.4358	0.5095
P_3	0.4269	0.6562	0.5784	**0.6729**

On the other hand, if we assign weights 0.10, 0.20, 0.30 and 0.40 respectively, then by applying correlation coefficient given in Equations (3) and (4), we can give the following values of the correlation coefficient:

Using Equations (3), we get the value of $\rho''(A, B)$

Table 5: M and N (Correlation Measure)

M	$Viral\ Fever$	$Malaria$	$Typhoid$	$Dengu$
P_1	**0.7233**	0.6496	0.4527	0.4623
P_2	0.4390	**0.5469**	0.4758	0.4194
P_3	0.5123	0.6606	0.7229	**0.7638**

Using Equations (4), we get the value of $\rho'''(A, B)$

Table 6: M and N (Correlation Measure)

M	$Viral\ Fever$	$Malaria$	$Typhoid$	$Dengu$
P_1	**0.6936**	0.5324	0.4280	0.4039
P_2	0.2812	**0.5316**	0.4245	0.4084
P_3	0.4321	0.6154	0.6727	**0.7518**

The highest correlation measure from the Tables 3,4,5,6 gives the proper medical diagnosis. Therefore, patient P_1 suffers from Viral Fever, patient P_2 suffers from Malaria and patient P_3 suffers from Dengu. Hence, we can see from the above four kinds of correlation coefficient indices that the results are same.

Conclusion

In this paper, we found the correlation measure of Pythagorean neutrosophic set with T and F are neutrosophic components (PNS) and proved some of their basic properties. Based on that the present paper have extended the theory of correlation coefficient from and neutrosophic sets (NS) to the Pythagorean neutrosophic set with T and F are neutrosophic components in which the constraint condition of sum of membership, non-membership and indeterminacy be less than two has been relaxed. Illustrate examples have handle the situation where the existing correlation coefficient in NS environment fails. Also to deal with the situations where the elements in a set are correlative, a weighted correlation coefficients has been defined. We studied an application of correlation measure of Pythagorean neutrosophic set with T and F are neutrosophic components in medical diagnosis.

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Operations of Single Valued Neutrosophic Coloring

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Abstract: Smarandache introduced the concept of Neutrosophic which deals with membership, non-membership and indeterminacy values. Wang discussed the Single Valued Neutrosophic sets in 2010. Single Valued Neutrosophic graph was introduced by Broumi and in 2019 Single Valued Neutrosophic coloring was introduced. In this paper, some properties of the Single Valued Neutrosophic Coloring of Strong Single Valued Neutrosophic graph, Complete Single Valued Neutrosophic graph and Complement of Single Valued Neutrosophic graphs are discussed.

Keywords: single-valued neutrosophic graphs; single-valued neutrosophic vertex coloring; strong single-valued neutrosophic graph; complete single-valued neutrosophic graph.

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1. Introduction

Francis Guthrie's four-color conjecture was reasoned for the development of the new branch of graph coloring in graph theory. Graph coloring is assigning labels to the vertices or edges or both vertices and edges. Distinct vertices received different colors are called proper coloring. Graph coloring technique used in many areas like telecommunication, scheduling, computer networks etc.

Most of the problems are not only deals the accurate values, sometimes handle vague values. Fuzzy sets were introduced by Zadeh [29] in 1965, dealt imprecise values in his work. Fuzzy graph theory concept was developed by Rosenfeld [25] in 1975. Munoz et al. [27] in 2004 and Eslahchi, Onagh [19] in 2006 discussed the fuzzy chromatic number and its properties.

Kassimir T. Atanassov [11] introduced the concept of intuitionistic fuzzy sets in 1986 and intuitionistic fuzzy graph in 1999. The intuitionistic graphs are handled membership and non-membership values. Vague set concept introduced by Gau and Buehrer [21] in 1993. In 2014, Akram et al. [9] discussed vague graphs and further work extended by Borzooei et al. [12, 13]. Vertex and Edge coloring of Vague graphs were introduced by Arindam Dey et al. [10] in 2018.

Neutrosophic set was introduced by F. Smarandache [25] in 1998, it's a generalization of the intuitionistic fuzzy set. It consists of membership value, indeterminacy value and non-membership value. Neutrosophic logic play a vital role in several of the real valued problems like law, medicine, industry, finance, engineering, IT, etc. Wang et al. [28] worked on Single valued neutrosophic sets in 2010. Strong Neutrosophic graph and its properties were introduced and discussed by Dhavaseelan et al. [20] in 2015 and Single valued neutrosophic concept introduced in 2016 by Akram and Shahzadi [6, 7, 8]. Broumi et al. [14, 15, 16, 17, 18] extended their works in single valued neutrosophic graphs, interval valued neutrosophic graphs (IVNG) and bipolar neutrosophic graphs. Abdel-Basset et al. used Neutrosophic concept in their papers [1, 2, 3, 4, 5] to find the decisions for some real-life operation research and IoT-based enterprises in 2019. In 2019, Jan et al. [23] have reviewed the following definitions: Interval-Valued Fuzzy Graphs (IVFG), Interval-Valued Intuitionistic Fuzzy Graphs (IVIFG), Complement of IVFG, SVNG, IVNG and the complement of SVNG and IVNG. They have modified those definitions, supported with some examples. Neutrosophic graphs happen to play a vital role in the building of neutrosophic models. Also, these graphs can be used in networking, Computer technology, Communication, Genetics, Economics, Sociology, Linguistics, etc., when the concept of indeterminacy is present.

In this research paper, the bounds of single valued neutrosophic vertex coloring for SVNG, Complement of SVNG are determined and discussed some more operations on SVNG.

Definition 1.1. [26] Let X be a space of points(objects). A neutrosophic set A in X is characterized by truth-membership function $t_A(x)$, an indeterminacy-membership function $i_A(x)$ and a falsity-membership function $f_A(x)$. The functions $t_A(x)$, $i_A(x)$, and $f_A(x)$, are real standard or non-standard subsets of $]0^-, 1^+[$. That is, $t_A(x): X \to]0^-, 1^+[$, $i_A(x): X \to]0^-, 1^+[$ and $f_A(x): X \to]0^-, 1^+[$ and $0^- \leq t_A(x) + i_A(x) + f_A(x) \leq 3^+$.

Definition 1.2. [7] A single-valued neutrosophic graphs (SVNG) G = (X, Y) is a pair where X: N → [0,1] is a single-valued neutrosophic set on N and Y: N × N → [0,1] is a single-valued neutrosophic relation on N such that

$$t_Y(xy) \leq \min\{t_X(x), t_X(y)\},$$

$$i_Y(xy) \leq \min\{i_X(x), i_X(y)\},$$

$$f_Y(xy) \leq \max\{f_X(x), f_X(y)\},$$

for all x, y ∈ N. X and Y are called the single-valued neutrosophic vertex set of G and the single-valued neutrosophic edge set of G, respectively. A single-valued neutrosophic relation Y is said to be symmetric if $t_Y(xy) = t_Y(yx)$, $i_Y(xy) = i_Y(yx)$ and $f_Y(xy) = f_Y(yx)$, for all x,y ∈ N. Single-valued neutrosophic be abbreviated here as SVN.

2. Single-Valued Neutrosophic Vertex Coloring (SVNVC)

In this section, discussed the bounds of SVNVC for the resultant SVNG by some operations on SVNG, CSVNG and complement of SVNG. Also discussed some theorems.

Definition 2.1. [24] A family $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ of SVN fuzzy set is called a k-SVNVC of a SVNG G = (X, Y) if

1. $\vee \gamma_i(x) = X,\ \forall x \in X$
2. $\gamma_i \wedge \gamma_j = 0$

3. For every incident vertices of edge xy of G, $\min\{\gamma_i(m_1(x)), \gamma_i(m_1(y))\} = 0$, $\min\{\gamma_i(i_1(x)), \gamma_i(i_1(y))\} = 0$ *and* $\max\{\gamma_i(n_1(x)), \gamma_i(n_1(y))\} = 1, (1 \le i \le k)$.
This k-SVNVC of G is denoted by $\chi_v(G)$, is called the SVN chromatic number of the SVNG G.

Definition 2.2 A SVNG G = (X, Y) is called complete single-valued neutrosophic graph (CSVNG) if the following conditions are satisfied:

$$t_Y(xy) = \min\{t_X(x), t_X(y)\},$$
$$i_Y(xy) = \min\{i_X(x), i_X(y)\},$$
$$f_Y(xy) = \max\{f_X(x), f_X(y)\},$$

for all x, y ∈ X.
For any single value neutrosophic subgraph H of SVNG G, $\chi_v(H) \le \chi_v(G)$
Theorem 2.3.
For any SVNG with n vertices $\chi_v(G) \le n$.
Proof:
By the observation that the CSVNG with n vertices has the SVNVC is n. All the other graphs with n vertices are subgraphs of the CSVNG, it is clear by the above observation. Hence $\chi_v(G) \le n$.

Definition 2.4 Let $G_1 = (X_1, Y_1)$ and $G_2 = (X_2, Y_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The union G1 ∪ G2 is defined as a pair (X, Y) such that

$$t_X(x) = \begin{cases} t_{X_1}(x), & if\ x \in V_1 and\ x \notin V_2, \\ t_{X_2}(x), & if\ x \in V_2 and\ x \notin V_1, \\ \max\left(t_{X_1}(x), t_{X_2}(x)\right), & if\ x \in V_1 \cap V_2. \end{cases}$$
$$i_X(x) = \begin{cases} i_{X_1}(x), & if\ x \in V_1 and\ x \notin V_2, \\ i_{X_2}(x), & if\ x \in V_2 and\ x \notin V_1, \\ \max\left(i_{X_1}(x), i_{X_2}(x)\right), & if\ x \in V_1 \cap V_2. \end{cases}$$

$$f_X(x) = \begin{cases} f_{X_1}(x), & if\ x \in V_1 and\ x \notin V_2, \\ f_{X_2}(x), & if\ x \in V_2 and\ x \notin V_1, \\ \min\left(f_{X_1}(x), f_{X_2}(x)\right), & if\ x \in V_1 \cap V_2. \end{cases}$$
$$t_Y(xy) = \begin{cases} t_{Y_1}(xy), & if\ xy \in E_1 and\ x \notin E_2, \\ t_{Y_2}(xy), & if\ xy \in E_2 and\ x \notin E_1, \\ \max\left(t_{Y_1}(x), t_{Y_2}(x)\right), & if\ x \in E_1 \cap E_2. \end{cases}$$
$$i_Y(xy) = \begin{cases} i_{Y_1}(xy), & if\ xy \in E_1 and\ x \notin E_2, \\ i_{Y_2}(xy), & if\ xy \in E_2 and\ x \notin E_1, \\ \max\left(i_{Y_1}(x), i_{Y_2}(x)\right), & if\ x \in E_1 \cap E_2. \end{cases}$$
$$f_Y(xy) = \begin{cases} f_{Y_1}(xy), & if\ xy \in E_1 and\ x \notin E_2, \\ f_{Y_2}(xy), & if\ xy \in E_2 and\ x \notin E_1, \\ \min\left(f_{Y_1}(x), f_{Y_2}(x)\right), & if\ x \in E_1 \cap E_2. \end{cases}$$

For any SVNGs $G_1 = (X_1, Y_1)$ and $G_2 = (X_2, Y_2)$, $\chi_v(G_1 \cup G_2) = max\{\chi_v(G_1), \chi_v(G_2)\}$.
Definition 2.5 [8] The complement of a SVNG G = (X, Y) is a SVNG $\bar{G} = (\bar{X}, \bar{Y})$, where

1. $\bar{X} = X$
2. $\bar{t_X}(x) = t_X(x), \bar{i_X}(x) = i_X(x), \bar{f_X}(x) = f_X(x)\ for\ all\ x \in X$

3. $\bar{t_X}(xy) = \begin{cases} \min\{t_X(x), t_X(y)\} & if \;\; t_Y(xy) = 0 \\ \min\{t_X(x), t_X(y)\} - t_Y(xy) & if \;\; t_Y(xy) > 0 \end{cases}$

$\bar{i_X}(xy) = \begin{cases} \min\{i_X(x), i_X(y)\} & if \;\; i_Y(xy) = 0 \\ \min\{i_X(x), i_X(y)\} - i_Y(xy) & if \;\; i_Y(xy) > 0 \end{cases}$

$\bar{f_X}(xy) = \begin{cases} \max\{f_X(x), f_X(y)\} & if \;\; f_Y(xy) = 0 \\ \max\{f_X(x), f_X(y)\} - f_Y(xy) & if \;\; f_Y(xy) > 0 \end{cases}$

for all $x, y \in X$.

Theorem 2.6. For any SVNG G with n vertices, $2\sqrt{n} \leq \chi_v(G) + \chi_v(\bar{G}) \leq 2n$ and $n \leq \chi_v(G)\chi_v(\bar{G}) \leq n^2$.

Let every vertex of G has n − 1 adjacent vertices, then by the definition of complement of SVNG each vertex of $\bar{G}$ has the lesser than or equal to n − 1 adjacent vertices. Hence, the inequalities true for all SVNG. Thus, $2\sqrt{n} \leq \chi_v(G) + \chi_v(\bar{G}) \leq 2n$ and $n \leq \chi_v(G)\chi_v(\bar{G}) \leq n^2$.

Definition 2.7.

A SVNG G = (X, Y) is called strong single-valued neutrosophic graph (SSVNG) if the following conditions are satisfied:

$$t_Y(xy) = \min\{t_X(x), t_X(y)\},$$
$$i_Y(xy) = \min\{i_X(x), i_X(y)\},$$
$$f_Y(xy) = \max\{f_X(x), f_X(y)\},$$

for all (x,y) ∈ Y .

Observation 2.8

For any SSVNG G with n vertices, $2\sqrt{n} \leq \chi_v(G) + \chi_v(\bar{G}) \leq n + 1$ and $n \leq \chi_v(G)\chi_v(\bar{G}) \leq (\frac{n+1}{2})^2$.

Given that G is SSVNG and the complement of G is defined by $\bar{G} = (\bar{X}, \bar{Y})$, where

1. $\bar{X} = X$
2. $\bar{t_X}(x) = t_X(x), \bar{i_X}(x) = i_X(x), \bar{f_X}(x) = f_X(x) \; for \; all \; x \in X$
3. $\bar{t_X}(xy) = \begin{cases} \min\{t_X(x), t_X(y)\} & if \;\; t_Y(xy) = 0 \\ 0 & if \;\; t_Y(xy) > 0 \end{cases}$

$\bar{i_X}(xy) = \begin{cases} \min\{i_X(x), i_X(y)\} & if \;\; i_Y(xy) = 0 \\ 0 & if \;\; i_Y(xy) > 0 \end{cases}$

$\bar{f_X}(xy) = \begin{cases} \max\{f_X(x), f_X(y)\} & if \;\; f_Y(xy) = 0 \\ 0 & if \;\; f_Y(xy) > 0 \end{cases}$

for all $x, y \in X$. Hence, the above inequalities hold.

Theorem 2.9. For a path graph P_n, $\chi_v(P_n) = 2$ where $n \geq 2$.

Let $\Gamma = \{\gamma_1, \gamma_2\}$ be a family of SVN fuzzy sets defined on V as follows:

$$\gamma_1(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for \; i = odd \\ (0,0,1) & for \; i = even \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for \; i = even \\ (0,0,1) & for \; i = odd \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2\}$ fulfilled the conditions of SVNVC of the graph G. Hence the SVN chromatic number of P_n is $\chi_v(P_n) = 2$.

Theorem 2.10. For a cycle graph C_n, $\chi_v(C_n) = \begin{cases} 2 \ if \ n = even \\ 3 \ if \ n = odd \end{cases}$ where $n \geq 3$.

For n is odd:

Let $\Gamma = \{\gamma_1, \gamma_2, , \gamma_3\}$ be a family of SVN fuzzy sets defined on V as follows:

$$\gamma_1(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for\ i = 1,3,5, \dots, n-2 \\ (0,0,1) & for\ others \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for\ i = 2,4,6, \dots, n-1 \\ (0,0,1) & for\ others \end{cases}$$

$$\gamma_3(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for\ i = n \\ (0,0,1) & for\ others \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, , \gamma_3\}$ fulfilled the conditions of SVNVC of the graph G. Hence the SVN chromatic number $\chi_v(C_n) = 3$.

For n is even:

Let $\Gamma = \{\gamma_1, \gamma_2\}$ be a family of SVN fuzzy sets defined on V as follows:

$$\gamma_1(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for\ i = odd \\ (0,0,1) & for\ i = even \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for\ i = even \\ (0,0,1) & for\ i = odd \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2\}$ fulfilled the conditions of SVNVC of the graph G. Hence the SVN chromatic number $\chi_v(C_n) = 2$.

Theorem 2.11. For any graph SVNG, $\chi_v(G) \leq \Delta(G) + 1$.

Here $\Delta(G)$ denotes the number of edges incident with a vertex of SVNG G, hence the result is true for all SVNG.

3. Conclusions

Graph Coloring is an useful technique to solve many real life problems which are easily converted as graph models. SVNG is dealt with vague and imprecise values. Single Valued Neutrosophic Coloring concept was introduced by the authors in [24]. In this paper, we discussed few more results of SVNVC using CSVNG and Complement of SVNG. We have an idea to extend the concept of SVNVC with irregular coloring and dominating coloring technique in future.

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Conflicts of Interest

The authors declare no conflict of interest.

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Multi-Aspect Decision-Making Process in Equity Investment Using Neutrosophic Soft Matrices

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Abstract: Neutrosophic theory alleviates the ambiguity situation more effectively than fuzzy sets. Neutrosophic soft set deals with the combination of truth, indeterminacy and falsity membership. This provides a space for the convention with multi-aspect decision-making (MADM) problems that involve these combinations. The main aim of this paper is to provide a unique ranking for the alternatives to overcome the existing drawbacks in the said environment. Initially, a new score function and the weighted neutrosophic vector are discussed. Secondly, to show the supremacy of the proposed score function a comparison analysis is discussed between the existing score method and the proposed approach. Thirdly, algorithm and flowchart are discussed for the case study. Lastly, a new technique for ranking the alternatives is discussed which enables us to determine the unique highest score. The working model is illustrated with suitable examples to authenticate the tool and to demonstrate the effectiveness of the planned approach.

Keywords: Single valued neutrosophic sets, Neutrosophic soft matrix (NSM), weighted neutrosophic vector, Score and value function, Multi-aspect decision-analysis.

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1. Introduction

Our world is complex and rapid changes keep occurring in the field of engineering, medical science, banking, modern education, social, economic, and various other fields. Complexity generally arises from ambiguity and to overcome these situations in day to day life, Zadeh (1965) introduced a fuzzy set (FS) [14] and an interval-valued fuzzy set (IVFS) [15]. Atanassov (1986) proposed the concept of intuitionistic fuzzy set (IFS) [1] and interval-valued intuitionistic fuzzy set [2] a combination of membership and non-membership functions. However, both fuzzy and intuitionistic fuzzy sets cannot treat the indeterminacy part in the day to day problems. To deal with indeterminacy situations, Smarandache (1998) grounded the neutrosophic set (NS) [10] theory which is an overview of FS and IFS. In plithogenic set (PS) elements are characterized by the attribute values. It was introduced by Smarandache [27] as a generalization of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets.

FS, IVFS, IFS, NS, PS and hybrid of these sets are used in various decision-making problems. Decision making plays a significant role in today’s social, scientific and economic endeavor. Most of the decision-making process is based on an objective to reduce the cost, reduce the production time, and increase the profit for the organization. However, considering today's environment the decision should include various objective sources to deal with uncertainty. It weighs the provided information and chooses the best criteria for subsequent action. The information provided in a complex world is likely ambiguous, hence the outcomes are vague, irrespective of the decision made on the criteria chosen. To explain this scenario, consider the criteria of taking a loan from a bank. The outcome can be ambiguous with the possibility of a loan getting approved or declined or undetermined. The primary issues in MADM are to rank the relative importance of each of the objectives. Despite our vast knowledge and experience in handling these objectives, we come across violations in our everyday life. A bank manager makes a decision in this complex environment and figures out that his/her decision becomes weird. We have come across many situations where the loan applicant fails to repay the loan amount despite following the scrutiny process. The said problem could be due to the change in information and condition according to the situation. The outcomes of these situations have nothing to do with the quality of the decisions made. The best we can do with our knowledge is that in the long run the `good decisions' will outplay the `bad decisions'.

Most of the researchers utilize NS as a significant tool to analyze MADM problems with the help of aggregation operators, information measures, score functions and machine learning algorithms. Abhishek et al. [28] developed a parametric divergence measure and initiated the concept of pattern recognition and medical diagnosis problem for neutrosophic sets. Abdel-Basset et al. [18] proposed a hybrid combination between analytical hierarchical process and neutrosophic theory to solve the uncertainty involved in the technology of the internet of things. Abhisek and Rakesh [29] proposed a notion for finding the threshold value in decision-making problems when the qualitative and quantitative information is outsized. Abdel-Basset et al. [20] proposed the concept of type 2 neutrosophic number TOPSIS method to deal with real case decision problems. Edalatpanah and Smarandache [30] found a new method to solve the data envelopment analysis using the weighted arithmetic average operator in neutrosophic sets. Abdel-Basset et al. [19] initiated a neutrosophic approach for evaluating green supply chain management to aid managers and decision-makers. Vakkas et al. [33] proposed a novel ranking method for decision-making problems in the bipolar neutrosophic environment. Pandy and Trinita [31] constructed a new approach to represent gray-scale (medical) images in the bipolar neutrosophic domain. Shazia et al. [32] presented the concept of the plithogenic hypersoft matrix and discussed some of its theoretical properties. Abdel-Basset et al. [17] developed the combination of quality function deployment with plithogenic operations and analyzed the case study of Thailand's sugar industry and also developed a novel evaluation approach to handle the hospital medical care systems based on plithogenic sets [16]. Azeddine et al. [34] introduced an improved method to map machine learning algorithms from crisp number to Neutrosophic environment. Wang and Smarandache (2010) focused on single-valued neutrosophic set [13] to magnetize on MADM problems. Chinnadurai et al., (2016) [3] discussed some of its theoretical properties. Smarandache and Teodorescu (2014) introduced the fusion of fuzzy data to neutrosophic data [11] with case studies. Garg and Nancy (2018) developed the neutrosophic Muirhead mean operators [5] for an aggregating single-valued neutrosophic set to solve MADM problems among the ambiguity. Gulistan et al., (2019) studied on neutrosophic cubic soft matrices [6] using max-min operations. Jun et al. presented elucidation to handle actual data which consists of crisp values using the neutrosophic analytic hierarchy process. Abdel-Basset et.al.

[12] developed the concept of Neutrosophic AHP-SWOT Analysis for MADM problems by analyzing a real case study.

The advantage of this proposed method is that it shortens the computation process and provides a better solution in decision-making. To establish the superiority of our improved score function a comparison study is illustrated with suitable examples. From the presented references [21, 22, 23, 24, 25, 26] it is clear that there are limitations in providing unique ranking using score function in neutrosophic MADM methods. The fact that we would like to enlighten in this manuscript is that there could always be a possibility of equal ranking among the alternatives. Hence, to our knowledge, a simple but effective way to determine the unique highest score for each object in a MADM is by including additional criteria from the parameter set which is not been discussed in any of the related literature works.

In this paper, we aim to discuss the weighted neutrosophic vector and value function of a neutrosophic soft matrix to combine the different components of truth, indeterminacy and falsity membership into a single membership value. An application of this matrix in MADM is also given by presenting the method, algorithm and numerical illustrations.

The structure of the manuscript is as follows. In section 2, some of the basic neutrosophic definitions are specified. In section 3, the notions of weighted neutrosophic vector and value functions are introduced. In section 4, an algorithm with a flowchart of NSM to MADM is developed. In section 5, case studies are presented to illustrate the working of the algorithm. This manuscript is concluded in section 6.

2. Preliminaries

In this section first we review some basic concepts and definitions.

Definition 2.1[9] Let U be the universal set and E be a set of parameters. The parameters represent some selected properties or characteristics of the elements of U. Let P(U) denote the power set of U. A pair (F, E) is called a **soft set** over U where F is a mapping $F: E \to P(U)$. It is clear that a soft set is a parameterized family of subsets of U.

Definition 2.2 [13] Let U be the universal set, then a set $\mathbb{A} = \{\langle x, T^{\mathbb{A}}(x), I^{\mathbb{A}}(x), F^{\mathbb{A}}(x)\rangle : x \in U\}$ is termed as **neutrosophic set** where $T^{\mathbb{A}}, I^{\mathbb{A}}, F^{\mathbb{A}}: X \to [0,1]$ with $0 \le T^{\mathbb{A}}(x) + I^{\mathbb{A}}(x) + F^{\mathbb{A}}(x) \le 3$ and the functions $T^{\mathbb{A}}, I^{\mathbb{A}}, F^{\mathbb{A}}$ are truth, indeterminacy and falsity membership degrees respectively.

Definition 2.3 [8] Let U be the universal set and E be a set of parameters. Consider $\mathbb{A} \subseteq E$. Let *NS(U)* denote the set of all neutrosophic sets of U. The collection $(F, \mathbb{A})$ is termed to be the **neutrosophic soft set** (NSS) over U, where F is a mapping given by $F: \mathbb{A} \to NS(U)$.

Definition 2.4 [4] Let $(N^{\mathbb{A}}, E)$ be a NSS over the universe U and E be a set of parameters and $\mathbb{A} \subseteq E$. Then a subset of $U \times E$ is uniquely defined by the relation $\{(x, e): e \in \mathbb{A}, x \in N^{\mathbb{A}}(e)\}$ and denoted by $R_{\mathbb{A}} = (N^{\mathbb{A}}, E)$. The relation $R_{\mathbb{A}}$ is characterized by truth function $T^{\mathbb{A}}: U \times E \to [0,1]$, indeterminacy $I^{\mathbb{A}}: U \times E \to [0,1]$ and the falsity function $F^{\mathbb{A}}: U \times E \to [0,1]$. $R_{\mathbb{A}}$ is represented as $R_{\mathbb{A}} = \{(T^{\mathbb{A}}(x,e), I^{\mathbb{A}}(x,e), F^{\mathbb{A}}(x,e)): 0 \le T^{\mathbb{A}} + I^{\mathbb{A}} + F^{\mathbb{A}} \le 3, (x,e) \in U \times E\}$. Now if the set of universe $U = \{x_1, x_2, \dots, x_m\}$ and the set of parameters $E = \{e_1, e_2, \dots, e_n\}$, then $R_{\mathbb{A}}$ can be represented by a matrix as follows:

$$R_{\mathbb{A}} = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where $a_{ij} = (T^{\mathbb{A}}(x,e), I^{\mathbb{A}}(x,e), F^{\mathbb{A}}(x,e)) = (T^{\mathbb{A}}_{ij}, I^{\mathbb{A}}_{ij}, F^{\mathbb{A}}_{ij})$.

The above matrix is called a **neutrosophic soft matrix** (NSM) of order $m \times$ n corresponding to the neutrosophic set $(N^{\mathbb{A}}, E)$ over U.

3. NSM theory in decision making

In this section, we define the concepts of weighted neutrosophic vector, score function and total score for a neutrosophic soft matrix. Later these notions will be used in MADM process.

Definition: 3.1 Let $\mathcal{M}$ be the collection of all neutrosophic values and $N = (n_1, n_2, \dots, n_n)$ be neutrosophic vector with components from $\mathcal{M}$. Thus the components of N are $N = ((n_1^T, n_1^I, n_1^F), (n_2^T, n_2^I, n_3^F), \dots, (n_n^T, n_n^I, n_n^F))$. Let $W = (w_1, w_2, \dots, w_n)$ be a weight vector associated with N. w_i can be considered as the significance attached to n_i; $i = 1,2,\dots,n$ with $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. Then the **weighted neutrosophic vector** corresponding to N and W denoted by WN is defined as $WN = (w_1 n_1, w_2 n_2, \dots, w_n n_n) = ((w_1 n_1^T, w_1 n_1^I, w_1 n_1^F), (w_2 n_2^T, w_2 n_2^I, w_2 n_2^F), \dots, (w_n n_n^T, w_n n_n^I, w_n n_n^F))$

Example:3.1 Let $N = ((0.4,0.3,0.6), (0.2,0.6,0.7), (0.7,0.1,0.5), (0.4,0.2,0.3))$ and $W = (0.1,0.4,0.2,0.3)$. Then $WN = ((0.04,0.03,0.06), (0.08,0.24,0.28), (0.14,0.02,0.10), (0.12,0.06,0.09))$

Definition: 3.2 Score function of a neutrosophic matrix helps to integrate the neutrosophic value into a single real number in order to bring out the importance of truth, indeterminacy and falsity membership values.

Let $A = [a_{ij}] = (T_{ij}^A, I_{ij}^A, F_{ij}^A)$. Then the score function for the element a_{ij} is defined as

$$s(a_{ij}) = s_{ij} = \frac{(T_{ij}^A + I_{ij}^A)}{2} + F_{ij}^A \ \forall\ i,j$$

Thus the **score function** for the NSM, $A = [a_{ij}]$ is given by

$$S_F(A) = \left[\frac{(T_{ij}^A + I_{ij}^A)}{2} + F_{ij}^A\right] = [s_{ij}].$$

$S_F(A)$ is also an $m \times n$ matrix, having the same dimension as A and has non-negative entries.

Definition 3.3 Let $N = [s_{ij}]$ be the matrix of score functions of a NSM N. The quantity $T_i = \sum_{j=1}^{n} s_{ij}$; $i = 1,2,\dots,m$ gives the **total of the score function** values for the i^{th} row of NSM. T_i represent the total value for the element x_i with representation to all the characteristics under consideration.

3.1 Comparison analysis with existing and proposed score functions

In this subsection, we compare and analyze the method developed in this paper with six of the recently developed score functions and methods. The below cited **Table 1** highlights the ranking difficulty of an existing score function in the neutrosophic environment. It also shows that the new score function can compute the rank of the alternatives even when the existing score function is unable to rank the alternatives.

Table 1. Comparison analysis of score values.

Neutrosophic environment	Existing & Proposed methods	Score value	Remarks
N_1=(0.6,0.2,0.6) & N_2=(0.6,0.4,0.2)	Sahin [25]	$S(N_1)$ = 0.3 & $S(N_2)$ = 0.3	$S(N_1) = S(N_2)$ unable to rank
	Proposed method	**$S(N_1)$ = 1** & $S(N_2)$ = 0.7	**$S(N_1) > S(N_2)$** able to rank
N_1=(0.7,0.3,0.1) & N_2=(0.9,0.4,0.2)	Peng et.al., [24]	$S(N_1)$ = 0.1 & $S(N_2)$ = 0.1	$S(N_1) = S(N_2)$ unable to rank
	Proposed method	$S(N_1)$ = 0.60 & **$S(N_2)$ = 0.85**	**$S(N_2) > S(N_1)$** able to rank
N_1=(0.9,0.6,0.3) & N_2=(0.6,0.4,0.2)	Garg and Nancy [23]	$S(N_1)$ = 0.26 & $S(N_2)$ = 0.26	$S(N_1) = S(N_2)$ unable to rank
	Proposed method	**$S(N_1)$ = 1.05** & $S(N_2)$ = 0.7	**$S(N_1) > S(N_2)$** able to rank
N_1=(0.4,0.2,0.6) & N_2=(0.7,0.6,0.7)	Arockiarani [21]	$S(N_1)$ = 0.28 & $S(N_2)$ = 0.28	$S(N_1) = S(N_2)$ unable to rank
	Proposed method	$S(N_1)$ = 0.9 & **$S(N_2)$ = 1.35**	**$S(N_2) > S(N_1)$** able to rank
N_1=(0.5,0.7,0.4) & N_2=(0.4,0.6,0.3)	Ye [26]	$S(N_1)$ = 0.55 & $S(N_2)$ = 0.55	$S(N_1) = S(N_2)$ unable to rank
	Proposed method	**$S(N_1)$ = 1** & $S(N_2)$ = 0.8	**$S(N_1) > S(N_2)$** able to rank
N_1=(0.8,0.3,0.2) & N_2=(0.6,0.3,0.7) N_3=(0.9,0.4,0.5) & N_4=(0.8,0.5,04)	Mondal [22]	$S(N_p)$ = 0.65, where p = 1,2 & $S(N_q)$ = 0.65 where q = 3,4	$S(N_p) = S(N_q)$ unable to rank
	Proposed method	$S(N_p)$ = 0.95, where p = 1,2 & **$S(N_q)$ = 1.1** where q = 3,4	**$S(N_q) > S(N_p)$** able to rank

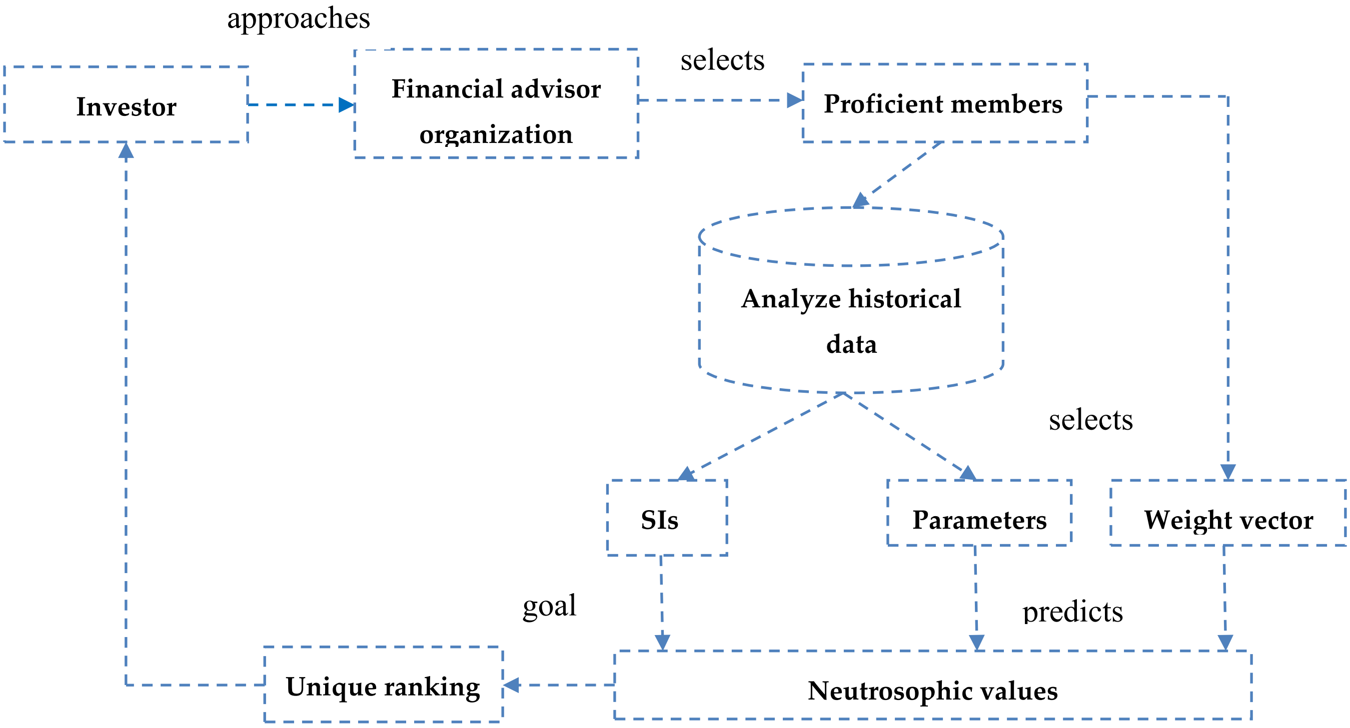
4. Application of NSM to MADM environment

In this section an application of NSM in MADM is explained. An algorithm is developed and the working of the same is illustrated with suitable examples.

4.1. Statement of the problem

Suppose a person is in the progression of stock investment (SI) in the equity market. Let's assume that person seeks the help of a financial advisor organization (FAO). FAO has a panel of highly-trained professionals to provide value-added services to the investors to ensure higher proficiency, consistency of charges and superior forecast of SI in equity market by analyzing the historical data. The FAO, in turn, selects a group of proficient members $P = \{p_1, p_2, \ldots, p_k\}$ to proceed with the same. Now according to the group let $C = \{c_1, c_2, \ldots, c_p\}$ be the list of selected SIs based on historical data analysis . Let $E = \{e_1, e_2, \ldots, e_q\}$ be the set of selected parameters based on which the SIs selection is to be finalized. Assume that weights are assigned for each criterion. Let $W = (w_1, w_2, \ldots, w_q)$ and $\sum_{i=1}^{q} w_i = 1$. Let's assume that the group assesses the SI based on a subset of the parameter set. Let $A = \{e_1, e_2, \ldots, e_l\}$ be the subset of the parameter set E, so that $l \le q$. Each of the personnel verifies the listed SI historical records based on the parameter set A and presents his forecast result in the form of neutrosophic soft matrices. The respective NSM's are denoted by $N^1, N^2, \ldots, N^K$. The crisis is to convert the NSM's into significant matrices which enables them to select the best SI for the investor. **Figure 1** illustrates the conceptual structure of the problem.

Figure 1. Conceptual structure of the statement



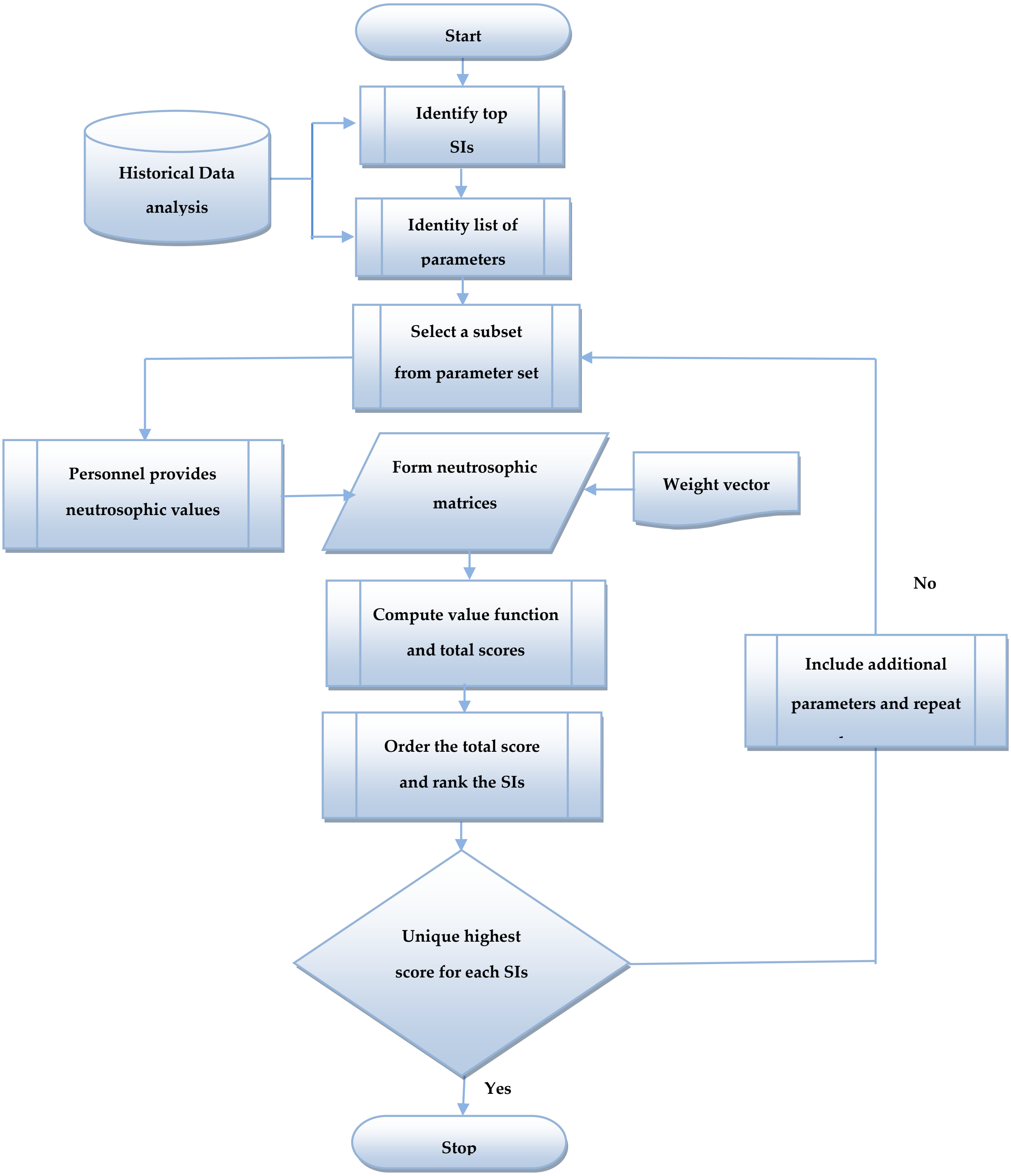
4.2. Methodology

Let's assume that the proficient members evaluate the SIs independently without any bias. Let $N^1, N^2, \ldots, N^K$ be the NSMs obtained from the members. Using **Definition 3.1**, and weight vector W the weighted neutrosophic matrices are calculated. The resultant of weighted neutrosophic matrices are denoted by $N_w^1, N_w^2, \ldots, N_w^k$ i.e., $N_w^r = WN^r = [n_{ij}^r]$ where $r = 1,2, \ldots, k$. Using **Definition 3.2**, convert each of the weighted neutrosophic matrix N_w^r value into corresponding score function as $S_F[N_w^r] = [s_{ij}^r] = \left[\frac{(T_{ij}^{rA} + I_{ij}^{rA})}{2} + F_{ij}^{rA}\right]$. Then using the **Definition 3.3** the score function for the i^{th} SI as evaluated by the r^{th} expert is calculated by adding the values of the i^{th} row of the score function matrix, ie., the i^{th} row of the weighted neutrosophic matrix N_w^r. Let us denote this sum by the symbol T_i^r. The total score ST_i for the i^{th} SI is obtained by summing T_i^r over r. That is the total score for the i^{th} SI $ST_i = \sum_{r=1}^{k} T_i^r = T_i^1 + T_i^2 + \cdots + T_i^k$. The total score is evaluated for all the SIs, $i = 1,2, \ldots, p$. Arrange the ST_i values in decreasing order. The SI with highest ST_i value is the most suitable one for the investor. If more than one SI are there with equal highest ST_i value, the entire process is repeated by adding one more parameter into the set *A*. This process is repeated until a unique SI with highest ST_i value is identified.

4.3. Algorithm

The algorithm for ranking the alternatives of MADM problem based on NSM is given below:
Step 1: Identify the list of SIs and the list of parameters.
Step 2: Select a subset of the parameter set.
Step 3: Present the result in the form of NSMs $(N^1, N^2, \ldots, N^K)$.
Step 4: Compute the weight order for the NSMs $\left(N_W^1, N_W^2, \ldots, N_W^k\right)$.
Step 5: Calculate the score function matrix $S_F[N_w^r] = \left[s_{ij}^r\right]$
Step 6: Calculate the total value T_i^r from each of the $S_F[N_w^r]$ matrices.
Step 7: Evaluate the ST_i for each SI.
Step 8: Order the ST_i values and select the SI with highest ST_i value as the most suitable one.
Step 9: If there are more than one SI with equal highest ST_i value, repeat the process by including another parameter into the set *A*. Continue the process until a unique SI with highest ST_i is identified.

4.4. Flowchart



5. Case studies

In this section we present two case studies to illustrate the working of the algorithm. In 5.1 we present an example where the ranking of the SIs are unique and processed based on a subset of the criteria set. In 5.2 an example is given where the initially selected set of parameters does not provide unique ranking and there are more than one SIs with equal highest total score. Addition of another parameter yields a clear ranking and the selection is performed by repeating some of the steps with enlarged parameter set.

5.1. Case study I

A person is in the process of selecting a suitable SI.

1. Let $C = (c_1, c_2, \ldots, c_7)$ be the set of listed SIs.
2. Let $E = (e_1, e_2, e_3, e_4)$ be the set of parameters which form the criteria for selection.

Here, e_1 = financial profitability projection, e_2 = asset-utilization, e_3 = conservative capital structure and e_4 = earnings momentum.

3. Let the personnel present his forecast result in the form of NSM- N^1, N^2 and N^3 for the subset of the criteria set (e_1, e_2, e_3) as

$$N^1 = \begin{bmatrix} (0.245,0.456,0.721) & (0.457,0.421,0.431) & (0.415,0.821,0.211) \\ (0.348,0.156,0.627) & (0.345,0.653,0.543) & (0.618,0.712,0.514) \\ (0.546,0.765,0.429) & (0.765,0.753,0.632) & (0.415,0.521,0.416) \\ (0.267,0.321,0.321) & (0.552,0.893,0.723) & (0.314,0.612,0.518) \\ (0.428,0.416,0.891) & (0.452,0.213,0.413) & (0.231,0.923,0.916) \\ (0.456,0.932,0.217) & (0.569,0.236,0.247) & (0.416,0.378,0.612) \\ (0.324,0.634,0.816) & (0.367,0.456,0.912) & (0.482,0.231,0.712) \end{bmatrix}$$

$$N^2 = \begin{bmatrix} (0.245,0.348,0.546) & (0.456,0.156,0.765) & (0.721,0.627,0.429) \\ (0.457,0.345,0.765) & (0.421,0.653,0.753) & (0.431,0.543,0.632) \\ (0.415,0.618,0.415) & (0.821,0.712,0.521) & (0.211,0.514,0.416) \\ (0.238,0.416,0.467) & (0.734,0.817,0.926) & (0.518,0.456,0.267) \\ (0.314,0.231,0.916) & (0.753,0.893,0.213) & (0.213,0.765,0.457) \\ (0.753,0.893,0.213) & (0.618,0.415,0.314) & (0.451,0.233,0.532) \\ (0.412,0.824,0.218) & (0.614,0.425,0.324) & (0.546,0.267,0.428) \end{bmatrix} \text{ and}$$

$$N^3 = \begin{bmatrix} (0.238,0.734,0.518) & (0.765,0.345,0.734) & (0.345,0.457,0.347) \\ (0.416,0.817,0.456) & (0.429,0.653,0.817) & (0.456,0.892,0.821) \\ (0.467,0.926,0.267) & (0.156,0.543,0.926) & (0.673,0.452,0.342) \\ (0.914,0.316,0.912) & (0.245,0.431,0.211) & (0.345,0.763,0.821) \\ (0.928,0.419,0.745) & (0.348,0.345,0.618) & (0.543,0.821,0.721) \\ (0.211,0.518,0.213) & (0.245,0.456,0.721) & (0.436,0.417,0.556) \\ (0.156,0.653,0.712) & (0.348,0.345,0.618) & (0.529,0.673,0.719) \end{bmatrix}$$

4. Let the weight order of neutrosophic soft sets be $W_1 = 0.3, W_2 = 0.4, W_3 = 0.3$. Using **Definition 3.1** the results are obtained as

$$N_w^1 = \begin{bmatrix} (0.074,0.137,0.216) & (0.183,0.168,0.172) & (0.125,0.246,0.063) \\ (0.104,0.047,0.188) & (0.138,0.261,0.217) & (0.185,0.214,0.154) \\ (0.164,0.230,0.129) & (0.306,0.301,0.253) & (0.125,0.156,0.125) \\ (0.080,0.096,0.096) & (0.221,0.357,0.289) & (0.094,0.184,0.155) \\ (0.128,0.125,0.267) & (0.181,0.085,0.165) & (0.069,0.277,0.275) \\ (0.137,0.280,0.065) & (0.228,0.094,0.099) & (0.125,0.113,0.184) \\ (0.097,0.190,0.245) & (0.147,0.182,0.365) & (0.145,0.069,0.214) \end{bmatrix},$$

$$N_w^2 = \begin{bmatrix} (0.074,0.104,0.164) & (0.182,0.062,0.306) & (0.216,0.188,0.129) \\ (0.137,0.104,0.230) & (0.168,0.261,0.301) & (0.129,0.163,0.190) \\ (0.125,0.185,0.125) & (0.328,0.285,0.208) & (0.063,0.154,0.125) \\ (0.071,0.125,0.140) & (0.294,0.327,0.370) & (0.155,0.137,0.080) \\ (0.094,0.069,0.275) & (0.301,0.357,0.085) & (0.064,0.230,0.137) \\ (0.226,0.268,0.064) & (0.247,0.166,0.126) & (0.135,0.070,0.160) \\ (0.124,0.247,0.065) & (0.246,0.170,0.130) & (0.164,0.080,0.128) \end{bmatrix} \ and$$

$$N_w^3 = \begin{bmatrix} (0.071,0.220,0.155) & (0.306,0.138,0.294) & (0.104,0.137,0.104) \\ (0.125,0.245,0.137) & (0.172,0.261,0.327) & (0.137,0.268,0.246) \\ (0.140,0.278,0.080) & (0.062,0.217,0.370) & (0.202,0.136,0.103) \\ (0.274,0.095,0.274) & (0.098,0.172,0.084) & (0.104,0.229,0.246) \\ (0.278,0.126,0.224) & (0.139,0.138,0.247) & (0.163,0.246,0.216) \\ (0.063,0.155,0.064) & (0.098,0.182,0.288) & (0.131,0.125,0.167) \\ (0.047,0.196,0.214) & (0.139,0.138,0.247) & (0.159,0.202,0.216) \end{bmatrix}$$

5. Using **Definition 3.2** the score function matrices are obtained as

$$S_F(N_w^1) = \begin{bmatrix} 0.321 & 0.348 & 0.249 \\ 0.264 & 0.417 & 0.354 \\ 0.325 & 0.556 & 0.265 \\ 0.185 & 0.578 & 0.294 \\ 0.394 & 0.298 & 0.448 \\ 0.273 & 0.260 & 0.303 \\ 0.389 & 0.529 & 0.321 \end{bmatrix} S_F(N_w^2) = \begin{bmatrix} 0.253 & 0.428 & 0.331 \\ 0.350 & 0.516 & 0.336 \\ 0.279 & 0.515 & 0.234 \\ 0.238 & 0.681 & 0.226 \\ 0.357 & 0.414 & 0.284 \\ 0.311 & 0.332 & 0.262 \\ 0.251 & 0.337 & 0.250 \end{bmatrix} S_F(N_w^3) =$$

$$\begin{bmatrix} 0.301 & 0.516 & 0.224 \\ 0.322 & 0.543 & 0.449 \\ 0.289 & 0.510 & 0.271 \\ 0.458 & 0.220 & 0.413 \\ 0.426 & 0.386 & 0.421 \\ 0.173 & 0.429 & 0.295 \\ 0.335 & 0.386 & 0.396 \end{bmatrix}$$

6. Applying **Definition 3.3** the total of the score functions are calculated as

$$T_i^1 = \begin{bmatrix} 0.918 \\ 1.034 \\ 1.147 \\ 1.057 \\ 1.140 \\ 0.836 \\ 1.238 \end{bmatrix}, T_i^2 = \begin{bmatrix} 1.012 \\ 1.202 \\ 1.028 \\ 1.145 \\ 1.055 \\ 0.905 \\ 1.839 \end{bmatrix} \ and \ T_i^3 = \begin{bmatrix} 1.041 \\ 1.313 \\ 1.071 \\ 1.090 \\ 1.232 \\ 0.897 \\ 1.117 \end{bmatrix}$$

7. The total value for each candidate is calculated and presented as

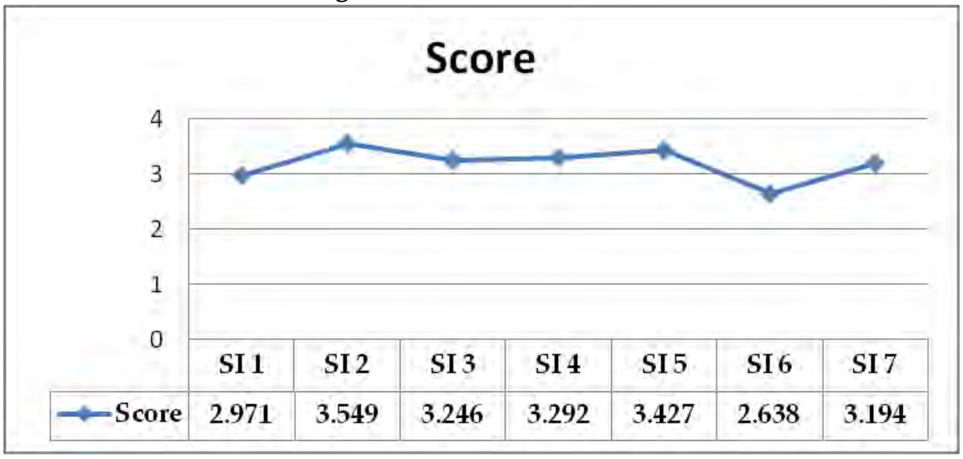
$$ST_i = \begin{bmatrix} 2.971 \\ 3.549 \\ 3.246 \\ 3.292 \\ 3.427 \\ 2.638 \\ 3.194 \end{bmatrix}$$

8. Arranging the SIs according to their total score values we obtain the ranking of the SIs as

Table 2. Tabular representation of SI's total score values.

c_i	Score	Rank
c_2	**3.549**	**1**
c_5	3.427	2
c_4	3.292	3
c_3	3.246	4
c_7	3.194	5
c_1	2.971	6
c_6	2.638	7

Figure 2. Score values of SIs.



From **Table 2** and **Figure 2**, we obtain the ranking of SIs as $c_2 > c_5 > c_4 > c_3 > c_7 > c_1 > c_6$. The SI c_2 ranks first and it is the most suitable SI for the investor.

5.2. Case study II

Consider the same example as in 5.1. A person would like to select the best SI.

1. Let $C = (c_1, c_2, \dots, c_7)$ be the set of top listed SIs.

2. Let $E = (e_1, e_2, e_3, e_4)$ be the set of parameters which form the criteria for selection. Here, e_1 = financial profitability projection, e_2 = asset-utilization, e_3 = conservative capital structure and e_4 = earnings momentum of the SI.

3. Let the personnel present his forecast result in the form of NSM- N^1, N^2 and N^3 for the subset of the criteria set (e_1, e_2, e_3) as

$$N^1 = \begin{bmatrix} (0.245,0.456,0.721) & (0.457,0.421,0.431) & (0.415,0.821,0.211) \\ (0.247,0.156,0.547) & (0.345,0.653,0.543) & (0.618,0.712,0.614) \\ (0.546,0.765,0.429) & (0.765,0.753,0.632) & (0.415,0.521,0.416) \\ (0.567,0.552,0.521) & (0.652,0.682,0.723) & (0.313,0.412,0.568) \\ (0.429,1.000,0.891) & (0.452,0.219,0.407) & (0.231,0.922,0.916) \\ (0.456,0.932,0.217) & (0.569,0.236,0.247) & (0.416,0.378,0.612) \\ (0.324,0.634,0.816) & (0.367,0.456,0.912) & (0.482,0.231,0.712) \end{bmatrix},$$

$$N^2 = \begin{bmatrix} (0.245,0.348,0.546) & (0.456,0.156,0.765) & (0.721,0.627,0.429) \\ (0.457,0.345,0.765) & (0.421,0.653,0.753) & (0.431,0.543,0.632) \\ (0.415,0.618,0.415) & (0.821,0.712,0.521) & (0.211,0.514,0.416) \\ (0.638,0.516,0.467) & (0.734,0.817,0.926) & (0.518,0.456,0.467) \\ (0.314,0.231,0.916) & (0.753,0.893,0.213) & (0.213,0.765,0.457) \\ (0.753,0.893,0.213) & (0.618,0.415,0.314) & (0.451,0.233,0.532) \\ (0.412,0.824,0.218) & (0.614,0.425,0.324) & (0.546,0.267,0.428) \end{bmatrix} and$$

$$N^3 = \begin{bmatrix} (0.238,0.734,0.518) & (0.765,0.345,0.734) & (0.345,0.457,0.347) \\ (0.416,0.817,0.456) & (0.429,0.753,0.817) & (0.456,0.892,0.821) \\ (0.467,0.926,0.267) & (0.156,0.543,0.926) & (0.673,0.452,0.342) \\ (0.714,0.716,0.912) & (0.245,0.431,0.211) & (0.345,0.763,0.821) \\ (0.928,0.419,0.745) & (0.348,0.345,0.616) & (0.543,0.821,0.721) \\ (0.211,0.518,0.213) & (0.245,0.456,0.721) & (0.436,0.417,0.556) \\ (0.156,0.653,0.712) & (0.348,0.345,0.618) & (0.529,0.673,0.719) \end{bmatrix}$$

4. Let the weight order of neutrosophic soft sets be $W_1 = 0.3, W_2 = 0.4, W_3 = 0.3$. Using Definition 3.1 the results are obtained as

$$N_w^1 = \begin{bmatrix} (0.074,0.137,0.216) & (0.183,0.168,0.172) & (0.125,0.246,0.063) \\ (0.074,0.047,0.164) & (0.138,0.261,0.217) & (0.184,0.214,0.184) \\ (0.164,0.230,0.129) & (0.306,0.301,0.253) & (0.125,0.156,0.125) \\ (0.070,0.166,0.156) & (0.261,0.273,0.289) & (0.094,0.124,0.170) \\ (0.129,0.300,0.267) & (0.181,0.088,0.163) & (0.069,0.277,0.275) \\ (0.137,0.280,0.065) & (0.228,0.094,0.099) & (0.125,0.113,0.184) \\ (0.097,0.190,0.245) & (0.147,0.182,0.365) & (0.145,0.069,0.213) \end{bmatrix},$$

$$N_w^2 = \begin{bmatrix} (0.074,0.104,0.164) & (0.182,0.062,0.306) & (0.216,0.188,0.129) \\ (0.137,0.104,0.230) & (0.168,0.261,0.301) & (0.129,0.163,0.190) \\ (0.125,0.185,0.125) & (0.328,0.285,0.208) & (0.063,0.154,0.125) \\ (0.091,0.155,0.140) & (0.294,0.327,0.370) & (0.155,0.137,0.140) \\ (0.094,0.069,0.275) & (0.301,0.357,0.085) & (0.064,0.230,0.137) \\ (0.226,0.268,0.064) & (0.247,0.166,0.126) & (0.135,0.070,0.160) \\ (0.124,0.247,0.065) & (0.246,0.170,0.130) & (0.164,0.080,0.128) \end{bmatrix} and$$

$$N_w^3 = \begin{bmatrix} (0.071,0.220,0.155) & (0.306,0.138,0.294) & (0.104,0.137,0.104) \\ (0.125,0.245,0.137) & (0.172,0.301,0.327) & (0.137,0.268,0.246) \\ (0.140,0.278,0.080) & (0.062,0.217,0.370) & (0.202,0.136,0.103) \\ (0.214,0.215,0.274) & (0.098,0.172,0.084) & (0.104,0.229,0.246) \\ (0.278,0.126,0.224) & (0.139,0.138,0.246) & (0.163,0.246,0.216) \\ (0.063,0.155,0.064) & (0.098,0.182,0.288) & (0.131,0.125,0.167) \\ (0.047,0.196,0.214) & (0.139,0.138,0.247) & (0.159,0.202,0.216) \end{bmatrix}$$

5. Using **Definition 3.2** the score function matrices are obtained as

$$V_F(N_w^1) = \begin{bmatrix} 0.321 & 0.348 & 0.249 \\ 0.225 & 0.417 & 0.384 \\ 0.325 & 0.556 & 0.265 \\ 0.324 & 0.556 & 0.279 \\ 0.482 & 0.297 & 0.448 \\ 0.273 & 0.260 & 0.303 \\ 0.389 & 0.529 & 0.321 \end{bmatrix} V_F(N_w^2) = \begin{bmatrix} 0.253 & 0.428 & 0.331 \\ 0.350 & 0.516 & 0.336 \\ 0.279 & 0.515 & 0.234 \\ 0.313 & 0.681 & 0.286 \\ 0.357 & 0.414 & 0.284 \\ 0.311 & 0.332 & 0.262 \\ 0.251 & 0.337 & 0.250 \end{bmatrix} V_F(N_w^3) = \begin{bmatrix} 0.301 & 0.516 & 0.224 \\ 0.322 & 0.563 & 0.449 \\ 0.289 & 0.510 & 0.271 \\ 0.488 & 0.220 & 0.413 \\ 0.426 & 0.385 & 0.421 \\ 0.173 & 0.429 & 0.295 \\ 0.335 & 0.386 & 0.396 \end{bmatrix}$$

6. Applying **Definition 3.3** the total of the score functions are calculated as

$$T_i^1 = \begin{bmatrix} 0.918 \\ 1.025 \\ 1.147 \\ 1.159 \\ 1.226 \\ 0.836 \\ 1.238 \end{bmatrix}, T_i^2 = \begin{bmatrix} 1.012 \\ 1.202 \\ 1.028 \\ 1.280 \\ 1.055 \\ 0.905 \\ 1.839 \end{bmatrix}, T_i^3 = \begin{bmatrix} 1.041 \\ 1.333 \\ 1.071 \\ 1.120 \\ 1.231 \\ 0.897 \\ 1.117 \end{bmatrix},$$

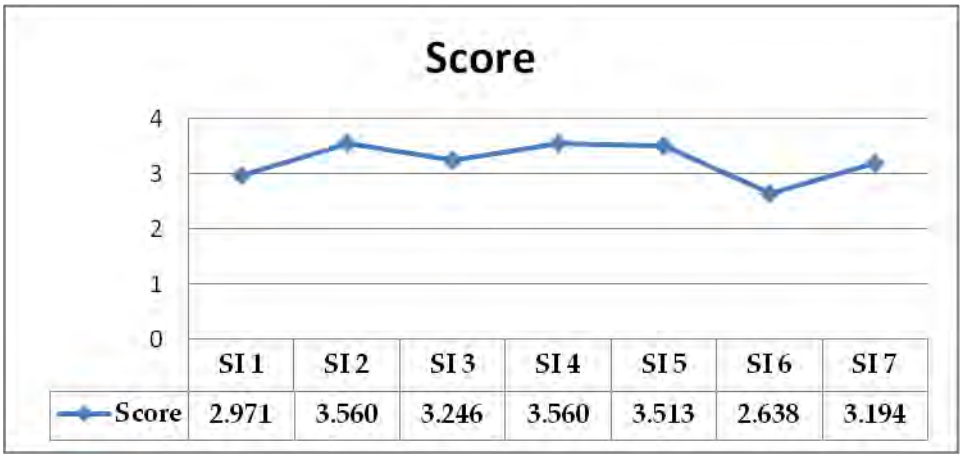
7. The total value for each SI is calculated and presented as

$$ST_i = \begin{bmatrix} 2.971 \\ 3.560 \\ 3.246 \\ 3.560 \\ 3.513 \\ 2.638 \\ 3.194 \end{bmatrix}$$

Table 3. Tabular representation of SI's total score values.

c_i	Score	Rank
$\boldsymbol{c_2}$	**3.560**	**1**
$\boldsymbol{c_4}$	**3.560**	**1**
c_5	3.513	3
c_3	3.246	4
c_7	3.194	5
c_1	2.971	6
c_6	2.638	7

Figure 3. Score values of SIs



From **Table 3** and **Figure 3**, we obtain the ranking of SIs as $c_2 = c_4 > c_5 > c_3 > c_7 > c_1 > c_6$. As there are more than one SI (c_2 and c_4) with the same ranking we add one more parameter e_4 in the list and repeat the process.

$$N^1 = \begin{bmatrix} (0.245,0.456,0.721) & (0.457,0.421,0.431) & (0.415,0.821,0.211) & (0.536,0.665,0.129) \\ (0.247,0.156,0.547) & (0.345,0.653,0.543) & (0.618,0.712,0.614) & (0.547,0.451,0.321) \\ (0.546,0.765,0.429) & (0.765,0.753,0.632) & (0.415,0.521,0.416) & (0.357,0.451,0.631) \\ (0.567,0.552,0.521) & (0.652,0.682,0.723) & (0.313,0.412,0.568) & (0.375,0.753,0.243) \\ (0.429,1.000,0.891) & (0.452,0.219,0.407) & (0.231,0.922,0.916) & (0.251,0.562,0.726) \\ (0.456,0.932,0.217) & (0.569,0.236,0.247) & (0.416,0.378,0.612) & (0.426,0.478,0.512) \\ (0.324,0.634,0.816) & (0.367,0.456,0.912) & (0.482,0.231,0.712) & (0.416,0.252,0.317) \end{bmatrix},$$

$$N^2 = \begin{bmatrix} (0.245,0.348,0.546) & (0.456,0.156,0.765) & (0.721,0.627,0.429) & (0.546,0.765,0.429) \\ (0.457,0.345,0.765) & (0.421,0.653,0.753) & (0.431,0.543,0.632) & (0.567,0.551,0.521) \\ (0.415,0.618,0.415) & (0.821,0.712,0.521) & (0.211,0.514,0.416) & (0.457,0.421,0.431) \\ (0.638,0.516,0.467) & (0.734,0.817,0.926) & (0.518,0.456,0.467) & (0.345,0.653,0.543) \\ (0.314,0.231,0.916) & (0.753,0.893,0.213) & (0.213,0.765,0.457) & (0.231,0.922,0.916) \\ (0.753,0.893,0.213) & (0.618,0.415,0.314) & (0.451,0.233,0.532) & (0.416,0.378,0.612) \\ (0.412,0.824,0.218) & (0.614,0.425,0.324) & (0.546,0.267,0.428) & (0.456,0.932,0.217) \end{bmatrix}$$

$$N^3 = \begin{bmatrix} (0.238,0.734,0.518) & (0.765,0.345,0.734) & (0.721,0.627,0.429) & (0.546,0.765,0.429) \\ (0.416,0.817,0.456) & (0.429,0.753,0.817) & (0.431,0.543,0.632) & (0.567,0.551,0.521) \\ (0.467,0.926,0.267) & (0.156,0.543,0.926) & (0.211,0.514,0.416) & (0.457,0.421,0.431) \\ (0.714,0.716,0.912) & (0.245,0.431,0.211) & (0.518,0.456,0.467) & (0.345,0.653,0.543) \\ (0.928,0.419,0.745) & (0.348,0.345,0.616) & (0.213,0.765,0.457) & (0.231,0.922,0.916) \\ (0.211,0.518,0.213) & (0.245,0.456,0.721) & (0.451,0.233,0.532) & (0.416,0.378,0.612) \\ (0.156,0.653,0.712) & (0.348,0.345,0.618) & (0.546,0.267,0.428) & (0.456,0.932,0.217) \end{bmatrix},$$

4. Let the weight order of neutrosophic soft sets be $W_1 = 0.3, W_2 = 0.4, W_3 = 0.15$ and $W_4 = 0.15$. Using **Definition 3.1** the resultant are obtained as

$$N_w^1 = \begin{bmatrix} (0.074,0.137,0.216) & (0.183,0.168,0.172) & (0.062,0.123,0.032) & (0.080,0.100,0.019) \\ (0.074,0.047,0.164) & (0.138,0.261,0.217) & (0.093,0.107,0.092) & (0.082,0.068,0.048) \\ (0.164,0.230,0.129) & (0.306,0.301,0.253) & (0.062,0.078,0.062) & (0.054,0.068,0.095) \\ (0.070,0.166,0.156) & (0.261,0.273,0.289) & (0.047,0.062,0.085) & (0.056,0.113,0.036) \\ (0.129,0.300,0.267) & (0.181,0.088,0.163) & (0.035,0.138,0.137) & (0.038,0.084,0.109) \\ (0.137,0.280,0.065) & (0.228,0.094,0.099) & (0.062,0.057,0.092) & (0.064,0.072,0.077) \\ (0.097,0.190,0.245) & (0.147,0.182,0.365) & (0.072,0.035,0.107) & (0.062,0.038,0.048) \end{bmatrix}$$

$$N_w^2 = \begin{bmatrix} (0.074,0.104,0.164) & (0.182,0.062,0.306) & (0.108,0.094,0.064) & (0.082,0.115,0.064) \\ (0.137,0.104,0.230) & (0.168,0.261,0.301) & (0.065,0.081,0.095) & (0.085,0.083,0.078) \\ (0.125,0.185,0.125) & (0.328,0.285,0.208) & (0.032,0.077,0.062) & (0.069,0.063,0.065) \\ (0.091,0.155,0.140) & (0.294,0.327,0.370) & (0.078,0.068,0.070) & (0.052,0.098,0.081) \\ (0.094,0.069,0.275) & (0.301,0.357,0.085) & (0.032,0.115,0.069) & (0.035,0.138,0.137) \\ (0.226,0.268,0.064) & (0.247,0.166,0.126) & (0.068,0.035,0.080) & (0.062,0.057,0.092) \\ (0.124,0.247,0.065) & (0.246,0.170,0.130) & (0.082,0.040,0.064) & (0.068,0.140,0.033) \end{bmatrix}$$

$$N_w^3 = \begin{bmatrix} (0.071,0.220,0.155) & (0.306,0.138,0.294) & (0.052,0.069,0.052) & (0.082,0.115,0.064) \\ (0.125,0.245,0.137) & (0.172,0.301,0.327) & (0.068,0.134,0.123) & (0.085,0.083,0.078) \\ (0.140,0.278,0.080) & (0.062,0.217,0.370) & (0.101,0.068,0.051) & (0.069,0.063,0.065) \\ (0.214,0.215,0.274) & (0.098,0.172,0.084) & (0.052,0.114,0.123) & (0.052,0.098,0.081) \\ (0.278,0.126,0.224) & (0.139,0.138,0.246) & (0.081,0.123,0.108) & (0.035,0.138,0.137) \\ (0.063,0.155,0.064) & (0.098,0.182,0.288) & (0.065,0.063,0.083) & (0.062,0.057,0.092) \\ (0.047,0.196,0.214) & (0.139,0.138,0.247) & (0.079,0.101,0.108) & (0.068,0.140,0.033) \end{bmatrix}$$

5. Using **Definition 3.2** the score function matrices are obtained as

$$V_F(N_w^1) = \begin{bmatrix} 0.321 & 0.348 & 0.124 & 0.109 \\ 0.225 & 0.417 & 0.192 & 0.123 \\ 0.325 & 0.556 & 0.133 & 0.155 \\ 0.324 & 0.556 & 0.140 & 0.121 \\ 0.482 & 0.297 & 0.224 & 0.170 \\ 0.273 & 0.260 & 0.151 & 0.145 \\ 0.389 & 0.529 & 0.160 & 0.098 \end{bmatrix}, V_F(N_w^2) = \begin{bmatrix} 0.253 & 0.428 & 0.165 & 0.163 \\ 0.350 & 0.516 & 0.168 & 0.162 \\ 0.279 & 0.515 & 0.117 & 0.131 \\ 0.313 & 0.681 & 0.143 & 0.156 \\ 0.357 & 0.414 & 0.142 & 0.224 \\ 0.311 & 0.332 & 0.131 & 0.151 \\ 0.251 & 0.337 & 0.125 & 0.137 \end{bmatrix}$$

$$V_F(N_w^3) = \begin{bmatrix} 0.301 & 0.516 & 0.112 & 0.163 \\ 0.322 & 0.563 & 0.224 & 0.162 \\ 0.289 & 0.510 & 0.136 & 0.131 \\ 0.488 & 0.220 & 0.206 & 0.156 \\ 0.426 & 0.385 & 0.210 & 0.224 \\ 0.173 & 0.429 & 0.147 & 0.151 \\ 0.335 & 0.386 & 0.198 & 0.137 \end{bmatrix}$$

6. Applying **Definition 3.3** the total of the score functions are calculated as

$$T_i^1 = \begin{bmatrix} 0.903 \\ 0.995 \\ 1.170 \\ 1.141 \\ 1.130 \\ 0.829 \\ 1.176 \end{bmatrix}, T_i^2 = \begin{bmatrix} 1.009 \\ 1.196 \\ 1.293 \\ 1.137 \\ 0.925 \\ 0.850 \end{bmatrix} and T_i^3 = \begin{bmatrix} 1.092 \\ 1.271 \\ 1.065 \\ 1.070 \\ 1.245 \\ 0.901 \\ 1.055 \end{bmatrix}$$

7. The total value for each SI is calculated and presented as

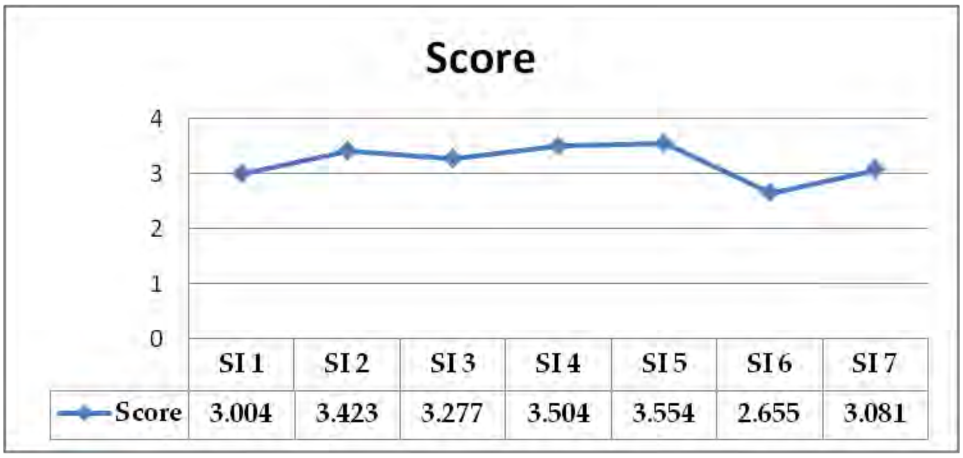
$$ST_i = \begin{bmatrix} 3.004 \\ 3.423 \\ 3.277 \\ 3.504 \\ 3.554 \\ 2.655 \\ 3.081 \end{bmatrix}$$

8. Arranging the SIs according to their total score values we obtain the ranking of the SIs as

Table 4. Tabular representation of SI's total score values.

c_i	Score	Rank
$\boldsymbol{c_5}$	**3.554**	**1**
c_4	3.504	2
c_2	3.423	3
c_3	3.277	4
c_7	3.081	5
c_1	3.004	6
c_6	2.655	7

Figure 4. Score values of SIs



From **Table 4** and **Figure 4**, we obtain the ranking of SIs as $c_5 > c_4 > c_2 > c_3 > c_7 > c_1 > c_6$. The SI c_5 ranks first and it is the most suitable SI for the investor.

6. Conclusions

The proposed NSM computational solution supports decision-makers in solving the complex decision-making problem faced in today's ambiguity situation. In this paper, the weight vector and score function are introduced with illustrative examples. By applying the score function we solve the MADM problems in the neutrosophic environment and transforming the values of truth, indeterminacy and falsity into a single membership value to obtain a more precise, efficient, and realistic solution. An application of NSM in MADM is also explained. An algorithm is developed for this purpose and two examples are provided to illustrate the working of the algorithm. Our future work is to extend the concept of MADM problems in real-life psychology applications by using standard or hybrid neutrosophic and plithogenic tools.

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Neutrosophic Fixed Point Theorems and Cone Metric Spaces

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Abstract. The intention of this paper is to give the general definition of cone metric space in the context of the neutrosophic theory. In this relation, we obtain some fundamental results concerting fixed points for weakly compatible mapping.

Keywords: neutrosophic theory, neutrosophic Fixed Point, neutrosophic topology, neutrosophic cone metric space, neutrosophic metric space.

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1. **Introduction**

Zadeh [13] introduced the notion of fuzzy sets. After that there have been a number of generalizations of this fundamental concept. The study of fuzzy topological spaces was first initiated by Chang [6] in the year 1968. Atanassov [12] introduced the notion of intuitionistic fuzzy sets. This notion was extended to intuitionistic L-fuzzy setting by Atanassov and Stoeva [20], which currently holds the name "intuitionistic L-topological spaces". Using the notion of intuitionistic fuzzy sets, Coker [7] introduced the notion of intuitionistic fuzzy topological space. The concept of generalized fuzzy closed set was introduced by G. Balasubramanian and P. Sundaram [11]. In various recent papers, F. Smarandache generalizes intuitionistic fuzzy sets (IFSs) and other kinds of sets to neutrosophic sets (NSs). F. Smarandache and A. Al Shumrani also defined the notion of neutrosophic topology on the non-standard interval [2, 9, 14, 16]. Also, ([8, 15, 17]) introduced the metric topology and neutrosophic geometric and studied various properties. Recently, Wadei Al-Omeri and Smarandache [18, 19] introduce and study the concepts of neutrosophic open sets and its complements in neutrosophic topological space, continuity in neutrosophic topology, and obtain some characterizations concerning neutrosophic connectedness and neutrosophic mapping.

This paper is arranged as follows. In Section 2, we will recall some notions which will be used throughout this paper. In Section 3, neutrosophic Cone Metric Space and investigate its basic properties. In Section 4, we study the neutrosophic Fixed Point Theorems and study some of their properties. Finally, Banach contraction theorem and some fixed point results on neutrosophic cone metric space are stated and proved.

2. **Preliminaries**

Definition 2.1. [4] Let Σ be a non-empty fixed set. A neutrosophic set (briefly NS) B is an object having the form $B = \{\langle r, \xi_B(r), \varrho_B(r), \eta_B(r)\rangle : r \in \Sigma\}$, where $\xi_B(r), \varrho_B(r)$, and $\eta_B(r)$ which represent the degree of membership function (namely $\xi_B(r)$), the degree of indeterminacy (namely $\varrho_B(r)$), and the degree of non-membership (namely $\eta_B(r)$) respectively, of each element $r \in \Sigma$ to the set B.

A neutrosophic set $B = \{\langle r, \xi_B(r), \varrho_B(r), \eta_B(r)\rangle : r \in \Sigma\}$ can be identified to an ordered triple $\langle \xi_B(r), \varrho_B(r)$
$, \eta_B(r)\rangle$ in $\rfloor 0^-, 1^+ \lfloor$ on Σ.

Remark 2.1. [4] For the sake of simplicity, we shall use the symbol $B = \{r, \xi_B(r),$
$\varrho_B(r), \eta_B(r)\}$ for the NS $B = \{\langle r, \xi_B(r), \varrho_B(r), \eta_B(r)\rangle : r \in \Sigma\}$.

Definition 2.2. [5] Let $B = \langle \xi_B(r), \varrho_B(r), \eta_B(r)\rangle$ be an NS on Σ. The complement of B(briefly$C(B)$), are defined as three types of complements

(1) $C(B) = \{\langle r, \eta_B(r), 1 - \varrho_B(r), \xi_B(r)\rangle : r \in \Sigma\}$,
(2) $C(B) = \{\langle r, 1 - \xi_B(r), 1 - \eta_B(r)\rangle : r \in \Sigma\}$
(3) $C(B) = \{\langle r, \eta_B(r), \varrho_B(r), \xi_B(r)\rangle : r \in \Sigma\}$

We have the following NSs (see [4]) which will be used in the sequel:

(1) $0_N = \{\langle r, 0, 0, 1\rangle : r \in \Sigma\}$ or
(2) $0_N = \{\langle r, 0, 1, 1\rangle : r \in \Sigma\}$ or
(3) $0_N = \{\langle r, 0, 0, 0\rangle : r \in \Sigma\}$ or
(4) $0_N = \{\langle r, 0, 1, 0\rangle : r \in \Sigma\}$

2- 1_N may be defined as four types:

(1) $1_N = \{\langle r, 1, 1, 1\rangle : r \in \Sigma\}$ or
(2) $1_N = \{\langle r, 1, 0, 0\rangle : r \in \Sigma\}$ or
(3) $1_N = \{\langle r, 1, 1, 0\rangle : r \in \Sigma\}$ or

(4) $1_N = \{\langle r, 1, 0, 1\rangle : r \in \Sigma\}$

Definition 2.3. [4] Let $x \neq \emptyset$, and generalized neutrosophic sets $(GNSs)$ B and Γ be in the form $B = \{r, \xi_B(r), \varrho_B(r), \eta_B(r)\}$, $\Gamma = \{r, \xi_\Gamma(r), \varrho_\Gamma(r), \eta_\Gamma(r)\}$. We think of two possible definitions $A \subseteq \Gamma$.

(1) $B \subseteq \Gamma \Leftrightarrow \xi_B(r) \leq \xi_\Gamma(r), \varrho_B(r) \geq \varrho_\Gamma(r), \text{and}\, \eta_B(r) \leq \eta_\Gamma(r)$
(2) $B \subseteq \Gamma \Leftrightarrow \xi_B(r) \leq \xi_\Gamma(r), \varrho_B(r) \geq \varrho_\Gamma(r), \text{and}\, \eta_B(r) \geq \eta_\Gamma(r)$.

Definition 2.4. [4] Let $\{B_j : j \in J\}$ be an arbitrary family of an NSs in Σ. Then

(1) $\cap B_j$ defined as two types:
- $\cap B_j = \langle r, \underset{j\in J}{\wedge} \xi_{Bj}(r), \underset{j\in J}{\wedge} \varrho_{Bj}(r), \underset{j\in J}{\vee} \eta_{Bj}(r)\rangle$ < Type 1 >
- $\cap B_j = \langle r, \underset{j\in J}{\wedge} \xi_{Bj}(r), \underset{j\in J}{\vee} \varrho_{Bj}(r), \underset{j\in J}{\vee} \eta_{Bj}(r)\rangle$ < Type 2 >.

(2) $\cup B_j$ defined as two types:
- $\cup B_j = \langle r, \underset{j\in J}{\vee} \xi_{Bj}(r), \underset{j\in J}{\vee} \varrho_{Bj}(r), \underset{j\in J}{\wedge} \eta_{Bj}(r)\rangle$ < Type 1 >
- $\cup B_j = \langle r, \underset{j\in J}{\vee} \xi_{Bj}(r), \underset{j\in J}{\wedge} \varrho_{Bj}(r), \underset{j\in J}{\wedge} \eta_{Bj}(r)\rangle$ < Type 2 >

Definition 2.5. [3] A neutrosophic topology (briefly NT) and a non empty set Σ is a family Υ of neutrosophic subsets of Σ satisfying the following axioms

(1) $0_N, 1_N \in \Upsilon$
(2) $S_1 \cap S_2 \in \Upsilon$ for any $S_1, S_2 \in \Upsilon$
(3) $\cup S_i \in \Upsilon$, $\forall \{S_i | i \in I\} \subseteq \Upsilon$.

The pair (Σ, Υ) is called a neutrosophic topological space (briefly NTS) and any neutrosophic set in Υ is defined as neutrosophic open set (NOS for short) in Σ. The elements of Υ are called open neutrosophic sets. A neutrosophic set S is closed if f its $C(S)$ is neutrosophic open. For any NTS A in (Σ, Υ) ([21]), we have $Int(A^c) = [Cl(A)]^c$ and $Cl(A^c) = [Int(A)]^c$.

Definition 2.6. A subset ω of Ω is called a cone if

(1) For non-empty ω is closed, and $\omega \neq 0$,
(2) If both $u \in \omega$ and $-u \in \omega$ then $u = 0$,
(3) If $u,\, v \in S,\, u,\, v \geq 0$ and $x,\, y \in \omega$ then $ux + vy \in \omega$.

Throughout this paper, we assume that all cones have non-empty interior. For any cone, $x \prec y$ will stand for $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in Int(\omega)$. a partial ordering $\preccurlyeq$ on Ω via ω is defined by $x \preccurlyeq y$ iff $y - x \in \omega$.

Definition 2.7. A cone metric space (briefly CMS) an ordered (Σ, d), where Σ is any set and $d : \Sigma \times \Sigma \longmapsto \Omega$ is a mapping satisfying:

(1) $d(s_1, s_2) = d(s_2, s_1)$ for all $s_1,\, s_2 \in \Sigma$,

(2) $d(s_1, s_2) = 0$ iff $s_1 = s_2$,

(3) $0 \preccurlyeq d(s_1, s_2)$ for all s_1, $s_2 \in \Sigma$,

(4) $d(s_1, s_3) \preccurlyeq d(s_1, s_2) + d(s_2, s_3)$ for all s_1, s_2, $s_3 \in \Sigma$.

Definition 2.8. Let (Σ, d) be a CMS. Then, for each $c_1 \gg 0$ and $c_2 \gg 0$, c_1, $c_2 \in \Omega$, there exists $c \gg 0$, $c \in \Omega$ such that $c \ll c_1$ and $c \ll c_2$.

Definition 2.9. A binary operation $\bigotimes : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if $\bigotimes$ satisfies the following conditions:

(1) $\bigotimes$ is continuous,

(2) $\bigotimes$ is commutative and associative,

(3) $m_1 \bigotimes m_2 \leq m_3 \bigotimes m_4$ whenever $m_1 \leq m_3$ and $m_2 \leq m_4$ $\forall m_1$, m_2, m_3, $m_4 \in [0,1]$,

(4) $m_1 \bigotimes 1 = m_1$ $\forall m_1 \in [0,1]$.

Definition 2.10. A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-conorm if $\diamond$ satisfies the following conditions:

(1) $\diamond$ is continuous,

(2) $\diamond$ is commutative and associative,

(3) $m_1 \diamond m_2 \leq m_3 \diamond m_4$ whenever $m_1 \leq m_3$ and $m_2 \leq m_4$ $\forall m_1$, m_2, m_3, $m_4 \in [0,1]$,

(4) $m_1 \diamond 1 = m_1$ $\forall m_1 \in [0,1]$.

Definition 2.11. Let Σ be a non-empty set. The mappings $\mathcal{G} : \Sigma \times \Sigma \longrightarrow \Sigma$ and $\mathcal{H} : \Sigma \longrightarrow \Sigma$ are called commutative if $\mathcal{H}(\mathcal{G}(x, y)) = \mathcal{G}(\mathcal{H}(x), \mathcal{H}(y))$ $\forall x$, $y \in \Sigma$.

Definition 2.12. Let $\Sigma \neq \emptyset$. An element $x \in \Sigma$ is called a common fixed point of mappings $\mathcal{G} : \Sigma \times \Sigma \longrightarrow \Sigma$ and $\mathcal{H} : \Sigma \longrightarrow \Sigma$ if $x = \mathcal{H}(x) = \mathcal{G}(x, x)$.

Definition 2.13. If U and V are two maps then, a pair of maps is called weakly compatible (briefly WCP) pair if they commute at (CP).

Definition 2.14. Let Σ be a set, $\mathcal{G}$, $\mathcal{H}$ self maps of Σ. A point x in Σ is called a coincidence point (briefly CP) of $\mathcal{G}$ and $\mathcal{H}$ if and only if $\mathcal{G}(x) = \mathcal{H}(x)$. We call $w = \mathcal{G}(x) = \mathcal{H}(x)$ a point of coincidence of $\mathcal{G}$ and $\mathcal{H}$.

Definition 2.15. Two self maps $\mathcal{G}$ and $\mathcal{H}$ of a set Σ are sporadically weakly compatible of Σ. If $\mathcal{G}$ and $\mathcal{H}$ have a unique point of coincidence, $z = \mathcal{G}(u) = \mathcal{H}(v)$, then z is the unique common fixed point of $\mathcal{G}$ and $\mathcal{H}$.

Lemma 2.2. *Two self maps $\mathcal{G}$ and $\mathcal{H}$ of a set Σ are sporadically weakly compatible of Σ. then z is the unique common fixed point of $\mathcal{G}$ and $\mathcal{H}$, if $z = \mathcal{G}(u) = \mathcal{H}(u)$ $\mathcal{G}$ and $\mathcal{H}$ have a unique point of coincidence.*

Definition 2.16. A pair of maps $\mathcal{G}$ and $\mathcal{H}$ which $\mathcal{G}$ and $\mathcal{H}$ commute of a set Σ are sporadically weakly compatible iff there is a point x in Σ which is a coincidence point of $\mathcal{G}$ and $\mathcal{H}$.

3. neutrosophic Cone Metric Space

Definition 3.1. A 3-tuple $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is said to be a neutrosophic CMS if ω is a neutrosophic cone metric (briefly NCMS) of Ω, Σ is an arbitrary set, $\diamond$ is a neutrosophic continuous t-conorm , $\bigotimes$ is a neutrosophic continuous t-norm, $\forall \epsilon_1, \epsilon_2, \epsilon_3 \in \Sigma$ and m, $n \in Int(\omega)$ (that is $n \gg 0_\Theta$, $s \gg 0_\Theta$), and Ξ, Θ are neutrosophic set on $\Sigma^2 \times Int(\omega)$ satisfying the following conditions:

(1) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) + \Theta(\epsilon_1, \epsilon_2, \epsilon_3) \leq 1_\Theta$;
(2) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) > 0_\Theta$;
(3) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) = 1$ iff $\epsilon_1 = \epsilon_2$;
(4) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) = \Xi(\epsilon_2, \epsilon_1, m)$;
(5) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) \bigotimes \Xi(\epsilon_2, \epsilon_3, n) \leq \Xi(\epsilon_1, \epsilon_3, m + n)$;
(6) $\Xi(\epsilon_1, \epsilon_2, .) : Int(\omega) \longrightarrow \rfloor 0^-, 1^+ \lfloor$ is neutrosophic continuous;
(7) $\Theta(\epsilon_1, \epsilon_2, \epsilon_3) < 0_\Theta$;
(8) $\Theta(\epsilon_1, \epsilon_2, \epsilon_3) = 0_\Theta$ if and only if $\epsilon_1 = \epsilon_2$;
(9) $\Theta(\epsilon_1, \epsilon_2, \epsilon_3) = \Theta(\epsilon_2, \epsilon_3, r)$;
(10) $\Theta(\epsilon_1, \epsilon_2, \epsilon_3) \diamond \Theta(\epsilon_2, \epsilon_3, n) \geq \Theta(\epsilon_1, \epsilon_3, m + n)$;
(11) $\Theta(\epsilon_1, \epsilon_2, .) : Int(\omega) \longrightarrow \rfloor 0^-, 1^+ \lfloor$ is neutrosophic continuous.

Then (Ξ, Θ) is called a neutrosophic cone metric on Σ. The functions $\Theta(\epsilon_1, \epsilon_2, m)$ and $\Xi(\epsilon_1, \epsilon_2, m)$ denote the degree of non-nearness and the degree of nearness between ϵ_1 and ϵ_2 with respect to n, respectively.

Example 3.2. Let $\Omega = R$, $\omega = [0, \infty)$ and $a \diamond b = max\{a, b\}$, $a \bigotimes b = min\{a, b\}$, then every neutrosophic metric space (Σ, Ξ, Θ) becomes a $NCMS$.

Example 3.3. If we take ω be an any cone, $a \bigotimes b = min\{a, b\}$, $\Sigma = \Theta$, $\Xi, \Theta : \Sigma^2 \times Int(\omega) \longrightarrow \rfloor 0^-, 1^+ \lfloor$ defined by

$$\Xi(\epsilon_1, \epsilon_2, t) = \begin{cases} \dfrac{\epsilon_1}{\epsilon_2}, & if \quad \epsilon_1 \leq \epsilon_2, \\ \dfrac{\epsilon_1}{\epsilon_2}, & if \quad \epsilon_2 \leq \epsilon_1, \end{cases}$$

$$\Theta(\epsilon_1, \epsilon_2, t) = \begin{cases} \dfrac{\epsilon_2 - \epsilon_1}{\epsilon_2}, & if \quad \epsilon_1 \leq \epsilon_2, \\ \dfrac{\epsilon_1 - \epsilon_2}{\epsilon_2}, & if \quad \epsilon_2 \leq \epsilon_1, \end{cases}$$

for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $r \gg 0_\Theta$. Then $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is a $NCMS$.

Definition 3.4. Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$, $\{\epsilon_{1n}\}$ be a sequence in Σ and $\epsilon_1 \in \Sigma$. Then $\{\epsilon_{1n}\}$ is said to converge to ϵ_1 if for any $s \in (0,1)$ and any $m \gg 0_\Theta$ $\exists$ a natural number n_0 such that $\Xi(\epsilon_{1n}, x, m) > 1-s, \Theta(\epsilon_{1n}, \epsilon_1, m) \leq s$ for all $n \geq n_0$. We denote this by $\lim_{\epsilon_{1n}\to\infty} = \epsilon_1$ or $\epsilon_{1n} \to \epsilon_1$ as $\to \infty$.

Definition 3.5. Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$. For $m \gg 0_\Theta$, the open ball $\Gamma(x, s, m)$ with radius $s \in (0,1)$ and center ϵ_1 is defined by $\Gamma(\epsilon_1, s, m) = \{\epsilon_2 \in \Sigma : \Xi(\epsilon_1, \epsilon_2, m) > 1-s, \Theta(\epsilon_1, \epsilon_2, m) < s\}$.

Definition 3.6. The neutrosophic cone metric CMS $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is called complete neutrosophic CMS if every Cauchy sequence in $NCMS$ (Σ, Ξ, Θ) is convergent.

Definition 3.7. Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$. A subset P of Σ is said to be $\mathcal{FC}$-bounded if $\exists$ $s \in (0,1)$ and $m \gg \theta$ such that $\Xi(\epsilon_1, \epsilon_2, t) > 1-m, \Theta(\epsilon_1, \epsilon_2, m) < s$ for all $\epsilon_1, \epsilon_2 \in P$.

Definition 3.8. Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a neutrosophic CMS and $h : \Sigma \to \Sigma$ is a self mapping. Then h is said to be neutrosophic cone contractive if there exists $c \in (0,1)$ such that
$\frac{1}{\Xi(h(\epsilon_1), h(\epsilon_2), m)} - 1 \leq c(\frac{1}{\Xi(\epsilon_1, \epsilon_2, m)} - 1)$
$\Theta(h(\epsilon_1), h(\epsilon_2), m) \leq c\Theta(\epsilon_1, \epsilon_2, m)$
for each $\epsilon_1, \epsilon_2 \in \Sigma$ and $m \gg 0_\Theta$. The constant c is called the contractive constant of h.

Lemma 3.9. *If for two points $\epsilon_1, \epsilon_2 \in \Sigma$ and $c \in (0,1)$ such that $\Xi(\epsilon_1, \epsilon_2, cm) \geq \Xi(\epsilon_1, \epsilon_2, m)$, $\Theta(\epsilon_1, \epsilon_2, cm) \geq \Theta(\epsilon_1, \epsilon_2, m)$ then $\epsilon_1 = \epsilon_2$.*

Theorem 3.10. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$. Define $\mathcal{T} = \{K \subseteq \Sigma : \epsilon_1 \in K \text{iff there exists} s \in (0,1) and m \gg 0_\Theta \text{such that} L(\epsilon_1, s, m) \subseteq K\}$, then $\mathcal{T}$ is a neutrosophic topology on Σ.*

Proof. If ϵ_1 is empty, then $\emptyset = L(\epsilon_1, s, m) \subseteq \emptyset$. Hence the empty set belong to $\mathcal{T}$ Since for any $\epsilon_1 \in \Sigma$, any $s \in (0,1)$ and any $m \gg 0_\Theta$, $L(\epsilon_1, s, m) \subseteq \Sigma$, then $\Sigma \in \mathcal{T}$.
Let $K, L \in \mathcal{T}$ and $\epsilon_1 \in K \cap L$. Then $\epsilon_1 \in K$ and $\epsilon_1 \in L$, so there exist $m_1 \gg 0_\Theta$; $m_2 \gg 0_\Theta$ and $m_1, m_2 \in (0,1)$ such that $L(\epsilon_1, s_1, m_1) \subseteq K$ and $L(\epsilon_1, s_2, m_2) \subseteq L$.
By Proposition 2.8, for $m_1 \gg 0$; $m_2 \gg 0$, there exists $m \gg 0_\Theta$ such that $m \gg m_1$; $r \gg m_2$ and take $s = min\{m_1, m_2\}$. Then $L(\epsilon_1, s, m) \subseteq \Sigma$ $L(\epsilon_1, s_1, m_1) \cap L(\epsilon_1, s_2, m_2) \subseteq K \cap L$. Thus $K \cap L \in \mathcal{T}$. Let $K_i \in \mathcal{T}$ for each $i \in I$ and $\epsilon_1 \in \cup_{i\in I} K_i$. Then there exists $i_0 \in I$ such that $\epsilon_1 \in K_{i0}$. So, there exist $r \gg 0_\Theta$ and $s \in (0,1)$ such that $L(\epsilon_1, s, m) \subseteq K_{i_0}$. Since$K_{i_0} \subseteq \cup_{i\in I} K_i$, $L(\epsilon_1, s, m) \subseteq \cup_{i\in I} K_i$. Thus $\cup_{i\in I} K_i \in \mathcal{T}$. Hence, $\mathcal{T}$ is a neutrosophic topology on Σ. □

Theorem 3.11. *If $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is a $NCMS$, then the neutrosophic topology $(\Sigma, \mathcal{T})$ is Hausdorff.*

Proof. Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a neutrosophic CMS. Let ϵ_1, ϵ_2 be two distinct points of Σ. Then $0 < \Xi(\epsilon_1, \epsilon_2, m) < 1_\Theta$ and $0 < \Theta(\epsilon_1, \epsilon_2, m) < 1_\Theta$. Let $\Xi(\epsilon_1, \epsilon_2, m) = s_1$, $\Theta(\epsilon_1, \epsilon_2, m) = s_2$ and $s = max\{s_1, s_2\}$. Then for each $s_0 \in (s, 1)$, there exists s_3 and s_4 such that $s_3 \bigotimes s_3 \geq s_0$ and $(1_\Theta - s_4) \diamond (1_\Theta - s_4) \leq (1_\Theta - s_0)$. Put $s_4 = max\{s_3, s_4\}$ and consider the open balls $L(\epsilon_1, 1_\Theta - s_5, m/2)$ and $L(\epsilon_2, 1_\Theta - s_5, m/2)$.

Then clearly $L(x, 1_\Theta - s_5, m = 2) \cap L(\epsilon_2, 1 - s_5, m/2) = \emptyset$

. Suppose that $L(x, 1_\Theta - s_5, m = 2) \cap L(\epsilon_2, 1 - s_5, m/2) \neq \emptyset$. Then there exists $\epsilon_3 \in L(x, 1_\Theta - s_5, m = 2) \cap L(\epsilon_2, 1_\Theta - s_5, m/2)$.

$$\begin{aligned}
s_1 =& \Xi(\epsilon_1, \epsilon_2, m) \\
\geq& \Xi(\epsilon_1, \epsilon_3, m/2) \bigotimes \Xi(\epsilon_3, \epsilon_2, m/2) \\
\geq& s_5 \bigotimes s_5 \\
\geq& s_3 \bigotimes s_3 \\
\geq& s_0 > s_1
\end{aligned}$$

and

$$\begin{aligned}
s_2 =& n(\epsilon_1, \epsilon_2, m) \\
\geq& n(\epsilon_1, \epsilon_3, m/2) \bigotimes n(\epsilon_3, \epsilon_2, m/2) \\
\geq& (1_\Theta - s_5) \diamond (1_\Theta - s_5) \\
\geq& (1_\Theta - s_4) \diamond (1_\Theta - s_4) \\
\leq& 1_\Theta - s_0 < s_2
\end{aligned}$$

This is a contradiction. Hence $((\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is Hausdorff. □

Theorem 3.12. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$, $\epsilon_1 \in \Sigma$ and (ϵ_{1n}) a sequence in Σ. Then (ϵ_{1n}) converges to ϵ_1 if and only if $\Xi(\epsilon_{1n}, \epsilon_1, m) \to 1$ and $\Theta(\epsilon_{1n}, \epsilon_1, m) \to 0$ as $n \to 1_\Theta$, for each $m \gg 0_\Theta$.*

Proof. Let $(\epsilon_{1n}) \to \epsilon_1$. Then, for each $m \gg 0_\Theta$ and $s \in (0, 1)$, there exists a natural number n_0 such that $\Xi(\epsilon_{1n}, \epsilon_1, m) > 1_\Theta - s$, $\Theta(\epsilon_{1n}, \epsilon_1, m) < s$ for all $n \gg n_0$. We have $1 - \Xi(\epsilon_{1n}, \epsilon_1, m) < m$ and $\Xi(\epsilon_{1n}, \epsilon_1, m) < m$. Hence $\Xi(\epsilon_{1n}, \epsilon_1, m) \to 1$ and $\Theta(\epsilon_{1n}, \epsilon_1, m) \to 0$ as $n \to 1$. Conversely, Suppose that $\Xi(\epsilon_{1n}, \epsilon_1, m) \to 1_\Theta$ as $n \to 1_\Theta$. Then, for each $m \gg 0_\Theta$ and $s \in (0, 1)$, there exists a natural number n_0 such that $1_\Theta - \Xi(\epsilon_{1n}, \epsilon_1, m) < s$ and $\Theta(\epsilon_{1n}, \epsilon_1, m) < s$ for all $n \geq n_0$. In that case, $\Xi(\epsilon_{1n}, \epsilon_1, m) > 1_\Theta - s$ and $\Theta(\epsilon_{1n}, \epsilon_1, m) < s$ Hence $(\epsilon_{1n}) \to \epsilon_1$ as $n \to 1_\Theta$. □

4. **Neutrosophic Fixed Point Theorems**

Theorem 4.1. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete NCMS in which neutrosophic cone contractive sequences are Cauchy. Let $\mathcal{H}$ a neutrosophic cone contractive mapping. Then $\mathcal{H}$ has a unique fixed point. Where $\mathcal{H} : \Sigma \to \Sigma$ with c as the contractive constant.*

Proof. Let $\epsilon_1 \in \Sigma$ and fix $\epsilon_{1n} = \mathcal{H}^n(x), n \in \Theta$ For $m \gg 0_\Theta$, we have

$$\frac{1}{\Xi(\mathcal{H}(\epsilon_1), \mathcal{H}^2(\epsilon_1), m)} - 1_\Theta \leq c(\frac{1}{\Xi(\epsilon_1, \epsilon_{11}, m)} - 1_\Theta),$$

$$\Theta(\mathcal{H}(\epsilon_1), \mathcal{H}^2(\epsilon_1), m) \leq c\Theta(\epsilon_1, \epsilon_{11}, m).$$

And by induction

$$\frac{1}{\Xi(\epsilon_{1n+1}, \epsilon_{1n+2}, m)} - 1 \leq c(\frac{1}{\Xi(\epsilon_1, \epsilon_{1n+1}, m)} - 1)$$

,

$$\Theta(\epsilon_{1n+1}, \epsilon_{1n+2}, m) \leq c\Theta(\epsilon_1, \epsilon_{1n+1}, m) \text{ for all } n \in \Theta.$$

Then (ϵ_{1n}) is a neutrosophic contractive sequence, by assumptions (ϵ_{1n}) converges to ϵ_2 and it is a Cauchy sequence, for some $\epsilon_2 \in \Sigma$. By Theorem 3.12, we have

$$\frac{1}{\Xi(\mathcal{H}(\epsilon_2), \mathcal{H}(\epsilon_{1n}), m)} - 1 \leq c(\frac{1}{\Xi(\epsilon_2, \epsilon_{1n}, m)} - 1) \to 0$$

$$\Theta(\mathcal{H}(\epsilon_2), \mathcal{H}(\epsilon_{1n}), m) \leq c\Theta(\epsilon_2, \epsilon_{1n}, m) \to o$$

as $n \to 1$. Then for each $m \gg 0_\Theta$,

$$\lim_{n\to\infty} \Xi(\mathcal{H}(\epsilon_2), \mathcal{H}(\epsilon_{1n}), m) = 1, \lim_{n\to\infty} \Theta(\mathcal{H}(\epsilon_2), \mathcal{H}(\epsilon_{1n}), m) = 0_\Theta,$$

and hence $\lim_{n\to\infty}\mathcal{H}(\epsilon_{1n}) = \mathcal{H}(\epsilon_2)$, i.e.,$\lim_{n\to\infty}\epsilon_{1n+1} = \mathcal{H}(\epsilon_2)$ and $\mathcal{H}(\epsilon_2) = \epsilon_2$. To show uniqueness. Let $\mathcal{H}(kkk) = \epsilon_3$ for some $\epsilon_3 \in W$. For $m \gg 0_\Theta$, we have

$$\begin{aligned}\frac{1}{\Xi(\epsilon_2,\epsilon_3,m)} - 1 =&\frac{1}{\Xi(\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)} - 1\\ \leq&\quad c(\frac{1}{\Xi(\epsilon_2,\epsilon_3,m)} - 1)\\ =&c(\frac{1}{\Xi(\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)} - 1)\\ \leq&c^2(\frac{1}{\Xi(\epsilon_2,\epsilon_3,m)} - 1)\\ \leq&... \leq c^n(\frac{1}{\Xi(\epsilon_2,\epsilon_3,m)} - 1) \to 0\,\text{as}\,n \to \infty.\end{aligned} \tag{4.1}$$

$$\begin{aligned}\Theta(\epsilon_2,\epsilon_3,m) =&\Theta(\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)\\ \leq&c(\Theta(\epsilon_2,\epsilon_3,m)\\ =&c\Theta(\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)\\ \leq&c^2\Theta(\epsilon_2,\epsilon_3,m)\\ \leq&... \leq c^n\Theta(\epsilon_2,\epsilon_3,m) \to 0\,\text{as}\,n \to \infty.\end{aligned} \tag{4.2}$$

Hence $\Xi(\epsilon_2,\epsilon_3,m) = 1_\Theta$ and $\Theta(\epsilon_2,\epsilon_3,m) = 0_\Theta$ and $\epsilon_2 = \epsilon_3$. □

Theorem 4.2. *Let $(\Sigma,\Xi,\Theta,\bigotimes,\diamond)$ be a complete NCMS, for $\mathcal{G}$ be self mappings of Σ and let K, L, G. Let $\{K, G\}$ and $\{L,\mathcal{G}\}$ are pairs be sporadically weakly compatible. If there exists $c \in (0, 1)$ such that*

$$\begin{aligned}\Xi(K_{\epsilon_1}, L_{\epsilon_2}, c(m)) \geq min\{&\Xi(G(\epsilon_1),\mathcal{G}(\epsilon_2),m),\Xi(G(\epsilon_1),K(\epsilon_1),m)\\ &\Xi(L(\epsilon_2),\mathcal{G}(\epsilon_2),m),\Xi(K(\epsilon_1),\mathcal{G}(\epsilon_2),m),\Xi(L(\epsilon_2),G(\epsilon_1),m)\}.\end{aligned} \tag{4.3}$$

$$\begin{aligned}\Theta(K_{\epsilon_1}, L_{\epsilon_2}, c(m)) \leq max\{&\Theta(G(\epsilon_1),\mathcal{G}(\epsilon_2),r),\Theta(G(\epsilon_1),K(\epsilon_1),m)\\ &\Theta(L(\epsilon_2),\mathcal{G}(\epsilon_2),m),\Theta(K(\epsilon_1),\mathcal{G}(\epsilon_2),r),\Theta(L(\epsilon_2),G(\epsilon_1),m)\}.\end{aligned} \tag{4.4}$$

for all $\epsilon_1,\epsilon_2 \in \Sigma$ and for all $r \gg 0_\Theta$, there exists a unique point $z \in \Sigma$ such that $K(z) = G(z) = z$ and a unique point $y \in \Sigma$ such that $L(y) = \mathcal{G}(y) = y$. Moreover $y = z$, so that there is a unique common fixed point of K, L, G and $\mathcal{G}$.

Proof. Let the pairs $\{K, G\}$ and $\{L,\mathcal{G}\}$ be sporadically weakly compatible, so there are points $\epsilon_1,\epsilon_2 \in \Sigma$ such that $K(\epsilon_1) = G(\epsilon_1)$ and $L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. We claim that $K(\epsilon_1) = L(\epsilon_2)$. By inequality 4.3,

$$\begin{aligned}\Xi(K_{\epsilon_1}, L_{\epsilon_2}, c(m)) \geq & min\{\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(G(\epsilon_1), K(\epsilon_1), m),\\ & \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(L(\epsilon_2), G(\epsilon_1), m)\}\\ = & min\{\Xi(K(\epsilon_1), L(\epsilon_2), r), \Xi(K(\epsilon_1), K(\epsilon_1), m),\\ & \Xi(L(\epsilon_2), L(\epsilon_2), m), \Xi(K(\epsilon_1), L(\epsilon_2), r), L(L(\epsilon_2), K(\epsilon_1), m)\}\\ = & \Xi(K_{\epsilon_1}, L_{\epsilon_2}, m). \qquad (4.5)\end{aligned}$$

$$\begin{aligned}\Theta(K_{\epsilon_1}, L_{\epsilon_2}, c(m)) \leq & max\{\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(G(\epsilon_1), K(\epsilon_1), m),\\ & \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(L(\epsilon_2), G(\epsilon_1), m)\}\\ = \quad & max\{\Theta(K(\epsilon_1), L(\epsilon_2), m), \Theta(K(\epsilon_1), K(\epsilon_1), m),\\ & \Theta(L(\epsilon_2), L(\epsilon_2), m), \Theta(K(\epsilon_1), L(\epsilon_2), m), \Theta(L(\epsilon_2), K(\epsilon_1), m)\}\\ = \quad & \Theta(K_{\epsilon_1}, L_{\epsilon_2}, m). \qquad (4.6)\end{aligned}$$

By Lemma 3.9, $K(\epsilon_1) = L(\epsilon_2)$, i.e. $K(\epsilon_1) = L(\epsilon_1) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Suppose that there is another point y such that $K(y) = G(y)$ and by 4.3, we have $K(y) = G(y) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Thus $K(\epsilon_1) = K(y)$ and $z = K(\epsilon_1) = G(\epsilon_1)$ is the unique point of coincidence of K and G. By Lemma 2.2, z is the unique common fixed point of K and G. Similarly there is a only point $y \in \Sigma$ such that $y = L(y) = \mathcal{G}(y)$. Assume that $z \neq y$, we have

$$\begin{aligned}\Xi(z, y, c(m)) = & \Xi(K(z), L(y), c(m))\\ \geq & min\{\Xi(G(z), \mathcal{G}(y), r), \Xi(G(z), K(y), m), \Xi(L(y), \mathcal{G}(y), m)\\ & \Xi(K(z), \mathcal{G}(y), m), \Xi(L(y), G(z), m)\}\\ = & min\{\Xi(z, y, m), \Xi(z, y, m), \Xi(y, y, m), \Xi(z, y, m), \Xi(y, z, m)\}\\ = & \Xi(z, y, m). \qquad (4.7)\end{aligned}$$

$$\begin{aligned}\Theta(z, y, c(r)) = & \Theta(K(z), L(y), c(m))\\ \geq & min\{\Theta(G(z), \mathcal{G}(y), m), \Theta(G(z), K(y), m), \Theta(L(y), \mathcal{G}(y), m)\\ & \Theta(K(z), \mathcal{G}(y), r), \Theta(L(y), G(z), m)\}\\ = & min\{\Theta(z, y, m), \Theta(z, y, m), \Theta(y, y, m), \Theta(z, y, m), \Theta(y, z, m)\}\\ = & \Theta(z, y, m). \qquad (4.8)\end{aligned}$$

by Lemma 2.2 and y is a common fixed point of K, L, G and $\mathcal{G}$. Then we have $y = z$. The uniqueness of the fixed point come from 4.6. □

Theorem 4.3. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete $NCMS$ and K, L, G and $\mathcal{G}$ be self-mappings of Σ. Let the pairs $\{K, G\}$ and $\{L, \mathcal{G}\}$ be sporadically weakly compatible. If there exists $c \in (0,1)$ such that*

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \phi[min\{&\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(G(\epsilon_1), K(\epsilon_1), m)\\ &\Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(L(\epsilon_2), G(\epsilon_1), m)\}].\end{aligned} \tag{4.9}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq \zeta[max\{&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(G(\epsilon_1), K(\epsilon_1), m)\\ &\Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(L(\epsilon_2), G(\epsilon_1), m)\}].\end{aligned} \tag{4.10}$$

for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $\phi, \zeta : \rfloor 0^-, 1^+ \lfloor \to \rfloor 0^-, 1^+ \lfloor$ such that $\zeta(m) < m$, $\phi(m) > m$, for all $0_\Theta \ll r < 1_\Theta$, thus there is a unique common fixed point of K, L, G and $\mathcal{G}$.

Proof. The proof follows from Theorem 4.4 □

Theorem 4.4. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete $NCMS$ and K, L, G and $\mathcal{G}$ be self-mappings of Σ. Let $\{K, G\}$ and $\{L, \mathcal{G}\}$ are pairs be sporadically weakly compatible. If $\exists c \in (0,1)$ such that*

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \phi(&\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(G(\epsilon_1), K(\epsilon_1), m)\\ &\Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(L(\epsilon_2), G(\epsilon_1), m)),\end{aligned} \tag{4.11}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq \zeta(&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(G(\epsilon_1), K(\epsilon_1), m)\\ &\Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(L(\epsilon_2), G(\epsilon_1), m)).\end{aligned} \tag{4.12}$$

for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $\phi, \zeta : \rfloor 0^-, 1^{+^5} \lfloor \to \rfloor 0^-, 1^+ \lfloor$ such that $\phi(r, 1_\Theta, 1_\Theta, m, m) > m$, $\zeta(m, 0_\Theta, 0_\Theta, m, m) < m$ for all $0 \ll m < 1$ then there exists a unique common fixed point of K, L, G and $\mathcal{G}$.

Proof. Let $\{K, G\}$ and $\{L, \mathcal{G}\}$ are pairs be sporadically weakly compatible. There are points $\epsilon_1, \epsilon_2 \in \Sigma$ such that $K(\epsilon_1) = G(\epsilon_1)$ and $L(\epsilon_2) = \mathcal{G}(\epsilon_2)$.
We claim that $K(\epsilon_1) = L(\epsilon_2)$. By inequalities (4.11) and (4.12), we have

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq{} & \phi(\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(G(\epsilon_1), K(\epsilon_1), m),\\ & \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), mr), \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(L(\epsilon_2), G(\epsilon_1), m))\\ ={} & \phi(\Xi(K(\epsilon_1), L(\epsilon_2), m), \Xi(K(\epsilon_1), K(\epsilon_1), m),\\ & \Xi(L(\epsilon_2), L(\epsilon_2), m), \Xi(K(\epsilon_1), L(\epsilon_2), r), L(L(\epsilon_2), K(\epsilon_1), m))\\ ={} & \phi((\Xi(K(\epsilon_1), L(\epsilon_2), m), 1_\Theta, 1_\Theta, \Xi(K(\epsilon_1), L(\epsilon_1), m), \Xi(L(\epsilon_2), K(\epsilon_2), m))\\ >{} & \Xi(K(\epsilon_1), L(\epsilon_2), m).\end{aligned}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq \zeta(&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(G(\epsilon_1), K(\epsilon_1), m),\\ &\Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(L(\epsilon_2), G(\epsilon_1), m))\\ =&\zeta(\Theta(K(\epsilon_1), L(\epsilon_2), m), \Theta(K(\epsilon_1), K(\epsilon_1), m),\\ &\Theta(L(\epsilon_2), L(\epsilon_2), m), \Theta(K(\epsilon_1), L(\epsilon_2), m), L(L(\epsilon_2), K(\epsilon_1), m))\\ =&\zeta((\Theta(K(\epsilon_1), L(\epsilon_2), m), 0_\Theta, 0_\Theta, \Theta(K(\epsilon_1), L(\epsilon_1), m), \Theta(L(\epsilon_2), K(\epsilon_2), m))\\ <&\Theta(K(\epsilon_1), L(\epsilon_2), m).\end{aligned}$$

a contradiction, therefore $K(\epsilon_1) = L(\epsilon_2)$, i.e. $K(\epsilon_1) = G(\epsilon_1) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Suppose that there is a another point y such that $K(y) = G(y)$. Then by 4.11 we have $K(y) = G(y) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$, so $K(\epsilon_1) = K(y)$ and $z = K(\epsilon_1) = \mathcal{G}(\epsilon_1)$ is the unique point of coincidence. z is a unique common fixed point of K and G, by Lemma 2.2. Similarly, for K and G there is a unique point $y \in \Sigma$ such that $y = L(y) = \mathcal{G}(y)$. Thus for K, L, G, y is a common fixed point and $\mathcal{G}$. For the uniqueness fixed point holds from (4.11). □

Theorem 4.5. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete NCMS and K, L, G and $\mathcal{G}$ be self-mappings of Σ. Let the pairs $\{K, G\}$ and $\{L, \mathcal{G}\}$ be sporadically weakly compatible. If there exists $c \in (0, 1)$ for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $m \gg 0_\Theta$ satisfying*

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m)\\ \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned} \tag{4.13}$$

$$\begin{aligned}\Xi\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Theta(K(\epsilon_1), G(\epsilon_1), m)\\ \bigotimes \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \bigotimes \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned} \tag{4.14}$$

then there exists a unique common fixed point of K, L, G and $\mathcal{G}$.

Proof. Let the pairs $\{K, G\}$ and $\{L, \mathcal{G}\}$ are sporadicallyweakly compatible, there are points $\epsilon_1, \epsilon_2 \in \Sigma$ such that $K(\epsilon_1) = G(\epsilon_1)$ and $L(\epsilon_2) = \mathcal{G}(\epsilon_2)$.
We claim that $K(\epsilon_1) = L(\epsilon_2)$. By inequalities (4.13) and (4.14), we have

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \Xi(&G(\epsilon_1), L(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m)\\ &\bigotimes \Xi(L(\epsilon_2), L(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), L(\epsilon_2), m)\\ =&\Xi(K(\epsilon_1), L(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), K(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), L(\epsilon_2), m)\\ &\bigotimes \Xi(K(\epsilon_1), L(\epsilon_2), m)\\ \geq&\Xi(K(\epsilon_1), L(\epsilon_2), m) \bigotimes 1_\Theta \bigotimes 1_\Theta \bigotimes \Xi(K(\epsilon_1), L(\epsilon_2), m)\\ \geq&\Xi(K(\epsilon_1), L(\epsilon_2), m)\end{aligned}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq\ & \Theta(G(\epsilon_1), L(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), G(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), L(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), L(\epsilon_2), m)\\ =&\Theta(K(\epsilon_1), L(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), K(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), L(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), L(\epsilon_2), m)\\ \leq&\Theta(K(\epsilon_1), L(\epsilon_2), m) \diamond 0_\Theta \diamond 0_\Theta \diamond \Theta(K(\epsilon_1), L(\epsilon_2), m)\\ \leq&\Theta(K(\epsilon_1), L(\epsilon_2), m)\end{aligned}$$

By Lemma 3.9, we have $K(\epsilon_1) = L(\epsilon_2)$, i.e. $K(\epsilon_1) = G(\epsilon_1) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Suppose that there is a another point y such that $K(y) = G(y)$. Then by (4.13, 4.14), we have $K(y) = G(y) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Thus $K(\epsilon_1) = K(y)$ and $z = K(\epsilon_1) = G(\epsilon_1)$ is the unique point of coincidence of K and G. Then there is a unique point $y \in \Sigma$ such that $y = L(y) = \mathcal{G}(y)$. Thus z is a common fixed point of K, L, G and $\mathcal{G}$. □

Theorem 4.6. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete neutrosophic CMS and $\mathcal{G}$ and K, L, G be self-mappings of Σ. Let $\{K, G\}$ and $\{L, \mathcal{G}\}$ are the pairs be sporadically weakly compatible. If $\exists c \in (0, 1)$ for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $r \gg 0_\Theta$ satisfying*

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq\ & \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \bigotimes \Xi(L(\epsilon_2), G(\epsilon_2), 2m) \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned} \tag{4.15}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq\ & \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), r) \bigotimes \Theta(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \bigotimes \Theta(L(\epsilon_2), G(\epsilon_2), 2m) \bigotimes \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned} \tag{4.16}$$

then for K, L, G and $\mathcal{G}$ there exists a unique common fixed point.

Proof. We have,

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq\ & \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \bigotimes \Xi(L(\epsilon_2), G(\epsilon_2), 2m) \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\\ =&\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \bigotimes \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_1), m) \bigotimes \Xi(\mathcal{H}(\epsilon_1), L(\epsilon_1), m) \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\\ \geq&\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq\ & \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), G(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \diamond \Theta(L(\epsilon_2), G(\epsilon_2), 2m) \diamond \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\\ =&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), G(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m)\\ & \diamond \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_1), m) \diamond \Theta(\mathcal{H}(\epsilon_1), L(\epsilon_1), m) \diamond \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\\ \leq&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), G(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned}$$

and therefore by Theorem 4.5, K, L, G and $\mathcal{G}$ have a common fixed point. □

Theorem 4.7. *Let* $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ *be a complete neutrosophic* CMS *and* K, L *be self-mappings of* Σ. *Let* K *and* L *be sporadically weakly compatible. If* $\exists$ *a point* $c \in (0,1)$ *for all* $\epsilon_1, \epsilon_2 \in \Sigma$ *and* $r \gg 0_\Theta$

$$\begin{aligned}\Xi(L(\epsilon_1), L(\epsilon_2), c(m)) \geq a\,\Xi(K(\epsilon_1), K(\epsilon_2), m) + b\,min\{\Xi(K(\epsilon_1), K(\epsilon_2), m),\\ \Xi(L(\epsilon_1), K(\epsilon_1), m), \Xi(L(\epsilon_2), K(\epsilon_2), m)\}\end{aligned} \tag{4.17}$$

$$\begin{aligned}\Theta(L(\epsilon_1), L(\epsilon_2), c(m)) \leq a\,\Theta(K(\epsilon_1), K(\epsilon_2), m) + b\,max\{\Theta(K(\epsilon_1), K(\epsilon_2), m),\\ \Theta(L(\epsilon_1), K(\epsilon_1), m), \Theta(L(\epsilon_2), K(\epsilon_2), m)\}\end{aligned} \tag{4.18}$$

for all $\epsilon_1, \epsilon_2 \in \Sigma$, *where* $a, b > 0_\Theta$, $a + b > 1_\Theta$. *Then* K *and* L *have a unique common fixed point.*

Proof. Let the pairs $\{K, L\}$ be sporadicallyweakly compatible, so there is a point $\epsilon_1 \in \Sigma$ such that $K(\epsilon_1) = L(\epsilon_1)$. Suppose that there exists another point $\epsilon_2 \in \Sigma$ for which $K(\epsilon_2) = L(\epsilon_2)$. We claim that $G(\epsilon_1) = L(\epsilon_2)$. By inequalities (4.17) and (4.18), we have

$$\begin{aligned}\Xi(L(\epsilon_1), L(\epsilon_2), c(m)) \geq\ & a\,\Xi(K(\epsilon_1), K(\epsilon_2), m) + b\,min\{\Xi(K(\epsilon_1), K(\epsilon_2), m),\\ & \Xi(L(\epsilon_1), K(\epsilon_1), r), \Xi(L(\epsilon_2), K(\epsilon_2), m)\}\\ =& a\Xi(L(\epsilon_1), L(\epsilon_2), m) + b\,min\{\Xi(L(\epsilon_1), L(\epsilon_2), m),\\ & \Xi(L(\epsilon_1), L(\epsilon_1), m), \Xi(L(\epsilon_2), L(\epsilon_2), m), \}\\ =& a + b\Xi(L(\epsilon_1), L(\epsilon_2), m)\\ \Theta(L(\epsilon_1), L(\epsilon_2), c(m)) \leq\ & a\,\Theta(K(\epsilon_1), K(\epsilon_2), m) + b\,max\{\Theta(K(\epsilon_1), K(\epsilon_2), m),\\ & \Theta(L(\epsilon_1), K(\epsilon_1), m), \Theta(L(\epsilon_2), K(\epsilon_2), r)\}\\ =& a\Theta(L(\epsilon_1), L(\epsilon_2), m) + b\,max\{\Theta(L(\epsilon_1), L(\epsilon_2), m),\\ & \Theta(L(\epsilon_1), L(\epsilon_1), m), \Theta(L(\epsilon_2), L(\epsilon_2), m), \}\\ =& a + b\Theta(L(\epsilon_1), L(\epsilon_2), m)\end{aligned}$$

a contradiction, since $a + b > 1_\Theta$. Therefore $L(\epsilon_1) = L(\epsilon_2)$. Therefore $K(\epsilon_1) = K(\epsilon_2)$ and $K(\epsilon_1)$ is unique. From Lemma 2.2, K and L have a unique fixed point. □

5. **Conclusion**

In this paper, the concept of neutrosophic CMS is introduced. Some fixed point theorems on neutrosophic CMS are stated and proved.

6. **Conflict of Interests**

Regarding this manuscript, the authors declare that there is no conflict of interests.

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Comment on "A Novel Method for Solving the Fully Neutrosophic Linear Programming Problems: Suggested Modifications"

Mohamed Abdel-Basset, Mai Mohamed, Florentin Smarandache

Mohamed Abdel-Basset, Mai Mohamed, Florentin Smarandache (2020). Comment on "A Novel Method for Solving the Fully Neutrosophic Linear Programming Problems: Suggested Modifications". *Neutrosophic Sets and Systems* 31, 305-309

Abstract. Some clarifications of a previous paper with the same title are presented here to avoid any reading conflict [1]. Also, corrections of some typo errors are underlined. Each modification is explained with details for making the reader able to understand the main concept of the paper. Also, some suggested modifications advanced by Singh et al. [3] (Journal of Intelligent & Fuzzy Systems, 2019, DOI:10.3233/JIFS-181541) are discussed. It is observed that Singh et al. [3] have constructed their modifications on several mathematically incorrect assumptions. Consequently, the reader must consider only the modifications which are presented in this research.

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1. Clarifications and Corrected Errors

In Section 5 and Step 3 of the proposed NLP method [1], the trapezoidal neutrosophic number was presented in the following form:

$\tilde{a}=\langle(a^l, a^{m1}, a^{m2}, a^u); T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}}\rangle$,

where a^l, a^{m1}, a^{m2}, a^u are the lower bound, the first and second median values and the upper bound for trapezoidal neutrosophic number, respectively. Also, $T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}}$ are the truth, indeterminacy and falsity degrees of the trapezoidal neutrosophic number. The ranking function for that trapezoidal neutrosophic number is as follows:

$$R(\tilde{a}) = \left|\left(\frac{-\frac{1}{3}(3a^l-9a^u)+2(a^{m1}-a^{m2})}{2}\right) \times (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})\right| \quad (8)$$

The previous ranking function is only for maximization problems.

But, if NLP problem is a minimization problem, then ranking function for that trapezoidal neutrosophic number is as follows:

$$R(\tilde{a}) = \left|\left(\frac{(a^l+a^u)-3(a^{m1}+a^{m2})}{-4}\right) \times (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})\right| \quad (9)$$

If reader deals with a symmetric trapezoidal neutrosophic number which has the following form:

$\tilde{a}=\langle(a^{m1}, a^{m2}); \alpha, \beta\rangle$,

where $\alpha = \beta,\ \alpha, \beta \geq 0$, then the ranking function for that number will be as follows:

$$R(\tilde{a}) = \left|\left(\frac{(a^{m1}+a^{m2})+2(\alpha+\beta)}{2}\right) \times (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})\right|. \quad (10)$$

We applied Eq. (10) directly in Example 1, but we did not illustrated it in the original work [1], and this caused a reading conflict. After handling typo errors in Example 1, the crisp model of the problem will be as follows:

Maximize $Z = 18x_1 + 19x_2 + 20x_3$

Subject to

$12x_1+13x_2+12x_3 \leq 502$,

$14x_1+13x_3 \leq 486$,

$12x_1+15x_2 \leq 490$,

$x_1, x_2, x_3 \geq 0$.

The initial simplex form will be as in Table 1.

Table 1 Initial simplex form

Basic variables	x_1	x_2	x_3	s_4	s_5	s_6	RHS
s_4	12	13	12	1	0	0	502
s_5	14	0	13	0	1	0	486
s_6	12	15	0	0	0	1	490
Z	-18	-19	-20	0	0	0	0

The optimal simplex form will be as in Table 2.

Table 2 Optimal form

Basic variables	x_1	x_2	x_3	s_4	s_5	s_6	RHS
x_2	-12/169	1	0	1/13	-12/169	0	694/169
x_3	14/13	0	1	0	1/13	0	486/13
s_6	2208/169	0	0	-15/13	180/169	1	72400/169
Z	370/169	0	0	19/13	32/169	0	139546/169

The obtained optimal solution is $x_1 = 0, x_2 = 4.11, x_3 = 37.38$.
The optimal value of the NLPP is $\tilde{z} \approx (13,15,2,2)x_1 + (12,14,3,3)x_2 + (15,17,2,2)x_3 = (13,15,2,2) * 0 + (12,14,3,3) * 4.11 + (15,17,2,2) * 37.38 =$
$(49.32,57.54,12.33,12.33) + (560.70,635.46,74.76,74.7) = (610.02,693,87.09,87.09)$.
$\tilde{z} \approx (610.02,693,87.09,87.09)$, which is in the symmetric trapezoidal neutrosophic number form. Since the traditional form of $\tilde{a} = \langle (a^{m1}, a^{m2}); \alpha, \beta \rangle$ is:
$\tilde{a} = \langle (a^{m1} - \alpha, a^{m1}, a^{m2}, a^{m2} + \beta) \rangle$,
where $a^{m1} - \alpha = a^l$, $a^{m2} + \beta = a^u$, then the optimal value of the NLPP can also be written as $\tilde{z} \approx (522.93,610.02,693,780.09)$.

The reader must also note that one can transform the symmetric trapezoidal neutrosophic numbers from Example 1 in [1] to its traditional form, and use Eq. (8) for solving the problem, obtaining the same result. By comparing the result with other existing models mentioned in the original research [1], the proposed model is the best.
By using Eq. (8) and solving Example 2 in [1], the crisp model will be as follows:
Maximize $Z = 25x_1 + 48x_2$
Subject to
$13x_1 + 28x_2 \leq 31559$,
$26x_1 + 9x_3 \leq 16835$,
$21x_1 + 15x_2 \leq 19624$,
$x_1, x_2 \geq 0$.
The initial simplex form will be as in Table 3.

Table 3 Initial simplex form

Basic variables	x_1	x_2	s_3	s_4	s_5	RHS
s_3	13	28	1	0	0	31559
s_4	26	9	0	1	0	16835
s_5	21	15	0	0	1	19624
Z	-25	-48	0	0	0	0

The optimal simplex form will be as in Table 4.

Table 4 Optimal simplex form

Basic variables	x_1	x_2	s_3	s_4	s_5	RHS
x_2	0	1	7/131	0	-13/393	407627/393
s_4	0	0	67/131	1	-611/393	969250/393
x_1	1	0	-5/131	0	28/393	76087/393
Z	0	0	211/131	0	76/393	21468271/393

The optimal value of objective function is 54627.
By using Eq. (9) and solving Example 3 in [1], the crisp model will be as follows:
Minimize $Z = 6x_1 + 10x_2$
Subject to
$2x_1 + 5x_2 \geq 6$,
$3x_1 + 4x_2 \geq 3$,
$x_1, x_2 \geq 0$.
The optimal simplex form will be as in Table 5.

Table 5 Optimal simplex form

Basic variables	x_1	x_2	s_3	s_4	RHS
s_4	-7/5	0	-4/5	1	0
x_2	2/5	1	-1/5	0	10
Z	-2	0	-2	0	12

Hence, the optimal solution has the value of variables:
$x_1 = 0, x_2 = 1.2$, Z = 12.
The obtained result is better than Saati et al. [2] method.
By correcting typo errors which percolated in the Case study in [1], the problem formulation model will be as follows:
Maximize $\tilde{Z} = \tilde{9}x_1 + \widetilde{12}x_2 + \widetilde{15}x_3 + \widetilde{11}x_4$
Subject to

$0.5x_1 + 1.5x_2 + 1.5x_3 + x_4 \leq \widetilde{1500}$,
$3x_1 + x_2 + 2x_3 + 3x_4 \leq \widetilde{2350}$,
$2x_1 + 4x_2 + x_3 + 2x_4 \leq \widetilde{2600}$,
$0.5x_1 + 1x_2 + 0.5x_3 + 0.5x_4 \leq \widetilde{1200}$,
$x_1 \leq \widetilde{150}$,
$x_2 \leq \widetilde{100}$,
$x_3 \leq \widetilde{300}$,
$x_4 \leq \widetilde{400}$,
$x_1, x_2, x_3, x_4 \geq 0$.

The values of each trapezoidal neutrosophic number remain the same [1].
By using Eq. (8) and solving the Case study, the crisp model will be as follows:
Maximize $\tilde{Z} = 10x_1 + 10x_2 + 12x_3 + 9x_4$
Subject to

$0.5x_1 + 1.5x_2 + 1.5x_3 + x_4 \leq 1225$,
$3x_1 + x_2 + 2x_3 + 3x_4 \leq 1680$,
$2x_1 + 4x_2 + x_3 + 2x_4 \leq 2030$,
$0.5x_1 + 1x_2 + 0.5x_3 + 0.5x_4 \leq 945$,
$x_1 \leq 122$,
$x_2 \leq 87$,
$x_3 \leq 227$,
$x_4 \leq 297$,
$x_1, x_2, x_3, x_4 \geq 0$.

By solving the previous model using simplex approach, the results are as follows:
$x_1 = 122, x_2 = 87,\ x_3 = 227, x_4 = \frac{773}{3},\ Z = 7133.$

2. A Note on the modifications suggested by Singh et al. [3]

This part illustrates how Singh et al. [3] constructed their modifications of Abdel-Basset et al.'s method [1] on wrong concepts. The errors in Singh et al.'s [3] modifications reflects the misunderstanding of Abdel-Basset et al.'s method [1].

In the second paragraph of the introductory section, Singh et al. [3] assert that "in Abdel-Basset et al.'s method [1], firstly, a neutrosophic linear programming problem (NLPP) is transformed into a crisp linear programming problem (LPP) by replacing each parameter of the NLPP, represented by a trapezoidal neutrosophic number with its equivalent defuzzified crisp value". However, this is not true, since the neutrosophic linear programming problem (NLPP) is transformed into a crisp linear programming problem (LPP) by replacing each parameter of the NLPP, represented by a trapezoidal neutrosophic number with its equivalent deneutrosophic crisp value. The deneutrosophication process means transforming a neutrosophic value to its equivalent crisp value. In Section 2, Step 1 Singh et al. [3] alleged that Abdel-Basset et al.'s method [1] for comparing two trapezoidal neutrosophic numbers is based on maximization and minimization of problem, which is again not true.
In Section 3 and Definition 4, Abdel-Basset et al. [1] illustrated that the method for comparing two trapezoidal neutrosophic numbers is as follows:

1. If $R(\tilde{A}) > R(\tilde{B})$ then $\tilde{A} > \tilde{B}$,
2. If $R(\tilde{A}) < R(\tilde{B})$ then $\tilde{A} < \tilde{B}$,
3. If $R(\tilde{A}) = R(\tilde{B})$ then $\tilde{A} = \tilde{B}$.

There is well known that if $a^l = a^{m1} = a^{m2} = a^u$, then the trapezoidal number $\tilde{a}$=$\langle (a^l, a^{m1}, a^{m2}, a^u\ ); 1\ ,0, 0\rangle$ will be transformed into a real number $a = \langle (a, a, a, a\ ); 1\ ,0, 0\rangle$, and hence in this case $R(a) = a$. We presented this fact to illustrate a great error in the suggested modifications of Singh et al. [3]
In the Suggested modifications section [3], the authors claimed that:

$$R\left(\sum_{i=1}^{m} \langle a_i^l, a_i^{m1}, a_i^{m2}, a_i^u, T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle\right) = \sum_{i=1}^{m} R\,\langle a_i^l, a_i^{m1}, a_i^{m2}, a_i^u, T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle - \sum_{i=1}^{m} T_{\tilde{a}_i}$$
$$+ \textstyle\sum_{i=1}^{m} I_{\tilde{a}_i} + \sum_{i=1}^{m} F_{\tilde{a}_i} + min_{1\le j\le n}\{T_{\tilde{c}_i}\} - max_{1\le j\le n}\{I_{\tilde{c}_i}\} - max_{1\le j\le n}\{F_{\tilde{c}_i}\} \qquad (11)$$

instead of ,
$R\left(\sum_{i=1}^{m} \langle a_i^l, a_i^{m1}, a_i^{m2}, a_i^u, T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle\right)$ $=\sum_{i=1}^{m} R\,\langle a_i^l, a_i^{m1}, a_i^{m2}, a_i^u, T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle$.

Let us consider the following example for proving the error in this suggestion [3]
Let $m = 3$, which are three trapezoidal neutrosophic numbers $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$; since $\tilde{a}_1$=$\langle (1, 1,1,1\ ); 1\ ,0, 0\rangle$, $\tilde{a}_2$ $= \langle (2, 2,2,2\ ); 1\ ,0, 0\rangle$, $\tilde{a}_3$= $\langle (3, 3,3,3\ ); 1\ ,0, 0\rangle$, then,
$R\left(\sum_{i=1}^{m} \langle a_i^l, a_i^{m1}, a_i^{m2}, a_i^u, T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle\right) = R(\langle (1, 1,1,1\ ); 1\ ,0, 0\rangle + \langle (2, 2,2,2\ ); 1\ ,0, 0\rangle + \langle (3, 3,3,3\ ); 1\ ,0, 0\rangle)$ $=\ R(\langle (6, 6,6,6\ ); 1\ ,0, 0\rangle)$, and according to the previously determined fact "if $a^l\ =\ a^{m1} = a^{m2} = a^u$ then the trapezoidal number $\tilde{a} = \langle (a^l, a^{m1}, a^{m2}, a^u\ ); 1\ ,0, 0\rangle$ will be transformed into a real number $a = \langle (a, a, a, a\ ); 1\ ,0, 0\rangle$ and hence in this case $R(a) = a$ ", the value of $R(\langle (6, 6,6,6\ ); 1\ ,0, 0\rangle) = 6$.

And by calculating the right hand side of Eq. (11), which is $\sum_{i=1}^{m} R\,\langle a_i^l, a_i^{m1}, a_i^{m2}, a_i^u, T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle -$ $\sum_{i=1}^{m} T_{\tilde{a}_i} + \sum_{i=1}^{m} I_{\tilde{a}_i} + \sum_{i=1}^{m} F_{\tilde{a}_i}$+$min_{1\le j\le n}\{T_{\tilde{c}_i}\} - max_{1\le j\le n}\{I_{\tilde{c}_i}\} - max_{1\le j\le n}\{F_{\tilde{c}_i}\}$, we note that,
$R\langle (1, 1,1,1\ ); 1\ ,0, 0\rangle + R\langle (2, 2,2,2\ ); 1\ ,0, 0\rangle + R\langle (3, 3,3,3\ ); 1\ ,0, 0\rangle - 3 + 0 + 0 + 1 - 0 - 0 = \quad 1 + 2 +$ $3 - 3 + 0 + 0 + 1 - 0 - 0 = 4.$

And then, the left hand side of Eq. (11) does not equal the right hand side, i.e. $6 \neq 4$.
Consequently, the authors [3] built their suggestions on a wrong concept.

Beside Eq. (11), the authors [3] used the expressions $R(a) = 3a + 1$ for maximization problems, and $R(a) = -2a + 1$ for minimization problems, and this shows peremptorily that their assumptions are scientifically incorrect.

There is also a repeated error in all corrected solutions suggested by Singh et al. [3] which contradicts with the basic operations of trapezoidal neutrosophic numbers. This error is iterated in Section 7, as in Example 1, in Step 6. Singh et al. [3] illustrated that the optimal value of the NLPP is calculated using the optimal solution obtained in Step 5 as follows:
$(11,13,15,17)x_1 + (9,12,14,17)x_2 + (13,15,17,19)x_3 = (11,13,15,17) * 0 + (9,12,14,17) * 0 + (13,15,17,19) * (\frac{245}{18}) = 13\,(\frac{245}{18}) + 15\,(\frac{245}{18}) + 17\,(\frac{245}{18}) + 19\,(\frac{245}{18}) = \frac{7840}{9}$, and because the basic operation of multiplying trapezoidal neutrosophic number by a constant value is as follows:
$\gamma\tilde{a} = \begin{cases} \langle(\gamma a_1, \gamma a_2, \gamma a_3, \gamma a_4); \mathrm{T}_{\tilde{a}}, \mathrm{I}_{\tilde{a}}, \mathrm{F}_{\tilde{a}}\rangle & if\,(\gamma \geq 0) \\ \langle(\gamma a_4, \gamma a_3, \gamma a_2, \gamma a_1); \mathrm{T}_{\tilde{a}}, \mathrm{I}_{\tilde{a}}, \mathrm{F}_{\tilde{a}}\rangle & if\,(\gamma < 0) \end{cases}$, then the value of $(11,13,15,17) * 0 + (9,12,14,17) * 0 + (13,15,17,19) * (\frac{245}{18}) = \left(\frac{3185}{18}, \frac{1225}{6}, \frac{4165}{18}, \frac{4655}{18}; 1,0,0\right)$.Then the optimal value of the NLPP is $\tilde{z} \approx = \left(\frac{3185}{18}, \frac{1225}{6}, \frac{4165}{18}, \frac{4655}{18}\right)$.

The same error appears in Example 4, where the optimal value of the NLPP is calculated by Singh et al. [3] using the optimal solution obtained in Step 5 as follows:
$(6,8,9,12)x_1(9,10,12,14)x_2 + (12,13,15,17)x_3 + (8,9,11,13)x_4 = (6,8,9,12)(\frac{3700}{21}) + (9,10,12,14)(0) + (12,13,15,17)(\frac{6200}{7}) + (8,9,11,13)(0) = 6(\frac{3700}{21}) + 8\left(\frac{3700}{21}\right) + 9\left(\frac{3700}{21}\right) + 12(\frac{3700}{21}) + 12\left(\frac{6200}{7}\right) + 13\,(\frac{6200}{7}) + 15(\frac{6200}{7}) + 17(\frac{6200}{7}) = \frac{1189700}{21}$, which is scientifically incorrect and reflects only the weak background of the authors in the neutrosophic field.
Therefore, we concluded that it is scientifically incorrect to use Singh et al.'s modifications [3].

3. Conclusions

Clarifications and corrections of some typo errors are presented here to avoid any reading conflict. Also, the correct results of NLPPs are presented. By using three modified functions for ranking process which were presented by Abdel-Basset et al. [1], the reader will be able to solve all types of linear programming problems with trapezoidal and symmetric trapezoidal neutrosophic numbers. Also, the mathematically incorrect assumptions used by Singh et al. [3] are discussed and rejected.

Conflict of interest

We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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Neutrosophic Soft Sets Applied on Incomplete Data

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Abstract: A neutrosophic set is a part of neutrosophy that studies the origin, nature and scope of neutralities as well as their interactions with different ideational spectra. In this present paper first we have introduced the concept of a neutrosophic soft set having incomplete data with suitable examples. Then we have tried to explain the consistent and inconsistent association between the parameters. We have introduced few new definitions, namely- consistent association number between the parameters, consistent association degree, inconsistent association number between the parameters and inconsistent association degree to measure these associations. Lastly we have presented a data filling algorithm. An illustrative example is employed to show the feasibility and validity of our algorithm in practical situation.

Keywords: Soft set, neutrosophic set, neutrosophic soft set, data filling.

1. Introduction

In 1999, Molodstov **[01]** initiated the concept of soft set theory as a new mathematical tool for modelling uncertainty, vague concepts and not clearly defined objects. Although various traditional tools, including but not limited to rough set theory **[02]**, fuzzy set theory **[03]**, intuitionistic fuzzy set theory **[04]** etc. have been used by many researchers to extract useful information hidden in the uncertain data, but there are immanent complications connected with each of these theories. Additionally, all these approaches lack in parameterizations of the tools and hence they couldn't be applied effectively in real life problems, especially in areas like environmental, economic and social problems. Soft set theory is standing uniquely in the sense that it is free from the above mentioned impediments and obliges approximate illustration of an object from the beginning, which makes this theory a natural mathematical formalism for approximate reasoning.

The Theory of soft set has excellent potential for application in various directions some of which are reported by Molodtsov in his pioneer work. Later on Maji et al. **[05]** introduced some new annotations on soft sets such as subset, complement, union and intersection of soft sets and discussed in detail its applications in decision making problems. Ali et al. **[06]** defined some new operations on soft sets and shown that De Morgan's laws holds in soft set theory with respect to these newly defined operations. Atkas and Cagman **[07]** compared soft sets with fuzzy sets and rough sets to show that every fuzzy set and every rough set may be considered as a soft set. Jun **[08]** connected soft sets to the theory of BCK/BCI-algebra and introduced the concept of soft BCK/BCI-algebras. Feng et al. **[09]** characterized soft semi rings and a few related notions to establish a relation between soft sets and semi rings. In 2001, Maji et al. **[10]** defined the concept of fuzzy soft set by combining of fuzzy sets and soft sets . Roy and Maji **[11]** proposed a fuzzy soft set based decision making method. Xiao et al. **[12]** presented a combined forecasting method based on fuzzy soft set. Feng et al. **[13]** discussed the validity of the

Roy-Maji method and presented an adjustable decision-making method based on fuzzy soft set. Yang et al. **[14]** initiated the idea of interval valued fuzzy soft set (IVFS-set) and analyzed a decision making method using the IVFS-sets. The notion of intuitionistic fuzzy set (IFS) was initiated by Atanassov as a significant generalization of fuzzy set. Intuitionistic fuzzy sets are very useful in situations when description of a problem by a linguistic variable, given in terms of a membership function only, seems too complicated. Recently intuitionistic fuzzy sets have been applied to many fields such as logic programming, medical diagnosis, decision making problems etc. Smarandache **[15]** introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. Thao and Smaran **[16]** proposed the concept of divergence measure on neutrosophic sets with an application to medical problem. Song et al. **[17]** applied neutrosophic sets to ideals in BCK/BCI algebras. Some recent applications of neutrosophic sets can be found in **[18]**, **[19]**, **[20]**, **[21]**, **[22]**, **[23]** and **[24]**. Maji **[25]** introduced the concept of neutrosophic soft set and established some operations on these sets. Mukherjee et al **[26]** introduced the concept of interval valued neutrosophic soft sets and studied their basic properties. In 2013, Broumi and Smarandache **[27, 28]** combined the intuitionistic neutrosophic and soft set which lead to a new mathematical model called "intuitionistic neutrosophic soft set". They studied the notions of intuitionistic neutrosophic soft set union, intuitionistic neutrosophic soft set intersection, complement of intuitionistic neutrosophic soft set and several other properties of intuitionistic neutrosophic soft set along with examples and proofs of certain results. Also, in **[29]** S. Broumi presented the concept of "generalized neutrosophic soft set" by combining the generalized neutrosophic sets and soft set models, studied some properties on it, and presented an application of generalized neutrosophic soft set in decision making problem. Recently, Deli **[30]** introduced the concept of interval valued neutrosophic soft set as a combination of interval neutrosophic set and soft set. In 2014, S. Broumi et al. **[31]** initiated the concept of relations on interval valued neutrosophic soft sets.

The soft sets mentioned above are based on complete information. However, incomplete information widely exists in various real life problems. Soft sets under incomplete information become incomplete soft sets. H. Qin et al **[32]** studied the data filling approach of incomplete soft sets. Y. Zou et al **[33]** investigated data analysis approaches of soft sets under incomplete information. In this paper first we have introduced the concept of a neutrosophic soft set with incomplete data supported by examples. Then we have introduced few new definitions to measure the consistent and inconsistent association between the parameters. Lastly we have presented a data filling algorithm supported by an illustrative example to show the feasibility and validity of our algorithm.

2. Preliminaries:

2.1 Definition: **[03]** Let U be a non empty set. Then a fuzzy set τ on U is a set having the form $\tau = \{(x, \mu_\tau(x)) : x \in U\}$ where the function $\mu_\tau : U \to [0, 1]$ is called the membership function and $\mu_\tau(x)$ represents the degree of membership of each element $x \in U$.

2.2 Definition: **[04]** Let U be a non empty set. Then an intuitionistic fuzzy set (IFS for short) τ is an object having the form $\tau = \{\langle x, \mu_\tau(x), \gamma_\tau(x)\rangle : x \in U\}$ where the functions $\mu_\tau : U \to [0, 1]$ and $\gamma_\tau : U \to [0, 1]$ are called membership function and non-membership function respectively.

$\mu_\tau(x)$ and $\gamma_\tau(x)$ represent the degree of membership and the degree of non-membership respectively of each element $x \in U$ and $0 \le \mu_\tau(x) + \gamma_\tau(x) \le 1$ for each $x \in U$. We denote the class of all intuitionistic fuzzy sets on U by IFS^U.

2.3 Definition: [01] Let U be a universe set and E be a set of parameters. Let $\mathrm{P}(\mathrm{U})$ denotes the power set of U and $\mathrm{A} \subseteq \mathrm{E}$. Then the pair (F, A) is called a soft set over U, where F is a mapping given by $\mathrm{F}: \mathrm{A} \rightarrow \mathrm{P}(\mathrm{U})$.

In other words, the soft set is not a kind of set, but a parameterized family of subsets of U. For $\mathrm{e} \in \mathrm{A}$, $F(\mathrm{e}) \subseteq \mathrm{U}$ may be considered as the set of e-approximate elements of the soft set (F, A).

2.4 Definition: [10] Let U be a universe set, E be a set of parameters and $\mathrm{A} \subseteq \mathrm{E}$. Then the pair (F, A) is called a fuzzy soft set over U, where F is a mapping given by $\mathrm{F}: \mathrm{A} \rightarrow \mathrm{FS}^{\mathrm{U}}$.

2.5 Definition: [34] Let U be a universe set, E be a set of parameters and $\mathrm{A} \subseteq \mathrm{E}$. Then the pair (F, A) is called an intuitionistic fuzzy soft set over U, where F is a mapping given by $\mathrm{F}: \mathrm{A} \rightarrow \mathrm{IFS}^{\mathrm{U}}$.

For $\mathrm{e} \in \mathrm{A}$, $F(\mathrm{e})$ is an intuitionistic fuzzy subset of U and is called the intuitionistic fuzzy value set of the parameter 'e'.

Let us denote $\mu_{F(\mathrm{e})}(\mathrm{x})$ by the membership degree that object 'x' holds parameter 'e' and $\gamma_{F(\mathrm{e})}(\mathrm{x})$ by the membership degree that object 'x' doesn't hold parameter 'e', where $\mathrm{e} \in \mathrm{A}$ and $\mathrm{x} \in \mathrm{U}$. Then $F(\mathrm{e})$ can be written as an intuitionistic fuzzy set such that $F(\mathrm{e})=\left\{\left(\mathrm{x}, \mu_{F(\mathrm{e})}(\mathrm{x}), \gamma_{F(\mathrm{e})}(\mathrm{x})\right): \mathrm{x} \in \mathrm{U}\right\}$.

2.6 Definition: [15] A neutrosophic set A on the universe of discourse U is defined as $A=\left\{\left\langle x, \mu_A(x), \gamma_A(x), \delta_A(x)\right\rangle : x \in U\right\}$, where $\mu_A, \gamma_A, \delta_A : U \rightarrow]^{-}0, 1^{+}[$ are functions such that the condition: $\forall x \in U, {}^{-}0 \leq \mu_A(x)+\gamma_A(x)+\delta_A(x) \leq 3^{+}$ is satisfied.

Here $\mu_A(x), \gamma_A(x), \delta_A(x)$ represent the truth-membership, indeterminacy-membership and falsity-membership respectively of the element $x \in U$.

Smarandache **[15]** applied neutrosophic sets in many directions after giving examples of neutrosophic sets. Then he introduced the neutrosophic set operations namely-complement, union, intersection, difference, Cartesian product etc.

2.7 Definition: [21] Let U be an initial universe, E be a set of parameters and $A \subseteq E$. Let $NP(U)$ denotes the set of all neutrosophic sets of U. Then the pair (f, A) is termed to be the neutrosophic soft set over U, where f is a mapping given by $f : A \rightarrow NP(U)$.

2.8 Example: Let us consider a neutrosophic soft set (f, A) which describes the "attractiveness of the house". Suppose $U=\{u_1, u_2, u_3, u_4, u_5, u_6\}$ be the set of six houses under consideration and $E=\{e_1$ (beautiful), e_2 (expensive), e_3 (cheap), e_4 (good location), e_5 (wooden)$\}$ be the set of parameters. Then a neutrosophic soft set (f, A) over U can be given by:

U	e_1	e_2	e_3	e_4	e_5
u_1	(0.8,0.5,0.2)	(0.3,0.4,0.6)	(0.1,0.6,0.4)	(0.7,0.3,0.6)	(0.3,0.4,0.6)
u_2	(0.4,0.1,0.7)	(0.8,0.2,0.4)	(0.4,0.1,0.7)	(0.2,0.4,0.4)	(0.1,0.1,0.3)
u_3	(0.2,0.6,0.4)	(0.5,0.5,0.5)	(0.8,0.1,0.7)	(0.5,0.3,0.5)	(0.5,0.5,0.5)
u_4	(0.3,0.4,0.4)	(0.1,0.3,0.3)	(0.3,0.4,0.4)	(0.6,0.6,0.6)	(0.1,0.1,0.5)

u_5	(0.1,0.1,0.7)	(0.2,0.6,0.7)	(0.4,0.2,0.1)	(0.8,0.6,0.1)	(0.6,0.7,0.7)
u_6	(0.5,0.3,0.9)	(0.3,0.6,0.6)	(0.1,0.5,0.5)	(0.3,0.6,0.5)	(0.4,0.4,0.4)

3. Neutrosophic soft sets with incomplete (missing) data:

Suppose that (f,E) is a neutrosophic soft set over U, such that $\$\ x_i$ Î U and e_j Î E so that none of $m_{f(e_j)}(x_i), g_{f(e_j)}(x_i)$ and $d_{f(e_j)}(x_i)$ is known. In this case, in the tabular representation of the neutrosophic soft set (f,E), we write $\left(m_{f(e_j)}(x_i), g_{f(e_j)}(x_i), d_{f(e_j)}(x_i)\right) = *$. Here we say that the data for $f(e_j)$ is missing and the neutrosophic soft set (f,E) over U has incomplete data.

3.1 Example: Suppose Tech Mahindra is recruiting some new Graduate Trainee for the session 2019-2020 and suppose that eight candidates have applied for the job. Assume that $U = \{u_1, u_2, u_3, ..., u_8\}$ be the set of candidates and $E = \{e_1$ (communication skill), e_2 (domain knowledge), e_3 (experienced), e_4 (young), e_5 (highest academic degree), e_6 (professional attitute)$\}$ be the set of parameters. Then a neutrosophic soft set over U having missing data can be given by Table-1.

Table-1

U	e_1	e_2	e_3	e_4	e_5	e_6
u_1	(0.8,0.5,0.2)	(0.3,0.4,0.6)	(0.1,0.6,0.4)	(0.7,0.3,0.6)	(0.3,0.4,0.6)	(0.2,0.5,0.5)
u_2	(0.4,0.1,0.7)	(0.8,0.2,0.4)	(0.4,0.1,0.7)	(0.2,0.4,0.4)	*	(0.6,0.6,0.4)
u_3	(0.2,0.6,0.4)	(0.5,0.5,0.5)	*	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.3,0.4,0.6)
u_4	(0.3,0.4,0.4)	(0.1,0.3,0.3)	(0.3,0.4,0.4)	(0.6,0.6,0.6)	(0.1,0.1,0.5)	(0.3,0.4,0.4)
u_5	(0.1,0.1,0.7)	*	(0.4,0.2,0.1)	(0.8,0.6,0.1)	(0.6,0.7,0.7)	(0.3,0.4,0.3)
u_6	(0.5,0.3,0.9)	(0.3,0.6,0.6)	(0.1,0.5,0.5)	(0.3,0.6,0.6)	(0.4,0.4,0.4)	(0.3,0.6,0.6)
u_7	(0.2,0.4,0.6)	(0.4,0.4,0.5)	(0.5,0.5,0.6)	*	(0.7,0.5,0.8)	(0.4,0.4,0.5)
u_8	(0.2,0.3,0.1)	(0.6,0.6,0.1)	(0.8,0.3,0.8)	(0.4,0.3,0.4)	(0.5,0.6,0.3)	(0.9,0.3,0.3)

In case of soft set theory, there always exist some obvious or hidden associations between parameters. Let us focus on this to find the associations between the parameters of a neutrosophic soft set.

In example **2.8**, one can easily find that if a house is expensive, the house is not cheap and vice versa. Thus there is an inconsistent association between the parameters 'expensive' and 'cheap'. Generally, if a house is beautiful or situated in a good location, the house is expensive. Thus there is a consistent association between the parameters 'beautiful' and 'expensive' or the parameters 'good location' and 'expensive'.

In example **3.1**, we find that if a candidate is experienced or have highest academic degree, he/she is not young. Thus there is an inconsistent association between parameters 'experienced' and 'young' or between 'highest academic degree' and 'young'.

The above two examples reveal the interior relations of parameters. In a neutrosophic soft set, these associations between parameters will be very useful for filling incomplete data. If it is found that the parameters e_i and e_j are associated and the data for $f(e_i)$ is missing, then we can fill the missing data according to the corresponding data in $f(e_j)$. To measure these associations, let us define the notion of association degree and some relevant concepts.

For the rest of the paper we shall assume that *U* be the universe set and *E* be the set of parameters.

Let U_{ij} denotes the set of objects that have specified values in the form of an ordered triplet (*a*, *b*, *c*) where *a*, *b*, *c*∈[0, 1] on both parameters e_i and e_j such that

$$U_{ij} = \left\{ x \in U : \left(m_{f(e_i)}(x), g_{f(e_i)}(x), d_{f(e_i)}(x)\right) \neq *, \left(m_{f(e_j)}(x), g_{f(e_j)}(x), d_{f(e_j)}(x)\right) \neq * \right\}$$

In other words U_{ij} is the collection of those objects that have known data both on e_i and e_j.

3.2 Definition: Let $e_i, e_j \in E$. Then the consistent association number between the parameters e_i and e_j is denoted by CAN_{ij} and is defined as:

$$CAN_{ij} = \left|\left\{ x \in U_{ij} : m_{f(e_i)}(x) = m_{f(e_j)}(x), g_{f(e_i)}(x) = g_{f(e_j)}(x), d_{f(e_i)}(x) = d_{f(e_j)}(x) \right\}\right|$$

where $|.|$ denotes the cardinality of a set.

3.3 Definition: Let $e_i, e_j \in E$. Then the consistent association degree between the parameters e_i and e_j is denoted by CAD_{ij} and is defined as: $CAD_{ij} = \dfrac{CAN_{ij}}{|U_{ij}|}$ where $|.|$ denotes the cardinality of a set.

It can be easily verified that the value of CAD_{ij} lies in [0, 1]. Actually CAD_{ij} measures the extent to which the value of parameter e_i keeps consistent with that of parameter e_j over U_{ij}. Next we define inconsistent association number and inconsistent association degree as follows:

3.4 Definition: Let $e_i, e_j \in E$. Then the inconsistent association number between the parameters e_i and e_j is denoted by $ICAN_{ij}$ and is defined as

$$ICAN_{ij} = \left|\left\{ x \in U_{ij} : m_{f(e_i)}(x) \neq m_{f(e_j)}(x) \text{ or } g_{f(e_i)}(x) \neq g_{f(e_j)}(x) \text{ or } d_{f(e_i)}(x) \neq d_{f(e_j)}(x) \right\}\right|$$

where $|.|$ denotes the cardinality of a set.

3.5 Definition: Let $e_i, e_j \in E$. Then the inconsistent association degree between the parameters e_i and e_j is denoted by $ICAD_{ij}$ and is defined as: $ICAD_{ij} = \dfrac{ICAN_{ij}}{|U_{ij}|}$ where $|.|$ denotes the cardinality of a set.

It can be easily verified that the value of $ICAD_{ij}$ lies in [0, 1]. Actually $ICAD_{ij}$ measures the extent to which the parameters e_i and e_j is inconsistent.

3.6 Definition: Let $e_i, e_j \in E$. Then the association degree between the parameters e_i and e_j is denoted by AD_{ij} and is defined by $AD_{ij} = \max\left\{CAD_{ij}, ICAD_{ij}\right\}$.

If $CAD_{ij} > ICAD_{ij}$, then $AD_{ij} = CAD_{ij}$, which means that most of the objects over U_{ij} have consistent values on parameters e_i and e_j . If $CAD_{ij} < ICAD_{ij}$, then $AD_{ij} = ICAD_{ij}$, which means that most of the objects over U_{ij} have inconsistent values on parameters e_i and e_j . Again if $CAD_{ij} = ICAD_{ij}$, then it means that there is the lowest association degree between the parameters e_i and e_j .

3.7 Theorem: For parameters e_i and e_j , $AD_{ij} \geq 0.5$ for all *i, j*.
Proof: Follows from the fact that $CAD_{ij} + ICAD_{ij} = 1$.

3.8 Definition: If $e_i \in E$, then the maximal association degree of parameter e_i is denoted by MAD_i and is defined by $MAD_i = \max_j AD_{ij}$.

4. DATA Filling Algorithm for a neutrosophic soft set:

<u>**Step-1:**</u> Input the neutrosophic soft set (f, E) which has incomplete data.

<u>**Step-2:**</u> Find all parameters e_i for which data is missing.

<u>**Step-3:**</u> Compute AD_{ij} for *j*=1,2,3….,*m* (where '*m*' is the number of parameters in *E*).

<u>**Step-4:**</u> Compute MAD_i .

<u>**Step-5:**</u> Find out all parameters e_j which have the maximal association degree MAD_i with the parameter e_i .

<u>**Step-6:**</u> In case of consistent association between the parameter e_i and e_j 's (*j*=1,2,3,….) $\left(m_{f(e_i)}(x), g_{f(e_i)}(x), d_{f(e_i)}(x)\right) = \left(\max_j m_{f(e_j)}(x), \max_j g_{f(e_j)}(x), \max_j d_{f(e_j)}(x)\right)$. In case of inconsistent association between the parameter e_i and e_j 's (*j*=1,2,3,….) $\left(m_{f(e_i)}(x), g_{f(e_i)}(x), d_{f(e_i)}(x)\right) = \left(1 - \max_j m_{f(e_j)}(x), 1 - \max_j g_{f(e_j)}(x), 1 - \max_j d_{f(e_j)}(x)\right)$.

<u>**Step-7:**</u> If all the missing data are filled then stop else go to step-2.

- **An Illustrative example:** Consider the neutrosophic soft set given in example 3.1.

<u>**Step-1:**</u>

U	e_1	e_2	e_3	e_4	e_5	e_6
u_1	(0.8,0.5,0.2)	(0.3,0.4,0.6)	(0.1,0.6,0.4)	(0.7,0.3,0.6)	(0.3,0.4,0.6)	(0.2,0.5,0.5)
u_2	(0.4,0.1,0.7)	(0.8,0.2,0.4)	(0.4,0.1,0.7)	(0.2,0.4,0.4)	*	(0.6,0.6,0.4)
u_3	(0.2,0.6,0.4)	(0.5,0.5,0.5)	*	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.3,0.4,0.6)
u_4	(0.3,0.4,0.4)	(0.1,0.3,0.3)	(0.3,0.4,0.4)	(0.6,0.6,0.6)	(0.1,0.1,0.5)	(0.3,0.4,0.4)
u_5	(0.1,0.1,0.7)	*	(0.4,0.2,0.1)	(0.8,0.6,0.1)	(0.6,0.7,0.7)	(0.3,0.4,0.3)
u_6	(0.5,0.3,0.9)	(0.3,0.6,0.6)	(0.1,0.5,0.5)	(0.3,0.6,0.6)	(0.4,0.4,0.4)	(0.3,0.6,0.6)
u_7	(0.2,0.4,0.6)	(0.4,0.4,0.5)	(0.5,0.5,0.6)	*	(0.7,0.5,0.8)	(0.4,0.4,0.5)
u_8	(0.2,0.3,0.1)	(0.6,0.6,0.1)	(0.8,0.3,0.8)	(0.4,0.3,0.4)	(0.5,0.6,0.3)	(0.9,0.3,0.3)

Step-2: Clearly there are missing data in $f(e_2), f(e_3), f(e_4), f(e_5)$. We shall fill these missing data.

Step-3:

(a) **For the parameter e_2.**

\ $U_{21} = \{u_1,u_2,u_3,u_4,u_6,u_7,u_8\}, U_{23} = \{u_1,u_2,u_4,u_6,u_7,u_8\}, U_{24} = \{u_1,u_2,u_3,u_4,u_6,u_8\}$,
$U_{25} = \{u_1,u_3,u_4,u_6,u_7,u_8\}, U_{26} = \{u_1,u_2,u_3,u_4,u_6,u_7,u_8\}$.

Now $CAN_{21} = |\{\ \}| = 0$ and so $CAD_{21} = 0$. Again $ICAN_{21} = |\{u_1,u_2,u_3,u_4,u_6,u_7,u_8\}| = 7$ and so $ICAD_{21} = \frac{ICAN_{21}}{|U_{21}|} = \frac{7}{7} = 1$. Hence $AD_{21} = \max\{CAD_{21}, ICAD_{21}\} = \max\{0,1\} = 1$.

$CAN_{23} = |\{\ \}| = 0$ and so $CAD_{23} = 0$. Again $ICAN_{23} = |\{u_1,u_2,u_4,u_6,u_7,u_8\}| = 6$ and so $ICAD_{23} = \frac{ICAN_{23}}{|U_{23}|} = \frac{6}{6} = 1$. Hence $AD_{23} = \max\{CAD_{23}, ICAD_{23}\} = \max\{0,1\} = 1$.

$CAN_{24} = |\{u_3,u_6\}| = 2$ and so $CAD_{24} = \frac{2}{6} = 0.33$. Again $ICAN_{24} = |\{u_1,u_2,u_4,u_8\}| = 4$ and so $ICAD_{24} = \frac{ICAN_{24}}{|U_{24}|} = \frac{4}{6} = 0.66$. Hence $AD_{24} = \max\{CAD_{24}, ICAD_{24}\} = \max\{0.33, 0.66\} = 0.66$.

$CAN_{25} = |\{u_3,u_1\}| = 2$ and so $CAD_{25} = \frac{2}{6} = 0.33$. Again $ICAN_{25} = |\{u_4,u_6,u_7,u_8\}| = 4$ and so $ICAD_{25} = \frac{ICAN_{24}}{|U_{24}|} = \frac{4}{6} = 0.66$. Hence $AD_{25} = \max\{CAD_{25}, ICAD_{25}\} = \max\{0.33, 0.66\} = 0.66$.

$CAN_{26} = |\{u_4\}| = 1$ and so $CAD_{26} = \frac{1}{7} = 0.14$. Again $ICAN_{26} = |\{u_1,u_2,u_3,u_6,u_7,u_8\}| = 6$ and so $ICAD_{26} = \frac{ICAN_{26}}{|U_{26}|} = \frac{6}{7} = 0.85$. Hence $AD_{26} = \max\{CAD_{26}, ICAD_{26}\} = \max\{0.14, 0.85\} = 0.85$.

Thus $MAD_2 = \max_j AD_{2j} = \max\{AD_{21}, AD_{23}, AD_{24}, AD_{25}, AD_{26}\} = \max\{1,1,0.66,0.66,0.85\} = 1$..

(b) **For the parameter e_3.**

\ $U_{31} = \{u_1,u_2,u_4,u_6,u_7,u_8\}, U_{32} = \{u_1,u_2,u_4,u_6,u_7,u_8\}, U_{34} = \{u_1,u_2,u_4,u_5,u_6,u_8\}$,
$U_{35} = \{u_1,u_4,u_5,u_6,u_7,u_8\}, U_{36} = \{u_1,u_2,u_4,u_5,u_6,u_7,u_8\}$.

Now $CAN_{31} = |\{u_2,u_4\}| = 2$ and so $CAD_{31} = \frac{2}{6} = 0.33$. Again $ICAN_{31} = |\{u_1,u_6,u_7,u_8\}| = 4$ and so $ICAD_{31} = \frac{ICAN_{31}}{|U_{31}|} = \frac{4}{6} = 0.66$. Hence $AD_{31} = \max\{CAD_{31}, ICAD_{31}\} = \max\{0.33, 0.66\} = 0.66$.

$CAN_{32} = |\{\ \}| = 0$ and so $CAD_{32} = 0$. Again $ICAN_{32} = |\{u_1,u_2,u_4,u_6,u_7,u_8\}| = 6$ and so $ICAD_{32} = \frac{ICAN_{32}}{|U_{32}|} = \frac{6}{6} = 1$. Hence $AD_{32} = \max\{CAD_{32}, ICAD_{32}\} = \max\{0,1\} = 1$.

$CAN_{34} = |\{\ \}| = 0$ and so $CAD_{34} = 0$. Again $ICAN_{34} = |\{u_1,u_2,u_4,u_5,u_6,u_8\}| = 6$ and so $ICAD_{34} = \frac{ICAN_{34}}{|U_{34}|} = \frac{4}{6} = 0.66$. Hence $AD_{34} = \max\{CAD_{34}, ICAD_{34}\} = \max\{0, 0.66\} = 0.66$.

$CAN_{35} = |\{\ \}| = 0$ and so $CAD_{35} = 0$. Again $ICAN_{35} = |\{u_1, u_4, u_5, u_6, u_7, u_8\}| = 6$ and so $ICAD_{35} = \frac{ICAN_{35}}{|U_{35}|} = \frac{6}{6} = 1$. Hence $AD_{35} = \max\{CAD_{35}, ICAD_{35}\} = \max\{0,1\} = 1$.

$CAN_{36} = |\{u_4\}| = 1$ and so $CAD_{36} = \frac{1}{7} = 0.14$. Again $ICAN_{36} = |\{u_1, u_2, u_5, u_6, u_7, u_8\}| = 6$ and so $ICAD_{36} = \frac{ICAN_{36}}{|U_{36}|} = \frac{6}{7} = 0.85$. Hence $AD_{36} = \max\{CAD_{36}, ICAD_{36}\} = \max\{0.14, 0.85\} = 0.85$.

Thus $MAD_3 = \max_j AD_{3j} = \max\{AD_{31}, AD_{32}, AD_{34}, AD_{35}, AD_{36}\} = \max\{0.66, 1, 0.66, 1, 0.85\} = 1$.

(c) **For the parameter** e_4 **.**

\ $U_{41} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_8\}, U_{42} = \{u_1, u_2, u_3, u_4, u_6, u_8\}, U_{43} = \{u_1, u_2, u_4, u_5, u_6, u_8\}$, $U_{45} = \{u_1, u_3, u_4, u_5, u_6, u_8\}, U_{46} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_8\}$.

Now $CAN_{41} = |\{\ \}| = 0$ and so $CAD_{41} = 0$. Again $ICAN_{41} = |\{u_1, u_2, u_3, u_4, u_5, u_6, u_8\}| = 7$ and so $ICAD_{41} = \frac{ICAN_{41}}{|U_{41}|} = \frac{7}{7} = 1$. Hence $AD_{41} = \max\{CAD_{41}, ICAD_{41}\} = \max\{0,1\} = 1$.

$CAN_{42} = |\{u_3, u_6\}| = 2$ and so $CAD_{42} = \frac{2}{6} = 0.33$. Again $ICAN_{42} = |\{u_1, u_2, u_4, u_8\}| = 4$ and so $ICAD_{42} = \frac{ICAN_{42}}{|U_{42}|} = \frac{4}{6} = 0.66$. Hence $AD_{42} = \max\{CAD_{42}, ICAD_{42}\} = \max\{0.33, 0.66\} = 0.66$.

$CAN_{43} = |\{\ \}| = 0$ and so $CAD_{43} = 0$. Again $ICAN_{43} = |\{u_1, u_2, u_4, u_5, u_6, u_8\}| = 6$ and so $ICAD_{43} = \frac{ICAN_{43}}{|U_{43}|} = \frac{6}{6} = 1$. Hence $AD_{43} = \max\{CAD_{43}, ICAD_{43}\} = \max\{0,1\} = 1$.

$CAN_{45} = |\{u_3\}| = 1$ and so $CAD_{45} = \frac{1}{6} = 0.16$. Again $ICAN_{45} = |\{u_1, u_4, u_5, u_6, u_8\}| = 5$ and so $ICAD_{45} = \frac{ICAN_{45}}{|U_{45}|} = \frac{5}{6} = 0.83$. Hence $AD_{45} = \max\{CAD_{45}, ICAD_{35}\} = \max\{0.16, 0.83\} = 0.83$.

$CAN_{46} = |\{u_6\}| = 1$ and so $CAD_{46} = \frac{1}{7} = 0.14$. Again $ICAN_{46} = |\{u_1, u_2, u_3, u_4, u_5, u_8\}| = 6$ and so $ICAD_{46} = \frac{ICAN_{46}}{|U_{46}|} = \frac{6}{7} = 0.85$. Hence $AD_{46} = \max\{CAD_{46}, ICAD_{46}\} = \max\{0.14, 0.85\} = 0.85$.

Thus $MAD_4 = \max_j AD_{4j} = \max\{AD_{41}, AD_{42}, AD_{43}, AD_{45}, AD_{46}\} = \max\{1, 0.66, 1, 0.83, 0.85\} = 1$.

(d) **For the parameter** e_5 **.**

\ $U_{51} = \{u_1, u_3, u_4, u_5, u_6, u_7, u_8\}, U_{52} = \{u_1, u_3, u_4, u_6, u_7, u_8\}, U_{53} = \{u_1, u_4, u_5, u_6, u_7, u_8\}$, $U_{54} = \{u_1, u_3, u_4, u_5, u_6, u_8\}, U_{56} = \{u_1, u_3, u_4, u_5, u_6, u_7, u_8\}$.

Now $CAN_{51} = |\{\ \}| = 0$ and so $CAD_{51} = 0$. Again $ICAN_{51} = |\{u_1, u_3, u_4, u_5, u_6, u_7, u_8\}| = 7$ and so $ICAD_{51} = \frac{ICAN_{51}}{|U_{51}|} = \frac{7}{7} = 1$. Hence $AD_{51} = \max\{CAD_{51}, ICAD_{51}\} = \max\{0,1\} = 1$.

$CAN_{52} = |\{u_1, u_3\}| = 2$ and so $CAD_{52} = \frac{2}{6} = 0.33$. Again $ICAN_{52} = |\{u_4, u_6, u_7, u_8\}| = 4$ and so $ICAD_{52} = \frac{ICAN_{52}}{|U_{52}|} = \frac{4}{6} = 0.66$. Hence $AD_{52} = \max\{CAD_{52}, ICAD_{52}\} = \max\{0.33, 0.66\} = 0.66$.

$CAN_{53} = |\{\ \}| = 0$ and so $CAD_{53} = 0$. Again $ICAN_{53} = |\{u_1, u_4, u_5, u_6, u_7, u_8\}| = 6$ and so $ICAD_{53} = \frac{ICAN_{53}}{|U_{53}|} = \frac{6}{6} = 1$. Hence $AD_{53} = \max\{CAD_{53}, ICAD_{53}\} = \max\{0,1\} = 1$.

$CAN_{54} = |\{u_3\}| = 1$ and so $CAD_{54} = \frac{1}{6} = 0.16$. Again $ICAN_{54} = |\{u_1, u_4, u_5, u_6, u_8\}| = 5$ and so $ICAD_{54} = \frac{ICAN_{54}}{|U_{54}|} = \frac{5}{6} = 0.83$. Hence $AD_{54} = \max\{CAD_{54}, ICAD_{54}\} = \max\{0.16, 0.83\} = 0.83$.

$CAN_{56} = |\{\ \}| = 0$ and so $CAD_{56} = 0$. Again $ICAN_{56} = |\{u_1, u_3, u_4, u_5, u_6, u_7, u_8\}| = 7$ and so $ICAD_{56} = \frac{ICAN_{56}}{|U_{56}|} = \frac{7}{7} = 1$. Hence $AD_{56} = \max\{CAD_{56}, ICAD_{56}\} = \max\{0,1\} = 1$.

Thus $MAD_5 = \max_j AD_{5j} = \max\{AD_{51}, AD_{52}, AD_{53}, AD_{54}, AD_{56}\} = \max\{1, 0.66, 1, 0.83, 1\} = 1$.

The association degree table for the neutrosophic soft set (f, E) is given below:

	e_1	e_2	e_3	e_4	e_5	e_6
e_2	1	_	1	0.66	0.66	0.85
e_3	0.66	1	_	0.66	1	0.85
e_4	1	0.66	1	_	0.83	0.85
e_5	1	0.66	1	0.83	_	1

Step-4: From step-3, we have, $MAD_2 = 1, MAD_3 = 1, MAD_4 = 1, MAD_5 = 1$.

Step-5: The parameters e_1 and e_3 have the maximal association degree AD_{21} and AD_{23} respectively with the parameter e_2.

The parameters e_2 and e_5 have the maximal association degree AD_{32} and AD_{35} respectively with the parameter e_3.

The parameters e_1 and e_3 have the maximal association degree AD_{41} and AD_{43} respectively with the parameter e_4.

The parameters e_1, e_3 and e_6 have the maximal association degree AD_{51}, AD_{53} and AD_{56} respectively with the parameter e_5.

Step-6: There is a consistent association between the parameters e_2 and e_1, e_2 and e_3, e_5 and e_1, e_3 and e_5; while there is an inconsistent association between the parameters e_4 and e_1, e_4 and e_3.So we have,

$$\left(m_{f(e_2)}(u_5), g_{f(e_2)}(u_5), d_{f(e_2)}(u_5)\right)$$
$$= \left(\max\left(m_{f(e_1)}(u_5), m_{f(e_3)}(u_5)\right),\ \max\left(g_{f(e_1)}(u_5), g_{f(e_3)}(u_5)\right),\ \max\left(d_{f(e_1)}(u_5), d_{f(e_3)}(u_5)\right)\right)$$
$$= (\max(0.1, 0.4),\ \max(0.1, 0.2),\ \max(0.7, 0.1)) = (0.4, 0.2, 0.7),$$
$$\left(m_{f(e_3)}(u_3), g_{f(e_3)}(u_3), d_{f(e_3)}(u_3)\right)$$
$$= \left(\max\left(m_{f(e_2)}(u_3), m_{f(e_5)}(u_3)\right),\ \max\left(g_{f(e_2)}(u_3), g_{f(e_5)}(u_3)\right),\ \max\left(d_{f(e_2)}(u_3), d_{f(e_5)}(u_3)\right)\right)$$
$$= (\max(0.5, 0.5),\ \max(0.5, 0.5),\ \max(0.5, 0.5)) = (0.5, 0.5, 0.5),$$
$$\left(m_{f(e_4)}(u_7), g_{f(e_4)}(u_7), d_{f(e_4)}(u_7)\right)$$
$$= \left(1-\ \max\left(m_{f(e_1)}(u_7), m_{f(e_3)}(u_7)\right),\ 1-\ \max\left(g_{f(e_1)}(u_7), g_{f(e_3)}(u_7)\right), 1-\ \max\left(d_{f(e_1)}(u_7), d_{f(e_3)}(u_7)\right)\right)$$
$$= (\max(0.2, 0.5),\ \max(0.4, 0.5),\ \max(0.6, 0.6)) = (0.5, 0.5, 0.6),$$
$$\left(m_{f(e_5)}(u_2), g_{f(e_5)}(u_2), d_{f(e_5)}(u_2)\right)$$
$$= \left(\max\left(m_{f(e_1)}(u_2), m_{f(e_3)}(u_2)\right),\ \max\left(g_{f(e_1)}(u_2), g_{f(e_3)}(u_2)\right),\ \max\left(d_{f(e_1)}(u_2), d_{f(e_3)}(u_2)\right)\right)$$
$$= (\max(0.4, 0.4),\ \max(0.1, 0.1),\ \max(0.7, 0.7)) = (0.4, 0.1, 0.7).$$

Thus we have the following table which gives the tabular representation of the filled neutrosophic soft set:

U	e_1	e_2	e_3	e_4	e_5	e_6
u_1	(0.8,0.5,0.2)	(0.3,0.4,0.6)	(0.1,0.6,0.4)	(0.7,0.3,0.6)	(0.3,0.4,0.6)	(0.2,0.5,0.5)
u_2	(0.4,0.1,0.7)	(0.8,0.2,0.4)	(0.4,0.1,0.7)	(0.2,0.4,0.4)	**(0.4,0.1,0.7)**	(0.6,0.6,0.4)
u_3	(0.2,0.6,0.4)	(0.5,0.5,0.5)	**(0.5,0.5,0.5)**	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.3,0.4,0.6)
u_4	(0.3,0.4,0.4)	(0.1,0.3,0.3)	(0.3,0.4,0.4)	(0.6,0.6,0.6)	(0.1,0.1,0.5)	(0.3,0.4,0.4)
u_5	(0.1,0.1,0.7)	**(0.4,0.2,0.7)**	(0.4,0.2,0.1)	(0.8,0.6,0.1)	(0.6,0.7,0.7)	(0.3,0.4,0.3)
u_6	(0.5,0.3,0.9)	(0.3,0.6,0.6)	(0.1,0.5,0.5)	(0.3,0.6,0.6)	(0.4,0.4,0.4)	(0.3,0.6,0.6)
u_7	(0.2,0.4,0.6)	(0.4,0.4,0.5)	(0.5,0.5,0.6)	**(0.5,0.5,0.6)**	(0.7,0.5,0.8)	(0.4,0.4,0.5)
u_8	(0.2,0.3,0.1)	(0.6,0.6,0.1)	(0.8,0.3,0.8)	(0.4,0.3,0.4)	(0.5,0.6,0.3)	(0.9,0.3,0.3)

Conclusion: Incomplete information or missing data in a neutrosophic soft set restricts the usage of the neutrosophic soft set. To make the neutrosophic soft set (with missing / incomplete data) more useful, in this paper, we have proposed a data filling approach, where missing data is filled in terms of the association degree between the parameters. We have validated the proposed algorithm by an example and drawn the conclusion that relation between parameters can be applied to fill the missing data.

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On Optimizing Neutrosophic Complex Programming Using Lexicographic Order

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Abstract: Neutrosophic sets are considered as a generalization of the crisp set, fuzzy set, and intuitionistic fuzzy set for representing the uncertainty, inconsistency, and incomplete knowledge about the real world problems. This paper aims to characterize the solution of complex programming (CP) problem with imprecise data instead of its prices information. The neutrosophic complex programming (NCP) problem is considered by incorporating single valued trapezoidal neutrosophic numbers in all the parameters of objective function and constraints. The score function corresponding to the neutrosophic number is used to transform the problem into the corresponding crisp CP. Here, lexicographic order is applied for the comparison between any two complex numbers. The comparison is developed between the real and imaginary parts separately. Through this manner, the CP problem is divided into two real sub-problems. In the last, a numerical example is solved for the illustration that shows the applicability of the proposed approach. The advantage of this approach is more flexible and makes a real-world situation more realistic.

Keywords: Complex programming; Neutrosophic numbers; Score function; Lexicographic order; Lingo software; Kuhn- Tucker conditions; Neutrosophic optimal solution.

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1. **Introduction**

In many earlier works in complex programming, the researchers considered the real part only of the complex objective function as the objective function. The constraints of the problem considered as a cone in complex space $\mathbb{C}^n$. Since the concept of complex fuzzy numbers was first introduced [17], many researchers studied the problems of the concept of fuzzy complex numbers. This branch subject will be widely applied in fuzzy system theory, especially in fuzzy mathematical programming, and also in complex programming too.

Complex programming problem was studied first by Levinson who studied the linear programming (LP) in complex space [39]. The duality theorem was extended to the quadratic complex programming by an adaption of the technique which was introduced by Dorn [27, 22]. The linear fractional programming in complex space was proposed [45]. Linear and nonlinear complex programming problems were treated by numerous authors [24, 33- 37, 41]. In applications, many practical problems related to complex variables, for instance, electrical engineering, filter theory, statistical signal processing, etc., were studied.

Some more general minimax fractional programming problem with complex variables was proposed with the establishment of the necessary and sufficient optimality conditions [36, 37]. A certain kind of linear programming with fuzzy complex numbers in the objective function coefficients also considered as complex fuzzy numbers [52]. The hyper complex neutrosophic similarity measure was proposed by numerous authors [29]. Also, they discussed its application in multicriteria decision making problem. There was proposed an interval neutrosophic multiple attribute decision-making method with credibility information [50]. Later, the multiple attribute group decision making based on interval neutrosophic uncertain linguistic variables was studied [51].

An extended TOPSIS for multi-attribute decision making problems with neutrosophic cubic information was proposed [42]. A single valued neutrosophic hesitant fuzzy computational algorithm was developed for multiple objective nonlinear optimization problem [9]. A computational algorithm was developed for the neutrosophic optimization model with an application to determine the optimal shale gas water management under uncertainty [10]. The interval complex neutrosophic set was studied by the formulation and applications in decision-making [11]. A group decision-making method was proposed under hesitant interval neutrosophic uncertain linguistic environment [40]. The neutrosophic complex topological spaces was studied, and introduced the concept of neutrosophic complex $\alpha\psi$ connectedness in neutrosophic complex topological spaces [30].

A computational algorithm based on the single-valued neutrosophic hesitant fuzzy was developed for multiple objective nonlinear optimization problems [9]. A neutrosophic optimization model was formulated and presented a computational algorithm for optimal shale gas water management under uncertainty [10]. A multiple objective programming approach was proposed to solve integer valued neutrosophic shortest path problems [32].

Some linguistic approaches were developed to study the interval complex neutrosophic sets in decision making applications [39].

Neutrosophic sets were studied to search some applications in the area of transportations and logistics. A multi-objective transportation model was studied under neutrosophic environment [43]. The multi-criteria decision making based on generalized prioritized aggregation operators was presented under simplified neutrosophic uncertain linguistic environment [46]. Some dynamic interval valued neutrosophic set were proposed by modeling decision making in dynamic environments [48]. A hybrid plithogenic decision-making approach was proposed with quality function deployment for selecting supply chain sustainability metrics [1]. Some applications of neutrosophic theory were studied to solve transition difficulties of IT-based enterprises [2].

Based on plithogenic sets, a novel model for the evaluation of hospital medical care systems was presented [3]. Some decision making applications of soft computing and IoT were proposed for a novel intelligent medical decision support model [4]. A novel neutrosophic approach was applied to evaluate the green supply chain management practices [5]. Numerous researchers studied the under type-2 neutrosophic numbers. An application of under type-2 neutrosophic number was presented for developing supplier selection with group decision making by using TOPSIS [6]. An application of hybrid neutrosophic multiple criteria group decision making approach for project selection was presented [7]. The Resource levelling problem was studied in construction projects under neutrosophic environment [8].

The N-valued interval neutrosophic sets with their applications in the field of medical diagnosis was presented [16]. Based on the pentagonal neutrosophic numbers, the de-neutrosophication technique was proposed with some applications in determining the minimal spanning tree [18]. The pentagonal fuzzy numbers were studied with their different representations, properties, ranking, defuzzification. The concept of pentagonal fuzzy neutrosophic numbers was proposed with some applications in game and transportation models [19- 20]. Various forms of linear as well as non-linear form of trapezoidal neutrosophic numbers, de-neutrosophication techniques were studied. Their application were also presented in time cost optimization technique and sequencing problems [21]. The parametric divergence measure of neutrosophic sets was studied with its application in decision-making situations [25]. A technique for reducing dimensionality of data in decision-making utilizing neutrosophic soft matrices was proposed [26].

In this paper, we aim to characterize the solution of complex programming (NCP) neutrosophic numbers. The score function corresponding to the neutrosophic number is used to convert the problem into the corresponding crisp CP, and hence lexicographic order used for comparing between any two complex numbers. The comparison developed between the real and imaginary parts separately. Through this manner, the CP problem is divided into two real sub-problems.

The outlay of the paper is organized as follows: In section 2; some preliminaries are presented. In section 3, a NCP problem is formulated. Section 4 characterizes a solution to the NCP problem to obtain neutrosophic optimal solution. In section 5, two numerical examples are given for illustration. Finally some concluding remarks are reported in section 6.

2. Preliminaries

In order to discuss our problem conveniently, basic concepts and results related to fuzzy numbers, trapezoidal fuzzy numbers, intuitionistic trapezoidal fuzzy numbers, neutrosophic set, and complex mathematical programming are recalled.

Definition 1. (Trapezoidal fuzzy numbers, Kaur and Kumar [31]).

A fuzzy number $\widetilde{A} = (r, s, t, u)$. is a trapezoidal fuzzy numbers where $r, s, t, u \in \mathbb{R}$ and its membership function is defined as

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x-r}{s-r}, & r \le x \le s, \\ 1, & s \le x \le t, \\ \frac{u-x}{u-t}, & t \le x \le u, \\ 0, & \text{otherwise}, \end{cases}$$

Definition 2. (Intuitionistic fuzzy set, Atanassov, [12]).

A fuzzy set $\widetilde{A}$ is said to be an intuitionistic fuzzy set $\widetilde{A}^{IN}$of a non empty set X if $\widetilde{A}^{IN} = \{\langle x, \mu_{\widetilde{B}^{IN}}, \rho_{\widetilde{B}^{IN}} \rangle : x \in X\}$, where $\mu_{\widetilde{A}^{IN}}$, and $\rho_{\widetilde{B}^{IN}}$ are membership and nonmembership functions such that $\mu_{\widetilde{A}^{IN}}, \rho_{\widetilde{A}^{IN}} : X \to [0, 1]$ and $0 \le \mu_{\widetilde{A}^{IN}} + \rho_{\widetilde{A}^{IN}} \le 1$, for all $x \in X$.

Definition 3. (Intuitionistic fuzzy number, Atanassov, [13]).

An intuitionistic fuzzy set $\widetilde{A}^{IN}$of a $\mathbb{R}$ is called an Intuitionistic fuzzy number if the following conditions hold:

1. There exists $c \in \mathbb{R}$: $\mu_{\widetilde{A}^{IN}}(c) = 1$, and $\rho_{\widetilde{A}^{IN}}(c) = 0$.

2. $\mu_{\widetilde{A}^{IN}} : \mathbb{R} \to [0, 1]$ is continuous function such that

   $0 \le \mu_{\widetilde{A}^{IN}} + \rho_{\widetilde{B}^{IN}} \le 1$, for all $x \in X$.

3. The membership and non-membership functions of $\widetilde{B}^{IN}$ are

$$\mu_{\tilde{B}^{IN}}(x) = \begin{cases} 0, & -\infty < x < r \\ h(x), & r \leq x \leq s \\ 1, & x = s \\ l(x), & s \leq x \leq t \\ 0, & t \leq x < \infty, \end{cases}$$

$$\rho_{\tilde{B}^{IN}}(x) = \begin{cases} 0, & -\infty < x < a \\ f(x), & a \leq x \leq s \\ 1, & x = s \\ g(x), & s \leq x \leq b \\ 0, & b \leq x < \infty, \end{cases}$$

Where $f, g, h, l: \mathbb{R} \rightarrow [0, 1]$, h and g are strictly increasing functions, l and f are strictly decreasing functions with the conditions $0 \leq f(x) + f(x) \leq 1$, and $0 \leq l(x) + g(x) \leq 1$.

Definition 4. (Trapezoidal intuitionistic fuzzy number, Jianqiang and Zhong, [28]).

A trapezoidal intuitionistic fuzzy number is denoted by $\tilde{B}^{IN} = (r, s, t, u), (a, s, t, b)$, where $a \leq r \leq s \leq t \leq u \leq b$ with membership and non-membership functions are defined as:

$$\mu_{\tilde{B}^{INT}}(x) = \begin{cases} \frac{x-r}{s-r}, & r \leq x < s \\ 1, & s \leq x \leq t \\ \frac{u-x}{u-t}, & t \leq x \leq u \\ 0, & \text{otherwise}, \end{cases}$$

$$\rho_{\tilde{B}^{INT}}(x) = \begin{cases} \frac{s-x}{s-a}, & a \leq x < s \\ 0, & s \leq x \leq t \\ \frac{x-t}{b-t}, & t \leq x \leq b \\ 1, & \text{otherwise}, \end{cases}$$

Definition 5 (Neutrosophic set, Smarandache, [44]).

A neutrosophic set $\bar{B}^N$ of non-empty set X is defined as

$$\bar{B}^N = \{\langle x, I_{\bar{B}^N}(x), J_{\bar{B}^N}(x), V_{\bar{B}^N}(x)\rangle : x \in X, I_{\bar{B}^N}(x), J_{\bar{B}^N}(x), V_{\bar{B}^N}(x) \in]0_-, 1^+[\ \},$$

where $I_{\bar{B}^N}(x), J_{\bar{B}^N}(x)$, and $V_{\bar{B}^N}(x)$ are truth membership function, an indeterminacy- membership function, and a falsity- membership function and there is no restriction on the sum of $I_{\bar{B}^N}(x), J_{\bar{B}^N}(x)$, and $V_{\bar{B}^N}(x)$, so $0^- \leq I_{\bar{B}^N}(x) + J_{\bar{B}^N}(x) + V_{\bar{B}^N}(x) \leq 3^+$, and $]0_-, 1^+[$ is a nonstandard unit interval.

Definition 6. (Single-valued neutrosophic set, Wang et al., [49]).

A Single-valued neutrosophic set $\bar{B}^{SVN}$of a non empty set X is defined as: $\bar{B}^{SVN} = \{\langle x, I_{\bar{B}^N}(x), J_{\bar{B}^N}(x), V_{\bar{B}^N}(x)\rangle : x \in X\}$,

where $I_{\bar{B}^N}(x), J_{\bar{B}^N}(x)$, and $V_{\bar{B}^N}(x) \in [0,1]$ for each $x \in X$ and $0 \leq I_{\bar{B}^N}(x) + J_{\bar{B}^N}(x) + V_{\bar{B}^N}(x) \leq 3$.

Definition 7. (Single-valued neutrosophic number, Thamariselvi and Santhi, [47]).

Let $\tau_{\tilde{b}}, \varphi_{\tilde{b}}, \omega_{\tilde{b}} \in [0,1]$ and $r, s, t, u \in \mathbb{R}$ such thatr $\leq s \leq t \leq u$. Then a single valued trapezoidal neutrosophic number, $\tilde{b}^N = \langle (r, s, t, u): \tau_{\tilde{b}}, \varphi_{\tilde{b}}, \omega_{\tilde{b}} \rangle$ is a special neutrosophic set on $\mathbb{R}$, whose truth-membership, indeterminacy-membership, and falsity-membership functions are

$$\mu_{\tilde{b}}{}^N(x) = \begin{cases} \tau_{\tilde{b}^N}\left(\frac{x-r}{s-r}\right), & r \leq x < s \\ \tau_{\tilde{b}}, & s \leq x \leq t \\ \tau_{\tilde{b}^N}\left(\frac{u-x}{u-t}\right), & t \leq x \leq u \\ 0, & \text{otherwise,} \end{cases}$$

$$\rho_{\tilde{b}}{}^N(x) = \begin{cases} \frac{s-x+\varphi_{\tilde{b}^N}(x-r)}{s-r}, & r \leq x < s \\ \varphi_{\tilde{b}^N}, & s \leq x \leq t \\ \frac{x-t+\varphi_{\tilde{b}^N}(u-x)}{u-t}, & t \leq x \leq u \\ 1, & \text{otherwise,} \end{cases}$$

$$\sigma_{\tilde{b}}{}^N(x) = \begin{cases} \frac{s-x+\omega_{\tilde{b}^N}(x-r)}{s-r}, & r \leq x < s \\ \omega_{\tilde{b}^N}, & s \leq x \leq t \\ \frac{x-t+\omega_{\tilde{b}^N}(u-x)}{u-t}, & t \leq x \leq u \\ 1, & \text{otherwise.} \end{cases}$$

Where $\tau_{\tilde{b}}, \varphi_{\tilde{b}}$, and $\omega_{\tilde{b}}$ denote the maximum truth, minimum-indeterminacy, and minimum falsity membership degrees, respectively. A single-valued trapezoidal neutrosophic number $\tilde{b}^N = \langle (r, s, t, u) \colon \tau_{\tilde{b}^N}, \varphi_{\tilde{b}^N}, \omega_{\tilde{b}^N} \rangle$ may express in ill-defined quantity about b, which is approximately equal to $[s, t]$.

Definition 8.

Let $\tilde{b}^N = \langle (r, s, t, u) \colon \tau_{\tilde{b}^N}, \varphi_{\tilde{b}^N}, \omega_{\tilde{b}^N} \rangle$, and $\tilde{d}^N = \langle (r', s', t', u') \colon \tau_{\tilde{d}^N}, \varphi_{\tilde{d}^N}, \omega_{\tilde{d}^N} \rangle$ be two single-valued trapezoidal neutrosophic numbers and $v \neq 0$. The arithematic operations on $\tilde{b}^N$, and $\tilde{d}^N$ are

1. $\tilde{b}^N \oplus \tilde{d}^N = \langle (r + r', s + s', t + t', u + u');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle$,

2. $\tilde{b}^N \ominus \tilde{d}^N = \langle (r - u', s - t', t - s', u' - r);\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle$,

3. $$\tilde{b}^N \otimes \tilde{d}^N = \begin{cases} \langle (rr', ss', tt', uu');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u,\ u' > 0 \\ \langle (ru', st', st', ru');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u < 0,\ u' > 0 \\ \langle (uu', ss', tt', rr');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u < 0,\ u' < 0, \end{cases}$$

4. $$\tilde{b}^N \oslash \tilde{d}^N = \begin{cases} \langle (r/u', s/t', t/s', u/r');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u,\ u' > 0 \\ \langle (u/u', t/t', s/s', r/r');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u < 0,\ u' > 0 \\ \langle (u/r', t/s', s/t', r/u');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u < 0,\ u' < 0, \end{cases}$$

5. $$k\tilde{d}^N = f(x) = \begin{cases} \langle (kr, ks, kt, k);\ \tau_{\tau_{\tilde{d}^N}},\ \varphi_{\tau_{\tilde{d}^N}}, \omega_{\tau_{\tilde{d}^N}} \rangle, k > 0, \\ \langle (ku, kt, ks, k\,r);\ \tau_{\tau_{\tilde{d}^N}},\ \varphi_{\tau_{\tilde{d}^N}}, \omega_{\tau_{\tilde{d}^N}} \rangle, k < 0, \end{cases}$$

6. $\tilde{d}^{N^{-1}} = \langle (1/u', 1/t', 1/s', 1/r');\ \tau_{\tau_{\tilde{d}^N}},\ \varphi_{\tau_{\tilde{d}^N}}, \omega_{\tau_{\tilde{d}^N}} \rangle, \tilde{d}^N \neq 0.$

Definition 9 (Score function of single-valued trapezoidal neutrosophic number, Thamaraiselvi and Santhi [47]).

A two single-valued trapezoidal neutrosophic numbers $\tilde{b}$, and $\tilde{d}$ can be compared based on the score function as

$$\text{Score function } SC(\tilde{b}^N) = \left(\frac{1}{16}\right)[r+s+t+u] * [\mu_{\tilde{b}^N} + (1-\rho_{\tilde{b}^N}(x) + (1-\sigma_{\tilde{b}^N}(x)].$$

Definition 10. (Thamaraiselvi and Santhi, [47]).

The order relations between $\tilde{b}^N$ and $\tilde{d}^N$ based on $SC(\tilde{b}^N)$ are defined as

1. If $SC(\tilde{b}^N) < SC(\tilde{d}^N)$, then $\tilde{b}^N \prec \tilde{d}^N$
2. If $SC(\tilde{b}^N) = SC(\tilde{d}^N)$, then $\tilde{b}^N \approx \tilde{d}^N$,
3. If $SC(\tilde{b}^N) > SC(\tilde{d}^N)$, then $\tilde{b}^N \succ \tilde{d}^N$,

3. Problem definition and solution concepts

Consider the following single -valued trapezoidal neutrosophic (NCP) problem

(NCP) $\quad \min \ \tilde{F}^N(x) = \tilde{v}^N(x) + i\,\tilde{w}^N(x)$

Subject to (1)

$$x \in \tilde{X}^N = \left\{ \begin{matrix} x \in \Re^n : \tilde{f}_r^{\ N}(x) = \tilde{p}_r^{\ N}(x) + i\,\tilde{q}_r^{\ N}(x) \le \tilde{l}_r^{\ N} + i\,\tilde{h}_r^{\ N}, \\ r = 1, 2, \dots, m \end{matrix} \right\}$$

Where, $\tilde{v}^N(x) = \sum_{j=1}^{n} \tilde{c}_j^{\ N} x_j$, $\tilde{w}^N(x) = \sum_{j=1}^{n} \tilde{d}_j^{\ N} x_j, j = 1, 2, \dots, n$; $\tilde{p}_r^{\ N}(x) = \sum_{j=1}^{n} x_j^T \tilde{a}_{rj}^{\ N} x_j$, $\tilde{q}_r^{\ N}(x) = \sum_{j=1}^{n} \tilde{e}_{rj}^{\ N} x_j$, are convex functions on $\tilde{X}^N$.

All of $\tilde{c}_j^{\ N}, \tilde{d}_j^{\ N}$, $\tilde{a}_{rj}^{\ N}, \tilde{e}_{rj}^{\ N}$ $\tilde{l}_r^{\ N} = (\tilde{l}_1^{\ N}, \tilde{l}_2^{\ N}, \dots, \tilde{l}_m^{\ N})^T$, $\tilde{h}_r^{\ N} = (\tilde{h}_1^{\ N}, \tilde{h}_2^{\ N}, \dots, \tilde{h}_m^{\ N})^T$ are single-valued trapezoidal neutrosophic numbers.

Definition 11.

Lexicographic order of two complex numbers $z_1 = a + i\,b$, and $z_2 = c + i\,d$ is defined as $z_1 \le z_2 \Leftrightarrow a \le c$ and $b \le d$.

Definition 12.

A neutrosophic feasible point $\mathbf{x}^{\circ}$ is called single-valued trapezoidal neutrosophic optimal solution to NCP problem if

$$\tilde{v}^N(x^{\circ}) \leq \tilde{v}^N(x), \text{ and } \tilde{w}^N(x^{\circ}) \leq \tilde{w}^N(x) \text{ for each } x \in \tilde{X}^N.$$

According to the score function in Definition 9, the NCP problem is converted into the following crisp CP problem as

$$\text{(CP)} \quad \text{Min } F(x) = v(x) + i\, w(x)$$

$$\text{Subject to} \qquad (2)$$

$$x \in X = \{x \in \mathbb{R}: f_r(x) = p_r(x) + i\, q_r(x) \leq l_r + i\, h_r,\ r = 1, 2, \dots, m\}.$$

4. Characterization of neutrosophic optimal solution for NCP problem

To characterize the neutrosophic optimal solution of NCP problem, let us divide the CP problem into the following two sub-problems

$$(P_v) \quad \text{Min } v(x)$$

$$\text{Subject to} \qquad (3)$$

$$x \in X = \{x \in \mathbb{R}^n: f_r(x) = p_r(x) + i\, q_r(x) \leq l_r + i\, h_r,\ r = 1, 2, \dots, m\}, \text{ and}$$

$$(P_w) \quad \text{Min } w(x)$$

$$\text{Subject to} \qquad (4)$$

$$x \in X = \{x \in \mathbb{R}^n: f_r(x) = p_r(x) + i\, q_r(x) \leq l_r + i\, h_r,\ r = 1, 2, \dots, m\}.$$

Definition 13.

$x^{\circ} \in X$ is said to be an optimal solution for P_{CP} if and only if $v(x^{\circ}) \leq v(x)$, and $w(x^{\circ}) \leq w(x)$ for each $x \in X$.

Let us denote S_v and S_w be the set of solution for P_v and P_w, respectively, i.e.,

$$S_v = \{x^{\circ} \in X: v(x^{\circ}) \leq v(x);\ \text{for all } x \in X\}, \text{ and} \qquad (5)$$

$$S_w = \{x^{*} \in X: v(x^{*}) \leq v(x);\ \text{for all } x \in X\} \qquad (6)$$

Lemma 1. For $S_v \cap S_w \neq \emptyset$, the solution of CP problem is embedded into $S_v \cap S_w$.

Proof. Assume that x° be a solution of CP, this leads to $v(x^\circ) \leq v(x);\ \forall\, x \in X$ (i.e., $x^\circ \in S_v$).

Similarly, $w(x^\circ) \leq w(x);\ \forall\, x \in X$ (i. e., $x^\circ \in S_w$). Then, $x^\circ \in S_v \cap S_w$.

In this paper, we focus on the case $S_v \cap S_w = \emptyset$.

Lemma 2. If S_v and S_w are open, $S_v \cap S_w = \varnothing$, and v, w are strictly convex functions on X then $\check{x} \in S_v$ is a solution of a conjugate function $\overline{F}(x) = v(x) - i\, w(x)$.

Proof. Since $x^\circ \in S_v$, then $v(x^\circ) \leq v(x)$ for all $x \in X$. Also,

$$v(x^\circ) \leq v(x^*) \text{ for all } x^* \in S_v \subset X \tag{7}$$

But $x^* \in S_w$ which means that $w(x^*) \leq v(x^\circ)$, for all $\check{x} \in S_v \subset X$ and $-i\,w(x^*) \geq -i\,w(x^\circ)$ i. e.,

$$-i\,w(x^\circ) \leq -i\,w(x^*) \tag{8}$$

From (7) and (8), we get

$v(x^\circ) - i\,w(x^\circ) \leq v(x^*) - i\,w(x^*)$, for all $x^* \in S_w$, (.i.e., $x^\circ \in S_v$)is a solution of a conjugate function $\overline{F}(x) = v(x) - i\,w(x)$. Now we will prove that there is no $\hat{x} \in X$ and $\hat{x} \notin S_v$ such that

$$\overline{F}(\hat{x}) = v(\hat{x}) - i\,w(\hat{x}) \leq \overline{F}(x^\circ) = v(x^\circ) - i\,w(x^\circ) \tag{9}$$

There are two cases:

Case 1: Assume that $x \in X$ $\hat{x} \notin S_v$, $\acute{x} \in S_v$ and $v(\hat{x}) - i\,w(\hat{x}) \leq v(x^\circ) - i\,w(x^\circ)$ i.e.,

$$w(x^\circ) \leq w(\hat{x})$$

Since the function $w(x)$ is strictly convex and S_w is open, then

$$w\left(\tau\hat{x} + (1-\tau)x^\circ\right) < \tau w(\hat{x}) + (1-\tau)w(x^\circ),\ 0 < \tau < 1. \text{ This leads to}$$

$$w\left(\tau\hat{x} + (1-\tau)x^\circ\right) < \tau w(\hat{x}) + (1-\tau)w(\hat{x}) \text{ i. e.,}$$

For certain τ such that $\tau\hat{x} + (1-\tau)x^\circ \in S_w$, we obtain

$$w\left(\tau \hat{x}+(1-\tau) x^{\circ}\right)<w(\hat{x}),$$

which contradicts to $\hat{x} \in S_w$ i.e., there is no $\hat{x} \in X$, $\hat{x} \notin S_v$, $\hat{x} \in S_w$ such that

$$\bar{F}(\hat{x})=v(\hat{x})-i\, w(\hat{x}) \leq \bar{F}\left(x^{\circ}\right)=v\left(x^{\circ}\right)-i\, w\left(x^{\circ}\right)$$

Case 2: Assume that $\hat{x} \in X, \hat{x} \notin S_v, \hat{x} \notin S_w$ and $v(\hat{x})-i\, w(\hat{x})<v\left(x^{\circ}\right)-i\, w\left(x^{\circ}\right)$ i.e.,

$$v(\hat{x})<v\left(x^{\circ}\right) \text{ and } w\left(x^{\circ}\right)<w(\hat{x}).$$

Since the function $v(x)$ is strictly convex and S_v is open, then,

$v\left(\tau x^{\circ}+(1-\tau) \hat{x}\right)<\tau v\left(x^{\circ}\right)+(1-\tau) v(\hat{x}), 0 \leq \tau \leq 1$. This leads to

$v\left(\tau x^{\circ}+(1-\tau) \hat{x}\right)<\tau v\left(x^{\circ}\right)(1-\tau) v(\hat{x})$, i.e., for certain τ, we have

$\tau x^{\circ}+(1-\tau) \hat{x} \in S_v$, such that $\tau x^{\circ}+(1-\tau) \hat{x} \in S_v$, we have

$$v\left(\tau x^{\circ}+(1-\tau) \hat{x}\right)<v\left(x^{\circ}\right),$$

which contradicts $x^{\circ} \in S_v$. Thus, there is no $\hat{x} \in X$ such that

$$v(\hat{x})-i\, w(\hat{x})<v\left(x^{\circ}\right)-i\, w\left(x^{\circ}\right)$$

5. Numerical examples

Example1. (Illustration of Lemma 1)

Consider the following complex problem

$$\min(\cos x + i \sin x)$$

Subject to (10)

$$x \in X=\{x \in \mathbb{R}: 0 \leq x \leq \pi\}.$$

Problem (10) is divided into the following two problems as:

(P_v) Min $\cos x$

Subject to (11)

$x \in X=\{x \in \mathbb{R}: 0 \leq x \leq \pi\}$, and

(P_w) Min $\sin x$

Subject to

$$x \in X = \{x \in \mathbb{R}: 0 \leq x \leq \pi\}, \tag{12}$$

The optimal solutions of problem (11) and (12) are $x = \pi$ (i. e., $S_v = \{\pi\}$), and $x = (0, \pi)$, (i. e., $S_w = \{0, \pi\}$), respectively. Thus, the optimal solution of problem (10) is $x = \pi \in S_v \cap S_w$.

Example2. (Illustration of Lemma 2)

Consider the following NCP problem:

$$\text{Min} \quad \tilde{F}^N(x) = \left(\tilde{c}_1{}^N x_1 + \tilde{c}_2{}^N x_2\right) + i\left(\tilde{d}_1{}^N x_1 - \tilde{d}_2{}^N x_2\right)$$

Subject to (13)

$$\left(\tilde{p}_{11}{}^N x_1^2 + \tilde{p}_{22}{}^N x_2^2\right) + i\left(\tilde{q}_1{}^N x_1 + \tilde{q}_2{}^N x_2\right) = \tilde{e}^N + i\tilde{g}^N.$$

Where,

$$\tilde{c}_1{}^N = \langle 5, 8, 10, 14; 0.3, 0.6, 0.6\rangle,$$

$$\tilde{c}_2{}^N = \langle 0, 1, 3, 6; 0.7, 0.5, 0.3\rangle,$$

$$\tilde{d}_1{}^N = \langle 4, 8, 11, 15; 0.6, 0.3, 0.2\rangle,$$

$$\tilde{d}_2{}^N = \langle 16, 18, 22, 30; 0.6, 0.2, 0.4\rangle,$$

$$\tilde{p}_{11}{}^N = \tilde{p}_{22}{}^N = \langle 0, 1, 3, 6; 0.7, 0.5, 0.3\rangle,$$

$$\tilde{q}_1{}^N = \langle 0, 1, 3, 6; 0.7, 0.5, 0.3\rangle = \tilde{q}_2{}^N,$$

$$\tilde{e}^N = \langle 4, 8, 11, 15; 0.6, 0.3, 0.2\rangle,$$

$$\tilde{g}^N = \langle 0, 1, 3, 6; 0.7, 0.5, 0.3\rangle.$$

Using the score function of the single- valued trapezoidal neutrosophic number introduced in Definition 9, the above problem become:

$$\text{Min} \quad F(x) = (3x_1 + x_2) + i(5x_1 - 11x_2)$$

Subject to (14)

$$x_1^2 + x_2^2 + i(x_1 - x_2) \leq 5 + i\,.$$

According to Lexicographic order, the problem is divided into the following two sub-problems as:

(P_v) Min $v(x) = 3x_1 + x_2$

Subject to (15)

$x_1^2 + x_2^2 \leq 5,$

$x_1 - x_2 \leq 1,$ and

(P_w) Min $w(x) = 5x_1 - 11x_2$

Subject to (16)

$x_1^2 + x_2^2 \leq 5,$

$x_1 - x_2 \leq 1.$

By applying the Kuhn-Tucker optimality conditions [14, 22], the optimal solutions of problems (15), and (16) are illustrated as in the following Tables 1 -2.

Table 1. The set of solution of $(\mathbf{P_v})$

S_v	Optimum value
$\{(-2,-1)\}$	$P_v = -7$ $\tilde{P}_v^{\,N} = \langle -34, -23, -17, -10; 0.3, 0.6, .06 \rangle$

Table 2. The set of solution of $(\mathbf{P_w})$

S_w	Optimum value
$\{(-2,1)\}$	$\mathrm{P_w} = -21$ $\tilde{P}_w{}^N = \langle -60, -44, -34, -24; 0.6, 0.3, 0.4\rangle$

Therefore, $\mathrm{S_v} \cap \mathrm{S_w} = \emptyset$ and S_v is not a solution of the conjugate function $\mathrm{v(x) - i\, w(x)}$, because of $\mathrm{v(x)}$, and $\mathrm{w(x)}$ are not strictly convex functions.

6. Concluding Remarks

In this paper, the solution of complex programming (NCP) with single valued trapezoidal neutrosophic numbers in all the parameters of objective function and constraints has been characterized. Through the use of the score function, the NCP has converted into the corresponding crisp CP problem and hence Lexicographic order has been used for comparing between any two complex numbers. The comparison was developed between the real and imaginary parts separately. Through this manner, the CP problem has divided into two real sub-problems. The main contribution of this approach is more flexible and makes a situation realistic to real world application. The obtained results are more significant to enhance the applicability of single-valued trapezoidal neutrosophic number in various new fields of decision-making situations. The future research scope is to apply the proposed approach to more complex and new applications. Another possibility is to work on the interval type complex neutrosophic sets for the applications in forecasting field.

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Analyzing Age Group and Time of the Day Using Interval Valued Neutrosophic Sets

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S. Broumi, M. Lathamaheswari, A. Bakali, M. Talea, F. Smarandache, D. Nagarajan, Kavikumar, Guennoun Asmae (2020). Analyzing Age Group and Time of the Day Using Interval Valued. *Neutrosophic Sets and Systems* 32, 361-371

Abstract: Human psychological behavior is always uncertain in nature with the truth, indeterminacy and falsity of the information and hence neutrosophic logic is able to deal with this kind of real world problems as it resembles human's attitude very closely. In this paper, age group analysis and time (day or night) analysis have been carried out using interval valued neutrosophic sets. Further, the impact of the present work is presented.

Keywords: Neutrosophic Logic; Human Psychological Behavior; Age Group; Day; Interval Valued Neutrosophic Set.

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1. Introduction

Uncertainty saturates our daily lives and period the entire range from index fluctuations of stock market to prediction of weather and car parking in a congested area to traffic control management. Hence almost all the area contains ambiguity or impression. For various real world problems, intelligent models with many types of mathematical designs of different logics have been modeled by the researchers. In the area of computational intelligence, fuzzy logic is one of the superior logic that provides appropriate representation of real world information and permits reasoning that are almost accurate in nature [1].

Generally the inputs conquered by the fuzzy logic are determinate and complete. Humans can able to take knowledgeable decisions in those situations, however it is difficult to express in proper terms. But fuzzy models need complete information. Due to basic non-linearity, huge erratic substantial disturbances, time varying nature, difficulties to find precise and predictable measurements, incompleteness and indeterminacy may arise in the data. All these problems can be dealt by neutrosophic logic proposed by Smarandache in the year 1999 [2-10]. Also this logic can able to represent mathematical structure of uncertainty, ambiguity, vagueness, imprecision, inconsistency, incompleteness and contradiction.

Also it is efficient in characterizing various attributes of data such as incompleteness and inaccuracy and hence gives proper estimation about the authenticity of the information. This approach proposes extending the proficiencies of representation of fuzzy logic and system of reasoning by introducing neutrosophic representation of the information and system of neutrosophic reasoning. Neutrosophic logic can exhibit various logical behaviors according to the nature of the problem to be solved and hence it influences its chance to be utilized and experimented for real world performance and simulations in human psychology [15].

Due to computational complexity of the neutrosophic sets, single valued neutrosophic sets have been introduced. It can deal with only exact numerical value of the three components truth, indeterminacy and falsity. While the data in the form of interval, then single valued neutrosophic sets unable to scope up and hence interval valued neutrosophic sets have been introduced. As it has lower and upper membership functions it can deal more uncertainty with less computational complexity than other types [25]. Neutrosophic set has been used in several areas like traffic control management, solving minimum spanning tree problem, analyzing failure modes and effect analysis, blockchain technology, resource leveling problem, medical diagnostic system, evaluating time-cost tradeoffs, analysis of criminal behavior, petal analysis, decision making problem etc. [26-40].

The major advantage of neutrosophic set and its types namely single valued neutrosophic sets and interval valued neutrosophic sets overrule other sets namely conventional set, fuzzy set, type-2 fuzzy, intuitionistic fuzzy and type-2 intuinistic fuzzy by their capability of dealing with indeterminacy which is missing with other types of sets. Since there is a possibility of having interval number than the exact number we consider interval valued neutrosophic set in this study of analyzing age group and time. Prediction of future trend is one of the interesting areas in the research field. Hence, in this paper, age group analysis and time (day or night) analysis have been done using interval valued neutrosophic sets. The remaining part of the paper is organized as follows. In section 2, review of literature is given. In section 3, preliminaries are given for better understanding of the paper. In section 4, age group and day and night time have been analyzed using the concept of interval valued neutrosophic sets. In section 5, impact of the present work is given. In section 6, concluded the present work with the future direction.

2. Review of Literature

The author in, [1] analyzed uncertainty exists in the project schedule using fuzzy logic. And the authors of, [2] analyzed power flow using fuzzy logic. [3] Examined specific seasonal prediction spatially under fuzzy environment for the group of long-term daily rainfall and temperature data spatiotemporally. [4] examined about the prediction of temperature flow of the atmosphere based on fuzzy knowledge–rule base for interior cities in India. [5] proposed a novel approach for intuitionistic fuzzy sets and its applications in the prediction area.

[6] proposed single-valued neutrosophic minimum spanning tree and its aggregation method. [7] proposed a new approach for the advisory of weather using fuzzy logic. [8] Proposed a method for prediction of weather under fuzzy neural network environment and Hierarchy particle swarm optimization algorithm. [9] Proposed various types of neutrosophic graphs and algebraic model and applied in the field of technology. [10] proposed single valued neutrosophic graphs (SVNGs).

[11] examined bipolar single valued neutrosophic graphs. [12] Proposed interval valued neutrosophic graphs. [13] proposed isolated SVNGs. [14] provided an introduction to the theory bipolar SVNG. [15] proposed the degree, size and order of SVNGs. 16] applied Dijkstra algorithm to solve shortest path problem under IVN environment. [17] solved minimum spanning tree problem under trapezoidal fuzzy neutrosophic environment.

[18] applied minimum spanning tree algorithm for shortest path (SP) problem using bipolar neutrosophic numbers. [19] proposed a novel matrix algorithm for solving MST for undirected interval value NG. [20] solved a spanning tree problem with neutrosophic edge weights. [21] proposed a new algorithm to solve MST problem with undirected NGs. [22] analyzed the role of SVNSs and rough sets with imperfect and incomplete information systems.

[23] Studied about neutrosophic set and its development . [24] studied about the prediction of long-term weather elements using adaptive neuro-fuzzy system using GIS approach in Jordan. [25] have done overview of neutrosophic sets. [26] proposed a methodology of traffic control management using triangular interval type-2 fuzzy sets and interval neutrosophic sets. [27] Solved MST problem using single valued trapezoidal neutrosophic numbers.

[28] estimated risk priority number in design failure modes and effect analysis using factor analysis. [29] have done edge detection on DICOM image using type-2 fuzzy logic. [30] made a review on the applications of type-2 fuzzy in the field of biomedicine. [31] have done image extraction on DICOM image usingtype-2 fuzzy. [32] made a review on application of type-2 fuzzy in control system. [33] proposed single and interval valued neutrosophic graphs using blockchain technology. [34] introduced interval valued neutrosophic graphs using Dombi triangular norms. [35] solved resource leveling problem under neutrosophic environment.

[36] introduced cosine similarity measures of bipolar neutrosophic sets and applied in diagnosis of disorder diseases. [37] introduced a methodology for petal analysis using neutrosophic cognitive maps. [38] analyzed criminal behavior using neutrosophic model. [39] presented assessments of linear time-cost tradeoffs using neutrosophic sets. [40] solved sustainable supply chain risk management problem using plithogenic TOPSIS-CRITIC methodology. In view of the literature, prediction of age group and day or night time under interval neutrosophic set are yet to be studied and which is the reason of the present study.

3. Preliminaries

In this section, preliminaries of the proposed concept are given

3. 1. Neutrosophic Set (NS) [25]

Consider the space X consists of universal elements characterized by e. The NS A is a phenomenon which has the structure $A = \left\{ \left(T_A(e), I_A(e), F_A(e) \right) / e \in X \right\}$ where the three grades of

memberships are from X to]−0,1+[of the element $e \in X$ to the set A, with the criterion:
$$^{-}0 \le T_A(e) + I_A(e) + F_A(e) \le 3^{+} \qquad (1)$$

The functions $T_A(e)$, $I_A(e)$ and $F_A(e)$ are the truth, indeterminate and falsity grades lies in real standard/non-standard subsets of] −0, 1+ [.

Since there is a complication of applying NSs to real issues, Samarandache and Wang et al. [11-12] proposed the notion of SVNS, which is a specimen of NS and it is useful for realistic applications of all the fields.

3.2. Single Valued Neutrosophic Set (SVNS) [25]

For the space X of objects contains global elements e. A SVNS is represented by degrees of bership grades mentioned in Def. 2.8. For all e in X, $T_A(e)$, $I_A(e)$, $F_A(e) \in [0, 1]$. A SVNS can be written as

$$A = \{\langle e : T_A(e), I_A(e), F_A(e)\rangle / e \in X\} \qquad (2)$$

3.3. Interval Valued Neutrosophic Set [12]

Let X be a space of objects with generic elements in X denoted by e. An interval valued neutrosophic set (IVNS) A in X is characterized by truth-membership function, $T_A(e)$, indeterminacy-membership function $I_A(e)$ and falsity membership function $F_A(e)$. For each point e in X, $T_A(e)$, $I_A(e), F_A(e) \in [0,1]$, and an IVNS A is defined by

$$A = \left\{\left\langle \left[T_A^L(e), T_A^U(e)\right], \left[I_A^L(e), I_A^U(e)\right], \left[F_A^L(e), F_A^U(e)\right]\right\rangle \mid e \in X\right\} \qquad (3)$$

Where, $T_A(e) = \left[T_A^L(e), T_A^U(e)\right]$, $I_A(e) = \left[I_A^L(e), I_A^U(e)\right]$ and $F_A(e) = \left[F_A^L(e), F_A^U(e)\right]$

Fig 1 shows the Pictorial Representation of the neutrosophic set [5]

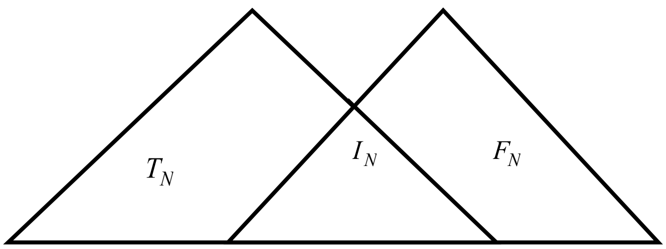

Fig.1. Neutrosophic set

4. Proposed Methodology

In this section, age group and time (day or night) have been analyzed using interval valued neutrosophic set.

4.1 Application of Interval Valued Neutrosophic Set in Age Group Analysis

As per our convenience, the age group is divided into three groups: young people, middle aged people and old people. Assume young people are a truth membership function, middle aged people are indeterminate membership function and old people are a falsity membership function. Here, the degree of middle aged people may provide either degree of old people or young people or both. Let us consider the age group is definitely young at and below 18-40, it is definitely old at and beyond 51-100 and in between the age group is middle. i.e., the level of the young age people decreases and the level of old age people increases. The age group is represented pictorially for young people, middle aged people and old people as in Fig. 2.

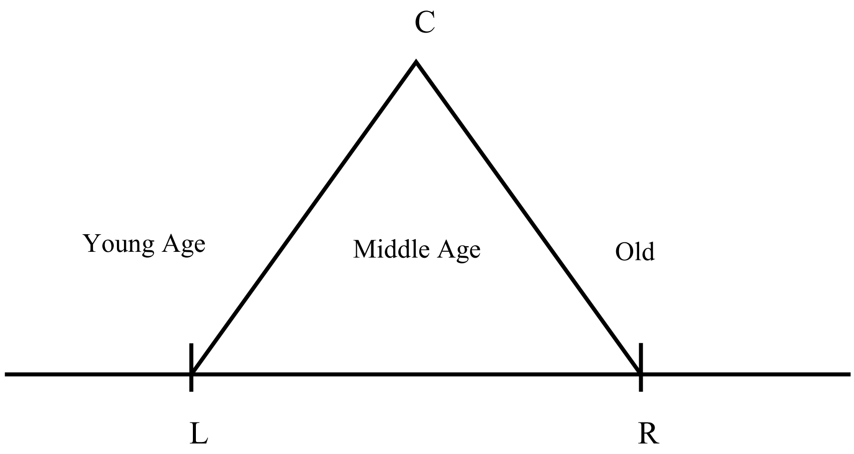

Fig.2. The degrees of 'young age', 'middle age' and 'old age' people.

Let A be the different age groups of the people and N be an interval valued neutrosophic set defined in the set A. Let $T_N(a)$ be the membership degree of the age group 'young age people' at a, here a denotes a numerical value. For example, $a = 20$. Similarly, indeterminate degree of 'middle age people' can be denoted by $I_N(a)$ and the falsity degree of 'old age people' denoted by $F_N(a)$ at a.

Consider $A = \{\langle [18,40],[41,50],[51,100] \rangle\}$ and

$$N = \{\langle T_N([18,40]), I_N([18,40]), F_N([18,40]) \rangle,$$

$$\langle T_N([41,50]), I_N([41,50]), F_N([41,50]) \rangle, \langle T_N([51,100]), I_N([51,100]), F_N([51,100]) \rangle\}.$$

Case (i). At and below [18, 40], there is no middle age people and old age people but there exist only young age people. Therefore the following values are obtained.

$\left[T_N^L, T_N^U\right]([18,40]) = [1,1]$, $\left[I_N^L, I_N^U\right]([18,40]) = [0,0]$ and

$\left[F_N^L, F_N^U\right]\left(\left[18,40\right]\right)=\left[0,0\right]$

i.e., the membership function of the interval valued neutrosophic set is $\left(\left[1,1\right],\left[0,0\right],\left[0,0\right]\right)$

Case (ii). At age [41, 50] (at the point C)

$\left[T_N^L, T_N^U\right]\left(\left[41,50\right]\right)=\left[0,0\right]$, $\left[I_N^L, I_N^U\right]\left(\left[41,50\right]\right)=\left[1,1\right]$ and

$\left[F_N^L, F_N^U\right]\left(\left[41,50\right]\right)=\left[0,0\right]$

i.e., the membership function of the interval valued neutrosophic set is $\left(\left[0,0\right],\left[1,1\right],\left[0,0\right]\right)$

Case (iii). At and above [51,100], there are no young age people and middle age people, but there exist only old age people.

$\left[T_N^L, T_N^U\right]\left(\left[51,100\right]\right)=\left[0,0\right]$, $\left[I_N^L, I_N^U\right]\left(\left[51,100\right]\right)=\left[0,0\right]$ and $\left[F_N^L, F_N^U\right]\left(\left[51,100\right]\right)=\left[1,1\right]$

i.e., the membership function of the interval valued neutrosophic set is $\left(\left[0,0\right],\left[0,0\right],\left[1,1\right]\right)$

Hence, $N=\left\{\left\langle\left[1,1\right],\left[0,0\right],\left[0,0\right]\right\rangle,\left\langle\left[0,0\right],\left[1,1\right],\left[0,0\right]\right\rangle,\left\langle\left[0,0\right],\left[0,0\right],\left[1,1\right]\right\rangle\right\}$

Also, young age people decreases and middle age people increases in between L and C.

i.e., $\left[1,1\right]>\left[T_N^L, T_N^U\right]>\left[0,0\right]$ and $\left[0,0\right]<\left[I_N^L, I_N^U\right]<\left[1,1\right]$

Further, middle age people decreases and old age people increases in between C and R.

i.e., $\left[1,1\right]>\left[I_N^L, I_N^U\right]>\left[0,0\right]$ and $\left[0,0\right]<\left[F_N^L, F_N^U\right]<\left[1,1\right]$

4.2 Application of Interval Valued Neutrosophic Set in Day and Night Time Analysis

As per our convenience, time of the day is divided into three groups: day, day or night (or both) and night. Assume day time is a truth membership function, day or night (or both) is an indeterminate membership function and night time is a falsity membership function. Here, the degree of day or night time may provide either degree of day time or night time or both. Let us consider the time of the day is definitely day time at and below 7 AM to 6 PM, it is definitely night at and beyond 7 PM and 5 AM and in between time is day or night. i.e., the level of the day time decreases and the level of night time increases. The time of the day is represented pictorially for day, day or night people and night as in Fig. 3.

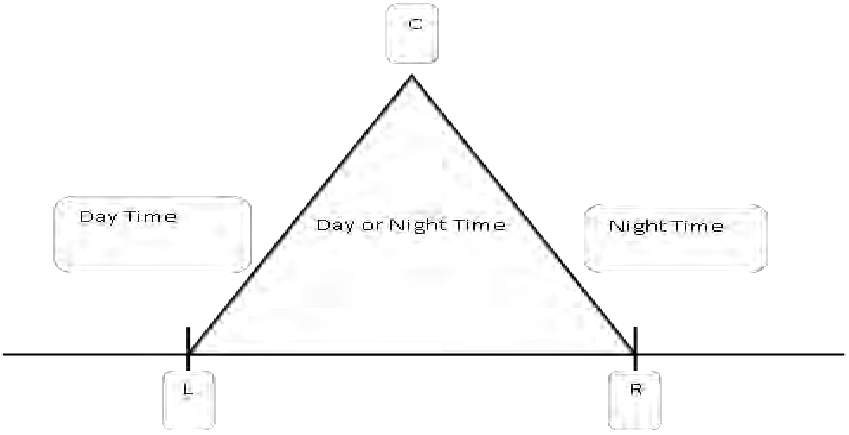

Fig.3. The degrees of time for 'day', 'day or night' and 'night'

Let B be the different times of the day, M an interval valued neutrosophic set defined in the set B. Let $T_M(b)$ be the membership degree of the time 'day' at b, here, b denotes a numerical value.

For example $b = 8$ AM or PM. Similarly, the indeterminate degree of the time $I_N(b)$ and the falsity degree of the time $F_M(b)$ can be represented by b.

Consider two cases.

$B = \{\langle [7AM, 6PM], [5AM, 6AM], [7PM, 5AM] \rangle\}$ and

$$M = \{\langle T_N([7AM,6PM]), I_N([7AM,6PM]), F_N([7AM,6PM])\rangle,$$

$$\langle T_N([5AM,6AM]), I_N([5AM,6AM]), F_N([5AM,6AM])\rangle,$$

$$\langle T_N([7PM,5AM]), I_N([7PM,5AM]), F_N([7PM,5AM])\rangle\}.$$

Also we can consider, $B = \{\langle [7AM, 6PM], [6PM, 7PM], [7PM, 5AM] \rangle\}$ and

$$M = \{\langle T_N([7AM,6PM]), I_N([7AM,6PM]), F_N([7AM,6PM])\rangle,$$

$$\langle T_N([6PM,7PM]), I_N([6PM,7PM]), F_N([6PM,7PM])\rangle,$$

$$\langle T_N([7PM,5AM]), I_N([7PM,5AM]), F_N([7PM,5AM])\rangle\}.$$

Case (i). At and below [7AM, 6 PM], there is no hesitation of day or night time and no night time but there exist only day time. Therefore the following values are obtained.

$[T_N^L, T_N^U]([7AM, 6PM]) = [1,1]$

$[I_N^L, I_N^U]([7AM, 6PM]) = [0,0]$ and

$\left[F_N^L, F_N^U\right]\left(\left[7AM, 6PM\right]\right) = \left[0,0\right]$

i.e., the membership function of the interval valued neutrosophic set is $\left(\left[1,1\right],\left[0,0\right],\left[0,0\right]\right)$

Case (ii). At [5AM, 6AM] (at the point C) and at [6 PM, 7PM]

$\left[T_N^L, T_N^U\right]\left(\left[5AM, 6AM\right]\right) = \left[0,0\right]$and $\left[T_N^L, T_N^U\right]\left(\left[6PM, 7PM\right]\right) = \left[0,0\right]$

$\left[I_N^L, I_N^U\right]\left(\left[5AM, 6AM\right]\right) = \left[1,1\right]$and $\left[I_N^L, I_N^U\right]\left(\left[6PM, 7PM\right]\right) = \left[1,1\right]$

$\left[F_N^L, F_N^U\right]\left(\left[5AM, 6AM\right]\right) = \left[0,0\right]$and $\left[F_N^L, F_N^U\right]\left(\left[6PM, 7PM\right]\right) = \left[0,0\right]$

i.e., the membership function of the interval valued neutrosophic set is $\left(\left[0,0\right],\left[1,1\right],\left[0,0\right]\right)$

Case (iii). At and above [7 PM, 5 PM], there is no day time and no hesitation of day or night time, but there exist only night time.

$\left[T_N^L, T_N^U\right]\left(\left[7PM, 5AM\right]\right) = \left[0,0\right]$

$\left[I_N^L, I_N^U\right]\left(\left[7PM, 5AM\right]\right) = \left[0,0\right]$and

$\left[F_N^L, F_N^U\right]\left(\left[7PM, 5AM\right]\right) = \left[1,1\right]$

i.e., the membership function of the interval valued neutrosophic set is $\left(\left[0,0\right],\left[0,0\right],\left[1,1\right]\right)$

Hence, $M = \left\{\left\langle\left[1,1\right],\left[0,0\right],\left[0,0\right]\right\rangle, \left\langle\left[0,0\right],\left[1,1\right],\left[0,0\right]\right\rangle, \left\langle\left[0,0\right],\left[0,0\right],\left[1,1\right]\right\rangle\right\}$

Also, day time decreases and day or night time increases in between L and C.

i.e., $\left[1,1\right] > \left[T_N^L, T_N^U\right] > \left[0,0\right]$ and $\left[0,0\right] < \left[I_N^L, I_N^U\right] < \left[1,1\right]$

Further, day or night time decreases and night time increases in between C and R.

i.e., $\left[1,1\right] > \left[I_N^L, I_N^U\right] > \left[0,0\right]$ and $\left[0,0\right] < \left[F_N^L, F_N^U\right] < \left[1,1\right]$

5. Impacts of the work

i). The proposed approach is the effective one in determining age group forecasting while the data is in the form of interval data with indeterminate information too.

ii). Time (day or night) analysis under interval neutrosophic environment will be very useful as it is the major scientific and technical problems.

iii). Analysing any future trend can be done easily by inferring the existing information into the future using interval neutrosophic sets as it has the capacity of addressing with the set of numbers in the real unit interval which is not just a determined number, it is efficient to deal with real world problems with various possible interval values

iv). The proposed methodology of age group analysis can be used in facial image analysis as age detection system.
v). The proposed methodology of time analysis can be utilized in time series analysis.

6. Conclusion

Since neutrosophic logic resembles human behavior for predicting age and time (day or night), it is suitable for this study. According to the knowledge of human, membership values of the truth, indeterminacy and falsity may be exact numbers or interval numbers. In this paper, analysis of age group and time(day or night) have been done using interval valued neutrosophic set with the detailed description and pictorial representation. Also the impact of the present work has been given. In future, the proposed concept can be done based on the concept of neutrosophic rough and soft sets.

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On Some Types of Neutrosophic Topological Groups with Respect to Neutrosophic Alpha Open Sets

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Abstract: In this article, we presented eight different types of neutrosophic topological groups, each of which depends on the conceptions of neutrosophic α-open sets and neutrosophic α-continuous functions. Also, we found the relation between these types, and we gave some properties on the other side.
Keywords: Neutrosophic α-open sets, neutrosophic α-continuous functions, neutrosophic topological groups, and neutrosophic topological groups of type (R), $R = 1,2,3, \dots ,8$.

1. Introduction

Smarandache [1,2] originally handed the theory of "neutrosophic set". Recently, Abdel-Basset et al. discussed a novel neutrosophic approach [3-6]. Salama et al. [7] gave the clue of neutrosophic topological space. Arokiarani et al. [8] added the view of neutrosophic α-open subsets of neutrosophic topological spaces. Dhavaseelan et al. [9] presented the idea of neutrosophic α^m-continuity. Banupriya et al. [10] investigated the notion of neutrosophic αgs continuity and neutrosophic αgs irresolute maps. Nandhini et al. [11] presented Nαg#ψ-open map, Nαg#ψ-closed map, and Nαg#ψ-homomorphism in neutrosophic topological spaces. Sumathi et al. [12] submitted the perception of neutrosophic topological groups. The target of this article is to perform eight different types of neutrosophic topological groups, each of which depends on the notions of neutrosophic α-open sets and neutrosophic α-continuous functions and also we found the relation between these types.

2. Preliminaries

In all this paper, $(\mathcal{G},\tau)$ and $(\mathcal{H},\sigma)$ (or briefly $\mathcal{G}$ and $\mathcal{H}$) frequently refer to neutrosophic topological spaces (or shortly NTSs). Suppose $\mathcal{A}$ be a neutrosophic open subset (or shortly Ne-OS) of $\mathcal{G}$, then its complement $\mathcal{A}^c$ is closed (or shortly Ne-CS). In addition, its interior and closure are denoted by $Nint(\mathcal{A})$and $Ncl(\mathcal{A})$, correspondingly.

Definition 2.1 [8]: Let $\boldsymbol{\mathcal{A}}$ be a Ne-OS in NTS $\boldsymbol{\mathcal{G}}$, then it is said that a neutrosophic **α**-open subset (or briefly Ne-**α**OS) if $\boldsymbol{\mathcal{A} \subseteq Nint(Ncl(Nint(\mathcal{A})))}$. Then $\boldsymbol{\mathcal{A}^c}$ is the so-called a neutrosophic **α**-closed (or briefly Ne-**α**CS). The collection of all such these Ne-**α**OSs (resp. Ne-**α**CSs) of $\boldsymbol{\mathcal{G}}$ is denoted by $\boldsymbol{N\alpha O(\mathcal{G})}$ (resp. $\boldsymbol{N\alpha C(\mathcal{G})}$).

Definition 2.2 [8]: Let $\mathcal{A}$ be a neurrosophic set in NTS $\mathcal{G}$. Then the union of all such these Ne-αOSs involved in $\mathcal{A}$(symbolized by $\alpha Nint(\mathcal{A})$) is said to be the neutrosophic α-interior of $\mathcal{A}$.

Definition 2.3 [8]: Let $\mathcal{A}$ be a neurrosophic set in NTS $\mathcal{G}$. Then the intersection of all such these Ne-αCSs that contain $\mathcal{A}$ (symbolized by $\alpha Ncl(\mathcal{A})$) is said to be the neutrosophic α-closure of $\mathcal{A}$.

Proposition 2.4 [13]: Let $\mathcal{A}$ be a neutrosophic set in NTS $\mathcal{G}$. Then $\mathcal{A} \in N\alpha O(\mathcal{B})$ iff there exists a Ne-αOS $\mathcal{B}$ where $\mathcal{B} \subseteq \mathcal{A} \subseteq Nint(Ncl(\mathcal{B}))$.

Proposition 2.5 [8]: In any NTS, the following claims hold, and not vice versa:

(i) For each, Ne-OS is a Ne-αOS.

(ii) For each, Ne-CS is a Ne-αCS.

Definition 2.6: Let $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ be a function, then $\hbar$ is called:

(i) a neutrosophic continuous (in short Ne-continuous) iff for each $\mathcal{A}$ Ne-OS in $\mathcal{H}$, then $\hbar^{-1}(\mathcal{A})$ is a Ne-OS in $\mathcal{G}$ [14].

(ii) a neutrosophic α-continuous (in short Ne-α-continuous) iff for each $\mathcal{A}$ Ne-OS in $\mathcal{H}$, then $\hbar^{-1}(\mathcal{A})$ is a Ne-αOS in $\mathcal{G}$ [8].

(iii) a neutrosophic α-irresolute (in short Ne-α-irresolute) iff for each $\mathcal{A}$ Ne-αOS in $\mathcal{H}$, then $\hbar^{-1}(\mathcal{A})$ is a Ne-αOS in $\mathcal{G}$.

Proposition 2.7 [8]: Every Ne-continuous function is a Ne-α-continuous, but the opposite is not valid in general.

Proposition 2.8: Every Ne-α-irresolute function is a Ne-α-continuous, but the opposite is not exact in general.

Proof: Let $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ be a Ne-α-irresolute function and let $\mathcal{A}$ be any Ne-OS in $\mathcal{H}$. From proposition 2.5, we get $\mathcal{A}$ is a Ne-αOS in $\mathcal{H}$. Since $\hbar$ is a Ne-α-irresolute, then $\hbar^{-1}(\mathcal{A})$ is a Ne-αOS in $\mathcal{G}$. Therefore $\hbar$ is a Ne-α-continuous. ■

Example 2.9: Let $\mathcal{G} = \{p, q\}$. Suppose the neutrosophic sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ be in $\mathcal{G}$ as follows:

$\mathcal{A} = \langle x, \left(\frac{p}{0.5}, \frac{q}{0.3}\right), \left(\frac{p}{0.5}, \frac{q}{0.3}\right), \left(\frac{p}{0.5}, \frac{q}{0.7}\right)\rangle$, $\mathcal{B} = \langle x, \left(\frac{p}{0.5}, \frac{q}{0.6}\right), \left(\frac{p}{0.5}, \frac{q}{0.6}\right), \left(\frac{p}{0.5}, \frac{q}{0.4}\right)\rangle$,

$\mathcal{C} = \langle x, \left(\frac{p}{0.6}, \frac{q}{0.3}\right), \left(\frac{p}{0.6}, \frac{q}{0.3}\right), \left(\frac{p}{0.4}, \frac{q}{0.7}\right)\rangle$ and $\mathcal{D} = \langle x, \left(\frac{p}{0.6}, \frac{q}{0.7}\right), \left(\frac{p}{0.6}, \frac{q}{0.7}\right), \left(\frac{p}{0.4}, \frac{q}{0.3}\right)\rangle$.

Then the families $\tau = \{0_N, \mathcal{A}, 1_N\}$ and $\sigma = \{0_N, \mathcal{D}, 1_N\}$ are neutrosophic topologies on $\mathcal{G}$.

Thus, $(\mathcal{G}, \tau)$ and $(\mathcal{G}, \sigma)$ are NTSs. Define $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{G}, \sigma)$ as $(p) = p$, $\hbar(q) = q$. Hence $\hbar$ is a Ne-α-continuous function, but not Ne-α-irresolute.

Definition 2.10: A function $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ is said to be $\mathcal{M}$-function iff $\hbar^{-1}(Nint(Ncl(\mathcal{B}))) \subseteq Nint(Ncl(\hbar^{-1}(\mathcal{B})))$, for every Ne-αOS $\mathcal{B}$ of $\mathcal{H}$.

Theorem 2.11: If $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ is a Ne-α-continuous function and $\mathcal{M}$-function, then $\hbar$ is a Ne-α-irresolute.

Proof: Let $\mathcal{A}$ be any Ne-αOS of $\mathcal{H}$, there exists a Ne-OS $\mathcal{B}$ of $\mathcal{H}$ where $\mathcal{B} \subseteq \mathcal{A} \subseteq Nint(Ncl(\mathcal{B}))$.

Since $\hbar$ is $\mathcal{M}$-function, we have $\hbar^{-1}(\mathcal{B}) \subseteq \hbar^{-1}(\mathcal{A}) \subseteq \hbar^{-1}(Nint(Ncl(\mathcal{B}))) \subseteq Nint(Ncl(\hbar^{-1}(\mathcal{B})))$.

By proposition 2.4, we have $\hbar^{-1}(\mathcal{A})$ is a Ne-αOS. Hence, $\hbar$ is a Ne-α-irresolute. ■

Definition 2.12 [8]: A function $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ is called a neutrosophic α-open (resp. neutrosophic α-closed) iff for each $\mathcal{A} \in N\alpha O(\mathcal{G})$ (resp. $\mathcal{A} \in N\alpha C(\mathcal{G})$), $\hbar(\mathcal{A}) \in N\alpha O(\mathcal{H})$ (resp. $\hbar(\mathcal{A}) \in N\alpha C(\mathcal{H})$).

Definition 2.13 [15]: A bijective function $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ is called a neutrosophic homeomorphism iff $\hbar$ and $\hbar^{-1}$ are Ne-continuous.

Definition 2.14 [12]: A neutrosophic topological group (briefly NTG) is a set $\mathcal{G}$ which carries a group structure and a neutrosophic topology with the following two postulates:
(i) The operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, given as $\mu(g, h) = g \cdot h$ is a Ne-continuous.
(ii) The inversion function $I: \mathcal{G} \to \mathcal{G}$, given as $I(g) = g^{-1}$ is a Ne-continuous.

Remark 2.15 [12]:
(i) The function $\gamma: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, given as $\gamma(g, h) = g \cdot h$ is a Ne-continuous iff for each Ne-OS $\mathcal{C}$ and $g \cdot h \in \mathcal{C}$, there exist Ne-OS $\mathcal{A}, \mathcal{B}$ such that $g \in \mathcal{A}, h \in \mathcal{B}$, and $\mathcal{A} \cdot \mathcal{B} \subseteq \mathcal{C}$.
(ii) The function $inv: \mathcal{G} \to \mathcal{G}$ is a Ne-continuous iff for each Ne-OS $\mathcal{A}$ and $g^{-1} \in \mathcal{A}$, there exists a Ne-OS $\mathcal{B}$ and $g \in \mathcal{B}$ where $\mathcal{B}^{-1} \subseteq \mathcal{A}$.

Definition 2.16 [16]: A group $\mathcal{G}$ is nice iff its operation is nice.

3. Different Types of Neutrosophic Topological Groups

In this section, we introduce eight types of neutrosophic topological groups, each of which depends on the notions of neutrosophic α-open sets and neutrosophic α-continuous functions.

Definition 3.1: Let $\mathcal{G}$ be a set that equips with a group structure and a neutrosophic topology. Then $\mathcal{G}$ is called:
(i) NTG of type (1) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and the inversion function $I: \mathcal{G} \to \mathcal{G}$ are both Ne-α-continuous.
(ii) NTG of type (2) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and the inversion function $I: \mathcal{G} \to \mathcal{G}$ are both Ne-α-irresolute.
(iii) NTG of type (3) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is Ne-α-continuous and the inversion function $I: \mathcal{G} \to \mathcal{G}$ is Ne-continuous.
(iv) NTG of type (4) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is Ne-α-irresolute and the inversion function $I: \mathcal{G} \to \mathcal{G}$ is Ne-continuous.
(v) NTG of type (5) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is Ne-α-irresolute and the inversion function $I: \mathcal{G} \to \mathcal{G}$ is Ne-α-continuous.
(vi) NTG of type (6) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is Ne-α-continuous and the inversion function $I: \mathcal{G} \to \mathcal{G}$ is Ne-α-irresolute.
(vii) NTG of type (7) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is Ne-continuous, and the inversion function $I: \mathcal{G} \to \mathcal{G}$ is Ne-α-continuous.

(viii) NTG of type (8) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ is Ne-continuous, and the inversion function $I: \mathcal{G} \rightarrow \mathcal{G}$ is Ne-α-irresolute.

Proposition 3.2:
(i) Every NTG is a NTG of type (R), where $R = 1,3,7$.
(ii) Every NTG of type (2) is a NTG of type (5).
(iii) Every NTG of type (2) is a NTG of type (6).
(iv) Every NTG of type (4) is a NTG of type (3).
(v) Every NTG of type (4) is a NTG of type (5).
(vi) Every NTG of type (R) is a NTG of type (1), where $R = 2,3,\dots,8$.
Proof:
(i) Let $\mathcal{G}$ be a NTG, then the operation function μ and the inversion function I are both Ne-continuous. By proposition 2.7, we have that the operation function μ and the inversion function I are both Ne-α-continuous. Hence, $\mathcal{G}$ is a NTG of type (R), where $R = 1,3,7$.
(ii) Let $\mathcal{G}$ be a NTG of type (2), then the operation function μ and the inversion function I are both Ne-α-irresolute. By proposition 2.8, we have that the inversion function I is a Ne-α-continuous. Hence, $\mathcal{G}$ is a NTG of type (5).
(iii) Let $\mathcal{G}$ be a NTG of type (2), then the operation function μ and the inversion function I are both Ne-α-irresolute. By proposition 2.8, we have that the operation function μ is a Ne-α-continuous. Hence, $\mathcal{G}$ is a NTG of type (6).
(iv) Let $\mathcal{G}$ be a NTG of type (4), then the operation function μ is a Ne-α-irresolute and the inversion function I is a Ne-continuous. By proposition 2.8, we have that the operation function μ is a Ne-α-continuous. Hence, $\mathcal{G}$ is a NTG of type (3).
(v) Let $\mathcal{G}$ be a NTG of type (4), then the operation function μ is a Ne-α-irresolute and the inversion function I is a Ne-continuous. By proposition 2.7, we have that the inversion function I is a Ne-α-continuous. Hence, $\mathcal{G}$ is a NTG of type (5).
(vi) Let $\mathcal{G}$ be a NTG of type (R), where $R = 2,3,\dots,8$. By proposition 2.7 and proposition 2.8, we have that the operation function μ and the inversion function I are both Ne-α-continuous. Hence, $\mathcal{G}$ is a NTG of type (1).

Proposition 3.3:
(i) A NTG of type (3) with $\mathcal{M}$-function operation μ is a NTG of type (4).
(ii) A NTG of type (1) with $\mathcal{M}$-function inversion I and $\mathcal{M}$-function operation μ is a NTG of type (2).
(iii) A NTG of type (1) with $\mathcal{M}$-function operation μ is a NTG of type (5).
(iv) A NTG of type (1) with $\mathcal{M}$-function inversion I is a NTG of type (6).
(v) A NTG of type (5) with $\mathcal{M}$-function inversion I is a NTG of type (2).
(vi) A NTG of type (6) with $\mathcal{M}$-function operation μ is a NTG of type (2).
(vii) A NTG of type (7) with $\mathcal{M}$-function inversion I is a NTG of type (8).
Proof:
(i) Let $\mathcal{G}$ be a NTG of type (3), then the operation function μ is a Ne-α-continuous and the inversion function I is a Ne-continuous. Since μ is $\mathcal{M}$-function. So by Theorem 2.11, we get that operation μ

is a Ne-α-irresolute. Hence, $\mathcal{G}$ is a NTG of type (4).

(ii) Let $\mathcal{G}$ be a NTG of type (1), then the operation function μ and the inversion function I are both Ne-α-continuous. Since μ, I are $\mathcal{M}$-function. So by Theorem 2.11, we get that the operation function μ and the inversion function I are both Ne-α-irresolute. Hence, $\mathcal{G}$ is a NTG of type (2). The proof is evident for others.

Remark 3.4: The next illustration displays relationship among different kinds of neutrosophic topological groups mentioned in this section and the neutrosophic topological group:

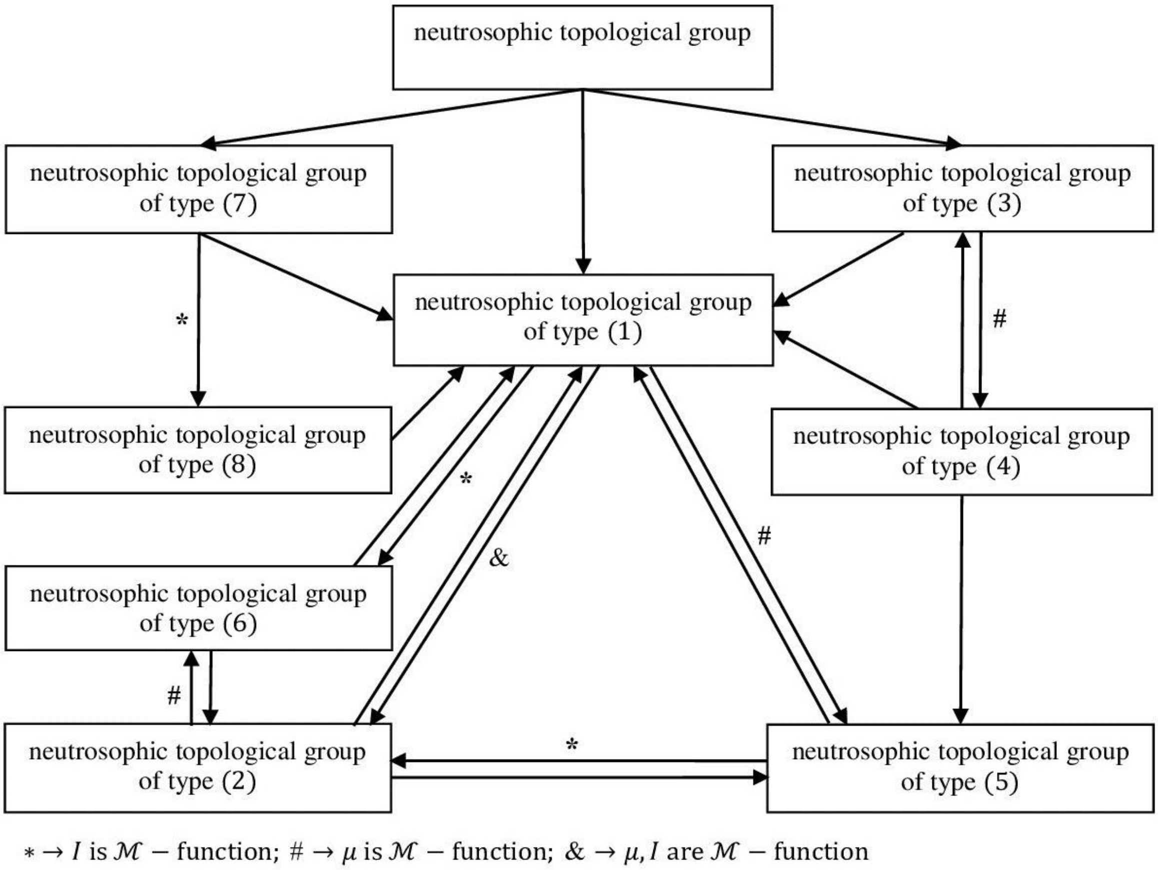

Fig. 3.1

Definition 3.5: A bijective function $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ is said to be:

(i) Neutrosophic α-homeomorphism iff $\hbar$ and $\hbar^{-1}$ are Ne-α-continuous.

(ii) Neutrosophic α-irresolute – homeomorphism iff $\hbar$ and $\hbar^{-1}$ are Ne-α-irresolute.

Definition 3.6: Let $(\mathcal{G}, \tau)$ be a NTS, then $\mathcal{G}$ is called neutrosophic α-homogeneous (resp. neutrosophic α-irresolute – homogeneous) iff for any two elements $g, h \in \mathcal{G}$, there exists a neutrosophic α-homeomorphism (resp. neutrosophic α-irresolute – homeomorphism) from $\mathcal{G}$ onto $\mathcal{G}$ which transforms g into h.

Proposition 3.7: The inversion function I in a NTG of type (R), where $R = 1,2, \dots \dots ,8$ is a neutrosophic α-homeomorphism.

Proof: Let $\mathcal{G}$ be a NTG of type (1). Since $\mathcal{G}$ is a group, $I(\mathcal{G}) = \mathcal{G}^{-1} = \mathcal{G}$ which implies I is onto, also for any $g \in \mathcal{G}$, there exists a unique inverse which is equal to $I(g)$ which implies, I is one-to-one. Now; we have I is a Ne-α-continuous and $I^{-1}: \mathcal{G} \to \mathcal{G}$ such that $I^{-1}(g) = g$, i.e $I^{-1}(g) = I(g)$ for each $g \in \mathcal{G}$, so, I^{-1} is a Ne-α-continuous. Thus, I is a neutrosophic α-homeomorphism. In the case of type (R), we have a similar proof, where $R = 2,3, \ldots, 8$.

Corollary 3.8: Let $\mathcal{G}$ be a NTG of type (1) and $\mathcal{A} \subseteq \mathcal{G}$. If $\mathcal{A} \in \tau$, then $\mathcal{A}^{-1} \in N\alpha O(\mathcal{G})$.
Proof: Since the inversion function I is a neutrosophic α-homeomorphism, then $I(\mathcal{A}) = \mathcal{A}^{-1}$ is a Ne-αOS in $\mathcal{G}$ for each $\mathcal{A} \in \tau$.

Proposition 3.9: The inversion function I in a NTG of type (3) [and type (4)] is a neutrosophic homeomorphism.
Proof: Suppose $\mathcal{G}$ be a NTG of type (3). Since $\mathcal{G}$ is a group, $I(\mathcal{G}) = \mathcal{G}^{-1} = \mathcal{G}$ which implies I is onto, also for any $g \in \mathcal{G}$, there exists a unique inverse which is equal to $I(g)$ which implies, I is one-to-one. Now; we have I is a Ne-continuous and $I^{-1}: \mathcal{G} \to \mathcal{G}$ such that $I^{-1}(g) = g$, i.e $I^{-1}(g) = I(g)$ for each $g \in \mathcal{G}$, so, I^{-1} is a Ne-continuous. Thus, I is a neutrosophic homeomorphism. In the case of type (4), we have similar proof.

Proposition 3.10: The inversion function I in a NTG of type (R), where $R = 2,6,8$ is a neutrosophic α-irresolute – homeomorphism.
Proof: Suppose $\mathcal{G}$ be a NTG of type (2). Since $\mathcal{G}$ is a group, $I(\mathcal{G}) = \mathcal{G}^{-1} = \mathcal{G}$ which implies I is onto, also for any $g \in \mathcal{G}$, there exists a unique inverse which is equal to $I(g)$ which implies, I is one-to-one. Now; we have I is a Ne-α-irresolute and $I^{-1}: \mathcal{G} \to \mathcal{G}$ such that $I^{-1}(g) = g$, i.e $I^{-1}(g) = I(g)$ for each $g \in \mathcal{G}$, so, I^{-1} is a Ne-α-irresolute. Thus, I is a neutrosophic α-irresolute – homeomorphism. In the case of type (6) and type (8), we have a similar proof.

Proposition 3.11: Let $\mathcal{G}$ be a set which carries a group structure and a neutrosophic topology, let $k_1, k_2 \in \mathcal{G}$. Then for each $g \in \mathcal{G}$ if one of the following functions:
(i) $l_{k_1}(g) = k_1 \cdot g$
(ii) $r_{k_1}(g) = g \cdot k_1$
(iii) $\hbar_{k_1 k_2}(g) = k_1 \cdot g \cdot k_2$
is a neutrosophic α-homeomorphism (resp. neutrosophic α-irresolute – homeomorphism), then so the others.
Proof: Since $k_1$and k_2 are arbitrary elements in $\mathcal{G}$, clear that $l_{k_1}$and r_{k_1}come from $\hbar_{k_1 k_2}$ by taking $k_2 = e$ or $k_1 = e$ respectively. Hence, when $\hbar_{k_1 k_2}$ is a neutrosophic α-homeomorphism, both l_{k_1} and r_{k_2} are neutrosophic α-homeomorphisms. Now; when l_{k_1} is a neutrosophic α-homeomorphism. Since $\mathcal{G}$ is a group, $\mathcal{G} \cdot k = \mathcal{G}$ for each $k \in \mathcal{G}$ then $\mathcal{G} \cdot k_2 = \mathcal{G}$. Hence, for each $h \in \mathcal{G} \cdot k_2$, $l_{k_1}(h) = k_1 \cdot h$, l_{k_1} is a neutrosophic α-homeomorphism. But $h = g \cdot k_2$ for some $g \in \mathcal{G}$, then for each $g \in \mathcal{G}$, $l_{k_1}(h) = l_{k_1}(g \cdot k_2) = k_1 \cdot g \cdot k_2 = \hbar_{k_1 k_2}(g)$, $\hbar_{k_1 k_2}$ is a neutrosophic α-homeomorphism. Then by the first part of the proof, r_{k_1}. And we have a similar proof if we are beginning with r_{k_1} is a neutrosophic α-homeomorphism. In the case of neutrosophic α-irresolute – homeomorphism, we have a similar proof.

Theorem 3.12: Let $\mathcal{G}$ be a nice NTG of type (R), where $R = 1,2,3,\dots,8$ and let $k_1, k_2 \in \mathcal{G}$. Then for each $g \in \mathcal{G}$ the following functions:

(i) $l_{k_1}(g) = k_1 \cdot g$

(ii) $r_{k_1}(g) = g \cdot k_1$

(iii) $\hbar_{k_1k_2}(g) = k_1 \cdot g \cdot k_2$

are neutrosophic α-homeomorphisms.

Proof: Let $\mathcal{G}$ be a nice NTG of type (1). It is clear that each of the functions l_{k_1}, r_{k_1} and $\hbar_{k_1k_2}$ is a bijective function. Let $\hbar$ be the operation of $\mathcal{G}$, then $\hbar$ is a Ne-α-continuous. Since $\mathcal{G}$ is a nice, so $l_{k_1} = \hbar/\{k_1\} \times \mathcal{G}$ is a Ne-α-continuous. Similarly, $l_{k_1}{}^{-1}(g) = k_1{}^{-1} \cdot g$, $l_{k_1}{}^{-1}$ is a Ne-α-continuous. Hence, l_{k_1} is a neutrosophic α-homeomorphism. Thus, because of the preceding proposition, r_{k_1} and $\hbar_{k_1k_2}$ are neutrosophic α-homeomorphisms. The case of type (R) has a similar proof, where $R = 2,3,\dots,8$.

Theorem 3.13: Let $\mathcal{G}$ be a nice NTG of type (R), where $R = 2,4,5$ and let $k_1, k_2 \in \mathcal{G}$. Then for each $g \in \mathcal{G}$ the following functions:

(i) $l_{k_1}(g) = k_1 \cdot g$

(ii) $r_{k_1}(g) = g \cdot k_1$

(iii) $\hbar_{k_1k_2}(g) = k_1 \cdot g \cdot k_2$

are neutrosophic α-irresolute – homeomorphisms.

Proof: Let $\mathcal{G}$ be a nice NTG of type (2). It is clear that each of the functions l_{k_1}, r_{k_1} and $\hbar_{k_1k_2}$ is a bijective function. Let $\hbar$ be the operation of $\mathcal{G}$, then $\hbar$ is a Ne-α-irresolute. Since $\mathcal{G}$ is a nice, so $l_{k_1} = \hbar/\{k_1\} \times \mathcal{G}$ is a Ne-α-irresolute. Similarly, $l_{k_1}{}^{-1}(g) = k_1{}^{-1} \cdot g$, $l_{k_1}{}^{-1}$ is a Ne-α-irresolute. Hence, l_{k_1} is a neutrosophic α-irresolute – homeomorphism. Thus, given the preceding proposition, r_{k_1} and $\hbar_{k_1k_2}$ are neutrosophic α-irresolute – homeomorphisms. The case of type (R) has a similar proof, where $R = 4,5$.

Corollary 3.14: Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be subsets of a nice NTG $\mathcal{G}$ of type (1) (resp. of type (4)) such that $\mathcal{A}$ is a Ne-CS (resp. Ne-αCS), and $\mathcal{B}$ is a Ne-OS (resp. Ne-αOS). Then for each $k \in \mathcal{G}, k \cdot \mathcal{A}$ and $\mathcal{A} \cdot k$ are Ne-α-CSs also $k \cdot \mathcal{B}, \mathcal{B} \cdot k, \mathcal{C} \cdot \mathcal{B}$ and $\mathcal{B} \cdot \mathcal{C}$ are Ne-αOSs.

Proof: Since $\mathcal{A}$ is a Ne-CS so in view of the theorem 3.12, $l_k(\mathcal{A}) = k \cdot \mathcal{A}$ and $r_k(\mathcal{A}) = \mathcal{A} \cdot k$ are Ne-αCSs.

Similarly, since $\mathcal{B}$ is a Ne-OS so in view of the theorem 3.12, $l_k(\mathcal{B}) = k \cdot \mathcal{B}$ and $r_k(\mathcal{B}) = \mathcal{B} \cdot k$ are Ne-αOSs. Also, $\mathcal{C} \cdot \mathcal{B} = \bigcup_{c \in \mathcal{C}} c \cdot \mathcal{B}$ but $c \cdot \mathcal{B}$ is a Ne-αOS for each $c \in \mathcal{C}$. Hence, $\mathcal{C} \cdot \mathcal{B}$ is a Ne-αOS. Similarly, $\mathcal{B} \cdot \mathcal{C}$ is a Ne-αOS. In the case of type (4), we have a similar proof.

Corollary 3.15: A nice NTG of type (R), where $R = 1,2,3,\dots,8$ is neutrosophic α-homogeneous.

Proof: Let $\mathcal{G}$ be a nice NTG of type (1) and $a, b \in \mathcal{G}$. Then for any fixed element $k \in \mathcal{G}, r_k$ is a neutrosophic α-homeomorphism, therefore, it is true when $k = a^{-1} \cdot b$. Thus, $r_{a^{-1}b}(g) = g \cdot a^{-1} \cdot b$ is a neutrosophic α-homeomorphism we need because $r_{a^{-1}b}(a) = b$. Therefore, $\mathcal{G}$ is a neutrosophic α-homogeneous. In the case of type (R), we have a similar proof, where $R = 2,3,\dots,8$.

Corollary 3.16: A nice NTG of type (R), where $R = 2,4,5$ is neutrosophic α-irresolute – homogeneous.

Proof: Let $\mathcal{G}$ be a nice NTG of type (2) and $a, b \in \mathcal{G}$. Then for any fixed element $k \in \mathcal{G}, r_k$ is a neutrosophic α-irresolute – homeomorphism, therefore, it is true when $k = a^{-1} \cdot b$. Thus, $r_{a^{-1}b}(g) = g \cdot a^{-1} \cdot b$ is a neutrosophic α-irresolute – homeomorphism. But $r_{a^{-1}b}(a) = b$, therefore $\mathcal{G}$ is a neutrosophic α-irresolute – homogeneous. In the case of type (R), we have a similar proof, where $R = 4,5$.

Definition 3.17: Let $\mathcal{G}$ be a NTG of type $(2), (5)$, and $\mathcal{F}$ be a fundamental system of neutrosophic α-open nhds of the identity element e. Then for any fixed element $k \in \mathcal{G}, r_k$ is a neutrosophic α-irresolute – homeomorphism. So $\mathcal{F}(k) = \{r_k(\mathcal{A}) = \mathcal{A} \cdot k : \mathcal{A} \in \mathcal{F}\}$ is a fundamental system of neutrosophic α-open nhds of k.

Proposition 3.18: Let $\mathcal{G}$ be a NTG of type $(2), (5)$. Any fundamental system $\mathcal{F}$ of neutrosophic α-open nhds of e in $\mathcal{G}$ has the below postulates:

(i) If $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, then $\exists \mathcal{C} \in \mathcal{F}$ such that $\mathcal{C} \subseteq \mathcal{A} \cap \mathcal{B}$.

(ii) If $g \in \mathcal{A} \in \mathcal{F}$, then $\exists \mathcal{B} \in \mathcal{F}$ such that $\mathcal{B} \cdot g \subseteq \mathcal{A}$.

(iii) If $\mathcal{A} \in \mathcal{F}$, then $\exists \mathcal{B} \in \mathcal{F}$ such that $\mathcal{B}^{-1} \cdot \mathcal{B} \subseteq \mathcal{A}$.

(iv) If $\mathcal{A} \in \mathcal{F}, k \in \mathcal{G}$, then $\exists \mathcal{B} \in \mathcal{F}$ such that $k^{-1} \cdot \mathcal{B} \cdot k \subseteq \mathcal{A}$.

(v) $\forall \mathcal{A} \in \mathcal{F}, \exists \mathcal{B} \in \mathcal{F}$ such that $\mathcal{B}^{-1} \subseteq \mathcal{A}$.

(vi) $\forall \mathcal{A} \in \mathcal{F}, \exists \mathcal{C} \in \mathcal{F}$ such that $\mathcal{C}^2 \subseteq \mathcal{A}$.

Proof:

(i) Let $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, then $\mathcal{A} \cap \mathcal{B} \in \mathcal{F}$, so $\exists \mathcal{C} \in \mathcal{F}$ such that $\mathcal{C} \subseteq \mathcal{A} \cap \mathcal{B}$.

(ii) Let $\mathcal{A} \in \mathcal{F}$ and $g \in \mathcal{A}$ implies $\mathcal{A} \cdot g^{-1} \in \mathcal{F}$, then $\exists \mathcal{B} \in \mathcal{F}$ such that $\mathcal{B} \subseteq \mathcal{A} \cdot g^{-1}$. Thus, $\mathcal{B} \cdot g \subseteq \mathcal{A}$.

(iii) The function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, given by $\mu(g, h) = g^{-1} \cdot h$ is a Ne-α-irresolute because $\mathcal{G}$ is a NTG of type $(2), (5)$. Thus $\mu^{-1}(\mathcal{A})$ is a neutrosophic α-open nhd in $\mathcal{G} \times \mathcal{G}$ contains (e, e) and hence includes a set of the from $\mathcal{U} \times \mathcal{V}$, where $\mathcal{U}, \mathcal{V}$ are neutrosophic α-open and provide e. But $\mathcal{U} \cap \mathcal{V}$ is a neutrosophic α-open contains e, so $\exists \mathcal{B} \in \mathcal{F}$ such that $\mathcal{B} \subseteq \mathcal{U} \cap \mathcal{V}$ then $\mathcal{B} \subseteq \mathcal{U}$ and $\mathcal{B} \subseteq \mathcal{V}$. Thus $\mathcal{B} \times \mathcal{B} \subseteq \mathcal{U} \times \mathcal{V} \subseteq \mu^{-1}(\mathcal{A})$, then $\mu(\mathcal{B} \times \mathcal{B}) \subseteq \mathcal{A}$ but $\mu(\mathcal{B} \times \mathcal{B}) = \mathcal{B}^{-1} \cdot \mathcal{B} \subseteq \mathcal{A}$.

(iv) The function $\hbar: \mathcal{G} \to \mathcal{G}$ given by $\hbar(g) = k^{-1} \cdot g \cdot k$ is a Ne-α-irresolute. Since $l_{k^{-1}}, r_k$ is Ne-α-irresolute. So $l_{k^{-1}} \circ r_k$ is a Ne-α-irresolute from $\mathcal{G}$ to $\mathcal{G}$ put $\hbar = l_{k^{-1}} \circ r_k, \hbar(g) = (l_{k^{-1}} \circ r_k)(g) = l_{k^{-1}}(r_k(g)) = l_{k^{-1}}(g \cdot k) = k^{-1} \cdot g \cdot k$.

$$\begin{tikzcd} \mathcal{G} \arrow[r, "r_k"] \arrow[dr, "\hbar"'] & \mathcal{G} \arrow[d, "l_{k^{-1}}"] \\ & \mathcal{G} \end{tikzcd}$$

So, $\hbar^{-1}(\mathcal{A})$ is a neutrosophic α-open nhd and contains e, hence $\exists \mathcal{B} \in \mathcal{F}$, $\mathcal{B} \subseteq \hbar^{-1}(\mathcal{A})$ then $\hbar(\mathcal{B}) \subseteq \mathcal{A}$. Thus, $\hbar(\mathcal{B}) = k^{-1} \cdot \mathcal{B} \cdot k \subseteq \mathcal{A}$.

(v) Since I the inverse function in a NTG of type (2) is a Ne-α-irresolute, then $I^{-1}(\mathcal{A})$ is a neutrosophic α-open contains e so $\exists\mathcal{B}\in\mathcal{F}$ such that $\mathcal{B}\subseteq I^{-1}(\mathcal{A})$ then $I(\mathcal{B})\subseteq\mathcal{A}$. Thus, $I(\mathcal{B})=\mathcal{B}^{-1}\subseteq\mathcal{A}$.

(vi) Since μ in a NTG of type (5) is a Ne-α-irresolute. So $\mu^{-1}(\mathcal{A})$ is a neutrosophic α-open contains (e,e) and thus contains a neutrosophic set of the from $\mathcal{U}\times\mathcal{V}$, where $\mathcal{U},\mathcal{V}$ are neutrosophic α-open and contain e then $\mathcal{U}\cap\mathcal{V}$ is a neutrosophic α-open and contain e $\exists\mathcal{C}\in\mathcal{F}$ such that $\mathcal{C}\subseteq\mathcal{U}\cap\mathcal{V}$, then $\mathcal{C}\times\mathcal{C}\subseteq\mathcal{U}\times\mathcal{V}\subseteq\mu^{-1}(\mathcal{A})$. Thus, $\mu(\mathcal{C}\times\mathcal{C})=\mathcal{C}\cdot\mathcal{C}=\mathcal{C}^2\subseteq\mathcal{A}$.

Definition 3.19: A neutrosophic α-open nhd $\mathcal{C}$ of g is called symmetric if $\mathcal{C}^{-1}=\mathcal{C}$.

Proposition 3.20: Let $\mathcal{G}$ be a NTG of type (R), where $R=1,2,\dots,8$, and let $\mathcal{B}$ be any neutrosophic α-open nhd of a point $g\in\mathcal{G}$. Then $\mathcal{B}\cup\mathcal{B}^{-1}$ is symmetric neutrosophic α-open nhd of g.

Proof: Let $\mathcal{B}$ is a neutrosophic α-open nhd of g, then $\mathcal{B}\cup\mathcal{B}^{-1}$ is a neutrosophic α-open nhd of g;

$\mathcal{B}\cup\mathcal{B}^{-1}=\{b: b\in\mathcal{B}\ or\ b\in\mathcal{B}^{-1}\}=\{b: b^{-1}\in\mathcal{B}\ or\ b^{-1}\in\mathcal{B}^{-1}\}$

$=\{b: b^{-1}\in\mathcal{B}\cup\mathcal{B}^{-1}\}=\{b: b\in(\mathcal{B}\cup\mathcal{B}^{-1})^{-1}\}=(\mathcal{B}\cup\mathcal{B}^{-1})^{-1}$.

That is, $\mathcal{B}\cup\mathcal{B}^{-1}$ is symmetric neutrosophic α-open nhd of g.

Proposition 3.21: Let $\mathcal{B}$ be any neutrosophic α-open nhd of e in a nice NTG of type (R), where $R=1,2,\dots\dots,8$. Then $\mathcal{B}\cdot\mathcal{B}^{-1}$ is symmetric neutrosophic α-open nhd of e.

Proof: Let $\mathcal{B}$ be a neutrosophic α-open nhd of e and since $\mathcal{G}$ is a nice, then $\mathcal{B}\cdot\mathcal{B}^{-1}$ is neutrosophic α-open nhd of e;

$\mathcal{B}\cdot\mathcal{B}^{-1}=\{x\cdot y^{-1}: x,y\in\mathcal{B}\}=\{(x^{-1})^{-1}\cdot y^{-1}: x,y\in\mathcal{B}\}=(\mathcal{B}^{-1})^{-1}\cdot\mathcal{B}^{-1}=(\mathcal{B}\cdot\mathcal{B}^{-1})^{-1}$.

That is, $\mathcal{B}\cdot\mathcal{B}^{-1}$ is symmetric neutrosophic α-open nhd of e.

4. Conclusion

In this work, we examined the conceptions of eight different types of neutrosophic topological groups, each of which, depending on the notions of neutrosophic α-open sets and neutrosophic α-continuous function. In the future, we plan to rsearch the ideas of neutrosophic topological subgroups and the neutrosophic topological quotient groups as well as defining the perception of neutrosophic topological product groups with some results.

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Neutrosophic Soft Rough Topology and its Applications to Multi-Criteria Decision-Making

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Abstract: In this manuscript, we introduce the notion of neutrosophic soft rough topology (NSR-topology) defined on neutrosophic soft rough set (NSR-set). We define certain properties of NSR-topology including NSR-interior, NSR-closure, NSR-exterior, NSR-neighborhood, NSR-limit point, and NSR-bases. Furthermore, we aim to develop some multi-criteria decision-making (MCDM) methods based on NSR-set and NSR-topology to deal with ambiguities in the real-world problems. For this purpose, we establish algorithm 1 for suitable brand selection and algorithm 2 to determine core issues to control crime rate based on NSR-lower approximations, NSR-upper approximations, matrices, core, and NSR-topology.

Keywords: Neutrosophic soft rough (NSR) set, NSR-topology, NSR-interior, NSR-closure, NSR-exterior, NSR-neighborhood, NSR-limit point, NSR-bases, Multi-criteria group decision making.

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1. Introduction

The limitations of existing research are recognized in the field of management, social sciences, operational research, medical, economics, artificial intelligence, and decision-making problems. These limitations can be dealt with the Fuzzy set [1], rough set [2, 3], neutrosophic set [4, 5], soft set [6], and different hybrid structures of these sets. Rough set theory was initiated by Pawlak [2], which is an effective mathematical model to deal with vagueness and imprecise knowledge. Its boundary region gives the concept of vagueness, which can be interpreted by using the vagueness of Frege's idea. He invented that vagueness can be dealt with the upper and lower approximations of precise set using any equivalence relation. In the real life, rough set theory has many applications in different fields such as social sciences, operational research, medical, economics, and artificial intelligence, etc. Many real-world problems have neutrosophy in their nature and cannot handle by using fuzzy or intuitionistic fuzzy set theory. For example, when we are dealing with conductors and non-conductors there must be a possibility having insulators. For this purpose, Smarandache [4, 5] inaugurated the neutrosophic set theory as a generalization of fuzzy and intuitionistic fuzzy set theory. The neutrosophic set yields the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$. It is difficult to utilize these values in daily life science and technology problems. Therefore, the concept of a single-valued neutrosophic set, which takes value from the subset of [0,1], as defined by Wang et al. [7]. The beauty of this set is that it gives the membership grades for truth, indeterminacy and falsity for the corresponding attribute. All the grades are independent of each other and provide information about the three shades of an arbitrary attribute. Smarandache [8] extended the neutrosophic set respectively to neutrosophic Overset (when some neutrosophic components are >1), Neutrosophic Underset (when some neutrosophic components are <0), and to Neutrosophic Offset (when some neutrosophic components are off the interval $[0,1]$, i.e. some neutrosophic components are >1 and other neutrosophic components <0). In 2016, Smarandache introduced the Neutrosophic Tripolar Set and Neutrosophic Multipolar Set, also the Neutrosophic Tripolar Graph and Neutrosophic Multipolar Graph [8].

The soft set is a mathematical model to deal ambiguities and imprecisions in parametric manners. This is another abstraction of the crisp set theory. In 1999, Molodtsov [9] worked on parametrizations of the universal set and invented a parameterized family of subsets of the universal set called soft set. In recent years, many mathematicians worked on different hybrid structures of the fuzzy and rough sets. Ali et al. [10, 11] established some novel operations in the soft sets, rough soft sets and, fuzzy soft set theory. Aktas and Çağman [12] introduced various results on soft sets and soft groups. Bakier et al. [13] introduced the idea of soft rough topology. Çağman et al. [14] introduced various results on soft topology. Chen [15] worked on parametrizations reduction of soft sets and gave its applications in decision-making. Feng et al. [16, 17] established various results on soft set, fuzzy set, rough set and soft rough sets with the help of illustrations. Hashmi et al. [18] introduced the notion of m-polar neutrosophic set and m-polar neutrosophic topology and their applications to multi-criteria decision-making (MCDM) in medical diagnosis and clustering analysis. Hashmi and Riaz [19] introduced a novel approach to the census process by using Pythagorean m-polar fuzzy Dombi's aggregation operators. Kryskiewicz [20] introduced the rough set approach to incomplete information systems. Karaaslan and Çağman [21] introduced bipolar soft rough sets and presented their applications in decision-making. Kumar and Garg [22] introduced the TOPSIS method based on the connection number of set pair analyses under an interval-valued intuitionistic fuzzy set environment. Maji et al. [23, 24, 25] worked on some results of a soft set and gave its applications in decision-making problems. He also invented the idea of a neutrosophic soft set and gave various results to intricate the concept with numerous applications. Naeem et al. [26] introduced the novel concept of Pythagorean m-polar fuzzy sets and the TOPSIS method for the selection of advertisement mode. Peng and Garg [27] introduced algorithms for interval-valued fuzzy soft sets in emergency decision making based on WDBA and CODAS with new information measures. Peng and Yang [28] presented some results for Pythagorean fuzzy sets. Peng et al. [29] introduced Pythagorean fuzzy information measures and their applications. Peng et al. [30] introduced a Pythagorean fuzzy soft set and its application. Peng and Dai [31] introduced certain approaches to single-valued neutrosophic MADM based on MABAC, TOPSIS and, new similarity measure with score function. Marei [32] invented some more results on neutrosophic soft rough sets and worked on its modifications. Pei and Miao [33] worked on the information system using the idea of a soft set. Quran et al. [34] introduced a novel approach to neutrosophic soft rough set under uncertainty. Riaz et al. [35] introduced soft rough topology with its applications to group decision making.

Riaz and Hashmi [36] introduced the notion of linear Diophantine fuzzy Set (LDFS) and its applications towards the MCDM problem. Linear Diophantine fuzzy Set (LDFS) is superior to IFS, PFS and, q-ROFS. Riaz and Hashmi [37] introduced novel concepts of soft rough Pythagorean m-Polar fuzzy sets and Pythagorean m-polar fuzzy soft rough sets with application to decision-making. Riaz and Tehrim [38] established the idea of cubic bipolar fuzzy ordered weighted geometric aggregation operators and, their application using internal and external cubic bipolar fuzzy data. They presented various illustrations and decision-making applications of these concepts by using different algorithms. Roy and Maji [39] introduced a fuzzy soft set-theoretic approach to decision-making problems. Salama [40] investigated some topological properties of rough sets with tools for data mining. Shabir and Naz [41] worked on soft topological spaces and presented their applications. Thivagar et al. [42] presented some mathematical innovations of a modern topology in medical events. Xueling et al. [43] presented some decision-making methods based on certain hybrid soft set models. Zhang et al. [44, 45, 46] established fuzzy soft β-covering based fuzzy rough sets, fuzzy soft coverings based fuzzy rough sets and, covering on generalized intuitionistic fuzzy rough sets with their applications to multi-attribute decision-making (MADM) problems. Broumi et al. [47] established the concept of rough neutrosophic sets. Christianto et al. [48] introduced the idea about the extension of standard deviation notion with neutrosophic interval and quadruple neutrosophic numbers. Adeleke et al. [49, 50] invented the concepts of refined eutrosophic rings I and refined neutrosophic rings II. Parimala et al. [51] worked on $\alpha\omega$-closed sets and its connectedness in terms of neutrosophic topological spaces. Ibrahim et al. [52] introduced the neutrosophic subtraction algebra and neutrosophic subraction semigroup.

The neutrosophic soft rough set and neutrosophic soft rough topology have many applications in MCDM problems. This hybrid erection is the most efficient and flexible rather than other constructions. It is constructed with a combination of neutrosophic, soft and, rough set theory. The interesting point in this structure is that by using this idea, we can deal with those type of models which have roughness, neutrosophy and, parameterizations in their nature.

The motivation of this extended and hybrid work is presented step by step in the whole manuscript. This model is generalized form and use to collect data at a large scale and applicable in medical, engineering, artificial intelligence, agriculture and, other daily life problems. In the future, this work can be gone easily for other approaches and different types of hybrid structures.

The layout of this paper is systematized as follows. Section 2, implies some basic ideas including soft set, rough set, neutrosophic set, neutrosophic soft set and, neutrosophic soft rough set. We elaborate on these ideas with the help of illustrations. In Section 3, we establish neutrosophic soft rough topology (NSR-topology) with some examples. We introduce some topological structures on NSR-topology named NSR-interior, NSR-closure, NSR-exterior, NSR-neighborhood, NSR-limit point and, NSR-bases. In Section 4 and 5, we present multi-criteria decision-making problems by using two different algorithms on NSR-set and NSR-topology. We use the idea of upper and lower approximations for NSR-set and construct algorithms using NSR-sets and NSR-bases We discuss the optimal results obtained from both algorithms and present a comparitive analysis of proposed approach with some existing approaches. Finally, the conclusion of this research is summarized in section 6.

2. Preliminaries

This section presents some basic definitions including soft set, rough set, neutrosophic soft set, and neutrosophic soft rough set .

Definition 2.1 [18]

Let U be the universal set. Let $\mathrm{I(U)}$ is collection of subsets of U. A pair $(\Theta, \mathfrak{A})$ is said to be a soft set over the universe U, where $\mathfrak{A} \subseteq \mathrm{E}$ and $\Theta: \mathfrak{A} \to \mathrm{I(U)}$ is a set-valued function. We denote soft set as $(\Theta, \mathfrak{A})$ or $\Theta_{\mathfrak{A}}$ and mathematically write it as

$$\Theta_{\mathfrak{A}} = \{(\xi, \Theta(\xi)) : \xi \in \mathfrak{A}, \Theta(\xi) \in \mathrm{I(U)}\}.$$

For any $\xi \in \mathfrak{A}$, $\Theta(\xi)$ is ξ-approximate elements of soft set $\Theta_{\mathfrak{A}}$.

Definition 2.2 [21]

Let U be the initial universe and $\mathrm{Y} \subseteq \mathrm{U}$. Then, lower, upper, and boundary approximations of Y are defined as

$$\mathfrak{R}_{\mathring{a}}(\mathrm{Y}) = \cup_{g \in \mathrm{U}} \{\mathfrak{R}(g) : \mathfrak{R}(g) \subseteq \mathrm{Y}\},$$

$$\mathfrak{R}^{\mathring{a}}(\mathrm{Y}) = \cup_{g \in \mathrm{U}} \{\mathfrak{R}(g) : \mathfrak{R}(g) \cap \mathrm{Y} \neq \emptyset\},$$

and

$$\mathrm{B}_{\mathfrak{R}}(\mathrm{Y}) = \mathfrak{R}^{\mathring{a}}(\mathrm{Y}) - \mathfrak{R}_{\mathring{a}}(\mathrm{Y}),$$

respectively. Where $\mathfrak{R}$ is an indiscernibility relation $\mathfrak{R} \subseteq \mathrm{U} \times \mathrm{U}$ which indicates our information about elements of U. The set Y is said to be defined if $\mathfrak{R}^{\mathring{a}}(\mathrm{Y}) = \mathfrak{R}_{\mathring{a}}(\mathrm{Y})$. If $\mathfrak{R}^{\mathring{a}}(\mathrm{Y}) \neq \mathfrak{R}_{\mathring{a}}(\mathrm{Y})$ i.e $\mathrm{B_R(Y)} \neq \emptyset$, the set Y is rough set w.r.t $\mathfrak{R}$.

Definition 2.3 [41] Let U be the initial universe. Then, a neutrosophic set N on the universe U is defined as

$$\mathrm{N} = \{< g, \mathfrak{T}_{\mathrm{N}}(g), \mathfrak{I}_{\mathrm{N}}(g), \mathfrak{F}_{\mathrm{N}}(g) >: g \in \mathrm{U}\}, \text{where}$$

$$^{-}0 \leq \mathfrak{T}_{\mathrm{N}}(g) + \mathfrak{I}_{\mathrm{N}}(g) + \mathfrak{F}_{\mathrm{N}}(g) \leq 3^{+}, \text{where}$$

$$\mathfrak{T}, \mathfrak{I}, \mathfrak{F}: \mathrm{U} \to]^{-}0, 1^{+}[.$$

Where $\mathfrak{T}, \mathfrak{I}$ and $\mathfrak{F}$ represent the degree of membership, degree of indeterminacy and degree of non-membership for some $g \in \mathrm{U}$, respectively.

Definition 2.4 [16] Let U be an initial universe and E be a set of parameters. Suppose $\mathfrak{A} \subset \mathrm{E}$, and let $\mathcal{J}(\mathrm{U})$ represents the set of all neutrosophic sets of U. The collection $(\Phi, \mathfrak{A})$ is said to be the neutrosophic soft set over U, where Φ is a mapping given by

$$\Phi: \mathfrak{A} \to \mathcal{J}(\mathrm{U}).$$

The set containing all neutrosophic soft sets over U is denoted by $\mathrm{NS_U}$.

Example 2.5 Consider $\mathrm{U} = \{g_1, g_2, g_3, g_4, g_5\}$ be set of objects and attribute set is given by $\mathfrak{A} = \{\xi_1, \xi_2, \xi_3, \xi_4\} = \mathrm{E} = \mathfrak{A}$, where

The neutrosophic soft set represented as $\Phi_{\mathfrak{A}}$. Consider a mapping $\Phi: \mathfrak{A} \to \mathrm{I(U)}$ such that

$$\Phi(\xi_1) = \{< g_1, 0.7,0.7,0.3 >, < g_2, 0.5,0.7,0.7 >, < g_3, 0.7,0.5,0.2 >, < g_4, 0.7,0.4,0.4 >, < g_5, 0.9,0.3,0.4 >\},$$
$$\Phi(\xi_2) = \{< g_1, 0.9,0.5,0.4 >, < g_2, 0.7,0.3,0.5 >, < g_3, 0.9,0.2,0.4 >, < g_4, 0.9,0.3,0.3 >, < g_5, 0.9,0.4,0.3 >\},$$
$$\Phi(\xi_3) = \{< g_1, 0.8,0.5,0.4 >, < g_2, 0.7,0.5,0.4 >, < g_3, 0.8,0.3,0.6 >, < g_4, 0.6,0.3,0.7 >, < g_5, 0.8,0.4,0.5 >\},$$
$$\Phi(\xi_4) = \{< g_1, 0.9,0.7,0.5 >, < g_2, 0.8,0.7,0.7 >, < g_3, 0.8,0.7,0.5 >, < g_4, 0.8,0.6,0.7 >, < g_5, 1.0,0.6,0.7 >\}.$$

The tabular representation of neutrosophic soft set $\mathrm{K} = (\Phi, \mathfrak{A})$ is given in Table 1.

$(\Phi,\mathfrak{A})$	g_1	g_2	g_3	g_4	g_5
ξ_1	(0.7,0.7,0.3)	(0.5,0.7,0.7)	(0.7,0.5,0.2)	(0.7,0.4,0.4)	(0.9,0.3,0.4)
ξ_2	(0.9,0.5,0.4)	(0.7,0.3,0.5)	(0.9,0.2,0.4)	(0.9,0.3,0.3)	(0.9,0.4,0.3)
ξ_3	(0.8,0.5,0.4)	(0.7,0.5,0.4)	(0.8,0.3,0.6)	(0.6,0.3,0.7)	(0.8,0.4,0.5)
ξ_4	(0.9,0.7,0.5)	(0.8,0.7,0.7)	(0.8,0.7,0.5)	(0.8,0.6,0.7)	(1.0,0.6,0.7)

Table 1: Neutrosophic soft set $(\Phi,\mathfrak{A})$

Definition 2.6 Let $(\Phi,\mathfrak{A})$ be a neutrosophic soft set on a universe U. For some elements $g \in U$, a neutrosophic right neighborhood, regarding $\xi \in \mathfrak{A}$ is interpreted as follows;

$$g_{\xi} = \{g_i \in U: \mathfrak{T}_{\xi}(g_i) \geq \mathfrak{T}_{\xi}(g), \mathfrak{I}_{\xi}(g_i) \geq \mathfrak{I}_{\xi}(g), \mathfrak{F}_{\xi}(g_i) \leq \mathfrak{F}_{\xi}(g)\}.$$

Definition 2.7 Let $(\Phi,\mathfrak{A})$ be a neutrosophic soft set over a universe U. For some elements $g \in U$, a neutrosophic right neighborhood regarding all parameters $\mathfrak{A}$ is interpreted as follows;

$$g]_{\mathfrak{A}} = \cap \{g_{\xi_i}: \xi_i \in \mathfrak{A}\}.$$

Example 2.8 Consider Example 2.5 then we find the following neutosophic right neighborhood regarding all parameters $\mathfrak{A}$ as

$$\begin{aligned} g_{1\xi_1} = g_{1\xi_2} = g_{1\xi_3} = g_{1\xi_4} = \{g_1\}, g_{2\xi_1} = g_{2\xi_3} = \{g_1, g_2\}, g_{2\xi_2} = \{g_1, g_2, g_4, g_5\}, g_{2\xi_4} = \{g_1, g_2, g_3\}, g_{3\xi_1} \\ = g_{3\xi_4} = \{g_1, g_3\}, g_{3\xi_2} = \{g_1, g_3, g_4, g_5\}, g_{3\xi_3} = \{g_1, g_3, g_5\}, g_{4\xi_1} = \{g_1, g_3, g_4\}, g_{4\xi_2} \\ = \{g_4, g_5\}, g_{4\xi_3} = U, g_{4\xi_4} = U, g_{5\xi_1} = g_{5\xi_2} = g_{5\xi_4} = \{g_5\}, g_{5\xi_3} = \{g_1, g_5\}. \end{aligned}$$

It follows that,

$g_1]_{\mathfrak{A}} = \{g_1\}$,

$g_2]_{\mathfrak{A}} = \{g_1, g_2\}$,

$g_3]_{\mathfrak{A}} = \{g_1, g_3\}$,

$g_4]_{\mathfrak{A}} = \{g_4\}$,

$g_5]_{\mathfrak{A}} = \{g_5\}$.

Definition 2.9 Let $(\Phi,\mathfrak{A})$ be a neutrosophic soft set over U. For any $X \subseteq U$, neutrosophic soft lower ($\underline{apr}_{NSR}$) approximation, neutrosophic soft upper ($\overline{apr}_{NSR}$) approximation, and neutrosophic soft boundary (B_{NSR}) approximation of X are defined as

$$\underline{apr}_{NSR}(X) = \cup \{g]_{\mathfrak{A}}: g \in U, g]_{\mathfrak{A}} \subseteq X\}$$

$$\overline{apr}_{NSR}(X) = \cup \{g]_{\mathfrak{A}}: g \in U, g]_{\mathfrak{A}} \cap X \neq \emptyset\}$$

$$B_{NSR}(X) = \overline{apr}_{NSR}(X) - \underline{apr}_{NSR}(X),$$

respectively. If $\underline{apr}_{NSR}(X) = \overline{apr}_{NSR}(X)$ then X is neutrosophic soft definable set.

Example 2.10 Consider Example 2.5, If $X = \{g_1\} \subseteq U$, then $\underline{apr}_{NSR}(X) = \{g_1\}$ and $\overline{apr}_{NSR}(X) = \{g_1, g_2, g_3\}$. Since its clear $\underline{apr}_{NSR}(X) \neq \overline{apr}_{NSR}(X)$, so X is neutrosophic soft rough set on U.

3 Neutrosophic Soft Rough Topology

In this section, we introduce and study the idea of neutrosophic soft rough topology and its related properties. Concepts of (NSR)-open set, (NSR)-closed set, (NSR)-closure, (NSR)-interior, (NSR)-exterior, (NSR)-neighborhood, (NSR)-limit point, and (NSR)-bases are defined.

Definition 3.1 Let U be the initial space, $\mathfrak{Y} \subseteq \mathrm{U}$ and $\mathrm{G} = (\mathrm{U}, \mathrm{K})$ be a neutrosophic soft approximation space, where $\mathrm{K} = (\Phi, \mathfrak{A})$ is a neutrosophic soft set. The upper and lower approximations are calculated on the basis of neutrosophic soft approximation space and neighborhoods. Then, the collection

$$\tau_{\mathrm{NSR}}(\mathfrak{Y}) = \{\mathrm{U}, \emptyset, \underline{\mathrm{apr}}_{\mathrm{NSR}}(\mathfrak{Y}), \overline{\mathrm{apr}}_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{B}_{\mathrm{NSR}}(\mathfrak{Y})\}$$

is called neutrosophic soft rough topology (NSR-topology) which guarantee the following postulates:

- U and $\emptyset$ belongs to $\tau_{\mathrm{NSR}}(\mathfrak{Y})$.
- Union of members of $\tau_{\mathrm{NSR}}(\mathfrak{Y})$ belongs to $\tau_{\mathrm{NSR}}(\mathfrak{Y})$.
- Finite Intersection of members of $\tau_{\mathrm{NSR}}(\mathfrak{Y})$ belongs to $\tau_{\mathrm{NSR}}(\mathfrak{Y})$.

Then $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ is said to be NSR-topological space, if $\tau_{\mathrm{NSR}}(\mathfrak{Y})$ is Neutrosophic soft rough topology.

Note that Neutrosophic soft rough topology is based on lower and upper approximations of neutrosophic soft rough set.

Example 3.2 From Example 2.5, if $\mathfrak{Y} = \{g_2, g_4\} \subseteq \mathrm{U}$, we obtain $\underline{\mathrm{apr}}_{\mathrm{NSR}}(\mathfrak{Y}) = \{g_4\}$, $\overline{\mathrm{apr}}_{\mathrm{NSR}}(\mathfrak{Y}) = \{g_1, g_2, g_4\}$ and $\mathrm{B}_{\mathrm{NSR}}(\mathfrak{Y}) = \{g_1, g_2\}$. Then,

$$\tau_{\mathrm{NSR}}(\mathfrak{Y}) = \{\mathrm{U}, \emptyset, \{g_4\}, \{g_1, g_2, g_4\}, \{g_1, g_2\}\}$$

is a NSR-topology.

Definition 3.3 Let $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ be an NSR-topological space. Then, the members of $\tau_{\mathrm{NSR}}(\mathfrak{Y})$ are called NSR-open sets. An NSR-set is said to be an NSR-closed set if its complement belongs to $\tau_{\mathrm{NSR}}(\mathfrak{Y})$.

Proposition 3.4 Consider $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ as NSR-space over U. Then,

- U and $\emptyset$ are NSR-closed sets.
- The intersection of any number of NSR-closed sets is an NSR-closed set over U.
- The finite union of NSR-closed sets is an NSR-closed set over U.

Proof. The proof is straightforward.

Definition 3.5 Let $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ be an NSR-space over U and $\tau_{\mathrm{NSR}}(\mathfrak{Y}) = \{\mathrm{U}, \emptyset\}$. Then, τ_{NSR} is called NSR-indiscrete topology on U w.r.t $\mathfrak{Y}$ and corresponding space is said to be an NSR-indiscrete space over U.

Definition 3.6 Let $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ is an NSR-topological space and $\mathrm{A} \subseteq \mathrm{B} \subseteq \mathrm{U}$. Then, the collection $\tau_{\mathrm{NSR}_\mathrm{A}} = \{\mathrm{B_i} \cap \mathrm{A} : \mathrm{B_i} \in \tau_{\mathrm{NSR}}, \mathrm{i} \in \mathrm{L} \subseteq \mathrm{N}\}$ is called NSR-subspace topology on A. Then, $(\mathrm{A}, \tau_{\mathrm{NSR}_\mathrm{A}})$ is called an NSR-topological subspace of $(\mathrm{B}, \tau_{\mathrm{NSR}})$.

Definition 3.7 Let $(\mathrm{U}, \tau_{\mathrm{NSR}'}(\mathfrak{Y}), \mathrm{E})$ and $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{X}), \mathrm{E})$ be two NSR-topological spaces. $\tau_{\mathrm{NSR}'}(\mathfrak{Y})$ is finer than $\tau_{\mathrm{NSR}}(\mathfrak{X})$, if $\tau_{\mathrm{NSR}'}(\mathfrak{Y}) \supseteq \tau_{\mathrm{NSR}}(\mathfrak{X})$.

Definition 3.8 Let $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ be a NSR-topological space and $\beta_{\mathrm{NSR}} \subseteq \tau_{\mathrm{NSR}}$. If we can write members of τ_{NSR} as the union of members of β_{NSR}, then β_{NSR} is called NSR-basis for the NSR-topology τ_{NSR}.

Proposition 3.9 If $\tau_{\mathrm{NSR}}(\mathfrak{Y})$ is an NSR-topology on U w.r.t $\mathfrak{Y}$ the the collection

$$\beta_{\mathrm{NSR}} = \{\mathrm{U}, \underline{\mathrm{apr}}_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{B}_{\mathrm{NSR}}(\mathfrak{Y})\}$$

is a base for $\tau_{\mathrm{NSR}}(\mathfrak{Y})$

Theorem 3.10 Let $(\mathrm{U}, \tau_{\mathrm{NSR}}(\mathfrak{Y}), \mathrm{E})$ and $(\mathrm{U}, \tau_{\mathrm{NSR}'}(\mathfrak{Y}'), \mathrm{E})$ be two NSR-topological spaces w.r.t $\mathfrak{Y}$ and $\mathfrak{Y}'$ respectively. Let β_{NSR} and $\beta_{\mathrm{NSR}'}$ be NSR-bases for τ_{NSR} and $\tau_{\mathrm{NSR}'}$, respectively. If $\beta_{\mathrm{NSR}'} \subseteq \beta_{\mathrm{NSR}}$ then τ_{NSR} is finer than $\tau_{\mathrm{NSR}'}$ and $\tau_{\mathrm{NSR}'}$ is weaker than τ_{NSR}.

Theorem 3.11 Let $(U, \tau_{NSR}(\mathfrak{Y}), E)$ be an NSR-topological space. If β_{NSR} is an NSR-basis for τ_{NSR}. Then, the collection $\beta_{NSR_B} = \{A_i \cap B : A_i \in \beta_{NSR}, i \in I \subseteq \mathbb{N}\}$ is an NSR-basis for the NSR-subspace topology on B.

Proof. Consider $A_i \in \tau_{NSR_B}$. By definition of NSR-subspace topology, $C = D \cap B$,where $D \in \tau_{NSR}$. Since $D \in \tau_{NSR}$, it follows that $D = \cup_{A_i \in \beta_{NSR}} A_i$. Therefore,

$$C = (\cup_{A_i \in \beta_{NSR}} A_i) \cap B = \cup_{A_i \in \beta_{NSR}} (A_i \cap B).$$

3.1 Main Results

We present some results of neutrosophic soft rough topology including NSR-interior, NSR-exterior, NSR-closure, NSR-frontier, NSR-neighbourhood and NSR-limit point. These are some topological properties of NSR-topology and can be used to prove various results related to NSR-topological spaces.

Definition 3.12 Let $(U, \tau_{NSR}(\mathfrak{Y}), E)$ be an NSR-topological space w.r.t $\mathfrak{Y}$, where $T \subseteq U$ be an arbitrary subset. The NSR-interior of T is union of all NSR-open subsets of T and we denote it as $Int_{NSR}(T)$.

We verify that $Int_{NSR}(T)$ is the largest NSR-open set contained by T.

Theorem 3.13 Let $(U, \tau_{NSR}(\mathfrak{Y}), E)$ be a NSR-topological space over U w.r.t $\mathfrak{Y}$, S and T are NSR-sets over U. Then

- $Int_{NSR}(\emptyset) = \emptyset$ and $Int_{NSR}(U) = U$,
- $Int_{NSR}(S) \subseteq S$,
- S is NSR-open set $\Leftrightarrow Int_{NSR}(S) = S$,
- $Int_{NSR}(Int_{NSR}(S)) = Int_{NSR}(S)$,
- $S \subseteq T$ implies $Int_{NSR}(S) \subseteq Int_{NSR}(T)$,
- $Int_{NSR}(S) \cup Int_{NSR}(T) \subseteq Int_{NSR}(S \cup T)$,
- $Int_{NSR}(S) \cap Int_{NSR}(T) = Int_{NSR}(S \cap T)$.

Proof. (i) and (ii) are obvious.

(iii) First, suppose that $Int_{NSR}(S) = S$. Since $Int_{NSR}(S)$ is an NSR-open set, it follows that S is NSR-open set. For the converse, if S is a NSR-open set, then the largest NSR-open set that is contained in S is S itself. Thus, $Int_{NSR}(S) = S$.

(iv) Since $Int_{NSR}(S)$ is an NSR-open set, by part (iii) we get $Int_{NSR}(Int_{NSR}(S)) = Int_{NSR}(S)$.

(v) Suppose that $S \subseteq T$. By (ii) $Int_{NSR}(S) \subseteq S$. Then $Int_{NSR}(S) \subseteq T$. Since $Int_{NSR}(S)$ is NSR-open set contained by T. So by definition of NSR-interior $Int_{NSR}(S) \subseteq Int_{NSR}(T)$.

(vi) By using (ii) $Int_{NSR}(S) \subseteq S$ and $Int_{NSR}(T) \subseteq T$. Then, $Int_{NSR}(S) \cup Int_{NSR}(T) \subseteq S \cup T$. Since $Int_{NSR}(S) \cup Int_{NSR}(T)$ is an NSR-open, it follows that $Int_{NSR}(S) \cup Int_{NSR}(T) \subseteq Int_{NSR}(S \cup T)$.

(vii) By using (ii) $Int_{NSR}(S) \subseteq S$ and $Int_{NSR}(T) \subseteq T$. Then, $Int_{NSR}(S) \cap Int_{NSR}(T) \subseteq S \cap T$. Since $Int_{NSR}(S) \cap Int_{NSR}(T)$ is NSR-open, it follows that $Int_{NSR}(S) \cap Int_{NSR}(T) \subseteq Int_{NSR}(S \cap T)$. For the converse, $S \cap T \subseteq S$ also $S \cap T \subseteq T$. Then, $Int_{NSR}(S \cap T) \subseteq Int_{NSR}(S)$ and $Int_{NSR}(S \cap T) \subseteq Int_{NSR}(T)$. Hence $Int_{NSR}(S \cap T) \subseteq Int_{NSR}(S) \cap Int_{NSR}(T)$.

Definition 3.14 Let $(U, \tau_{NSR}(\mathfrak{Y}), E)$ be an NSR-topological space w.r.t $\mathfrak{Y}$, where $\mathfrak{Y} \subseteq U$. Let $T \subseteq U$. Then, NSR-exterior of T is defined as $Int_{NSR}(T^c)$, where T^c is complement of T. NSR-exterior of T is denoted by $Ext_{NSR}(T)$.

Definition 3.15 Let $(U, \tau_{NSR}(\mathfrak{Y}), E)$ be an NSR-topological space w.r.t $\mathfrak{Y}$, where $\mathfrak{Y} \subseteq U$. Let $T \subseteq U$. Then, NSR-closure of T is defined to be intersection of all NSR-closed supersets of T and is denoted by $Cl_{NSR}(T)$.

Example 3.16 Consider the NSR-topology given in Example 3.2, taking $T = \{g_1, g_2, g_3\}$, so $T^c = \{g_4, g_5\}$. Then $Int_{NSR}(T) = \{g_1, g_2\}$, $Ext_{NSR(T)} = Int_{NSR}(T^c) = \{g_4\}$ and $Cl_{NSR}(\mathrm{T}) = \{g_1, g_2, g_3, g_5\}$.

Theorem 3.17 Let $(U, \tau_{NSR}(\mathfrak{Y}), E)$ be a NSR-topological space over U w.r.t $\mathfrak{Y}$, S and T are NSR-sets over U. Then

- $Cl_{NSR}(\emptyset) = \emptyset$ and $Cl_{NSR}(U) = U$,
- $S \subseteq Cl_{NSR}(S)$,
- S is NSR-closed set $\Leftrightarrow S = Cl_{NSR}(S)$,
- $Cl_{NSR}(Cl_{NSR}(S)) = Cl_{NSR}(S)$,
- $S \subseteq T$ implies $Cl_{NSR}(S) \subseteq Cl_{NSR}(T)$,
- $Cl_{NSR}(S \cup T) = Cl_{NSR}(S) \cup Cl_{NSR}(T)$,
- $Cl_{NSR}(S \cap \mathrm{T}) \subseteq Cl_{NSR}(S) \cap Cl_{NSR}(T)$.

Proof. (i) and (ii) are straightforward.

(iii) First, consider $S = Cl_{NSR}(S)$. Since $Cl_{NSR}(S)$ is an NSR-closed set, so S is an NSR-closed set over U. For the converse, suppose that S be an NSR-closed set over U. Then, S is NSR-closed superset of S. So that $S = Cl_{NSR}(S)$.

(iv) By definition $Cl_{NSR}(S)$ is always NSR-closed set. Therefore, by part (iii) we have $Cl_{NSR}(Cl_{NSR}(S)) = Cl_{NSR}(S)$.

(v) Let $S \subseteq T$. By (ii) $T \subseteq Cl_{NSR}(T)$. Then, $S \subseteq Cl_{NSR}(T)$. Since $Cl_{NSR}(T)$ is a NSR-closed superset of S, it follows that $Cl_{NSR}(S) \subseteq Cl_{NSR}(T)$.

(vi) Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, by part (v), $Cl_{NSR}(S) \subseteq Cl_{NSR}(S \cup T)$ and $Cl_{NSR}(T) \subseteq Cl_{NSR}(S \cup T)$. Hence $Cl_{NSR}(S) \cup Cl_{NSR}(S) \subseteq Cl_{NSR}(S \cup T)$. For the converse, let $S \subseteq Cl_{NSR}(S)$ and $T \subseteq Cl_{NSR}(T)$. Then, $S \cup T \subseteq Cl_{NSR}(S) \cup CL_{NSR}(T)$. Since $Cl_{NSR}(S) \cup Cl_{NSR}(T)$ is a NSR-closed superset of $S \cup T$. Thus, $Cl_{NSR}(S \cup T) = Cl_{NSR}(S) \cup Cl_{NSR}(T)$.

(vii) Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, by part(5) $Cl_{NSR}(S \cap T) \subseteq Cl_{NSR}(S)$ and $Cl_{NSR}(S \cap T) \subseteq Cl_{NSR}(T)$. Thus, we obtain $Cl_{NSR}(S \cap T) \subseteq Cl_{NSR}(S) \cap Cl_{NSR}(T)$.

Definition 3.18 Let $(U, \tau_{NR}(\mathfrak{Y}), E)$ be a NSR-topological space w.r.t $\mathfrak{Y}$, where $\mathfrak{Y} \subseteq U$. Let $T \subseteq U$. Then, NSR-frontier or NSR-boundary of T is denoted by $F_{r_{NSR}}(T)$ or $b_{NSR}(T)$ and mathematically defined as

$$F_{r_{NSR}}(T) = Cl_{NSR}(T) \cap Cl_{NSR}(T^c).$$

Clearly NSR-frontier $F_{r_{NSR}}(T)$ is an NSR-closed set.

Example 3.19 Consider the NSR-topology given in Example 3.2, taking $T = \{g_1, g_2, g_3\}$, so $T^c = \{g_4, g_5\}$. Then, $Cl_{NSR}(T) = \{g_1, g_2, g_3, g_5\}$ and $Cl_{NSR}(T^c) = \{g_3, g_4, g_5\}$.

$$F_{r_{NSR}}(T) = Cl_{NSR}(T) \cap Cl_{NSR}(T^c) = \{g_3, g_5\}$$

Definition 3.20 Let $(U, \tau_{NR}(\mathfrak{Y}), E)$ be an NSR-topological space. A subset X of U is said to be NSR-neighborhood of $g \in U$ if there exist an NSR-open set W_g containing g so that

$$g \in W_g \subseteq X.$$

Definition 3.21 The set of all the NSR-limit points of S is known as NSR-derived set of S and is denoted by S^d_{NSR}.

4 NSR-set in multi-criteria decision-making

In this section, we present an idea for multi-criteria decision-making method based on the neutrosophic soft rough sets $NSR-set$.

Let $U = \{g_1, g_2, g_3, \dots, g_m\}$ is the set of objects under observation, E be the set of criteria to analyze the objects in U. Let $\mathfrak{A} = \{\xi_1, \xi_2, \xi_3, \dots, \xi_n\} \subseteq E$ and $(\Phi, \mathfrak{A})$ be a neutrosophic soft set over U. Suppose that $H = \{P_1, P_2, \dots, P_k\}$ be a set of experts, $\mathfrak{Y}_1, \mathfrak{Y}_2, \dots, \mathfrak{Y}_k$ are subsets of U which indicate results of initial evaluations of experts $P_1, P_2, \dots, P_k$, respectively and $\mathfrak{T}_1, \mathfrak{T}_2, \dots \mathfrak{T}_r \in NS_U$ are real results that previously obtained for same or similar problems in different times or different places.

Definition 4.1 Let $\underline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j), \overline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j)$ be neutrosophic soft lower and upper approximations of $\mathfrak{Y}_j (j = 1,2, \dots, k)$ related to $\mathfrak{T}_q (q = 1,2, \dots, r)$. Then,

$$\underline{a} = \begin{pmatrix} \underline{n}_1^1 & \underline{n}_2^1 & \cdots & \underline{n}_k^1 \\ \underline{n}_1^2 & \underline{n}_2^2 & \cdots & \underline{n}_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{n}_1^r & \underline{n}_2^r & \cdots & \underline{n}_k^r \end{pmatrix} \quad (1)$$

$$\overline{a} = \begin{pmatrix} \overline{n}_1^1 & \overline{n}_2^1 & \cdots & \overline{n}_k^1 \\ \overline{n}_1^2 & \overline{n}_2^2 & \cdots & \overline{n}_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{n}_1^r & \overline{n}_2^r & \cdots & \overline{n}_k^r \end{pmatrix} \quad (2)$$

are called neutrosophic soft lower and neutrosophic upper approximations matrices, respectively, and represented by $\underline{a}$ and $\overline{a}$. Here

$$\underline{n}_j^q = (\underline{g}_{1j}^q, \underline{g}_{2j}^q, \dots, \underline{g}_{nj}^q) \quad (3)$$

$$\overline{n}_j^q = (\overline{g}_{1j}^q, \overline{g}_{2j}^q, \dots, \overline{g}_{nj}^q) \quad (4)$$

Where

$$\underline{g}_{ij}^q = \begin{cases} 1, g_i \in \underline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j) \\ 0, g_i \notin \underline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j) \end{cases}$$

and

$$\overline{g}_{ij}^q = \begin{cases} 1, g_i \in \overline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j) \\ 0, g_i \notin \overline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j) \end{cases}$$

Definition 4.2 Let $\underline{n}$ and $\overline{n}$ be neutrosophic soft lower and neutrosophic upper approximations matrices based on $\underline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y})_j$, $\overline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y})_j$ for $q = 1,2,\ldots r$ and $j = 1,2,\ldots,k$. Neutrosophic soft lower approximation vector represented by $(\underline{n})$ and neutrosophic soft upper approximation vector represented by $(\overline{n})$ are defined by, respectively,

$$\underline{n} = \bigoplus_{j=1}^{k} \bigoplus_{q=1}^{r} \underline{n}_j^q \tag{5}$$

$$\overline{n} = \bigoplus_{j=1}^{k} \bigoplus_{q=1}^{r} \overline{n}_j^q \tag{6}$$

Here the operation $\oplus$ represents the vector summation.

Definition 4.3 Let $\underline{n}$ and $\overline{n}$ be neutrosophic soft $\mathfrak{T}_q -$ lower approximation vector and neutrosophic soft $\mathfrak{T}_q -$ upper approximation vector, respectively. Then, vector summation $\underline{n} \oplus \overline{n} = (w_1, w_2, \ldots, w_n)$ is called decision vector.

Definition 4.4 Let $\underline{n} \oplus \overline{n} = (w_1, w_2, \ldots, w_n)$ be the decision vector. Then, each w_i is called a weighted number of $g_i \in U$ and g_i is called an optimum element of U if it weighted number is maximum of $w_i \forall i \in I_n$. In this case, if there are more then one optimum elements of U, select one of them.

Algorithm 1 for neutrosophic soft rough set:

Input

Step-1: Take initial evaluations $\mathfrak{Y}_1, \mathfrak{Y}_2, \ldots, \mathfrak{Y}_k$ of experts $P_1, P_2, \ldots, P_k$.

Step-2: Construct $\mathfrak{T}_1, \mathfrak{T}_2, \ldots \mathfrak{T}_r$ neutrosophic soft sets using real results.

Step-3: Compute $\underline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j)$ and $\overline{apr}_{NSR_{\mathfrak{T}_q}}(\mathfrak{Y}_j)$ for each $q = 1,2,\ldots,r$ and $j = 1,2,\ldots,k$.

Step-4: Construct neutrosophic soft lower and neutrosophic soft upper approximations matrices $\underline{a}$ and $\overline{a}$.

Step-5: Compute $\underline{n}$ and $\overline{n}$,

Step-6: Compute $\underline{n} \oplus \overline{n}$,

Output

Step-7: Select $max_{i \in I_n} w_i$.

The flow chart of proposed algorithm 1 is represented in Figure.1

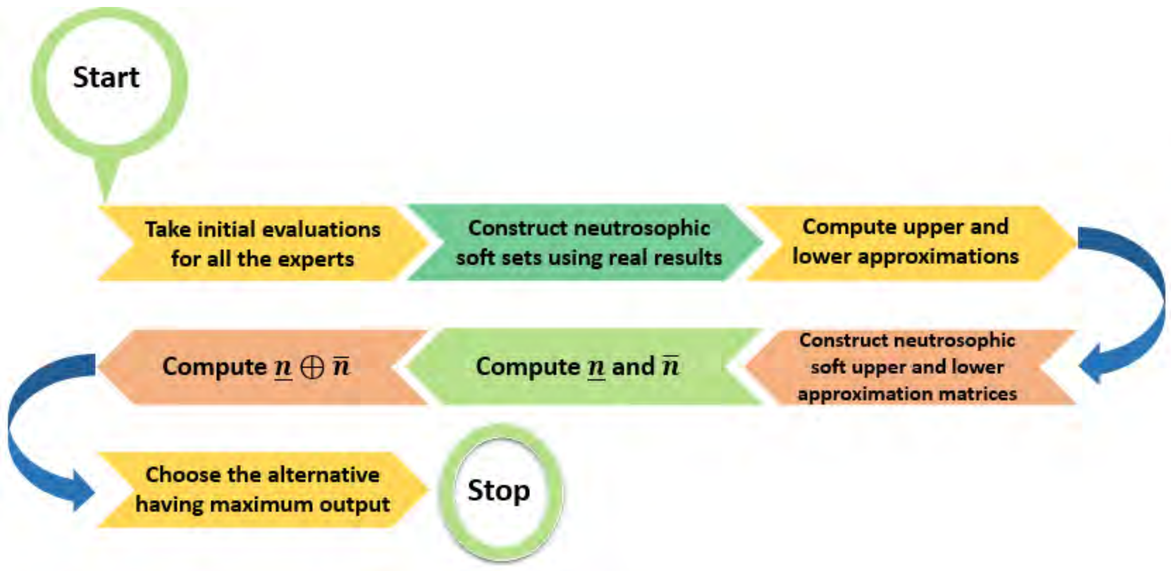

Fig 1: Flow chart diagram of proposed algorithm 1 for NSR-set.

Example 4.5 In finance company three finance experts P_1, P_2, P_3 want to make investment one of the clothing brand

$$\{g_1 = Jor, g_2 = Aero, g_3 = Chan, g_4 = Li, g_5 = Srk\}.$$

The set of parameters include the following parameters

$$\mathfrak{A} = \{\xi_1 = Market\ Share, \xi_2 = Acknowledgement, \xi_3 = Uniqueness, \xi_4 = Economical\ Magnification\}$$

*Step*1: $\mathfrak{Y}_1 = \{g_1, g_2, g_4\}, \mathfrak{Y}_2 = \{g_1, g_3, g_5\}, \mathfrak{Y}_3 = \{g_2, g_4, g_5\}$ are primary evaluations of experts P_1, P_2, P_3, respectively.

*Step*2: Neutrosophic soft sets $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ are the actual results in individual three periods and tabular representations of these neutrosophic soft sets are given in Table 2, Table 3 and Table 4, respectively.

$\mathfrak{T}_1$	ξ_1	ξ_2	ξ_3	ξ_4
g_1	(0.6,0.6,0.2)	(0.8,0.4.0.3)	(0.7,0.4,0.3)	(0.8,0.6,0.4)
g_2	(0.4,0.6,0.6)	(0.6,0.2,0.4)	(0.6,0.4,0.3)	(0.7,0.6,0.6)
g_3	(0.6,0.4,0.2)	(0.8,0.1,0.3)	(0.7,0.2,0.5)	(0.7,0.6,0.4)
g_4	(0.6,0.3,0.3)	(0.8,0.2,0.2)	(0.5,0.2,0.6)	(0.7,0.5,0.6)
g_5	(0.8,0.2,0.3)	(0.8,0.3,0.2)	(0.7,0.3,0.4)	(0.9,0.5,0.7)

Table 2: Neutrosophic soft set $\mathfrak{T}_1$

$\mathfrak{T}_2$	ξ_1	ξ_2	ξ_3	ξ_4
g_1	(0.6,0.4,0.2)	(0.8,0.1,0.3)	(0.7,0.2,0.5)	(0.7,0.6,0.4)
g_2	(0.4,0.6,0.6)	(0.6,0.2,0.4)	(0.6,0.4,0.3)	(0.7,0.6,0.6)
g_3	(0.8,0.2,0.3)	(0.8,0.3,0.2)	(0.7,0.3,0.4)	(0.9,0.5,0.7)
g_4	(0.6,0.3,0.3)	(0.8,0.2,0.2)	(0.5,0.2,0.6)	(0.7,0.5,0.6)
g_5	(0.6,0.6,0.2)	(0.8,0.4.0.3)	(0.7,0.4,0.3)	(0.8,0.6,0.4)

Table 3: Neutrosophic soft set $\mathfrak{T}_2$

$\mathfrak{T}_3$	ξ_1	ξ_2	ξ_3	ξ_4
g_1	(0.6,0.6,0.2)	(0.8,0.4.0.3)	(0.7,0.4,0.3)	(0.8,0.6,0.4)
g_2	(0.6,0.3,0.3)	(0.8,0.2,0.2)	(0.5,0.2,0.6)	(0.7,0.5,0.6)
g_3	(0.6,0.4,0.2)	(0.8,0.1,0.3)	(0.7,0.2,0.5)	(0.7,0.6,0.4)
g_4	(0.4,0.6,0.6)	(0.6,0.2,0.4)	(0.6,0.4,0.3)	(0.7,0.6,0.6)
g_5	(0.8,0.2,0.3)	(0.8,0.3,0.2)	(0.7,0.3,0.4)	(0.9,0.5,0.7)

Table 4: Neutrosophic soft set $\mathfrak{T}_3$

The tabular representation of the neutrosophic right neighborhoods of $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ are given in Table5, Table 6 and Table 7 respectively.

Neighborhoods of $\mathfrak{T}_1$	
$g_1]_{\mathfrak{A}}$	$\{g_1\}$
$g_2]_{\mathfrak{A}}$	$\{g_1, g_2\}$
$g_3]_{\mathfrak{A}}$	$\{g_1, g_3\}$
$g_4]_{\mathfrak{A}}$	$\{g_4\}$
$g_5]_{\mathfrak{A}}$	$\{g_5\}$

Table 5: Neutrosophic right neighborhoods of $\mathfrak{T}_1$ w.r.t set $\mathfrak{A}$

Neighborhoods of $\mathfrak{T}_2$	
$g_1]_{\mathfrak{A}}$	$\{g_1, g_5\}$
$g_2]_{\mathfrak{A}}$	$\{g_2, g_5\}$
$g_3]_{\mathfrak{A}}$	$\{g_3\}$
$g_4]_{\mathfrak{A}}$	$\{g_4\}$
$g_5]_{\mathfrak{A}}$	$\{g_5\}$

Table 6: Neutrosophic right neighborhoods of $\mathfrak{T}_2$ w.r.t set $\mathfrak{A}$

Neighborhoods of $\mathfrak{T}_3$	
$g_1]_{\mathfrak{A}}$	$\{g_1\}$
$g_2]_{\mathfrak{A}}$	$\{g_2\}$
$g_3]_{\mathfrak{A}}$	$\{g_1, g_3\}$
$g_4]_{\mathfrak{A}}$	$\{g_1, g_4\}$
$g_5]_{\mathfrak{A}}$	$\{g_5\}$

Table 7: Neutrosophic right neighborhoods of $\mathfrak{T}_3$ w.r.t set $\mathfrak{A}$

$Step3$: Next we find $\underline{apr}_{NSR_{\mathfrak{T}_1}}$ and $\overline{apr}_{NSR_{\mathfrak{T}_1}}$ for each $\mathfrak{Y}_j$, where $j = 1,2,3$.

$$\underline{apr}_{NSR_{\mathfrak{T}_1}}(\mathfrak{Y}_1) = \{g_1, g_2, g_4\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_1}}(\mathfrak{Y}_1) = \{g_1, g_2, g_3, g_4\},$$
$$\underline{apr}_{NSR_{\mathfrak{T}_1}}(\mathfrak{Y}_2) = \{g_1, g_3, g_5\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_1}}(\mathfrak{Y}_2) = \{g_1, g_2, g_3, g_5\},$$
$$\underline{apr}_{NSR_{\mathfrak{T}_1}}(\mathfrak{Y}_3) = \{g_4, g_5\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_1}}(\mathfrak{Y}_3) = \{g_1, g_2, g_3, g_4, g_5\}$$

Similarly we find $\underline{apr}_{NSR_{\mathfrak{T}_2}}$,$\overline{apr}_{NSR_{\mathfrak{T}_2}}$ and $\underline{apr}_{NSR_{\mathfrak{T}_3}}$, $\overline{apr}_{NSR_{\mathfrak{T}_3}}$ corresponding to each $\mathfrak{Y}_j$, where $j = 1,2,3$.

$$\underline{apr}_{NSR_{\mathfrak{T}_2}}(\mathfrak{Y}_1) = \{g_4\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_2}}(\mathfrak{Y}_1) = \{g_1, g_2, g_4, g_5\},$$
$$\underline{apr}_{NSR_{\mathfrak{T}_2}}(\mathfrak{Y}_2) = \{g_1, g_3, g_5\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_2}}(\mathfrak{Y}_2) = \{g_1, g_2, g_3, g_5\},$$
$$\underline{apr}_{NSR_{\mathfrak{T}_2}}(\mathfrak{Y}_3) = \{g_4, g_5\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_2}}(\mathfrak{Y}_3) = \{g_1, g_2, g_4, g_5\}$$

and

$$\underline{apr}_{NSR_{\mathfrak{T}_3}}(\mathfrak{Y}_1) = \{g_1, g_2, g_4\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_3}}(\mathfrak{Y}_1) = \{g_1, g_2, g_3, g_4\},$$
$$\underline{apr}_{NSR_{\mathfrak{T}_3}}(\mathfrak{Y}_2) = \{g_1, g_3, g_5\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_3}}(\mathfrak{Y}_2) = \{g_1, g_3, g_4, g_5\},$$
$$\underline{apr}_{NSR_{\mathfrak{T}_3}}(\mathfrak{Y}_3) = \{g_2, g_5\},$$
$$\overline{apr}_{NSR_{\mathfrak{T}_3}}(\mathfrak{Y}_3) = \{g_1, g_2, g_4, g_5\}$$

$Step4$: Neutrosophic soft lower approximation matrix and neutrosophic soft upper approximation matrix are obtained as follows:

$$\underline{a} = \begin{pmatrix} (1,1,0,1,0) & (1,0,1,0,1) & (0,0,0,1,1) \\ (0,0,0,1,0) & (1,0,1,0,1) & (0,0,0,1,1) \\ (1,1,0,1,0) & (1,0,1,0,1) & (0,1,0,0,0) \end{pmatrix} \tag{7}$$

$$\overline{a} = \begin{pmatrix} (1,1,1,1,0) & (1,1,1,0,1) & (1,1,1,1,1) \\ (1,1,0,1,1) & (1,1,1,0,1) & (1,1,0,1,1) \\ (1,1,1,1,0) & (1,0,1,1,1) & (1,1,0,1,1) \end{pmatrix} \tag{8}$$

$Step5$: Using Eqs. 7 and 8, neutrosophic soft lower approximation vector and neutrosophic soft upper approximation vector are obtained as follows:

$$\underline{n} = (5,3,3,5,5)$$

$$\overline{n} = (9,8,6,7,7)$$

$Step6$: Decision vector is obtained as $\underline{n} \oplus \overline{n} = (14,11,9,12,12)$.

$Step7$: Since $max_{i \in I_n} w_i = w_1 = 14$, optimal clothing brand is $g_1 = Jor$.

5 NSR-topology in multi-criteria decision-making

In this section, we use the concept of NSR-topology in multi-criteria decision-making. The idea of core in the picking of attributes to the rough set was introduced by Thivagar in [45]. In the following definition, we develop this idea of core to the NSR-set.

Definition 5.1 Let U be the set of objects, $K = (\Phi, \mathfrak{A})$ is the neutrosophic soft set and $G = (U, K)$ is the the corresponding neutrosophic soft approximation space. Let $\mathfrak{R}$ be an indiscernibility relation. Let τ_{NSR} be an NSR-topology on U and β_{NSR} be the basis defined for τ_{NSR}. Let $\mathfrak{N}$ be the subset of $\mathfrak{A}$, is said to be core of $\mathfrak{R}$ if $\beta_{\mathfrak{N}} \neq \beta_{NSR-(s)}$ for each 's' in $\mathfrak{N}$. i.e. a core of $\mathfrak{R}$ is the subset of attributes with the condition that if we remove any element from $\mathfrak{N}$ it will affect the classification power of the attributes.

Algorithm 2 for neutrosophic soft rough topology:

Input

Step-1: Consider initial universe U, set of attributes $\mathfrak{A}$ which can be classified into division $\mathbb{D}$ of decision attributes, $\mathbb{C}$ of condition attributes and an indiscernibility relation $\mathfrak{R}$ on U. Construct the neutrosophic soft set in tabular form corresponding to $\mathbb{C}$ condition attributes and a subset $\mathfrak{Y}$ of U. The columns indicate the elements of universe, rows represent the attributes and entries of table give attribute values.

Output

Step-2: Classify set $\mathfrak{Y}$ and find the NSR-approximation subsets $\left(\underline{\mathfrak{R}}_G(\mathfrak{Y}), \overline{\mathfrak{R}}_G(\mathfrak{Y})\right) and\ B_G(\mathfrak{Y})$ w.r.t $\mathfrak{R}$.

Step-3: Define Neutrosophic Soft Rough Topology $\tau_{\mathfrak{R}}$ on U and find basis β_{NSR}.

Step-4: By removing an attribute ξ from $\mathbb{C}$, find again the NSR-approximation subsets $(\underline{\mathfrak{R}}_G(\mathfrak{Y}), \overline{\mathfrak{R}}_G(\mathfrak{Y})), B_G(\mathfrak{Y}))$ w.r.t $\mathfrak{R} on \mathbb{C} - (\xi)$.

Step-5: Generate $NSR - topology$ $\tau_{NSR-(\xi)}$ on U,define its basis $\beta_{NSR-(\xi)}$.

Step-6: Repeat step 4 and step 5 for each attribute in $\mathbb{C}$.

Step-7: The attributes for which $\beta_{NSR-(\xi)} \neq \beta_{NSR}$ gives the $core(\mathfrak{R})$.

The flow chart diagram of proposed algorithm 2 is represented as Figure 2.

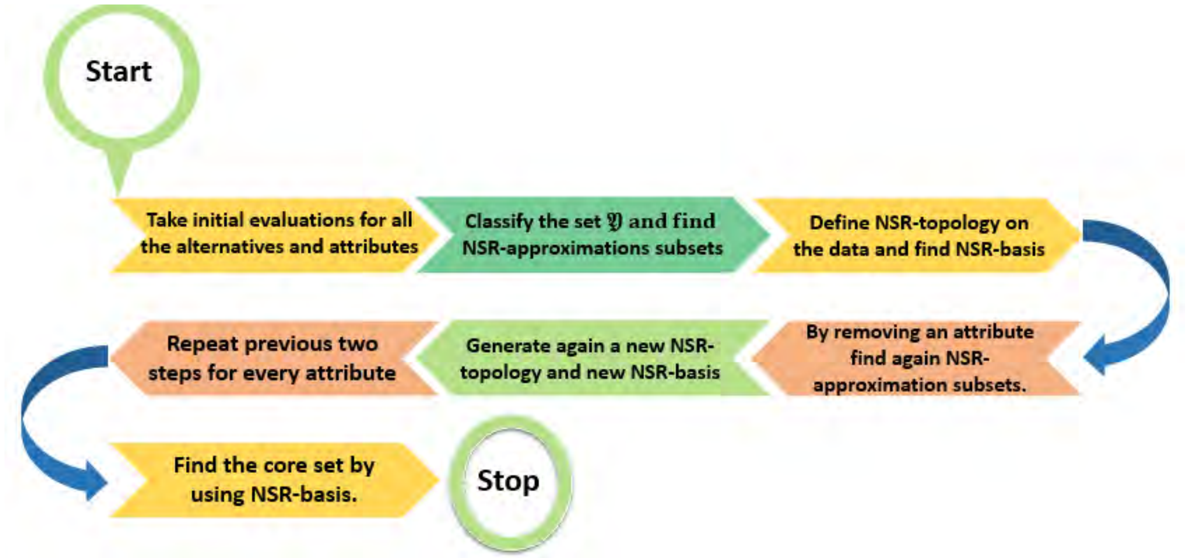

Fig 2: The flow chart diagram of algorithm 2 for NSR-topology.

Example 5.2 Here we consider the problem of Crime rate in developing countries of Asia, Crime is an unlawful act punishable by a state or other authority. In other words, we can say that a crime is an act harmful not only to some individual but also to a community, society or the state. A developing country is a country with a less developed industrial base and a low Human Development Index (HDI) relative to other countries. Developing countries are facing so many issues including high crime rate. This is the fundamental reason of emerging questions in our mind, that why the crime rate is higher in developing countries?

We apply the concept of NSR-topology in Crime rate of developing countries of Asia. Consider the following information table which shows data about 5 developing countries. The rows of the table represent the objects(countries). Let $U = \{g_1 = Bangladesh, g_2 = Afghanistan, g_3 = SriLanka, g_4 = Nepal, g_5 = Pakistan\}$ be the set of developing countries and $\mathfrak{A} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$, where ξ_1 stands for `corruption', $\xi_2$ stands for `poverty ', ξ_3 stands for `self actualization' and $\xi_4$ stands for `lack of education'. Let $K = (\Phi, \mathfrak{A})$ is the neutrosophic soft set over U shown by Table 8,corresponding soft approximation space $G = (U, K)$.

K	ξ_1	ξ_2	ξ_3	ξ_4	Crime Rate
g_1	(0.6,0.6,0.2)	(0.8,0.4.0.3)	(0.7,0.4,0.3)	(0.8,0.6,0.4)	High
g_2	(0.4,0.6,0.6)	(0.6,0.2,0.4)	(0.6,0.4,0.3)	(0.7,0.6,0.6)	Medium
g_3	(0.6,0.4,0.2)	(0.8,0.1,0.3)	(0.7,0.2,0.5)	(0.7,0.6,0.4)	Medium
g_4	(0.6,0.3,0.3)	(0.8,0.2,0.2)	(0.5,0.2,0.6)	(0.7,0.5,0.6)	High
g_5	(0.8,0.2,0.3)	(0.8,0.3,0.2)	(0.7,0.3,0.4)	(0.9,0.5,0.7)	High

Table 8: Neutrosophic soft set $K = (\Phi, \mathfrak{A})$

The tabular representation of neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A}$ is given Table 9.

Neighborhoods of K	
$g_1]_{\mathfrak{A}}$	$\{g_1\}$
$g_2]_{\mathfrak{A}}$	$\{g_1, g_2\}$
$g_3]_{\mathfrak{A}}$	$\{g_1, g_3\}$

$g_4]_{\mathfrak{A}}$	$\{g_4\}$
$g_5]_{\mathfrak{A}}$	$\{g_5\}$

Table 9: Neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A}$

For $\mathfrak{Y} = \{g_1, g_3, g_5\}$ and indiscernibility relation 'Crime rate' we have $\underline{\mathfrak{R}}_G(\mathfrak{Y}) = \{g_1, g_3, g_5\}$, $\overline{\mathfrak{R}}_G(\mathfrak{Y}) = \{g_1, g_2, g_3, g_5\}$ and $B_G(\mathfrak{Y}) = \{g_2\}$.
So we define NSR-topology as $\tau_{NSR}(\mathfrak{Y}) = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_1, g_2, g_3, g_5\}, \{g_2\}\}$ and its basis $\beta_{NSR} = \{U, \{g_1, g_3, g_5\}, \{g_2\}\}$.

If we remove the attribute `Corruption', then the tabular representation of neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_1$ is given Table 10.

Neighborhoods of K	
$g_1]_{\mathfrak{A}-\xi_1}$	$\{g_1\}$
$g_2]_{\mathfrak{A}-\xi_1}$	$\{g_1, g_2\}$
$g_3]_{\mathfrak{A}-\xi_1}$	$\{g_1, g_3\}$
$g_4]_{\mathfrak{A}-\xi_1}$	$\{g_4\}$
$g_5]_{\mathfrak{A}-\xi_1}$	$\{g_5\}$

Table 10: Neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_1$

we have

$$\tau_{NSR-\xi_1}(\mathfrak{Y}) = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_1, g_2, g_3, g_5\}, \{g_2\}\}$$

is a NSR-topology and its basis is

$$\beta_{NSR} - \xi_1 = \{U, \{g_1, g_3, g_5\}, \{g_2\}\} = \beta_{NSR}.$$

If we remove the attribute `poverty', then the tabular representation of neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_2$ is given Table 11.

Neighborhoods of K	
$g_1]_{\mathfrak{A}-\xi_2}$	$\{g_1\}$
$g_2]_{\mathfrak{A}-\xi_2}$	$\{g_1, g_2\}$
$g_3]_{\mathfrak{A}-\xi_2}$	$\{g_1, g_3\}$
$g_4]_{\mathfrak{A}-\xi_2}$	$\{g_1, g_3, g_4\}$
$g_5]_{\mathfrak{A}-\xi_2}$	$\{g_5\}$

Table 11: Neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_2$

We have an NSR-topology and its base as follows:

$$\tau_{NSR-\xi_2}(Y) = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_2, g_4\}\}$$

and $$\beta_{NSR} - \xi_2 = \{U, \{g_1, g_3, g_5\}, \{g_2, g_4\}\} \neq \beta_{NSR},$$

respectively. If we remove the attribute 'self actualization', then the tabular representation of neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A} - \xi_3$ is given Table 12.

Neighborhoods of K	
$g_1]_{\mathfrak{A}-\xi_3}$	$\{g_1\}$
$g_2]_{\mathfrak{A}-\xi_3}$	$\{g_1, g_2\}$

$g_3]_{\mathfrak{A}-\xi_3}$	$\{g_1, g_3\}$
$g_4]_{\mathfrak{A}-\xi_3}$	$\{g_4\}$
$g_5]_{\mathfrak{A}-\xi_3}$	$\{g_5\}$

Table 12: Neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A}-\xi_3$

We have an NSR-topology and its base as follows:

$$\tau_{NSR-\xi_3}(Y) = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_1, g_2, g_3, g_5\}, \{g_2\}\}$$

and

$$\beta_{NSR} - \xi_3 = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_2\} = \beta_{NSR}\},$$

respectively. If we remove the attribute `lack of education', then the tabular representation of neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A}-\xi_4$ is given Table 13.

Neighborhoods of K	
$g_1]_{\mathfrak{A}-\xi_4}$	$\{g_1\}$
$g_2]_{\mathfrak{A}-\xi_4}$	$\{g_1, g_2\}$
$g_3]_{\mathfrak{A}-\xi_4}$	$\{g_1, g_3\}$
$g_4]_{\mathfrak{A}-\xi_4}$	$\{g_4\}$
$g_5]_{\mathfrak{A}-\xi_4}$	$\{g_5\}$

Table 13: Neutrosophic right neighborhoods of K w.r.t set $\mathfrak{A}-\xi_4$

We have an NSR-topology and its base as follows:

$$\tau_{NSR-\xi_4}(Y) = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_1, g_2, g_3, g_5\}, \{g_2\}\}$$

and

$$\beta_{NSR} - \xi_4 = \{U, \emptyset, \{g_1, g_3, g_5\}, \{g_2\} = \beta_{NSR}\},$$

respectively. Thus, $CORE(NSR) = \{\xi_2\}$, i.e., `poverty' is the deciding attributes of the Crime Rate in developing countries of Asia.

Discussion and comparitive analysis 5.3 In this section, we discuss our results obtained from both numerical examples and present a comparative analysis of proposed topological space to some existing topological spaces. Table 14 describes the comparison of both proposed algorithms based on NSR-sets and NSR-topology. The algorithm 1 is used to find the optimal decision about the set of alternatives and establish the ranking order between them. We can choose the best and worst alternative from the given input information. While algorithm 2 is used to choose the most relevant and significant attribute to which one can observe the specific characteristic of the alternatives. This is called the CORE of the problem, which is an essential part of the decision-making difficulty. Both algorithms have their own merits and can be used to solve decision-making problems in medical, artificial intelligence, business, agriculture, engineering, etc.

Proposed Algorithms	Choice values	Final Decision	Selection criteria
Algorithm 1 (NSR-sets)	$g_1 \succ g_4 \succ g_5 \succ g_2 \succ g_3$	g_1	Based on alternatives
Algorithm 2 (NSR-topology)	$CORE(NSR) = \{\xi_2\}$	$\xi_2 =$ poverty	Based on attributes

Table 14: Comparison of prooposed algorithms

Now we present a soft comparative analysis of proposed approach with some existing approaches. In Table 15, we describe the comparison and discuss about their advantages and limitations.

Set theories	Information about Indeterminacy part	Upper and lower approximations with boundary region	Parameterizations	Advantages	Limitations
Fuzzy sets [1]	No	No	No	Deal with the hesitations.	Do not collect any information about the indeterminacy of input data.
Neutrosophic sets [4, 5]	Yes	No	No	Deal with the data having indeterminacy information.	Do not deal with the roughness and parameterizations.
Rough sets [2, 3]	No	Yes	No	Deal with the roughness of input information and create upper, lower and boundary regions.	Do not give any information about the parameterizations.
Soft sets [6]	No	No	Yes	Deal with the uncertainity with parameterizations.	Do not provide information about the roughness of data.
Soft rough sets [17]	No	Yes	Yes	Deal with uncertainities and roughness of data.	Do not give information about the indeterminacy part of problem.
Rough neutrosophic sets [47]	Yes	Yes	No	Deal with the roughness having indeterminacy information.	Do not deal with the parameterizations.
Neutrosophic soft rough sets and topology (proposed)	Yes	Yes	Yes	Provide the data of indeterminacy part and remove roughness under parameterizations without any loss of information.	Effective but heavy calculations as compared to above existing theories.

Table 15: Comparitive analysis of proposed approach with some exsting theories.

6. Conclusion

Most of the issues in decision-making problems are associated with uncertain, imprecise and, multipolar information, which cannot be tackled properly through the fuzzy set. So to overcome this particular deficiency rough set was introduced by Pawlak, which deals with the vagueness of input data. This research implies the novel approach of neutrosophic soft rough set (NSR-set) with neutrosophic soft rough topology (NSR-topology). We presented various topological structures of NSR-topology named as NSR-interior, NSR-closure, NSR-exterior, NSR-neighborhood, NSR-limit point and, NSR-bases with numerous examples. We established two novel algorithms to deal with multi-criteria decision-making (MCDM) problems under NSR-data. One is based on NSR-sets and the other is based on NSR-topology with NSR-bases. This research is more efficient and flexible than the other approaches. The proposed algorithms are simple and easy to understand which can be applied easily on whatever type of alternatives and measures. Both algorithms are flexible and easily altered according to the different situations, inputs and, outputs. In the future, we will extend our work to solve the MCDM problems by using TOPSIS, AHP, VIKOR, ELECTRE family and, PROMETHEE family using different hybrid structures of fuzzy and rough sets.

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Exploration of the Factors Causing Autoimmune Diseases using Fuzzy Cognitive Maps with Concentric Neutrosophic Hypergraphic Approach

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Nivetha Martin, Florentine Smarandache, I. Pradeepa, N. Ramila Gandhi, P. Pandiammal (2020). Exploration of the Factors Causing Autoimmune Diseases using Fuzzy Cognitive Maps with Concentric Neutrosophic Hypergraphic Approach. *Neutrosophic Sets and Systems* 35, 232-238

Abstract: Neutrosophic sets are comprehensively used in decision making environment. The manifestation of neutrosophic sets in concentric hypergraphs is proposed in this research work. The intention of developing a decision making model using the combination of Fuzzy Cognitive Maps and concentric neutrosophic hypergraph is to rank the core factors of decision making problem and find the inter relational impacts. This proposed model is validated with the exploration of the causative factors of autoimmune diseases. The proposed model is highly compatible as it assists in determining the core factors and their inter association. This model will certainly benefit the decision maker at all managerial levels to design optimal decisions.

Keywords: Autoimmune disease, fuzzy cognitive maps, neutrosophic hypergraphs, optimal decision making

1. Introduction

Westernization the cause of modernization has unlocked the portals of cultural, behavioural and environmental changes of the people which greatly influence the biological system of human and this also lays the core reason for the outbreak of novel diseases. Presently the people of the world are characterized by multicultural and multi technological adoption. The integration and the association between people of varied culture have brought diverse implications on the external and internal environment of the human. Not just the social interactions contribute to such modifications; also the technological advancement and the work space of an individual cause a varied range of changes in the mankind. The tendency of manhood repelling from indigenous practices is the gateway for several health woes. The health system of the human is getting affected by several factors and especially the vulnerable target group is the women. In recent days, the people are chained by diseases of various kinds, even the economy of the nation face huge falls due to the effect of epidemic diseases, amidst such miserable situations, the immunity of the human is the only armed force against these viruses, but if the immune system fails to be defensive in nature and if it joins hand with the external invaders the entire human health system collapses and it ends in fatality. This is the characterization of auto immune diseases and the women are greatly affected by these diseases. It is highly a dreadful circumstance to tackle the consequences of these self-destructing diseases. The autoimmune diseases predominantly affecting the women are Rheumatoid Arthritis, Multiple Sclerosis, Systemic lupus Erythematosus, Grave's disease, Hashimoto's thyroiditis and Myasthenia gravis. Presently the rate of occurrences of such diseases is at its pinnacle and the medical experts are investigating the ways and means of its mitigation. [1]

Generally the women are highly susceptible to these autoimmune diseases as the immune system gets weakened during pre and post pregnancy stages. This scenario has gained the medical concerns and medical researchers are on their study, to render support to it, this paper aims to underlie the core factors contributing to autoimmune diseases in women and to find the inter association between the core factors. Optimal decisions can be made by applying scientific methods in the process of decision making process. The entire scenario of decision making must be modeled based on decisive factors of the study. One of the realistic tools of decision making is fuzzy cognitive maps (FCM), introduced by Kosko [2], later several academicians extended this FCM tool based on the requirements. FCM is a directed graph representing the casual relationship between factors considered for study. The nodes and the edges of the graph represent the study factors and their association. The weights [-1,1] represent the nature of the association. The integration of FCM with other graphic structures was initiated by Nivetha and Pradeepa [3]. The hypergraphic and fuzzy hypergraphic approaches with FCM unlocked the construction of concentric fuzzy hypergraphs and its integration with FCM [4,5]. This field of integrated FCM with fuzzy hypergraphs has made the researchers explore by introducing various types of concentric fuzzy hypergraphs.

In this research work, a fuzzy cognitive map with concentric neutrosophic hypergraphic approach is introduced. The notion of neutrosophic fuzzy sets and neutrosophic logic was first coined by Smarandache [6] and presently many researchers are highly interested to carry out their research in this field, the concepts of neutrosophic is applied in almost all types of decision making tools. Neutrosophic sets, play significant role in making decisions in uncertain environment as it provides space for the pragmatic representation of the expert's opinion. Abdel Basset et al [7]developed a decision making model for evaluating the framework for smart disaster response system in an uncertain environment, neutrosophic sets are used for uncertainty assessments of linear time-cost tradeoffs [8]; resource levelling problem[9] in construction project was modeled under neutrosophic environment. The concept of neutrosophic sets was extended to bipolar neutrosophic representation [10] and it is used in multi criteria decision making framework for professional selection. Das et al [11] developed neutrosophic fuzzy matrices and algebraic operation that had some utility in decision making. Plithogenic sets, the extension of neutrosophic sets are used in solving supply chain problem with the development of a novel plithogenic model [12]. Such massive applications of neutrosophic sets in decision making and its robust nature triggered the idea of integrating neutrosophic sets to concentric hypergraphs. To the best of our knowledge, the integration of neutrosophic concentric fuzzy hypergraphs with FCM has not been instituted and so this is a new arena of research towards optimal decision making.

Fuzzy Cognitive Maps are more useful in determining the association between study factors, if the number of study factors is less, FCM's are highly compatible, but if the number of factors is more, then comparative analysis between the factors is difficult and tedious, to resolve such crisis, the core factors of the problem are to be decided and then the inter association between the core factors can be determined easily. To find the core factors, the intervention of various experts is mandatory, based on which the factors can be ranked and the core factors are decided based on the rank positions of the factors. This eases the process of making decisions as it helps in filtering the non- core factors. Generally in medicinal environment, the medical experts analyze the factors contributing to diseases, initially the causative factors taken for study will be more in number, but the factors have to drop at each stage of their research to find the prime causative factors. In the process of factor filtration, the expert's opinions play a vital role. The role of each causative factor of a disease cannot be certainly express but representation using neutrosophic sets makes it possible and more meaningful. Thus the integration of FCM with concentric neutrosophic hypergraph will help to tackle the difficulties in handling large number of study factors.

The paper is structured as follows: section 2consists of the methodology in which the algorithm of finding optimal decision is presented; section 3 comprises of the adaptation of the proposed model to the decision making problem; section 4 discusses the results and the last section summarizes the research work.

2. Methodology and its application

The steps in making optimal decisions is presented as an algorithm as follows,

Step 1: The expert's opinion of the study factors are represented by concentric fuzzy hypergraphs with neutrosophic fuzzy sets representations of the envelope.

Step 2: The score values of the neutrosophic fuzzy sets are determined.

Step 3: The factors are ranked based on the score values.

Step 4: The core factors are determined based on the ranking positions.

Step5: The inter association between the core factors is obtained based on the conventional FCM procedure.

The case histories of patients belonging to women gender suffering from autoimmune diseases are taken as the source of data collection and the factors contributing to the occurrence of auto immune disease in women [13] are presented below based on the medical expert's opinion and data obtained from questionnaire.

F1. Excess presence of VGLL3 (Vestigial Like Family Member 3) in skin cells
F2. Changes in the gene system
F3. Exposure to ultraviolet radiation from sunlight
F4. Acquaintance with organic mercury
F5. Alteration in food habits
F6. Gene-Environment interface
F7. Fluctuations in sex hormones
F8. Modifications in Nutritional diet
F9. Post pregnancy impacts
F10. Genetic vulnerability
F11. Genetic differences in immunity

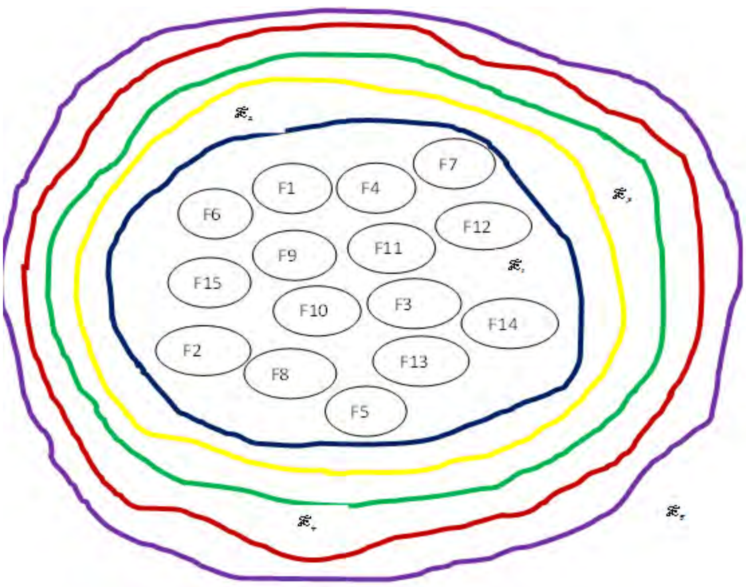

Fig.3.1.Concentric Neutrosophic Fuzzy Hypergraphic representation

The concentric neutrosophic fuzzy hyper envelopes with neutrosophic representations of the expert's opinion are presented below in Table 3.1.

Table 3.1 Representations of Expert's opinion

Experts	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11
E1	(0.3,0.2, 0.8)	(0.5,0.2, 0.3)	(0.4,0.1, 0.5)	(0.3,0.4, 0.6)	(0.8,0.1, 0.2)	(0.7,0.2, 0.3)	(0.7,0.3, 0.4)	(0.7,0.2, 0.3)	(0.3,0.2, 0.8)	(0.5,0.2, 0.3)	(0.5,0.2, 0.3)
E2	(0.2,0.2, 0.9)	(0.4,0.3, 0.5)	(0.5,0.2, 0.3)	(0.2,0.2, 0.9)	(0.7,0.2, 0.3)	(0.6,0.2, 0.3)	(0.7,0.5, 0.4)	(0.6,0.2, 0.3)	(0.4,0.3, 0.5)	(0.6,0.2, 0.3)	(0.8,0.3, 0.2)
E3	(0.3,0.4, 0.6)	(0.3,0.5, 0.6)	(0.4,0.3, 0.5)	(0.3,0.2, 0.8)	(0.8,0.3, 0.2)	(0.9,0.2, 0.3)	(0.9,0.1, 0.3)	(0.6,0.2, 0.3)	(0.3,0.5, 0.6)	(0.7,0.3, 0.4)	(0.6,0.2, 0.3)
E4	(0.5,0.2, 0.3)	(0.2,0.2, 0.9)	(0.5,0.2, 0.3)	(0.4,0.4, 0.6)	(0.7,0.1, 0.2)	(0.7,0.3, 0.4)	(0.6,0.2, 0.3)	(0.7,0.1, 0.2)	(0.2,0.2, 0.9)	(0.6,0.2, 0.3)	(0.4,0.3, 0.5)
E5	(0.2,0.5 ,0.6)	(0.3,0.2, 0.8)	(0.6,0.2, 0.3)	(0.5,0.2, 0.3)	(0.6,0.2, 0.3)	(0.8,0.1, 0.2)	(0.6,0.2, 0.3)	(0.9,0.2, 0.3)	(0.4,0.4, 0.6)	(0.5,0.2, 0.3)	(0.7,0.3, 0.4)

The score values of the factors are presented in Table 3.2 and it is represented graphically in Fig.3.2

Table 3.2 Score values of the Factors

F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11
0.571	0.571	0.546	0.538	0.667	0.783	0.756	0.573	0.445	0.636	0.667
7	7	8	9	5	1	2	6	10	3	4

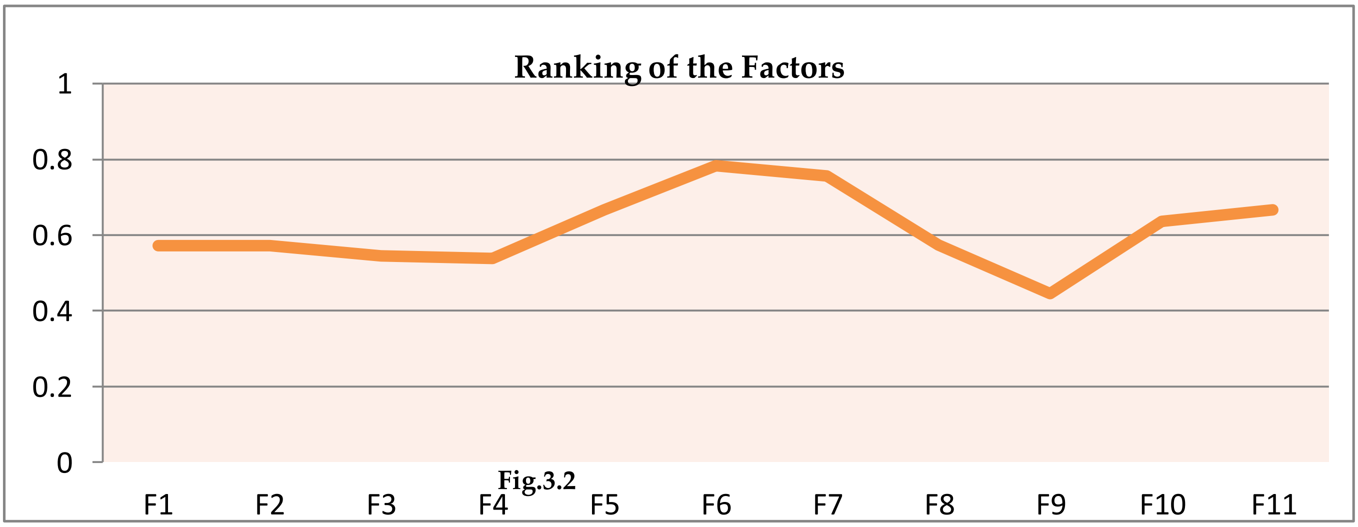

Fig.3.2

Based on the scores, the following factors are considered as the core factors and their inter association is expressed as linguistic variables, which then later quantified by heptagonal fuzzy numbers.

HP1. Alteration in food habits

HP2. Gene-Environment interface

HP3. Fluctuations in sex hormones
HP4. Genetic vulnerability
HP5. Genetic differences in immunity
The connection matric between the factors, based on the expert's opinion

$$\begin{array}{c} \\ HP1 \\ HP2 \\ HP3 \\ HP4 \\ HP5 \end{array} \begin{array}{c} \begin{array}{ccccc} HP1 & HP2 & HP3 & HP4 & HP5 \end{array} \\ \begin{pmatrix} 0 & M & H & L & L \\ L & 0 & M & H & H \\ L & M & 0 & M & L \\ L & M & H & 0 & M \\ L & M & M & H & 0 \end{pmatrix} \end{array}$$

The modified matrix based on the values of quantification in Table 3.3

Linguistic Variable	**Heptagonal Weight**	**Membership value**
Low	(0,0.1,0.2,0.3,0.35,0.4,0.45)	**0.26**
Medium	(0.4,0.45,0.5,0.55,0.6,0.65,0.7)	**0.55**
High	(0.65,0.7,0.8,0.9,1,1,1)	**0.86**

$$\begin{array}{c} \\ HP1 \\ HP2 \\ HP3 \\ HP4 \\ HP5 \end{array} \begin{array}{c} \begin{array}{ccccc} HP1 & HP2 & HP3 & HP4 & HP5 \end{array} \\ \begin{pmatrix} 0 & 0.55 & 0.86 & 0.26 & 0.26 \\ 0.26 & 0 & 0.55 & 0.86 & 0.86 \\ 0.26 & 0.55 & 0 & 0.55 & 0.26 \\ 0.26 & 0.55 & 0.86 & 0 & 0.55 \\ 0.26 & 0.55 & 0.55 & 0.86 & 0 \end{pmatrix} \end{array}$$

The interrelationship between the factors is determined by the similar application of FCM methodology [9-10] and it is presented graphically in Fig 3.2

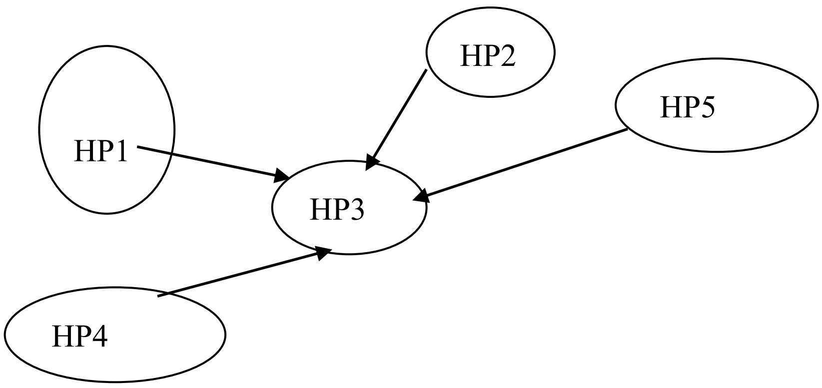

Fig.3.2 FCM representation of the inter association of the core factors

4. Results and Discussion

Fig. 3.2 clearly states the factor, fluctuations in sex hormone is the core causative factor of auto immune diseases. The findings of this research will certainly assist the medical experts to ascertain the causes of the auto immune disease in women and give treatment in accordance to it. Hormonal imbalance is quite common in the life of the women as they undergo various stages of puberty, maternity, menopause, but still proper medications has to be given to avoid the risks of such fatal diseases. The representation of the imprecise data in the form neutrosophic sets is the pragmatic reflection of the expert's opinion, as the factors contributing to the diseases are quite uncertain. The degree of truth values, indeterminacy and false values are indeed very essential in making optimal decisions.

5. Conclusion

The proposed decision making tool with the integration of FCM and concentric neutrosophic fuzzy hypergraphs is a highly feasible tool to obtain optimal decisions. The difficulty in handling several factors in FCM is reduced and this integrated approach facilitate the determination of inter association between the factors. This method of decision making can be extended to other kinds of concentric fuzzy hypergraphs with various representations. Plithogenic sets representation is the future extension of this proposed research work.

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Translative and Multiplicative Interpretation of Neutrosophic Cubic Set

Mohsin Khalid, Florentin Smarandache, Neha Andalleb Khalid, Said Broumi

Mohsin Khalid, Florentin Smarandache, Neha Andalleb Khalid, Said Broumi (2020). Translative and Multiplicative Interpretation of Neutrosophic Cubic Set. *Neutrosophic Sets and Systems* 35, 299-239

Abstract: In this paper, we introduce the idea of neutrosophic cubic translation (NCT) and neutrosophic cubic multiplication (NCM) and provide entirely new type of conditions for neutrosophic cubic translation and neutrosophic cubic multiplication on BF-algebra. This is the new kind of approach towards translation and multiplication which involves the indeterminacy membership function. We also define neutrosophic cubic magnified translation (NCMT) on BF-algebra which handles the neutrosophic cubic translation and neutrosophic cubic multiplication at the same time on membership function, indeterminacy membership function and non-membership function. We present the examples for better understanding of neutrosophic cubic translation, neutrosophic cubic multiplication, and neutrosophic cubic magnified translation, and investigate significant results of BF-ideal and BF-subalgebra by applying the ideas of NCT, NCM and NCMT. Intersection and union of neutrosophic cubic BF-ideals are also explained through this new type of translation and multiplication.

Keywords: BF-algebra, neutrosophic cubic translation, neutrosophic cubic multiplication, neutrosophic cubic BF ideal, neutrosophic cubic BF subalgebra, neutrosophic cubic magnified translation.

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1. Introduction

Zadeh [1] presented the theory of fuzzy set in 1965. Fuzzy idea has been applied to different algebraic structures like groups, rings, modules, vector spaces and topologies. In this way, Iseki and Tanaka [2] introduced the idea of BCK-algebra in 1978. Iseki [3] introduced the idea of BCI-algebra in 1980 and it is obvious that the class of BCK-algebra is a proper sub class of the class of BCI-algebra. Lee et al. [4] studied the fuzzy translation, (normalized, maximal) fuzzy extension and fuzzy multiplication of fuzzy subalgebra in BCK/BCI-algebra. Link among fuzzy translation, (normalized, maximal) fuzzy extension and fuzzy multiplication are also discussed. Ansari and Chandramouleeswaran [5] introduced the idea of fuzzy translation, fuzzy extension and fuzzy multiplication of fuzzy β ideal of β-algebra and investigated some of their properties. Satyanarayana [6] introduced the concepts of fuzzy ideals, fuzzy implicative ideals and fuzzy p-ideals in BF-algebras and investigate some of its properties. Andrzej [7] defined the BF-algebra. Lekkoksung [8] focused on fuzzy magnified translation in ternary hemirings, which is a extension of BCI / BCK/Q / KU / d-algebra. Senapati et al. [9] have done much work on intuitionistic fuzzy H-ideal in BCK/BCI-algebra. Jana et al. [10] wrote on intuitionistic fuzzy G-algebra. Senapati et al. [11] studied fuzzy translations of fuzzy H-ideals in BCK/BCI-algebra. Atanassov [12] introduced the intuitionistic fuzzy sets. Senapati [13] investigated the relationship among intuitionistic fuzzy translation, intuitionistic fuzzy extension and intuitionistic fuzzy multiplication in B-algebra. Kim and Jeong [14] defined the intuitionistic fuzzy structure of B-algebra. Senapati et al. [15] introduced the cubic subalgebras and cubic closed ideals of B-algebras. Senapati et al. [16] discussed the fuzzy dot subalgebra and fuzzy dot ideal of B-algebras. Priya and Ramachandran [17] worked on fuzzy translation and fuzzy multiplication in PS-algebra. Chandramouleeswaran et al. [18] worked on fuzzy translation and fuzzy multiplication in BF/BG-algebra. Jana et al. [19] studied the cubic G-subalgebra of G-algebra. Smarandache [20,21] extended the intuitionistic fuzzy set, paraconsistent set, and intuitionistic set to the neutrosophic set through Several examples. Jun et al. [22] studied the Cubic set and apply the idea of cubic sets in group and gave the definition of cubic subgroups. Saeid and Rezvani [23] introduced and studied fuzzy BF-algebra, fuzzy BF-subalgebras, level subalgebras,fuzzy topological BF-algebra. Jun et al. [24] defined the neutrosophic cubic set introduced truth-internal and truth-external and discuss the many properties. Jun et al. [25] investigated the commutative falling neutrosophic ideals in BCK-algebra. C. H. Park [26] defined the neutrosophic ideal in subtraction algebra and studied it through several properties, he also discussed conditions for a neutrosophic set to be a neutrosophic ideal along with properties of neutrosophic ideal. Khalid et al. [27] investigated the neutrosophic soft cubic subalgebra through significant characteristic like P-union, R-intersection etc. Khalid et al. [28] interestinly investigated the intuitionistic fuzzy translation and multiplication through subalgebra and ideals. Khalid et al. [29] defined the T-neutrosophic cubic set and studied this set through ideals and subalgebras and investigated many results.

The purpose of this paper is to introduce the idea of neutrosophic cubic translation (**NCT**), neutrosophic cubic multiplication **(NCM)** and neutrosophic cubic magnified translation **(NCMT)** on BF-algebra. In second section we discuss some fundamental definitions which are used to develop the paper. In third's first subsection we discuss the neutrosophic cubic translation (**NCT)** and neutrosophic cubic multiplication **(NCM)** of BF subalgebra. In second subsection we discuss the neutrosophic cubic translation (**NCT)** and neutrosophic cubic multiplication **(NCM)** of BF ideal. In third subsection we discuss the neutrosophic cubic magnified translation **(NCMT)** of BF ideal and BF subalgebra.

2 Preliminaries

First we discuss some definitions which are used to present this paper.

Definition 2.1 [3] An algebra $(Y,*,0)$ of type (2,0) is called a BCI-algebra if it satisfies the following conditions:

i) $(t_1 * t_2) * (t_1 * t_3) \leq (t_3 * t_2)$,

ii) $t_1 * (t_1 * t_2) \leq t_2$,

iii) $t_1 \leq t_1$,

iv) $t_1 \leq t_2$ and $t_2 \leq t_1 \Rightarrow t_1 = t_2$,

v) $t_1 \leq 0 \Rightarrow t_1 = 0$, where $t_1 \leq t_2$ is defined by $t_1 * t_2 = 0$, for all $t_1, t_2, t_3 \in Y$.

Definition 2.2 [1] An algebra $(Y,*,0)$ of type (2,0) is called a BCK-algebra if it satisfies the following conditions:

i) $(t_1 * t_2) * (t_1 * t_3) \leq (t_3 * t_2)$,

ii) $t_1 * (t_1 * t_2) \leq t_2$,

iii) $t_1 \leq t_1$,

iv) $t_1 \leq t_2$ and $t_2 \leq t_1 \Rightarrow t_1 = t_2$,

v) $0 \leq t_1 \Rightarrow t_1 = 0$, where $t_1 \leq t_2$ is defined by $t_1 * t_2 = 0$, for all $t_1, t_2, t_3 \in Y$.

Definition 2.3 [7] A nonempty set Y with a constant 0 and a binary operation $*$ is called BF–algebra when it fulfills these axioms.

i) $t_1 * t_1 = 0$

ii) $t_1 * 0 = 0$

iii) $0 * (t_1 * t_2) = t_2 * t_1$ for all $t_1, t_2 \in Y$.

A BF-algebra is denoted by $(Y,*,0)$.

Definition 2.4 [7] Let S be a nonempty subset of BF-algebra Y, then S is called a BF-subalgebra of Y if $t_1 * t_2 \in S$, for all $t_1, t_2 \in S$.

Definition 2.5 [6] Let Y ba a BF-algebra and I is a subset of Y, then I is called a BF ideal of Y if it satisfies the following conditions:

i) $0 \in I$,

ii) $t_2 * t_1 \in I$ and $t_2 \in I \rightarrow t_1 \in I$.

Definition 2.6 [6] Let Y be a BF-algebra. A fuzzy set B of Y is called a fuzzy BF ideal of Y if it satisfies the following conditions:

i) $\kappa(0) \geq \kappa(t_1)$,

ii) $\kappa(t_1) \geq \min\{\kappa(t_2 * t_1), \kappa(t_2)\}$, for all $t_1, t_2 \in Y$.

Definition 2.7 [1] Let Y be a group of objects denoted generally by t_1. Then a fuzzy set B of Y is defined as $B = \{< t_1, \kappa_B(t_1) > | t_1 \in Y\}$, where $\kappa_B(t_1)$ is called the membership value of t_1 in B and $\kappa_B(t_1) \in [0,1]$.

Definition 2.8 [23] A fuzzy set B of BF-algebra Y is called a fuzzy PS subalgebra of Y if $\kappa(t_1 * t_2) \geq \min\{\kappa(t_1), \kappa(t_2)\}$, for all $t_1, t_2 \in Y$.

Definition 2.9 [4,5] Let a fuzzy subset B of Y and $\alpha \in [0,1 - \sup\{\kappa_B(t_1) | t_1 \in Y\}]$. A mapping $(\kappa_B)^T_\alpha | Y \in [0,1]$ is said to be a fuzzy α translation of κ_B if it satisfies $(\kappa_B)^T_\alpha(t_1) = \kappa_B(t_1) + \alpha$, for all $t_1 \in Y$.

Definition 2.9 [4,5] Let a fuzzy subset B of Y and $\alpha \in [0,1]$. A mapping $(\kappa_B)^M_\alpha : Y \to [0,1]$ is said to be a fuzzy α multiplication of B if it satisfies $(\kappa_B)^M_\alpha(t_1) = \alpha.(\kappa_B)(t_1)$, for all $t_1 \in Y$.

Definition 2.10 [12] An intuitionistic fuzzy set **(IFS)** B over Y is an object having the form $B = \{\langle t_1, \kappa_B(t_1), \upsilon_B(t_1)\rangle | t_1 \in Y\}$, where $\kappa_B(t_1): Y \to [0,1]$ and $\upsilon_B(t_1) | Y \to [0,1]$, with the condition $0 \leq \kappa_B(t_1) + \upsilon_B(t_1) \leq 1$, for all $t_1 \in Y$. $\kappa_B(t_1)$ and $\upsilon_B(t_1)$ represent the degree of membership and the degree of non-membership of the element t_1 in the set B respectively.

Definition 2.11 [12] Let $B = \{\langle t_1, \kappa_B(t_1), \upsilon_B(t_1)\rangle | t_1 \in Y\}$ and $B = \{\langle t_1, \kappa_B(t_1), \upsilon_B(t_1)\rangle | t_1 \in Y\}$ be two **IFSs** on Y. Then intersection and union of A and B are indicated by $A \cap B$ and $A \cup B$ respectively and are given by

$$A \cap B = \{\langle t_1, \min(\kappa_A(t_1), \kappa_B(t_1)), \max(\upsilon_A(t_1), \upsilon_B(t_1))\rangle | t_1 \in Y\},$$

$$A \cup B = \{\langle t_1, \max(\kappa_A(t_1), \kappa_B(t_1)), \min(\upsilon_A(t_1), \upsilon_B(t_1))\rangle | t_1 \in Y\}.$$

Definition 2.12 [14] An **IFS** $B = \{\langle t_1, \kappa_B(t_1), \upsilon_B(t_1)\rangle | t_1 \in Y\}$ of Y is called an **IFSU** of Y if it satisfies these two conditions:

(i) $\kappa_B(t_1 * t_2) \geq \min\{\kappa_B(t_1), \kappa_B(t_2)\}$,

(ii) $\upsilon_B(t_1 * t_2) \leq \max\{\upsilon_B(t_1), \upsilon_B(t_2)\}$, for all $t_1, t_2 \in Y$.

Definition 2.13 An **IFS** $B = \{\langle t_1, \kappa_B(t_1), \upsilon_B(t_1)\rangle | t_1 \in Y\}$ of Y is said to be an **IFID** of Y if it satisfies these three conditions:

(i) $\kappa_B(0) \geq \kappa_B(t_1), \upsilon_B(0) \leq \upsilon_B(t_1)$,

(ii) $\kappa_B(t_1) \geq \min\{\kappa_B(t_1 * t_2), \kappa_B(t_2)\}$,

(iii) $\upsilon_B(t_1) \leq \max\{\upsilon_B(t_1 * t_2), \upsilon_B(t_2)\}$, for all $t_1, t_2 \in Y$.

Definition 2.14 [8] Let κ be a fuzzy subset of Y, $\alpha \in [0,T]$ and $\beta \in [0,1]$. A mapping $\kappa^{M\,T}_{\beta\,\alpha} | Y \to [0,1]$ is said to be a fuzzy magnified $\beta\alpha$ translation of κ if it satisfies: $\kappa^{M\,T}_{\beta\,\alpha}(t_1) = \beta.\kappa(t_1) + \alpha$ for all $t_1 \in Y$.

Jun et al. [22,24]introduced neutrosophic cubic set and investigated several properties.

Definition 2.15 [24] Suppose X be a nonempty set. A neutrosophic cubic set in X is pair $\mathcal{C} = (\kappa, \sigma)$ where $\kappa = \{\langle t_1; \kappa_E(t_1), \kappa_I(t_1), \kappa_N(t_1)\rangle \,|t_1 \in X\}$ is an interval neutrosophic set in X and $\sigma = \{\langle t_1; \sigma_E(t_1), \sigma_I(t_1), \sigma_N(t_1)\rangle \,|t_1 \in X\}$ is a neutrosophic set in X.

Definition 2.16 [15] Let $C = \{\langle t_1, \kappa(t_1), \sigma(t_1)\rangle\}$ be a cubic set, where $\kappa(t_1)$ is an interval-valued fuzzy set in X, $\sigma(t_1)$ is a fuzzy set in X. Then C is cubic subalgebra under binary operation "$*$", if following axioms are satisfied:

i) $\kappa(t_1 * t_2) \geq \text{rmin}\{\kappa(t_1), \kappa(t_2)\}$,

ii) $\sigma(t_1 * t_2) \leq \max\{\sigma(t_1), \sigma(t_2)\}\ \forall\ t_1, t_2 \in X$.

Definition 2.17 [28] Let $A = (\kappa_A, \upsilon_A)$ be an **IFS** of G-algebra and let $\alpha \in [0, ¥]$. An object of the form $A^T_\alpha = ((\kappa_A)^T_\alpha, (\upsilon_A)^T_\alpha)$ is called an intuitionistic fuzzy α-translation (**IFAT**) of A when $(\kappa_A)^T_\alpha(t_1) = \kappa_A(t_1) + \alpha$ and $(\upsilon_A)^T_\alpha(t_1) = \upsilon_A(t_1) - \alpha$ for all $t_1 \in Y$.

3 Translative and Multiplicative Interpretation of Neutrosophic Cubic Set

For our simplicity, we use the notation $B = (\kappa_{T,I,F}, \upsilon_{T,I,F})$ for the NCS $B = \{\langle t_1, \kappa_{T,I,F}(t_1), \upsilon_{T,I,F}(t_1)\rangle | t_1 \in Y\}$. In this paper, we used $ℸ = [1,1] - \mathrm{rsup}\{\kappa_{\{T,I\}}(t_1) | t_1 \in Y\}$, $¥ = \mathrm{rinf}\{\kappa_F(t_1) | t_1 \in Y\}$, $\Gamma = 1 - \sup\{\upsilon_{\{T,I\}}(t_1) | t_1 \in Y\}$, $£ = \inf\{\upsilon_F(t_1) | t_1 \in Y$ for any NCS $B = (\kappa_{T,I,F}, \upsilon_{T,I,F})$ of Y.

3.1 Translative and Multiplicative Interpretation of Neutrosophic Cubic Subalgebra

Definition 3.1.1 Let $B = (\kappa_{T,I,F}, \upsilon_{T,I,F})$ be a **NCS** of Y and for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. An object of the form $B^T_{\alpha,\beta,\gamma} = ((\kappa_{T,I,F})^T_{\alpha,\beta,\gamma}, (\upsilon_{T,I,F})^T_{\alpha,\beta,\gamma})$ is called a **NCT** of B, when $(\kappa_T)^T_\alpha(t_1) = \kappa_B(t_1) + \alpha$, $(\kappa_I)^T_\beta(t_1) = \kappa_B(t_1) + \beta$, $(\kappa_F)^T_\gamma(t_1) = \kappa_F(t_1) - \gamma$ and $(\upsilon_T)^T_\alpha(t_1) = \upsilon_T(t_1) + \alpha$, $(\upsilon_I)^T_\beta(t_1) = \upsilon_B(t_1) + \beta$, $(\upsilon_F)^T_\gamma(t_1) = \upsilon_B(t_1) - \gamma$ for all $t_1 \in Y$.

Example 3.1.1 Let $Y = \{0,1,2\}$ be a BF-algebra with the following Cayley table:

*	0	1	2
0	0	1	2
1	0	0	1
2	0	2	0

Let $B = (\kappa_{T,I,F}, \upsilon_{T,I,F})$ be a **NCS** of Y is defined as

$$\kappa_T(t_1) = \begin{cases} [0.1, 0.3] & \text{if } t_1 = 0 \\ [0.4, 0.7] & \text{if otherwise} \end{cases}$$

$$\kappa_I(t_1) = \begin{cases} [0.2, 0.4] & \text{if } t_1 = 0 \\ [0.5, 0.7] & \text{if otherwise} \end{cases}$$

$$\kappa_F(t_1) = \begin{cases} [0.4, 0.6] & \text{if } t_1 = 0 \\ [0.3, 0.8] & \text{if otherwise} \end{cases}$$

and

$$\upsilon_T(t_1) = \begin{cases} 0.1 & \text{if } t_1 = 0 \\ 0.4 & \text{if otherwise} \end{cases}$$

$$\upsilon_I(t_1) = \begin{cases} 0.2 & \text{if } t_1 = 0 \\ 0.3 & \text{if otherwise} \end{cases}$$

$$\upsilon_F(t_1) = \begin{cases} 0.5 & \text{if } t_1 = 0 \\ 0.7 & \text{if otherwise.} \end{cases}$$

Then B is a neutrosophic cubic subalgebra. Here we choose for $\upsilon_{T,I,F}$, $\alpha = 0.01, \beta = 0.02$, $\gamma = 0.03$, and for $\kappa_{T,I,F}$, $\alpha = [0.1,0.2]$, $\beta = [0.2,0.25]$, $\gamma = [0.2,0.3]$ then the mapping $B^T: Y \to [0,1]$ is given by

$$(\kappa_T)^T_{[0.1,0.2]}(t_1) = \begin{cases} [0.2, 0.5] & \text{if } t_1 = 0 \\ [0.5, 0.9] & \text{if otherwise} \end{cases}$$

$$(\kappa_I)^T_{[0.2,0.25]}(t_1) = \begin{cases} [0.4, 0.7] & \text{if } t_1 = 0 \\ [0.7, 0.95] & \text{if otherwise} \end{cases}$$

$$(\kappa_F)^T_{[0.2,0.3]}(t_1) = \begin{cases} [0.2, 0.3] & \text{if } t_1 = 0 \\ [0.1, 0.5] & \text{if otherwise} \end{cases}$$

and

$$(\upsilon_T)^T_{0.01}(t_1) = \begin{cases} 0.11 & \text{if } t_1 = 0 \\ 0.41 & \text{if otherwise.} \end{cases}$$

$$(\upsilon_I)^T_{0.02}(t_1) = \begin{cases} 0.22 & \text{if } t_1 = 0 \\ 0.32 & \text{if otherwise.} \end{cases}$$

$$(\upsilon_F)^T_{0.03}(t_1) = \begin{cases} 0.47 & \text{if } t_1 = 0 \\ 0.67 & \text{if otherwise,} \end{cases}$$

which imply $(\kappa_T)^T_{[0.1,0.2]}(t_1) = \kappa_T(t_1) + [0.1,0.2]$, $(\kappa_I)^T_{[0.2,0.25]}(t_1) = \kappa_I(t_1) + [0.2,0.25\ ]$, $(\kappa_F)^T_{[0.2,0.3]}(t_1) = \kappa_F(t_1) - [0.2,0.3]$ and $(\upsilon_T)^T_{0.01}(t_1) = \upsilon_T(t_1) + 0.01$, $(\upsilon_I)^T_{0.02}(t_1) = \upsilon_I(t_1) + 0.02$, $(\upsilon_F)^T_{0.03}(t_1) = \upsilon_F(t_1) - 0.03$ for all $t_1 \in Y$. Hence B^T is a neutrosophic cubic translation.

Theorem 3.1.1 Let B be a **NCSU** of Y and for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], \daleth]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a **NCSU** of Y.

Proof. Assume $t_1, t_2 \in Y$. Then

$$(\kappa_T)^T_\alpha(t_1 * t_2) = \kappa_T(t_1 * t_2) + \alpha$$

$$\geq \text{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\} + \alpha$$

$$= \text{rmin}\{\kappa_T(t_1) + \alpha, \kappa_T(t_2) + \alpha\}$$

$$(\kappa_T)^T_\alpha(t_1 * t_2) = \text{rmin}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_\alpha(t_2)\},$$

$$(\kappa_I)^T_\beta(t_1 * t_2) = \kappa_I(t_1 * t_2) + \beta$$

$$\geq \text{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\} + \beta$$

$$= \text{rmin}\{\kappa_I(t_1) + \beta, \kappa_I(t_2) + \beta\}$$

$$(\kappa_I)^T_\beta(t_1 * t_2) = \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_\beta(t_2)\},$$

$$(\kappa_F)^T_\gamma(t_1 * t_2) = \kappa_F(t_1 * t_2) - \gamma$$

$$\geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\} - \gamma$$

$$= \mathrm{rmin}\{\kappa_F(t_1) - \gamma, \kappa_F(t_2) - \gamma\}$$

$$(\kappa_F)^T_\gamma(t_1 * t_2) = \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1), (\kappa_F)^T_\gamma(t_2)\}$$

and

$$(\upsilon_T)^T_\alpha(t_1 * t_2) = \upsilon_T(t_1 * t_2) + \alpha$$

$$\leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\} + \alpha$$

$$= \max\{\upsilon_T(t_1) + \alpha, \upsilon_T(t_2) + \alpha\}$$

$$(\upsilon_T)^T_\alpha(t_1 * t_2) = \max\{(\upsilon_T)^T_\alpha(t_1), (\upsilon_T)^T_\alpha(t_2)\},$$

$$(\upsilon_I)^T_\beta(t_1 * t_2) = \upsilon_I(t_1 * t_2) + \beta$$

$$\leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\} + \beta$$

$$= \max\{\upsilon_I(t_1) + \beta, \upsilon_I(t_2) + \beta\}$$

$$(\upsilon_I)^T_\beta(t_1 * t_2) = \max\{(\upsilon_I)^T_\beta(t_1), (\upsilon_I)^T_\beta(t_2)\},$$

$$(\upsilon_F)^T_\gamma(t_1 * t_2) = \upsilon_F(t_1 * t_2) - \gamma$$

$$\leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\} - \gamma$$

$$= \max\{\upsilon_F(t_1) - \gamma, \upsilon_F(t_2) - \gamma\}$$

$$(\upsilon_F)^T_\gamma(t_1 * t_2) = \max\{(\upsilon_F)^T_\gamma(t_1), (\upsilon_F)^T_\gamma(t_2)\}.$$

Hence **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a **NCSU** of Y.

Theorem 3.1.2 Let B be a **NCS** of Y such that **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a **NCSU** of Y for some $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ך]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then B is a **NCSU** of Y.

Proof. Let $B^T_{\alpha,\beta,\gamma} = ((\kappa_{T,I,F})^T_{\alpha,\beta,\gamma}, (\upsilon_{T,I,F})^T_{\alpha,\beta,\gamma})$ be a **NCSU** of Y for some $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ך]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $t_1, t_2 \in Y$. Then

$$\kappa_T(t_1 * t_2) + \alpha = (\kappa_T)^T_\alpha(t_1 * t_2)$$

$$\geq \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_\alpha(t_2)\}$$

$$= \mathrm{rmin}\{\kappa_T(t_1) + \alpha, \kappa_T(t_2) + \alpha\}$$

$$\kappa_T(t_1 * t_2) + \alpha = \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\} + \alpha,$$

$$\kappa_I(t_1 * t_2) + \beta = (\kappa_I)^T_\beta(t_1 * t_2)$$

$$\geq \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_\beta(t_2)\}$$

$= \mathrm{rmin}\{\kappa_I(t_1) + \beta, \kappa_I(t_2) + \beta\}$

$\kappa_I(t_1 * t_2) + \beta = \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\} + \beta,$

$\kappa_F(t_1 * t_2) - \gamma = (\kappa_F)^T_\gamma(t_1 * t_2)$

$\geq \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1), (\kappa_F)^T_\gamma(t_2)\}$

$= \mathrm{rmin}\{\kappa_F(t_1) - \gamma, \kappa_F(t_2) - \gamma\}$

$\kappa_F(t_1 * t_2) - \gamma = \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\} - \gamma$

and

$\upsilon_T(t_1 * t_2) + \alpha = (\upsilon_T)^T_\alpha(t_1 * t_2)$

$\leq \max\{(\upsilon_T)^T_\alpha(t_1), (\upsilon_T)^T_\alpha(t_2)\}$

$= \max\{\upsilon_T(t_1) + \alpha, \upsilon_B(t_2) + \alpha\}$

$\upsilon_T(t_1 * t_2) + \alpha = \max\{\upsilon_T(t_1), \upsilon_T(t_2)\} + \alpha,$

$\upsilon_I(t_1 * t_2) + \beta = (\upsilon_I)^T_\beta(t_1 * t_2)$

$\leq \max\{(\upsilon_I)^T_\beta(t_1), (\upsilon_I)^T_\beta(t_2)\}$

$= \max\{\upsilon_I(t_1) + \beta, \upsilon_B(t_2) + \beta\}$

$\upsilon_I(t_1 * t_2) + \beta = \max\{\upsilon_I(t_1), \upsilon_I(t_2)\} + \beta,$

$\upsilon_F(t_1 * t_2) - \gamma = (\upsilon_F)^T_\gamma(t_1 * t_2)$

$\leq \max\{(\upsilon_F)^T_\gamma(t_1), (\upsilon_F)^T_\gamma(t_2)\}$

$= \max\{\upsilon_F(t_1) - \gamma, \upsilon_B(t_2) - \gamma\}$

$\upsilon_F(t_1 * t_2) - \gamma = \max\{\upsilon_F(t_1), \upsilon_F(t_2)\} - \gamma,$

which imply $\kappa_T(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}$, $\kappa_I(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$, $\kappa_F(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$, and $\upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$, $\upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$, $\upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$, for all $t_1, t_2 \in Y$. Hence B is a **NCSU** of Y.

Definition 3.1.2 Let B be a **NCS** of Y and $\delta \in [0,1]$. An object having the form $B^M_\delta = (((\kappa_T)^M_\delta, (\kappa_I)^M_\delta, (\kappa_F)^M_\delta), ((\upsilon_T)^M_\delta, (\upsilon_I)^M_\delta, (\upsilon_F)^M_\delta))$ is called a **NCM** of B, when $(\kappa_T)^M_\delta(t_1) = \delta.\kappa_T(t_1)$, $(\kappa_I)^M_\delta(t_1) = \delta.\kappa_I(t_1)$, $(\kappa_F)^M_\delta(t_1) = \delta.\kappa_F(t_1)$ and $(\upsilon_T)^M_\delta(t_1) = \delta.\upsilon_T(t_1)$, $(\upsilon_I)^M_\delta(t_1) = \delta.\upsilon_I(t_1)$,$(\upsilon_F)^M_\delta(t_1) = \delta.\upsilon_F(t_1)$ for all $t_1 \in Y$.

Example 3.1.2 Let $Y = \{0,1,2\}$ be a BF-algebra with the following Cayley table:

*	0	1	2
0	0	1	2
1	0	0	1
2	0	2	0

Let $B = (\kappa_{T,I,F}, \upsilon_{T,I,F})$ be a **NCS** of Y is defined as

$$\kappa_T(t_1) = \begin{pmatrix} [0.1, 0.3] & \text{if } t_1 = 0 \\ [0.4, 0.7] & \text{if otherwise} \end{pmatrix}$$

$$\kappa_I(t_1) = \begin{pmatrix} [0.2, 0.4] & \text{if } t_1 = 0 \\ [0.5, 0.7] & \text{if otherwise} \end{pmatrix}$$

$$\kappa_F(t_1) = \begin{pmatrix} [0.4, 0.6] & \text{if } t_1 = 0 \\ [0.3, 0.8] & \text{if otherwise} \end{pmatrix}$$

and

$$\upsilon_T(t_1) = \begin{pmatrix} 0.1 & \text{if } t_1 = 0 \\ 0.4 & \text{if otherwise} \end{pmatrix}$$

$$\upsilon_I(t_1) = \begin{pmatrix} 0.2 & \text{if } t_1 = 0 \\ 0.3 & \text{if otherwise} \end{pmatrix}$$

$$\upsilon_F(t_1) = \begin{pmatrix} 0.5 & \text{if } t_1 = 0 \\ 0.7 & \text{if otherwise.} \end{pmatrix}$$

Then B is a neutrosophic cubic subalgebra, choose $\delta = 0.01$ for υ and $\delta = [0.1,0.2]$ for κ then the mapping $B_\delta^M|Y \to [0,1]$ is given by

$$(\kappa_T)^M_{[0.1,0.2]}(t_1) = \begin{pmatrix} [0.01, 0.06] & \text{if } t_1 = 1 \\ [0.04, 0.14] & \text{if otherwise,} \end{pmatrix}$$

$$(\kappa_I)^M_{[0.1,0.2]}(t_1) = \begin{pmatrix} [0.02, 0.08] & \text{if } t_1 = 1 \\ [0.05, 0.14] & \text{if otherwise,} \end{pmatrix}$$

$$(\kappa_F)^M_{[0.1,0.2]}(t_1) = \begin{pmatrix} [0.04, 0.12] & \text{if } t_1 = 1 \\ [0.03, 0.16] & \text{if otherwise} \end{pmatrix}$$

and

$$(\upsilon_T)^M_{0.01}(t_1) = \begin{pmatrix} 0.001 & \text{if } t_1 = 0 \\ 0.004 & \text{if otherwise,} \end{pmatrix}$$

$$(\upsilon_I)^M_{0.01}(t_1) = \begin{pmatrix} 0.002 & \text{if } t_1 = 0 \\ 0.003 & \text{if otherwise,} \end{pmatrix}$$

$$(\upsilon_F)^M_{0.01}(t_1) = \begin{pmatrix} 0.005 & \text{if } t_1 = 0 \\ 0.007 & \text{if otherwise,} \end{pmatrix}$$

which imply $(\kappa_T)^M_{[0.1,0.2]}(t_1) = \kappa_T(t_1).[0.1,0.2]$, $(\kappa_I)^M_{[0.1,0.2]}(t_1) = \kappa_I(t_1).[0.1,0.2]$, $(\kappa_F)^M_{[0.1,0.2]}(t_1) = \kappa_F(t_1).[0.1,0.2]$ and $(\upsilon_T)^M_{0.01}(t_1) = \upsilon_T(t_1).(0.01)$, $(\upsilon_I)^M_{0.01}(t_1) = \upsilon_I(t_1).(0.01)$, $(\upsilon_F)^M_{0.01}(t_1) = \upsilon_F(t_1).(0.01)$ for all $t_1 \in Y$. Hence B^M_δ is a neutrosophic cubic multiplication.

Theorem 3.1.3 Let B be a **NCS** of Y such that **NCM** B^M_δ of B is a **NCSU** of Y for some $\delta \in [0,1]$. Then B is a **NCSU** of Y.

Proof. Assume B^M_δ of B is a **NCSU** of Y for some $\delta \in [0,1]$. Now for all $t_1, t_2 \in Y$, we have

$$\kappa_T(t_1 * t_2).\delta = (\kappa_T)^M_\delta(t_1 * t_2)$$
$$\geq \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1), (\kappa_T)^M_\delta(t_2)\}$$
$$= \mathrm{rmin}\{\kappa_T(t_1).\delta, \kappa_T(t_2).\delta\}$$
$$\kappa_T(t_1 * t_2).\delta = \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}.\delta,$$

$$\kappa_I(t_1 * t_2).\delta = (\kappa_I)^M_\delta(t_1 * t_2)$$
$$\geq \mathrm{rmin}\{(\kappa_I)^M_\delta(t_1), (\kappa_I)^M_\delta(t_2)\}$$
$$= \mathrm{rmin}\{\kappa_I(t_1).\delta, \kappa_I(t_2).\delta\}$$
$$\kappa_I(t_1 * t_2).\delta = \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}.\delta,$$

$$\kappa_F(t_1 * t_2).\delta = (\kappa_F)^M_\delta(t_1 * t_2)$$
$$\geq \mathrm{rmin}\{(\kappa_F)^M_\delta(t_1), (\kappa_F)^M_\delta(t_2)\}$$
$$= \mathrm{rmin}\{\kappa_F(t_1).\delta, \kappa_F(t_2).\delta\}$$
$$\kappa_F(t_1 * t_2).\delta = \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}.\delta$$

and

$$\upsilon_T(t_1 * t_2).\delta = (\upsilon_T)^M_\delta(t_1 * t_2)$$
$$\leq \max\{(\upsilon_T)^M_\delta(t_1), (\upsilon_T)^M_\delta(t_2)\}$$
$$= \max\{\upsilon_T(t_1).\delta, \upsilon_T(t_2).\delta\}$$
$$\upsilon_T(t_1 * t_2).\delta = \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}.\delta,$$

$$\upsilon_I(t_1 * t_2).\delta = (\upsilon_I)^M_\delta(t_1 * t_2)$$
$$\leq \max\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_\delta(t_2)\}$$
$$= \max\{\upsilon_I(t_1).\delta, \upsilon_I(t_2).\delta\}$$
$$\upsilon_I(t_1 * t_2).\delta = \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}.\delta,$$

$$\upsilon_F(t_1 * t_2).\delta = (\upsilon_F)^M_\delta(t_1 * t_2)$$

$\leq \max\{(\upsilon_F)_\delta^M(t_1), (\upsilon_F)_\delta^M(t_2)\}$

$= \max\{\upsilon_F(t_1).\delta, \upsilon_F(t_2).\delta\}$

$\upsilon_F(t_1 * t_2).\delta = \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}.\delta,$

which imply $\kappa_T(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}$, $\kappa_I(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$, $\kappa_F(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$ and $\upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$, $\upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$, $\upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$ for all $t_1, t_2 \in Y$. Hence B is a **NCSU** of Y.

Theorem 3.1.4 Let B be a **NCSU** of Y for $\delta \in [0,1]$. Then **NCM** B_δ^M of B is a **NCSU** of Y.

Proof. Assume $t_1, t_2 \in Y$. Then

$(\kappa_T)_\delta^M(t_1 * t_2) = \delta.\kappa_T(t_1 * t_2)$

$\geq \delta.\mathrm{rmin}\{(\kappa_T)(t_1), (\kappa_T)(t_2)\}$

$= \mathrm{rmin}\{\delta.\kappa_T(t_1), \delta.\kappa_T(t_2)\}$

$= \mathrm{rmin}\{(\kappa_T)_\delta^M(t_1), (\kappa_T)_\delta^M(t_2)\}$

$(\kappa_T)_\delta^M(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_T)_\delta^M(t_1), (\kappa_T)_\delta^M(t_2)\},$

$(\kappa_I)_\delta^M(t_1 * t_2) = \delta.\kappa_I(t_1 * t_2)$

$\geq \delta.\mathrm{rmin}\{(\kappa_I)(t_1), (\kappa_I)(t_2)\}$

$= \mathrm{rmin}\{\delta.\kappa_I(t_1), \delta.\kappa_I(t_2)\}$

$= \mathrm{rmin}\{(\kappa_I)_\delta^M(t_1), (\kappa_I)_\delta^M(t_2)\}$

$(\kappa_I)_\delta^M(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_I)_\delta^M(t_1), (\kappa_I)_\delta^M(t_2)\},$

$(\kappa_F)_\delta^M(t_1 * t_2) = \delta.\kappa_F(t_1 * t_2)$

$\geq \delta.\mathrm{rmin}\{(\kappa_F)(t_1), (\kappa_F)(t_2)\}$

$= \mathrm{rmin}\{\delta.\kappa_F(t_1), \delta.\kappa_F(t_2)\}$

$= \mathrm{rmin}\{(\kappa_F)_\delta^M(t_1), (\kappa_F)_\delta^M(t_2)\}$

$(\kappa_F)_\delta^M(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_F)_\delta^M(t_1), (\kappa_F)_\delta^M(t_2)\}$

and

$(\upsilon_T)_\delta^M(t_1 * t_2) = \delta.\upsilon_T(t_1 * t_2)$

$\leq \delta.\max\{(\upsilon_T)(t_1), (\upsilon_T)(t_2)\}$

$= \max\{\delta.\upsilon_T(t_1), \delta.\upsilon_T(t_2)\}$

$= \max\{(\kappa_B)_\delta^M(t_1), (\kappa_B)_\delta^M(t_2)\}$

$(\upsilon_T)_\delta^M(t_1 * t_2) \leq \max\{(\upsilon_T)_\delta^M(t_1), (\upsilon_T)_\delta^M(t_2)\},$

$(\upsilon_I)^M_\delta(t_1 * t_2) = \delta.\upsilon_I(t_1 * t_2)$

$\leq \delta.\max\{(\upsilon_I)(t_1), (\upsilon_I)(t_2)\}$

$= \max\{\delta.\upsilon_I(t_1), \delta.\upsilon_I(t_2)\}$

$= \max\{(\kappa_B)^M_\delta(t_1), (\kappa_B)^M_\delta(t_2)\}$

$(\upsilon_I)^M_\delta(t_1 * t_2) \leq \max\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_\delta(t_2)\},$

$(\upsilon_F)^M_\delta(t_1 * t_2) = \delta.\upsilon_F(t_1 * t_2)$

$\leq \delta.\max\{(\upsilon_F)(t_1), (\upsilon_F)(t_2)\}$

$= \max\{\delta.\upsilon_F(t_1), \delta.\upsilon_F(t_2)\}$

$= \max\{(\kappa_B)^M_\delta(t_1), (\kappa_B)^M_\delta(t_2)\}$

$(\upsilon_F)^M_\delta(t_1 * t_2) \leq \max\{(\upsilon_F)^M_\delta(t_1), (\upsilon_F)^M_\delta(t_2)\},$

which imply $\kappa_T(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}$, $\kappa_I(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$, $\kappa_F(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$ and $\upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$, $\upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$, $\upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$ for all $t_1, t_2 \in Y$. Hence B^M_δ is a **NCSU** of Y.

3.2 Translative and Multiplicative Interpretation of Neutrosophic Cubic Ideal

In this section, neutrosophic cubic translation of **NCID**, neutrosophic cubic multiplication of **NCID**, union and intersection of neutrosophic cubic translation of **NCID** are investigated through some results.

Theorem 3.2.1 If **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a neutrosophic cubic BF ideal, then it fulfills the conditions $(\kappa_T)^T_\alpha(t_1 * (t_2 * t_1)) \geq (\kappa_T)^T_\alpha(t_2)$, $(\kappa_I)^T_\beta(t_1 * (t_2 * t_1)) \geq (\kappa_I)^T_\beta(t_2)$, $(\kappa_F)^T_\gamma(t_1 * (t_2 * t_1)) \geq (\kappa_F)^T_\gamma(t_2)$ and $(\upsilon_T)^T_\alpha(t_1 * (t_2 * t_1)) \leq (\upsilon_T)^T_\alpha(t_2)$, $(\upsilon_I)^T_\beta(t_1 * (t_2 * t_1)) \leq (\upsilon_I)^T_\beta(t_2)$, $(\upsilon_F)^T_\gamma(t_1 * (t_2 * t_1)) \leq (\upsilon_F)^T_\gamma(t_2)$.

Proof. Let **NCT** $B^T_{\alpha,\beta,\gamma}$ of B be a neutrosophic cubic BF ideal. Then

$(\kappa_T)^T_\alpha(t_1 * (t_2 * t_1)) = \kappa_T(t_1 * (t_2 * t_1)) + \alpha$

$\geq \mathrm{rmin}\{\kappa_T(t_2 * (t_1 * (t_2 * t_1))) + \alpha, \kappa_T(t_2) + \alpha\}$

$= \mathrm{rmin}\{\kappa_T(0) + \alpha, \kappa_T(t_2) + \alpha\}$

$= \mathrm{rmin}\{(\kappa_T)^T_\alpha(0), (\kappa_T)^T_\alpha(t_2)\}$

$(\kappa_T)^T_\alpha(t_1 * (t_2 * t_1)) = (\kappa_T)^T_\alpha(t_2),$

$(\kappa_I)^T_\alpha(t_1 * (t_2 * t_1)) = \kappa_I(t_1 * (t_2 * t_1)) + \beta$

$\geq \mathrm{rmin}\{\kappa_I(t_2 * (t_1 * (t_2 * t_1))) + \beta, \kappa_I(t_2) + \beta\}$

$= \mathrm{rmin}\{\kappa_I(0) + \beta, \kappa_I(t_2) + \beta\}$

$$= \mathrm{rmin}\{(\kappa_I)^T_\beta(0), (\kappa_I)^T_\beta(t_2)\}$$
$$(\kappa_I)^T_\beta(t_1 * (t_2 * t_1)) = (\kappa_I)^T_\beta(t_2),$$

$$(\kappa_F)^T_\gamma(t_1 * (t_2 * t_1)) = \kappa_F(t_1 * (t_2 * t_1)) - \gamma$$
$$\geq \mathrm{rmin}\{\kappa_F(t_2 * (t_1 * (t_2 * t_1))) - \gamma, \kappa_F(t_2) - \gamma\}$$
$$= \mathrm{rmin}\{\kappa_F(0) - \gamma, \kappa_F(t_2) - \gamma\}$$
$$= \mathrm{rmin}\{(\kappa_F)^T_\gamma(0), (\kappa_F)^T_\gamma(t_2)\}$$
$$(\kappa_F)^T_\gamma(t_1 * (t_2 * t_1)) = (\kappa_F)^T_\gamma(t_2)$$

and

$$(\upsilon_T)^T_\alpha(t_1 * (t_2 * t_1)) = \upsilon_T(t_1 * (t_2 * t_1)) + \alpha$$
$$\leq \max\{\upsilon_T(t_2 * (t_1 * (t_2 * t_1))) + \alpha, \upsilon_T(t_2) + \alpha\}$$
$$= \max\{\upsilon_T(0) + \alpha, \upsilon_T(t_2) + \alpha\}$$
$$= \max\{(\upsilon_T)^T_\alpha(0), (\upsilon_T)^T_\alpha(t_2)\}$$
$$(\upsilon_T)^T_\alpha(t_1 * (t_2 * t_1)) = (\upsilon_T)^T_\alpha(t_2),$$

$$(\upsilon_I)^T_\alpha(t_1 * (t_2 * t_1)) = \upsilon_I(t_1 * (t_2 * t_1)) + \beta$$
$$\leq \max\{\upsilon_I(t_2 * (t_1 * (t_2 * t_1))) + \beta, \upsilon_I(t_2) + \beta\}$$
$$= \max\{\upsilon_I(0) + \beta, \upsilon_I(t_2) + \beta\}$$
$$= \max\{(\upsilon_I)^T_\beta(0), (\upsilon_I)^T_\beta(t_2)\}$$
$$(\upsilon_I)^T_\beta(t_1 * (t_2 * t_1)) = (\upsilon_I)^T_\beta(t_2),$$

$$(\upsilon_F)^T_\gamma(t_1 * (t_2 * t_1)) = \upsilon_F(t_1 * (t_2 * t_1)) - \gamma$$
$$\leq \max\{\upsilon_F(t_2 * (t_1 * (t_2 * t_1))) - \gamma, \upsilon_F(t_2) - \gamma\}$$
$$= \max\{\upsilon_F(0) - \gamma, \upsilon_F(t_2) - \gamma\}$$
$$= \max\{(\upsilon_F)^T_\gamma(0), (\upsilon_F)^T_\gamma(t_2)\}$$
$$(\upsilon_F)^T_\gamma(t_1 * (t_2 * t_1)) = (\upsilon_F)^T_\gamma(t_2).$$

Hence $(\kappa_T)^T_\alpha(t_1 * (t_2 * t_1)) \geq (\kappa_T)^T_\alpha(t_2)$, $(\kappa_I)^T_\beta(t_1 * (t_2 * t_1)) \geq (\kappa_I)^T_\beta(t_2)$, $(\kappa_F)^T_\gamma(t_1 * (t_2 * t_1)) \geq (\kappa_F)^T_\gamma(t_2)$ and $(\upsilon_T)^T_\alpha(t_1 * (t_2 * t_1)) \leq (\upsilon_T)^T_\alpha(t_2)$, $(\upsilon_I)^T_\beta(t_1 * (t_2 * t_1)) \leq (\upsilon_I)^T_\beta(t_2)$, $(\upsilon_F)^T_\gamma(t_1 * (t_2 * t_1)) \leq (\upsilon_F)^T_\gamma(t_2)$.

Theorem 3.2.2 Let B be a **NCID** of Y and for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a **NCID** of Y.

Proof. Let B be a **NCID** of Y and for $\kappa_{T,I,F}$, $\alpha, \beta \in \left[[0,0], ℸ\right]$ and $\gamma \in \left[[0,0], ¥\right]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then $(\kappa_T)^T_\alpha(0) = \kappa_T(0) + \alpha \geq \kappa_T(t_1) + \alpha = (\kappa_T)^T_\alpha(t_1)$, $(\kappa_I)^T_\beta(0) = \kappa_I(0) + \beta \geq$

$\kappa_I(t_1) + \beta = (\kappa_I)^T_\beta(t_1)$, $(\kappa_F)^T_\gamma(0) = \kappa_F(0) - \gamma \geq \kappa_F(t_1) - \gamma = (\kappa_F)^T_\gamma(t_1)$ and $(\upsilon_T)^T_\alpha(0) = \upsilon_T(0) + \alpha \leq \upsilon_T(t_1) + \alpha = (\upsilon_T)^T_\alpha(t_1)$, $(\upsilon_I)^T_\beta(0) = \upsilon_I(0) + \beta \leq \upsilon_I(t_1) + \beta = (\upsilon_I)^T_\beta(t_1)$, $(\upsilon_F)^T_\gamma(0) = \upsilon_F(0) - \gamma \leq \upsilon_F(t_1) - \gamma = (\upsilon_F)^T_\gamma(t_1)$. So

$$(\kappa_T)^T_\alpha(t_1) = \kappa_T(t_1) + \alpha$$
$$\geq \mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\} + \alpha$$
$$= \mathrm{rmin}\{\kappa_T(t_1 * t_2) + \alpha, \kappa_T(t_2) + \alpha\}$$
$$(\kappa_T)^T_\alpha(t_1) = \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1 * t_2), (\kappa_T)^T_\alpha(t_2)\},$$
$$(\kappa_I)^T_\beta(t_1) = \kappa_I(t_1) + \beta$$
$$\geq \mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\} + \beta$$
$$= \mathrm{rmin}\{\kappa_I(t_1 * t_2) + \beta, \kappa_I(t_2) + \beta\}$$
$$(\kappa_I)^T_\beta(t_1) = \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1 * t_2), (\kappa_I)^T_\beta(t_2)\},$$
$$(\kappa_F)^T_\alpha(t_1) = \kappa_F(t_1) - \gamma$$
$$\geq \mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\} - \gamma$$
$$= \mathrm{rmin}\{\kappa_F(t_1 * t_2) - \gamma, \kappa_F(t_2) - \gamma\}$$
$$(\kappa_F)^T_\gamma(t_1) = \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1 * t_2), (\kappa_F)^T_\gamma(t_2)\}$$

and

$$(\upsilon_T)^T_\alpha(t_1) = \upsilon_T(t_1) + \alpha$$
$$\leq \max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\} + \alpha$$
$$= \max\{\upsilon_T(t_1 * t_2) + \alpha, \upsilon_T(t_2) + \alpha\}$$
$$(\upsilon_T)^T_\alpha(t_1) = \max\{(\upsilon_T)^T_\alpha(t_1 * t_2), (\upsilon_T)^T_\alpha(t_2)\},$$
$$(\upsilon_I)^T_\beta(t_1) = \upsilon_I(t_1) + \beta$$
$$\leq \max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\} + \beta$$
$$= \max\{\upsilon_I(t_1 * t_2) + \beta, \upsilon_I(t_2) + \beta\}$$
$$(\upsilon_I)^T_\beta(t_1) = \max\{(\upsilon_I)^T_\beta(t_1 * t_2), (\upsilon_I)^T_\beta(t_2)\},$$
$$(\upsilon_F)^T_\gamma(t_1) = \upsilon_F(t_1) - \gamma$$
$$\leq \max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\} - \gamma$$
$$= \max\{\upsilon_F(t_1 * t_2) - \gamma, \upsilon_F(t_2) - \gamma\}$$
$$(\upsilon_F)^T_\gamma(t_1) = \max\{(\upsilon_F)^T_\gamma(t_1 * t_2), (\upsilon_F)^T_\gamma(t_2)\},$$

for all $t_1, t_2 \in Y$ and for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ٦]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Hence $B^{T}_{\alpha,\beta,\gamma}$ of B is a **NCID** of Y.

Theorem 3.2.3 Let B be a neutrosophic cubic set of Y such that **NCT** $B^{T}_{\alpha,\beta,\gamma}$ of B is a **NCID** of Y for all $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ٦]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then B is a **NCID** of Y.

Proof. Suppose $B^{T}_{\alpha,\beta,\gamma}$ is a **NCID** of Y, where for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ٦]$ and $\gamma \in [[0,0], ¥]$, and for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $t_1, t_2 \in Y$. Then

$$\kappa_T(0) + \alpha = (\kappa_T)^T_\alpha(0) \geq (\kappa_T)^T_\alpha(t_1) = \kappa_T(t_1) + \alpha,$$
$$\kappa_I(0) + \beta = (\kappa_I)^T_\beta(0) \geq (\kappa_I)^T_\beta(t_1) = \kappa_I(t_1) + \beta,$$
$$\kappa_F(0) - \gamma = (\kappa_F)^T_\gamma(0) \geq (\kappa_F)^T_\gamma(t_1) = \kappa_F(t_1) - \gamma,$$

and

$$\upsilon_T(0) + \alpha = (\upsilon_T)^T_\alpha(0) \leq (\upsilon_T)^T_\alpha(t_1) = \upsilon_T(t_1) + \alpha,$$
$$\upsilon_I(0) + \beta = (\upsilon_I)^T_\beta(0) \leq (\upsilon_I)^T_\beta(t_1) = \upsilon_I(t_1) + \beta$$
$$\upsilon_F(0) - \gamma = (\upsilon_F)^T_\gamma(0) \leq (\upsilon_F)^T_\gamma(t_1) = \upsilon_F(t_1) - \gamma,$$

which imply $\kappa_T(0) \geq \kappa_T(t_1)$, $\kappa_I(0) \geq \kappa_I(t_1)$, $\kappa_F(0) \geq \kappa_F(t_1)$ and $\upsilon_T(0) \leq \upsilon_T(t_1)$, $\upsilon_I(0) \leq \upsilon_I(t_1)$, $\upsilon_F(0) \leq \upsilon_F(t_1)$, now

$$\kappa_T(t_1) + \alpha = (\kappa_T)^T_\alpha(t_1) \geq \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1 * t_2), (\kappa_T)^T_\alpha(t_2)\}$$
$$= \mathrm{rmin}\{\kappa_T(t_1 * t_2) + \alpha, \kappa_T(t_2) + \alpha\}$$
$$\kappa_T(t_1) + \alpha = \mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\} + \alpha,$$

$$\kappa_I(t_1) + \beta = (\kappa_I)^T_\beta(t_1) \geq \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1 * t_2), (\kappa_I)^T_\beta(t_2)\}$$
$$= \mathrm{rmin}\{\kappa_I(t_1 * t_2) + \beta, \kappa_I(t_2) + \beta\}$$
$$\kappa_I(t_1) + \beta = \mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\} + \beta,$$

$$\kappa_F(t_1) - \gamma = (\kappa_F)^T_\gamma(t_1) \geq \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1 * t_2), (\kappa_F)^T_\gamma(t_2)\}$$
$$= \mathrm{rmin}\{\kappa_F(t_1 * t_2) - \gamma, \kappa_F(t_2) - \gamma\}$$
$$\kappa_F(t_1) - \gamma = \mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\} - \gamma,$$

and

$$\upsilon_T(t_1) + \alpha = (\upsilon_T)^T_\alpha(t_1) \leq \max\{(\upsilon_T)^T_\alpha(t_1 * t_2), (\upsilon_T)^T_\alpha(t_2)\}$$
$$= \max\{\upsilon_T(t_1 * t_2) + \alpha, \upsilon_T(t_2) + \alpha\}$$

$\upsilon_T(t_1) + \alpha = \max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\} + \alpha,$

$\upsilon_I(t_1) + \beta = (\upsilon_I)^T_\beta(t_1) \le \max\{(\upsilon_I)^T_\beta(t_1 * t_2), (\upsilon_I)^T_\beta(t_2)\}$

$= \max\{\upsilon_I(t_1 * t_2) + \beta, \upsilon_I(t_2) + \beta\}$

$\upsilon_I(t_1) + \beta = \max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\} + \beta,$

$\upsilon_F(t_1) - \gamma = (\upsilon_F)^T_\gamma(t_1) \le \max\{(\upsilon_F)^T_\gamma(t_1 * t_2), (\upsilon_F)^T_\gamma(t_2)\}$

$= \max\{\upsilon_F(t_1 * t_2) - \gamma, \upsilon_F(t_2) - \gamma\}$

$\upsilon_F(t_1) - \gamma = \max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\} - \gamma,$

which imply $\kappa_T(t_1) \ge \mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\}$, $\kappa_I(t_1) \ge \mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\}$, $\kappa_F(t_1) \ge \mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\}$ and $\upsilon_T(t_1) \le \max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\}$, $\upsilon_I(t_1) \le \max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\}$, $\upsilon_F(t_1) \le \max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\}$ for all $t_1, t_2 \in Y$. Hence B is a **NCID** of Y.

Theorem 3.2.4 Let B be a **NCID** of Y for some $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ٦]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then **NCT** $B^T_{\alpha,\beta,\gamma}$ of B is a **NCSU** of Y.

Proof. Assume $t_1, t_2 \in Y$. Then

$(\kappa_T)^T_\alpha(t_1 * t_2) = \kappa_T(t_1 * t_2) + \alpha$

$\ge \mathrm{rmin}\{\kappa_T(t_2 * (t_1 * t_2)), \kappa_T(t_2)\} + \alpha$

$= \mathrm{rmin}\{\kappa_T(0), \kappa_T(t_2)\} + \alpha$

$\ge \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\} + \alpha$

$= \mathrm{rmin}\{\kappa_T(t_1) + \alpha, \kappa_T(t_2) + \alpha\}$

$(\kappa_T)^T_\alpha(t_1 * t_2) = \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_\alpha(t_2)\}$

$(\kappa_T)^T_\alpha(t_1 * t_2) \ge \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_\alpha(t_2)\},$

$(\kappa_I)^T_\beta(t_1 * t_2) = \kappa_I(t_1 * t_2) + \beta$

$\ge \mathrm{rmin}\{\kappa_I(t_2 * (t_1 * t_2)), \kappa_I(t_2)\} + \beta$

$= \mathrm{rmin}\{\kappa_I(0), \kappa_I(t_2)\} + \beta$

$\ge \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\} + \beta$

$= \mathrm{rmin}\{\kappa_I(t_1) + \beta, \kappa_I(t_2) + \beta\}$

$(\kappa_I)^T_\beta(t_1 * t_2) = \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_\beta(t_2)\}$

$(\kappa_I)^T_\beta(t_1 * t_2) \ge \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_\beta(t_2)\},$

$(\kappa_F)^T_\gamma(t_1 * t_2) = \kappa_F(t_1 * t_2) - \gamma$

$$\geq \mathrm{rmin}\{\kappa_F(t_2 * (t_1 * t_2)), \kappa_F(t_2)\} - \gamma$$

$$= \mathrm{rmin}\{\kappa_F(0), \kappa_F(t_2)\} - \gamma$$

$$\geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\} - \gamma$$

$$= \mathrm{rmin}\{\kappa_F(t_1) - \gamma, \kappa_F(t_2) - \gamma\}$$

$$(\kappa_F)^T_\gamma(t_1 * t_2) = \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1), (\kappa_F)^T_\gamma(t_2)\}$$

$$(\kappa_F)^T_\gamma(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1), (\kappa_F)^T_\gamma(t_2)\}$$

and

$$(\upsilon_T)^T_\alpha(t_1 * t_2) = \upsilon_T(t_1 * t_2) + \alpha$$

$$\leq \max\{\upsilon_T(t_2 * (t_1 * t_2)), \upsilon_T(t_2)\} + \alpha$$

$$= \max\{\upsilon_T(0), \upsilon_T(t_2)\} + \alpha$$

$$\leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\} + \alpha$$

$$= \max\{\upsilon_T(t_1) + \alpha, \upsilon_T(t_2) + \alpha\}$$

$$(\upsilon_T)^T_\alpha(t_1 * t_2) = \max\{(\upsilon_T)^T_\alpha(t_1), (\upsilon_T)^T_\alpha(t_2)\}$$

$$(\upsilon_T)^T_\alpha(t_1 * t_2) \leq \max\{(\upsilon_T)^T_\alpha(t_1), (\upsilon_T)^T_\alpha(t_2)\},$$

$$(\upsilon_I)^T_\beta(t_1 * t_2) = \upsilon_I(t_1 * t_2) + \beta$$

$$\leq \max\{\upsilon_I(t_2 * (t_1 * t_2)), \upsilon_I(t_2)\} + \beta$$

$$= \max\{\upsilon_I(0), \upsilon_I(t_2)\} + \beta$$

$$\leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\} + \beta$$

$$= \max\{\upsilon_I(t_1) + \beta, \upsilon_I(t_2) + \beta\}$$

$$(\upsilon_I)^T_\beta(t_1 * t_2) = \max\{(\upsilon_I)^T_\beta(t_1), (\upsilon_I)^T_\beta(t_2)\}$$

$$(\upsilon_I)^T_\beta(t_1 * t_2) \leq \max\{(\upsilon_I)^T_\beta(t_1), (\upsilon_I)^T_\beta(t_2)\},$$

$$(\upsilon_F)^T_\gamma(t_1 * t_2) = \upsilon_F(t_1 * t_2) - \gamma$$

$$\leq \max\{\upsilon_F(t_2 * (t_1 * t_2)), \upsilon_F(t_2)\} - \gamma$$

$$= \max\{\upsilon_F(0), \upsilon_F(t_2)\} - \gamma$$

$$\leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\} - \gamma$$

$$= \max\{\upsilon_F(t_1) - \gamma, \upsilon_F(t_2) - \gamma\}$$

$$(\upsilon_F)^T_\gamma(t_1 * t_2) = \max\{(\upsilon_F)^T_\gamma(t_1), (\upsilon_F)^T_\gamma(t_2)\}$$

$$(\upsilon_F)^T_\gamma(t_1 * t_2) \leq \max\{(\upsilon_F)^T_\gamma(t_1), (\upsilon_F)^T_\gamma(t_2)\}.$$

Hence $B^T_{\alpha,\beta,\gamma}$ is a **NCSU** of Y.

Theorem 3.2.5 If **NCT** $B^{T}_{\alpha,\beta,\gamma}$ of B is a **NCID** of Y for some $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$ and $\gamma \in [[0,0], ¥]$, and for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$. Then B is a **NCSU** of Y.

Proof. Suppose $B^{T}_{\alpha,\beta,\gamma}$ of B is a **NCID** of Y. Then

$$(\kappa_T)(t_1 * t_2) + \alpha = (\kappa_T)^T_\alpha(t_1 * t_2)$$

$$\geq \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_2 * (t_1 * t_2)), (\kappa_T)^T_\alpha(t_2)\}$$

$$= \mathrm{rmin}\{(\kappa_T)^T_\alpha(0), (\kappa_T)^T_\alpha(t_2)\}$$

$$\geq \mathrm{rmin}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_\alpha(t_2)\}$$

$$= \mathrm{rmin}\{\kappa_T(t_1) + \alpha, \kappa_T(t_2) + \alpha\}$$

$$(\kappa_T)(t_1 * t_2) + \alpha = \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\} + \alpha,$$

$$(\kappa_I)(t_1 * t_2) + \beta = (\kappa_I)^T_\beta(t_1 * t_2)$$

$$\geq \mathrm{rmin}\{(\kappa_I)^T_\beta(t_2 * (t_1 * t_2)), (\kappa_I)^T_\beta(t_2)\}$$

$$= \mathrm{rmin}\{(\kappa_I)^T_\beta(0), (\kappa_I)^T_\beta(t_2)\}$$

$$\geq \mathrm{rmin}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_\beta(t_2)\}$$

$$= \mathrm{rmin}\{\kappa_I(t_1) + \beta, \kappa_I(t_2) + \beta\}$$

$$(\kappa_I)(t_1 * t_2) + \beta = \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\} + \beta,$$

$$(\kappa_F)(t_1 * t_2) - \gamma = (\kappa_F)^T_\gamma(t_1 * t_2)$$

$$\geq \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_2 * (t_1 * t_2)), (\kappa_F)^T_\gamma(t_2)\}$$

$$= \mathrm{rmin}\{(\kappa_F)^T_\gamma(0), (\kappa_F)^T_\gamma(t_2)\}$$

$$\geq \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1), (\kappa_F)^T_\gamma(t_2)\}$$

$$= \mathrm{rmin}\{\kappa_F(t_1) - \gamma, \kappa_F(t_2) - \gamma\}$$

$$(\kappa_F)(t_1 * t_2) - \gamma = \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\} - \gamma$$

$\Rightarrow \kappa_T(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}, \kappa_I(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$ and $\kappa_F(t_1 * t_2)$ $\geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$ and now

$$(\upsilon_T)(t_1 * t_2) + \alpha = (\upsilon_T)^T_\alpha(t_1 * t_2)$$

$$\leq \max\{(\upsilon_T)^T_\alpha(t_2 * (t_1 * t_2)), (\upsilon_T)^T_\alpha(t_2)\}$$

$$= \max\{(\upsilon_T)^T_\alpha(0), (\upsilon_T)^T_\alpha(t_2)\}$$

$$\leq \max\{(\upsilon_T)^T_\alpha(t_1), (\upsilon_T)^T_\alpha(t_2)\}$$

$$= \max\{\upsilon_T(t_1) + \alpha, \upsilon_T(t_2) + \alpha\}$$

$$(\upsilon_T)(t_1 * t_2) + \alpha = \max\{\upsilon_T(t_1), \upsilon_T(t_2)\} + \alpha,$$

$$(\upsilon_I)(t_1 * t_2) + \beta = (\upsilon_I)^T_\beta(t_1 * t_2)$$
$$\leq \max\{(\upsilon_I)^T_\beta(t_2 * (t_1 * t_2)), (\upsilon_I)^T_\beta(t_2)\}$$
$$= \max\{(\upsilon_I)^T_\beta(0), (\upsilon_I)^T_\beta(t_2)\}$$
$$\leq \max\{(\upsilon_I)^T_\beta(t_1), (\upsilon_I)^T_\beta(t_2)\}$$
$$= \max\{\upsilon_I(t_1) + \beta, \upsilon_I(t_2) + \beta\}$$
$$(\upsilon_I)(t_1 * t_2) + \beta = \max\{\upsilon_I(t_1), \upsilon_I(t_2)\} + \beta,$$

$$(\upsilon_F)(t_1 * t_2) - \gamma = (\upsilon_F)^T_\gamma(t_1 * t_2)$$
$$\leq \max\{(\upsilon_F)^T_\gamma(t_2 * (t_1 * t_2)), (\upsilon_F)^T_\gamma(t_2)\}$$
$$= \max\{(\upsilon_F)^T_\gamma(0), (\upsilon_F)^T_\gamma(t_2)\}$$
$$\leq \max\{(\upsilon_F)^T_\gamma(t_1), (\upsilon_F)^T_\gamma(t_2)\}$$
$$= \max\{\upsilon_F(t_1) - \gamma, \upsilon_F(t_2) - \gamma\}$$
$$(\upsilon_F)(t_1 * t_2) - \gamma = \max\{\upsilon_F(t_1), \upsilon_F(t_2)\} - \gamma$$

$\Rightarrow \upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}, \upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$ and $\upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$. Hence B is a **NCSU** of Y.

Theorem 3.2.6 Intersection of any two neutrosophic cubic translations of a neutrosophic cubic BF ideals B of Y is a neutrosophic cubic BF ideal of Y.

Proof. Suppose $B^T_{\alpha,\beta,\gamma}$ and $B^T_{\alpha',\beta',\gamma'}$ are two neutrosophic cubic translations of neutrosophic cubic BF ideal B and C of Y respectively, where for $B^T_{\alpha,\beta,\gamma}$, for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$, $\gamma \in [[0,0], ¥]$, for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$, $\gamma \in [0, £]$ and for $B^T_{\alpha',\beta',\gamma'}$, for $\kappa_{T,I,F}$ $\alpha', \beta' \in [[0,0], ℸ]$, $\gamma' \in [[0,0], ¥]$, for $\upsilon_{T,I,F}$, $\alpha', \beta' \in [0, \Gamma]$, $\gamma' \in [0, £]$ and $\alpha \leq \alpha'$, $\beta \leq \beta'$, $\gamma \leq \gamma'$ as we know that, $B^T_{\alpha,\beta,\gamma}$ and $B^T_{\alpha',\beta',\gamma'}$ are neutrosophic cubic BF ideals of Y. So

$$((\kappa_T)^T_\alpha \cap (\kappa_T)^T_{\alpha'})(t_1) = \text{rmin}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_{\alpha'}(t_1)\}$$
$$= \text{rmin}\{\kappa_T(t_1) + \alpha, \kappa_T(t_1) + \alpha'\}$$
$$= \kappa_T(t_1) + \alpha$$
$$((\kappa_T)^T_\alpha \cap (\kappa_T)^T_{\alpha'})(t_1) = (\kappa_T)^T_\alpha(t_1),$$

$$((\kappa_I)^T_\beta \cap (\kappa_I)^T_{\beta'})(t_1) = \text{rmin}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_{\beta'}(t_1)\}$$
$$= \text{rmin}\{\kappa_I(t_1) + \beta, \kappa_I(t_1) + \beta'\}$$
$$= \kappa_I(t_1) + \beta$$
$$((\kappa_I)^T_\beta \cap (\kappa_I)^T_{\beta'})(t_1) = (\kappa_I)^T_\beta(t_1),$$

$$((\kappa_F)^T_\gamma \cap (\kappa_F)^T_{\gamma'})(t_1) = \mathrm{rmin}\{(\kappa_F)^T_\gamma(t_1), (\kappa_F)^T_{\gamma'}(t_1)\}$$
$$= \mathrm{rmin}\{\kappa_F(t_1) - \gamma, \kappa_F(t_1) - \gamma'\}$$
$$= \kappa_F(t_1) - \gamma'$$
$$((\kappa_F)^T_\gamma \cap (\kappa_F)^T_{\gamma'})(t_1) = (\kappa_F)^T_{\gamma'}(t_1)$$

and

$$((\upsilon_T)^T_\alpha \cap (\upsilon_T)^T_{\alpha'})(t_1) = \max\{(\upsilon_T)^T_\alpha(t_1), (\upsilon_T)^T_{\alpha'}(t_1)\}$$
$$= \max\{\upsilon_T(t_1) + \alpha, \upsilon_T(t_1) + \alpha'\}$$
$$= \upsilon_T(t_1) + \alpha'$$
$$((\upsilon_T)^T_\alpha \cap (\upsilon_T)^T_{\alpha'})(t_1) = (\upsilon_T)^T_{\alpha'}(t_1),$$

$$((\upsilon_I)^T_\beta \cap (\upsilon_I)^T_{\beta'})(t_1) = \max\{(\upsilon_I)^T_\beta(t_1), (\upsilon_I)^T_{\beta'}(t_1)\}$$
$$= \max\{\upsilon_I(t_1) + \beta, \upsilon_I(t_1) + \beta'\}$$
$$= \upsilon_I(t_1) + \beta'$$
$$((\upsilon_I)^T_\beta \cap (\upsilon_I)^T_{\beta'})(t_1) = (\upsilon_I)^T_{\beta'}(t_1),$$

$$((\upsilon_F)^T_\gamma \cap (\upsilon_F)^T_{\gamma'})(t_1) = \max\{(\upsilon_F)^T_\gamma(t_1), (\upsilon_F)^T_{\gamma'}(t_1)\}$$
$$= \max\{\upsilon_F(t_1) - \gamma, \upsilon_F(t_1) - \gamma'\}$$
$$= \upsilon_F(t_1) - \gamma$$
$$((\upsilon_F)^T_\gamma \cap (\upsilon_F)^T_{\gamma'})(t_1) = (\upsilon_F)^T_\gamma(t_1).$$

Hence $B^T_{\alpha,\beta,\gamma} \cap B^T_{\alpha',\beta',\gamma'}$ is a neutrosophic cubic BF ideal of Y.

Theorem 3.2.7 Union of any two neutrosophic cubic translations of a neutrosophic cubic BF ideals B of Y is a neutrosophic cubic BF ideal of Y.

Proof. Suppose $B^T_{\alpha,\beta,\gamma}$ and $B^T_{\alpha',\beta',\gamma'}$ are two neutrosophic cubic translations of neutrosophic cubic BF ideal B of Y respectively, where for $B^T_{\alpha,\beta,\gamma}$, for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$, $\gamma \in [[0,0], ¥]$, for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$, $\gamma \in [0, £]$ and for $B^T_{\alpha\prime,\beta\prime,\gamma\prime}$, for $\kappa_{T,I,F}$ $\alpha', \beta' \in [[0,0], ℸ]$, $\gamma' \in [[0,0], ¥]$, for $\upsilon_{T,I,F}$, $\alpha', \beta' \in [0, \Gamma]$, $\gamma' \in [0, £]$ and $\alpha \geq \alpha'$, $\beta \geq \beta'$, $\gamma \geq \gamma'$ as we know that, $B^T_{\alpha,\beta,\gamma}$ and $B^T_{\alpha',\beta',\gamma'}$ are neutrosophic cubic BF ideals of Y. Then

$$((\kappa_T)^T_\alpha \cup (\kappa_T)^T_{\alpha'})(t_1) = \mathrm{rmax}\{(\kappa_T)^T_\alpha(t_1), (\kappa_T)^T_{\alpha'}(t_1)\}$$
$$= \mathrm{rmax}\{\kappa_T(t_1) + \alpha, \kappa_T(t_1) + \alpha'\}$$
$$= \kappa_T(t_1) + \alpha$$
$$((\kappa_T)^T_\alpha \cup (\kappa_T)^T_{\alpha'})(t_1) = (\kappa_T)^T_\alpha(t_1),$$

$$((\kappa_I)^T_\beta \cup (\kappa_I)^T_{\beta'})(t_1) = \mathrm{rmax}\{(\kappa_I)^T_\beta(t_1), (\kappa_I)^T_{\beta'}(t_1)\}$$
$$= \mathrm{rmax}\{\kappa_I(t_1) + \beta, \kappa_I(t_1) + \beta'\}$$
$$= \kappa_I(t_1) + \beta$$
$$((\kappa_I)^T_\beta \cup (\kappa_I)^T_{\beta'})(t_1) = (\kappa_I)^T_\beta(t_1),$$

$$((\kappa_F)^T_\gamma \cup (\kappa_F)^T_{\gamma'})(t_1) = \mathrm{rmax}\{(\kappa_F)^T_\gamma(t_1), (\kappa_F)^T_{\gamma'}(t_1)\}$$
$$= \mathrm{rmax}\{\kappa_F(t_1) - \gamma, \kappa_F(t_1) - \gamma'\}$$
$$= \kappa_F(t_1) - \gamma'$$
$$((\kappa_F)^T_\gamma \cup (\kappa_F)^T_{\gamma'})(t_1) = (\kappa_F)^T_{\gamma'}(t_1)$$

and

$$((\upsilon_T)^T_\alpha \cup (\upsilon_T)^T_{\alpha'})(t_1) = \min\{(\upsilon_T)^T_\alpha(t_1), (\upsilon_T)^T_{\alpha'}(t_1)\}$$
$$= \min\{\upsilon_T(t_1) + \alpha, \upsilon_T(t_1) + \alpha'\}$$
$$= \upsilon_T(t_1) + \alpha'$$
$$((\upsilon_T)^T_\alpha \cup (\upsilon_T)^T_{\alpha'})(t_1) = (\upsilon_T)^T_{\alpha'}(t_1),$$

$$((\upsilon_I)^T_\beta \cup (\upsilon_I)^T_{\beta'})(t_1) = \min\{(\upsilon_I)^T_\beta(t_1), (\upsilon_I)^T_{\beta'}(t_1)\}$$
$$= \min\{\upsilon_I(t_1) + \beta, \upsilon_I(t_1) + \beta'\}$$
$$= \upsilon_I(t_1) + \beta'$$
$$((\upsilon_I)^T_\beta \cup (\upsilon_I)^T_{\beta'})(t_1) = (\upsilon_I)^T_\beta(t_1),$$

$$((\upsilon_F)^T_\gamma \cup (\upsilon_F)^T_{\gamma'})(t_1) = \min\{(\upsilon_F)^T_\gamma(t_1), (\upsilon_F)^T_{\gamma'}(t_1)\}$$
$$= \min\{\upsilon_F(t_1) - \gamma, \upsilon_F(t_1) - \gamma'\}$$
$$= \upsilon_F(t_1) - \gamma$$
$$((\upsilon_F)^T_\gamma \cup (\upsilon_F)^T_{\gamma'})(t_1) = (\upsilon_F)^T_\gamma(t_1)$$

Hence $B^T_{\alpha,\beta,\gamma} \cup B^T_{\alpha',\beta',\gamma'}$ is a neutrosophic cubic BF ideal of Y.

Theorem 3.2.8 Let B be a **NCS** of Y such that **NCM** B^M_δ of B is a **NCID** of Y for $\delta \in (0,1]$ then B is a **NCID** of Y.

Proof. Suppose that B^M_δ is a NCID of Y for $\delta \in (0,1]$ and $t_1, t_2 \in Y$. Then $\delta.\kappa_T(0) = (\kappa_T)^M_\delta(0) \geq (\kappa_T)^M_\delta(t_1) = \delta.\kappa_T(t_1)$, so $\kappa_T(0) \geq \kappa_T(t_1)$,$\delta.\kappa_I(0) = (\kappa_I)^M_\delta(0) \geq (\kappa_I)^M_\delta(t_1) = \delta.\kappa_I(t_1)$, so $\kappa_I(0) \geq \kappa_I(t_1)$, $\delta.\kappa_F(0) = (\kappa_F)^M_\delta(0) \geq (\kappa_F)^M_\delta(t_1) = \delta.\kappa_F(t_1)$, so $\kappa_F(0) \geq \kappa_F(t_1)$ and $\delta.\upsilon_T(0) = (\upsilon_T)^M_\delta(0) \leq (\upsilon_T)^M_\delta(t_1) = \delta.\upsilon_T(t_1)$, so $\upsilon_T(0) \leq \upsilon_T(t_1)$, $\delta.\upsilon_I(0) = (\upsilon_I)^M_\delta(0) \leq (\upsilon_I)^M_\delta(t_1) = \delta.\upsilon_I(t_1)$, so $\upsilon_I(0) \leq \upsilon_I(t_1)$, $\delta.\upsilon_F(0) = (\upsilon_F)^M_\delta(0) \leq (\upsilon_F)^M_\delta(t_1) = \delta.\upsilon_F(t_1)$, so $\upsilon_F(0) \leq \upsilon_F(t_1)$. Now

$$\delta.\kappa_T(t_1) = (\kappa_T)^M_\delta(t_1)$$
$$\geq \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1 * t_2), (\kappa_T)^M_\delta(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_T(t_1 * t_2), \delta.\kappa_T(t_2)\}$$
$$\delta.\kappa_T(t_1) = \delta.\mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\},$$

$$\delta.\kappa_I(t_1) = (\kappa_I)^M_\delta(t_1)$$

$\geq \mathrm{rmin}\{(\kappa_I)_\delta^M(t_1 * t_2), (\kappa_I)_\delta^M(t_2)\}$

$= \mathrm{rmin}\{\delta.\kappa_I(t_1 * t_2), \delta.\kappa_I(t_2)\}$

$\delta.\kappa_I(t_1) = \delta.\mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\},$

$\delta.\kappa_F(t_1) = (\kappa_F)_\delta^M(t_1)$

$\geq \mathrm{rmin}\{(\kappa_F)_\delta^M(t_1 * t_2), (\kappa_F)_\delta^M(t_2)\}$

$= \mathrm{rmin}\{\delta.\kappa_F(t_1 * t_2), \delta.\kappa_F(t_2)\}$

$\delta.\kappa_F(t_1) = \delta.\mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\},$

so $\kappa_T(t_1) \geq \mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\}, \kappa_I(t_1) \geq \mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\}$ and $\kappa_F(t_1) \geq \mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\}$ and also

$\delta.\upsilon_T(t_1) = (\upsilon_T)_\delta^M(t_1)$

$\leq \max\{(\upsilon_T)_\delta^M(t_1 * t_2), (\upsilon_T)_\delta^M(t_2)\}$

$= \max\{\delta.\upsilon_T(t_1 * t_2), \delta.\upsilon_T(t_2)\}$

$\delta.\upsilon_T(t_1) = \delta.\max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\},$

$\delta.\upsilon_I(t_1) = (\upsilon_I)_\delta^M(t_1)$

$\leq \max\{(\upsilon_I)_\delta^M(t_1 * t_2), (\upsilon_I)_\delta^M(t_2)\}$

$= \max\{\delta.\upsilon_I(t_1 * t_2), \delta.\upsilon_I(t_2)\}$

$\delta.\upsilon_I(t_1) = \delta.\max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\},$

$\delta.\upsilon_F(t_1) = (\upsilon_F)_\delta^M(t_1)$

$\leq \max\{(\upsilon_F)_\delta^M(t_1 * t_2), (\upsilon_F)_\delta^M(t_2)\}$

$= \max\{\delta.\upsilon_F(t_1 * t_2), \delta.\upsilon_F(t_2)\}$

$\delta.\upsilon_F(t_1) = \delta.\max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\},$

so $\upsilon_T(t_1) \leq \max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\}, \upsilon_I(t_1) \leq \max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\}$ and $\upsilon_F(t_1) \leq \max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\}$. Hence B is a **NCID** of Y.

Theorem 3.2.9 If B is a **NCID** of Y, then **NCM** B_δ^M of B is a **NCID** of Y, for all $\delta \in (0,1]$.

Proof. Let B be a **NCID** of Y and $\delta \in (0,1]$. Then we have $(\kappa_T)_\delta^M(0) = \delta.\kappa_T(0) \geq \delta.\kappa_T(t_1) \rightarrow (\kappa_T)_\delta^M(0) = (\kappa_T)_\delta^M(t_1)$, $(\kappa_I)_\delta^M(0) = \delta.\kappa_I(0) \geq \delta.\kappa_I(t_1) \rightarrow (\kappa_I)_\delta^M(0) = (\kappa_I)_\delta^M(t_1)$, $(\kappa_F)_\delta^M(0) = \delta.\kappa_F(0) \geq \delta.\kappa_F(t_1) \rightarrow (\kappa_F)_\delta^M(0) = (\kappa_F)_\delta^M(t_1)$ and $(\upsilon_T)_\delta^M(0) = \delta.\upsilon_T(0) \leq \delta.\upsilon_T(t_1) \rightarrow (\upsilon_T)_\delta^M(0) = (\upsilon_T)_\delta^M(t_1)$,

$(\upsilon_I)^M_\delta(0) = \delta.\upsilon_I(0) \leq \delta.\upsilon_I(t_1) \to (\upsilon_I)^M_\delta(0) = (\upsilon_I)^M_\delta(t_1)$, $(\upsilon_F)^M_\delta(0) = \delta.\upsilon_F(0) \leq \delta.\upsilon_F(t_1) \to (\upsilon_F)^M_\delta(0) = (\upsilon_F)^M_\delta(t_1)$.

Now

$$(\kappa_T)^M_\delta(t_1) = \delta.\kappa_T(t_1)$$
$$\geq \delta.\mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_T(t_1 * t_2), \delta.\kappa_T(t_2)\}$$
$$(\kappa_T)^M_\delta(t_1) = \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1 * t_2), (\kappa_T)^M_\delta(t_2)\}$$
$$(\kappa_T)^M_\delta(t_1) \geq \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1 * t_2), (\kappa_T)^M_\delta(t_2)\},$$

$$(\kappa_I)^M_\delta(t_1) = \delta.\kappa_I(t_1)$$
$$\geq \delta.\mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_I(t_1 * t_2), \delta.\kappa_I(t_2)\}$$
$$(\kappa_I)^M_\delta(t_1) = \mathrm{rmin}\{(\kappa_I)^M_\delta(t_1 * t_2), (\kappa_I)^M_\delta(t_2)\}$$
$$(\kappa_I)^M_\delta(t_1) \geq \mathrm{rmin}\{(\kappa_I)^M_\delta(t_1 * t_2), (\kappa_I)^M_\delta(t_2)\},$$

$$(\kappa_F)^M_\delta(t_1) = \delta.\kappa_F(t_1)$$
$$\geq \delta.\mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_F(t_1 * t_2), \delta.\kappa_F(t_2)\}$$
$$(\kappa_F)^M_\delta(t_1) = \mathrm{rmin}\{(\kappa_F)^M_\delta(t_1 * t_2), (\kappa_F)^M_\delta(t_2)\}$$
$$(\kappa_F)^M_\delta(t_1) \geq \mathrm{rmin}\{(\kappa_F)^M_\delta(t_1 * t_2), (\kappa_F)^M_\delta(t_2)\}$$

and

$$(\upsilon_T)^M_\delta(t_1) = \delta.\upsilon_T(t_1)$$
$$\leq \delta.\max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\}$$
$$= \max\{\delta.\upsilon_T(t_1 * t_2), \delta.\upsilon_T(t_2)\}$$
$$(\upsilon_T)^M_\delta(t_1) = \max\{(\upsilon_T)^M_\delta(t_1 * t_2), (\upsilon_T)^M_\delta(t_2)\}$$
$$(\upsilon_T)^M_\delta(t_1) \leq \max\{(\upsilon_T)^M_\delta(t_1 * t_2), (\upsilon_T)^M_\delta(t_2)\},$$

$$(\upsilon_I)^M_\delta(t_1) = \delta.\upsilon_I(t_1)$$
$$\leq \delta.\max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\}$$
$$= \max\{\delta.\upsilon_I(t_1 * t_2), \delta.\upsilon_I(t_2)\}$$
$$(\upsilon_I)^M_\delta(t_1) = \max\{(\upsilon_I)^M_\delta(t_1 * t_2), (\upsilon_I)^M_\delta(t_2)\}$$
$$(\upsilon_I)^M_\delta(t_1) \leq \max\{(\upsilon_I)^M_\delta(t_1 * t_2), (\upsilon_I)^M_\delta(t_2)\},$$

$$(\upsilon_F)^M_\delta(t_1) = \delta.\upsilon_F(t_1)$$
$$\leq \delta.\max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\}$$
$$= \max\{\delta.\upsilon_F(t_1 * t_2), \delta.\upsilon_F(t_2)\}$$
$$(\upsilon_F)^M_\delta(t_1) = \max\{(\upsilon_F)^M_\delta(t_1 * t_2), (\upsilon_F)^M_\delta(t_2)\}$$
$$(\upsilon_F)^M_\delta(t_1) \leq \max\{(\upsilon_F)^M_\delta(t_1 * t_2), (\upsilon_F)^M_\delta(t_2)\}.$$

Hence B^M_δ of B is a **NCID** of Y, for all $\delta \in (0,1]$.

Theorem 3.2.10 Let B be a **NCID** of Y and $\delta \in [0,1]$ then **NCM** B^M_δ of B is a **NCSU** of Y.

Proof. Suppose $t_1, t_2 \in Y$. Then

$$(\kappa_T)^M_\delta(t_1 * t_2) = \delta.\kappa_T(t_1 * t_2)$$
$$\geq \delta.\mathrm{rmin}\{\kappa_T(t_2 * (t_1 * t_2)), \kappa_T(t_2)\}$$
$$= \delta.\mathrm{rmin}\{\kappa_T(0), \kappa_T(t_2)\}$$
$$\geq \delta.\mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_T(t_1), \delta.\kappa_T(t_2)\}$$
$$(\kappa_T)^M_\delta(t_1 * t_2) = \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1), (\kappa_T)^M_\delta(t_2)\}$$
$$(\kappa_T)^M_\delta(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1), (\kappa_T)^M_\delta(t_2)\},$$

$$(\kappa_I)^M_\delta(t_1 * t_2) = \delta.\kappa_I(t_1 * t_2)$$
$$\geq \delta.\mathrm{rmin}\{\kappa_I(t_2 * (t_1 * t_2)), \kappa_I(t_2)\}$$
$$= \delta.\mathrm{rmin}\{\kappa_I(0), \kappa_I(t_2)\}$$
$$\geq \delta.\mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_I(t_1), \delta.\kappa_I(t_2)\}$$
$$(\kappa_I)^M_\delta(t_1 * t_2) = \mathrm{rmin}\{(\kappa_I)^M_\delta(t_1), (\kappa_I)^M_\delta(t_2)\}$$
$$(\kappa_I)^M_\delta(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_I)^M_\delta(t_1), (\kappa_I)^M_\delta(t_2)\},$$

$$(\kappa_F)^M_\delta(t_1 * t_2) = \delta.\kappa_F(t_1 * t_2)$$
$$\geq \delta.\mathrm{rmin}\{\kappa_F(t_2 * (t_1 * t_2)), \kappa_F(t_2)\}$$
$$= \delta.\mathrm{rmin}\{\kappa_F(0), \kappa_F(t_2)\}$$
$$\geq \delta.\mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$$
$$= \mathrm{rmin}\{\delta.\kappa_F(t_1), \delta.\kappa_F(t_2)\}$$
$$(\kappa_F)^M_\delta(t_1 * t_2) = \mathrm{rmin}\{(\kappa_F)^M_\delta(t_1), (\kappa_F)^M_\delta(t_2)\}$$

$$(\kappa_F)^M_\delta(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_F)^M_\delta(t_1), (\kappa_F)^M_\delta(t_2)\}$$

and

$$(\upsilon_T)^M_\delta(t_1 * t_2) = \delta.\upsilon_T(t_1 * t_2)$$
$$\leq \delta.\max\{\upsilon_T(t_2 * (t_1 * t_2)), \upsilon_T(t_2)\}$$
$$= \delta.\max\{\upsilon_T(0), \upsilon_T(t_2)\}$$
$$\leq \delta.\max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$$
$$= \max\{\delta.\upsilon_T(t_1), \delta.\upsilon_T(t_2)\}$$
$$(\upsilon_T)^M_\delta(t_1 * t_2) = \max\{(\upsilon_T)^M_\delta(t_1), (\upsilon_T)^M_\delta(t_2)\}$$
$$(\upsilon_T)^M_\delta(t_1 * t_2) \leq \max\{(\upsilon_T)^M_\delta(t_1), (\upsilon_T)^M_\delta(t_2)\},$$

$$(\upsilon_I)^M_\delta(t_1 * t_2) = \delta.\upsilon_I(t_1 * t_2)$$
$$\leq \delta.\max\{\upsilon_I(t_2 * (t_1 * t_2)), \upsilon_I(t_2)\}$$
$$= \delta.\max\{\upsilon_I(0), \upsilon_I(t_2)\}$$
$$\leq \delta.\max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$$
$$= \max\{\delta.\upsilon_I(t_1), \delta.\upsilon_I(t_2)\}$$
$$(\upsilon_I)^M_\delta(t_1 * t_2) = \max\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_\delta(t_2)\}$$
$$(\upsilon_I)^M_\delta(t_1 * t_2) \leq \max\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_\delta(t_2)\},$$

$$(\upsilon_F)^M_\delta(t_1 * t_2) = \delta.\upsilon_F(t_1 * t_2)$$
$$\leq \delta.\max\{\upsilon_F(t_2 * (t_1 * t_2)), \upsilon_F(t_2)\}$$
$$= \delta.\max\{\upsilon_F(0), \upsilon_F(t_2)\}$$
$$\leq \delta.\max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$$
$$= \max\{\delta.\upsilon_F(t_1), \delta.\upsilon_F(t_2)\}$$
$$(\upsilon_F)^M_\delta(t_1 * t_2) = \max\{(\upsilon_F)^M_\delta(t_1), (\upsilon_F)^M_\delta(t_2)\}$$
$$(\upsilon_F)^M_\delta(t_1 * t_2) \leq \max\{(\upsilon_F)^M_\delta(t_1), (\upsilon_F)^M_\delta(t_2)\}.$$

Hence B^M_δ is a **NCSU** of Y.

Theorem 3.2.11 If the **NCM** B^M_δ of B is a **NCID** of Y, for $\delta \in (0,1]$. Then B is a neutrosophic cubic BF-subalgebra of Y.

Proof. Assume B^M_δ of B is a **NCID** of Y. Then

$$\delta.(\kappa_T)(t_1 * t_2) = (\kappa_T)^M_\delta(t_1 * t_2)$$
$$\geq \mathrm{rmin}\{(\kappa_T)^M_\delta(t_2 * (t_1 * t_2)), (\kappa_T)^M_\delta(t_2)\}$$
$$= \mathrm{rmin}\{(\kappa_T)^M_\delta(0), (\kappa_T)^M_\delta(t_2)\}$$

$\geq \mathrm{rmin}\{(\kappa_T)_\delta^M(t_1), (\kappa_T)_\delta^M(t_2)\}$

$= \mathrm{rmin}\{\delta.\kappa_T(t_1), \delta.\kappa_T(t_2)\}$

$\delta.(\kappa_T)(t_1 * t_2) = \delta.\mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}$

$\Rightarrow \kappa_T(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}$,

$\delta.(\kappa_I)(t_1 * t_2) = (\kappa_I)_\delta^M(t_1 * t_2)$

$\geq \mathrm{rmin}\{(\kappa_I)_\delta^M(t_2 * (t_1 * t_2)), (\kappa_I)_\delta^M(t_2)\}$

$= \mathrm{rmin}\{(\kappa_I)_\delta^M(0), (\kappa_I)_\delta^M(t_2)\}$

$\geq \mathrm{rmin}\{(\kappa_I)_\delta^M(t_1), (\kappa_I)_\delta^M(t_2)\}$

$= \mathrm{rmin}\{\delta.\kappa_I(t_1), \delta.\kappa_I(t_2)\}$

$\delta.(\kappa_I)(t_1 * t_2) = \delta.\mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$

$\Rightarrow \kappa_I(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$,

$\delta.(\kappa_F)(t_1 * t_2) = (\kappa_F)_\delta^M(t_1 * t_2)$

$\geq \mathrm{rmin}\{(\kappa_F)_\delta^M(t_2 * (t_1 * t_2)), (\kappa_F)_\delta^M(t_2)\}$

$= \mathrm{rmin}\{(\kappa_F)_\delta^M(0), (\kappa_F)_\delta^M(t_2)\}$

$\geq \mathrm{rmin}\{(\kappa_F)_\delta^M(t_1), (\kappa_F)_\delta^M(t_2)\}$

$= \mathrm{rmin}\{\delta.\kappa_F(t_1), \delta.\kappa_F(t_2)\}$

$\delta.(\kappa_F)(t_1 * t_2) = \delta.\mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$

$\Rightarrow \kappa_F(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$

and

$\delta.(\upsilon_T)(t_1 * t_2) = (\upsilon_T)_\delta^M(t_1 * t_2)$

$\leq \max\{(\upsilon_T)_\delta^M(t_2 * (t_1 * t_2)), (\upsilon_T)_\delta^M(t_2)\}$

$= \max\{(\upsilon_T)_\delta^M(0), (\upsilon_T)_\delta^M(t_2)\}$

$\leq \max\{(\upsilon_T)_\delta^M(t_1), (\upsilon_T)_\delta^M(t_2)\}$

$= \max\{\delta.\upsilon_T(t_1), \delta.\upsilon_T(t_2)\}$

$\delta.(\upsilon_T)(t_1 * t_2) = \delta.\max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$

$\Rightarrow \upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$,

$\delta.(\upsilon_I)(t_1 * t_2) = (\upsilon_I)_\delta^M(t_1 * t_2)$

$\leq \max\{(\upsilon_I)_\delta^M(t_2 * (t_1 * t_2)), (\upsilon_I)_\delta^M(t_2)\}$

$= \max\{(\upsilon_I)_\delta^M(0), (\upsilon_I)_\delta^M(t_2)\}$

$\leq \max\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_\delta(t_2)\}$

$= \max\{\delta.\upsilon_I(t_1), \delta.\upsilon_I(t_2)\}$

$\delta.(\upsilon_I)(t_1 * t_2) = \delta.\max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$

$\Rightarrow \upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\},$

$\delta.(\upsilon_F)(t_1 * t_2) = (\upsilon_F)^M_\delta(t_1 * t_2)$

$\leq \max\{(\upsilon_F)^M_\delta(t_2 * (t_1 * t_2)), (\upsilon_F)^M_\delta(t_2)\}$

$= \max\{(\upsilon_F)^M_\delta(0), (\upsilon_F)^M_\delta(t_2)\}$

$\leq \max\{(\upsilon_F)^M_\delta(t_1), (\upsilon_F)^M_\delta(t_2)\}$

$= \max\{\delta.\upsilon_F(t_1), \delta.\upsilon_F(t_2)\}$

$\delta.(\upsilon_F)(t_1 * t_2) = \delta.\max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$

$\Rightarrow \upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}.$

Hence B is a **NCSU** of Y.

Theorem 3.2.12 Intersection of any two neutrosophic cubic multiplications of a **NCID** B of Y is a **NCID** of Y.

Proof. Suppose B^M_δ and $B^M_{\delta'}$ are neutrosophic cubic multiplications of **NCID** B of Y, where $\delta, \delta' \in (0,1]$ and $\delta \leq \delta'$, as we know that B^M_δ and $B^M_{\delta'}$ are **NCIDs** of Y. Then

$((\kappa_T)^M_\delta \cap (\kappa_T)^M_{\delta'})(t_1) = \mathrm{rmin}\{(\kappa_T)^M_\delta(t_1), (\kappa_T)^M_{\delta'}(t_1)\}$

$= \mathrm{rmin}\{\kappa_T(t_1).\delta, \kappa_T(t_1).\delta'\}$

$= \kappa_T(t_1).\delta$

$((\kappa_T)^M_\delta \cap (\kappa_T)^M_{\delta'})(t_1) = (\kappa_T)^M_\delta(t_1),$

$((\kappa_I)^M_\delta \cap (\kappa_I)^M_{\delta'})(t_1) = \mathrm{rmin}\{(\kappa_I)^M_\delta(t_1), (\kappa_I)^M_{\delta'}(t_1)\}$

$= \mathrm{rmin}\{\kappa_I(t_1).\delta, \kappa_I(t_1).\delta'\}$

$= \kappa_I(t_1).\delta$

$((\kappa_I)^M_\delta \cap (\kappa_I)^M_{\delta'})(t_1) = (\kappa_I)^M_\delta(t_1),$

$((\kappa_F)^M_\delta \cap (\kappa_F)^M_{\delta'})(t_1) = \mathrm{rmin}\{(\kappa_F)^M_\delta(t_1), (\kappa_F)^M_{\delta'}(t_1)\}$

$= \mathrm{rmin}\{\kappa_F(t_1).\delta, \kappa_F(t_1).\delta'\}$

$= \kappa_F(t_1).\delta$

$((\kappa_F)^M_\delta \cap (\kappa_F)^M_{\delta'})(t_1) = (\kappa_F)^M_\delta(t_1)$

and

$$((\upsilon_T)^M_\delta \cap (\upsilon_T)^M_{\delta'})(t_1) = \max\{(\upsilon_T)^M_\delta(t_1), (\upsilon_T)^M_{\delta'}(t_1)\}$$
$$= \max\{\upsilon_T(t_1).\delta, \upsilon_T(t_1).\delta'\}$$
$$= \upsilon_T(t_1).\delta'$$
$$((\upsilon_T)^M_\delta \cap (\upsilon_T)^M_{\delta'})(t_1) = (\upsilon_T)^M_{\delta'}(t_1),$$

$$((\upsilon_I)^M_\delta \cap (\upsilon_I)^M_{\delta'})(t_1) = \max\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_{\delta'}(t_1)\}$$
$$= \max\{\upsilon_I(t_1).\delta, \upsilon_I(t_1).\delta'\}$$
$$= \upsilon_I(t_1).\delta'$$
$$((\upsilon_I)^M_\delta \cap (\upsilon_I)^M_{\delta'})(t_1) = (\upsilon_I)^M_{\delta'}(t_1),$$

$$((\upsilon_F)^M_\delta \cap (\upsilon_F)^M_{\delta'})(t_1) = \max\{(\upsilon_F)^M_\delta(t_1), (\upsilon_F)^M_{\delta'}(t_1)\}$$
$$= \max\{\upsilon_F(t_1).\delta, \upsilon_F(t_1).\delta'\}$$
$$= \upsilon_F(t_1).\delta'$$
$$((\upsilon_F)^M_\delta \cap (\upsilon_F)^M_{\delta'})(t_1) = (\upsilon_F)^M_{\delta'}(t_1).$$

Hence $B^M_\delta \cap B^M_{\delta'}$ is **NCID** of Y.

Theorem 3.2.13 Union of any two neutrosophic cubic multiplications of a **NCID** B of Y is a **NCID** of Y.

Proof. Suppose B^M_δ and $B^M_{\delta'}$ are neutrosophic cubic multiplications of **NCID** B of Y, where $\delta, \delta' \in (0,1]$ and $\delta \leq \delta'$, as we know that B^M_δ and $B^M_{\delta'}$ are **NCIDs** of Y. Then

$$((\kappa_T)^M_\delta \cup (\kappa_T)^M_{\delta'})(t_1) = \mathrm{rmax}\{(\kappa_T)^M_\delta(t_1), (\kappa_T)^M_{\delta'}(t_1)\}$$
$$= \mathrm{rmax}\{\kappa_T(t_1).\delta, \kappa_T(t_1).\delta'\}$$
$$= \kappa_T(t_1).\delta'$$
$$((\kappa_T)^M_\delta \cup (\kappa_T)^M_{\delta'})(t_1) = (\kappa_T)^M_{\delta'}(t_1),$$

$$((\kappa_I)^M_\delta \cup (\kappa_I)^M_{\delta'})(t_1) = \mathrm{rmax}\{(\kappa_I)^M_\delta(t_1), (\kappa_I)^M_{\delta'}(t_1)\}$$
$$= \mathrm{rmax}\{\kappa_I(t_1).\delta, \kappa_I(t_1).\delta'\}$$
$$= \kappa_I(t_1).\delta'$$
$$((\kappa_I)^M_\delta \cup (\kappa_I)^M_{\delta'})(t_1) = (\kappa_I)^M_{\delta'}(t_1),$$

$$((\kappa_F)^M_\delta \cup (\kappa_F)^M_{\delta'})(t_1) = \mathrm{rmax}\{(\kappa_F)^M_\delta(t_1), (\kappa_F)^M_{\delta'}(t_1)\}$$
$$= \mathrm{rmax}\{\kappa_F(t_1).\delta, \kappa_F(t_1).\delta'\}$$
$$= \kappa_F(t_1).\delta'$$
$$((\kappa_F)^M_\delta \cup (\kappa_F)^M_{\delta'})(t_1) = (\kappa_F)^M_{\delta'}(t_1)$$

and

$$((\upsilon_T)^M_\delta \cup (\upsilon_T)^M_{\delta'})(t_1) = \min\{(\upsilon_T)^M_\delta(t_1), (\upsilon_T)^M_{\delta'}(t_1)\}$$
$$= \min\{\upsilon_T(t_1).\delta, \upsilon_T(t_1).\delta'\}$$
$$= \upsilon_T(t_1).\delta$$
$$((\upsilon_T)^M_\delta \cup (\upsilon_T)^M_{\delta'})(t_1) = (\upsilon_T)^M_\delta(t_1),$$

$$((\upsilon_I)^M_\delta \cup (\upsilon_I)^M_{\delta'})(t_1) = \min\{(\upsilon_I)^M_\delta(t_1), (\upsilon_I)^M_{\delta'}(t_1)\}$$
$$= \min\{\upsilon_I(t_1).\delta, \upsilon_I(t_1).\delta'\}$$
$$= \upsilon_I(t_1).\delta$$
$$((\upsilon_I)^M_\delta \cup (\upsilon_I)^M_{\delta'})(t_1) = (\upsilon_I)^M_\delta(t_1),$$

$$((\upsilon_F)^M_\delta \cup (\upsilon_F)^M_{\delta'})(t_1) = \min\{(\upsilon_F)^M_\delta(t_1), (\upsilon_F)^M_{\delta'}(t_1)\}$$
$$= \min\{\upsilon_F(t_1).\delta, \upsilon_F(t_1).\delta'\}$$
$$= \upsilon_F(t_1).\delta$$
$$((\upsilon_F)^M_\delta \cup (\upsilon_F)^M_{\delta'})(t_1) = (\upsilon_F)^M_\delta(t_1).$$

Hence $B^M_\delta \cup B^M_{\delta'}$ is **NCID** of Y.

3.3 Magnified Translative Interpretation of Neutrosophic Cubic Subalgebra and Neutrosophic Cubic Ideal

In this section, we define the notion of neutrosophic cubic magnified translation **NCMT** and investigate some results.

Definition 3.3.1 Let $B = (\kappa_{T,I,F}, \upsilon_{T,I,F})$ be a **NCS** of Y and for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], \urcorner]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and for all $\delta \in [0,1]$. An object having the form $B^{M\,T}_{\delta\,\alpha,\beta,\gamma} = \{(\kappa_{T,I,F})^{M\,T}_{\delta\,\alpha,\beta,\gamma}, (\upsilon_{T,I,F})^{M\,T}_{\delta\,\alpha,\beta,\gamma}\}$ is said to be a **NCMT** of B, when $(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1) = \delta.\kappa_T(t_1) + \alpha$, $(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1) = \delta.\kappa_I(t_1) + \beta$, $(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1) = \delta.\kappa_F(t_1)$ - γ and $(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1) = \delta.\upsilon_T(t_1) + \alpha$, $(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1) = \delta.\upsilon_I(t_1) + \beta$, $(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1) = \delta.\upsilon_F(t_1)$-$\gamma$ for all $t_1 \in Y$.

Example 3.3.1 Let $Y = \{0,1,2\}$ be a BF-algebra as defined in Example 3.2.1. A **NCS** $B = (\kappa_{T,I,F}, \upsilon_{T,I,F})$ of Y is defined as

$$\kappa_T(t_1) = \begin{cases} [0.1, 0.3] & \text{if } t_1 = 0 \\ [0.4, 0.7] & \text{if otherwise} \end{cases}$$

$$\kappa_I(t_1) = \begin{cases} [0.2, 0.4] & \text{if } t_1 = 0 \\ [0.5, 0.7] & \text{if otherwise} \end{cases}$$

$$\kappa_F(t_1) = \begin{cases} [0.4, 0.6] & \text{if } t_1 = 0 \\ [0.5, 0.8] & \text{if otherwise} \end{cases}$$

and

$$\upsilon_T(t_1) = \begin{cases} 0.1 & \text{if } t_1 = 0 \\ 0.4 & \text{if otherwise} \end{cases}$$

$$\upsilon_I(t_1) = \begin{cases} 0.2 & \text{if } t_1 = 0 \\ 0.3 & \text{if otherwise} \end{cases}$$

$$\upsilon_F(t_1) = \begin{cases} 0.5 & \text{if } t_1 = 0 \\ 0.7 & \text{if otherwise.} \end{cases}$$

Then B is a neutrosophic cubic subalgebra, for $\upsilon_{T,I,F}$ choose $\delta = 0.1, \alpha = 0.02, \beta = 0.03, \gamma = 0.04$ and for $\kappa_{T,I,F}$ choose $\delta = [0.1,0.4], \alpha = [0.03,0.07], \beta = [0.04,0.08], \gamma = [0.02,0.06]$ then the mapping $B^{M\,T}_{(0.1)\,(\alpha,\beta,\gamma)}|Y \to [0,1]$ is given by

$$(\kappa_T)^{M}_{[0.1,0.4]\,[0.03,0.07]}(t_1) = \begin{cases} [0.04, 0.19] & \text{if } t_1 = 1 \\ [0.07, 0.35] & \text{if otherwise} \end{cases}$$

$$(\kappa_I)^{M\,T}_{[0.1,0.4]\,[0.04,0.08]}(t_1) = \begin{cases} [0.06, 0.24] & \text{if } t_1 = 1 \\ [0.09, 0.36] & \text{if otherwise} \end{cases}$$

$$(\kappa_F)^{M\,T}_{[0.1,0.4]\,[0.02,0.06]}(t_1) = \begin{cases} [0.02, 0.18] & \text{if } t_1 = 1 \\ [0.03, 0.26] & \text{if otherwise} \end{cases}$$

and

$$(\upsilon_T)^{M}_{0.1,0.02}(t_1) = \begin{cases} 0.03 & \text{if } t_1 = 1 \\ 0.06 & \text{if otherwise} \end{cases}$$

$$(\upsilon_I)^{M\,T}_{0.1,0.03}(t_1) = \begin{cases} 0.05 & \text{if } t_1 = 1 \\ 0.06 & \text{if otherwise} \end{cases}$$

$$(\upsilon_F)^{M\,T}_{0.1.0.04}(t_1) = \begin{cases} 0.01 & \text{if } t_1 = 1 \\ 0.03 & \text{if otherwise,} \end{cases}$$

which imply $(\kappa_T)^{M\,T}_{[0.1,0.4][0.03,0.07]}(t_1) = [0.1,0.4].\kappa_T(t_1) + [0.03,0.07]$, $(\kappa_I)^{M\,T}_{[0.1,0.4][0.04,0.08]}(t_1) = [0.1,0.4].\kappa_T(t_1) + [0.04,0.08]$, $(\kappa_F)^{M\,T}_{[0.1,0.4][0.02,0.06]}(t_1) = [0.1,0.4].\kappa_F(t_1) - [0.02,0.06]$ and $(\upsilon_T)^{M\,T}_{(0.1)(0.02)}(t_1) = (0.1).\upsilon_T(t_1) + 0.02$, $(\upsilon_I)^{M\,T}_{(0.1)(0.03)}(t_1) = (0.1).\upsilon_T(t_1) + 0.03$, $(\nu_F)^{M\,T}_{(0.1)\,(0.04)}(t_1) = (0.1).\nu_F(t_1) - 0.04$ for all $t_1 \in Y$. Hence $B^{M\,T}$ is a neutrosophic cubic magnified translation.

Theorem 3.3.1 Let B be a neutrosophic cubic subset of Y such that for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ꒶]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $\delta \in [0,1]$ and a mapping $B^{M\,T,I,F}_{\delta\,\alpha,\beta,\gamma}|Y \to [0,1]$ be a **NCMT** of B. If B is **NCSU** of Y, then $B^{M\,T,I,F}_{\delta\,\alpha,\beta,\gamma}$ is a **NCSU** of Y.

Proof. Let B be a neutrosophic cubic subset of Y such that for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ꒶]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $\delta \in [0,1]$ and a mapping $B^{M\,T,I,F}_{\delta\,\alpha,\beta,\gamma}|Y \to [0,1]$ be a **NCMT** of B. Suppose B is a **NCSU** of Y. Then $\kappa_T(t_1 * t_2) \geq \text{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}, \kappa_I(t_1 * t_2) \geq \text{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}, \kappa_F(t_1 * t_2) \geq \text{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$ and $\upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}, \upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}, \upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$. Now

$$(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2) = \delta.\kappa_T(t_1 * t_2) + \alpha$$

$$\geq \delta.\text{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\} + \alpha$$

$= \mathrm{rmin}\{\delta.\kappa_T(t_1) + \alpha, \delta.\kappa_T(t_2) + \alpha\}$

$(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2) = \mathrm{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1), (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$

$(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1), (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$,

$(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2) = \delta.\kappa_I(t_1 * t_2) + \beta$

$\geq \delta.\mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\} + \beta$

$= \mathrm{rmin}\{\delta.\kappa_I(t_1) + \beta, \delta.\kappa_I(t_2) + \beta\}$

$(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2) = \mathrm{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1), (\kappa_I)^{M\,T}_{\delta\,\beta}(t_2)\}$

$(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1), (\kappa_I)^{M\,T}_{\delta\,\beta}(t_2)\}$,

$(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2) = \delta.\kappa_F(t_1 * t_2) - \gamma$

$\geq \delta.\mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\} - \gamma$

$= \mathrm{rmin}\{\delta.\kappa_F(t_1) - \gamma, \delta.\kappa_F(t_2) - \gamma\}$

$(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2) = \mathrm{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1), (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$

$(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2) \geq \mathrm{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1), (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$

and

$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2) = \delta.\upsilon_T(t_1 * t_2) + \alpha$

$\leq \delta.\max\{\upsilon_T(t_1), \upsilon_T(t_2)\} + \alpha$

$= \max\{\delta.\upsilon_T(t_1) + \alpha, \delta.\upsilon_T(t_2) + \alpha\}$

$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2) = \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1), (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$

$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2) \leq \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1), (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$,

$(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2) = \delta.\upsilon_I(t_1 * t_2) + \beta$

$\leq \delta.\max\{\upsilon_I(t_1), \upsilon_I(t_2)\} + \beta$

$= \max\{\delta.\upsilon_I(t_1) + \beta, \delta.\upsilon_I(t_2) + \beta\}$

$(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2) = \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1), (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_2)\}$

$(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2) \leq \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1), (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_2)\}$,

$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2) = \delta.\upsilon_F(t_1 * t_2) - \gamma$

$\leq \delta.\max\{\upsilon_F(t_1), \upsilon_F(t_2)\} - \gamma$

$= \max\{\delta.\upsilon_F(t_1) - \gamma, \delta.\upsilon_F(t_2) - \gamma\}$

$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2) = \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1), (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$

$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2) \leq \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1), (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$.

Hence **NCMT** $B^{MT}_{\delta\,\alpha,\beta,\gamma}$ is a **NCSU** of Y.

Theorem 3.3.2 Let B be a **NCS** of Y such that and for $\kappa_{T,I,F}$, $\alpha,\beta \in [[0,0],ℸ]$ and $\gamma \in [[0,0],¥]$, where for $\upsilon_{T,I,F}$, $\alpha,\beta \in [0,\Gamma]$ and $\gamma \in [0,£]$ and $\delta \in [0,1]$ and a mapping $B^{MT}_{\delta\,\alpha,\beta,\gamma}: Y \to [0,1]$ be a **NCMT** of B. If $B^{MT}_{\delta\,\alpha,\beta,\gamma}$ is **NCSU** of Y. Then B is a **NCSU** of Y.

Proof. Let B be a neutrosophic cubic subset of Y, where $\alpha,\beta,\gamma \in [0,¥]$, $\delta \in [0,1]$ and a mapping $B^{MT}_{\delta\,\alpha,\beta,\gamma}: Y \to [0,1]$ be a **NCMT** of B. Suppose $B^{MT}_{\delta\,\alpha,\beta,\gamma} = \{(\kappa_B)^{MT,I,F}_{\delta\,\alpha,\beta,\gamma}, (\upsilon_B)^{MT,I,F}_{\delta\,\alpha,\beta,\gamma}\}$ is a **NCSU** of Y, then

$$\delta.\kappa_T(t_1 * t_2) + \alpha = (\kappa_T)^{MT}_{\delta\,\alpha}(t_1 * t_2)$$

$$\geq \text{rmin}\{(\kappa_T)^{MT}_{\delta\,\alpha}(t_1), (\kappa_T)^{MT}_{\delta\,\alpha}(t_2)\}$$

$$= \text{rmin}\{\delta.\kappa_T(t_1) + \alpha, \delta.\kappa_T(t_2) + \alpha\}$$

$$\delta.\kappa_T(t_1 * t_2) + \alpha = \delta.\text{rmin}\{\kappa_T(t_2), \kappa_T(t_1)\} + \alpha,$$

$$\delta.\kappa_I(t_1 * t_2) + \beta = (\kappa_I)^{MT}_{\delta\,\beta}(t_1 * t_2)$$

$$\geq \text{rmin}\{(\kappa_I)^{MT}_{\delta\,\beta}(t_1), (\kappa_I)^{MT}_{\delta\,\beta}(t_2)\}$$

$$= \text{rmin}\{\delta.\kappa_I(t_1) + \beta, \delta.\kappa_I(t_2) + \beta\}$$

$$\delta.\kappa_I(t_1 * t_2) + \beta = \delta.\text{rmin}\{\kappa_I(t_2), \kappa_I(t_1)\} + \beta,$$

$$\delta.\kappa_F(t_1 * t_2) - \gamma = (\kappa_F)^{MT}_{\delta\,\gamma}(t_1 * t_2)$$

$$\geq \text{rmin}\{(\kappa_F)^{MT}_{\delta\,\gamma}(t_1), (\kappa_F)^{MT}_{\delta\,\gamma}(t_2)\}$$

$$= \text{rmin}\{\delta.\kappa_F(t_1) - \gamma, \delta.\kappa_F(t_2) - \gamma\}$$

$$\delta.\kappa_F(t_1 * t_2) - \gamma = \delta.\text{rmin}\{\kappa_F(t_2), \kappa_F(t_1)\} - \gamma,$$

and

$$\delta.\upsilon_T(t_1 * t_2) + \alpha = (\upsilon_T)^{MT}_{\delta\,\alpha}(t_1 * t_2)$$

$$\leq \max\{(\upsilon_T)^{MT}_{\delta\,\alpha}(t_1), (\upsilon_T)^{MT}_{\delta\,\alpha}(t_2)\}$$

$$= \max\{\delta.\upsilon_T(t_1) + \alpha, \delta.\upsilon_T(t_2) + \alpha\}$$

$$\delta.\upsilon_T(t_1 * t_2) + \alpha = \delta.\max\{\upsilon_T(t_2), \upsilon_T(t_1)\} + \alpha,$$

$$\delta.\upsilon_I(t_1 * t_2) + \beta = (\upsilon_I)^{MT}_{\delta\,\beta}(t_1 * t_2)$$

$$\leq \max\{(\upsilon_I)^{MT}_{\delta\,\beta}(t_1), (\upsilon_I)^{MT}_{\delta\,\beta}(t_2)\}$$

$$= \max\{\delta.\upsilon_I(t_1) + \beta, \delta.\upsilon_I(t_2) + \beta\}$$

$$\delta.\upsilon_I(t_1 * t_2) + \beta = \delta.\max\{\upsilon_I(t_2), \upsilon_I(t_1)\} + \beta,$$

$\delta.\upsilon_F(t_1 * t_2) - \gamma = (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2)$

$\leq \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1), (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$

$= \max\{\delta.\upsilon_F(t_1) - \gamma, \delta.\upsilon_F(t_2) - \gamma\}$

$\delta.\upsilon_F(t_1 * t_2) - \gamma = \delta.\max\{\upsilon_F(t_2), \upsilon_F(t_1)\} - \gamma,$

which imply $\kappa_T(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_T(t_1), \kappa_T(t_2)\}$, $\kappa_I(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_I(t_1), \kappa_I(t_2)\}$, $\kappa_F(t_1 * t_2) \geq \mathrm{rmin}\{\kappa_F(t_1), \kappa_F(t_2)\}$ and $\upsilon_T(t_1 * t_2) \leq \max\{\upsilon_T(t_1), \upsilon_T(t_2)\}$,$\upsilon_I(t_1 * t_2) \leq \max\{\upsilon_I(t_1), \upsilon_I(t_2)\}$,$\upsilon_F(t_1 * t_2) \leq \max\{\upsilon_F(t_1), \upsilon_F(t_2)\}$ for all $t_1, t_2 \in Y$. Hence B is a **NCSU** of Y.

Theorem 3.3.3 If B is a **NCID** of Y. Then **NCMT** $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ of B is a **NCID** of Y for all $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $\delta \in (0,1]$.

Proof. Suppose $B = (\kappa_{T,I,F}, \upsilon_{T,I,F})$ is a **NCID** of Y. Then

$(\kappa_T)^{M\,T}_{\delta\,\alpha}(0) = \delta.\kappa_T(0) + \alpha$

$\geq \delta.\kappa_T(t_1) + \alpha$

$(\kappa_T)^{M\,T}_{\delta\,\alpha}(0) = (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1),$

$(\kappa_I)^{M\,T}_{\delta\,\beta}(0) = \delta.\kappa_I(0) + \beta$

$\geq \delta.\kappa_I(t_1) + \beta$

$(\kappa_I)^{M\,T}_{\delta\,\beta}(0) = (\kappa_I)^{M\,T}_{\delta\,\beta}(t_1),$

$(\kappa_F)^{M\,T}_{\delta\,\gamma}(0) = \delta.\kappa_F(0) - \gamma$

$\geq \delta.\kappa_F(t_1) - \gamma$

$(\kappa_F)^{M\,T}_{\delta\,\gamma}(0) = (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1)$

and

$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(0) = \delta.\upsilon_T(0) + \alpha$

$\leq \delta.\upsilon_T(t_1) + \alpha$

$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(0) = (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1),$

$(\upsilon_I)^{M\,T}_{\delta\,\beta}(0) = \delta.\upsilon_I(0) + \beta$

$\leq \delta.\upsilon_I(t_1) + \beta$

$(\upsilon_I)^{M\,T}_{\delta\,\beta}(0) = (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1),$

$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(0) = \delta.\upsilon_F(0) - \gamma$

$\leq \delta.\upsilon_F(t_1) - \gamma$

$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(0) = (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1)$

Now

$$(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1) = \delta.\kappa_T(t_1) + \alpha$$

$$\geq \delta.\,\text{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\} + \alpha$$

$$= \text{rmin}\{\delta.\kappa_T(t_1 * t_2) + \alpha, \delta.\kappa_T(t_2) + \alpha\}$$

$$(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1) = \text{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2), (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$$

$$\Rightarrow (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1) \geq \text{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2), (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_2)\},$$

$$(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1) = \delta.\kappa_I(t_1) + \beta$$

$$\geq \delta.\,\text{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\} + \beta$$

$$= \text{rmin}\{\delta.\kappa_I(t_1 * t_2) + \beta, \delta.\kappa_I(t_2) + \beta\}$$

$$(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1) = \text{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2), (\kappa_I)^{M\,T}_{\delta\,\beta}(t_2)\}$$

$$\Rightarrow (\kappa_I)^{M\,T}_{\delta\,\beta}(t_1) \geq \text{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2), (\kappa_I)^{M\,T}_{\delta\,\beta}(t_2)\},$$

$$(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1) = \delta.\kappa_F(t_1) - \gamma$$

$$\geq \delta.\,\text{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\} - \gamma$$

$$= \text{rmin}\{\delta.\kappa_F(t_1 * t_2) - \gamma, \delta.\kappa_F(t_2) - \gamma\}$$

$$(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1) = \text{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2), (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$$

$$\Rightarrow (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1) \geq \text{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2), (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$$

and

$$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1) = \delta.\upsilon_T(t_1) + \alpha$$

$$\leq \delta.\max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\} + \alpha$$

$$= \max\{\delta.\upsilon_T(t_1 * t_2) + \alpha, \delta.\upsilon_T(t_2) + \alpha\}$$

$$(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1) = \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2), (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$$

$$\Rightarrow (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1) \leq \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2), (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_2)\},$$

$$(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1) = \delta.\upsilon_I(t_1) + \beta$$

$$\leq \delta.\max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\} + \beta$$

$$= \max\{\delta.\upsilon_I(t_1 * t_2) + \beta, \delta.\upsilon_I(t_2) + \beta\}$$

$$(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1) = \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2), (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_2)\}$$

$$\Rightarrow (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1) \leq \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2), (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_2)\},$$

$$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1) = \delta.\upsilon_F(t_1) - \gamma$$

$$\leq \delta.\max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\} - \gamma$$

$$= \max\{\delta.\upsilon_F(t_1 * t_2) - \gamma, \delta.\upsilon_F(t_2) - \gamma\}$$

$$(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1) = \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2), (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$$

$$\Rightarrow (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1) \leq \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2), (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_2)\},$$

for all $t_1, t_2 \in Y$ and all for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ˥]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $\delta \in (0,1]$. Hence $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ of B is a **NCID** of Y.

Theorem 3.3.3 If B is a neutrosophic cubic set of Y such that **NCMT** $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ of B is a **NCID** of Y for all for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ˥]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $\delta \in (0,1]$, then B is a **NCID** of Y.

Proof. Suppose **NCMT** $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ is a **NCID** of Y for some $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ˥]$ and $\gamma \in [[0,0], ¥]$, where for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$ and $\gamma \in [0, £]$ and $\delta \in (0,1]$ and $t_1, t_2 \in Y$. Then

$$\delta.\kappa_T(0) + \alpha = (\kappa_T)^{M\,T}_{\delta\,\alpha}(0)$$

$$\geq (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1)$$

$$\delta.\kappa_T(0) + \alpha = \delta.\kappa_T(t_1) + \alpha,$$

$$\delta.\kappa_I(0) + \beta = (\kappa_I)^{M\,T}_{\delta\,\beta}(0)$$

$$\geq (\kappa_I)^{M\,T}_{\delta\,\beta}(t_1)$$

$$\delta.\kappa_I(0) + \beta = \delta.\kappa_I(t_1) + \beta,$$

$$\delta.\kappa_F(0) - \gamma = (\kappa_F)^{M\,T}_{\delta\,\gamma}(0)$$

$$\geq (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1)$$

$$\delta.\kappa_F(0) - \gamma = \delta.\kappa_F(t_1) - \gamma,$$

and

$$\delta.\upsilon_T(0) + \alpha = (\upsilon_T)^{M\,T}_{\delta\,\alpha}(0)$$

$$\leq (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1)$$

$$\delta.\upsilon_T(0) + \alpha = \delta.\upsilon_T(t_1) + \alpha,$$

$$\delta.\upsilon_I(0) + \beta = (\upsilon_I)^{M\,T}_{\delta\,\beta}(0)$$

$$\leq (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1)$$

$$\delta.\upsilon_I(0) + \beta = \delta.\upsilon_I(t_1) + \beta,$$

$$\delta.\upsilon_F(0) - \gamma = (\upsilon_F)^{M\,T}_{\delta\,\gamma}(0)$$
$$\leq (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1)$$
$$\delta.\upsilon_F(0) - \gamma = \delta.\upsilon_F(t_1) - \gamma,$$

which imply $\kappa_T(0) \geq \kappa_T(t_1), \kappa_I(0) \geq \kappa_I(t_1), \kappa_F(0) \geq \kappa_F(t_1)$ and $\upsilon_T(0) \leq \upsilon_T(t_1)$, $\upsilon_I(0) \leq \upsilon_I(t_1)$, $\upsilon_F(0) \leq \upsilon_F(t_1)$. Now, we have

$$\delta.\kappa_T(t_1) + \alpha = (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1)$$
$$\geq \text{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2), (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$$
$$= \text{rmin}\{\delta.\kappa_T(t_1 * t_2) + \alpha, \delta.\kappa_T(t_2) + \alpha\}$$
$$\delta.\kappa_T(t_1) + \alpha = \delta.\text{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\} + \alpha,$$

$$\delta.\kappa_I(t_1) + \beta = (\kappa_I)^{M\,T}_{\delta\,\beta}(t_1)$$
$$\geq \text{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2), (\kappa_I)^{M\,T}_{\delta\,\beta}(t_2)\}$$
$$= \text{rmin}\{\delta.\kappa_I(t_1 * t_2) + \beta, \delta.\kappa_I(t_2) + \beta\}$$
$$\delta.\kappa_I(t_1) + \beta = \delta.\text{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\} + \beta,$$

$$\delta.\kappa_F(t_1) - \gamma = (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1)$$
$$\geq \text{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2), (\kappa_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$$
$$= \text{rmin}\{\delta.\kappa_F(t_1 * t_2) - \gamma, \delta.\kappa_F(t_2) - \gamma\}$$
$$\delta.\kappa_F(t_1) - \gamma = \delta.\text{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\} - \gamma$$

and

$$\delta.\upsilon_T(t_1) + \alpha = (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1)$$
$$\leq \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1 * t_2), (\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_2)\}$$
$$= \max\{\delta.\upsilon_T(t_1 * t_2) + \alpha, \delta.\upsilon_T(t_2) + \alpha\}$$
$$\delta.\upsilon_T(t_1) + \alpha = \delta.\max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\} + \alpha,$$

$$\delta.\upsilon_I(t_1) + \beta = (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1)$$
$$\leq \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1 * t_2), (\upsilon_I)^{M\,T}_{\delta\,\beta}(t_2)\}$$
$$= \max\{\delta.\upsilon_I(t_1 * t_2) + \beta, \delta.\upsilon_I(t_2) + \beta\}$$
$$\delta.\upsilon_I(t_1) + \beta = \delta.\max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\} + \beta,$$

$$\delta.\upsilon_F(t_1) - \gamma = (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1)$$
$$\leq \max\{(\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_1 * t_2), (\upsilon_F)^{M\,T}_{\delta\,\gamma}(t_2)\}$$
$$= \max\{\delta.\upsilon_F(t_1 * t_2) - \gamma, \delta.\upsilon_F(t_2) - \gamma\}$$

$$\delta.\upsilon_F(t_1) - \gamma = \delta.\max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\} - \gamma$$

which imply $\kappa_T(t_1) \geq \mathrm{rmin}\{\kappa_T(t_1 * t_2), \kappa_T(t_2)\}$, $\kappa_I(t_1) \geq \mathrm{rmin}\{\kappa_I(t_1 * t_2), \kappa_I(t_2)\}$, $\kappa_F(t_1) \geq \mathrm{rmin}\{\kappa_F(t_1 * t_2), \kappa_F(t_2)\}$ and $\upsilon_T(t_1) \leq \max\{\upsilon_T(t_1 * t_2), \upsilon_T(t_2)\}$, $\upsilon_I(t_1) \leq \max\{\upsilon_I(t_1 * t_2), \upsilon_I(t_2)\}$, $\upsilon_F(t_1) \leq \max\{\upsilon_F(t_1 * t_2), \upsilon_F(t_2)\}$ for all $t_1, t_2 \in Y$. Hence B is a **NCID** of Y.

Theorem 3.3.4 Intersection of any two **NCMT** of a **NCID** B of Y is a **NCID** of Y.

Proof. Suppose $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ and $B^{M\,T}_{\delta'\alpha',\beta',\gamma'}$ are two **NCMTs** of **NCID** B ofY, where for $B^{M\,T}_{\alpha,\beta,\gamma}$, for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ]$, $\gamma \in [[0,0], ¥]$, for $\upsilon_{T,I,F}$, $\alpha, \beta \in [0, \Gamma]$, $\gamma \in [0, £]$ and for $B^{T}_{\alpha',\beta',\gamma'}$, for $\kappa_{T,I,F}$ $\alpha', \beta' \in [[0,0], ℸ]$, $\gamma' \in [[0,0], ¥]$, for $\upsilon_{T,I,F}$, $\alpha', \beta' \in [0, \Gamma]$, $\gamma' \in [0, £]$. Assume $\alpha \leq \alpha', \beta \leq \beta', \gamma \leq \gamma'$ and $\delta = \delta'$. Then by Theorem 3.3.3, $B^{M\,T}_{\delta\,\alpha,\beta,\gamma}$ and $B^{M\,T}_{\delta'\,\alpha',\beta',\gamma'}$ are **NCIDs** of Y. So

$$\begin{aligned}
&((\kappa_T)^{M\,T}_{\delta\,\alpha} \cap (\kappa_T)^{M\,T}_{\delta'\,\alpha'})(t_1) = \mathrm{rmin}\{(\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1), (\kappa_T)^{M\,T}_{\delta'\,\alpha'}(t_1)\}\\
&= \mathrm{rmin}\{\delta.\kappa_T(t_1) + \alpha, \delta'.\kappa_T(t_1) + \alpha'\}\\
&= \delta.\kappa_T(t_1) + \alpha\\
&((\kappa_T)^{M\,T}_{\delta\,\alpha} \cap (\kappa_T)^{M\,T}_{\delta'\,\alpha'})(t_1) = (\kappa_T)^{M\,T}_{\delta\,\alpha}(t_1),
\end{aligned}$$

$$\begin{aligned}
&((\kappa_I)^{M\,T}_{\delta\,\beta} \cap (\kappa_I)^{M\,T}_{\delta'\,\beta'})(t_1) = \mathrm{rmin}\{(\kappa_I)^{M\,T}_{\delta\,\beta}(t_1), (\kappa_I)^{M\,T}_{\delta'\,\beta'}(t_1)\}\\
&= \mathrm{rmin}\{\delta.\kappa_I(t_1) + \beta, \delta'.\kappa_I(t_1) + \beta'\}\\
&= \delta.\kappa_I(t_1) + \beta\\
&((\kappa_I)^{M\,T}_{\delta\,\beta} \cap (\kappa_I)^{M\,T}_{\delta'\,\beta'})(t_1) = (\kappa_I)^{M\,T}_{\delta\,\beta}(t_1),
\end{aligned}$$

$$\begin{aligned}
&((\kappa_F)^{M\,T}_{\delta\,\gamma} \cap (\kappa_F)^{M\,T}_{\delta'\,\gamma'})(t_1) = \mathrm{rmin}\{(\kappa_F)^{M\,T}_{\delta\,\gamma}(t_1), (\kappa_F)^{M\,T}_{\delta'\,\gamma'}(t_1)\}\\
&= \mathrm{rmin}\{\delta.\kappa_F(t_1) - \gamma, \delta'.\kappa_F(t_1) - \gamma'\}\\
&= \delta'.\kappa_F(t_1) - \gamma'\\
&((\kappa_F)^{M\,T}_{\delta\,\gamma} \cap (\kappa_F)^{M\,T}_{\delta'\,\gamma'})(t_1) = (\kappa_F)^{M\,T}_{,\delta'\,\gamma'}(t_1)
\end{aligned}$$

and

$$\begin{aligned}
&((\upsilon_T)^{M\,T}_{\delta\,\alpha} \cap (\upsilon_T)^{M\,T}_{\delta'\,\alpha'})(t_1) = \max\{(\upsilon_T)^{M\,T}_{\delta\,\alpha}(t_1), (\upsilon_T)^{M\,T}_{\delta'\,\alpha'}(t_1)\}\\
&= \max\{\delta.\upsilon_T(t_1) + \alpha, \delta'.\upsilon_T(t_1) + \alpha'\}\\
&= \delta'.\upsilon_T(t_1) + \alpha'\\
&((\upsilon_T)^{M\,T}_{\delta\,\alpha} \cap (\upsilon_T)^{M\,T}_{\delta'\,\alpha'})(t_1) = (\upsilon_T)^{M\,T}_{\delta'\alpha'}(t_1),
\end{aligned}$$

$$\begin{aligned}
&((\upsilon_I)^{M\,T}_{\delta\,\beta} \cap (\upsilon_I)^{M\,T}_{\delta'\,\beta'})(t_1) = \max\{(\upsilon_I)^{M\,T}_{\delta\,\beta}(t_1), (\upsilon_I)^{M\,T}_{\delta'\,\beta'}(t_1)\}\\
&= \max\{\delta.\upsilon_I(t_1) + \beta, \delta'.\upsilon_I(t_1) + \beta'\}\\
&= \delta'.\upsilon_I(t_1) + \beta'\\
&((\upsilon_I)^{M\,T}_{\delta\,\beta} \cap (\upsilon_I)^{M\,T}_{\delta'\beta'})(t_1) = (\upsilon_I)^{M\,T}_{\delta'\,\beta'}(t_1),
\end{aligned}$$

$((\upsilon_F)_{\delta\,\gamma}^{M\,T} \cap (\upsilon_F)_{\delta'\,\gamma'}^{M\,T})(t_1) = \max\{(\upsilon_F)_{\delta\,\gamma}^{M\,T}(t_1), (\upsilon_F)_{\delta'\,\gamma'}^{M\,T}(t_1)\}$

$= \max\{\delta.\upsilon_F(t_1) - \gamma, \delta'.\upsilon_F(t_1) - \gamma'\}$

$= \delta.\upsilon_F(t_1) - \gamma$

$((\upsilon_F)_{\delta\,\gamma}^{M\,T} \cap (\upsilon_F)_{\delta'\,\gamma'}^{M\,T})(t_1) = (\upsilon_F)_{\delta\,\gamma}^{M\,T}(t_1).$

Hence $B_{\delta\,\alpha,\beta,\gamma}^{M\,T} \cap B_{\delta'\,\alpha',\beta',\gamma'}^{M\,T}$ is **NCID** of Y.

Theorem 3.3.5 Union of any two **NCMT** $B_{\delta\,\alpha,\beta,\gamma}^{M\,T}$ of a **NCID** B of Y is a **NCID** of Y.

Proof. Suppose $B_{\delta\,\alpha,\beta,\gamma}^{M\,T}$ and $B_{\delta'\,\alpha',\beta',\gamma'}^{M\,T}$ are two NCMTs of NCID B of Y , where for $B_{\alpha,\beta,\gamma}^{M\,T}$, for $\kappa_{T,I,F}$, $\alpha, \beta \in [[0,0], ℸ], \gamma \in [[0,0], ¥]$, for $\upsilon_{T,I,F}, \alpha, \beta \in [0, \Gamma], \gamma \in [0, £]$ and for $B_{\alpha',\beta',\gamma'}^{T}$, for $\kappa_{T,I,F}$ $\alpha', \beta' \in [[0,0], ℸ], \gamma' \in [[0,0], ¥]$, for $\upsilon_{T,I,F}, \alpha', \beta' \in [0, \Gamma]$, $\gamma' \in [0, £]$. Assume $\alpha \geq \alpha', \beta \geq \beta', \gamma \geq \gamma'$ and $\delta = \delta'$. Then by Theorem 3.3.3, $B_{\delta\,\alpha,\beta,\gamma}^{M\,T}$ and $B_{\delta'\,\alpha',\beta',\gamma'}^{M\,T}$ are NCIDs of Y. So

$((\kappa_T)_{\delta\,\alpha}^{M\,T} \cup (\kappa_T)_{\delta'\,\alpha'}^{M\,T})(t_1) = \mathrm{rmax}\{(\kappa_T)_{\delta\,\alpha}^{M\,T}(t_1), (\kappa_T)_{\delta'\,\alpha'}^{M\,T}(t_1)\}$

$= \mathrm{rmax}\{\delta.\kappa_T(t_1) + \alpha, \delta'.\kappa_T(t_1) + \alpha'\}$

$= \delta.\kappa_T(t_1) + \alpha$

$((\kappa_T)_{\delta\,\alpha}^{M\,T} \cup (\kappa_T)_{\delta'\,\alpha'}^{M\,T})(t_1) = (\kappa_T)_{\delta\,\alpha}^{M\,T}(t_1),$

$((\kappa_I)_{\delta\,\beta}^{M\,T} \cup (\kappa_I)_{\delta'\,\beta'}^{M\,T})(t_1) = \mathrm{rmax}\{(\kappa_I)_{\delta\,\beta}^{M\,T}(t_1), (\kappa_I)_{\delta'\,\beta'}^{M\,T}(t_1)\}$

$= \mathrm{rmax}\{\delta.\kappa_I(t_1) + \beta, \delta'.\kappa_I(t_1) + \beta'\}$

$= \delta.\kappa_I(t_1) + \beta$

$((\kappa_I)_{\delta\,\beta}^{M\,T} \cup (\kappa_I)_{\delta'\,\beta'}^{M\,T})(t_1) = (\kappa_I)_{\delta\,\beta}^{M\,T}(t_1),$

$((\kappa_F)_{\delta\,\gamma}^{M\,T} \cup (\kappa_F)_{\delta'\,\gamma'}^{M\,T})(t_1) = \mathrm{rmax}\{(\kappa_F)_{\delta\,\gamma}^{M\,T}(t_1), (\kappa_F)_{\delta'\,\gamma'}^{M\,T}(t_1)\}$

$= \mathrm{rmax}\{\delta.\kappa_F(t_1) - \gamma, \delta'.\kappa_F(t_1) - \gamma'\}$

$= \delta'.\kappa_F(t_1) - \gamma'\}$

$((\kappa_F)_{\delta\,\gamma}^{M\,T} \cup (\kappa_F)_{\delta'\,\gamma'}^{M\,T})(t_1) = (\kappa_F)_{\delta'\,\gamma'}^{M\,T}(t_1)$

and

$((\upsilon_T)_{\delta\,\alpha}^{M\,T} \cup (\upsilon_T)_{\delta'\,\alpha'}^{M\,T})(t_1) = \min\{(\upsilon_T)_{\delta\,\alpha}^{M\,T}(t_1), (\upsilon_T)_{\delta'\,\alpha'}^{M\,T}(t_1)\}$

$= \min\{\delta.\upsilon_T(t_1) + \alpha, \delta'.\upsilon_T(t_1) + \alpha'\}$

$= \delta'.\upsilon_T(t_1) + \alpha'$

$((\upsilon_T)_{\delta\,\alpha}^{M\,T} \cup (\upsilon_T)_{\delta'\,\alpha'}^{M\,T})(t_1) = (\upsilon_T)_{\delta'\,\alpha'}^{M\,T}(t_1),$

$((\upsilon_I)_{\delta\,\beta}^{M\,T} \cup (\upsilon_I)_{\delta'\,\beta'}^{M\,T})(t_1) = \min\{(\upsilon_I)_{\delta\,\beta}^{M\,T}(t_1), (\upsilon_I)_{\delta'\,\beta'}^{M\,T}(t_1)\}$

$= \min\{\delta.\upsilon_I(t_1) + \beta, \delta'.\upsilon_I(t_1) + \beta'\}$

$= \delta'.\upsilon_I(t_1) + \beta'$

$$((\upsilon_I)^{MT}_{\delta\beta} \cup (\upsilon_I)^{MT}_{\delta'\beta'})(t_1) = (\upsilon_I)^{MT}_{\delta'\beta'}(t_1),$$

$$((\upsilon_F)^{MT}_{\delta\gamma} \cup (\upsilon_F)^{MT}_{\delta'\gamma'})(t_1) = \min\{(\upsilon_F)^{MT}_{\delta\gamma}(t_1), (\upsilon_F)^{MT}_{\delta'\gamma'}(t_1)\}$$
$$= \min\{\delta.\upsilon_F(t_1) - \gamma, \delta'.\upsilon_F(t_1) - \gamma'\}$$
$$= \delta.\upsilon_F(t_1) - \gamma$$
$$((\upsilon_F)^{MT}_{\delta\gamma} \cup (\upsilon_F)^{MT}_{\delta'\gamma'})(t_1) = (\upsilon_F)^{MT}_{\delta\gamma}(t_1).$$

Hence $B^{MT}_{\delta\,\alpha,\beta,\gamma} \cup B^{MT}_{\delta'\,\alpha',\beta',\gamma'}$ is **NCID** of Y.

4. Conclusion

In this paper, we defined neutrosophic cubic translation,, neutrosophic cubic multiplication and neutrosophic cubic magnified translation for neutrosophic cubic set on BF-algebra. We provided the new sort of different conditions for neutrosophic cubic translation, neutrosophic cubic multiplication and neutrosophic cubic magnified translation and proved with examples. Moreover, for better understanding we investigated many results for NCT, NCM and NCMT using the subalgebra and ideals. For future work, translation and multiplication can be applied on neutrosophic cubic soft set and T-neutrosophic cubic set.

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A new distance measure for trapezoidal fuzzy neutrosophic numbers based on the centroids

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Abstract: Distance measure is a numerical measurement of the distance between any two objects. The aim of this paper is to propose a new distance measure for trapezoidal fuzzy neutrosophic numbers based on the centroids with graphical representation. In addition, the metric properties of the proposed measure are examined in detail. A decision making problem also has been solved using the proposed distance measure for a software selection process. comparative analysis has been done with the existing methods to show the potential of the proposed distance measure and various forms of trapezoidal fuzzy neutrosophic number have been listed out to show the uniqueness of the proposed graphical representation. Further, advantages of the proposed distance measure have been given.

Keywords: trapezoidal fuzzy neutrosophic numbers; centroids; distance measure

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1-Introduction

Zadeh introduced a mathematical frame work called fuzzy set [43] which plays a very significant role in many aspects of science. Intuitionistic fuzzy set is the generalization of the Zadeh's fuzzy set which was presented by Atanassov [3]. Later, triangular intuitionistic fuzzy sets was developed by Liu and Yuan [22] which is based on the combination of triangular fuzzy numbers and intuitionistic fuzzy sets. The fundamental characteristic of the triangular intuitionistic fuzzy set is that the values of its membership function and non-membership function are triangular fuzzy numbers rather than exact numbers. Furthermore, Ye [38] extended the triangular intuitionistic fuzzy set to the trapezoidal intuitionistic fuzzy set, where its fundamental characteristic is that the values of its membership function and non-membership function are trapezoidal fuzzy numbers rather than triangular fuzzy numbers, and proposed the trapezoidal intuitionistic fuzzy prioritized weighted averaging (TIFPWA) operator and trapezoidal intuitionistic fuzzy prioritized weighted geometric (TIFPWG) operator and their multi-criteria decision-making method, in which the criteria are in different priority level. Recently, Wang et al. [35] introduced a single-valued neutrosophic set, which is a subclass of a neutrosophic set presented by Smarandache [30], as a generalization of the classic set, fuzzy set and intuitionistic fuzzy set. The single-valued neutrosophic set can independently express truth-membership degree, indeterminacy-membership degree and falsity-membership degree and deal with incomplete, indeterminate and inconsistent information. All the factors described by the single-valued neutrosophic set are very suitable for human thinking due to the imperfection of knowledge that human receives or observes from the external world. For example, for a given proposition ''Movie X would be hit,'' in this situation human brain certainly cannot generate precise answers in terms of yes or no, as indeterminacy is the sector of unawareness of a proposition's value between truth and falsehood. Obviously, the neutrosophic components are best fit in the representation of indeterminacy and inconsistent information, while the intuitionistic fuzzy set cannot represent and handle indeterminacy and inconsistent information. Hence, the single-valued neutrosophic set has been a rapid development and a wide range of applications [39, 40]. Ye [42] introduced the trapezoidal neutrosophic set and its application to multiple attribute decision-making. Cui and Ye [10], Donghai et al. [16], Ebadi et al. [17], Guha and Chakraborty [18], Hajjari [19], Nayagam et al. [25], Rouhparvar et al. [29], Wu [37], Ye [40], Zou et al. [45] and more researchers have shown interest on decision making problem using distance measures. Weighted projection measure, the combination of angle cosine and weighted projection measure,similarity measure, hybrid vector similarity measure of single valued neutrosophic set and interval valued neutrosophic set, outranking strategy, complete ranking, new ranking function have been introduced so far under fuzzy, intuitionistic fuzzy and neutrosophic environments and applied in decision making problem. The rest of the paper is organized as follows. In section 2, literature review is given. In section 3, basic concepts are presented for better understanding. In section 4, proposed a new distnace measure and its graphical representation, and derived its properties in detail. In section 5, new methodology is described for a decision making process using the proposed measure. In section 6, a numerical example is using the proposed methodology to choose the best software system. In section 7, comparative analysis has been done with the existing methods and various forms of trapezoidal fuzzy neutrosophic numbers have been listed out to ahow the uniqueness of the proposed graphical representation. In section 8, advantages of the proposed measure are given. In section 9, conclusion of the present work is given with the future direction.

2-Literature Review

The authors of, Ahmad et al. [1] proposed a similarity measure based on the distance and set theory for generalized trapezoidal fuzzy numbers. Allahviranloo et al. [2] contributed a new distance measure and ranking method for generalized trapezoidal fuzzy numbers. Atanassov [3] introduced intuitionistic fuzzy sets. Azman and Abdullah [4] proposed a novel centroid method for trapezoidal fuzzy numbers for ranking. Biswas et al. [6] solved a decision making problem using expected value of neutrosophic trapezoidal numbers. Biswas et al. [6] solved a decision making problem using distance measure under interval trapezoidal neutrosophic numbers. Bolos et al. [7] designed the performance indicators of financial assets using neutrosophic fuzzy numbers. Bora and Gupta [8] studied the reaction of distance measure on the work of K-Means algorithm Matlab. Chakraborty et al. [9] presented different forms of trapezoidal neutrosophic number and deneutrosophication techniques. Cui and Ye [10] proposed logarithmic similarity measure and applied in medical diagnosis under dynamic neutrosophic cubic setting. Darehmiraki [11] introduced a new ranking methodology to solve linear programming problem. Das and De [12] introduced a new distance measure for the ranking IFNs. Das and Guha [13] introduced a ranking method for IFN using the point of centroid. Deli and Oztaurk [14] introduced a defuzzification method and applied in a decision-making problem for single valued trapezoidal neutrosophic numbers. Dhar et al. [15] indicated square neutrosophic fuzzy matrices. Donghai et al. [16] proposed a new similarity measure and distance measure between hesitant linguisticterm sets and applied the proposed concepts in a decision making problem. Ebadi et al. [17] proposed a novel distance measure for trapezoidal fuzzy numbers. Guha and Chakraborty [18] contributed a theoretical development of distance measure for intuitionistic fuzzy numbers (IFNs). Hajjari [19] conferred a new distance measure for Trapezoidal fuzzy numbers. Huang and Wu [20] presented equivalent forms of the triangle inequalities in fuzzy metric spaces. Liang et al. [21] proposed an integrated approach under a single valued trapezoidal neutrosophic environment. Liu and Yuan [22] prospected fuzzy number of intuitionistic fuzzy set. Llopis and Micheli [23] rectified a state of conflict in the sequence of input images. Minculete and Paltanea [24] introduced an enhanced estimates for the triangle inequality. Nayagam et al. [25] contributed a complete ranking of IFNs. Pardha Saradhi et al. [26] presented ordering of IFNs using centroids of centroids. Ravi Shankar et al. [27] developed a new ranking formula using centroid of centroids for fuzzy numbers and applied in a fuzzy critical path method. Rezvani [28] proposed a new ranking exponential formula using median value for trapezoidal fuzzy numbers. Rouhparvar et al. [29] introduced a novel fuzzy distance measure. Uppada [31] examined clustering algorithm using centroid clearly. Varghese and Kuriakose [32] proposed a formula to find the centroid of the fuzzy number. Wang [33] introduced geometric aggregation operator and applied in a decision making problem under intuitionistic fuzzy environment. Wang [34] proposed arithmetic aggregation operators. Wang et al. [35] introduced single valued neutrosophic sets. Wei et al. [36] introduced some persuaded aggregation operators under intuinistic fuzzy setting and applied in a group decision making problem. Wu [37] explained about distance metrics and their role in data transformations.Ye [38] proposed prioritized aggregation operators based on trapezoidal intuitionistic fuzzy concept and applied in a multi-criteria decision making problem. Ye [39] solved minimum spanning tree problem under single valued neutrosophic setting and its clustering method. Ye [40] proposed single valued neutrosophic cross entropy measure and applied in a decision making problem. Ye [41] introduced the expected Dice similarity measure and applied in a decision making problem. Ye [42] projected trapezoidal neutrosophic set and applied in a multiple attribute decision making. Zhang et al. [44] introduced interval neutrosophic sets and used in multi criteria decision making problem. Zou et al. [45] introduced a distance measure between neutrosophic sets as an evidential approach. From the literature, it is found that distance measure for trapezoidal neutrosophic numbers using centroids with its properties has not yet been studied so far. Hence the motivation of the present study.

Hence, in this paper a new distance measure for trapezoidal fuzzy neutrosophic numbers based on centroids has been proposed with its metric properties in detail. Also the graphical representation is presented for trapezoidal fuzzy neutrosophic number. Comparative study also have been made with the existing cases for both proposed distance measure and proposed graphical representation. Further advantages of the proposed distance measure are presented.

3-Preliminaries

Definition 1. [38] Let X be a space of discourse, a trapezoidal intuitionistic fuzzy set B in X is defined as: $B=\{\langle y,\alpha_B(y),\beta_B(y)\rangle | y\in X\}$, where $\alpha_B(y)\subset[0,1]$ and $\beta_B(y)\subset[0,1]$ are two trapezoidal fuzzy numbers $\alpha_B(y)=(\alpha_B^1(y),\alpha_B^2(y),\alpha_B^3(y),\alpha_B^4(y)):Y\rightarrow[0,1]$ and $\beta_B(y)=(\beta_B^1(y),\beta_B^2(y),\beta_B^3(y),\beta_B^4(y)):Y\rightarrow[0,1]$ with the condition that $0\le\alpha_B^4(y)+\beta_B^4(y)\le 1,\ \forall\, y\in Y$.

For Convenience, let $\alpha_B(y)=(a,b,c,d)$ and $\beta_B(y)=(e,f,g,h)$ be two trapezoidal fuzzy numbers, thus a trapezoidal intuitionistic fuzzy number (TrIFN) can be denoted by $j=\langle(a,b,c,d),(e,f,g,h)\rangle$, which is basic element in a trapezoidal intuitionistic fuzzy set.

If $b=c$ and $f=g$ hold in a TrIFN j, which is a special case of the TrIFN.

Definition 2. [38] Let $j_1=\langle(a_1,b_1,c_1,d_1),(e_1,f_1,g_1,h_1)\rangle$ and

$j_2=\langle(a_2,b_2,c_2,d_2),(e_2,f_2,g_2,h_2)\rangle$, be two TrIFNs. Then there are the following operational rules:

1. $$j_1\oplus j_2=\left\langle\begin{matrix}(a_1+a_2-a_1a_2,b_1+b_2-b_1b_2,c_1+c_2-c_1c_2,d_1+d_2-d_1d_2),\\(e_1e_2,f_1f_2,g_1g_2,h_1h_2)\end{matrix}\right\rangle$$

2. $$j_1\otimes j_2=\left\langle\begin{matrix}(a_1a_2,b_1b_2,c_1c_2,d_1d_2),\\(e_1+e_2-e_1e_2,f_1+f_2-f_1f_2,g_1+g_2-g_1g_2,h_1+h_2-h_1h_2)\end{matrix}\right\rangle$$

3. $$\lambda j_1=\left\langle\begin{matrix}\left(1-(1-a_1)^\lambda,1-(1-b_1)^\lambda,1-(1-c_1)^\lambda,1-(1-d_1)^\lambda\right),\\(e_1^\lambda,f_1^\lambda,g_1^\lambda,h_1^\lambda)\end{matrix}\right\rangle,\lambda>0;$$

4. $$m_1^\lambda=\left\langle\begin{matrix}(a_1^\lambda,b_1^\lambda,c_1^\lambda,d_1^\lambda),\left(1-(1-e_1)^\lambda,1-(1-f_1)^\lambda,1-(1-g_1)^\lambda,1-(1-h_1)^\lambda\right),\\\left(1-(1-i_1)^\lambda,1-(1-j_1)^\lambda,1-(1-k_1)^\lambda,1-(1-l_1)^\lambda\right)\end{matrix}\right\rangle,\lambda\ge 0$$

Definition 3. [30] From philosophical point of view, Smarandache [30] originally presented the concept of a neutrosophic set B in a universal set Y, which is characterized independently by a truth-membership function $T_B(y)$, an indeterminacy membership function $I_B(y)$ and a falsity-membership function $F_B(y)$. The function $T_B(y)$, $I_B(y)$ and $F_B(y)$ in Y are real standard or nonstandard subsets of $]^-0,1^+[$, such that $T_B(y): Y \to]^-0,1^+[$, $I_B(y): Y \to]^-0,1^+[$, and $F_B(y): Y \to]^-0,1^+[$ .Then, the sum of $T_B(y), I_B(y)$ and $F_B(y)$ satisfies the condition $^-0 \le \sup T_B(y) + \sup I_B(y) + \sup F_B(y) \le 3^+$. Obviously, it is difficult to apply the neutrosophic set to practical problems. To easily apply it in science and engineering fields, Wang et al. [35] introduced the concept of a single-valued neutrosophic set as a subclass of the neutrosophic set and gave the following definition.

Definition 4. [35] A single-valued neutrosophic set B in a universal set Y is characterized by a truth-membership function $T_B(y)$, an indeterminacy-membership function $I_B(y)$ and a falsity-membership function $F_B(y)$. Then, a single-valued neutrosophic set B can be denoted by

$$B = \{\langle y, T_B(y), I_B(y), F_B(y)\rangle \mid y \in Y\}$$

where, $T_B(y), I_B(y), F_B(y) \in [0,1]$ for each $y \in Y$. Therefore, the sum of $T_B(y), I_B(y)$ and $F_B(y)$ satisfies $0 \le T_B(y) + I_B(y) + F_B(y) \le 3$.

Let $M = \{\langle y, T_M(y), I_M(y), F_M(y)\rangle \mid y \in Y\}$ and $N = \{\langle y, T_N(y), I_N(y), F_N(y)\rangle \mid y \in Y\}$ be two single-valued neutrosophic sets, then we the following relations [8,11]:

1. Complement: $M^C = \{\langle y, F_M(y), 1 - I_M(y), T_M(y)\rangle \mid y \in Y\}$;
2. Inclusion: $M \subseteq N$ if and only if $T_M(y) \le T_N(y)$, $I_M(y) \ge I_N(y)$ and $F_M(y) \ge F_N(y)$ for any $y \in Y$;
3. Equality: $M = N$ if and only if $M \subseteq N$ and $N \subseteq M$;
4. Union: $M \cup N = \{\langle y, T_M(y) \vee T_N(y), I_M(y) \wedge I_N(y), F_M(y) \wedge F_N(y)\rangle \mid y \in Y\}$;
5. Intersection: $M \cap N = \{\langle y, T_M(y) \wedge T_N(y), I_M(y) \vee I_N(y), F_M(y) \vee F_N(y)\rangle \mid y \in Y\}$;

6. Addition: $M \oplus N = \left\{ \left\langle \begin{array}{l} y, T_M(y)+T_N(y)-T_M(y)T_N(y), I_M(y)I_N(y), \\ F_M(y)F_N(y) \end{array} \right\rangle \middle| y \in Y \right\}$;

7. Multiplication: $M \otimes N = \left\{ \left\langle \begin{array}{l} y, T_M(y)T_N(y), I_M(y)+I_N(y)-I_M(y)I_N(y), \\ F_M(y)+F_N(y)-F_M(y)F_N(y) \end{array} \right\rangle \middle| y \in Y \right\}$.

Definition 5. [42] Let Y be a space of discourse, a trapezoidal neutrosophic set H in Y is defined as follow:

$H = \{\langle y, T_H(y), I_H(y), F_H(y)\rangle | y \in Y\}$, where $T_H(y) \subset [0,1]$, $I_H(y) \subset [0,1]$ and $F_H(y) \subset [0,1]$ are three trapezoidal fuzzy numbers $T_H(y) = (t_H^1(y), t_H^2(y), t_H^3(y), t_H^4(y)) : Y \to [0,1]$,

$I_H(y) = (i_H^1(y), i_H^2(y), i_H^3(y), i_H^4(y)) : Y \to [0,1]$ and

$F_H(y) = (f_H^1(y), f_H^2(y), f_H^3(y), f_H^4(y)) : Y \to [0,1]$ with the condition

$0 \le t_H^4(y) + i_H^4(y) + f_H^4(y) \le 3,\ y \in Y$.

For convenience, the three trapezoidal fuzzy numbers are denoted by $T_H(y) = (a,b,c,d)$, $I_H(y) = (e,f,g,h)$ and $F_H(y) = (i,j,k,l)$. Thus, a trapezoidal neutrosophic numbers is denoted by $m = \langle (a,b,c,d),(e,f,g,h),(i,j,k,l)\rangle$, which is a basic element in the trapezoidal neutrosophic set.

If $b = c$, $f = g$ and $j = k$ hold in a trapezoidal neutrosophic number j_1, it reduces to the triangular neutrosophic number, which is considered as a special case of the trapezoidal neutrosophic number.

Definition 6. [42] Let $m_1 = \langle (a_1,b_1,c_1,d_1),(e_1,f_1,g_1,h_1),(i_1,j_1,k_1,l_1)\rangle$, and $m_2 = \langle (a_2,b_2,c_2,d_2),(e_2,f_2,g_2,h_2),(i_2,j_2,k_2,l_2)\rangle$ be two trapezoidal neutrosophic numbers. Then there are the following operational rules:

1. $m_1 \oplus m_2 = \left\langle \begin{array}{l} (a_1+a_2-a_1a_2, b_1+b_2-b_1b_2, c_1+c_2-c_1c_2, d_1+d_2-d_1d_2), \\ (e_1e_2, f_1f_2, g_1g_2, h_1h_2), (i_1i_2, j_1j_2, k_1k_2, l_1l_2) \end{array} \right\rangle$,

2. $$m_1 \otimes m_2 = \left\langle \begin{array}{l} (a_1a_2, b_1b_2, c_1c_2, d_1d_2), \\ (e_1+e_2-e_1e_2, f_1+f_2-f_1f_2, g_1+g_2-g_1g_2, h_1+h_2-h_1h_2), \\ (i_1+i_2-i_1i_2, j_1+j_2-j_1j_2, k_1+k_2-k_1k_2, l_1+l_2-l_1l_2) \end{array} \right\rangle;$$

3. $$\lambda m_1 = \left\langle \begin{array}{l} (1-(1-a_1)^\lambda, 1-(1-b_1)^\lambda, 1-(1-c_1)^\lambda, 1-(1-d_1)^\lambda), \\ (e_1^\lambda, f_1^\lambda, g_1^\lambda, h_1^\lambda), (i_1^\lambda, j_1^\lambda, k_1^\lambda, l_1^\lambda) \end{array} \right\rangle, \lambda > 0;$$

4. $$m_1^\lambda = \left\langle \begin{array}{l} (a_1^\lambda, b_1^\lambda, c_1^\lambda, d_1^\lambda), \\ (1-(1-e_1)^\lambda, 1-(1-f_1)^\lambda, 1-(1-g_1)^\lambda, 1-(1-h_1)^\lambda), \\ (1-(1-i_1)^\lambda, 1-(1-j_1)^\lambda, 1-(1-k_1)^\lambda, 1-(1-l_1)^\lambda) \end{array} \right\rangle, \lambda \geq 0$$

Definition 7. [18] Let P and Q be the intuitionistic fuzzy sets with membership functions $\mu_P(x), \mu_Q(x)$, non-membership functions $\nu_P(x), \nu_Q(x)$ and hesitation degree $\pi_P(x), \pi_Q(x)$. Then the normalized Hamming distance is

$$D(P,Q) = \frac{1}{2n}\sum_{i=1}^{n}\left[|\mu_P(x_i)-\mu_Q(x_i)| + |\nu_P(x_i)-\nu_Q(x_i)| + |\pi_P(x_i)-\pi_Q(x_i)|\right]$$

And the normalized Euclidean distance is

$$D_E(P,Q) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[(\mu_P(x_i)-\mu_Q(x_i))^2 + (\nu_P(x_i)-\nu_Q(x_i))^2 + (\pi_P(x_i)-\pi_Q(x_i))^2\right]}$$

Definition 8. [17] Consider the real values r_i, $i = 1,2,3,\ldots,6$ and if $r_1 \leq r_2, r_3 \leq r_4, r_5 \leq r_6$ then the following results are true.

1. $\max\{r_1, r_3, r_5\} \leq \max\{r_2, r_4, r_6\}$

2. $\max\{r_1+r_2, r_3+r_4, r_5+r_6\} \leq \max\{r_1, r_3, r_5\} + \max\{r_2, r_4, r_6\}$

Definition 9. [34] For any real numbers $r_i, s_i \geq 0, i = 1,2,\ldots,d$, the Euclidean distance is defined as, $D(r,s) = \sqrt{\sum_{i=1}^{d}(r_i - s_i)^2}$ and satisfies the condition that $\left[\sum_{i=1}^{d}(r_i+s_i)^p\right]^{1/p} \leq \left(\sum_{i=1}^{d}(r_i)^p\right)^{1/p} + \left(\sum_{i=1}^{d}(s_i)^p\right)^{1/p}$.

Definition 10. [42] Let $m_p = \langle (a_p, b_p, c_p, d_p), (e_p, f_p, g_p, h_p), (i_p, j_p, k_p, l_p) \rangle, p = 1, 2, 3, ..., n$ be the trapezoidal fuzzy neutrosophic numbers then the trapezoidal fuzzy neurosophic weighted geometric operator is defined by

$$TFNWG(m_1, m_2, ..., m_n) = m_1^{\omega_1} \otimes m_2^{\omega_2} \otimes m_3^{\omega_3} \otimes ... \otimes m_n^{\omega_n}$$

$$= \left\langle \left(\prod_{p=1}^{n} a_p^{\omega_p}, \prod_{p=1}^{n} b_p^{\omega_p}, \prod_{p=1}^{n} c_p^{\omega_p}, \prod_{p=1}^{n} d_p^{\omega_p} \right), \left(1 - \prod_{p=1}^{n} (1-e_p)^{\omega_p}, 1 - \prod_{p=1}^{n} (1-f_p)^{\omega_p}, 1 - \prod_{p=1}^{n} (1-g_p)^{\omega_p}, 1 - \prod_{p=1}^{n} (1-h_p)^{\omega_p} \right), \right.$$

$$\left. \left(1 - \prod_{p=1}^{n} (1-i_p)^{\omega_p}, 1 - \prod_{p=1}^{n} (1-j_p)^{\omega_p}, 1 - \prod_{p=1}^{n} (1-k_p)^{\omega_p}, 1 - \prod_{p=1}^{n} (1-l_p)^{\omega_p} \right) \right\rangle$$

where, $\omega_1, \omega_2, ..., \omega_n$ are the weight vectors and the sum of the weight vectors is 1.

Definition 11. [9] Graphical representation of trapezoidal neutrosophic number

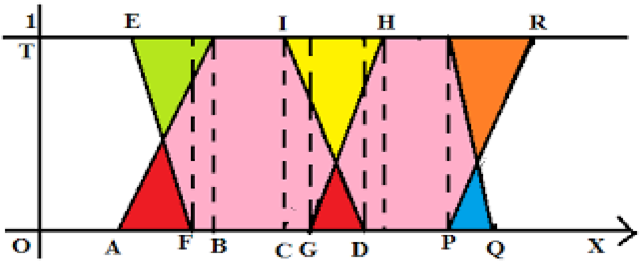


Figure 1. Graphical representation of Trapezoidal neutrosophic number

Figure 1 shows that graphical representation of trapezoidal fuzzy neutrosophic number can be done in different ways. It is a linear trapezoidal neutrosophic number.

4-Proposed Distance Measure for Trapezoidal Fuzzy Neutrosophic Number

Here we propose a new distance measure for trapezoidal fuzzy neutrosophic number based on centroids. Firstly, individual graphical representation proposed measure is presented here with the individual representation of truth, indeterminacy, falsity membership functions and trapezoidal fuzzy neutrosophic fuzzy number described by Figure 2-Figure 6.

Centre point of the object is called centroid. It should lie inside the object. At this point, the three medians of the triangle intersect and is termed point of intersection. Centroid is the average of coordinate points in X axis and Y axis of each vertex of the triangle. Centroid is the fixed point of all linear transformation which maintains length in translation, rotation, glides and reflection.

The centroid of the truth, indeterminacy and falsity trapezoid is treated as a balance point for the trapezoid. The centroid of each part are estimated using the calculation of centroid and the simple area and this combination will generate a triangle. Also the distance is measured from the centroid of all the parts to X axis and Y axis. Here the area of all the parts are multiplied by the distance and find their sum to get the total value. And the sum of the products of the area and distances is divided by the total area and obtain the centroid of circumcentre described by x and y point. Since centroid based distance measure may be derived using Euclidean measure, here it is obtained from the circumcentre of the centroids and the authentic point for the trapezoidal fuzzy neutrosophic number.

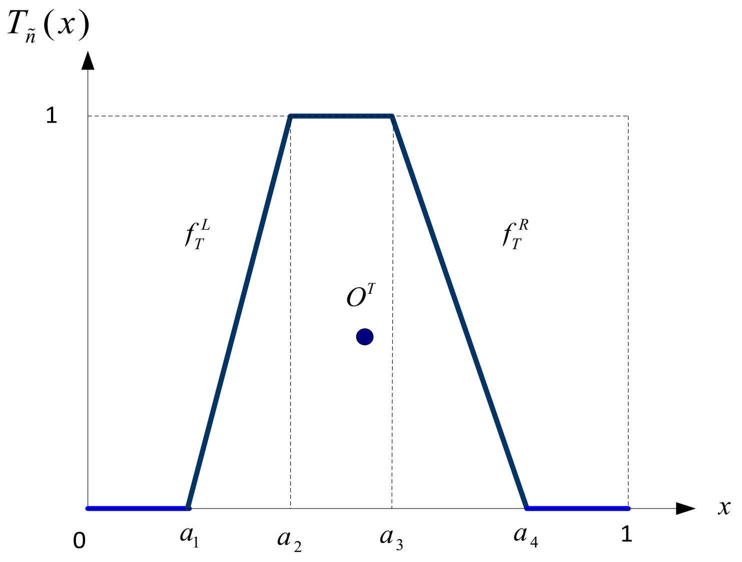


Figure 2. Truth membership function of trapezoidal fuzzy neutrosophic set with centroid

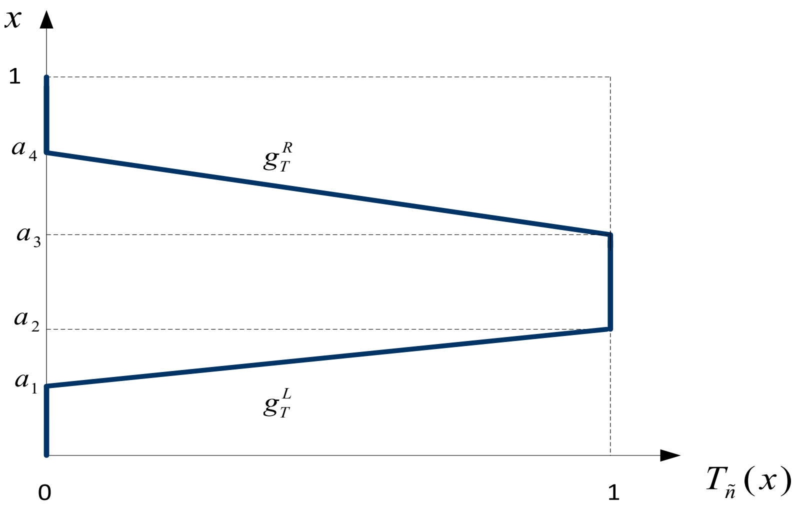


Figure 3. Truth membership function of trapezoidal fuzzy neutrosophic set

Suppose $\tilde{n} = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ be a trapezoidal fuzzy neutrosophic number. Based on the literature (Y. M. Wang et al. On the centroids of fuzzy numbers), we can get the centroid point $O^T = (x_o^T(\tilde{n}), y_o^T(\tilde{n}))$ of the truth membership function of trapezoidal fuzzy neutrosophic number $\tilde{n}$.

$$x_o^T(\tilde{n}) = \frac{\int_{a_1}^{a_2} x f_T^L dx + \int_{a_2}^{a_3} x \cdot 1 dx + \int_{a_3}^{a_4} x f_T^R dx}{\int_{a_1}^{a_2} f_T^L dx + \int_{a_2}^{a_3} 1 dx + \int_{a_3}^{a_4} f_T^R dx} = \frac{1}{3}[a_1 + a_2 + a_3 + a_4 - \frac{a_4 a_3 - a_1 a_2}{(a_4 + a_3) - (a_1 + a_2)}],$$

$$y_o^T(\tilde{n}) = \frac{\int_0^1 y(g_T^L - g_T^R)dy}{\int_0^1 (g_T^L - g_T^R)dy} = \frac{1}{3}[1 + \frac{a_3 - a_2}{(a_4 + a_3) - (a_1 + a_2)}].$$

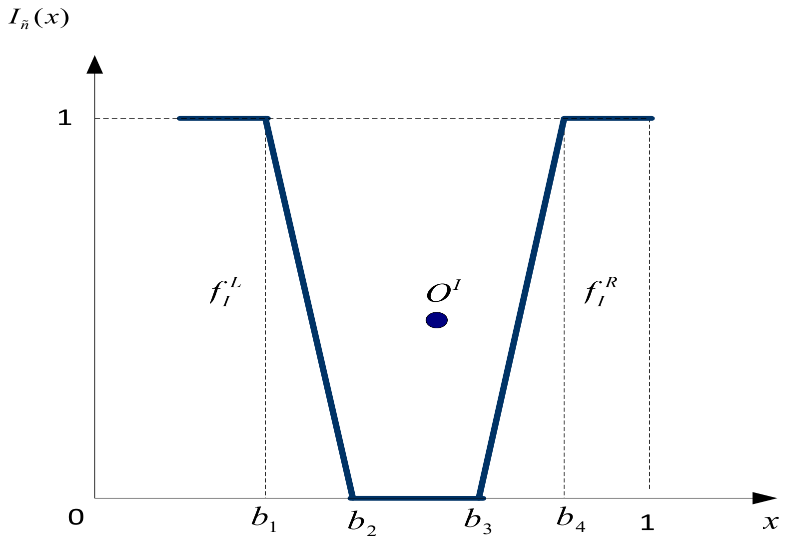


Figure 4. Indeterminate membership function of trapezoidal fuzzy neutrosophic set with centroid

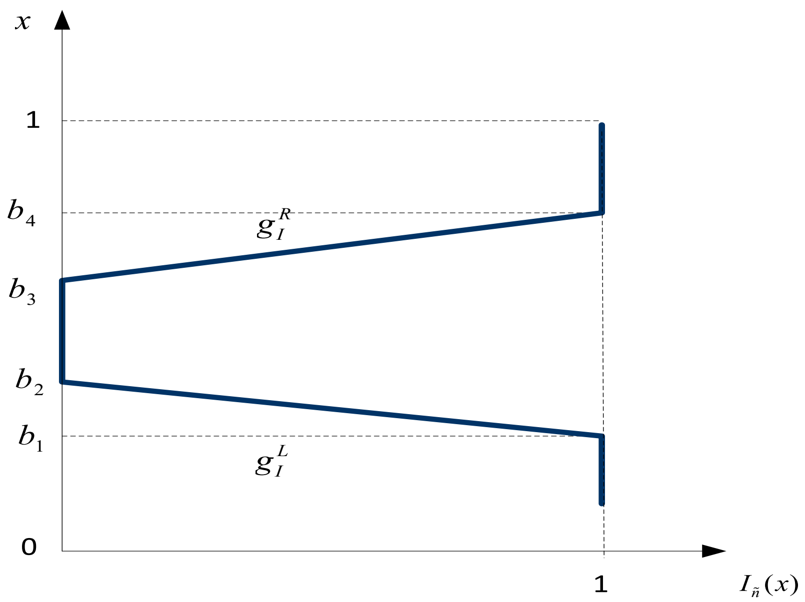


Figure 5. Indeterminate membership function of trapezoidal fuzzy neutrosophic number

we can get the centroid point $O^I = (x_o^I(\tilde{n}), y_o^I(\tilde{n}))$ of indeterminacy membership function of trapezoidal fuzzy neutrosophic number $\tilde{n}$.

$$x_o^I(\tilde{n}) = \frac{\int_{b_1}^{b_2} x f_I^L dx + \int_{b_2}^{b_3} x \cdot 1 dx + \int_{b_3}^{b_4} x f_I^R dx}{\int_{b_1}^{b_2} f_I^L dx + \int_{b_2}^{b_3} 1 dx + \int_{b_3}^{b_4} f_I^R dx} = \frac{1}{3}[b_1 + b_2 + b_3 + b_4 - \frac{b_4 b_3 - b_1 b_2}{(b_4 + b_3) - (b_1 + b_2)}],$$

$$y_o^I(\tilde{n}) = \frac{\int_0^1 y(g_I^L - g_I^R) dy}{\int_0^1 (g_I^L - g_I^R) dy} = \frac{1}{3}[1 + \frac{b_3 - b_2}{(b_4 + b_3) - (b_1 + b_2)}].$$

Similarly, we can get the centroid point $O^F = (x_o^F(\tilde{n}), y_o^F(\tilde{n}))$ of falsity membership function of trapezoidal fuzzy neutrosophic number $\tilde{n}$.

$$x_o^F(\tilde{n}) = \frac{\int_{c_1}^{c_2} x f_F^L dx + \int_{c_2}^{c_3} x \cdot 1 dx + \int_{c_3}^{c_4} x f_F^R dx}{\int_{c_1}^{c_2} f_F^L dx + \int_{c_2}^{c_3} 1 dx + \int_{c_3}^{c_4} f_F^R dx} = \frac{1}{3}[c_1 + c_2 + c_3 + c_4 - \frac{c_4 c_3 - c_1 c_2}{(c_4 + c_3) - (c_1 + c_2)}],$$

$$y_o^F(\tilde{n}) = \frac{\int_0^1 y(g_F^L - g_F^R) dy}{\int_0^1 (g_F^L - g_F^R) dy} = \frac{1}{3}[1 + \frac{c_3 - c_2}{(c_4 + c_3) - (c_1 + c_2)}].$$

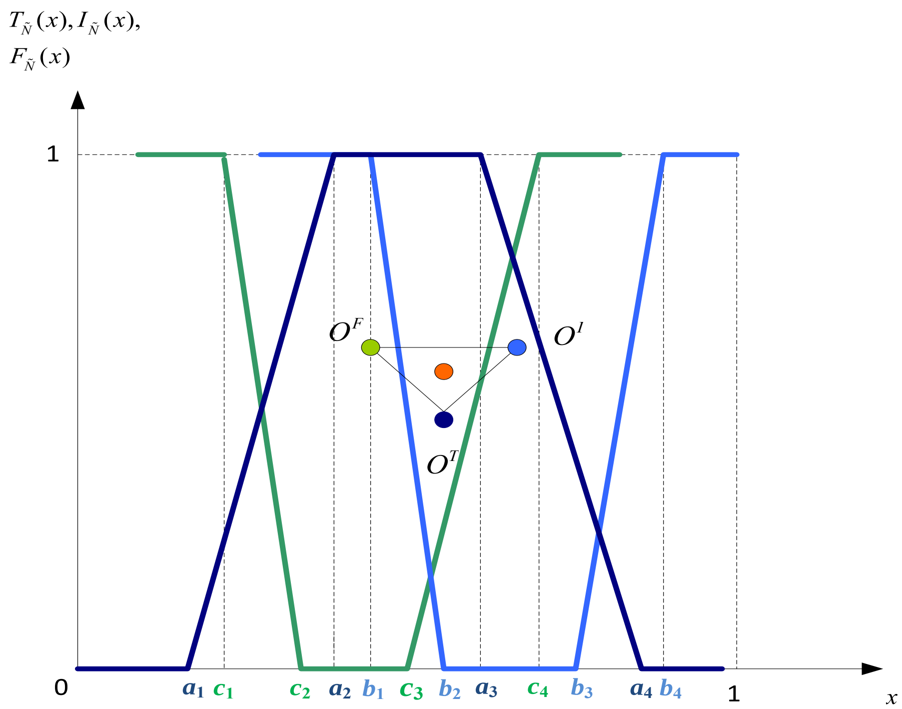


Figure 6. Trapezoidal fuzzy neutrosophic number with circumcentre of Centroids

In the above figure 5, the red dot represents the center of gravity of the triangle consisting of O^T, O^I, and O^F. According to the coordinate formula of the center of gravity of the triangle, we can get the coordinates of red dots $O=(x(\tilde{n}),y(\tilde{n}))$.

$$x(\tilde{n})=\frac{x_o^T(\tilde{n})+x_o^I(\tilde{n})+x_o^F(\tilde{n})}{3}$$

$$=\frac{1}{3}\left\{\begin{array}{l}\frac{1}{3}[a_1+a_2+a_3+a_4-\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}] \\ +\frac{1}{3}[b_1+b_2+b_3+b_4-\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}]+\frac{1}{3}[c_1+c_2+c_3+c_4-\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}]\end{array}\right\}$$

$$=\frac{1}{9}[\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}]$$

$$y(\tilde{n})=\frac{y_o^T(\tilde{n})+y_o^I(\tilde{n})+y_o^F(\tilde{n})}{3}$$

$$=\frac{\frac{1}{3}[1+\frac{a_3-a_2}{(a_4+a_3)-(a_1+a_2)}]+\frac{1}{3}[1+\frac{b_3-b_2}{(b_4+b_3)-(b_1+b_2)}]+\frac{1}{3}[1+\frac{c_3-c_2}{(c_4+c_3)-(c_1+c_2)}]}{3}$$

$$=\frac{1}{9}[3+\frac{a_3-a_2}{(a_4+a_3)-(a_1+a_2)}+\frac{b_3-b_2}{(b_4+b_3)-(b_1+b_2)}+\frac{c_3-c_2}{(c_4+c_3)-(c_1+c_2)}]$$

Definition1: Let $\tilde{n}_1=\langle(a_1,a_2,a_3,a_4),(b_1,b_2,b_3,b_4),(c_1,c_2,c_3,c_4)\rangle$ and $\tilde{n}_2=\langle(e_1,e_2,e_3,e_4),(f_1,f_2,f_3,f_4),(g_1,g_2,g_3,g_4)\rangle$ be two trapezoidal fuzzy neutrosophic numbers, and their centroids are $O_1=(x(\tilde{n}_1),y(\tilde{n}_1))$, $O_2=(x(\tilde{n}_2),y(\tilde{n}_2))$ respectively, then the distance between $\tilde{n}_1$ and $\tilde{n}_2$ is

$$D(\tilde{n}_1,\tilde{n}_2)=\frac{1}{9}\sqrt{\begin{array}{l}[\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}) \\ -(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2 \\ +[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}) \\ +(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2\end{array}}$$

Theorem 1: This distance $D(\tilde{n}_1,\tilde{n}_2)$ of $\tilde{n}_1$ and $\tilde{n}_2$ fulfills the following properties:

1. $0 \le D(\tilde{n}_1, \tilde{n}_2) \le 1$;

2. $D(\tilde{n}_1, \tilde{n}_2) = 0$ if and only if $\tilde{n}_1 = \tilde{n}_2$, i.e., $a_i = e_i$, $b_i = f_i$ and $c_i = g_i$ hold for $i = 1, 2, 3, 4$;

3. $D(\tilde{n}_1, \tilde{n}_2) = D(\tilde{n}_2, \tilde{n}_1)$.

4. If $\tilde{n}_1, \tilde{n}_2 \,\&\, \tilde{n}_3$ are the trapezoidal fuzzy neutrosophic numbers then

$$D(\tilde{n}_1, \tilde{n}_3) \le D(\tilde{n}_1, \tilde{n}_2) + D(\tilde{n}_2, \tilde{n}_3)$$

Proof

1. It is easy to prove $0 \le D(\tilde{n}_1, \tilde{n}_2)$. In addition, it can be seen from figure 1, the maximum distance is the distance between the point $(0,0)$ and the point $(1,1)$, or the point $(0,1)$ and the point $(1,0)$, assume the coordinates of centroids of $\tilde{n}_1$ and $\tilde{n}_2$ are O_1 and O_2, and $O_1 = (0,1)$ and $O_2 = (1,0)$, or $O_1 = (1,0)$ and $O_2 = (0,1)$, or $O_1 = (0,0)$ and $O_2 = (1,1)$, or $O_1 = (1,1)$ and $O_2 = (0,0)$, then the $D(\tilde{n}_1, \tilde{n}_2) = 1$, otherwise, $D(\tilde{n}_1, \tilde{n}_2) < 1$, thus $0 \le D(\tilde{n}_1, \tilde{n}_2) \le 1$.

2. if $\tilde{n}_1 = \tilde{n}_2$, i.e., $a_i = e_i$, $b_i = f_i$ and $c_i = g_i$, then

$$D(\tilde{n}_1,\tilde{n}_2)=\frac{1}{9}\sqrt{\begin{array}{l}[\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}a_i-\sum_{i=1}^{4}b_i-\sum_{i=1}^{4}c_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)})\\ -(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)})]^2\\ +[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)})\\ +(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)})]^2\end{array}}$$
$$=0.$$

if $D\left(\tilde{n}_1,\tilde{n}_2\right)=0$, then

$$\begin{array}{l}[\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})\\ -(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2\\ +[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})\\ +(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2=0,\end{array}$$

Thus,

$$\begin{array}{l}\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})\\ -(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})\\ =0,\end{array}$$

$$\begin{array}{l}(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})\\ +(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})\\ =0,\end{array}$$

thus

$$\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}=0,$$
$$\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}=0,$$
$$\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}=0,$$

thus

$a_i=e_i$, $b_i=f_i$, $c_i=g_i$, that is $\tilde{n}_1=\tilde{n}_2$.

3. Since,

$$\begin{aligned}&[\sum_{i=1}^4 a_i+\sum_{i=1}^4 b_i+\sum_{i=1}^4 c_i-\sum_{i=1}^4 e_i-\sum_{i=1}^4 f_i-\sum_{i=1}^4 g_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})\\&-(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2\\&+[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})\\&+(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2\end{aligned}$$

$$\begin{aligned}&=[\sum_{i=1}^4 e_i+\sum_{i=1}^4 f_i+\sum_{i=1}^4 g_i-\sum_{i=1}^4 a_i-\sum_{i=1}^4 b_i-\sum_{i=1}^4 c_i-(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)})\\&-(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)})-(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)})]^2\\&+[(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)})+(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)})\\&+(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)})]^2\end{aligned}$$

then $D(\tilde{n}_1,\tilde{n}_2)=D(\tilde{n}_2,\tilde{n}_1)$.

4. Using Def. 8, we can prove (4).

Let $\tilde{n}_1=\langle(a_1,a_2,a_3,a_4),(b_1,b_2,b_3,b_4),(c_1,c_2,c_3,c_4)\rangle$,

$\tilde{n}_2=\langle(e_1,e_2,e_3,e_4),(f_1,f_2,f_3,f_4),(g_1,g_2,g_3,g_4)\rangle$ and

$\tilde{n}_3 = \left\langle \left(j_1, \mathrm{j}_2, \mathrm{j}_3, \mathrm{j}_4 \right), \left(k_1, \mathrm{k}_2, \mathrm{k}_3, \mathrm{k}_4 \right), \left(l_1, l_2, l_3, l_4 \right) \right\rangle$ are the three trapezoidal fuzzy neutrosophic numbers then $D\left(\tilde{n}_1, \tilde{n}_3\right) \le D\left(\tilde{n}_1, \tilde{n}_2\right) + D\left(\tilde{n}_2, \tilde{n}_3\right)$

Using the results we have,

$$D(\tilde{n}_1, \tilde{n}_3)$$

$$= \frac{1}{9}\sqrt{\begin{array}{l}[\sum_{i=1}^{4} a_i + \sum_{i=1}^{4} b_i + \sum_{i=1}^{4} c_i - \sum_{i=1}^{4} j_i - \sum_{i=1}^{4} k_i - \sum_{i=1}^{4} l_i - (\frac{a_4 a_3 - a_1 a_2}{(a_4 + a_3) - (a_1 + a_2)} - \frac{j_4 j_3 - j_1 j_2}{(\mathrm{j}_4 + j_3) - (\mathrm{j}_1 + j_2)}) \\ -(\frac{b_4 b_3 - b_1 b_2}{(b_4 + b_3) - (b_1 + b_2)} - \frac{k_4 k_3 - k_1 k_2}{(\mathrm{k}_4 + k_3) - (\mathrm{k}_1 + k_2)}) - (\frac{c_4 c_3 - c_1 c_2}{(c_4 + c_3) - (c_1 + c_2)} - \frac{l_4 l_3 - l_1 l_2}{(l_4 + l_3) - (l_1 + l_2)})]^2 \\ +[(\frac{a_4 a_3 - a_1 a_2}{(a_4 + a_3) - (a_1 + a_2)} - \frac{j_4 j_3 - j_1 j_2}{(\mathrm{j}_4 + j_3) - (\mathrm{j}_1 + j_2)}) + (\frac{b_4 b_3 - b_1 b_2}{(b_4 + b_3) - (b_1 + b_2)} - \frac{k_4 k_3 - k_1 k_2}{(\mathrm{k}_4 + k_3) - (\mathrm{k}_1 + k_2)}) \\ +(\frac{c_4 c_3 - c_1 c_2}{(c_4 + c_3) - (c_1 + c_2)} - \frac{l_4 l_3 - l_1 l_2}{(l_4 + l_3) - (l_1 + l_2)})]^2\end{array}}$$

$$\le \frac{1}{9}\sqrt{\begin{aligned}&\{\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i+\sum_{i=1}^{4}e_i+\sum_{i=1}^{4}f_i+\sum_{i=1}^{4}g_i-\sum_{i=1}^{4}j_i-\sum_{i=1}^{4}k_i-\sum_{i=1}^{4}l_i\\&-[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})-(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})]\\&-[(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})]\\&-[(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})-(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]\}^2\\&+\{[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})]\\&-[(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})+(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})]\\&-[(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})+(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]\}^2\end{aligned}}$$

$$\le \frac{1}{9}\left[\sqrt{\begin{aligned}&\{\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i+\sum_{i=1}^{4}e_i+\sum_{i=1}^{4}f_i+\sum_{i=1}^{4}g_i-\sum_{i=1}^{4}j_i-\sum_{i=1}^{4}k_i-\sum_{i=1}^{4}l_i\\&-[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})-(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})]\\&-[(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})]\\&-[(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})-(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]\}^2\\&+\{[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})]\\&-[(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})+(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})]\\&-[(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})+(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]\}^2\end{aligned}}\right]^2$$

$$\le \frac{1}{9}\left(\begin{aligned}&[\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})\\&-(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2\\&+[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})\\&+(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2\end{aligned}\right)^{1/2}$$

$$
+\frac{1}{9}\left(\begin{aligned}
&[\sum_{i=1}^{4}e_i+\sum_{i=1}^{4}f_i+\sum_{i=1}^{4}g_i-\sum_{i=1}^{4}j_i-\sum_{i=1}^{4}k_i-\sum_{i=1}^{4}l_i-(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})\\
&-(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})-(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]^2\\
&+[(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})+(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})\\
&+(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]^2
\end{aligned}\right)^{1/2}
$$

Using Def.9 we have,

$$
\leq\frac{1}{9}\sqrt{\begin{aligned}
&[\sum_{i=1}^{4}a_i+\sum_{i=1}^{4}b_i+\sum_{i=1}^{4}c_i-\sum_{i=1}^{4}e_i-\sum_{i=1}^{4}f_i-\sum_{i=1}^{4}g_i-(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})\\
&-(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})-(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2\\
&+[(\frac{a_4a_3-a_1a_2}{(a_4+a_3)-(a_1+a_2)}-\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)})+(\frac{b_4b_3-b_1b_2}{(b_4+b_3)-(b_1+b_2)}-\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)})\\
&+(\frac{c_4c_3-c_1c_2}{(c_4+c_3)-(c_1+c_2)}-\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)})]^2
\end{aligned}}
$$

$$
+\frac{1}{9}\sqrt{\begin{aligned}
&[\sum_{i=1}^{4}e_i+\sum_{i=1}^{4}f_i+\sum_{i=1}^{4}g_i-\sum_{i=1}^{4}j_i-\sum_{i=1}^{4}k_i-\sum_{i=1}^{4}l_i-(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})\\
&-(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})-(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]^2\\
&+[(\frac{e_4e_3-e_1e_2}{(e_4+e_3)-(e_1+e_2)}-\frac{j_4j_3-j_1j_2}{(j_4+j_3)-(j_1+j_2)})+(\frac{f_4f_3-f_1f_2}{(f_4+f_3)-(f_1+f_2)}-\frac{k_4k_3-k_1k_2}{(k_4+k_3)-(k_1+k_2)})\\
&+(\frac{g_4g_3-g_1g_2}{(g_4+g_3)-(g_1+g_2)}-\frac{l_4l_3-l_1l_2}{(l_4+l_3)-(l_1+l_2)})]^2
\end{aligned}}
$$

$\leq D(\tilde{n}_1,\tilde{n}_2)+D(\tilde{n}_2,\tilde{n}_3)$ and hence the result (4).

5- Decision Making method based on new distance measure based on centroids

In this section, we establish an approach based an trapezoidal fuzzy neutrosophic number weighted geometric arithmetic operator and a new distance measure based on centroid to deal with trapezoidal fuzzy neutrosophic information. The proposed approach is described as follows.

Step 1: Apply trapezoidal fuzzy neutrosophic number weighted geometric arithmetic operator [39] to find the aggregated trapezoidal fuzzy neutrosophic numbers for all the alternatives.

Step 2: Use the proposed distance measure, find the distances between all the alternatives and the ideal trapezoidal fuzzy neutrooshic number

Step 3: Rank the alternatives in which smaller value of distance indicate the best one.
Step 4: End

6- Numerical Example for the application of the proposed distance measure

In this section, a numerical example of a software selection problem and the aggregation operator called trapezoidal neutrosophic number weighted geometric averaging operator are get used from Ye [39] for a multiple attribute decision making problem is contributed to exhibit the application and effectiveness of the proposed distance measure under trapezoidal fuzzy neutrosophic environment. For a software selection process, consider candidate software systems are given as the set of five alternatives S_1, S_2, S_3, S_4, S_5 and the investment company need to take a decision according to four criteria: (i). the contribution to organization performance, (ii). The effort totranform from current system, (iii). The costs of hardware/software investment, (iv). The outsourcing software deneloper reliability denoted by C_1, C_2, C_3, C_4 respectively with the weight vector $\omega = (0.25, 0.25, 0.3, 0.2)^T$. The experts evaluate the five alternatives with repect to the four criteions under trapezoidal fuzzy neutrosophic environment and thus we can form the trapezoidal fuzzy neutrosophic decision matrix:

Table 1: Decision matrix using trapezoidal fuzzy neutrosophic numbers

$$D = \begin{bmatrix} \langle (0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1) \rangle & \langle (0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3),(0.2,0.3,0.4,0.5) \rangle & \langle (0.3,0.4,0.5,0.6),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1) \rangle & \langle (0.3,0.4,0.5,0.6),(0.1,0.1,0.1,0.1),(0.1,0.2,0.3,0.4) \rangle \\ \langle (0.3,0.4,0.5,0.5),(0.1,0.2,0.3,0.4),(0.0,0.1,0.1,0.1) \rangle & \langle (0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3) \rangle & \langle (0.0,0.1,0.1,0.2),(0.1,0.1,0.1,0.1),(0.5,0.6,0.7,0.8) \rangle & \langle (0.3,0.4,0.5,0.6),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.2) \rangle \\ \langle (0.1,0.1,0.1,0.1),(0.1,1.1,0.1,0.1),(0.6,0.7,0.8,0.9) \rangle & \langle (0.0,0.1,0.1,0.2),(0.0,0.1,0.2,0.3),(0.3,0.4,0.5,0.6) \rangle & \langle (0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.1,0.2,0.2,0.3) \rangle & \langle (0.1,0.2,0.3,0.4),(0.1,0.1,0.1,0.1),(0.3,0.4,0.5,0.6) \rangle \\ \langle (0.7,0.7,0.7,0.7),(0.0,0.1,0.2,0.3),(0.1,0.1,0.1,0.1) \rangle & \langle (0.4,0.5,0.6,0.7),(0.1,0.1,0.1,0.1),(0.0,0.1,0.2,0.2) \rangle & \langle (0.2,0.3,0.4,0.5),(0.0,0.1,0.2,0.3),(0.1,0.2,0.3,0.3) \rangle & \langle (0.1,0.2,0.3,0.4),(0.1,0.1,0.1,0.1),(0.1,0.1,0.1,0.1) \rangle \\ \langle (0.0,0.1,0.2,0.2),(0.1,0.1,0.1,0.1),(0.5,0.6,0.7,0.8) \rangle & \langle (0.4,0.4,0.4,0.4),(0.0,0.1,0.2,0.3),(0.0,0.1,0.2,0.3) \rangle & \langle (0.6,0.7,0.7,0.8),(0.1,0.1,0.1,0.1),(0.0,0.1,0.1,0.2) \rangle & \langle (0.1,0.2,0.3,0.3),(0.1,0.2,0.3,0.4),(0.2,0.3,0.4,0.5) \rangle \end{bmatrix}$$

Here we used the developed method to obtain the best software system(s) and it is described as follows:

Step 1: Using trapezoidal fuzzy neutrosophic weighted geometric operator in Definition 10, get the aggregated trapezoidal fuzzy neutrosophic numbers of $n_i,\ i = 1,2,3,4,5$ for the software system $S_i,\ i = 1,2,3,4,5$ as follows:

$n_1 = \langle (0.0000, 0.2985, 0.4162, 0.5244), (0.0209, 0.1003, 0.1809, 0.2639), (0.1261, 0.1745, 0.2266, 0.2836) \rangle$
$n_2 = \langle (0.0000, 0.2458, 0.2919, 0.3798), (0.0563, 0.1262, 0.1984, 0.2739), (0.1879, 0.2944, 0.3717, 0.4743) \rangle$
$n_3 = \langle (0.0000, 0.1599, 0.1888, 0.2545), (0.0464, 0.1000, 0.1566, 0.2162), (0.3437, 0.4502, 0.5424, 0.6655) \rangle$
$n_4 = \langle (0.2833, 0.3885, 0.4807, 0.5658), (0.0464, 0.1000, 0.1566, 0.2162), (0.1480, 0.2276, 0.3109, 0.3109) \rangle$
$n_5 = \langle (0.0000, 0.2912, 0.3756, 0.3910), (0.0760, 0.1210, 0.1690, 0.2208), (0.1958, 0.3012, 0.3877, 0.5020) \rangle$

Step 2: Use the proposed distance measure and find the distance between all $n_i,\ i = 1,2,3,4,5$ and the ideal trapezoidal fuzzy neutrosophic number $n_{Ideal} = \langle (1,1,1,1), (0,0,0,0), (0,0,0,0) \rangle$.

The obtained distances are as follows:

$D(n_1, I) = 0.1712 = D_1$
$D(n_2, I) = 0.1276 = D_2$
$D(n_3, I) = 0.1000 = D_3$
$D(n_4, I) = 0.1280 = D_4$
$D(n_5, I) = 0.1246 = D_5$

Step 3: Find the best alternative by considering the smaller value of the distance as the smaller value of distance indicates the best one.

Using step 2 it is found that, $D_3 > D_5 > D_2 > D_4 > D_1$ and from the ranking order, S_3 is the best is the best software system.

7- Comparative analysis for the proposed distance measure and graphical representation

In this section, a comparative study is made to show the effectiveness of the proposed distance measure with the existing methods and to show the uniqueness of the proposed graphical representation.

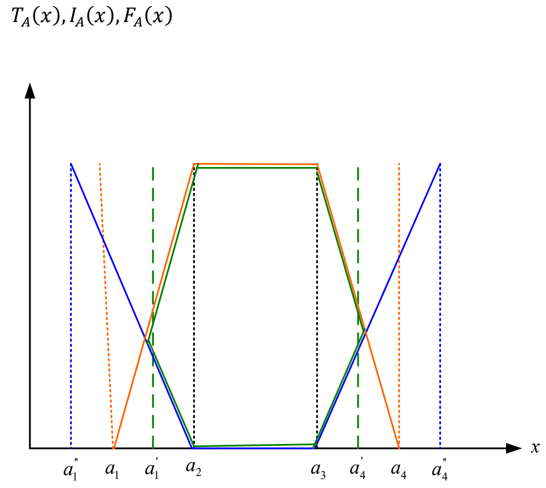
Table 2: Comparative analysis with the existing methods

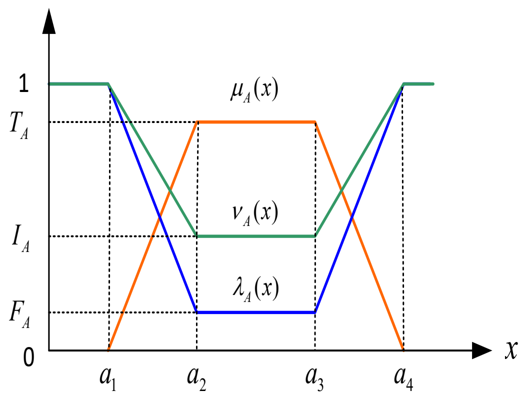
Existing Methods	Score/ distance values					Ranking
	D_1	D_2	D_3	D_4	D_5	
[6]	0.6092	0.4512	0.6039	0.6121	0.6321	$S_2 > S_3 > S_1 > S_4 > S_5$
[16]	0.2788	0.6790	0.9394	0.6564	0.4014	$S_3 > S_2 > S_4 > S_5 > S_1$
[42]	0.6553	0.5779	0.5069	0.6835	0.5904	$S_4 > S_1 > S_5 > S_2 > S_3$
[45]	0.7716	0.7798	0.7349	0.8124	0.8201	$S_3 > S_1 > S_2 > S_4 > S_5$

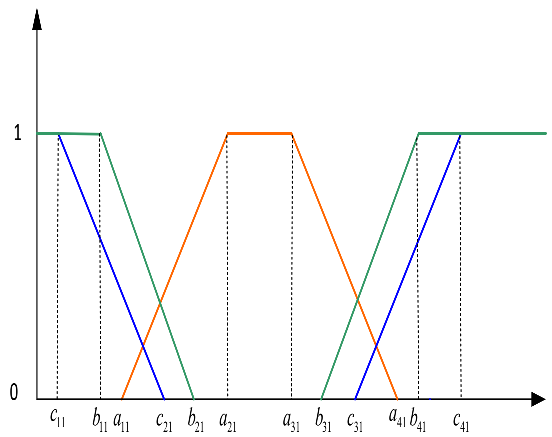
From the Table 2, it is found that, the third software system is the best one among the five alternatives. The results in the existing methods overlaps the proposed result. Theresore the proposed methodology using the proposed under trapezoidal fuzzy neutrosophic environment to solve the decision making problem suitably in comparision with the existing methods.

Table 3 represents the various forms of trapezoidal fuzzy neutrosophic numbers (TrFNN) have been listed out and it shows the uniqueness of the proposed graphical representation among the existing graphical representations.

Table 3: Comparative analysis with the existing graphical representation

<table>
<tr><th>Trapezoidal fuzzy neutrosophic forms</th><th>Graphical representation</th></tr>
<tr><td>
Darehmiraki [11]; A is a TrFNN,
$a_1^{''}, a_1, a_1^{'}, a_2, a_3, a_4^{'}, a_4, a_4^{''} \in R$ such that
$a_1^{''} \le a_1 \le a_1^{'} \le a_2 \le a_3 \le a_4^{'} \le a_4 \le a_4^{''}$
$A = \left\langle \left(a_1^{''}, a_1, a_1^{'}, a_2, a_3, a_4^{'}, a_4, a_4^{''}\right), T_A, I_A, F_A \right\rangle$
</td><td>


</td></tr>
<tr><td>
Liang [21]; A is a TrFNN,
$a_1, a_2, a_3, a_4 \in [0,1]$ such that
$0 \le a_1 \le a_2 \le a_3 \le a_4 \le 1$
$A = \left\langle [a_1, a_2, a_3, a_4], (T_A, I_A, F_A) \right\rangle$
</td><td>


</td></tr>
<tr><td>
Biswas [5]; A is a TpFNN,
$(a_{41}, a_{21}, a_{31}, a_{41}), (b_{41}, b_{21}, b_{31}, b_{41}),$
$(c_{41}, c_{21}, c_{31}, c_{41}) \in R$
such that
$c_{11} \le b_{11} \le a_{11} \le c_{21} \le b_{21} \le a_{21}$
$\le a_{31} \le b_{31} \le c_{31} \le a_{41} \le b_{41} \le c_{41}$
and
$A = \left\langle (a_{11}, a_{21}, a_{31}, a_{41}), (b_{11}, b_{21}, b_{31}, b_{41}),\right.$
$\left.(c_{11}, c_{21}, c_{31}, c_{41}) \right\rangle$
</td><td>


</td></tr>
</table>

8-Advantages of the proposed measure

An efficient distance measure boosts the performance of task analysis or clustering. Also centroid method is specific and location based one and acquire the best geographical location in consideration of the distance between all the competences. Though the existing methods namely Euclidean measure, Manhattan measure Minkowski measure and Hamming distance measure have been applied in many real time problems they could not provide good results for the indeterminate data. Hence in this paper, we proposed a new distance measure for trapezoidal neutrosophic fuzzy numbers based on centroids and the significant advantages of the proposed measure are given as follows.

(i). Trapezoidal fuzzy neutrosophic number is a simple design of arithmetic operations and easy and perceptive interpretation as well. Therefore the proposed measure is an easy and effective one under neutrosophic environment.

(ii). Distance measure can be estimated with simple algorithm and significant level of accuracy can be acquired as well.

(iii). While taking the important decision of choosing the method to measure a distance it can be used due its simplicity.

(iv). The proposed distance measure is based on centroid and hence estimation of the distance between all objects of the data set is possible and indeterminacy also can be addressed.

(v). It is derived using Euclidean distance and hence it is very useful in correlation analysis.

(vi). Also it can be applied in location planning, operations management, Neutrosophic Statistics, clustering, medical diagnosis, Optimization and image processing to get more accurate results without any computational complexity.

9-Conclusion and Future Research

The concept of distance measure of trapezoidal fuzzy neutrosophic number has sufficient scope of utilization in different studies in various domain. In this paper, we proposed a new distance measure for the trapezoidal fuzzy neutrosophic number based on centroid with the graphical representation. Also, the properties of the proposed measure have been derived in detail. In addition, a decision making problem has been solved using the proposed measure as a numerical example. Further, comparative analysis has been done with the existing methods to show the potential of the proposed distance measure and various forms of trapezoidal fuzzy neutrosophic number have been listed and shown the uniqueness of the proposed graphical representation. Furthermore, advantages of the proposed measure are given. In future, the present work may be extended to other special types of neutrosophic set like pentagonal neutrosophic set, neutrosophic rough set, interval valued neutrosophic set and plithogeneic environments.

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Neutrosophic Soft Fixed Points

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Abstract. In a wide spectrum of mathematical issues, the presence of a fixed point (FP) is equal to the presence of a appropriate map solution. Thus in several fields of math and science, the presence of a fixed point is important. Furthermore, an interesting field of mathematics has been the study of the existence and uniqueness of common fixed point (CFP) and coincidence points of mappings fulfilling the contractive conditions. Therefore, the existence of a FP is of significant importance in several fields of mathematics and science. Results of the FP, coincidence point (CP) contribute conditions under which maps have solutions. The aim of this paper is to explore these conditions (mappings) used to obtain the FP, CP and CFP of a neutrosophic soft set. We study some of these mappings (conditions) such as contraction map, L-lipschitz map, non-expansive map, compatible map, commuting map, weakly commuting map, increasing map, dominating map, dominated map of a neutrosophic soft set. Moreover we introduce some new points like a coincidence point, common fixed point and periodic point of neutrosophic soft mapping. We establish some basic results, particular examples on these mappings and points. In these results we show the link between FP and CP. Moreover we show the importance of mappings for obtaining the FP, CP and CFP of neutrosophic soft mapping.

Keyword. Neutrosophic set, fuzzy neutrosophic soft mapping, fixed point, coincidence point.

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1. Introduction

It is well known fact that fuzzy sets (FS) [1], complex fuzzy sets (CFS) [2], intuitionistic fuzzy sets (IFSs), the soft sets [3], fuzzy soft sets (FSS) and the fuzzy parameterized fuzzy soft sets (FPFS-sets) [4], [5] have been used to model the real life problems in various fields like in medical science, environments, economics, engineering, quantum physics and psychology etc.

In 1965, L. A. Zadeh [1] introduced a FS, which is the generalization of a crisp set. A grade value of a crisp set is either 1 or 0 but a grade value of fuzzy set has all the values in closed interval $[0,1]$. A FS plays a central role in modeling of real world problems. There are a lot of applications of FS theory in various branches of science such as in engineering, economics, medical science, mathematical chemistry, image processing, non-equilibrium thermodynamics etc. The concept for IFSs is provided in [3] which are generalizations of FS. An IFS P can be expressed as $P=\{\langle \nu,, \beta_P(\nu), \gamma_P(\nu)\rangle \;:\; \nu \in X\}$, where $\beta_P(\nu)$ represents the degree of membership, $\gamma_P(\nu)$ represents the degree of non-membership of the element $\nu \in X$. FPFS-sets is the extension of a FS and soft set proposed in [4], [5]. FPFS-sets maintain a proper degree of membership to both elements and parameters.

The notion of a complex CFS, the extension of the fuszy set, was introduced by Ramot et, al., [2]. A CFS membership function has all the values in the unit disk. A complex fuzzy set is used for representing two-dimensional phenomena and plays an important role in periodic phenomena. Complex fuzzy set is used in signals and systems to identify a reference signal out of large signals detected by a digital receiver. Moreover it is used for expressing complex fuzzy solar activity (solar maximum and solar minimum) through the average number of sunspot.

Smarandache [6], [7] has given the notion of a neutrosophic set (NS). A NS is the extension of a crisp set, FS and IFS. In NS, truth membership (TM), falsity membership (FM) and indeterminacy membership (IM) are independent. In decision-making problems, the indeterminacy function is very significant. A NS and its extensions plays a vital role in many fields such as decision making problems, educational problems, image processing, medical diagnosis and conflict resolution. Moreover the field of neutrosophic probability, statistics, measures and logic have been developed in [8]. The generalization of fuzzy logic (FL) has been suggested by Smarandache in [8] and is termed as neutrosophic logic (NL). A proposition in NL is true (t), indeterminate (i) and false (f) are real values from the ranges T, I, F. T, I, F and also the sum of t, i, f are not restricted. In neutrosophic logic, there is indeterminacy term, which have no other logics, such as intuitionistic logic (IL), FL, boolean logic (BL) etc. Neutrosophic probability (NP) [8] is the extension of imprecise probability and classical probability. In NP, the chance occurs by an event is $t\%$ true, $i\%$ indeterminate and $f\%$ false where t, i, f varies in the subsets T, I and F respectively. Dynamically these subsets are functions based on parameters, but they are subsets on a static basis. In NP $n\_\sup \leq 3^{+}$, while in classical probability $n\_\sup \leq 1$. The extension of classical statistics is neutrosophic statistics [8] which is the analysis of events described by NP. There are twenty seven new definitions derived from NS, neutrosophic statistics and a neutrosophic probability. Each of these are independent. The sets derived from NS are intuitionistic set, paradoxist set, paraconsistent set, nihilist set, faillibilist set, trivialist set, and dialetheist set. Intuitionistic probability and statistics, faillibilist probability and statistics,tautological probability and statistics, dialetheist probabilityand statistics, paraconsistent probability and statistics, nihilist probability and statistics and trivialist probability and statistics are derived from neutosophic probability and statistics. N. A. Nabeeh [9] suggested a technique that would promote a personal selection process by integrating the neutrosophic analytical hierarchy process to show the ideal solution among distinct options with order preference tevhnique similar to an ideal solution (TOPSIS). M. A. Baset [10] introduced a new type of neutrosophy technique called type 2 neutrosophic numbers. By combining type 2 neutrosophic number and TOPSIS, they suggested a novel method T2NN-TOPSIS which is very useful in group decision making. They researched a multi criteria group decision making technique of the analytical network process method and Visekriterijusmska Optmzacija I Kommpromisno Resenje method under neutrosophic environment that deals high order imprecision and incomplete information [11]. M. A. Baet suggested a new strategy for estimating the smart medical device selecting process in a GDM in a vague decision environment. Neutrosophic with TOPSIS strategy is used in decision-making processes to deal with incomplete information, vagueness and uncertainty, taking into account the decision requirements in the information gathered by decision-makers [12]. They suggested the robust ranking method with NS to manage supply chain management (GSCM) performance and methods that have been widely employed to promote environmental efficiency and gain competitive benefits. The NS theory was used to manage imprecise understanding, linguistic imprecision, vague data and incomplete information [13]. Moreover M. A. Baset [14] et, al., used NS for assessment technique and decision-making to determine and evaluate the factors affecting supplier selection of supply chain management. T. Bera [15] et, al., defined a neutrosophic norm on a soft linear space known as neutrosophic soft linear space. They also modified the concept of neutrosophic soft (Ns) prime ideal over a ring. They presented the notion of Ns completely semi prime ideals, Ns completely prime ideals and Ns prime K-ideals [16]. Moreover T. Bera [17] introduced the concept of compactness and connectedness on Ns topological space along with their several characteristics. R. A. Cruz [18] et, al., discussed P-intersection, P- union, P-AND and P-OR of neutrosophic cubic sets and their related properties. N. Shah [19] et, al., studied neutrosophic soft graphs. They presented a link between neutorosophic soft sets and graphs. Moreover they also discussed the notion of strong neutosophic soft graphs.

Smarandache [20] discussed the idea of a single valued neutrosophic set (SVNS). A SVNS defined as for any space of points set U' with u in U', a SVNS W in U', the truth membership, false memebership and indeterminac membership functions denoetd as T_A, F_A and I_A respectively with T_A, F_A, $I_A \in [0,1]$ for each u in U'. A SVNS W is expressed as $W = \int_X \langle T_W(\nu), I_W(\nu), F_W(\nu) \rangle / \nu, \nu \in X,$ when X is continous. For a discrete case, a SVNS can be expressed as $W = \sum_{i=1}^{n} \langle T(\nu i), I(\nu i), F(\nu i) \rangle / \nu i,$ $\nu i \in X.$ Later, Maji [21] gave a new concept neutrosophic soft set (NSS). For any initial universal set W and any parameters set E with $A \subset E$ and $P(W)$ represents all the NS of W. The order set (ϕ, A) is said to be the soft NS over W where $\phi : A \to P(W)$. Arockiarani et al., [22] introduced fuzzy neutrosophic soft topological space and presents main results of fuzzy neutrosophic soft topological space. Later on the researchers linked the above theories with different field of sciences.

The purpose of this paper is to study the mappings such as contraction mapping, expansive mapping, non-expansive mapping, commuting mapping, and weakly commuting mapping used to attain the FP, CP and CFP of a neutrosophic soft set. We present some basic resultsnd particular examples of fixed points, coincidence points, common fixed points in which contraction mapping, expansive mapping, non-expansive mapping, commuting mapping, and weakly commuting mapping are used.

2. Preliminaries

We will discuss here the basic notions of NS and neutrosophic soft sets. We will also discuss some new neutrosophic soft mappings such as contraction mapping, increasing mapping, dominated mapping, dominating mapping, K-lipschitz mapping, non-expansive mapping, commuting mapping, weakly compatible mapping. Moreover we will study periodic point, common fixed point, coinciding point of neutrosophic soft-mapping. Here $\hat{N}S(U'_E)$ is the collection of all neutrosophic soft points.

Definition 2.1 [7] Let U be any universal set, with generic element $\nu \in U'$. A NS $\hat{N}$ is defined by $\hat{N} = \{\langle \nu, T_{\hat{N}}(\nu), I_{\hat{N}}(\nu), F_{\hat{N}}(\nu) \rangle, \nu \in U'\}$, where $T, I, F : U \to]^{-}0, 1^{+}[$ and

$$^{-}0 \le T_{\hat{N}}(\nu) + I_{\hat{N}}(\nu) + F_{\hat{N}}(\nu) \le 3^{+}.$$

$T_{\hat{N}}(\nu), I_{\hat{N}}(\nu)$ and $F_{\hat{N}}(\nu)$ denote TM, IM and FM functions respectively. In $]^{-}0, 1^{+}[, 1^{+} = 1 + \varepsilon,$ where ε is it's non-standard part and 1 is it's standard part. Likely $^{-}0 = 0 - \varepsilon,$ ε is it's non-standard part and 0 is it's standard part. It is difficult to employ these values in real life applications. Hence we take all the values of neutrosophic set from subset $[0,1]$.

Definition 2.2 [23] Let E and W be the set of parameters and initial universal set respectively. Let the power set of W is denoted by $P(W)$. Then a pair (β, A) is called soft set (SS) over W, where $A \subseteq E$ and $\beta : A \to P(W)$.

Definition 2.3 [21] Let E and W be the set of parameters and initial universal set respectively. Suppose that the set of all neutrosophic soft set (NSS) is denoted as $\hat{N}S(W)$. Then for $\mathrm{P} \subseteq E$, a pair (β, P) is called a $\hat{N}SS$ over W, where $\beta : \mathrm{P} \to \hat{N}S(W)$ is a mapping.

Definition 2.4 [24] Let E and W be the set of parameters and initial universal set respectively. Suppose that the set of all NSS is denoted as $\hat{N}S(W)$. A NSS $\hat{N}$ over W is a set which defined by a set valued function $\mathbf{P}_{\hat{N}}$ representing a mapping $\mathbf{P}_{\hat{N}} : E \to \hat{N}S(W)$. $\mathbf{P}_{\hat{N}}$ is known as approximate function of the $\hat{N}S(W)$. The neutrosophic soft set can be written as:

$$N = \{(e, \{\langle \upsilon, T_{\mathrm{P}_{\hat{N}}(e)}(\upsilon), I_{\mathrm{P}_{\hat{N}}(e)}(\upsilon), F_{\mathrm{P}_{\hat{N}}(e)}(\upsilon) \rangle \ : \ \upsilon \in W\}) : \ e \in E\}$$

where $T_{N'}(\nu), I_{N'}(\nu), F_{N'}(\nu)$ represents the TM, IM and FM functions of $\mathrm{P}_{\hat{N}}(e)$ respectively and has values in $[0,1]$. Also

$$0 \le T_{\mathrm{P}_{\hat{N}}(e)}(\nu), I_{\mathrm{P}_{\hat{N}}(e)}(\nu), F_{\mathrm{P}_{\hat{N}}(e)}(\nu) \le 3.$$

Definition 2.5 [22] Let U' be any universal set. The fuzzy neutrosophic set (fn-s) N' is defined as

$$N' = \{\langle \alpha, T_{N'}(\alpha), I_{N'}(\alpha), F_{N'}(\alpha) \rangle, \alpha \in X\}$$

where $T_{N'}(\alpha), I_{N'}(\alpha), F_{N'}(\alpha)$ represents the TM, IM and FM functions respectively and $T, I, F \ : \ N' \to [0,1]$. Also $0 \le T_{N'}(\alpha) + I_{N'}(\alpha) + F_{N'}(\alpha) \le 3.$

Definition 2.6 [22] Let E and W be the set of parameters and initial universal set respectively. Suppose that the set of all fuzzy neutrosophic soft set (FNS-set) is denoted as $\hat{FN}\, S(U'_E)$. Then for $\mathrm{P} \subseteq E$, a pair (β, P) is said to be a FNS-set over W , where $\beta \ : \ \mathrm{P} \to \hat{N}\, S(W)$ is a mapping.

Definition 2.7 [25] Let $\Lambda_{A'}, \Lambda_{B'}$ be two fuzzy neutrosophic soft set. An fuzzy neutrosophic soft (FNS) relation ξ from $\Lambda_{A'}$ to $\Lambda_{B'}$ is known as FNS mapping if the two conditions are fulfilled.

(i) For every $\Lambda^{\alpha}_{A'_1} \in \Lambda_{A'}$, there exists $\Lambda^{\alpha}_{B'_1} \in \Lambda_{B'}$, where $\Omega^{\alpha}_{A'_1}, \Omega^{\alpha}_{B'_1}$ are FNS elements.

(ii) For empty fuzzy FNS element in $\Lambda_{A'}$, the $\xi(\Lambda_{A'})$ is also empty FNS element.

Definition 2.8 [25] Let $\Lambda_{A'} \in FNS(W, R)$ be a FNS-set and $\phi \ : \ \Lambda_{A'} \to \Lambda_{A'}$ an FNS-mapping. A fuzzy neutrosophic element $\Lambda^{\alpha}_{A'}$ is called a fixed point of ϕ if $\phi(\Lambda^{\alpha}_{A'}) = \Lambda^{\alpha}_{A'}$.

Criterion [26], [27] Let $\hat{N}\, S(W)$ be the set of all neutrosophic points over (W, E). Then the neutrosophic soft metric on based of neutrosophic points is defined as $d \ : \ \hat{N}\, S(W_E) \to \hat{N}\, S(W_E)$ having the following properties.

M_1). $d(\Lambda^{\alpha}_{A'}, \Lambda^{\alpha}_{B'}) \ge 0$ for all $\Lambda^{\alpha}_{A'}, \Lambda^{\alpha}_{B'} \in \hat{N}\, S(W_E)$.

M_2). $d(\Lambda^{\alpha}_{A'}, \Lambda^{\alpha}_{B'}) = 0 \ \Leftrightarrow \ \Lambda^{\alpha}_{A'} = \Lambda^{\alpha}_{B'}$.

M_3). $d(\Lambda^{\alpha}_{A'}, \Lambda^{\alpha}_{B'}) = d(\Lambda^{\alpha}_{B'}, \Lambda^{\alpha}_{A'})$.

M_4). $d(\Lambda^{\alpha}_{A'}, \Lambda^{\alpha}_{B'}) \le d(\Lambda^{\alpha}_{A'}, \Lambda^{\alpha}_{C'}) + d(\Lambda^{\alpha}_{C'}, \Lambda^{\alpha}_{B'})$.

Then $(\hat{N}\, S(U'_E), d)$ is said to be neutrosophic soft metric space. Here $\Lambda^{\alpha}_{A'} = \Lambda^{\alpha}_{B'}$ implies $T_{\Lambda^{\alpha}_{A'}} = T_{\Lambda^{\alpha}_{B'}}, I_{\Lambda^{\alpha}_{A'}} = I_{\Lambda^{\alpha}_{B'}}$ and $F_{\Lambda^{\alpha}_{A'}} = F_{\Lambda^{\alpha}_{B'}}$.

3. Mappings on Neutrosophic Soft Set

Here, we introduced some new neutrosophic soft mappings such as contraction mapping, increasing mapping, dominated mapping, dominating mapping, K-lipschitz mapping, non-expansive mapping, commuting mapping, weakly compatible mapping. Also we introduced periodic point, common fixed point, coinciding point of neutrosophic soft-mapping. Here $\hat{N}\, S(U'_E)$ is the collection of all neutrosophic soft points.

Definition 3.1 Let ϕ be a mapping from $\hat{N}S(U'_E)$ to $\hat{N}S(U'_E)$. Then ϕ is called neutrosophic soft contraction if $d(\phi(\Lambda^{\alpha}_{A'}),\phi(\Lambda^{\alpha}_{B'}))\le kd(\Lambda^{\alpha}_{A'},\Lambda^{\alpha}_{B'})$ for all $\Lambda^{\alpha}_{A'},\Lambda^{\alpha}_{B'}\in F\hat{N}S(U'_E)$ and $k\in[0,1)$. Where k is called contraction factor.

Example 3.1 Let $U'=\{\theta_1,\theta_1,\theta_3\}$ be any initial universal set and $R=A'=B'=\{\alpha_1,\alpha_2\}$. Define a NSS $\Lambda^{\alpha}_{A'}$ and $\Lambda^{\alpha}_{B'}$ as below:

$$\Lambda^{\alpha}_{A'}=\{(\alpha_1,\{\langle\theta_1,0.8,0.1,0.3\rangle,\langle\theta_2,0.6,0.7,0.4\rangle,\langle\theta_3,1,0.2,0.4\rangle\}),\\ (\alpha_2,\{\langle\theta_1,0.3,0.7,0.6\rangle,\langle\theta_2,0.1,0.9,0.3\rangle,\langle\theta_3,0.1,0.8,0.7\rangle\})\}$$

and

$$\Lambda^{\alpha}_{B'}=\{(\alpha_1,\{\langle\theta_1,0.9,0.7,0.1\rangle,\langle\theta_2,1,0.8,0.6\rangle,\langle\theta_3,1,0.2,0.4\rangle\}\}),\\ (\alpha_2,\{\langle\theta_1,0.1,0.3,0.6\rangle,\langle\theta_2,0.2,0.3,0.9\rangle,\langle\theta_3,0.1,0.8,0.7\rangle\})\}.$$

The distance defined [27] as

$$d(\phi(\Lambda^{\alpha_1}_{A'}),\phi(\Lambda^{\alpha_2}_{A'}))=\min_{\theta_i}\{(|T_{\Lambda^{\alpha_1}_{B'}}(\theta_i)-T_{\Lambda^{\alpha_2}_{B'}}(\theta_i)|^p+|I_{\Lambda^{\alpha_1}_{B'}}(\theta_i)-I_{\Lambda^{\alpha_2}_{B'}}(\theta_i)|^p+|T_{\Lambda^{\alpha_1}_{B'}}(\theta_i)-T_{\Lambda^{\alpha_2}_{B'}}(\theta_i)|^p)^{\frac{1}{p}}\}$$

$(p\ge 1)$.

In this example, we take $p=1$, now

$$\begin{aligned} d(\phi(\Lambda^{\alpha_1}_{A'_1}),\phi((\Lambda^{\alpha_2}_{A'_1}))&=\min_{\theta_i}\{|T_{\Lambda^{\alpha_1}_{B'}}(\theta_i)-T_{\Lambda^{\alpha_2}_{B'}}(\theta_i)|+|I_{\Lambda^{\alpha_1}_{B'}}(\theta_i)-I_{\Lambda^{\alpha_2}_{B'}}(\theta_i)|\\ &\quad+|F_{\Lambda^{\alpha_1}_{B'}}(\theta_i)-F_{\Lambda^{\alpha_2}_{B'}}(\theta_i)|\}\\ &=|T_{\Lambda^{\alpha_1}_{B'}}(\theta_2)-T_{\Lambda^{\alpha_2}_{B'}}(\theta_2)|+|I_{\Lambda^{\alpha_1}_{B'}}(\theta_2)-I_{\Lambda^{\alpha_2}_{B'}}(\theta_2)|\\ &\quad+|F_{\Lambda^{\alpha_1}_{B'}}(\theta_2)-F_{\Lambda^{\alpha_2}_{B'}}(\theta_2)|\\ &=|1-0.2|+|0.8-0.3|+|0.6-0.9|\\ &=0.8+0.5+0.3\\ &=0.16\\ &=(0.2)(0.8)\\ &=0.2d(\Lambda^{\alpha_1}_{A'_1},\Lambda^{\alpha_2}_{A'_1}). \end{aligned}$$

Here $k=0.2$, so ϕ is a contraction.

Definition 3.2 Let ϕ be a mapping from $\hat{N}S(W_E)$ to $F\hat{N}S(W_E)$. Then ϕ is called neutrosophic soft non-expansive mapping if $d(\phi(\Lambda^{\alpha}_{A'}),\phi(\Lambda^{\alpha}_{B'}))\le kd(\Lambda^{\alpha}_{A'},\Lambda^{\alpha}_{B'})$ for all $\Lambda^{\alpha}_{A'},\Lambda^{\alpha}_{B'}\in\hat{N}S(W_E)$ and $k=1$.

Example 3.2 Let $W=\{\upsilon_1,\upsilon_2,\upsilon_3\}$ and $R=A'=B'=\{\alpha_1,\alpha_2\}$. Define a neutrosophic soft sets $\Lambda^{\alpha}_{A'}$ and $\Lambda^{\alpha}_{B'}$ as follows:

$$\Lambda^{\alpha}_{A'}=\{(\alpha_1,\{\langle\upsilon_1,1,0.1,0.2\rangle,\langle\upsilon_2,0.6,0.7,0.4\rangle,\langle\upsilon_3,0.2,0.4,0.6\rangle\}),\\ (\alpha_2,\{\langle\upsilon_1,0.3,0.7,0.6\rangle,\langle\upsilon_2,0.1,0.9,0.3\rangle,\langle\upsilon_3,0.4,0.6,0.7\rangle\})\}$$

and

$$\Lambda_{B'}^{\alpha} = \{(\alpha_1, \{\langle \upsilon_1, 1, 0.5, 0.2\rangle, \langle \upsilon_2, 1, 0.5, 0.6\rangle, \langle \upsilon_3, 0.2, 0.4, 0.6\rangle\}\}),$$
$$(\alpha_2, \{\langle \upsilon_1, 0.1, 0.3, 0.6\rangle, \langle \upsilon_2, 0.2, 0.3, 0.9\rangle, \langle \upsilon_3, 0.4, 0.6, 0.7\rangle\})\}.$$

$$\begin{aligned} d(\phi(\Lambda_{A_1'}^{\alpha_1}), \phi((\Lambda_{A_1'}^{\alpha_2})) &= \min_{x_i}\{| T_{\Lambda_{B'}^{\alpha_1}}(\upsilon_i) - T_{\Lambda_{B'}^{\alpha_2}}(\upsilon_i) | + | I_{\Lambda_{B'}^{\alpha_1}}(\upsilon_i) - I_{\Lambda_{B'}^{\alpha_2}}(\upsilon_i) | \\ &+ | F_{\Lambda_{B'}^{\alpha_1}}(\upsilon_i) - F_{\Lambda_{B'}^{\alpha_2}}(\upsilon_i) |\} \\ &= | T_{\Lambda_{B'}^{\alpha_1}}(\upsilon_3) - T_{\Lambda_{B'}^{\alpha_2}}(\upsilon_3) | + | I_{\Lambda_{A'}^{\alpha_1}}(\upsilon_3) - I_{\Lambda_{B'}^{\alpha_2}}(\upsilon_3) | \\ &+ | F_{\Lambda_{A'}^{\alpha_1}}(\upsilon_3) - F_{\Lambda_{B'}^{\alpha_2}}(\upsilon_3) | \\ &= | 0.2 - 0.4 | + | 0.4 - 0.6 | + | 0.6 - 0.7 | \\ &= 0.2 + 0.2 + 0.1 \\ &= 0.5 \\ &= (1)(0.5) \\ &= 1d(\Lambda_{A_1'}^{\alpha_1}, \Lambda_{A_1'}^{\alpha_2}). \end{aligned}$$

Here $k = 1$, so ϕ is non-expansive.

Definition 3.3 Let ϕ be a mapping from $\hat{N}S(W_E)$ to $\hat{N}S(W_E)$. Then ϕ is called neutrosophic soft k-Lipschitz mapping if $d(\phi(\Lambda_{A'}^{\alpha}), \phi(\Lambda_{B'}^{\alpha})) \leq kd(\Lambda_{A'}^{\alpha}, \Lambda_{B'}^{\alpha})$ for all $\Lambda_{A'}^{\alpha}, \Lambda_{B'}^{\alpha} \in F\hat{N}S(W_E)$ and $k > 0$.

Example 3.3 Let $W = \{\upsilon_1, \upsilon_2, \upsilon_3\}$ and $R = A' = B' = \{\alpha_1, \alpha_2\}$. Define a NSS $\Lambda_{A'}^{\alpha}$ and $\Lambda_{B'}^{\alpha}$ as below:

$$\Lambda_{A'}^{\alpha} = \{(\alpha_1, \{\langle \upsilon_1, 0.3, 0.4, 0.3\rangle, \langle \upsilon_2, 0.6, 0.7, 0.4\rangle, \langle \upsilon_3, 0.2, 0.4, 0.6\rangle\}),$$
$$(\alpha_2, \{\langle \upsilon_1, 0.5, 0.6, 0.4\rangle, \langle \upsilon_2, 0.1, 0.9, 0.3\rangle, \langle \upsilon_3, 0.4, 0.6, 0.7\rangle\})\}$$

and

$$\Lambda_{B'}^{\alpha} = \{(\alpha_1, \{\langle \upsilon_1, 1, 0.4, 0.3\rangle, \langle \upsilon_2, 1, 0.6, 0.3\rangle, \langle \upsilon_3, 0.2, 0.4, 0.6\rangle\}\}),$$
$$(\alpha_2, \{\langle \upsilon_1, 0.5, 0.7, 0.5\rangle, \langle \upsilon_2, 0.3, 0.2, 0.9\rangle, \langle \upsilon_3, 1, 0.3, 0.9\rangle\})\}.$$

$$
\begin{aligned}
d(\phi(\Lambda^{\alpha_1}_{A_1'}),\phi((\Lambda^{\alpha_2}_{A_1'})) &= \min_{?_i}\{|T_{\Lambda^{\alpha_1}_{B'}}(\upsilon_i)-T_{\Lambda^{\alpha_2}_{B'}}(\upsilon_i)|+|I_{\Lambda^{\alpha_1}_{B'}}(\upsilon_i)-I_{\Lambda^{\alpha_2}_{B'}}(\upsilon_i)| \\
&\quad +|F_{\Lambda^{\alpha_1}_{B'}}(\upsilon_i)-F_{\Lambda^{\alpha_2}_{B'}}(\upsilon_i)|\} \\
&=|T_{\Lambda^{\alpha_1}_{B'}}(\upsilon_1)-T_{\Lambda^{\alpha_2}_{B'}}(\upsilon_1)|+|I_{\Lambda^{\alpha_1}_{A'}}(\upsilon_1)-I_{\Lambda^{\alpha_2}_{B'}}(\upsilon_1)| \\
&\quad +|F_{\Lambda^{\alpha_1}_{A'}}(\upsilon_1)-F_{\Lambda^{\alpha_2}_{B'}}(\upsilon_1)| \\
&=|1-0.5|+|0.4-0.7|+|0.3-0.5| \\
&= 0.5+0.3+0.2 \\
&= 1 \\
&= (2)(0.5) \\
&= 2d(\Lambda^{\alpha_1}_{A_1'},\Lambda^{\alpha_2}_{A_1'}).
\end{aligned}
$$

Here $k=2,$ so ϕ is k-lipschitz.

Note: Every neutrosophic soft contraction mapping is neutrosophic soft K-lipschitz mapping but its converse does not hold.

Definition 3.4 Let ϕ be a mapping from $\hat{N}S(W_E)$ to $\hat{N}S(W_E)$. Then ϕ is said to be neutrosophic soft kanan contraction if $d(\phi(\Lambda^{\alpha}_{A'}),\phi(\Lambda^{\alpha}_{B'}))\le k[d(\Lambda^{\alpha}_{A'},\phi(\Lambda^{\alpha}_{A'}))+d(\Lambda^{\alpha}_{B'},\phi(\Lambda^{\alpha}_{B'}))]$ for all $\Lambda^{\alpha}_{A'},\Lambda^{\alpha}_{B'}\in\hat{N}S(W_E)$ and $k\in[0,\frac{1}{2})$. Where k is called contraction factor.

Definition 3.5 Let ϕ and ψ be two mappings from $\hat{N}S(U'_E)$ to $\hat{N}S(U'_E)$. Then ϕ and ψ are called neutrosophic soft commuting mapping if $\phi(\psi(\Omega^{\alpha}_{A'}))=\psi(\phi(\Omega^{\alpha}_{A'}))$ for all $\Omega^{\alpha}_{A'}\in\hat{N}S(U'_E)$.

Definition 3.6 Let ϕ and ψ be two mappings from $\hat{N}S(U'_E)$ to $\hat{N}S(U'_E)$. Then ϕ and ψ are called neutrosophic soft weakly commuting mapping if $d(\phi(\psi(\Lambda^{\alpha}_{A'})),\psi(\phi(\Lambda^{\alpha}_{A'})))\le d(\phi(\Lambda^{\alpha}_{A'}),\psi(\Lambda^{\alpha}_{A'}))$ for all $\Lambda^{\alpha}_{A'}\in\hat{N}S(U'_E)$.

Definition 3.7 Let ϕ and ψ be two mappings from $\hat{N}S(U'_E)$ to $\hat{N}S(U'_E)$. If for $\phi(\Omega^{\alpha}_{A_n'})\to\Omega^{\alpha}_{A_0'}$ and $\psi(\Omega^{\alpha}_{A_n'})\to\Omega^{\alpha}_{A_0'}$ as $n\to\infty$ and $\Omega^{\alpha}_{A_n'},\Omega^{\alpha}_{A_0'}\in\hat{N}S(U'_E)$. Then it is called neutrosophic soft compatible mapping if $\lim_{n\to\infty}d(\phi(\psi(\Omega^{\alpha}_{A'})),\psi(\phi(\Omega^{\alpha}_{A'})))\to 0$.

Definition 3.8 Let ϕ,ψ : $\hat{N}S(U'_E)\to\hat{N}S(U'_E)$ be two mappings. If there is $\Omega^{\alpha}_{A'}\in\hat{N}S(U'_E)$ such that $\phi(\Omega^{\alpha}_{A'})=\psi(\Omega^{\alpha}_{A'})=\Omega^{\alpha}_{A'}$, then $\Omega^{\alpha}_{A'}\in\hat{N}S(U'_E)$ is called common fixed point neutrosophic soft mappings.

Definition 3.9 If $\Omega^{\alpha}_{A'}$ is a fixed point of ϕ : $\hat{N}S(U'_E)\to\hat{N}S(U'_E)$, then $\Omega^{\alpha}_{A'}$ is also a fixed point ϕ^k that is $\phi^k(\Omega^{\alpha}_{A'})=\Omega^{\alpha}_{A'}$ for all $\Omega^{\alpha}_{A'}\in\hat{N}S(U'_E)$. So $\Omega^{\alpha}_{A'}$ is called periodic point of neutrosophic soft mapping ϕ and k is called period of ϕ.

Remark Every fixed point of neutrosophic soft mapping is a periodic point but every periodic point of neutrosophic soft mapping is not a fixed point.

Definition 3.9 Let ϕ, ψ be two mappings from $\hat{N}S(U'_E)$ to $\hat{N}S(U'_E)$. If $\phi(\Omega^{\alpha}_{A'})=\psi(\Omega^{\alpha}_{A'})=\Omega^{\alpha}_{B'}$, for all

$\Omega^{\alpha}_{A'}, \Omega^{\alpha}_{B'} \in F\hat{N}S(U'_E)$. Then $\Omega^{\alpha}_{A'}$ is called coincidence point of ϕ and ψ and $\Omega^{\alpha}_{B'}$ is called point of coincidence for ϕ and ψ.

Definition 3.10 Let $\phi : \hat{N}S(U'_E) \to \hat{N}S(U'_E)$ be a mapping. Then ϕ is said to be neutrosophic soft increasing map if for any $\Omega^{\alpha}_{A'} \le \Omega^{\alpha}_{B'}$ implies $\phi(\Omega^{\alpha}_{A'}) \le \phi(\Omega^{\alpha}_{B'})$ for all $\Omega^{\alpha}_{A'}, \Omega^{\alpha}_{B'} \in \hat{N}S(U'_E)$.

Definition 3.11 Let $\phi : \hat{N}S(U'_E) \to \hat{N}S(U'_E)$ be a mapping. Then ϕ is said to be neutrosophic soft dominated map if $\phi(\Omega^{\alpha}_{A'}) \le \Omega^{\alpha}_{A'}$ for all $\Omega^{\alpha}_{A'} \in \hat{N}S(U'_E)$.

Definition 3.12 Let $\phi : \hat{N}S(U'_E) \to \hat{N}S(U'_E)$ be a mapping. Then ϕ is said to be neutrosophic soft dominating map if $\Omega^{\alpha}_{A'} \le \phi(\Omega^{\alpha}_{A'})$ for all $\Omega^{\alpha}_{A'} \in \hat{N}S(U'_E)$.

4. Main Results

Banach Contraction Theorem

Proposition 1 Let $\hat{N}S(U'_E)$ be a non-empty set of neutrosophic points and $(\hat{N}S(U'_E), d)$ be a complete neutrosophic soft metric space. Suppose ϕ is a mapping from $\hat{N}S(U'_E)$ to $\hat{N}S(U'_E)$ be contraction. Then fixed point of ϕ exists and unique.

Proof Let $\Omega^{\alpha}_{A'_0} \in \hat{N}S(U'_E)$ be arbitrary. Define $\Omega^{\alpha}_{A'_1} = \phi(\Omega^{\alpha}_{A'_0})$ and by continuing we have a sequence in the form $\Omega^{\alpha}_{A'_{n+1}} = \phi(\Omega^{\alpha}_{A'_n})$. Now

$$\begin{aligned}
d(\Omega^{\alpha}_{A'_{n+1}}, \Omega^{\alpha}_{A'_n}) &= d(\phi(\Omega^{\alpha}_{A'_n}), \phi(\Omega^{\alpha}_{A'_{n-1}})) \\
&\le kd(\Omega^{\alpha}_{A'_n}, \Omega^{\alpha}_{A'_{n-1}}) \\
&= kd(\phi(\Omega^{\alpha}_{A'_{n-1}}), \phi(\Omega^{\alpha}_{A'_{n-2}})) \\
&\le k^2 d(\Omega^{\alpha}_{A'_{n-1}}, \Omega^{\alpha}_{A'_{n-2}}) \\
&= k^2 d(\phi(\Omega^{\alpha}_{A'_{n-2}}), \phi(\Omega^{\alpha}_{A'_{n-3}})) \\
&\le k^3 d(\Omega^{\alpha}_{A'_{n-2}}, \Omega^{\alpha}_{A'_{n-3}}) \\
&\quad . \\
&\quad . \\
&\quad . \\
&\le k^n d(\Omega^{\alpha}_{A'_1}, \Omega^{\alpha}_{A'_0}).
\end{aligned}$$

Now for $m, n > n_0$, we have

$$
\begin{aligned}
d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_{n}}) &\leq d(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha}_{A'_{n+1}}) + d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_{n+2}}) + \ldots + d(\Omega^{\alpha}_{A'_{m-1}},\Omega^{\alpha}_{A'_{m}}) \\
&\leq k^{n} d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}}) + k^{n+1} d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}}) + \ldots + k^{m-1} d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}}) \\
&= k^{n}[1 + k + k^{2} + \ldots + k^{m-n-1}] d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}}) \\
&= \frac{k^{n}}{1-k} d(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{A'_{0}})
\end{aligned}
$$

$$d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_{n}}) \to 0 \text{ as } n \to \infty.$$

So $\Omega^{\alpha}_{A'_{n}}$ is a cauchy sequence in $(\hat{N}S(U'_{E}),d)$, but $(\hat{N}S(U'_{E}),d)$ is complete, so there exists $\Omega^{\alpha*}_{A'} \in \hat{N}S(U'_{E})$ such that $d(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha*}_{A'}) \to 0$ as $n \to \infty$. Now

$$
\begin{aligned}
d(\Omega^{\alpha}_{A'_{n+1}},\phi(\Omega^{\alpha*}_{A'})) &= d(\phi(\Omega^{\alpha}_{A'_{n}}),\phi(\Omega^{\alpha*}_{A'})) \\
&\leq kd(\Omega^{\alpha}_{A'_{n}},\Omega^{\alpha*}_{A'}).
\end{aligned}
$$

On taking limit as $n \to \infty$, we get

$$d(\phi(\Omega^{\alpha*}_{A'}),\Omega^{\alpha*}_{A'}) \leq 0.$$

But

$$d(\phi(\Omega^{\alpha*}_{A'}),\Omega^{\alpha*}_{A'}) \geq 0.$$

So

$$
\begin{aligned}
d(\phi(\Omega^{\alpha*}_{A'}),\Omega^{\alpha*}_{A'}) &= 0 \\
\phi(\Omega^{\alpha*}_{A'}) &= \Omega^{\alpha*}_{A'}.
\end{aligned}
$$

So $\Omega^{\alpha*}_{A'}$ is the FP of ⩟

Now we have to show that $\Omega^{\alpha*}_{A'}$ is unique. Suppose there exists another FP $\Omega^{\alpha*}_{B'} \in \hat{N}S(U'_{E})$ such that $\phi(\Omega^{\alpha*}_{B'}) = \Omega^{\alpha*}_{B'}$. Now

$$
\begin{aligned}
d(\Omega^{\alpha*}_{A'},\Omega^{\alpha*}_{B'}) &= d(\phi(\Omega^{\alpha*}_{A'}),\phi(\Omega^{\alpha*}_{B'})) \\
&\leq kd(\Omega^{\alpha*}_{A'},\Omega^{\alpha*}_{B'})
\end{aligned}
$$

$$(1-k)d(\Omega^{\alpha*}_{A'},\Omega^{\alpha*}_{B'}) \leq 0.$$

Here $(1-k) \nleq 0$, so

$$d(\Omega^{\alpha ?}_{A'},\Omega^{\alpha ?}_{B'}) \leq 0.$$

But

$$
\begin{aligned}
d(\Omega^{\alpha*}_{A'},\Omega^{\alpha*}_{B'}) &\geq 0 \\
d(\Omega^{\alpha*}_{A'},\Omega^{\alpha*}_{B'}) &= 0.
\end{aligned}
$$

Hence $\Omega^{\alpha*}_{A'} = \Omega^{\alpha*}_{B'}$, so the fixed point is unique.

Proposition 2 Let $(\hat{N}S(U'_{E}),d)$ be a complete neutrosophic soft metric space. Suppose ϕ be a mapping from $F\hat{N}S(U'_{E})$ to $F\hat{N}S(U'_{E})$ satisfies the contraction $d(\phi^{m}(\Omega^{\alpha}_{A'_{1}}),\phi^{m}(\Omega^{\alpha}_{B'_{1}})) \leq kd(\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{B'_{1}})$ for all $\Omega^{\alpha}_{A'_{1}},\Omega^{\alpha}_{B'_{1}} \in \hat{N}S(U'_{E})$, where $k \in [0,1)$ and m is any natural number. Then ϕ has a FP.

Proof It follows from banach contraction theorem that ϕ^m has unique a FP that is $\phi^m(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{A_1'}$. Now

$$\begin{aligned}\phi^m(\phi(\Omega^{\alpha}_{A_1'})) &= \phi^{m+1}(\Omega^{\alpha}_{A_1'}) \\ &= \phi(\phi^m(\Omega^{\alpha}_{A_1'})) \\ &= \phi(\Omega^{\alpha}_{A_1'}).\end{aligned}$$

By the uniqueness of FP, we have $\phi(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{A_1'}$.

Proposition 3 Let $(\hat{N}S(U_E'), d)$ be a complete neutrosophic soft metric space. Suppose ϕ , ψ satisfy $d(\phi(\Omega^{\alpha}_{A_1'}), \psi(\Omega^{\alpha}_{B_1'})) \le \alpha d(\Omega^{\alpha}_{A_1'}, \phi(\Omega^{\alpha}_{A_1'})) + \beta d(\Omega^{\alpha}_{B_1'}, \psi(\Omega^{\alpha}_{B_1'})) + \gamma[d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{B_1'})) + d(\Omega^{\alpha}_{B_1'}, \phi(\Omega^{\alpha}_{A_1'})]$ for all $\Omega^{\alpha}_{A_1'}, \Omega^{\alpha}_{B_1'} \in F\hat{N}S(U_E')$ with α, β, γ are non-negative and $\alpha + \beta + \gamma < 1$. Then ϕ and ψ have a unique FP.

Proof Let $\Omega^{\alpha}_{A_1'} \in \hat{N}S(U_E')$ be a fixed point of ϕ that is $\phi(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{A_1'}$. We need to show that $\psi(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{A_1'}$.
Now

$$\begin{aligned}d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) &= d(\phi(\Omega^{\alpha}_{A_1'}), \psi(\Omega^{\alpha}_{A_1'})) \\ &\le \alpha d(\Omega^{\alpha}_{A_1'}, \phi(\Omega^{\alpha}_{A_1'})) + \beta d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) + \gamma[d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) + d(\Omega^{\alpha}_{A_1'}, \phi(\Omega^{\alpha}_{A_1'}))] \\ &= \alpha d(\Omega^{\alpha}_{A_1'}, \Omega^{\alpha}_{A_1'}) + \beta d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) + \gamma[d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) + d(\Omega^{\alpha}_{A_1'}, \Omega^{\alpha}_{A_1'})] \\ &= \beta d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'}) + \gamma d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'}))\end{aligned}$$

$$(1-\beta-\gamma) d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) \le 0$$

Since $(1-\beta-\gamma) \not\le 0$, so

$$d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) \le 0.$$

But

$$d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) \ge 0$$

hence

$$d(\Omega^{\alpha}_{A_1'}, \psi(\Omega^{\alpha}_{A_1'})) = 0.$$

Thus $\psi(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{A_1'}$.

Proposition 4 Let $\hat{N}S(U_E')$ be a non-empty set of neutrosophic points and $(\hat{N}S(U_E'), d)$ be a complete neutrosophic soft metric space. Suppose ϕ is a mapping from $\hat{N}S(U_E')$ to $\hat{N}S(U_E')$ be kanan contraction. Then fixed point of ϕ exists and unique.

Proof Let $\Omega^{\alpha}_{A_0'} \in \hat{N}S(U_E')$ be arbitrary. Define $\Omega^{\alpha}_{A_1'} = \phi(\Omega^{\alpha}_{A_0'})$ and by continuing we have a sequence in the form $\Omega^{\alpha}_{A_{n+1}'} = \phi(\Omega^{\alpha}_{A_n'})$. Now

$$\begin{aligned}
d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_n}) &= d(\phi(\Omega^{\alpha}_{A'_n}),\phi(\Omega^{\alpha}_{A'_{n-1}}))\\
&\le k[d(\Omega^{\alpha}_{A'_n},\phi(\Omega^{\alpha}_{A'_n}))+d(\Omega^{\alpha}_{A'_{n-1}},\phi(\Omega^{\alpha}_{A'_{n-1}}))\\
&= k[d(\Omega^{\alpha}_{A'_n},\Omega^{\alpha}_{A'_{n+1}})+d(\Omega^{\alpha}_{A'_{n-1}},\Omega^{\alpha}_{A'_n})]\\
&= kd(\Omega^{\alpha}_{A'_n},\Omega^{\alpha}_{A'_{n+1}})+kd(\Omega^{\alpha}_{A'_{n-1}},\Omega^{\alpha}_{A'_n})\\
(1-k)d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_n}) &\le kd(\Omega^{\alpha}_{A'_n},\Omega^{\alpha}_{A'_{n-1}})\\
d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_n}) &\le \frac{k}{1-k}d(\Omega^{\alpha}_{A'_n},\Omega^{\alpha}_{A'_{n-1}})\\
&= hd(\Omega^{\alpha}_{A'_n},\Omega^{\alpha}_{A'_{n-1}})
\end{aligned}$$

for $h=\frac{k}{1-k}$

$$\begin{aligned}
d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_n}) &\le hd(\Omega^{\alpha}_{A'_n},\Omega^{\alpha}_{A'_{n-1}})\\
&\le h^2 d(\Omega^{\alpha}_{A'_{n-1}},\Omega^{\alpha}_{A'_{n-2}})\\
&\le h^3 d(\Omega^{\alpha}_{A'_{n-2}},\Omega^{\alpha}_{A'_{n-3}})\\
&\quad .\\
&\quad .\\
&\quad .\\
&\le h^n d(\Omega^{\alpha}_{A'_1},\Omega^{\alpha}_{A'_0}).
\end{aligned}$$

For $m>n$

$$\begin{aligned}
d(\Omega^{\alpha}_{A'_n},\Omega^{\alpha}_{A'_m}) &\le d(\Omega^{\alpha}_{A'_n},\Omega^{\alpha}_{A'_{n+1}})+d(\Omega^{\alpha}_{A'_{n+1}},\Omega^{\alpha}_{A'_{n+2}})+...+d(\Omega^{\alpha}_{A'_{m-1}},\Omega^{\alpha}_{A'_m})\\
&\le h^n d(\Omega^{\alpha}_{A'_1},\Omega^{\alpha}_{A'_0})+h^{n+1}d(\Omega^{\alpha}_{A'_1},\Omega^{\alpha}_{A'_0})+...+h^{m-1}d(\Omega^{\alpha}_{A'_1},\Omega^{\alpha}_{A'_0})\\
&= h^n[1+h+h^2+...+h^{m-n-1}]d(\Omega^{\alpha}_{A'_1},\Omega^{\alpha}_{A'_0})\\
&= h^n(\frac{1}{1+h})d(\Omega^{\alpha}_{A'_1},\Omega^{\alpha}_{A'_0})\\
d(\Omega^{\alpha}_{A'_n},\Omega^{\alpha}_{A'_m}) &\to 0 \text{ as } n\to\infty.
\end{aligned}$$

The sequence $\Omega^{\alpha}_{A'_n}$ is a cauchy sequence in $(\tilde{N}S(U'_E),d)$. Since $(\tilde{N}S(U'_E),d)$ is complete, so $\Omega^{\alpha}_{A'_n}$ converges to any $\Omega^{\alpha*}_{A'}\in\tilde{N}S(U'_E)$. Now

$$\begin{aligned}
d(\phi(\Omega^{\alpha*}_{A'}),\Omega^{\alpha}_{A'_{n+1}}) &= d(\phi(\Omega^{\alpha*}_{A'}),\phi(\Omega^{\alpha}_{A'_n}))\\
&\le h[d(\Omega^{\alpha*}_{A'},\phi(\Omega^{\alpha*}_{A'}))+d(\Omega^{\alpha}_{A'_n},\phi(\Omega^{\alpha}_{A'_n}))].
\end{aligned}$$

Taking limit as $n\to\infty$, we have

$$\begin{aligned}
d(\phi(\Omega^{\alpha*}_{A'}),\Omega^{\alpha*}_{A'}) &\le h[d(\Omega^{\alpha*}_{A'},\phi(\Omega^{\alpha*}_{A'}))+d(\Omega^{\alpha*}_{A'},\phi(\Omega^{\alpha*}_{A'}))]\\
&= 2hd(\Omega^{\alpha*}_{A'},\phi(\Omega^{\alpha*}_{A'}))\\
(1-2h)d(\phi(\Omega^{\alpha*}_{A'}),\Omega^{\alpha*}_{A'}) &\le 0
\end{aligned}$$

As $(1-2h)\not\le 0$, so

$$d(\phi(\Omega_{A'}^{\alpha*}),\Omega_{A'}^{\alpha*})\leq 0$$

but

$$d(\phi(\Omega_{A'}^{\alpha*}),\Omega_{A'}^{\alpha*})\geq 0$$

thus

$$d(\phi(\Omega_{A'}^{\alpha*}),\Omega_{A'}^{\alpha*})=0.$$

Hence $\Omega_{A'}^{\alpha*}\in \hat{N}S(U_E^{'})$ is a FP of ϕ.

Suppose $\Omega_{B'}^{\alpha*}\in \hat{N}S(U_E^{'})$ be another FP. Now

$$\begin{aligned}d(\Omega_{A'}^{\alpha*},\Omega_{B'}^{\alpha*}) &= d(\phi(\Omega_{A'}^{\alpha*}),\phi(\Omega_{B'}^{\alpha*}))\\ &\leq h[d(\Omega_{A'}^{\alpha*},\phi(\Omega_{A'}^{\alpha*}))+d(\Omega_{B'}^{\alpha*},\phi(\Omega_{B'}^{\alpha*}))]\\ &\leq h[d(\Omega_{A'}^{\alpha*},\Omega_{A'}^{\alpha*})+d(\Omega_{B'}^{\alpha*},\Omega_{B'}^{\alpha*})]\end{aligned}$$

$$d(\Omega_{A'}^{\alpha*},\Omega_{B'}^{\alpha*})\leq 0 \qquad (1)$$

but

$$d(\Omega_{A'}^{\alpha*},\Omega_{B'}^{\alpha*})\geq 0. \qquad (2)$$

From (1) and (2) we have

$$d(\Omega_{A'}^{\alpha*},\Omega_{B'}^{\alpha*})=0.$$

Hence $\Omega_{A'}^{\alpha*}=\Omega_{B'}^{\alpha*}$.

Proposition 5 Let ϕ,ψ : $\hat{N}S(U_E^{'})\rightarrow \hat{N}S(U_E^{'})$ be weakly compatible maps. If ϕ and ψ have unique coincidence point. Then ϕ and ψ have unique common fixed point (CFP).

Proof Suppose there is $\Omega_{A_1'}^{\alpha}\in \hat{N}S(U_E^{'})$ such that $\phi(\Omega_{A_1'}^{\alpha})=\psi(\Omega_{A_1'}^{\alpha})=\Omega_{B_1'}^{\alpha}$. Since ϕ and ψ are weakly compatible, so $\phi(\psi(\Omega_{A_1'}^{\alpha}))=\psi(\phi(\Omega_{A_1'}^{\alpha}))$ for all $\Omega_{A_1'}^{\alpha}\in \hat{N}S(U_E^{'})$. Now

$$\phi(\Omega_{B_1'}^{\alpha})=\phi(\Omega_{B_1'}^{\alpha})=\phi(\psi(\Omega_{A_1'}^{\alpha}))=\psi(\phi(\Omega_{A_1'}^{\alpha})).$$

So $\Omega_{B_1'}^{\alpha}$ is also coincidence point (CP) of ϕ and ψ, but $\Omega_{A_1'}^{\alpha}$ is the unique CP of ϕ and ψ, so

$$\phi(\Omega_{A_1'}^{\alpha})=\psi(\Omega_{A_1'}^{\alpha})=\phi(\Omega_{B_1'}^{\alpha})=\psi(\Omega_{B_1'}^{\alpha})$$

$$\Omega_{B_1'}^{\alpha}=\phi(\Omega_{B_1'}^{\alpha})=\psi(\Omega_{B_1'}^{\alpha}).$$

So $\Omega_{B_1'}^{\alpha}\in \hat{N}S(U_E^{'})$ is CFP.

Proposition 6 Let $(\hat{N}S(U_E^{'}),d)$ be a complete metric space and ϕ : $\hat{N}S(U_E^{'})\rightarrow \hat{N}S(U_E^{'})$ be a mapping satisfies $d(\phi^2(\Omega_{A_1'}^{\alpha}),\phi(\Omega_{A_1'}^{\alpha}))\leq kd(\phi(\Omega_{A_1'}^{\alpha}),\Omega_{A_1'}^{\alpha})$ for all $\Omega_{A_1'}^{\alpha}\in \hat{N}S(U_E^{'})$ and $k\in[0,1)$. Then fixed point of ϕ is singleton.

Proof Let $\Omega_{A_0'}^{\alpha}\in \hat{N}S(U_E^{'})$ be arbitrary and defines $\Omega_{A_{n+1}'}^{\alpha}=\phi^n(\Omega_{A_0'}^{\alpha})=\phi(\Omega_{A_n'}^{\alpha})$. Now

$$
\begin{aligned}
d(\phi^{n+1}(\Omega^{\alpha}_{A_0'}),\phi^{n}(\Omega^{\alpha}_{A_0'})) &\leq kd(\phi^{n}(\Omega^{\alpha}_{A_0'}),\phi^{n-1}(\Omega^{\alpha}_{A_0'}))\\
&\leq k^2 d(\phi^{n-1}(\Omega^{\alpha}_{A_0'}),\phi^{n-2}(\Omega^{\alpha}_{A_0'}))\\
&\leq k^3 d(\phi^{n-3}(\Omega^{\alpha}_{A_0'}),\phi^{n-4}(\Omega^{\alpha}_{A_0'}))\\
&\ \ .\\
&\ \ .\\
&\ \ .\\
&\leq k^n d(\phi(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha}_{A_0'}).
\end{aligned}
$$

Now for $m > n$

$$
\begin{aligned}
d(\phi^{m}(\Omega^{\alpha}_{A_0'}),\phi^{n}(\Omega^{\alpha}_{A_0'})) &\leq d(\phi^{n}(\Omega^{\alpha}_{A_0'}),\phi^{n+1}(\Omega^{\alpha}_{A_0'})) + d(\phi^{n+1}(\Omega^{\alpha}_{A_0'}),\phi^{n+2}(\Omega^{\alpha}_{A_0'}))\\
&\quad +\ldots+ d(\phi^{m-1}(\Omega^{\alpha}_{A_0'}),\phi^{m}(\Omega^{\alpha}_{A_0'}))\\
&\leq k^n d(\phi(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha}_{A_0'}) + k^{n+1} d(\phi(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha}_{A_0'})\\
&\quad +\ldots+ k^{m-1} d(\phi(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha}_{A_0'})\\
&\leq k^n[1+k+k^2+\ldots+k^{m-n-1}] d(\phi(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha}_{A_0'})\\
&\leq \frac{k^n}{1-k} d(\phi(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha}_{A_0'})
\end{aligned}
$$

$$d(\phi^{m}(\Omega^{\alpha}_{A_0'}),\phi^{n}(\Omega^{\alpha}_{A_0'})) \to 0 \text{ as } n \to \infty.$$

So $\phi^{n}(\Omega^{\alpha}_{A_0'})$ is a cauchy sequence in $(\hat{N}S(U'_E),d)$, but $(\hat{N}S(U'_E),d)$ is complete, so every cauchy sequence is convergent that is $\phi^{n}(\Omega^{\alpha}_{A_0'}) \to \Omega^{\alpha *}_{A_0'}$ as $n \to \infty$. Now

$$d(\phi^{n+1}(\Omega^{\alpha}_{A_0'}),\phi(\Omega^{\alpha *}_{A_0'})) \leq kd(\phi^{n}(\Omega^{\alpha}_{A_0'}),\Omega^{\alpha *}_{A_0'})$$

taking limit as $n \to \infty$, we have

$$
\begin{aligned}
&d(\Omega^{\alpha *}_{A_0'},\phi(\Omega^{\alpha *}_{A_0'})) \leq kd(\Omega^{\alpha *}_{A_0'},\Omega^{\alpha *}_{A_0'})\\
&d(\Omega^{\alpha *}_{A_0'},\phi(\Omega^{\alpha *}_{A_0'})) \leq 0
\end{aligned}
$$

but

$$
\begin{aligned}
&d(\Omega^{\alpha *}_{A_0'},\phi(\Omega^{\alpha *}_{A_0'})) \geq 0\\
&d(\Omega^{\alpha *}_{A_0'},\phi(\Omega^{\alpha *}_{A_0'})) = 0\\
&\qquad\qquad \Rightarrow \phi(\Omega^{\alpha *}_{A_0'}) = \Omega^{\alpha *}_{A_0'}.
\end{aligned}
$$

Hence $\Omega^{\alpha *}_{A_0'} \in \hat{N}S(U'_E)$ is the FP of ϕ.

Now suppose $\Omega^{\alpha *}_{B_0'} \in \hat{N}S(U'_E)$ is another FP of with $\phi(\Omega^{\alpha *}_{B_0'}) = \Omega^{\alpha *}_{B_0'}$, then

$$d(\Omega^{\alpha*}_{B_0'},\Omega^{\alpha*}_{A_0'}) = d(\phi(\Omega^{\alpha*}_{B_0'}),\phi(\Omega^{\alpha*}_{A_0'}))$$
$$= d(\phi(\phi(\Omega^{\alpha*}_{B_0'})),\phi(\Omega^{\alpha*}_{A_0'}))$$
$$\le kd(\phi(\Omega^{\alpha*}_{B_0'})),\phi(\Omega^{\alpha*}_{A_0'}))$$
$$= kd(\Omega^{\alpha*}_{B_0'},\Omega^{\alpha*}_{A_0'})$$
$$(1-k)d(\Omega^{\alpha*}_{B_0'},\Omega^{\alpha*}_{A_0'}) \le 0.$$

As $(1-k) \not\le 0,$ so

$$d(\Omega^{\alpha ?}_{B_0'},\Omega^{\alpha ?}_{A_0'}) \le 0 \qquad (1)$$

but

$$d(\Omega^{\alpha*}_{B_0'},\Omega^{\alpha*}_{A_0'}) \ge 0. \qquad (2)$$

From (1) and (2) we have

$$d(\Omega^{\alpha*}_{B_0'},\Omega^{\alpha*}_{A_0'}) = 0$$
$$\Rightarrow \Omega^{\alpha*}_{B_0'} = \Omega^{\alpha*}_{A_0'}.$$

Hence the FP is unique.

Proposition 7 Let $\phi,\psi\ :\ \hat{N}S(U_E') \to \hat{N}S(U_E')$ be commuting maps. If ϕ and ψ have unique coincidence point. Then ϕ and ψ have unique common fixed point.

Proof Suppose there is $\Omega^{\alpha}_{A_1'} \in \hat{N}S(U_E')$ such that $\phi(\psi(\Omega^{\alpha}_{A_1'})) = \psi(\phi(\Omega^{\alpha}_{A_1'}))$. Since ϕ and ψ have unique coincidence point, so let $\phi(\Omega^{\alpha}_{A_1'}) = \psi(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{B_1'}$. Now

$$\phi(\Omega^{\alpha}_{B_1'}) = \phi(\psi(\Omega^{\alpha}_{A_1'})) = \psi(\phi(\Omega^{\alpha}_{A_1'})) = \psi(\Omega^{\alpha}_{B_1'}).$$

Here $\Omega^{\alpha}_{B_1'} \in \hat{N}S(U_E')$ is also a coincidence point, but $\Omega^{\alpha}_{A_1'} \in \hat{N}S(U_E')$ is unique coincidence point, so

$$\phi(\Omega^{\alpha}_{B_1'}) = \psi(\Omega^{\alpha}_{B_1'}) = \psi(\Omega^{\alpha}_{A_1'}) = \phi(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{B_1'}.$$

Hence $\Omega^{\alpha}_{B_1'} \in \hat{N}S(U_E')$ is also a fixed point.

Proposition 8 Every neutrosophic soft identity map is non-expansive.

Proof Suppose that I from $\hat{N}S(U_E')$ to $\hat{N}S(U_E')$ be a neutrosophic soft identity map such that $I(\Omega^{\alpha}_{A_1'}) = \Omega^{\alpha}_{A_1'}$ for all $\Omega^{\alpha}_{A_1'} \in \hat{N}S(U_E')$. Now

$$d(I(\Omega^{\alpha}_{A_1'}),I(\Omega^{\alpha}_{B_1'})) = d(\Omega^{\alpha}_{A_1'},\Omega^{\alpha}_{B_1'})$$

Here $k=1$, so I is non-expansive map.

5. Conclusion

In this paper, we have discussed some new mappings of NSS and some basic results and particular examples. Like fixed point, here also present some new concepts of points that is coincidence point, periodic point and CFP.

FP theory has a lot of applications in control and communicating system. FP theory is an important mathematical instrument used to demonstrate the existence of a solution in mathematical economics and game theory. So the notion of a neutrosophic soft fixed point can be used in these areas. For stabilization of dynamic systems, neutrosophic soft fixed point can be used. In addition, dynamic programming may employ the notion of presence and uniqueness of the common solution of neutrosophic soft set.

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Soft Subring Theory Under Interval-Valued Neutrosophic Environment

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Abstract. The primary goal of this article is to establish and investigate the idea of interval-valued neutrosophic soft subring. Again, we have introduced function under interval-valued neutrosophic soft environment and investigated some of its homomorphic attributes. Additionally, we have established product of two interval-valued neutrosophic soft subrings and analyzed some of its fundamental attributes. Furthermore, we have presented the notion of interval-valued neutrosophic normal soft subring and investigated some of its algebraic properties and homomorphic attributes.

Keywords: Neutrosophic set; Interval-valued neutrosophic soft set; Interval-valued neutrosophic soft subring; Interval-valued neutrosophic normal soft subring

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ABBREVIATIONS

TN indicates "T-norm".

SN indicates "S-norm".

IVTN indicates "Interval-valued T-norm".

IVSN indicates "Interval-valued S-norm".

CS indicates "Crisp set".

US indicates "Universal set".

FS indicates "Fuzzy set".

IFS indicates "Intuitionistic fuzzy set".

NS indicates "Neutrosophic set".

PS indicates "Plithogenic set".

SS indicates "Soft set".

IVFS indicates "Interval-valued fuzzy set".

IVIFS indicates "Interval-valued intuitionistic fuzzy set".

IVNS indicates "Interval-valued neutrosophic set".

NSSR indicates "Neutrosophic soft subring".

NNSSR indicates "Neutrosophic normal soft subring".

IVNSR indicates "Interval-valued neutrosophic subring".

IVNSSR indicates "Interval-valued neutrosophic soft subring".

IVNNSSR indicates "Interval-valued neutrosophic normal soft subring".

DMP indicates "Decision making problem".

$\boldsymbol{\phi(F)}$ indicates "Power set of F".

$\boldsymbol{K}$ indicates "The set $[0,1]$".

1. Introduction

Uncertainty plays a huge part in different economical, sociological, biological, as well as other scientific fields. It is not always possible to tackle ambiguous data using CS theory. To cope with its limitations Zadeh introduced the groundbreaking concept of FS [1] theory. Which was further generalized by Atanassov as IFS [2] theory. Later on, Smarandache extended these notions by introducing NS [3] theory, which became more reasonable for managing indeterminate situations. From the beginning, NS theory became very popular among various researchers. Nowadays, it is heavily utilized in numerous research domains. PS [4] theory is another innovative concept introduced by Smarandache, which is more general than all the previously mentioned notions. In NS and PS theory some of Smarandache's remarkable contributions are the notions of neutrosophic robotics [5], neutrosophic psychology [6], neutrosophic measure [7], neutrosophic calculus [8], neutrosophic statistics [9], neutrosophic probability [10], neutrosophic triplet group [11], plithogenic logic, probability [12], plithogenic subgroup [13], plithogenic aggregation operators [14], plithogenic hypersoft set [15], plithogenic fuzzy whole hypersoft set [16], plithogenic hypersoft subgroup [17], etc. Moreover, NS and PS theory has several contributions in various other scientific fields, for instance, in selection of suppliers [18], professional selection [19], fog and mobile-edge computing [20], fractional programming [21], linear programming [22], shortest path problem [23–30], supply chain problem [31], DMP [32–37], healthcare [38, 39], etc.

Interval-valued versions of FS [40], IFS [41], and NS [42] are further generalizations of their previously discussed counterparts. Since the beginning, various researchers have carried out this concepts and explored them in different research domains. For instance, nowadays in logic [42], abstract algebra [43–46], graph theory [47, 48], DMPs [49–51], etc., these concepts are widely used.

Another set theory of utmost importance is SS [52] theory. It was introduced by Molodtsov to deal with uncertainty more conveniently and easily. At present, it is extensively used in different scientific areas, like in DMPs [53–57], abstract algebra [58–61], stock treading [62], etc. Furthermore, to achieve higher uncertainty handling potentials researchers have implemented SS theory in different interval-valued environments. The following Table 1 comprises some momentous aspects of different interval-valued soft notions.

TABLE 1. Significance of different interval-valued soft notions in various fields.

Author & references	**Year**	**Contributions in various fields**
Yang et al. [63]	2009	Introduced soft IVFS and defined complement, "and" and "or" operations on them.
Jiang et al. [64]	2010	Proposed soft IVIFS and defined complement, "and", "or", union, intersection, necessity, and possibility operations on them.
Feng et al. [65]	2010	Introduced soft reduct fuzzy sets of soft IVFS and utilizing soft versions of reduct fuzzy sets and level sets, proposed flexible strategy for DMP.
Broumi et al. [66]	2014	Presented generalized soft IVNS, analyzed some set operations and further, applied it in DMP.
Mukherje et al. [67]	2014	Proposed relation on soft IVIFSs and presented a solution to a DMP.
Broumi et al. [68]	2014	Proposed relation on soft IVNSs and studied reflexivity, symmetry, transitivity of it.
Mukherje and Sarkar [69]	2015	Defined Euclidean and Hamming distances between two soft IVNSs and presented similarity measures according to distances within them.
Deli [70]	2017	Defined soft IVNS and introduced some operations. Further, implemented this in DMP.
Garg and Arora [71]	2018	Solved DMP with soft IVIFS information.

Group theory and ring theory are essential parts of abstract algebra, which have various applications in different research domains. But these were initially introduced under the crisp environment, which has certain limitations. From the year 1971, various mathematicians started implementing uncertainty theories to generalize these notions. Some noteworthy contributions in the field of group theory under uncertainty can be found on [72–76]. In ring theory under uncertainty, the following articles [77–80] are some important developments. Again, several researchers introduced these notions under soft environments. For instance, researchers have introduced the concepts of ring theory under soft fuzzy [81], soft intuitionistic fuzzy [82], and soft neutrosophic [83] environments. Also, some more articles which can be helpful to different researchers are [84–91], etc. Now, by mixing interval-valued environment with soft neutrosophic environment, we can introduce a more general version of NSSR, which will be called IVNSSR. Also, their homomorphic attributes can be studied. Again, their product and normal versions can be introduced and studied. Based on these perceptions, the followings are our primary objectives for this article:

- Introducing the concept of IVNSSR and a analyzing its homomorphic attributes.
- Introducing the product of IVNSSRs.
- Introducing subring of a IVNSSR.
- Introducing the concept of IVNNSSR and a analyzing its homomorphic properties.

The arrangement our article is: in Section 2, some desk researches of IVTN, IVSN, NS, IVNS, IVNSS, NSR, NSSR, etc., are discussed. In Section 3, the concept of IVNSSR has been introduced and some fundamental theories are provided. Also, their product and normal versions are defined and some theories are given to understand their different algebraic characteristics. Lastly, in Section 4, mentioning some future scopes, the concluding segment is given.

2. Literature Review

Definition 2.1. [92] A function $T : K \to K$ is known as a TN iff $\forall g, n, z \in K$, the followings can be concluded

(i) $T(g, 1) = g$
(ii) $T(g, n) = T(n, g)$
(iii) $T(g, n) \leq T(z, n)$ if $g \leq z$
(iv) $T(g, T(n, z)) = T(T(g, n), z)$

Definition 2.2. [93] A function $\bar{T} : \phi(K) \times \phi(K) \to \phi(K)$ defined as $\bar{T}(\bar{g}, \bar{n}) = [T(g^-, n^-), T(g^+, n^+)]$ (T is a TN) is known as an IVTN.

Definition 2.3. [92] A function $S : K \to K$ is known as SN iff $\forall g, n, z \in K$, the followings can be concluded

(i) $S(g, 0) = g$
(ii) $S(g, n) = S(n, g)$
(iii) $S(g, n) \leq S(z, n)$ if $g \leq z$
(iv) $S(g, S(n, z)) = S(S(g, n), z)$

Definition 2.4. [93] The function $\bar{S} : \phi(K) \times \phi(K) \to \phi(K)$ defined as $\bar{S}(\bar{g}, \bar{n}) = [S(g^-, n^-), S(g^+, n^+)]$ (S is a SN) is called an IVSN.

Definition 2.5. [3] A NS σ of a CS Q is denoted as $\sigma = \{(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)) : g \in Q\}$. Here $\forall g \in Q$, $t_\sigma(g), i_\sigma(g)$, and $f_\sigma(g)$ are known as degree of truth, indeterminacy, and falsity which satisfy the inequality $^{-}0 \leq t_\sigma(g) + i_\sigma(g) + f_\sigma(g) \leq 3^{+}$.

The set of all NSs of Q will be expressed as NS(Q).

Definition 2.6. [52] Let Q be a US and A be a set of parameters. Also, let $L \subseteq A$. Then the ordered pair (f, L) is called a SS over Q, where $f : L \to \phi(Q)$ is a function.

Definition 2.7. [94] Let Q be a US and A be a set of parameters. Also, let $M \subseteq A$. Then a NSS over Q is denoted as (f, M) where $f : M \to \mathrm{NS}(Q)$ is a function.

The following Definition 2.7 is a redefined version of NSS, which we have adopted in this article.

Definition 2.8. [56] Let Q be a US and A be a set of parameters. Then a NSS δ of Q is denoted as $\delta = \{(r, l_\delta(r)) : r \in A\}$ where $l_\delta : A \to \mathrm{NS}(Q)$ is a function which is also known as an approximate function of NSS δ and $l_\delta(r) = \{(g, t_{l_\delta(r)}(g), i_{l_\delta(r)}(g), f_{l_\delta(r)}(g)) : g \in Q\}$. Here, $\forall g \in Q$, $t_{l_\delta(r)}(g)$, $i_{l_\delta(r)}(g)$, and $f_{l_\delta(r)}(g) \in [0, 1]$ and they satisfy the inequality $3 \geq t_{l_\delta(r)}(g) + i_{l_\delta(r)}(g) + f_{l_\delta(r)}(g) \geq 0$.

The set of all NSSs of a set Q will be expressed as NSS(Q).

Definition 2.9. [42] An IVNS of Q is defined as the mapping $\bar{\sigma} : Q \to \phi(K) \times \phi(K) \times \phi(K)$, where $\bar{\sigma}(g) = \{(g, \bar{t}_{\bar{\sigma}}(g), \bar{i}_{\bar{\sigma}}(g), \bar{f}_{\bar{\sigma}}(g)) : g \in Q\}$, where $\forall g \in Q$, $\bar{t}_{\bar{\sigma}}(g)$, $\bar{i}_{\bar{\sigma}}(g)$, and $\bar{f}_{\bar{\sigma}}(g) \subseteq [0, 1]$.

The set of all IVNSs of a set Q will be expressed as IVNS(Q).

Definition 2.10. [70] Let Q be a US and A be a set of parameters. Then a IVNSS Ψ of Q is denoted as $\Psi = \{(r, l_\Psi(r)) : r \in A\}$, where $l_\Psi : A \to \mathrm{IVNS}(Q)$ is a function which is also known as an approximate function of IVNSS Ψ and $l_\Psi(r) = \{(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)) : g \in Q\}$. Here, $\forall g \in Q$, $\bar{t}_{l_\Psi(r)}(g)$, $\bar{i}_{l_\Psi(r)}(g)$, and $\bar{f}_{l_\Psi(r)}(g) \subseteq [0, 1]$.

The set of all IVNSSs of a set Q will be expressed as IVNSS(Q).

Definition 2.11. [70] $\Psi_1 = \{(r, l_{\Psi_1}(r)) : r \in A\}$ and $\Psi_2 = \{(r, l_{\Psi_2}(r)) : r \in A\}$ be two IVNSSs of Q. Then $\Psi = \Psi_1 \cup \Psi_2 = \{(r, l_\Psi(r)) : r \in A\}$ is defined as

$$\bar{t}_{l_\Psi(r)} = \left[\max\{\bar{t}^-_{l_{\Psi_1}(r)}, \bar{t}^-_{l_{\Psi_2}(r)}\}, \max\{\bar{t}^+_{l_{\Psi_1}(r)}, \bar{t}^+_{l_{\Psi_2}(r)}\}\right]$$
$$\bar{t}_{l_\Psi(r)} = \left[\min\{\bar{i}^-_{l_{\Psi_1}(r)}, \bar{i}^-_{l_{\Psi_2}(r)}\}, \min\{\bar{i}^+_{l_{\Psi_1}(r)}, \bar{i}^+_{l_{\Psi_2}(r)}\}\right]$$
$$\bar{t}_{l_\Psi(r)} = \left[\min\{\bar{f}^-_{l_{\Psi_1}(r)}, \bar{f}^-_{l_{\Psi_2}(r)}\}, \min\{\bar{f}^+_{l_{\Psi_1}(r)}, \bar{f}^+_{l_{\Psi_2}(r)}\}\right]$$

Definition 2.12. [70] $\Psi_1 = \{(r, l_{\Psi_1}(r)) : r \in A\}$ and $\Psi_2 = \{(r, l_{\Psi_2}(r)) : r \in A\}$ be two IVNSSs of Q. Then $\Psi = \Psi_1 \cap \Psi_2 = \{(r, l_{\Psi}(r)) : r \in A\}$ is defined as

$$\bar{t}_{l_\Psi(r)} = \left[\min\left\{\bar{t}^-_{l_{\Psi_1}(r)}, \bar{t}^-_{l_{\Psi_2}(r)}\right\}, \min\left\{\bar{t}^+_{l_{\Psi_1}(r)}, \bar{t}^+_{l_{\Psi_2}(r)}\right\}\right]$$
$$\bar{t}_{l_\Psi(r)} = \left[\max\left\{\bar{i}^-_{l_{\Psi_1}(r)}, \bar{i}^-_{l_{\Psi_2}(r)}\right\}, \max\left\{\bar{i}^+_{l_{\Psi_1}(r)}, \bar{i}^+_{l_{\Psi_2}(r)}\right\}\right]$$
$$\bar{t}_{l_\Psi(r)} = \left[\max\left\{\bar{f}^-_{l_{\Psi_1}(r)}, \bar{f}^-_{l_{\Psi_2}(r)}\right\}, \max\left\{\bar{f}^+_{l_{\Psi_1}(r)}, \bar{f}^+_{l_{\Psi_2}(r)}\right\}\right]$$

2.1. *Neutrosophic subring*

Definition 2.13. [80] Let $(Q, +, \cdot)$ be a crisp ring. A NS $\sigma = \{(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)) : g \in Q\}$ is called a NSR of F, iff $\forall g, n \in Q$,

(i) $t_\sigma(g+n) \geq T(t_\sigma(g), t_\sigma(n))$, $i_\sigma(g+n) \geq I(i_\sigma(g), i_\sigma(n))$, $f_\sigma(g+n) \leq F(f_\sigma(g), f_\sigma(n))$
(ii) $t_\sigma(-g) \geq t_\sigma(g)$, $i_\sigma(-g) \geq i_\sigma(g)$, $f_\sigma(-g) \leq f_\sigma(g)$
(iii) $t_\sigma(g \cdot n) \geq T(t_\sigma(g), t_\sigma(n))$, $i_\sigma(g \cdot n) \geq I(i_\sigma(g), i_\sigma(n))$, $f_\sigma(g \cdot n) \leq S(f_\sigma(g), f_\sigma(n))$.

Here, T and I are two TNs and S is a SN.

The set of all NSR of a crisp ring $(Q, +, \cdot)$ will be expressed as NSR(Q).

Proposition 2.1. *[80] A NS $\sigma = \{(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)) : g \in Q\}$ is called a NSR of Q, iff $\forall g, n \in Q$,*

(i) $t_\sigma(g-n) \geq T(t_\sigma(g), t_\sigma(n))$, $i_\sigma(g-n) \geq I(i_\sigma(g), i_\sigma(n))$, $f_\sigma(g-n) \leq F(f_\sigma(g), f_\sigma(n))$
(ii) $t_\sigma(g \cdot n) \geq T(t_\sigma(g), t_\sigma(n))$, $i_\sigma(g \cdot n) \geq I(i_\sigma(g), i_\sigma(n))$, $f_\sigma(g \cdot n) \leq S(f_\sigma(g), f_\sigma(n))$.

Here, T and I are two TNs and S is a SN.

Proposition 2.2. *[80] Let $\sigma_1, \sigma_2 \in NSR(Q)$. Then $\sigma_1 \cap \sigma_2 \in NSR(Q)$.*

Theorem 2.3. *[80] Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a homomorphism. If σ is a NSR of Q then $h(\sigma)$ is a NSR of Y.*

Theorem 2.4. *[80] Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a homomorphism. If σ' is a NSR of Y then $h^{-1}(\sigma')$ is a NSR of Q.*

Definition 2.14. [80] Let $\sigma = \{(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)) : g \in Q\}$ be a NSR of Q. Then $\forall s \in [0, 1]$ the s-level sets of Q are defined as

(i) $(t_\sigma)_s = \{g \in Q : t_\sigma(g) \geq s\}$,
(ii) $(i_\sigma)_s = \{g \in Q : i_\sigma(g) \geq s\}$, and
(iii) $(f_\sigma)^s = \{g \in Q : f_\sigma(g) \leq s\}$.

Proposition 2.5. *[80] A NS $\sigma = \{(g, t_\sigma(g), i_\sigma(g), f_\sigma(g)) : g \in Q\}$ of a crisp ring $(Q, +, \cdot)$ is a NSR of Q iff $\forall s \in [0, 1]$ the s-level sets of Q, i.e. $(t_\sigma)_s$, $(i_\sigma)_s$, and $(f_\sigma)^s$ are crisp rings of Q.*

2.2. *Neutrosophic soft subring*

Definition 2.15. [83] Let $(Q, +, \cdot)$ be a crisp ring and A be a set of parameters. Then a NSS $\delta = \{(r, l_\delta(r)) : r \in A\}$ with $l_\delta : A \to \mathrm{NS}(Q)$ is called a NSSR if $\forall r \in A$, $l_\delta(r) \in \mathrm{NSR}(Q)$.

The set of all NSSR of a crisp ring $(Q, +, \cdot)$ will be expressed as $\mathrm{NSSR}(Q)$.

Proposition 2.6. *[83] A NSS* $\delta = \Big\{\Big(r, \{(g, t_{l_\delta(r)}(g), i_{l_\delta(r)}(g), f_{l_\delta(r)}(g)) : g \in Q\}\Big) : r \in A\Big\}$ *over a crisp ring* $(Q, +, \cdot)$ *is called a NSSR iff the following conditions hold:*

(i) $t_{l_\delta(r)}(g - n) \geq T\big(t_{l_\delta(r)}(g), t_{l_\delta(r)}(n)\big)$, $i_{l_\delta(r)}(g - n) \geq I\big(i_{l_\delta(r)}(g), i_{l_\delta(r)}(n)\big)$, $f_{l_\delta(r)}(g - n) \leq F\big(f_{l_\delta(r)}(g), f_{l_\delta(r)}(n)\big)$ *and*

(ii) $t_{l_\delta(r)}(g \cdot n) \geq T\big(t_{l_\delta(r)}(g), t_{l_\delta(r)}(n)\big)$, $i_{l_\delta(r)}(g \cdot n) \geq I\big(i_{l_\delta(r)}(g), i_{l_\delta(r)}(n)\big)$, $f_{l_\delta(r)}(g \cdot n) \leq S\big(f_{l_\delta(r)}(g), f_{l_\delta(r)}(n)\big)$.

Proposition 2.7. *[83] Let* $\delta_1, \delta_2 \in NSSR(Q)$*. Then* $\delta_1 \cap \delta_2 \in NSSR(Q)$*.*

Theorem 2.8. *[83] Let* $(Q, +, \cdot)$ *and* $(Y, +, \cdot)$ *be two crisp rings. Also, let* $h : Q \to Y$ *be an isomorphism. If* δ *is a NSSR of* Q *then* $h(\delta)$ *is a NSSR of* Y*.*

Theorem 2.9. *[83] Let* $(Q, +, \cdot)$ *and* $(Y, +, \cdot)$ *be two crisp rings. Also, let* $h : Q \to Y$ *be a homomorphism. If* δ' *is a NSSR of* Y *then* $h^{-1}(\delta')$ *is a NSSR of* Q*.*

Theorem 2.10. $\delta_1 \in NSSR(Q)$ *and* $\delta_2 \in NSSR(Y)$*, then their cartesian product* $\delta_1 \times \delta_2 \in NSSR(Q \times Y)$*.*

Definition 2.16. [83] A NSSR $\delta = \{(r, l_\delta(r)) : r \in A\}$ of a crisp ring $(Q, +, \cdot)$ is known as a NNSSR of Q iff $t_{l_\delta(r)}(g \cdot n) = t_{l_\delta(r)}(n \cdot g)$, $i_{l_\delta(r)}(g \cdot n) = i_{l_\delta(r)}(n \cdot g)$, and $f_{l_\delta(r)}(g \cdot n) = f_{l_\delta(r)}(n \cdot g)$.

The set of all NNSSR of Q will be expressed as $\mathrm{NNSSR}(Q)$.

Proposition 2.11. *[83] Let* $\delta_1, \delta_2 \in NNSSR(Q)$*. Then* $\delta_1 \cap \delta_2 \in NNSSR(Q)$*.*

Theorem 2.12. *[83] Let* $(Q, +, \cdot)$ *and* $(Y, +, \cdot)$ *be two crisp rings. Also, let* $h : Q \to Y$ *be an isomorphism. If* δ *is a NNSSR of* Q *then* $h(\delta)$ *is a NNSSR of* Y*.*

Theorem 2.13. *[83] Let* $(Q, +, \cdot)$ *and* $(Y, +, \cdot)$ *be two crisp rings. Also, let* $h : Q \to Y$ *be a ring homomorphism. If* δ' *is a NNSSR of* Y *then* $h^{-1}(\delta')$ *is a NNSSR of* Q*.*

3. Proposed notion of interval-valued neutrosophic soft subring

Definition 3.1. Let $(Q, +, \cdot)$ be a crisp ring and A be a set of parameters. An IVNSS $\Psi = \Big\{\Big(r, \{(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)) : g \in Q\}\Big) : r \in A\Big\}$ is called an IVNSSR of $(Q, +, \cdot)$ if $\forall g, n \in Q$, and $\forall r \in A$, the followings can be concluded:

$$\text{(i)} \begin{cases} \bar{t}_{l_\Psi(r)}(g+n) \geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big), \\ \bar{i}_{l_\Psi(r)}(g+n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \\ \bar{f}_{l_\Psi(r)}(g+n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big) \end{cases}$$

$$\text{(ii)} \begin{cases} \bar{t}_{l_\Psi(r)}(-r) \geq t_{l_\Psi(r)}(g), \\ \bar{i}_{l_\Psi(r)}(-r) \leq i_{l_\Psi(r)}(g), \\ \bar{f}_{l_\Psi(r)}(-r) \leq f_{l_\Psi(r)}(g) \end{cases}$$

$$\text{(iii)} \begin{cases} \bar{t}_{l_\Psi(r)}(g \cdot n) \geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big), \\ \bar{i}_{l_\Psi(r)}(g \cdot n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \\ \bar{f}_{l_\Psi(r)}(g \cdot n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big), \end{cases}$$

The set of all IVNSSR of a crisp ring $(Q, +, \cdot)$ will be expressed as IVNSSR(Q).

Example 3.2. Let $(\mathbb{Z}, +, \cdot)$ be the ring and $\mathbb{N}$ be a set of parameters. Also, let $\Psi = \Big\{\Big(r, \big\{\big(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\big) \ : \ g \ \in \ \mathbb{Z}\big\}\Big) \ : \ e \ \in \ \mathbb{N}\Big\}$ be an IVNSS of $\mathbb{Z}$, where $l_\Psi : \mathbb{N} \to \text{IVNS}(Q)$ and $\forall g \in \mathbb{Z}$, $\forall r \in \mathbb{N}$ corresponding memberships are

$$\bar{t}_{l_\Psi(r)}(g) = \begin{cases} \Big[\dfrac{1}{r+1}, \dfrac{1}{r}\Big] & \text{if } g \in 2\mathbb{Z} \\ [0,0] & \text{if } g \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{l_\Psi(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \Big[\dfrac{1}{2r+2}, \dfrac{1}{2r}\Big] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{l_\Psi(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \Big[\dfrac{r-1}{r}, \dfrac{r}{r+1}\Big] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}.$$

Here, considering minimum TN and maximum SNs $\forall r \in \mathbb{N}$, $\Psi \in$ IVNSSR$(\mathbb{Z})$.

Example 3.3. Let $(\mathbb{Z}_4, +, \cdot)$ be the ring of integers modulo 4 and $A = \{r_1, r_2, r_3\}$ be a set of parameters. Also, let $\Psi = \Big\{\Big(r, \big\{\big(r, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\big) : g \in \mathbb{Z}_4\big\}\Big) : r \in A\Big\}$ be an IVNSS of $\mathbb{Z}_4$, where $l_\Psi : A \to \text{IVNS}(Q)$. Again, let the membership values of the elements belonging to Ψ are specified in Table 2, 3, and 4.

TABLE 2. Membership values of elements with respect to parameter r_1

$\Psi(r_1)$	$\bar{t}_{l_\Psi(r_1)}$	$\bar{i}_{l_\Psi(r_1)}$	$\bar{f}_{l_\Psi(r_1)}$
$\bar{0}$	$[0.64, 0.66]$	$[0.33, 0.35]$	$[0.13, 0.14]$
$\bar{1}$	$[0.7, 0.72]$	$[0.21, 0.23]$	$[0.77, 0.79]$
$\bar{2}$	$[0.74, 0.76]$	$[0.24, 0.26]$	$[0.51, 0.53]$
$\bar{3}$	$[0.66, 0.68]$	$[0.31, 0.33]$	$[0.28, 0.3]$

TABLE 3. Membership values of elements with respect to parameter r_2

$\Psi(r_2)$	$\bar{t}_{l_\Psi(r_2)}$	$\bar{i}_{l_\Psi(r_2)}$	$\bar{f}_{l_\Psi(r_2)}$
$\bar{0}$	$[0.68, 0.7]$	$[0.3, 0.32]$	$[0.31, 0.33]$
$\bar{1}$	$[0.61, 0.63]$	$[0.31, 0.33]$	$[0.41, 0.43]$
$\bar{2}$	$[0.57, 0.59]$	$[0.4, 0.42]$	$[0.65, 0.67]$
$\bar{3}$	$[0.7, 0.72]$	$[0.26, 0.28]$	$[0.52, 0.54]$

TABLE 4. Membership values of elements with respect to parameter r_3

$\Psi(r_3)$	$\bar{t}_{l_\Psi(r_3)}$	$\bar{i}_{l_\Psi(r_3)}$	$\bar{f}_{l_\Psi(r_3)}$
$\bar{0}$	$[0.71, 0.73]$	$[0.2, 0.23]$	$[0.15, 0.17]$
$\bar{1}$	$[0.83, 0.85]$	$[0.15, 0.17]$	$[0.24, 0.26]$
$\bar{2}$	$[0.68, 0.7]$	$[0.3, 0.32]$	$[0.38, 0.4]$
$\bar{3}$	$[0.78, 0.8]$	$[0.18, 0.2]$	$[0.4, 0.43]$

Here, considering the Łukasiewicz TN ($T(g,n) = \max\{0, g+n-1\}$) and bounded sum SNs ($S(g,n) = \min\{g+n, 1\}$), $\forall r \in A$, $\Psi \in \text{IVNSSR}(\mathbb{Z}_4)$.

Proposition 3.1. *An IVNSS* $\Psi = \Big\{\Big(r, \big\{\big(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\big) : g \in Q\big\}\Big) : r \in A\Big\}$ *of a crisp ring* $(Q, +, \cdot)$ *is an IVNSSR iff the following conditions hold (considering idempotent IVTN and IVSNs):*

(i) $\bar{t}_{l_\Psi(r)}(g-n) \geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big)$, $\bar{i}_{l_\Psi(r)}(g-n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big)$, $\bar{f}_{l_\Psi(r)}(g-n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big)$ *and*

(ii) $\bar{t}_{l_\Psi(r)}(g\cdot n) \geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big)$, $\bar{i}_{l_\Psi(r)}(g\cdot n) \leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big)$, $\bar{f}_{l_\Psi(r)}(g\cdot n) \leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big)$.

Proof. Let $\Psi \in \text{IVNSSR}(Q)$. Then

$$\begin{aligned}\bar{t}_{l_\Psi(r)}(g-n) &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(-n)\big) \text{ [by Definition 3.1]}\\ &\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big) \text{ [by Definition 3.1]}\end{aligned}$$

Similary, we will have

$$\begin{aligned}\bar{i}_{l_\Psi(r)}(g-n) &\leq \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \text{ and}\\ \bar{f}_{l_\Psi(r)}(g-n) &\leq \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big),\end{aligned}$$

Again, (ii) follows immediately from condition (iii) of Definition 3.1.

Conversely, let conditions (i) and (ii) of Proposition 3.1 hold. Assuming θ_Q as the additive neutral member of $(Q, +, \cdot)$, we have

$$\begin{aligned} \bar{t}_{l_{\Psi}(r)}(\theta_Q) &= \bar{t}_{l_{\Psi}(r)}(g-g) \\ &\geq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(g)\big) \\ &= \bar{t}_{l_{\Psi}(r)}(g) \end{aligned} \tag{3.1}$$

Similaly,

$$\bar{i}_{l_{\Psi}(r)}(\theta_Q) \leq \bar{i}_{l_{\Psi}(r)}(g) \tag{3.2}$$

$$\bar{f}_{l_{\Psi}(r)}(\theta_Q) \leq \bar{f}_{l_{\Psi}(r)}(g) \tag{3.3}$$

Now,

$$\begin{aligned} \bar{t}_{l_{\Psi}(r)}(-g) &= \bar{t}_{l_{\Psi}(r)}(\theta_Q - g) \\ &\geq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(\theta_Q), \bar{t}_{l_{\Psi}(r)}(g)\big) \\ &\geq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(g)\big) \text{ [by 3.1]} \\ &= \bar{t}_{l_{\Psi}(r)}(g) \text{ [since } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.4}$$

Similarly,

$$\bar{i}_{l_{\Psi}(r)}(-g) \leq \bar{i}_{l_{\Psi}(r)}(g) \text{ [since } \bar{I} \text{ is idempotent]} \tag{3.5}$$

$$\bar{f}_{l_{\Psi}(r)}(-g) \leq \bar{f}_{l_{\Psi}(r)}(g) \text{ [since } \bar{F} \text{ is idempotent]} \tag{3.6}$$

Hence,

$$\begin{aligned} \bar{t}_{l_{\Psi}(r)}(g+n) &= \bar{t}_{l_{\Psi}(r)}(g-(-n)) \\ &\geq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(-n)\big) \\ &\geq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(n)\big) \text{ [by 3.4]} \end{aligned} \tag{3.7}$$

Similarly,

$$\bar{i}_{l_{\Psi}(r)}(g+n) \leq \bar{I}\big(\bar{i}_{l_{\Psi}(r)}(g), \bar{i}_{l_{\Psi}(r)}(n)\big) \text{ [by 3.5]} \tag{3.8}$$

$$\bar{f}_{l_{\Psi}(r)}(g+n) \leq \bar{F}\big(\bar{f}_{l_{\Psi}(r)}(g), \bar{f}_{l_{\Psi}(r)}(n)\big) \text{ [by 3.6]} \tag{3.9}$$

Hence, Equations 3.7, 3.8, and 3.9 prove part (i) of Proposition 3.1. Again, part (ii) of Proposition 3.1 is similar to condition (iii) of Definition 3.1. So, $\Psi \in$ IVNSSR(Q). □

Theorem 3.2. *Let $(Q, +, \cdot)$ be a crisp ring. If $\Psi_1, \Psi_2 \in IVNSSR(Q)$, then $\Psi_1 \cap \Psi_2 \in IVNSSR(Q)$ (considering idempotent IVTN and IVSNs).*

Proof. Let $\Psi = \Psi_1 \cap \Psi_2$. Now, $\forall g, n \in Q$ and $\forall r \in A$

$$\begin{aligned}
\bar{t}_{l_{\Psi}(r)}(g+n) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g+n), \bar{t}_{l_{\Psi_2}(r)}(g+n)\big) \\
&\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g), \bar{t}_{l_{\Psi_1}(r)}(n)\big), \bar{T}\big(\bar{t}_{l_{\Psi_2}(r)}(g), \bar{t}_{l_{\Psi_2}(r)}(n)\big)\Big) \\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g), \bar{t}_{l_{\Psi_1}(r)}(n)\big), \bar{T}\big(\bar{t}_{l_{\Psi_2}(r)}(n), \bar{t}_{l_{\Psi_2}(r)}(g)\big)\Big) \text{ [as } \bar{T} \text{ is commutative]} \\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Psi_1}(g), \bar{t}_{l_{\Psi_2}(r)}(g)\big), \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(n), \bar{t}_{l_{\Psi_2}(r)}(n)\big)\Big) \text{ [as } \bar{T} \text{ is associative]} \\
&= \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(n)\big) \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
\bar{t}_{l_{\Psi}(r)}(-g) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(-g), \bar{t}_{l_{\Psi_2}(r)}(-g)\big) \\
&\geq \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g), \bar{t}_{l_{\Psi_2}(r)}(g)\big) \text{ [by Definition 3.1]} \\
&= \bar{t}_{l_{\Psi}(r)}(g) \qquad (3.11)
\end{aligned}$$

Similarly, we can show

$$\bar{i}_{l_{\Psi}(r)}(g+n) \leq \bar{I}\big(\bar{i}_{l_{\Psi}(r)}(g), \bar{i}_{l_{\Psi}(r)}(n)\big) \qquad (3.12)$$

$$\bar{f}_{l_{\Psi}(r)}(g+n) \leq \bar{F}\big(\bar{f}_{l_{\Psi}(r)}(g), \bar{f}_{l_{\Psi}(r)}(n)\big) \qquad (3.13)$$

and

$$\bar{i}_{l_{\Psi}(r)}(-g) \leq \bar{i}_{l_{\Psi}(r)}(g) \qquad (3.14)$$

$$\bar{f}_{l_{\Psi}(r)}(-g) \leq \bar{f}_{l_{\Psi}(r)}(g) \qquad (3.15)$$

Also, we can show that

$$\bar{t}_{l_{\Psi}(r)}(g \cdot n) \geq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(n)\big), \qquad (3.16)$$

$$\bar{i}_{l_{\Psi}(r)}(g \cdot n) \leq \bar{I}\big(\bar{i}_{l_{\Psi}(r)}(g), \bar{i}_{l_{\Psi}(r)}(n)\big), \text{ and} \qquad (3.17)$$

$$\bar{f}_{l_{\Psi}(r)}(g \cdot n) \leq \bar{F}\big(\bar{f}_{l_{\Psi}(r)}(g), \bar{f}_{l_{\Psi}(r)}(n)\big) \qquad (3.18)$$

So, from Equations 3.10–3.18 $\Psi \in$ IVNSSR(Q). □

Remark 3.3. *In general, if $\Psi_1, \Psi_2 \in IVNSSR(Q)$, then $\Psi_1 \cup \Psi_2$ may not always be an IVNSSR of $(Q, +, \cdot)$.*

The following Example 3.4 will prove Remark 3.3.

Example 3.4. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers and $\mathbb{N}$ be a set of parameters. Again, let $\Psi_1 = \Big\{\Big(r, \big\{\big(g, \bar{t}_{l_{\Psi_1}(r)}(g), \bar{i}_{l_{\Psi_1}(r)}(g), \bar{f}_{l_{\Psi_1}(r)}(g)\big) : g \in \mathbb{Z}\big\}\Big) : r \in \mathbb{N}\Big\}$ and $\Psi_2 =$

$\Big\{\Big(r, \big\{\big(g, \bar{t}_{l_{\Psi_2}(r)}(g), \bar{i}_{l_{\Psi_2}(r)}(g), \bar{f}_{l_{\Psi_2}(r)}(g)\big) : g \in \mathbb{Z}\big\}\Big) : r \in \mathbb{N} \setminus \{1\}\Big\}$ be two IVNSSs of $\mathbb{Z}$, where $l_{\Psi_1} : \mathbb{N} \to \text{IVNSS}(Q)$ be defined as

$$\bar{t}_{l_{\Psi_1}(r)}(g) = \begin{cases} \Big[\dfrac{1}{r+1}, \dfrac{1}{r}\Big] & \text{if } g \in 2\mathbb{Z} \\ [0,0] & \text{if } g \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{l_{\Psi_1}(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \Big[\dfrac{1}{2r+2}, \dfrac{1}{2r}\Big] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{l_{\Psi_1}(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \Big[\dfrac{r-1}{r}, \dfrac{r}{r+1}\Big] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}.$$

and $l_{\Psi_2} : \mathbb{N} \setminus \{1\} \to \text{IVNSS}(Q)$ be defined as

$$\bar{t}_{l_{\Psi_2}(r)}(g) = \begin{cases} \Big[\dfrac{1}{r}, \dfrac{1}{r-1}\Big] & \text{if } g \in 3\mathbb{Z} \\ [0,0] & \text{if } g \in 3\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{l_{\Psi_2}(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 3\mathbb{Z} \\ \Big[\dfrac{1}{2r}, \dfrac{1}{2r-2}\Big] & \text{if } g \in 3\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{l_{\Psi_2}(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 3\mathbb{Z} \\ \Big[\dfrac{r-2}{r-1}, \dfrac{r-1}{r}\Big] & \text{if } g \in 3\mathbb{Z}+1 \end{cases}.$$

Here, considering minimum TN and maximum SNs $\Psi_1, \Psi_2 \in \text{IVNSSR}(\mathbb{Z})$. Let $\Psi = \Psi_1 \cup \Psi_2$. Now considering $r = 3$ we will have

$$\bar{t}_{l_{\Psi_1}(3)}(g) = \begin{cases} \Big[\dfrac{1}{4}, \dfrac{1}{3}\Big] & \text{if } g \in 2\mathbb{Z} \\ [0,0] & \text{if } g \in 2\mathbb{Z}+1 \end{cases} \text{ and}$$

$$\bar{t}_{l_{\Psi_2}(3)}(g) = \begin{cases} \Big[\dfrac{1}{3}, \dfrac{1}{2}\Big] & \text{if } g \in 3\mathbb{Z} \\ [0,0] & \text{if } g \in 3\mathbb{Z}+1 \end{cases}$$

Now, taking $g = 10$ and $n = 15$, we will have

$$\begin{aligned} \bar{t}_{l_{\Psi}(3)}(g+n) &= \bar{t}_{l_{\Psi}(3)}(10+15) \\ &= \bar{t}_{l_{\Psi}(3)}(25) \\ &= \max\{\bar{t}_{l_{\Psi_1}(3)}(25), \bar{t}_{l_{\Psi_2}(3)}(25)\} \\ &= \max\{[0,0],[0,0]\} \\ &= [0,0] \end{aligned}$$

Again, if $\Psi \in$ IVNSSR(Q), then $\forall g, n \in Q$, $\bar{t}_{l_\Psi(3)}(g+n) \geq \min\{\bar{t}_{l_\Psi(3)}(g), \bar{t}_{l_\Psi(3)}(n)\}$. But, here for $g = 10$ and $n = 15$, $\min\{\bar{t}_{l_\Psi(3)}(10), \bar{t}_{l_\Psi(3)}(15)\} = \min\left\{\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right]\right\} = \left[\frac{1}{4}, \frac{1}{3}\right] \nleq [0,0] = \bar{t}_{l_\Psi(3)}(10+15)$. So, $\Psi \notin$ IVNSSR(Q).

Corollary 3.4. *If $\Psi_1, \Psi_2 \in$ IVNSSR(Q), then $\Psi_1 \cup \Psi_2 \in$ IVNSSR(Q) iff one is a subset of other.*

Definition 3.5. let $\Psi = \left\{\left(r, \left\{\left(g, \bar{t}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(g)\right) : g \in \mathbb{Z}_4\right\}\right) : r \in A\right\}$ be an IVNSS of a crisp ring $(Q, +, \cdot)$. Also, let $[g_1, n_1]$, $[g_2, n_2]$, and $[g_3, n_3] \in \phi(K)$. Then the CS $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is called a level set of IVNSSR Ψ, where for any $g \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ the following inequalities will hold: $\bar{t}_{l_\Psi(r)}(g) \geq [g_1, n_1]$, $\bar{i}_{l_\Psi(r)}(g) \leq [g_2, n_2]$, and $\bar{f}_{l_\Psi(r)}(g) \leq [g_3, n_3]$.

Theorem 3.5. *Let $(Q, +, \cdot)$ be a crisp ring. Then $\Psi \in$ IVNSSR(Q) iff $\forall [g_1, n_1], [g_2, n_2], [g_3, n_3] \in \phi(K)$ with $\bar{t}_{l_\Psi(r)}(\theta_Q) \geq [g_1, n_1]$, $\bar{i}_{l_\Psi(r)}(\theta_Q) \leq [g_2, n_2]$, and $\bar{f}_{l_\Psi(r)}(\theta_Q) \leq [g_3, n_3]$, $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is a crisp subring of $(Q, +, \cdot)$ (considering idempotent IVTN and IVSNs).*

Proof. Since, $\bar{t}_{l_\Psi(r)}(\theta_Q) \geq [g_1, n_1]$, $\bar{i}_{l_\Psi(r)}(\theta_Q) \leq [g_2, n_2]$, and $\bar{f}_{l_\Psi(r)}(\theta_Q) \leq [g_3, n_3]$, $\theta_Q \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$, i.e., $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is non-empty. Now, let $\Psi \in$ IVNSSR(Q) and $g, n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$. To show that, $(g-n)$ and $g \cdot n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$. Here,

$$\begin{aligned}
\bar{t}_{l_\Psi(r)}(g-n) &\geq \bar{T}\left(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\right) \text{ [by Proposition 3.1]} \\
&\geq \bar{T}\left([g_1, n_1], [g_1, n_1]\right) \left[\text{as } g, n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}\right] \\
&\geq [g_1, n_1] \text{ [as } \bar{T} \text{ is idempotent]}
\end{aligned} \tag{3.19}$$

Again,

$$\begin{aligned}
\bar{t}_{l_\Psi(r)}(g \cdot n) &\geq \bar{T}\left(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\right) \text{ [by Proposition 3.1]} \\
&\geq \bar{T}\left([g_1, n_1], [g_1, n_1]\right) \left[\text{as } g, n \in \Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}\right] \\
&\geq [g_1, n_1] \text{ [as } \bar{T} \text{ is idempotent]}
\end{aligned} \tag{3.20}$$

Similarly, as $\bar{I}$ and $\bar{F}$ are idempotent, we can prove that

$$\bar{i}_{l_\Psi(r)}(g-n) \leq [g_2, n_2], \tag{3.21}$$

$$\bar{i}_{l_\Psi(r)}(g \cdot n) \leq [g_2, n_2], \tag{3.22}$$

$$\bar{f}_{l_\Psi(r)}(g-n) \leq [g_3, n_3], \text{ and} \tag{3.23}$$

$$\bar{f}_{l_\Psi(r)}(g \cdot n) \leq [g_3, n_3]. \tag{3.24}$$

So, from Equations 3.19–3.24 $(g-n)$ and $g\cdot n \in \Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$, i.e., $\Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$ is a crisp subring of $(Q,+,\cdot)$.

Conversely, let $\Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$ is a crisp subring of $(Q,+,\cdot)$. To show that, $\Psi \in$ IVNSSR(Q).

Let $g,n \in Q$, then there exists $[g_1,n_1] \in \phi(K)$ such that $\bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big) = [g_1,n_1]$. Wherefrom $\bar{t}_{l_\Psi(r)}(g) \geq [g_1,n_1]$ and $\bar{t}_{l_\Psi(r)}(n) \geq [g_1,n_1]$. Also, let there exist $[g_2,n_2],[g_3,n_3] \in \phi(K)$ such that $\bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big) = [g_2,n_2]$ and $\bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big) = [g_3,n_3]$. Then $g,n \in \Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$.

Now, as $\Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$ is a crisp subring, $g-n \in \Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$ and $g\cdot n \in \Psi_{\big([g_1,n_1],[g_2,n_2],[g_3,n_3]\big)}$.

Hence,

$$\begin{aligned} \bar{t}_{l_\Psi(r)}(g-n) &\geq [k_1,s_1] \\ &= \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big) \text{ and} \end{aligned} \tag{3.25}$$

$$\begin{aligned} \bar{t}_{l_\Psi(r)}(g\cdot n) &\geq [k_1,s_1] \\ &= \bar{T}\big(\bar{t}_{l_\Psi(r)}(g), \bar{t}_{l_\Psi(r)}(n)\big) \end{aligned} \tag{3.26}$$

Similarly, we can prove that

$$\begin{aligned} \bar{i}_{l_\Psi(r)}(g-n) &\leq [k_2,s_2] \\ &= \bar{I}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \end{aligned} \tag{3.27}$$

$$\begin{aligned} \bar{i}_{l_\Psi(r)}(g\cdot n) &\leq [k_2,s_2] \\ &= \bar{T}\big(\bar{i}_{l_\Psi(r)}(g), \bar{i}_{l_\Psi(r)}(n)\big), \end{aligned} \tag{3.28}$$

$$\begin{aligned} \bar{f}_{l_\Psi(r)}(g-n) &\leq [k_3,s_3] \\ &= \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big), \text{ and} \end{aligned} \tag{3.29}$$

$$\begin{aligned} \bar{f}_{l_\Psi(r)}(g\cdot n) &\leq [k_3,s_3] \\ &= \bar{F}\big(\bar{f}_{l_\Psi(r)}(g), \bar{f}_{l_\Psi(r)}(n)\big) \end{aligned} \tag{3.30}$$

Hence, from Equations 3.25–3.30 $\Psi \in$ IVNSSR(Q). □

Definition 3.6. Let Ψ and Ψ' be two IVNSSs of two CSs Q and Y, respectively. Also, let $h: Q \to Y$ be a function. Then

(i) image of Ψ under h will be
$h(\Psi) = \Big\{\Big(r, \big\{\big(n, \bar{t}_{h(l_\Psi(r))}(n), \bar{i}_{h(l_\Psi(r))}(n), \bar{f}_{h(l_\Psi(r))}(n)\big) : n \in Y\big\}\Big) : r \in A\Big\}$,
where $\bar{t}_{h(l_\Psi(r))}(n) = \underset{s\in h^{-1}(n)}{\vee} \bar{t}_{l_\Psi(r)}(s)$, $\bar{i}_{h(l_\Psi(r))}(n) = \underset{s\in h^{-1}(n)}{\wedge} \bar{i}_{l_\Psi(r)}(s)$, and $\bar{f}_{h(l_\Psi(r))}(v) =$

$\bigwedge\limits_{s\in h^{-1}(n)} \bar{f}_{l_\Psi(r)}(s)$. Wherefrom, if h is injective then $\bar{t}_{h(l_\Psi(r))}(n) = \bar{t}_{l_\Psi(r)}\big(h^{-1}(n)\big)$, $\bar{i}_{h(l_\Psi(r))}(n) = \bar{i}_{l_\Psi(r)}\big(h^{-1}(n)\big)$, $\bar{f}_{h(l_\Psi(r))}(n) = \bar{f}_{l_\Psi(r)}\big(h^{-1}(n)\big)$.

(i) preimage of Ψ' under h will be
$h^{-1}(\Psi') = \Big\{\Big(r, \big\{\big(g, \bar{t}_{h^{-1}(l_{\Psi'}(r))}(g), \bar{i}_{h^{-1}(l_{\Psi'}(r))}(g), \bar{f}_{h^{-1}(l_{\Psi'}(r))}(g)\big) : g \in Q\big\}\Big) : r \in A\Big\}$,
where $\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g) = \bar{t}_{l_{\Psi'}(r)}\big(h(g)\big)$, $\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g) = \bar{i}_{l_{\Psi'}(r)}\big(h(g)\big)$, $\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g) = \bar{f}_{l_{\Psi'}(r)}\big(h(g)\big)$.

Theorem 3.6. *Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be an isomorphism. If Ψ is an IVNSSR of Q then $h(\Psi)$ is an IVNSSR of Y.*

Proof. Let $n_1 = h(g_1)$ and $n_2 = h(g_2)$, where $g_1, g_2 \in Q$ and $n_1, n_2 \in Y$. Now,

$$\begin{aligned}
\bar{t}_{h(l_\Psi(r))}(n_1 - n_2) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1 - n_2)\big) \text{ [as } h \text{ is injective]}\\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1) - h^{-1}(n_2)\big) \text{ [as } h^{-1} \text{ is a homomorphism]}\\
&= \bar{t}_{l_\Psi(r)}(g_1 - g_2)\\
&\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g_1), \bar{t}_{l_\Psi(r)}(g_2)\big)\\
&= \bar{T}\Big(\bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1)\big), \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_2)\big)\Big)\\
&= \bar{T}\big(\bar{t}_{h(l_\Psi(r))}(n_1), \bar{t}_{h(l_\Psi(r))}(n_2)\big) \qquad (3.31)
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{t}_{h(l_\Psi(r))}(n_1 \cdot n_2) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1 \cdot n_2)\big) \text{ [as } h \text{ is injective]}\\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1) \cdot h^{-1}(n_2)\big) \text{ [as } h^{-1} \text{ is a homomorphism]}\\
&= \bar{t}_{l_\Psi(r)}(g_1 \cdot g_2)\\
&\geq \bar{T}\big(\bar{t}_{l_\Psi(r)}(g_1), \bar{t}_{l_\Psi(r)}(g_2)\big)\\
&= \bar{T}\Big(\bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1)\big), \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_2)\big)\Big)\\
&= \bar{T}\big(\bar{t}_{h(l_\Psi(r))}(n_1), \bar{t}_{h(l_\Psi(r))}(n_2)\big) \qquad (3.32)
\end{aligned}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(n_1 - n_2) \leq \bar{I}\big(\bar{i}_{h(l_\Psi(r))}(n_1), \bar{i}_{h(l_\Psi(r))}(n_2)\big), \qquad (3.33)$$

$$\bar{i}_{h(l_\Psi(r))}(n_1 \cdot n_2) \leq \bar{I}\big(\bar{i}_{h(l_\Psi(r))}(n_1), \bar{i}_{h(l_\Psi(r))}(n_2)\big), \qquad (3.34)$$

$$\bar{f}_{h(l_\Psi(r))}(n_1 - n_2) \leq \bar{F}\big(\bar{f}_{h(l_\Psi(r))}(n_1), \bar{f}_{h(l_\Psi(r))}(n_2)\big), \text{ and} \qquad (3.35)$$

$$\bar{f}_{h(l_\Psi(r))}(n_1 \cdot n_2) \leq \bar{F}\big(\bar{f}_{h(l_\Psi(r))}(n_1), \bar{f}_{h(l_\Psi(r))}(n_2)\big) \qquad (3.36)$$

So, from Equations 3.31–3.36 $h(\Psi)$ is an IVNSSR of Y. □

Theorem 3.7. *Let $(Q,+,\cdot)$ and $(Y,+,\cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a homomorphism. If Ψ' is an IVNSSR of Y then $h^{-1}(\Psi')$ is an IVNSSR of Q. (Note that, h^{-1} may not be an inverse function but $h^{-1}(\Psi')$ is an inverse image of Ψ').*

Proof. Let $n_1 = h(g_1)$ and $n_2 = h(g_2)$, where $g_1, g_2 \in Q$ and $n_1, n_2 \in Y$. Now,

$$\begin{aligned}
\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_1 - g_2) &= \bar{t}_{l_{\Psi'}(r)}\big(h(g_1 - g_2)\big) \\
&= \bar{t}_{l_{\Psi'}(r)}\big(h(g_1) - h(g_2)\big) \text{ [as } h \text{ is a homomorphism]} \\
&= \bar{t}_{l_{\Psi'}(r)}(n_1 - n_2) \\
&\geq \bar{T}\big(\bar{t}_{l_{\Psi'}(r)}(n_1), \bar{t}_{l_{\Psi'}(r)}(n_2)\big) \\
&= \bar{T}\Big(\bar{t}_{l_{\Psi'}(r)}\big(h(g_1)\big), \bar{t}_{l_{\Psi'}(r)}\big(h(g_2)\big)\Big) \\
&= \bar{T}\big(\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \qquad (3.37)
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_1 \cdot g_2) &= \bar{t}_{l_{\Psi'}(r)}\big(h(g_1 \cdot g_2)\big) \\
&= \bar{t}_{l_{\Psi'}(r)}\big(h(g_1) \cdot h(g_2)\big) \text{ [as } h \text{ is a homomorphism]} \\
&= \bar{t}_{l_{\Psi'}(r)}(n_1 \cdot n_2) \\
&\geq \bar{T}\big(\bar{t}_{l_{\Psi'}(r)}(n_1), \bar{t}_{l_{\Psi'}(r)}(n_2)\big) \\
&= \bar{T}\Big(\bar{t}_{l_{\Psi'}(r)}\big(h(g_1)\big), \bar{t}_{l_{\Psi'}(r)}\big(h(g_2)\big)\Big) \\
&= \bar{T}\big(\bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{t}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \qquad (3.38)
\end{aligned}$$

Similarly,

$$\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_1 - g_2) \leq \bar{I}\big(\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \qquad (3.39)$$

$$\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_1 \cdot g_2) \leq \bar{I}\big(\bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{i}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \qquad (3.40)$$

$$\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_1 - g_2) \leq \bar{F}\big(\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \qquad (3.41)$$

$$\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_1 \cdot g_2) \leq \bar{F}\big(\bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_1), \bar{f}_{h^{-1}(l_{\Psi'}(r))}(g_2)\big) \qquad (3.42)$$

So, from Equations 3.37–3.42 $h^{-1}(\Psi')$ is an IVNSSR of Q. □

Definition 3.7. Let $(Q,+,\cdot)$ be a crisp ring and $\Psi \in \text{IVNSSR}(Q)$. Again, let $\bar{\alpha} = [\alpha_1, \alpha_2], \bar{\nu} = [\nu_1, \nu_2], \bar{\chi} = [\chi_1, \chi_2] \in \phi(K)$. Then

(i) Ψ is called a $(\bar{\alpha}, \bar{\nu}, \bar{\chi})-$identity IVNSSR over Q, if $\forall g \in Q$

$$\bar{t}_{l_\Psi(r)}(g) = \begin{cases} \bar{\alpha} & \text{if } g = \theta_Q \\ [0,0] & \text{if } g \neq \theta_Q \end{cases},$$

$$\bar{i}_{l_\Psi(r)}(g) = \begin{cases} \bar{\nu} & \text{if } g = \theta_Q \\ [1,1] & \text{if } g \neq \theta_Q \end{cases}, \text{ and}$$

$$\bar{f}_{l_\Psi(r)}(g) = \begin{cases} \bar{\chi} & \text{if } g = \theta_Q \\ [1,1] & \text{if } g \neq \theta_Q \end{cases},$$

where θ_Q is the additive zero element of Q.

(ii) Ψ is called a $(\bar{\alpha}, \bar{\nu}, \bar{\chi})-$absolute IVNSSR over Q, if $\forall g \in Q$, $\bar{t}_{l_\Psi(r)}(g) = \bar{\alpha}$, $\bar{i}_{l_\Psi(r)}(g) = \bar{\nu}$, and $\bar{f}_{l_\Psi(r)}(g) = \bar{\chi}$.

Theorem 3.8. *Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings and $\Psi \in$IVNSSR (Q). Again, let $h : Q \to Y$ be a homomorphism. Then*

(i) *$h(\Psi)$ will be a $(\bar{\alpha}, \bar{\nu}, \bar{\chi})-$identity IVNSSR over Y, if $\forall g \in Q$*

$$\bar{t}_{l_\Psi(r)}(g) = \begin{cases} \bar{\alpha} & \textit{if } g \in Ker(h) \\ [0,0] & \textit{otherwise} \end{cases},$$

$$\bar{i}_{l_\Psi(r)}(g) = \begin{cases} \bar{\nu} & \textit{if } g \in Ker(h) \\ [1,1] & \textit{otherwise} \end{cases}, \textit{ and}$$

$$\bar{f}_{l_\Psi(r)}(g) = \begin{cases} \bar{\chi} & \textit{if } g \in Ker(h) \\ [1,1] & \textit{otherwise} \end{cases},$$

(ii) *$h(\Psi)$ will be a $(\bar{\alpha}, \bar{\nu}, \bar{\chi})-$absolute IVNSSR over Y, if Ψ is a $(\bar{\alpha}, \bar{\nu}, \bar{\chi})-$absolute IVNSSR over Q.*

Proof. (i) Clearly, by Theorem 3.6 $h(\Psi) \in$ IVNSSR(Y). Let $g \in Ker(h)$, then $h(g) = \theta_Y$. So,

$$\begin{aligned} \bar{t}_{h(l_\Psi(r))}(\theta_Y) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(\theta_Y)\big) \\ &= \bar{t}_{l_\Psi(r)}(g) \\ &= \bar{\alpha} \end{aligned} \tag{3.43}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(\theta_Y) = \bar{\nu}, \text{ and} \tag{3.44}$$

$$\bar{f}_{h(l_\Psi(r))}(\theta_Y) = \bar{\chi} \tag{3.45}$$

Again, let $g \in Q \setminus Ker(h)$ and $h(g) = n$. Then

$$\begin{aligned} \bar{t}_{h(l_\Psi(r))}(n) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n)\big) \\ &= \bar{t}_{l_\Psi(r)}(g) \\ &= [0,0] \end{aligned} \tag{3.46}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(n) = [1,1] \text{ and} \tag{3.47}$$

$$\bar{f}_{h(l_\Psi(r))}(n) = [1,1] \tag{3.48}$$

So, from the Equations 3.43–3.48 $h(\Psi)$ is a $(\bar{\alpha}, \bar{\nu}, \bar{\chi})-$identity IVNSSR over Y.

(ii) Let $h(g) = n$, for $g \in Q$ and $n \in Y$. Then

$$\begin{aligned} \bar{t}_{h(l_\Psi(r))}(n) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n)\big) \\ &= \bar{t}_{l_\Psi(r)}(g) \\ &= \bar{\alpha} \end{aligned} \tag{3.49}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(n) = \bar{\nu} \text{ and} \tag{3.50}$$

$$\bar{f}_{h(l_\Psi(r))}(n) = \bar{\chi} \tag{3.51}$$

So, from the Equations 3.48–3.51 $h(\Psi)$ is a $(\bar{\alpha}, \bar{\nu}, \bar{\chi})-$absolute IVNSSR over Y. □

3.1. *Product of interval-valued neutrosophic subrings*

Definition 3.8. Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Again, let $\Psi_1 \in$ IVNSSR(Q) and $\Psi_2 \in$ IVNSSR(Y), where $\Psi_1 = \Big\{\Big(r_1, \big\{\big(g, \bar{t}_{l_{\Psi_1}(r_1)}(g), \bar{i}_{l_{\Psi_1}(r_1)}(g), \bar{f}_{l_{\Psi_1}(r_1)}(g)\big) : g \in Q\big\}\Big) : r_1 \in A\Big\}$ and $\Psi_2 = \Big\{\Big(r_2, \big\{\big(v, \bar{t}_{l_{\Psi_2}(r_2)}(n), \bar{i}_{l_{\Psi_2}(r_2)}(n), \bar{f}_{l_{\Psi_2}(r_2)}(n)\big) : n \in Y\big\}\Big) : r_2 \in A\Big\}$. Then cartesian product of Ψ_1 and Ψ_2 will be

$$\begin{aligned} \Psi &= \Psi_1 \times \Psi_2 \\ &= \big\{\big((r_1, r_2), l_{\Psi_1 \times \Psi_2}(r_1, r_2)\big) : (r_1, r_2) \in A \times A\big\} \end{aligned}$$

where the approximate function $l_{\Psi_1 \times \Psi_2} : A \times A \to$ IVNS$(Q \times Y)$ is defined as

$$\begin{aligned} \bar{t}_{l_{\Psi_1 \times \Psi_2}(r_1, r_2)}(g, n) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r_1)}(g), \bar{t}_{l_{\Psi_2}(r_2)}(n)\big), \\ \bar{i}_{l_{\Psi_1 \times \Psi_2}(r_1, r_2)}(g, n) &= \bar{I}\big(\bar{i}_{l_{\Psi_1}(r_1)}(g), \bar{i}_{l_{\Psi_2}(r_2)}(n)\big), \text{ and} \\ \bar{f}_{l_{\Psi_1 \times \Psi_2}(r_1, r_2)}(g, n) &= \bar{F}\big(\bar{f}_{l_{\Psi_1}(r_1)}(g), \bar{f}_{l_{\Psi_2}(r_2)}(n) \end{aligned}$$

Similarly, product of 3 or more IVNSSRs can be defined.

Theorem 3.9. *Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings with $\Psi_1 \in$ IVNSSR(Q) and $\Psi_2 \in$ IVNSSR(Y). Then $\Psi_1 \times \Psi_2 \in$ IVNSSR$(Q \times Y)$.*

Proof. Let $\Psi = \Psi_1 \times \Psi_2$ and $(g_1, n_1), (g_2, n_2) \in Q \times R$. Then

$$\begin{aligned}
&\bar{t}_{l_{\Psi(r_1,r_2)}}\big((g_1, n_1) - (g_2, n_2)\big)\\
&\quad = \bar{t}_{l_{\Psi_1\times\Psi_2(r_1,r_2)}}\big((g_1 - g_2, n_1 - n_2)\big)\\
&\quad = \bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_1 - g_2), \bar{t}_{l_{\Psi_2(r_2)}}(n_1 - n_2)\big)\\
&\quad \geq \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_1), \bar{t}_{l_{\Psi_1(r_1)}}(g_2)\big), \bar{T}\big(\bar{t}_{l_{\Psi_2(r_2)}}(n_1), \bar{t}_{l_{\Psi_2(r_2)}}(n_2)\big)\Big)\\
&\quad = \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_1), \bar{t}_{l_{\Psi_2(r_2)}}(n_1)\big), \bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_2), \bar{t}_{l_{\Psi_2(r_2)}}(n_2)\big)\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{as } \bar{T} \text{ is associative}]\\
&\quad = \bar{T}\big(\bar{t}_{l_{\Psi(r_1,r_2)}}(g_1, n_1), \bar{t}_{l_{\Psi(r_1,r_2)}}(g_2, n_2)\big) \qquad (3.52)
\end{aligned}$$

Again,

$$\begin{aligned}
&\bar{t}_{l_{\Psi(r_1,r_2)}}\big((g_1, n_1) \cdot (g_2, n_2)\big)\\
&\quad = \bar{t}_{l_{\Psi_1\times\Psi_2(r_1,r_2)}}\big((g_1 \cdot g_2, n_1 \cdot n_2)\big)\\
&\quad = \bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_1 \cdot g_2), \bar{t}_{l_{\Psi_2(r_2)}}(n_1 \cdot n_2)\big)\\
&\quad \geq \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_1), \bar{t}_{l_{\Psi_1(r_1)}}(g_2)\big), \bar{T}\big(\bar{t}_{l_{\Psi_2(r_2)}}(n_1), \bar{t}_{l_{\Psi_2(r_2)}}(n_2)\big)\Big)\\
&\quad = \bar{T}\Big(\bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_1), \bar{t}_{l_{\Psi_2(r_2)}}(n_1)\big), \bar{T}\big(\bar{t}_{l_{\Psi_1(r_1)}}(g_2), \bar{t}_{l_{\Psi_2(r_2)}}(n_2)\big)\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{as } \bar{T} \text{ is associative}]\\
&\quad = \bar{T}\big(\bar{t}_{l_{\Psi(r_1,r_2)}}(g_1, n_1), \bar{t}_{l_{\Psi(r_1,r_2)}}(g_2, n_2)\big) \qquad (3.53)
\end{aligned}$$

Similary,

$$\bar{i}_{l_{\Psi(r_1,r_2)}}\big((g_1, n_1) - (g_2, n_2)\big) \leq \bar{I}\big(\bar{i}_{l_{\Psi(r_1,r_2)}}(g_1, n_1), \bar{i}_{l_{\Psi(r_1,r_2)}}(g_2, n_2)\big), \qquad (3.54)$$

$$\bar{i}_{l_{\Psi(r_1,r_2)}}\big((g_1, n_1) \cdot (g_2, n_2)\big) \leq \bar{I}\big(\bar{i}_{l_{\Psi(r_1,r_2)}}(g_1, n_1), \bar{i}_{l_{\Psi(r_1,r_2)}}(g_2, n_2)\big), \qquad (3.55)$$

$$\bar{f}_{l_{\Psi(r_1,r_2)}}\big((g_1, n_1) - (g_2, n_2)\big) \leq \bar{F}\big(\bar{f}_{l_{\Psi(r_1,r_2)}}(g_1, n_1), \bar{f}_{l_{\Psi(r_1,r_2)}}(g_2, n_2)\big), \text{ and} \qquad (3.56)$$

$$\bar{f}_{l_{\Psi(r_1,r_2)}}\big((g_1, n_1) \cdot (g_2, n_2)\big) \leq \bar{F}\big(\bar{f}_{l_{\Psi(r_1,r_2)}}(g_1, n_1), \bar{f}_{l_{\Psi(r_1,r_2)}}(g_2, n_2)\big) \qquad (3.57)$$

So, by Proposition 3.1 and from Equations 3.52–3.57 $\Psi_1 \times \Psi_2 \in$ IVNSSR$(Q \times Y)$. □

Corollary 3.10. *Let $\forall i \in \{1, 2, ..., n\}$, $(Q_i, +, \cdot)$ are crisp rings and $\Psi_i \in$ IVNSSR(Q_i). Then $\Psi_1 \times \Psi_2 \times \cdots \times \Psi_n$ is a IVNSSR of $Q_1 \times Q_2 \times \cdots \times Q_n$, where $n \in \mathbb{N}$.*

3.2. *Subring of a interval-valued neutrosophic soft subgring*

Definition 3.9. Let $(Q, +, \cdot)$ be a crisp ring and $\Psi_1, \Psi_2 \in$ IVNSSR(Q), where $\Psi_1 = \Big\{\big(r, \{(g, \bar{t}_{l_{\Psi_1}(r)}(g), \bar{i}_{l_{\Psi_1}(r)}(g), \bar{f}_{l_{\Psi_1}(r)}(g)) : g \in Q\}\big) : r \in A\Big\}$ and $\Psi_2 = \Big\{\big(r, \{(g, \bar{t}_{l_{\Psi_2}(r)}(g), \bar{i}_{l_{\Psi_2}(r)}(g), \bar{f}_{l_{\Psi_2}(r)}(g)) : g \in Q\}\big) : r \in A\Big\}$. Then Ψ_1 is called a subring of Ψ_2 if $\forall g \in Q$, $\bar{t}_{l_{\Psi_1}(r)}(g) \leq \bar{t}_{l_{\Psi_2}(r)}(g)$, $\bar{i}_{l_{\Psi_1}(r)}(g) \geq \bar{i}_{l_{\Psi_2}(r)}(g)$, and $\bar{f}_{l_{\Psi_1}(r)}(g) \geq \bar{f}_{l_{\Psi_2}(r)}(g)$.

Theorem 3.11. *Let $(Q, +, \cdot)$ be a crisp ring and $\Psi \in$ IVNSSR(Q). Again, let Ψ_1 and Ψ_2 be two subrings of Ψ. Then $\Psi_1 \cap \Psi_2$ is also a subring of Ψ, considering all the IVTN and IVSNs as idempotent.*

Proof. Here, $\forall g \in Q$

$$\begin{aligned} \bar{t}_{l_{\Psi_1 \cap \Psi_2}(r)}(g) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g), \bar{t}_{l_{\Psi_2}(r)}(g)\big) \\ &\leq \bar{T}\big(\bar{t}_{l_{\Psi}(r)}(g), \bar{t}_{l_{\Psi}(r)}(g)\big) \\ &= \bar{t}_{l_{\Psi}(r)}(g) \text{ [as } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.58}$$

Similarly, since $\bar{I}$ and $\bar{F}$ are idempotent we have,

$$\bar{i}_{l_{\Psi_1 \cap \Psi_2}(r)}(g) \geq \bar{i}_{l_{\Psi}(r)}(g) \text{ and} \tag{3.59}$$

$$\bar{f}_{l_{\Psi_1 \cap \Psi_2}(r)}(g) \geq \bar{f}_{l_{\Psi}(r)}(g) \tag{3.60}$$

So, from Equations 3.58–3.60 $\Psi_1 \cap \Psi_2$ is a subring of Ψ. □

Theorem 3.12. *Let $(Q, +, \cdot)$ be a crisp ring and $\Psi_1, \Psi_2 \in$ IVNSSR(Q) such that Ψ_1 is a subring of Ψ_2. Let $(Y, +, \cdot)$ is another crisp ring and $h : Q \to Y$ be an isomorphism. Then*

(i) *$h(\Psi_1)$ and $h(\Psi_2)$ are two IVNSSRs over Y and*

(i) *$h(\Psi_1)$ is a subring of $h(\Psi_2)$.*

Proof. (i) can be proved by using Theorem 3.6.
(ii) Let $n = h(g)$, where $g \in Q$ and $n \in Y$. Then

$$\begin{aligned} &\bar{t}_{l_{\Psi_1}(r)}(g) \leq \bar{t}_{l_{\Psi_2}(r)}(g) \text{ [as } \Psi_1 \text{ is a subring of } \Psi_2] \\ \Rightarrow &\bar{t}_{l_{\Psi_1}(r)}\big(h^{-1}(n)\big) \leq \bar{t}_{l_{\Psi_2}(r)}\big(h^{-1}(n)\big) \\ \Rightarrow &\bar{t}_{h(l_{\Psi_1}(r))}(n) \leq \bar{t}_{h(l_{\Psi_2}(r))}(n) \end{aligned} \tag{3.61}$$

Similarly,

$$\bar{i}_{h(l_{\Psi_1}(r))}(n) \geq \bar{i}_{h(l_{\Psi_2}(r))}(n) \text{ and} \tag{3.62}$$

$$\bar{f}_{h(l_{\Psi_1}(r))}(n) \geq \bar{f}_{h(l_{\Psi_2}(r))}(n) \tag{3.63}$$

So, from Equations 3.61–3.63 $h(\Psi_1)$ is a subring of $h(\Psi_2)$. □

3.3. *Interval-valued neutrosophic normal soft subrings*

Definition 3.10. Let $(Q,+,\cdot)$ be a crisp ring and Ψ is an IVNSS of Q, where $\Psi = \Big\{\big(r,\big\{\big(g,\bar{t}_{l_\Psi(r)}(g),\bar{i}_{l_\Psi(r)}(g),\bar{f}_{l_\Psi(r)}(g)\big) : g \in Q\big\}\big) : r \in A\Big\}$. Then Ψ is called an IVNNSSR over Q if

(i) Ψ is an IVNSSR of Q and
(ii) $\forall g,n \in Q$, $\bar{t}_{l_\Psi(r)}(g\cdot n) = \bar{t}_{l_\Psi(r)}(n\cdot g)$, $\bar{i}_{l_\Psi(r)}(g\cdot n) = \bar{i}_{l_\Psi(r)}(n\cdot g)$, and $\bar{f}_{l_\Psi(r)}(g\cdot n) = \bar{f}_{l_\Psi(r)}(n\cdot g)$.

The set of all IVNNSSR of $(Q,+,\cdot)$ will be expressed as IVNNSSR(Q).

Example 3.11. Let $(\mathbb{Z},+,\cdot)$ be the ring and $\mathbb{N}$ be the set of parameters. Also, let $\Psi = \Big\{\Big(r,\big\{\big(g,\bar{t}_{l_\Psi(r)}(g),\bar{i}_{l_\Psi(r)}(g),\bar{f}_{l_\Psi(r)}(g)\big) : g \in \mathbb{Z}\big\}\Big) : r \in \mathbb{N}\Big\}$ be an IVNSS of $\mathbb{Z}$, where $l_\Psi(r) : \mathbb{N} \to$ IVNSS(Q) and $\forall g \in \mathbb{Z}$, $\forall r \in \mathbb{N}$ corresponding membership values are

$$\bar{t}_{l_\Psi(r)}(g) = \begin{cases} \left[\dfrac{1}{r+1},\dfrac{1}{r-1}\right] & \text{if } g \in 2\mathbb{Z} \\ [0,0] & \text{if } g \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{l_\Psi(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \left[\dfrac{1}{2r+2},\dfrac{1}{2r-2}\right] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{l_\Psi(r)}(g) = \begin{cases} [0,0] & \text{if } g \in 2\mathbb{Z} \\ \left[\dfrac{r-2}{r-1},\dfrac{r}{r+1}\right] & \text{if } g \in 2\mathbb{Z}+1 \end{cases}.$$

Here, considering minimum TN and maximum SNs $\forall r \in \mathbb{N}$, $\Psi \in$ IVNNSSR$(\mathbb{Z})$.

Theorem 3.13. *Let $(Q,+,\cdot)$ be a crisp ring. If $\Psi_1,\Psi_2 \in$ IVNNSSR(Q), then $\Psi_1 \cap \Psi_2 \in$ IVNNSSR(Q).*

Proof. As $\Psi_1,\Psi_2 \in$ IVNSSR(Q) by Theorem 3.2 $\Psi_1 \cap \Psi_2 \in$ IVNSSR(Q). Again,

$$\begin{aligned} \bar{t}_{l_{\Psi_1\cap\Psi_2}(r)}(g\cdot n) &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(g\cdot n),\bar{t}_{l_{\Psi_2}(r)}(g\cdot n)\big) \\ &= \bar{T}\big(\bar{t}_{l_{\Psi_1}(r)}(n\cdot g),\bar{t}_{l_{\Psi_2}(r)}(n\cdot g)\big) \text{ [as } \Psi_1,\Psi_2 \in \text{IVNNSSR}(Q)] \\ &= \bar{t}_{\Psi_1\cap\Psi_2}(n\cdot g) \end{aligned} \tag{3.64}$$

Similarly,

$$\bar{i}_{l_{\Psi_1\cap\Psi_2}(r)}(g\cdot n) = \bar{i}_{l_{\Psi_1\cap\Psi_2}(r)}(n\cdot g) \tag{3.65}$$

$$\bar{f}_{l_{\Psi_1\cap\Psi_2}(r)}(g\cdot n) = \bar{f}_{l_{\Psi_1\cap\Psi_2}(r)}(n\cdot g) \tag{3.66}$$

Hence, $\Psi_1 \cap \Psi_2 \in$ IVNNSSR(Q). □

Remark 3.14. *In general, if $\Psi_1, \Psi_2 \in IVNNSSR(Q)$, then $\Psi_1 \cup \Psi_2$ may not always be an IVNNSSR of $(Q, +, \cdot)$.*

Remark 3.14 can be shown by Example 3.4.

Theorem 3.15. *Let $(Q, +, \cdot)$ be a crisp ring. Then $\Psi \in IVNNSSR(Q)$ iff $\forall [g_1, n_1], [g_2, n_2], [g_3, n_3] \in \phi(K)$ with $\bar{t}_{l_\Psi(r)}(\theta_Q) \geq [g_1, n_1]$, $\bar{i}_{l_\Psi(r)}(\theta_Q) \leq [g_2, n_2]$, and $\bar{f}_{l_\Psi(r)}(\theta_Q) \leq [g_3, n_3]$, $\Psi_{\left([g_1,n_1],[g_2,n_2],[g_3,n_3]\right)}$ is a crisp normal subring of $(Q, +, \cdot)$ (considering idempotent IVTN and IVSNs).*

Proof. This can be proved using Theorem 3.5. □

Theorem 3.16. *Let $(Q, +, \cdot)$ and $(Y, +, \cdot)$ be two crisp rings. Also, let $h : Q \to Y$ be a ring isomorphism. If Ψ is an IVNNSSR of Q then $h(\Psi)$ is an IVNNSSR of Y.*

Proof. As Ψ is an IVNSSR of Q, by Theorem 3.6 $h(\Psi)$ is an IVNSSR of Y. Let $h(g_1) = n_1$ and $h(g_2) = n_2$, where $g_1, g_2 \in Q$ and $n_1, n_2 \in Y$. Then

$$\begin{aligned}
\bar{t}_{h(l_\Psi(r))}(n_1 \cdot n_2) &= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1 \cdot n_2)\big) \text{ [as } h \text{ is injective]} \\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_1) \cdot h^{-1}(n_2)\big) \text{ [as } h^{-1} \text{ is a homomorphism]} \\
&= \bar{t}_{l_\Psi(r)}(g_1 \cdot g_2) \\
&= \bar{t}_{l_\Psi(r)}(g_2 \cdot g_1) \text{ [as } \Psi \text{ is an IVNNSSR of } Q] \\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_2) \cdot h^{-1}(n_1)\big) \\
&= \bar{t}_{l_\Psi(r)}\big(h^{-1}(n_2 \cdot n_1)\big) \\
&= \bar{t}_{h(l_\Psi(r))}(n_2 \cdot n_1)
\end{aligned} \tag{3.67}$$

Similarly,

$$\bar{i}_{h(l_\Psi(r))}(n_1 \cdot n_2) = \bar{i}_{h(l_\Psi(r))}(n_2 \cdot n_1) \text{ and} \tag{3.68}$$

$$\bar{f}_{h(l_\Psi(r))}(n_1 \cdot n_2) = \bar{f}_{h(l_\Psi(r))}(n_2 \cdot n_1) \tag{3.69}$$

So, from Equations 3.67–3.69 $h(\Psi)$ is an IVNNSSR of Y. □

4. **Conclusions**

Interval-valued neutrosophic field is a dynamic research domain. Under soft environment, it becomes more general and productive. For this reason, we have adopted this mixed environment and defined the notions of interval-valued neutrosophic soft subring along with its normal version. Also, we have studied several homomorphic attributes of these newly introduced notions. Again, we have introduced the product of two interval-valued neutrosophic soft subrings. Furthermore, we have given several fundamental theories to understand some of its algebraic characteristics. These newly introduced notions have the potentials to become fruitful research domains. In future, for generalizing this concepts one can introduce them under the hypersoft set environment.

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Introduction to Interval-valued Neutrosophic Subring

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Abstract. The main purpose of this article is to develop and study the notion of interval-valued neutrosophic subring. Also, we have studied some homomorphic characteristics of interval-valued neutrosophic subring. Again, we have defined the concept of product of two interval-valued neutrosophic subrings and analyzed some of its important properties. Furthermore, we have developed the notion of interval-valued neutrosophic normal subring and studied some of its basic characteristics and homomorphic properties.

Keywords: Neutrosophic set; Interval-valued neutrosophic set; Interval-valued neutrosophic subring; Interval-valued neutrosophic normal subring

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ABBREVIATIONS

TN signifies "T-norm".

SN signifies "S-norm".

IVTN signifies "interval-valued T-norm".

IVSN signifies "interval-valued S-norm".

CS signifies "crisp set".

FS signifies "fuzzy set".

IFS signifies "intuitionistic fuzzy set".

NS signifies "neutrosophic set".

PS signifies "plithogenic set".

FSG signifies "fuzzy subgroup".

IFSG signifies "intuitionistic fuzzy subgroup".

NSG signifies "neutrosophic subgroup".

CR signifies "crisp ring".

FSR signifies "fuzzy subring".

IFSR signifies "intuitionistic fuzzy subring".

NSR signifies "neutrosophic subring".

IVFSR signifies "interval-valued fuzzy subring".

IVIFSR signifies "interval-valued intuitionistic fuzzy subring".

IVNSR signifies "interval-valued neutrosophic subring".

IVNNSR signifies "interval-valued neutrosophic normal subring".

DMP signifies "decision making problem".

$\boldsymbol{\psi(P)}$ signifies "power set of P".

$\boldsymbol{L}$ signifies "the set $[0,1]$".

1. Introduction

Zadeh's vision behind introducing the revolutionary concept of FS [1] theory was to tackle uncertainty in a better way than CS theory, which has certain drawbacks. Later on, following his vision Atanassov introduced a more general version of it, which is known as IFS [2] theory. These IFSs are a little step ahead in managing ambiguities and hence are welcomed by numerous researchers. Furthermore, following their footsteps Smarandache introduced NS [3] theory, which is more capable of handling vague situations. It is a significant generalization over CS, FS, and IFS theories. Smarandache has also initiated the concept of PS [4] theory which has broader aspects than those previously discussed concepts. In NS and PS theory, he has also developed the notions of neutrosophic calculus [5], neutrosophic probability [6], neutrosophic statistics [7], integral, measure [8], neutrosophic psychology [9], neutrosophic robotics [10], neutrosophic triplet group [11], plithogenic hypersoft set [12], plithogenic fuzzy whole hypersoft set [13], plithogenic logic, probability [14], plithogenic subgroup [15], plithogenic hypersoft subgroup [16], etc. Again, NS theory has various other contributions in different scientific researches, like in linear programming [17–20], decision making [21–27], healthcare [28,29], shortest path problem [30–37], neutrosophic forecasting [38], resource leveling [39], transportation problem [40, 41], project scheduling [42], brain processing [43], etc.

Gradually, interval-valued versions of FS [44], IFS [45], and NS [46] were introduced, which are further generalizations of their CS, FS, IFS, and NS counterparts. Presently, these set theories are extensively used in different scientific domains. From the very start, various researchers have carried out this concepts and explored them in different dimensions. In the subsequent Table 1 we have referred some significant aspects of these notions.

Table 1. Importance of interval-valued notions in different domains.

Author & references	**Year**	**Contributions in various fields**
Biswas [47]	1994	Introduced interval-valued FSG.

continued ...

Author & references	**Year**	**Contributions in various fields**
Atanassov [45]	1999	Studied basic definition and some properties of IVFS.
Mondal & Samanta [48]	2001	Defined and studied topology of IVIFSs.
Wang et al. [46]	2005	Proposed and studied IVNS and interval-valued neutrosophic logic.
Ye [49]	2009	Worked on multi-criteria DMP under IVIFSs.
Kang & Hur [50]	2010	Introduced and studied the notion of IVFSR.
Akram & Dudek [51]	2011	Defined some basic operations on interval-valued fuzzy graphs and studied some of their properties.
Aygünoğlu et al. [52]	2012	Introduced interval-valued IFSG and studied some homomorphic properties of it.
Moorthy & Arjunan [53]	2014	Introduced and studied some properties of IVIFSR.
Aiwu et al. [54]	2015	Worked on multi-attribute DMP under IVNSs.
Broumi et al. [56]	2016	Worked on interval-valued neutrosophic graph theory.
Deli [55]	2017	Applied soft version of IVNS in DMP.
Broumi et al. [56]	2019	Studied some properties of interval-valued neutrosophic graphs.

Group theory and ring theory are fundamental building blocks of abstract algebra, which are utilized in different scientific domains. But, initially, these concepts were introduced upon crisp environment. Gradually, from 1971 on-wards researchers started introducing these concepts under various uncertain environments. Some significant developments of these notions under uncertainty are the concepts of FSG [57], IFSG [58], NSG [59], FSR [60, 61], IFSR [62], NSR [63], etc. Again some researchers have introduced these concepts under interval-valued environments and initiated the notions of interval-valued FSG [47], interval-valued IFSG [52], interval-valued NSG [64], interval-valued FSR [50], interval-valued IFSR [53], etc. Some more articles which can be helpful to different researchers are [65–71], etc. But, still, the notion of interval-valued NSR is undefined. Hence, by mixing interval-valued environment with neutrosophic environment, we can introduce a more general version of NSR, which will be called IVNSR. Also, their homomorphic properties can be studied. Again, their product and normal forms can be developed and analyzed. Based on these observations, the followings are some of our main objectives for this article:

- Introducing the notion of IVNSR and a analyzing its homomorphic properties.
- Introducing the product of IVNSRs.
- Introducing subring of a IVNSR.

- Introducing the notion of IVNNSR and a analyzing its homomorphic attributes.

The subsequent arrangement of this article is: in Section 2, some desk researches of FS, IFS, NS, IVFS, IVIFS, IVNS, FSR, IFSR, NSR, IVFSR, IVIFSR, etc., are discussed. In Section 3, the idea of IVNSR has been introduced and some basic theories are provided. Also, their product and normal versions are defined. Also, some theories are given to understand their algebraic attributes. Lastly, in Section 4, the concluding segment is given and also some opportunities for further studies are mentioned.

2. **Literature Review**

Definition 2.1. [1] A FS of a CS P is defined as the function $\nu : P \to L$.

Definition 2.2. [2] An IFS ρ of a CS P is defined as $\rho = \{(r, t_\rho(r), f_\gamma(r)) : r \in P\}$, where $\forall r \in P$, $t_\rho(r)$ and $f_\rho(r)$ known as the degree of membership and non-membership which satisfy the inequality $0 \leq t_\rho(r) + f_\rho(r) \leq 1$.

Definition 2.3. [3] A NS κ of a CS P is defined as $\kappa = \{(r, t_\kappa(r), i_\kappa(r), f_\kappa(r)) : r \in P\}$, where $\forall r \in P$, $t_\kappa(r), i_\kappa(r),$ and $f_\kappa(r)$ are known as degree of truth, indeterminacy, and falsity which satisfy the inequality $^{-}0 \leq t_\kappa(r) + i_\kappa(r) + f_\kappa(r) \leq 3^{+}$.

Definition 2.4. [52] An interval number of $L = [0, 1]$ is denoted as $\bar{k} = [k^{-}, k^{+}]$, where $1 \geq k^{+} \geq k^{-} \geq 0$.

Definition 2.5. [44] An IVFS of P is defined as the mapping $\nu : P \to \psi(L)$.

Definition 2.6. [45] An IVIFS of P is defined as the mapping $\bar{\rho} : P \to \psi(L) \times \psi(L)$, It is denoted as $\bar{\rho} = \{(r, \bar{t}_{\bar{\rho}}(r), \bar{f}_{\bar{\rho}}(r)) : r \in P\}$, where $\bar{t}_{\bar{\rho}}(r),\ \bar{f}_{\bar{\rho}}(r) \subseteq [0, 1]$.

Definition 2.7. [46] An IVNS of P is defined as the mapping $\bar{\kappa} : P \to \psi(L) \times \psi(L) \times \psi(L)$, It is denoted as $\bar{\kappa} = \{(r, \bar{t}_{\bar{\kappa}}(r), \bar{i}_{\bar{\kappa}}(r), \bar{f}_{\bar{\kappa}}(r)) : r \in P\}$ where $\forall r \in P$, $\bar{t}_{\bar{\kappa}}(r),\ \bar{i}_{\bar{\kappa}}(r),$ and $\bar{f}_{\bar{\kappa}}(r) \subseteq L$.

Definition 2.8. [46] Let $\bar{\kappa_1} = \{(r, \bar{t}_{\bar{\kappa_1}}(r), \bar{i}_{\bar{\kappa_1}}(r), \bar{f}_{\bar{\kappa_1}}(r)) : r \in P\}$ and $\bar{\kappa_2} = \{(r, \bar{t}_{\bar{\kappa_2}}(r), \bar{i}_{\bar{\kappa_2}}(r), \bar{f}_{\bar{\kappa_2}}(r)) : r \in P\}$ be two IVNSs of P. Then union of $\bar{\kappa_1}$ and $\bar{\kappa_2}$ is defined as

$$\bar{t}_{\bar{\kappa_1} \cup \bar{\kappa_2}} = \left[\max\{\bar{t}^{-}_{\bar{\kappa_1}}, \bar{t}^{-}_{\bar{\kappa_2}}\}, \max\{\bar{t}^{+}_{\bar{\kappa_1}}, \bar{t}^{+}_{\bar{\kappa_2}}\}\right]$$
$$\bar{t}_{\bar{\kappa_1} \cup \bar{\kappa_2}} = \left[\min\{\bar{i}^{-}_{\bar{\kappa_1}}, \bar{i}^{-}_{\bar{\kappa_2}}\}, \min\{\bar{i}^{+}_{\bar{\kappa_1}}, \bar{i}^{+}_{\bar{\kappa_2}}\}\right]$$
$$\bar{t}_{\bar{\kappa_1} \cup \bar{\kappa_2}} = \left[\min\{\bar{f}^{-}_{\bar{\kappa_1}}, \bar{f}^{-}_{\bar{\kappa_2}}\}, \min\{\bar{f}^{+}_{\bar{\kappa_1}}, \bar{f}^{+}_{\bar{\kappa_2}}\}\right]$$

Then intersection of $\bar{\kappa_1}$ and $\bar{\kappa_2}$ is defined as

$$\bar{t}_{\bar{\kappa_1} \cap \bar{\kappa_2}} = \left[\min\{\bar{t}^{-}_{\bar{\kappa_1}}, \bar{t}^{-}_{\bar{\kappa_2}}\}, \min\{\bar{t}^{+}_{\bar{\kappa_1}}, \bar{t}^{+}_{\bar{\kappa_2}}\}\right]$$
$$\bar{t}_{\bar{\kappa_1} \cap \bar{\kappa_2}} = \left[\max\{\bar{i}^{-}_{\bar{\kappa_1}}, \bar{i}^{-}_{\bar{\kappa_2}}\}, \max\{\bar{i}^{+}_{\bar{\kappa_1}}, \bar{i}^{+}_{\bar{\kappa_2}}\}\right]$$
$$\bar{t}_{\bar{\kappa_1} \cap \bar{\kappa_2}} = \left[\max\{\bar{f}^{-}_{\bar{\kappa_1}}, \bar{f}^{-}_{\bar{\kappa_2}}\}, \max\{\bar{f}^{+}_{\bar{\kappa_1}}, \bar{f}^{+}_{\bar{\kappa_2}}\}\right]$$

Definition 2.9. [72] A function $T : L \to L$ is called a TN iff $\forall r, v, z \in L$, the followings can be concluded

(i) $T(r, 1) = r$
(ii) $T(r, v) = T(v, r)$
(iii) $T(r, v) \leq T(z, v)$ if $r \leq z$
(iv) $T(r, T(v, z)) = T(T(r, v), z)$

Definition 2.10. [73] A function $\bar{T} : \psi(L) \times \psi(L) \to \psi(L)$ defined as $\bar{T}(\bar{k}, \bar{w}) = [T(k^-, w^-), T(k^+, w^+)]$, where T is a TN is known as an IVTN.

Definition 2.11. [72] A function $S : L \to L$ is called a SN iff $\forall r, v, z \in L$, the followings can be concluded

(i) $S(r, 0) = r$
(ii) $S(r, v) = S(v, r)$
(iii) $S(r, v) \leq S(z, v)$ if $r \leq z$
(iv) $S(r, S(v, z)) = S(S(r, v), z)$

Definition 2.12. [73] The function $\bar{S} : \psi(L) \times \psi(L) \to \psi(L)$ defined as $\bar{S}(\bar{k}, \bar{w}) = [S(k^-, w^-), S(k^+, w^+)]$, where S is a SN is called an IVSN.

2.1. *Fuzzy, Intuitionistic fuzzy & Neutrosophic subrings*

Definition 2.13. [60] Let $(P, +, \cdot)$ be a crisp ring. A FS λ is called a FSR of P, iff $\forall r, v \in P$,

(i) $\lambda(r - v) \geq \min\{\lambda(r), \lambda(v)\}$,
(ii) $\lambda(r \cdot v) \geq \min\{\lambda(r), \lambda(v)\}$

The set of all FSR of a crisp ring $(P, +, \cdot)$ will be denoted as FSR(P).

Theorem 2.1. *[61] Any FS λ of a ring $(P, +, \cdot)$ is a FSR of P iff the level sets λ_s ($\lambda(\theta_P) \geq s \geq 0$) are crisp subrings of P, where θ_P is the zero element of P.*

Definition 2.14. [61] Let λ be a FSR of $(P, +, \cdot)$ and $\lambda(\theta_P) \geq s \geq 0$, where θ_P is the zero element of P. Then λ_s is called a level subring of λ.

Proposition 2.2. *[61] Let $\lambda_1, \lambda_2 \in FSR(P)$. Then $\lambda_1 \cap \lambda_2 \in FSR(P)$.*

Theorem 2.3. *[61] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If λ is a FSR of P then $l(\lambda)$ is a FSR of R.*

Theorem 2.4. *[61] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If λ' is a FSR of R then $l^{-1}(\lambda')$ is a FSR of P.*

Definition 2.15. [62] Let $(P, +, \cdot)$ be a crisp ring. An IFS $\gamma = \{(r, t_\gamma(r), f_\gamma(r)) : r \in P\}$ is called an IFSR of P, iff $\forall r, v \in P$,

(i) $t_\gamma(r+v) \geq T(t_\gamma(r), t_\gamma(v)),\ f_\gamma(r+v) \leq S(f_\gamma(r), i_\gamma(v))$

(ii) $t_\gamma(-r) \geq t_\gamma(r),\ f_\gamma(-r) \leq f_\gamma(r)$

(iii) $t_\gamma(r \cdot v) \geq T(t_\gamma(r), t_\gamma(v)),\ f_\gamma(r \cdot v) \leq S(f_\gamma(r), i_\gamma(v))$.

Here, T is a TN and S is a SN.

The set of all IFSR of a crisp ring $(P, +, \cdot)$ will be denoted as IFSR(P).

Proposition 2.5. *[62] Let $\gamma \in$ IFSR(P). Then the followings will hold*

(i) $t_\gamma(-r) = t_\gamma(r),\ f_\gamma(-r) = f_\gamma(r)$ *and*

(ii) $t_\gamma(\theta_P) \geq t_\gamma(r),\ f_\gamma(\theta_P) \leq f_\gamma(r)$, *where θ_P is the zero element of P.*

Proposition 2.6. *[62] An IFS $\gamma = \{(r, t_\gamma(r), f_\gamma(r)) : r \in P\}$ is called an IFSR of P, iff $\forall r, v \in P$,*

(i) $t_\gamma(r-v) \geq T(t_\gamma(r), t_\gamma(v)),\ f_\gamma(r-v) \leq S(f_\gamma(r), f_\gamma(v))$

(ii) $t_\gamma(r \cdot v) \geq T(t_\gamma(r), t_\gamma(v)),\ f_\gamma(r \cdot v) \leq S(f_\gamma(r), f_\gamma(v))$

Proposition 2.7. *[62] Let $\gamma_1, \gamma_2 \in$ IFSR(P). Then $\gamma_1 \cap \gamma_2 \in$ IFSR(P).*

Theorem 2.8. *[62] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \rightarrow R$ be a homomorphism. If γ is an IFSR of P then $l(\gamma)$ is an IFSR of R.*

Theorem 2.9. *[62] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \rightarrow R$ be a homomorphism. If γ' is an IFSR of R then $l^{-1}(\gamma')$ is an IFSR of P.*

Definition 2.16. [63] Let $(P, +, \cdot)$ be a crisp ring. A NS $\omega = \{(r, t_\omega(r), i_\omega(r), f_\omega(r)) : r \in P\}$ is called a NSR of P, iff $\forall r, v \in P$,

(i) $t_\omega(r+v) \geq T(t_\omega(r), t_\omega(v)),\ i_\omega(r+v) \geq I(i_\omega(r), i_\omega(v)),\ f_\omega(r+v) \leq F(f_\omega(r), f_\omega(v))$

(ii) $t_\omega(-r) \geq t_\omega(r),\ i_\omega(-r) \geq i_\omega(r),\ f_\omega(-r) \leq f_\omega(r)$

(iii) $t_\omega(r \cdot v) \geq T(t_\omega(r), t_\omega(v)),\ i_\omega(r \cdot v) \geq I(i_\omega(r), i_\omega(v)),\ f_\omega(r \cdot v) \leq S(f_\omega(r), f_\omega(v))$.

Here, T and I are two TNs and S is a SN.

The set of all NSR of a crisp ring $(P, +, \cdot)$ will be denoted as NSR(P).

Proposition 2.10. *[63] A NS $\omega = \{(r, t_\omega(r), i_\omega(r), f_\omega(r)) : r \in P\}$ is called a NSR of P, iff $\forall r, v \in P$,*

(i) $t_\omega(r-v) \geq T(t_\omega(r), t_\omega(v)),\ i_\omega(r-v) \geq I(i_\omega(r), i_\omega(v)),\ f_\omega(r-v) \leq F(f_\omega(r), f_\omega(v))$

(ii) $t_\omega(r \cdot v) \geq T(t_\omega(r), t_\omega(v)),\ i_\omega(r \cdot v) \geq I(i_\omega(r), i_\omega(v)),\ f_\omega(r \cdot v) \leq S(f_\omega(r), f_\omega(v))$.

Here, T and I are two TNs and S is a SN.

Proposition 2.11. *[63] Let $\omega_1, \omega_2 \in NSR(P)$. Then $\omega_1 \cap \omega_2 \in NSR(P)$.*

Theorem 2.12. *[63] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If ω is a NSR of P then $l(\omega)$ is a NSR of R.*

Theorem 2.13. *[63] Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a homomorphism. If ω' is a NSR of R then $l^{-1}(\omega')$ is a NSR of P.*

Definition 2.17. [63] Let $\omega = \{(r, t_\omega(r), i_\omega(r), f_\omega(r)) : r \in P\}$ be a NSR of P. Then $\forall s \in [0,1]$ the s-level sets of P are defined as

(i) $(t_\omega)_s = \{r \in P : t_\omega(r) \geq s\}$,
(ii) $(i_\omega)_s = \{r \in P : i_\omega(r) \geq s\}$, and
(iii) $(f_\omega)^s = \{r \in P : f_\omega(r) \leq s\}$.

Proposition 2.14. *[63] A NS $\omega = \{(r, t_\omega(r), i_\omega(r), f_\omega(r)) : r \in P\}$ of a crisp ring $(P, +, \cdot)$ is a NSR of P iff $\forall s \in [0,1]$ the s-level sets of P, i.e. $(t_\omega)_s$, $(i_\omega)_s$, and $(f_\omega)^s$ are crisp rings of P.*

2.2. *Interval-valued Fuzzy and intuitionistic fuzzy subrings*

Definition 2.18. [50] Let $(P, +, \cdot)$ be a crisp ring. An IVFS $\Lambda = \{(r, \bar{t}_\Lambda(r)) : r \in P\}$ is called an IVFSR of $(P, +, \cdot)$ with respect to IVTN $\bar{T}$ if $\forall r, v \in P$, the followings can be concluded:

(i) $\bar{t}_\Lambda(r + v) \geq \bar{T}(\bar{t}_\Lambda(r), \bar{t}_\Lambda(v))$,
(ii) $\bar{t}_\Lambda(-r) \geq \Lambda(r)$, and
(iii) $\bar{t}_\Lambda(r \cdot v) \geq \bar{T}(\bar{t}_\Lambda(r), \bar{t}_\Lambda(v))$,

The set of all IVFSR of a crisp ring $(P, +, \cdot)$ with respect to an IVTN $\bar{T}$ will be denoted as IVFSR$(P, \bar{T})$.

Proposition 2.15. *[50] Let $\lambda = \{(r, t_\lambda(r)) : r \in P\}$ be a FSR of $(P, +, \cdot)$. Then $\Lambda = [t_\lambda, t_\lambda]$ is an IVFSR of P.*

Proposition 2.16. *[50] Let $\Lambda = \{(r, \bar{t}_\Lambda(r)) : r \in P\}$ be an IVFSR of $(P, +, \cdot)$. Then $\Lambda^- = \{(r, \bar{t}_\Lambda^-(r)) : r \in P\}$ and $\Lambda^+ = \{(r, \bar{t}_\Lambda^+(r)) : r \in P\}$ are FSRs of P.*

Definition 2.19. [53] Let $(P, +, \cdot)$ be a crisp ring. An IVIFS $\Gamma = \{(r, \bar{t}_\Gamma(r), \bar{f}_\Gamma(r)) : r \in P\}$ is called an IVIFSR of $(P, +, \cdot)$ if $\forall r, v \in P$, the followings can be concluded:

(i) $\bar{t}_\Gamma(r + v) \geq \bar{T}(\bar{t}_\Gamma(r), \bar{t}_\Gamma(v))$, $\bar{f}_\Gamma(r + v) \leq \bar{F}(\bar{f}_\Gamma(r), \bar{f}_\Gamma(v))$,
(ii) $\bar{t}_\Gamma(-r) \geq \bar{t}_\Gamma(r)$, $\bar{f}_\Gamma(-r) \leq \bar{f}_\Gamma(r)$, and
(iii) $\bar{t}_\Gamma(r \cdot v) \geq \bar{T}(\bar{t}_\Gamma(r), \bar{t}_\Gamma(v))$, $\bar{f}_\Gamma(r \cdot v) \leq \bar{F}(\bar{f}_\Gamma(r), \bar{f}_\Gamma(v))$.

The set of all IVIFSR of a crisp ring $(P, +, \cdot)$ will be denoted as IVIFSR(P).

Theorem 2.17. *[53] If* $\Gamma = \{(r, \bar{t}_\Gamma(r), \bar{f}_\Gamma(r)) : r \in P\} \in$ *IVIFSR*(P)*, then* $\bar{t}_\Gamma(r) \leq \bar{t}_\Gamma(\theta_P)$ *and* $\bar{f}_\Gamma(r) \geq \bar{f}_\Gamma(\theta_P)$*.*

Theorem 2.18. *[53] If* Γ_1 *and* $\Gamma_2 \in$ *IVIFSR*(P)*, then* $\Gamma_1 \cap \Gamma_2 \in$ *IVIFSR*(P)*.*

Theorem 2.19. *[53] Let* $\Gamma = \{(r, \bar{t}_\Gamma(r), \bar{f}_\Gamma(r)) : r \in P\} \in$ *IVIFSR*(P)*, then* $\forall r, v \in P$

(i) $\bar{t}_\Gamma(r - v) = \bar{t}_\Gamma(\theta_P)$ *implies that* $\bar{t}_\Gamma(r) = \bar{t}_\Gamma(v)$*.*
(ii) $\bar{f}_\Gamma(r - v) = \bar{f}_\Gamma(\theta_P)$ *implies that* $\bar{f}_\Gamma(r) = \bar{f}_\Gamma(v)$*.*

3. Proposed notion of interval-valued neutrosophic subring

Definition 3.1. Let $(P, +, \cdot)$ be a crisp ring. An IVNS $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in P\}$ is called an IVNSR of $(P, +, \cdot)$ if $\forall r, v \in P$, the followings can be concluded:

$$\text{(i)} \begin{cases} \bar{t}_\Omega(r+v) \geq \bar{T}(\bar{t}_\Omega(r), \bar{t}_\Omega(v)), \\ \bar{i}_\Omega(r+v) \leq \bar{I}(\bar{i}_\Omega(r), \bar{i}_\Omega(v)), \\ \bar{f}_\Omega(r+v) \leq \bar{F}(\bar{f}_\Omega(r), \bar{f}_\Omega(v)) \end{cases}$$

$$\text{(ii)} \begin{cases} \bar{t}_\Omega(-r) \geq t_\Omega(r), \\ \bar{i}_\Omega(-r) \leq i_\Omega(r), \\ \bar{f}_\Omega(-r) \leq f_\Omega(r) \end{cases}$$

$$\text{(iii)} \begin{cases} \bar{t}_\Omega(r \cdot v) \geq \bar{T}(\bar{t}_\Omega(r), \bar{t}_\Omega(v)), \\ \bar{i}_\Omega(r \cdot v) \leq \bar{I}(\bar{i}_\Omega(r), \bar{i}_\Omega(v)), \\ \bar{f}_\Omega(r \cdot v) \leq \bar{F}(\bar{f}_\Omega(r), \bar{f}_\Omega(v)), \end{cases}$$

where $\bar{T}$ is an IVTN, $\bar{I}$ and $\bar{F}$ are two IVSNs.

The set of all IVNSR of a crisp ring $(P, +, \cdot)$ will be denoted as IVNSR(P).

Example 3.2. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers with respect to usual addition and multiplication. Let $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in \mathbb{Z}\}$ be an IVNS of $\mathbb{Z}$, where $\forall r \in \mathbb{Z}$

$$\bar{t}_\Omega(r) = \begin{cases} [0.2, 0.25] & \text{if } r \in 2\mathbb{Z} \\ [0, 0] & \text{if } r \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_\Omega(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.1, 0.12] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_\Omega(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.75, 0.8] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}.$$

Now, if we consider minimum TN and maximum SNs, then $\Omega \in$ IVNSR$(\mathbb{Z})$.

Example 3.3. Let $(\mathbb{Z}_4, +, \cdot)$ be the ring of integers modulo 4 with usual addition and multiplication. Let $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in \mathbb{Z}_4\}$ be an IVNS of $\mathbb{Z}_4$, where interval-valued memberships of elements belonging to Ω are mentioned in Table 2.

TABLE 2. Membership values of elements belonging to Ω

Ω	$\bar{t}_\Omega$	$\bar{i}_\Omega$	$\bar{f}_\Omega$
$\bar{0}$	$[0.6, 0.7]$	$[0.33, 0.35]$	$[0.2, 0.3]$
$\bar{1}$	$[0.7, 0.8]$	$[0.21, 0.23]$	$[0.5, 0.6]$
$\bar{2}$	$[0.75, 0.85]$	$[0.24, 0.26]$	$[0.3, 0.7]$
$\bar{3}$	$[0.75, 0.9]$	$[0.31, 0.33]$	$[0.5, 0.7]$

Now, if we consider the Łukasiewicz T-norm ($T(r, v) = \max\{0, r + v - 1\}$) and bounded sum S-norms ($S(r, v) = \min\{r + v, 1\}$), then $\Omega \in \text{IVNSR}(\mathbb{Z}_4)$.

Proposition 3.1. *An IVNS $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in P\}$ of a crisp ring $(P, +, \cdot)$ is an IVNSR iff the followings can be concluded (assuming that all the IVTN and IVSNs are idempotent):*

$$\text{(i)} \begin{cases} \bar{t}_\Omega(r - v) \geq \bar{T}(\bar{t}_\Omega(r), \bar{t}_\Omega(v)), \\ \bar{i}_\Omega(r - v) \leq \bar{I}(\bar{i}_\Omega(r), \bar{i}_\Omega(v)), \\ \bar{f}_\Omega(r - v) \leq \bar{F}(\bar{f}_\Omega(r), \bar{f}_\Omega(v)) \end{cases}$$

$$\text{(ii)} \begin{cases} \bar{t}_\Omega(r \cdot v) \geq \bar{T}(\bar{t}_\Omega(r), \bar{t}_\Omega(v)), \\ \bar{i}_\Omega(r \cdot v) \leq \bar{I}(\bar{i}_\Omega(r), \bar{i}_\Omega(v)), \\ \bar{f}_\Omega(r \cdot v) \leq \bar{F}(\bar{f}_\Omega(r), \bar{f}_\Omega(v)). \end{cases}$$

Proof. Let $\Omega \in$ IVNSR(P). Then we have

$$\begin{aligned} \bar{t}_\Omega(r - v) &\geq \bar{T}(\bar{t}_\Omega(r), \bar{t}_\Omega(-v)) \text{ [by condition (i) of Definition 3.1]} \\ &\geq \bar{T}(\bar{t}_\Omega(r), \bar{t}_\Omega(v)) \text{ [by condition (ii) of Definition 3.1]} \end{aligned}$$

Similary, we will have

$$\begin{aligned} \bar{i}_\Omega(r - v) &\leq \bar{I}(\bar{i}_\Omega(r), \bar{i}_\Omega(v)), \text{ and} \\ \bar{f}_\Omega(r - v) &\leq \bar{F}(\bar{f}_\Omega(r), \bar{f}_\Omega(v)), \end{aligned}$$

which proves (i).

Again, (ii) follows immediately from condition (iii) of Definition 3.1.

Conversely, let (i) and (ii) of Proposition 3.1 hold. Also, let θ_P be the additive neutral element in $(P, +, \cdot)$. Then

$$\begin{aligned}\bar{t}_\Omega(\theta_P) &= \bar{t}_\Omega(r - r)\\ &\geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(r)\big)\\ &= \bar{t}_\Omega(r) \end{aligned} \tag{3.1}$$

Similaly, we can show that

$$\bar{i}_\Omega(\theta_P) \leq \bar{i}_\Omega(r) \tag{3.2}$$

$$\bar{f}_\Omega(\theta_P) \leq \bar{f}_\Omega(r) \tag{3.3}$$

Now,

$$\begin{aligned}\bar{t}_\Omega(-r) &= \bar{t}_\Omega(\theta_P - r)\\ &\geq \bar{T}\big(\bar{t}_\Omega(\theta_P), \bar{t}_\Omega(r)\big)\\ &\geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(r)\big) \text{ [by 3.1]}\\ &= \bar{t}_\Omega(r) \text{ [since } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.4}$$

Similarly, we can prove

$$\bar{i}_\Omega(-r) \leq \bar{i}_\Omega(r) \text{ [since } \bar{I} \text{ is idempotent]} \tag{3.5}$$

$$\bar{f}_\Omega(-r) \leq \bar{f}_\Omega(r) \text{ [since } \bar{F} \text{ is idempotent]} \tag{3.6}$$

Hence,

$$\begin{aligned}\bar{t}_\Omega(r + v) &= \bar{t}_\Omega(r - (-v))\\ &\geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(-v)\big)\\ &\geq \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \text{ [by 3.4]} \end{aligned} \tag{3.7}$$

Similarly,

$$\bar{i}_\Omega(r + v) \leq \bar{I}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \text{ [by 3.5]} \tag{3.8}$$

$$\bar{f}_\Omega(r + v) \leq \bar{F}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \text{ [by 3.6]} \tag{3.9}$$

So, by Equations 3.7, 3.8, and 3.9 condition (i) of Proposition 3.1 has been proved. Also, condition (ii) of Proposition 3.1 is same as condition (iii) of Definition 3.1. Hence, $\Omega \in$ IVNSR(P). □

Theorem 3.2. *Let $(P, +, \cdot)$ be a crisp ring. If $\Omega_1, \Omega_2 \in IVNSR(P)$, then $\Omega_1 \cap \Omega_2 \in IVNSR(P)$ (assuming all the IVTN and IVSNs are idempotent).*

Proof. Let $\Omega = \Omega_1 \cap \Omega_2$. Now, $\forall r, v \in P$

$$\begin{aligned}\bar{t}_{\Omega}(r+v) &= \bar{T}\big(\bar{t}_{\Omega_1}(r+v), \bar{t}_{\Omega_2}(r+v)\big)\\ &\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_1}(v)\big), \bar{T}\big(\bar{t}_{\Omega_2}(r), \bar{t}_{\Omega_2}(v)\big)\Big)\\ &= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_1}(v)\big), \bar{T}\big(\bar{t}_{\Omega_2}(v), \bar{t}_{\Omega_2}(r)\big)\Big) \text{ [as } \bar{T} \text{ is commutative]}\\ &= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_2}(r)\big), \bar{T}\big(\bar{t}_{\Omega_1}(v), \bar{t}_{\Omega_2}(v)\big)\Big) \text{ [as } \bar{T} \text{ is associative]}\\ &= \bar{T}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(v)\big) \qquad (3.10)\end{aligned}$$

Similarly, as both $\bar{I}$ and $\bar{S}$ are commutative as well as associative, we will have

$$\bar{i}_{\Omega}(r+v) \leq \bar{I}\big(\bar{i}_{\Omega}(r), \bar{i}_{\Omega}(v)\big) \qquad (3.11)$$

$$\bar{f}_{\Omega}(r+v) \leq \bar{F}\big(\bar{f}_{\Omega}(r), \bar{f}_{\Omega}(v)\big) \qquad (3.12)$$

Again,

$$\begin{aligned}\bar{t}_{\Omega}(-r) &= \bar{T}\big(\bar{t}_{\Omega_1}(-r), \bar{t}_{\Omega_2}(-r)\big)\\ &\geq \bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_2}(r)\big) \text{ [by Definition 3.1]}\\ &= \bar{t}_{\Omega}(r) \qquad (3.13)\end{aligned}$$

Also,

$$\bar{i}_{\Omega}(-r) \leq \bar{i}_{\Omega}(r) \qquad (3.14)$$

$$\bar{f}_{\Omega}(-r) \leq \bar{f}_{\Omega}(r) \qquad (3.15)$$

Similarly, we can show that

$$\bar{t}_{\Omega}(r \cdot v) \geq \bar{T}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(v)\big), \qquad (3.16)$$

$$\bar{i}_{\Omega}(r \cdot v) \leq \bar{I}\big(\bar{i}_{\Omega}(r), \bar{i}_{\Omega}(v)\big), \text{ and} \qquad (3.17)$$

$$\bar{f}_{\Omega}(r \cdot v) \leq \bar{F}\big(\bar{f}_{\Omega}(r), \bar{f}_{\Omega}(v)\big) \qquad (3.18)$$

Hence, by Equations 3.10–3.18 $\Omega = \Omega_1 \cap \Omega_2 \in \text{IVNSR}(P)$. □

Remark 3.3. *In general, if $\Omega_1, \Omega_2 \in IVNSR(P)$, then $\Omega_1 \cup \Omega_2$ may not always be an IVNSR of $(P, +, \cdot)$.*

The following Example 3.4 will prove our claim.

Example 3.4. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers with respect to usual addition and multiplication. Let $\Omega_1 = \big\{\big(r, \bar{t}_{\Omega_1}(r), \bar{i}_{\Omega_1}(r), \bar{f}_{\Omega_1}(r)\big) : r \in \mathbb{Z}\big\}$ and $\Omega_2 = \big\{\big(r, \bar{t}_{\Omega_2}(r), \bar{i}_{\Omega_2}(r), \bar{f}_{\Omega_2}(r)\big) :$

$r \in \mathbb{Z}\big\}$ be two IVNSs of $\mathbb{Z}$, where $\forall r \in \mathbb{Z}$

$$\bar{t}_{\Omega_1}(r) = \begin{cases} [0.25, 0.4] & \text{if } r \in 2\mathbb{Z} \\ [0, 0] & \text{if } r \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{\Omega_1}(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.17, 0.2] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{\Omega_1}(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.33, 0.4] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}.$$

and

$$\bar{t}_{\Omega_2}(r) = \begin{cases} [0.5, 0.67] & \text{if } r \in 3\mathbb{Z} \\ [0, 0] & \text{if } r \in 3\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_{\Omega_2}(r) = \begin{cases} [0, 0] & \text{if } r \in 3\mathbb{Z} \\ [0.2, 0.25] & \text{if } r \in 3\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_{\Omega_2}(r) = \begin{cases} [0, 0] & \text{if } r \in 3\mathbb{Z} \\ [0.33, 0.5] & \text{if } r \in 3\mathbb{Z}+1 \end{cases}.$$

Now, if we consider minimum TN and maximum SNs, then $\Omega_1, \Omega_2 \in \text{IVNSR}(\mathbb{Z})$.
Now let $\Omega = \Omega_1 \cup \Omega_2$. Then for $r = 4$ and $v = 9$

$$\begin{aligned} \bar{t}_\Omega(r+v) &= \bar{t}_\Omega(4+9) \\ &= \bar{t}_\Omega(13) \\ &= \max\{\bar{t}_{\Omega_1}(13), \bar{t}_{\Omega_2}(13)\} \\ &= \max\{[0,0], [0,0]\} \\ &= [0,0] \end{aligned}$$

Again, if $\Omega \in$ IVNSR(P), then $\forall r, v \in P$, $\bar{t}_\Omega(r+v) \geq \min\{\bar{t}_\Omega(r), \bar{t}_\Omega(v)\}$. But, here for $r = 4$ and $v = 9$, $\min\{\bar{t}_\Omega(4), \bar{t}_\Omega(9)\} = \min\{[0.25, 0.4], [0.5, 0.67]\} = [0.25, 0.4] \not\leq [0, 0] = \bar{t}_\Omega(4+9)$. Hence, $\Omega \notin$ IVNSR(P).

Corollary 3.4. *If $\Omega_1, \Omega_2 \in$ IVNSR(P), then $\Omega_1 \cup \Omega_2 \in$ IVNSR(P) iff one is contained in other.*

Definition 3.5. Let $\Omega = \big\{\big(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)\big) : r \in P\big\}$ be an IVNS of a crisp ring $(P, +, \cdot)$. Also, let $[k_1, s_1]$, $[k_2, s_2]$ and $[k_3, s_3] \in \Psi(L)$. Then the crisp set $\Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ is called a level set of IVNSR Ω, where for any $r \in \Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ the following inequalities will hold: $\bar{t}_\Omega(r) \geq [k_1, s_1]$, $\bar{i}_\Omega(r) \leq [k_2, s_2]$, and $\bar{f}_\Omega(r) \leq [k_3, s_3]$.

Theorem 3.5. *Let $(P,+,\cdot)$ be a crisp ring. Then $\Omega \in IVNSR(P)$ iff $\forall [k_1,s_1],[k_2,s_2],[k_3,s_3] \in \Psi(L)$ with $\bar{t}_\Omega(\theta_P) \geq [k_1,s_1]$, $\bar{i}_\Omega(\theta_P) \leq [k_2,s_2]$, and $\bar{f}_\Omega(\theta_P) \leq [k_3,s_3]$, $\Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$ is a crisp subring of $(P,+,\cdot)$ (assuming all the IVTN and IVSNs are idempotent).*

Proof. Since, $\bar{t}_\Omega(\theta_P) \geq [k_1,s_1]$, $\bar{i}_\Omega(\theta_P) \leq [k_2,s_2]$, and $\bar{f}_\Omega(\theta_P) \leq [k_3,s_3]$, $\theta_P \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$, i.e., $\Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$ is non-empty.
Now, let $\Omega \in$ IVNSR(P) and $r,v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$. To show that, $(r-v)$ and $r\cdot v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$. Here,

$$\begin{aligned}\bar{t}_\Omega(r-v) &\geq \bar{T}\big(\bar{t}_\Omega(r),\bar{t}_\Omega(v)\big) \text{ [by Proposition 3.1]}\\ &\geq \bar{T}\big([k_1,s_1],[k_1,s_1]\big) \left[\text{as } r,v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}\right]\\ &\geq [k_1,s_1] \text{ [as } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.19}$$

Again,

$$\begin{aligned}\bar{t}_\Omega(r\cdot v) &\geq \bar{T}\big(\bar{t}_\Omega(r),\bar{t}_\Omega(v)\big) \text{ [by Proposition 3.1]}\\ &\geq \bar{T}\big([k_1,s_1],[k_1,s_1]\big) \left[\text{as } r,v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}\right]\\ &\geq [k_1,s_1] \text{ [as } \bar{T} \text{ is idempotent]} \end{aligned} \tag{3.20}$$

Similarly, we can show that

$$\bar{i}_\Omega(r-v) \leq [k_2,s_2], \tag{3.21}$$

$$\bar{i}_\Omega(r\cdot v) \leq [k_2,s_2], \tag{3.22}$$

$$\bar{f}_\Omega(r-v) \leq [k_3,s_3], \text{ and} \tag{3.23}$$

$$\bar{f}_\Omega(r\cdot v) \leq [k_3,s_3] \tag{3.24}$$

Hence, by Equations 3.19–3.24 $(r-v)$ and $r\cdot v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$, i.e., $\Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$ is a crisp subring of $(P,+,\cdot)$.
Conversely, let $\Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$ is a crisp subgroup of $(P,+,\cdot)$. To show that, $\Omega \in$ IVNSR(P).
Let $r,v \in P$, then there exists $[k_1,s_1] \in \Psi(L)$ such that $\bar{T}\big(\bar{t}_\Omega(r),\bar{t}_\Omega(v)\big) = [k_1,s_1]$. So, $\bar{t}_\Omega(r) \geq [k_1,s_1]$ and $\bar{t}_\Omega(v) \geq [k_1,s_1]$. Also, let there exist $[k_2,s_2],[k_3,s_3] \in \Psi(L)$ such that $\bar{I}\big(\bar{i}_\Omega(r),\bar{i}_\Omega(v)\big) = [k_2,s_2]$ and $\bar{F}\big(\bar{f}_\Omega(r),\bar{f}_\Omega(v)\big) = [k_3,s_3]$. Then $r,v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$.
Again, as $\Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$ is a crisp subring, $r-v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$ and $r\cdot v \in \Omega_{\left([k_1,s_1],[k_2,s_2],[k_3,s_3]\right)}$.

Hence,

$$\begin{aligned}\bar{t}_\Omega(r-v) &\geq [k_1, s_1]\\ &= \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \text{ and} \qquad (3.25)\\ \bar{t}_\Omega(r\cdot v) &\geq [k_1, s_1]\\ &= \bar{T}\big(\bar{t}_\Omega(r), \bar{t}_\Omega(v)\big) \qquad (3.26)\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}\bar{i}_\Omega(r-v) &\leq [k_2, s_2]\\ &= \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \qquad (3.27)\\ \bar{i}_\Omega(r\cdot v) &\leq [k_2, s_2]\\ &= \bar{I}\big(\bar{i}_\Omega(r), \bar{i}_\Omega(v)\big), \qquad (3.28)\\ \bar{f}_\Omega(r-v) &\leq [k_3, s_3]\\ &= \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big), \text{ and} \qquad (3.29)\\ \bar{f}_\Omega(r\cdot v) &\leq [k_3, s_3]\\ &= \bar{F}\big(\bar{f}_\Omega(r), \bar{f}_\Omega(v)\big) \qquad (3.30)\end{aligned}$$

So, Equations 3.25–3.30 imply that Ω follows Proposition 3.1, i.e., $\Omega \in \mathrm{IVNSR}(P)$. □

Definition 3.6. Let Ω and Ω' be two IVNSs of two CSs P and R, respectively. Also, let $l : P \to R$ be a function. Then

(i) image of Ω under l will be $l(\Omega) = \big\{\big(v, \bar{t}_{l(\Omega)}(v), \bar{i}_{l(\Omega)}(v), \bar{f}_{l(\Omega)}(v)\big) : v \in R\big\}$, where $\bar{t}_{l(\Omega)}(v) = \underset{s\in l^{-1}(v)}{\vee} \bar{t}_\Omega(s)$, $\bar{i}_{l(\Omega)}(v) = \underset{s\in l^{-1}(v)}{\wedge} \bar{i}_\Omega(s)$, $\bar{f}_{l(\Omega)}(v) = \underset{s\in l^{-1}(v)}{\wedge} \bar{f}_\Omega(s)$. Wherefrom, if l is injective then $\bar{t}_{l(\Omega)}(v) = \bar{t}_\Omega\big(l^{-1}(v)\big)$, $\bar{i}_{l(\Omega)}(v) = \bar{i}_\Omega\big(l^{-1}(v)\big)$, $\bar{f}_{l(\Omega)}(v) = \bar{f}_\Omega\big(l^{-1}(v)\big)$, and

(ii) preimage of Ω' under l will be $l^{-1}(\Omega') = \big\{\big(r, \bar{t}_{l^{-1}(\Omega')}(r), \bar{i}_{l^{-1}(\Omega')}(r), \bar{f}_{l^{-1}(\Omega')}(r)\big) : r \in R\big\}$, where $\bar{t}_{l^{-1}(\Omega')}(r) = \bar{t}_{\Omega'}\big(l(r)\big)$, $\bar{i}_{l^{-1}(\Omega')}(r) = \bar{i}_{\Omega'}\big(l(r)\big)$, $\bar{f}_{l^{-1}(\Omega')}(r) = \bar{f}_{\Omega'}\big(l(r)\big)$.

Theorem 3.6. *Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a ring isomorphism. If Ω is an IVNSR of P then $l(\Omega)$ is an IVNSR of R.*

Proof. Let $v_1 = l(r_1)$ and $v_2 = l(r_2)$, where $r_1, r_2 \in P$ and $v_1, v_2 \in R$. Now,

$$\begin{aligned}
\bar{t}_{l(\Omega)}(v_1 - v_2) &= \bar{t}_\Omega\big(l^{-1}(v_1 - v_2)\big) \text{ [as } l \text{ is injective]} \\
&= \bar{t}_\Omega\big(l^{-1}(v_1) - l^{-1}(v_2)\big) \text{ [as } l^{-1} \text{ is a homomorphism]} \\
&= \bar{t}_\Omega(r_1 - r_2) \\
&\geq \bar{T}\big(\bar{t}_\Omega(r_1), \bar{t}_\Omega(r_2)\big) \\
&= \bar{T}\Big(\bar{t}_\Omega\big(l^{-1}(v_1)\big), \bar{t}_\Omega\big(l^{-1}(v_2)\big)\Big) \\
&= \bar{T}\big(\bar{t}_{l(\Omega)}(v_1), \bar{t}_{l(\Omega)}(v_2)\big) \qquad (3.31)
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{t}_{l(\Omega)}(v_1 \cdot v_2) &= \bar{t}_\Omega\big(l^{-1}(v_1 \cdot v_2)\big) \text{ [as } l \text{ is injective]} \\
&= \bar{t}_\Omega\big(l^{-1}(v_1) \cdot l^{-1}(v_2)\big) \text{ [as } l^{-1} \text{ is a homomorphism]} \\
&= \bar{t}_\Omega(r_1 \cdot r_2) \\
&\geq \bar{T}\big(\bar{t}_\Omega(r_1), \bar{t}_\Omega(r_2)\big) \\
&= \bar{T}\Big(\bar{t}_\Omega\big(l^{-1}(v_1)\big), \bar{t}_\Omega\big(l^{-1}(v_2)\big)\Big) \\
&= \bar{T}\big(\bar{t}_{l(\Omega)}(v_1), \bar{t}_{l(\Omega)}(v_2)\big) \qquad (3.32)
\end{aligned}$$

Similarly,

$$\bar{i}_{l(\Omega)}(v_1 - v_2) \leq \bar{I}\big(\bar{i}_{l(\Omega)}(v_1), \bar{i}_{l(\Omega)}(v_2)\big), \qquad (3.33)$$

$$\bar{i}_{l(\Omega)}(v_1 \cdot v_2) \leq \bar{I}\big(\bar{i}_{l(\Omega)}(v_1), \bar{i}_{l(\Omega)}(v_2)\big), \qquad (3.34)$$

$$\bar{f}_{l(\Omega)}(v_1 - v_2) \leq \bar{F}\big(\bar{f}_{l(\Omega)}(v_1), \bar{f}_{l(\Omega)}(v_2)\big), \text{ and} \qquad (3.35)$$

$$\bar{f}_{l(\Omega)}(v_1 \cdot v_2) \leq \bar{F}\big(\bar{f}_{l(\Omega)}(v_1), \bar{f}_{l(\Omega)}(v_2)\big) \qquad (3.36)$$

Hence, Equations 3.31–3.36 imply that $l(\Omega)$ follows Proposition 3.1, i.e., $l(\Omega)$ is an IVNSR of R. □

Theorem 3.7. *Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a ring homomorphism. If Ω' is an IVNSR of R then $l^{-1}(\Omega')$ is an IVNSR of P (Note that, l^{-1} may not be an inverse mapping but $l^{-1}(\Omega')$ is an inverse image of Ω').*

Proof. Let $v_1 = l(r_1)$ and $v_2 = l(r_2)$, where $r_1, r_2 \in P$ and $v_1, v_2 \in R$. Now,

$$\begin{aligned} \bar{t}_{l^{-1}(\Omega')}(r_1 - r_2) &= \bar{t}_{\Omega'}\big(l(r_1 - r_2)\big) \\ &= \bar{t}_{\Omega'}\big(l(r_1) - l(r_2)\big) \text{ [as } l \text{ is a homomorphism]} \\ &= \bar{t}_{\Omega'}(v_1 - v_2) \\ &\geq \bar{T}\big(\bar{t}_{\Omega'}(v_1), \bar{t}_{\Omega'}(v_2)\big) \\ &= \bar{T}\Big(\bar{t}_{\Omega'}\big(l(r_1)\big), \bar{t}_{\Omega'}\big(l(r_2)\big)\Big) \\ &= \bar{T}\big(\bar{t}_{l^{-1}(\Omega')}(r_1), \bar{t}_{l^{-1}(\Omega')}(r_2)\big) \end{aligned} \tag{3.37}$$

Again,

$$\begin{aligned} \bar{t}_{l^{-1}(\Omega')}(r_1 \cdot r_2) &= \bar{t}_{\Omega'}\big(l(r_1 \cdot r_2)\big) \\ &= \bar{t}_{\Omega'}\big(l(r_1) \cdot l(r_2)\big) \text{ [as } l \text{ is a homomorphism]} \\ &= \bar{t}_{\Omega'}(v_1 \cdot v_2) \\ &\geq \bar{T}\big(\bar{t}_{\Omega'}(v_1), \bar{t}_{\Omega'}(v_2)\big) \\ &= \bar{T}\Big(\bar{t}_{\Omega'}\big(l(r_1)\big), \bar{t}_{\Omega'}\big(l(r_2)\big)\Big) \\ &= \bar{T}\big(\bar{t}_{l^{-1}(\Omega')}(r_1), \bar{t}_{l^{-1}(\Omega')}(r_2)\big) \end{aligned} \tag{3.38}$$

Similarly,

$$\bar{i}_{l^{-1}(\Omega')}(r_1 - r_2) \leq \bar{I}\big(\bar{i}_{l^{-1}(\Omega')}(r_1), \bar{i}_{l^{-1}(\Omega')}(r_2)\big) \tag{3.39}$$

$$\bar{i}_{l^{-1}(\Omega')}(r_1 \cdot r_2) \leq \bar{I}\big(\bar{i}_{l^{-1}(\Omega')}(r_1), \bar{i}_{l^{-1}(\Omega')}(r_2)\big) \tag{3.40}$$

$$\bar{f}_{l^{-1}(\Omega')}(r_1 - r_2) \leq \bar{F}\big(\bar{f}_{l^{-1}(\Omega')}(r_1), \bar{f}_{l^{-1}(\Omega')}(r_2)\big) \tag{3.41}$$

$$\bar{f}_{l^{-1}(\Omega')}(r_1 \cdot r_2) \leq \bar{F}\big(\bar{f}_{l^{-1}(\Omega')}(r_1), \bar{f}_{l^{-1}(\Omega')}(r_2)\big) \tag{3.42}$$

Hence, Equations 3.37–3.42 imply that $l^{-1}(\Omega')$ follows Proposition 3.1, i.e., $l^{-1}(\Omega')$ is an IVNSR of P. □

Definition 3.7. Let $(P, +, \cdot)$ be a crisp ring and $\Omega \in$ IVNSR(P). Again, let $\bar{\sigma} = [\sigma_1, \sigma_2], \bar{\tau} = [\tau_1, \tau_2], \bar{\delta} = [\delta_1, \delta_2] \in \Psi(L)$. Then

(i) Ω is called a $(\bar{\sigma}, \bar{\tau}, \bar{\delta})-$identity IVNSR over P, if $\forall r \in P$

$$\bar{t}_\Omega(r) = \begin{cases} \bar{\sigma} & \text{if } r = \theta_P \\ [0,0] & \text{if } r \neq \theta_P \end{cases},$$

$$\bar{i}_\Omega(r) = \begin{cases} \bar{\tau} & \text{if } r = \theta_P \\ [1,1] & \text{if } \ \mathrm{r} \neq \theta_P \end{cases}, \text{ and}$$

$$\bar{f}_\Omega(r) = \begin{cases} \bar{\delta} & \text{if } r = \theta_P \\ [1,1] & \text{if } r \neq \theta_P \end{cases},$$

where θ_P is the zero element of P.

(ii) Ω is called a $(\bar{\sigma}, \bar{\tau}, \bar{\delta})-$absolute IVNSR over P, if $\forall r \in P$, $\bar{t}_\Omega(r) = \bar{\sigma}$, $\bar{i}_\Omega(r) = \bar{\tau}$, and $\bar{f}_\Omega(r) = \bar{\delta}$.

Theorem 3.8. *Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings and $\Omega \in$IVNSR (P). Again, let $l : P \to R$ be a ring homomorphism. Then*

(i) *$l(\Omega)$ will be a $(\bar{\sigma}, \bar{\tau}, \bar{\delta})-$identity IVNSR over R, if $\forall r \in P$*

$$\bar{t}_\Omega(r) = \begin{cases} \bar{\sigma} & \textit{if } r \in Ker(l) \\ [0,0] & \textit{otherwise} \end{cases},$$

$$\bar{i}_\Omega(r) = \begin{cases} \bar{\tau} & \textit{if } r \in Ker(l) \\ [1,1] & \textit{otherwise} \end{cases}, \ \textit{and}$$

$$\bar{f}_\Omega(r) = \begin{cases} \bar{\delta} & \textit{if } r \in Ker(l) \\ [1,1] & \textit{otherwise} \end{cases},$$

(ii) *$l(\Omega)$ will be a $(\bar{\sigma}, \bar{\tau}, \bar{\delta})-$absolute IVNSR over R, if Ω is a $(\bar{\sigma}, \bar{\tau}, \bar{\delta})-$absolute IVNSR over P.*

Proof. (i) Clearly, by Theorem 3.6 $l(\Omega) \in$ IVNSR(R). Let $r \in Ker(l)$, then $l(r) = \theta_R$.
So,

$$\begin{aligned} \bar{t}_{l(\Omega)}(\theta_R) &= \bar{t}_\Omega\big(l^{-1}(\theta_R)\big) \\ &= \bar{t}_\Omega(r) \\ &= \bar{\sigma} \end{aligned} \tag{3.43}$$

Similarly, we can show that

$$\bar{i}_{l(\Omega)}(\theta_R) = \bar{\tau}, \text{ and} \tag{3.44}$$

$$\bar{f}_{l(\Omega)}(\theta_R) = \bar{\delta} \tag{3.45}$$

Again, let $r \in P \setminus Ker(l)$ and $l(r) = v$. Then

$$\begin{aligned} \bar{t}_{l(\Omega)}(v) &= \bar{t}_{\Omega}\big(l^{-1}(v)\big) \\ &= \bar{t}_{\Omega}(r) \\ &= [0,0] \end{aligned} \tag{3.46}$$

Similarly, we can show that

$$\bar{i}_{l(\Omega)}(v) = [1,1] \text{ and} \tag{3.47}$$

$$\bar{f}_{l(\Omega)}(v) = [1,1] \tag{3.48}$$

Hence, by the Equations 3.43–3.48 $l(\Omega)$ is a $(\bar{\sigma}, \bar{\tau}, \bar{\delta})-$identity IVNSR over R.
(ii) Let $l(r) = v$, for $r \in P$ and $v \in R$. Then

$$\begin{aligned} \bar{t}_{l(\Omega)}(v) &= \bar{t}_{\Omega}\big(l^{-1}(v)\big) \\ &= \bar{t}_{\Omega}(r) \\ &= \bar{\sigma} \end{aligned} \tag{3.49}$$

Similarly, we can show that

$$\bar{i}_{l(\Omega)}(v) = \bar{\tau} \text{ and} \tag{3.50}$$

$$\bar{f}_{l(\Omega)}(v) = \bar{\delta} \tag{3.51}$$

Hence, by the Equations 3.48–3.51 $l(\Omega)$ is a $(\bar{\sigma}, \bar{\tau}, \bar{\delta})-$absolute IVNSR over R. □

3.1. *Product of interval-valued neutrosophic subrings*

Definition 3.8. Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Again, let $\Omega_1 = \big\{\big(r, \bar{t}_{\Omega_1}(r), \bar{i}_{\Omega_1}(r), \bar{f}_{\Omega_1}(r)\big) : r \in P\big\}$ and $\Omega_2 = \big\{\big(v, \bar{t}_{\Omega_2}(v), \bar{i}_{\Omega_2}(v), \bar{f}_{\Omega_2}(v)\big) : v \in R\big\}$ are IVNSRs of P and R respectively. Then Cartesian product of Ω_1 and Ω_2 will be

$$\begin{aligned} \Omega &= \Omega_1 \times \Omega_2 \\ &= \Big\{\Big((r,v), \bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_2}(v)\big), \bar{I}\big(\bar{i}_{\Omega_1}(r), \bar{i}_{\Omega_2}(v)\big), \bar{F}\big(\bar{f}_{\Omega_1}(r), \bar{f}_{\Omega_2}(v)\big)\Big) : (r,v) \in P \times R\Big\} \end{aligned}$$

Similarly, product of 3 or more IVNSRs can be defined.

Theorem 3.9. *Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings with $\Omega_1 \in$ IVNSR(P) and $\Omega_2 \in$ IVNSR(R). Then $\Omega_1 \times \Omega_2$ is a IVNSR of $P \times R$.*

Proof. Let $\Omega = \Omega_1 \times \Omega_2$ and $(r_1, v_1), (r_2, v_2) \in P \times R$. Then

$$\begin{aligned}
\bar{t}_\Omega\big((r_1,v_1)-(r_2,v_2)\big) &= \bar{t}_{\Omega_1\times\Omega_2}\big((r_1-r_2, v_1-v_2)\big)\\
&= \bar{T}\big(\bar{t}_{\Omega_1}(r_1-r_2), \bar{t}_{\Omega_2}(v_1-v_2)\big) \text{ [by Definition 3.8]}\\
&\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r_1), \bar{t}_{\Omega_1}(r_2)\big), \bar{T}\big(\bar{t}_{\Omega_2}(v_1), \bar{t}_{\Omega_2}(v_2)\big)\Big) \text{ [by Proposition 3.1]}\\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r_1), \bar{t}_{\Omega_2}(v_1)\big), \bar{T}\big(\bar{t}_{\Omega_1}(r_2), \bar{t}_{\Omega_2}(v_2)\big)\Big) \text{ [as } \bar{T} \text{ is associative]}\\
&= \bar{T}\big(\bar{t}_\Omega(r_1,v_1), \bar{t}_\Omega(r_2,v_2)\big) \qquad (3.52)
\end{aligned}$$

Again,

$$\begin{aligned}
\bar{t}_\Omega\big((r_1,v_1)\cdot(r_2,v_2)\big) &= \bar{t}_{\Omega_1\times\Omega_2}\big((r_1\cdot r_2, v_1\cdot v_2)\big)\\
&= \bar{T}\big(\bar{t}_{\Omega_1}(r_1\cdot r_2), \bar{t}_{\Omega_2}(v_1\cdot v_2)\big) \text{ [by Definition 3.8]}\\
&\geq \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r_1), \bar{t}_{\Omega_1}(r_2)\big), \bar{T}\big(\bar{t}_{\Omega_2}(v_1), \bar{t}_{\Omega_2}(v_2)\big)\Big) \text{ [by Proposition 3.1]}\\
&= \bar{T}\Big(\bar{T}\big(\bar{t}_{\Omega_1}(r_1), \bar{t}_{\Omega_2}(v_1)\big), \bar{T}\big(\bar{t}_{\Omega_1}(r_2), \bar{t}_{\Omega_2}(v_2)\big)\Big) \text{ [as } \bar{T} \text{ is associative]}\\
&= \bar{T}\big(\bar{t}_\Omega(r_1,v_1), \bar{t}_\Omega(r_2,v_2)\big) \qquad (3.53)
\end{aligned}$$

Similary, the followings can be shown

$$\bar{i}_\Omega\big((r_1,v_1)-(r_2,v_2)\big) \leq \bar{I}\big(\bar{i}_\Omega(r_1,v_1), \bar{i}_\Omega(r_2,v_2)\big), \qquad (3.54)$$

$$\bar{i}_\Omega\big((r_1,v_1)\cdot(r_2,v_2)\big) \leq \bar{I}\big(\bar{i}_\Omega(r_1,v_1), \bar{i}_\Omega(r_2,v_2)\big), \qquad (3.55)$$

$$\bar{f}_\Omega\big((r_1,v_1)-(r_2,v_2)\big) \leq \bar{F}\big(\bar{f}_\Omega(r_1,v_1), \bar{f}_\Omega(r_2,v_2)\big), \text{ and} \qquad (3.56)$$

$$\bar{f}_\Omega\big((r_1,v_1)\cdot(r_2,v_2)\big) \leq \bar{F}\big(\bar{f}_\Omega(r_1,v_1), \bar{f}_\Omega(r_2,v_2)\big) \qquad (3.57)$$

Hence, using Proposition 3.1 and by Equations 3.52–3.57 $\Omega_1 \times \Omega_2 \in$ IVNSR$(P \times R)$. □

Corollary 3.10. *Let $\forall i \in \{1, 2, ..., n\}$, $(P_i, +, \cdot)$ are crisp rings and $\Omega_i \in$ IVNSR(P_i). Then $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ is a IVNSR of $P_1 \times P_2 \times \cdots \times P_n$, where $n \in \mathbb{N}$.*

3.2. *Subring of a interval-valued neutrosophic subgring*

Definition 3.9. Let $(P, +, \cdot)$ be a crisp ring and $\Omega_1, \Omega_2 \in$ IVNSR(P), where $\Omega_1 = \big\{\big(r, \bar{t}_{\Omega_1}(r), \bar{i}_{\Omega_1}(r), \bar{f}_{\Omega_1}(r)\big) : r \in P\big\}$ and $\Omega_2 = \big\{\big(r, \bar{t}_{\Omega_2}(r), \bar{i}_{\Omega_2}(r), \bar{f}_{\Omega_2}(r)\big) : r \in P\big\}$. Then Ω_1 is called a subring of Ω_2 if $\forall r \in P$, $\bar{t}_{\Omega_1}(r) \leq \bar{t}_{\Omega_2}(r)$, $\bar{i}_{\Omega_1}(r) \geq \bar{i}_{\Omega_2}(r)$, and $\bar{f}_{\Omega_1}(r) \geq \bar{f}_{\Omega_2}(r)$.

Theorem 3.11. *Let $(P, +, \cdot)$ be a crisp ring and $\Omega \in$ IVNSR(P). Again, let Ω_1 and Ω_2 be two subrings of Ω. Then $\Omega_1 \cap \Omega_2$ is also a subring of Ω, assuming that all the IVTN and IVSNs are idempotent.*

Proof. Here, $\forall r \in P$

$$\begin{aligned}\bar{t}_{\Omega_1 \cap \Omega_2}(r) &= \bar{T}\big(\bar{t}_{\Omega_1}(r), \bar{t}_{\Omega_2}(r)\big)\\ &\leq \bar{T}\big(\bar{t}_{\Omega}(r), \bar{t}_{\Omega}(r)\big)\\ &= \bar{t}_{\Omega}(r) \text{ [as } \bar{T} \text{ is idempotent]}\end{aligned} \tag{3.58}$$

Similarly, as $\bar{I}$ and $\bar{F}$ are idempotent we can show that,

$$\bar{i}_{\Omega_1 \cap \Omega_2}(r) \geq \bar{i}_{\Omega}(r) \text{ and} \tag{3.59}$$

$$\bar{f}_{\Omega_1 \cap \Omega_2}(r) \geq \bar{f}_{\Omega}(r) \tag{3.60}$$

Hence, by Equations 3.58–3.60 $\Omega_1 \cap \Omega_2$ is a subring of Ω. □

Theorem 3.12. *Let $(P, +, \cdot)$ be a crisp ring and $\Omega_1, \Omega_2 \in IVNSR(P)$ such that Ω_1 is a subring of Ω_2. Let $(R, +, \cdot)$ is another crisp ring and $l : P \to R$ be a ring isomorphism. Then*

(i) *$l(\Omega_1)$ and $l(\Omega_2)$ are two IVNSRs over R and*
(ii) *$l(\Omega_1)$ is a subring of $l(\Omega_2)$.*

Proof. (i) can be proved by using Theorem 3.6.
(ii) Let $v = l(r)$, where $r \in P$ and $v \in R$. Then

$$\begin{aligned}&\bar{t}_{\Omega_1}(r) \leq \bar{t}_{\Omega_2}(r) \text{ [as } \Omega_1 \text{ is a subring of } \Omega_2]\\ \Rightarrow&\bar{t}_{\Omega_1}\big(l^{-1}(v)\big) \leq \bar{t}_{\Omega_2}\big(l^{-1}(v)\big)\\ \Rightarrow&\bar{t}_{l(\Omega_1)}(v) \leq \bar{t}_{l(\Omega_2)}(v)\end{aligned} \tag{3.61}$$

Similarly, we can prove that

$$\bar{i}_{l(\Omega_1)}(v) \geq \bar{i}_{l(\Omega_2)}(v) \text{ and} \tag{3.62}$$

$$\bar{f}_{l(\Omega_1)}(v) \geq \bar{f}_{l(\Omega_2)}(v) \tag{3.63}$$

Hence, by Equations 3.61–3.63 $l(\Omega_1)$ is a subring of $l(\Omega_2)$. □

3.3. *Interval-valued neutrosophic normal subrings*

Definition 3.10. Let $(P, +, \cdot)$ be a crisp ring and Ω is an IVNS of P, where $\Omega = \big\{\big(r, \bar{t}_{\Omega}(r), \bar{i}_{\Omega}(r), \bar{f}_{\Omega}(r)\big) : r \in P\big\}$. Then Ω is called an IVNNSR over P if

(i) Ω is an IVNSR of P and
(ii) $\forall r, v \in P$, $\bar{t}_{\Omega}(r \cdot v) = \bar{t}_{\Omega}(v \cdot r)$, $\bar{i}_{\Omega}(r \cdot v) = \bar{i}_{\Omega}(v \cdot r)$, and $\bar{f}_{\Omega}(r \cdot v) = \bar{f}_{\Omega}(v \cdot r)$.

The set of all IVNNSR of a crisp ring $(P, +, \cdot)$ will be denoted as IVNNSR(P).

Example 3.11. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integers with respect to usual addition and multiplication. Let $\Omega = \{(r, \bar{t}_\Omega(r), \bar{i}_\Omega(r), \bar{f}_\Omega(r)) : r \in \mathbb{Z}\}$ be an IVNS of $\mathbb{Z}$, where $\forall r \in \mathbb{Z}$

$$\bar{t}_\Omega(r) = \begin{cases} [0.67, 1] & \text{if } r \in 2\mathbb{Z} \\ [0, 0] & \text{if } r \in 2\mathbb{Z}+1 \end{cases},$$

$$\bar{i}_\Omega(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0.33, 0.5] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}, \text{ and}$$

$$\bar{f}_\Omega(r) = \begin{cases} [0, 0] & \text{if } r \in 2\mathbb{Z} \\ [0, 0.33] & \text{if } r \in 2\mathbb{Z}+1 \end{cases}.$$

Now, if we consider minimum TN and maximum SNs, then $\Omega \in \text{IVNNSR}(\mathbb{Z})$.

Theorem 3.13. *Let $(P, +, \cdot)$ be a crisp ring. If $\Omega_1, \Omega_2 \in IVNNSR(P)$, then $\Omega_1 \cap \Omega_2 \in IVNNSR(P)$.*

Proof. As $\Omega_1, \Omega_2 \in \text{IVNSR}(P)$ by Theorem 3.2 $\Omega_1 \cap \Omega_2 \in \text{IVNSR}(P)$. Again,

$$\begin{aligned} \bar{t}_{\Omega_1 \cap \Omega_2}(r \cdot v) &= \bar{T}\big(\bar{t}_{\Omega_1}(r \cdot v), \bar{t}_{\Omega_2}(r \cdot v)\big) \\ &= \bar{T}\big(\bar{t}_{\Omega_1}(v \cdot r), \bar{t}_{\Omega_2}(v \cdot r)\big) \text{ [as } \Omega_1, \Omega_2 \in \text{IVNNSR}(P)] \\ &= \bar{t}_{\Omega_1 \cap \Omega_2}(v \cdot r) \end{aligned} \tag{3.64}$$

Similarly,

$$\bar{i}_{\Omega_1 \cap \Omega_2}(r \cdot v) = \bar{i}_{\Omega_1 \cap \Omega_2}(v \cdot r) \tag{3.65}$$

$$\bar{f}_{\Omega_1 \cap \Omega_2}(r \cdot v) = \bar{f}_{\Omega_1 \cap \Omega_2}(v \cdot r) \tag{3.66}$$

Hence, $\Omega_1 \cap \Omega_2 \in \text{IVNNSR}(P)$. □

Remark 3.14. *In general, if $\Omega_1, \Omega_2 \in IVNNSR(P)$, then $\Omega_1 \cup \Omega_2$ may not always be an IVNNSR of $(P, +, \cdot)$.*

Remark 3.14 can be proved by Example 3.4.

Theorem 3.15. *Let $(P, +, \cdot)$ be a crisp ring. Then $\Omega \in IVNNSR(P)$ iff $\forall [k_1, s_1], [k_2, s_2], [k_3, s_3] \in \Psi(L)$ with $\bar{t}_\Omega(\theta_P) \geq [k_1, s_1]$, $\bar{i}_\Omega(\theta_P) \leq [k_2, s_2]$, and $\bar{f}_\Omega(\theta_P) \leq [k_3, s_3]$, $\Omega_{\big([k_1,s_1],[k_2,s_2],[k_3,s_3]\big)}$ is a crisp normal subring of $(P, +, \cdot)$ (assuming all the IVTN and IVSNs are idempotent).*

Proof. This can be proved using Theorem 3.5. □

Theorem 3.16. *Let $(P, +, \cdot)$ and $(R, +, \cdot)$ be two crisp rings. Also, let $l : P \to R$ be a ring isomorphism. If Ω is an IVNNSR of P then $l(\Omega)$ is an IVNNSR of R.*

Proof. As Ω is an IVNSR of P by Theorem 3.6 $l(\Omega)$ is an IVNSR of R. Let $l(r_1) = v_1$ and $l(r_2) = v_2$, where $r_1, r_2 \in P$ and $v_1, v_2 \in R$. Then

$$\begin{aligned}
\bar{t}_{l(\Omega)}(v_1 \cdot v_2) &= \bar{t}_\Omega\big(l^{-1}(v_1 \cdot v_2)\big) \text{ [as } l \text{ is injective]} \\
&= \bar{t}_\Omega\big(l^{-1}(v_1) \cdot l^{-1}(v_2)\big) \text{ [as } l^{-1} \text{ is a homomorphism]} \\
&= \bar{t}_\Omega(r_1 \cdot r_2) \\
&= \bar{t}_\Omega(r_2 \cdot r_1) \text{ [as } \Omega \text{ is an IVNNSR of } P] \\
&= \bar{t}_\Omega\big(l^{-1}(v_2) \cdot l^{-1}(v_1)\big) \\
&= \bar{t}_\Omega\big(l^{-1}(v_2 \cdot v_1)\big) \\
&= \bar{t}_{l(\Omega)}(v_2 \cdot v_1) \qquad (3.67)
\end{aligned}$$

Similarly,

$$\bar{i}_{l(\Omega)}(v_1 \cdot v_2) = \bar{i}_{l(\Omega)}(v_2 \cdot v_1) \text{ and} \qquad (3.68)$$

$$\bar{f}_{l(\Omega)}(v_1 \cdot v_2) = \bar{f}_{l(\Omega)}(v_2 \cdot v_1) \qquad (3.69)$$

Hence, by Equations 3.67–3.69 $l(\Omega)$ is an IVNNSR of R. □

4. Conclusions

As interval-valued neutrosophic environment is more general than regular one, we have adopted it and defined the notions of interval-valued neutrosophic subring and its normal version. Also, we have analyzed some homomorphic properties of these newly defined notions. Again, we have studied product of two interval-valued neutrosophic subrings. Furthermore, we have provided some essential theories to study some of their algebraic structures. These newly introduced notions have potentials to become fruitful research areas. For instance, soft set theory can be implemented and the notion of interval-valued neutrosophic soft subring can be defined.

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Generalized Aggregate Operators on Neutrosophic Hypersoft Set

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Abstract: Multi-criteria decision making (MCDM) is concerned about coordinating as well as looking after selection as well as planning problems which included multi-criteria. The neutrosophic soft set cannot handle the environment which involved more than one attribute. To overcome those hurdles neutrosophic hypersoft set (NHSS) is defined. In this paper, we proposed the generalized aggregate operators on NHSS such as extended union, extended intersection, OR-operation, AND-operation, etc. with their properties. Finally, the necessity and possibility operations on NHSS with suitable examples and properties are presented in the following research.

Keywords: Soft set; Neutrosophic Set; Neutrosophic soft set; Hypersoft set; Neutrosophic hypersoft set.

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1. Introduction

Zadeh developed the notion of fuzzy sets [1] to solve those problems which contain uncertainty and vagueness. It is observed that in some cases circumstances cannot be handled by fuzzy sets, to overcome such types of situations Turksen [2] gave the idea of interval-valued fuzzy set. In some cases, we must deliberate membership unbiassed as the non- membership values for the suitable representation of an object in uncertain and indeterminate conditions that could not be handled by fuzzy sets nor interval-valued fuzzy sets. To overcome these difficulties Atanassov presented the notion of Intuitionistic fuzzy sets in [3]. The theory which was presented by Atanassov only deals the insufficient data considering both the membership and non-membership values, but the intuitionistic fuzzy set theory cannot handle the incompatible and imprecise information. To deal with such incompatible and imprecise data the idea of the neutrosophic set (NS) was developed by Smarandache [4].

A general mathematical tool was proposed by Molodtsov [5] to deal with indeterminate, fuzzy, and not clearly defined substances known as a soft set (SS). Maji et al. [6] extended the work on SS and defined some operations and their properties. In [7], they also used the SS theory for decision making. Ali et al. [8] revised the Maji approach to SS and developed some new operations with their properties. De Morgan's Law on SS theory was proved in [9] by using different operators. Cagman and Enginoglu [10] developed the concept of soft matrices with operations and discussed their properties, they also introduced a decision-making method to resolve those problems which contain uncertainty. In [11], they revised the operations proposed by Molodtsov's SS. In [12], the author's proposed some new operations on soft matrices such as soft difference product, soft restricted difference product, soft extended difference product, and soft weak-extended difference product with their properties.

Maji [13] offered the idea of a neutrosophic soft set (NSS) with necessary operations and properties. The idea of the possibility NSS was developed by Karaaslan [14] and introduced a possibility of neutrosophic soft decision-making method to solve those problems which contain uncertainty based on And-product. Broumi [15] developed the generalized NSS with some operations and properties and used the proposed concept for decision making. To solve MCDM problems with single-valued Neutrosophic numbers presented by Deli and Subas in [16], they constructed the concept of cut sets of single-valued Neutrosophic numbers. On the base of the correlation of intuitionistic fuzzy sets, the term correlation coefficient of SVNSs [17] was introduced. In [18], the idea of simplified NSs introduced with some operational laws and aggregation operators such as real-life Neutrosophic weighted arithmetic average operator and weighted geometric average operator. They constructed an MCDM method on the base of proposed aggregation operators.

Smarandache [19] generalized the SS to hypersoft set (HSS) by converting the function to a multi-attribute function to deal with uncertainty. Saqlain et al. [20] developed the generalization of TOPSIS for the NHSS, by using the accuracy function they transformed the fuzzy neutrosophic numbers to crisp form. In [21],s the author's proposed the fuzzy plithogenic hypersoft set in matrix form with some basic operations and properties. Martin and Smarandache developed the plithogenic hypersoft set by combining the plithogenic sets and hypersoft set in [22]. Saqlain et al. [23] proposed the aggregate operators and similarity measure [24] on NHSS. In [25], Abdel basset et al. applied TODIM and TOPSIS methods based on the best-worst method to increase the accuracy of evaluation under uncertainty according to the neutrosophic set. They also used the plithogenic set theory to solve the uncertain information and evaluate the financial performance of manufacturing industries, they used the AHP method to find the weight vector of the financial ratios to achieve this goal after that they used the VIKOR and TOPSIS methods to utilized the companies ranking in [26].

In the following paragraph, we explain some positive impacts of this research. The main focus of this study is too generalized the aggregate operators of the neutrosophic hypersoft set. We will use the proposed aggregate operators to solve multi-criteria decision-making problems after developing distance-based similarity measures. Saqlain et al. [23], developed the aggregate operators on NHSS but in some cases, we face some limitations such as in union and intersection. To overcome these limitations we develop the generalized version of aggregate operators on NHSS.

The following research is organized as follows: In section 2, we recall some basic definitions used in the following research such as SS, NS, NSS, HSS, and NHSS. We develop the generalized aggregate operators on NHSS such as extended union, extended intersection, And-operation, etc. in section 3 with properties. In section 4, the necessity and possibility of operations are presented with examples and properties.

2. Preliminaries

In this section, we recall some basic definitions such as SS, NSS, and NHSS which use in the following sequel.

Definition 2.1 [5] **Soft Set**

The soft set is a pair (F, Λ) over Ű if and only if F: Λ → P(Ű) is a mapping. That is the parameterized family of subsets of Ű known as a SS.

Definition 2.2 [4] **Neutrosophic Set**

Let Ű be a universe and Λ be an NS on Ű is defined as Λ = $\{< u, T_\Lambda(u), I_\Lambda(u), F_\Lambda(u) >: u \in$ Ű$\}$, where T, I, F: Ű → $]0^-, 1^+[$ and $0^- \leq T_\Lambda(u) + I_\Lambda(u) + F_\Lambda(u) \leq 3^+$.

Definition 2.3 [13] **Neutrosophic Soft Set**

Let Ű and Ḝ are universal set and set of attributes respectively. Let P(Ű) be the set of Neutrosophic values of Ű and Λ ⊆ Ḝ. A pair (F, Λ) is called an NSS over Ű and its mapping is given as

F: Λ → P(Ű)

Definition 2.4 [19] **Hypersoft Set**

Let Ű be a universal set and P(Ű) be a power set of Ű and for $n \geq 1$, there are n distinct attributes such as $k_1, k_2, k_3, \dots, k_n$ and $K_1, K_2, K_3, \dots, K_n$ are sets for corresponding values attributes respectively with following conditions such as $K_i \cap K_j = \emptyset$ $(i \neq j)$ and $i, j \in \{1,2,3 \dots n\}$. Then the pair (F, $K_1 \times K_2 \times K_3 \times \dots \times K_n$) is said to be Hypersoft set over Ű where F is a mapping from $K_1 \times K_2 \times K_3 \times \dots \times K_n$ to P(Ű).

Definition 2.5 [22] **Neutrosophic Hypersoft Set (NHSS)**

Let Ű be a universal set and P(Ű) be a power set of Ű and for $n \geq 1$, there are n distinct attributes such as $k_1,\ k_2,\ k_3, \dots,\ k_n$ and $K_1,\ K_2,\ K_3, \dots,\ K_n$ are sets for corresponding values attributes respectively with following conditions such as $K_i \cap K_j = \emptyset$ $(i \neq j)$ and $i, j \in \{1,2,3 \dots n\}$. Then the pair (F, Λ) is said to be NHSS over Ű if there exists a relation $K_1 \times K_2 \times K_3 \times \dots \times K_n = \Lambda$. F is a mapping from $K_1 \times K_2 \times K_3 \times \dots \times K_n$ to P(Ű) and F$(K_1 \times K_2 \times K_3 \times \dots \times K_n) = \{< u, T_\Lambda(u), I_\Lambda(u), F_\Lambda(u) >: u \in$ Ű$\}$ where T, I, F are membership values for truthness, indeterminacy, and falsity respectively such that T, I, F: Ű $\to\ ]0^-,\ 1^+[$ and $0^- \leq T_\Lambda(u) + I_\Lambda(u) + F_\Lambda(u) \leq 3^+$.

Example 2.6 Assume that a person examines the attractiveness of a living house. Let Ű be a universe which consists of three choices Ű = $\{u_1,\ u_2\}$ and E = {έ$_1$, έ$_2$, έ$_3$} be a set of decision parameters. Then, the NHSS is given as

$$F_\Lambda = \{< \mathrm{u}_1, (έ_1\{0.4, 0.7, 0.5\}, έ_2\{0.8, 0.5, 0.3\}, έ_3\{0.6, 0.5, 0.9\}) >$$
$$< \mathrm{u}_2, (έ_1\{0.1,0.5,0.7\}, έ_2\{0.5, 0.6, 0.2\}, έ_3\{0.7, 0.4, 0.6\}) >\}$$

3. Generalized Aggregate Operators on Neutrosophic Hypersoft Set and Properties

In this section, we present the generalized aggregate operations on NHSS with examples. We prove commutative and associative laws by using proposed aggregate operators in the following section.

Definition 3.1

Let $F_\Lambda \in$ NHSS, then its complement, is written as $(F_\Lambda)^c = F^c(\Lambda)$ and defined as

$F^c(\Lambda) = \{< u, T(F^c(\Lambda)), I(F^c(\Lambda)), F(F^c(\Lambda)) >: u \in \mathrm{U}\}$ such that

$$T(F^c(\Lambda)) = 1 - T_\Lambda(u),$$
$$I(F^c(\Lambda)) = 1 - I_\Lambda(u),$$
$$F(F^c(\Lambda)) = 1 - F_\Lambda(u).$$

Example 3.2 Reconsider example 2.6

$$F^c(\Lambda) = \{< \mathrm{u}_1, (έ_1\{0.6, 0.3, 0.5\}, έ_2\{0.2, 0.5, 0.7\}, έ_3\{0.4, 0.5, 0.1\}) >$$
$$< \mathrm{u}_2, (έ_1\{0.9, 0.5, 0.3\}, έ_2\{0.5, 0.4, 0.8\}, έ_3\{0.3, 0.6, 0.4\}) >\}$$

Proposition 3.3

If $F_\Lambda \in$ NHSS, then $(F^c(\Lambda))^c = F_\Lambda$.

Proof

By using definition 3.1, we have

$$\begin{aligned} F^c(\Lambda) &= \{< u, T(F^c(\Lambda)), I(F^c(\Lambda)), F(F^c(\Lambda)) >: u \in \mathrm{U}\} \\ &= \{< u, 1 - T\ (F_\Lambda), 1 - I\ (F_\Lambda), 1 - F\ (F_\Lambda) >: u \in \mathrm{U}\}, \end{aligned}$$

Thus

$$(F^c(\Lambda))^c = \{< u, 1 - (1 - T(F_\Lambda)), 1 - (1 - I(F_\Lambda)), 1 - (1 - F(F_\Lambda)) >: u \in \mathrm{U}\},$$
$$(F^c(\Lambda))^c = \{< u, T(F_\Lambda), I(F_\Lambda), F(F_\Lambda) >: u \in \mathrm{U}\} = F_\Lambda.$$

Which completes the proof.

Definition 3.4 Extended Union of Two Neutrosophic Hypersoft Set

Let $F_{\Lambda_1},\ F_{\Lambda_2} \in$ NHSS, then their extended union is

$$T\ (F_{\Lambda_1} \cup F_{\Lambda_2}) = \begin{cases} T(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ T(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Max\left(T(F_{\Lambda_1}), T(F_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$I\ (F_{\Lambda_1} \cup F_{\Lambda_2}) = \begin{cases} I(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\left(I(F_{\Lambda_1}), I(F_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$F\ (F_{\Lambda_1} \cup F_{\Lambda_2}) = \begin{cases} F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\left(F(F_{\Lambda_1}), F(F_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

Example 3.5 Let U = $\{u_1, u_2, u_3, u_4\}$ be a universal set and E = $\{\acute{\varepsilon}_1, \acute{\varepsilon}_2, \acute{\varepsilon}_3, \acute{\varepsilon}_4\}$ be a set of decision parameters and F_{Λ_1} = $\{u_1, u_4\}$ and F_{Λ_2} = $\{u_2, u_4\}$

$$F_{\Lambda_1} = \{< u_1, (\acute{\varepsilon}_1\{0.4, 0.7, 0.5\}, \acute{\varepsilon}_2\{0.8, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.6, 0.5, 0.9\}, \acute{\varepsilon}_4\{0.3, 0.7, 0.2\}) >$$
$$< u_4, (\acute{\varepsilon}_1\{0.4, 0.7, 0.2\}, \acute{\varepsilon}_2\{0.6, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.8, 0.4, 0.7\}, \acute{\varepsilon}_4\{0.6, 0.4, 0.3\}) >\}$$
$$F_{\Lambda_2} = \{< u_2, (\acute{\varepsilon}_1\{0.7, 0.4, 0.6\}, \acute{\varepsilon}_2\{0.4, 0.6, 0.9\}, \acute{\varepsilon}_3\{0.7, 0.4, 0.6\}, \acute{\varepsilon}_4\{0.7, 0.6, 0.3\}) >$$
$$< u_4, (\acute{\varepsilon}_1\{0.6, 0.2, 0.7\}, \acute{\varepsilon}_2\{0.5, 0.7, 0.3\}, \acute{\varepsilon}_3\{0.4, 0.8, 0.5\}, \acute{\varepsilon}_4\{0.5, 0.6, 0.4\}) >\}$$
$$F_{\Lambda_1} \cup F_{\Lambda_2} = \{< u_1, (\acute{\varepsilon}_1\{0.4, 0.7, 0.5\}, \acute{\varepsilon}_2\{0.8, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.6, 0.5, 0.9\}, \acute{\varepsilon}_4\{0.3, 0.7, 0.2\}) >$$
$$< u_2, (\acute{\varepsilon}_1\{0.7, 0.4, 0.6\}, \acute{\varepsilon}_2\{0.4, 0.6, 0.9\}, \acute{\varepsilon}_3\{0.7, 0.4, 0.6\}, \acute{\varepsilon}_4\{0.7, 0.6, 0.3\}) >$$
$$< u_4, (\acute{\varepsilon}_1\{0.6, 0.7, 0.7\}, \acute{\varepsilon}_2\{0.6, 0.7, 0.3\}, \acute{\varepsilon}_3\{0.8, 0.8, 0.7\}, \acute{\varepsilon}_4\{0.6, 0.6, 0.4\}) >\}$$

Proposition 3.6

Let F_{Λ_1}, F_{Λ_2} and F_{Λ_3} are NHSSs than

1. $\left(F_{\Lambda_1} \cup F_{\Lambda_2}\right) = \left(F_{\Lambda_2} \cup F_{\Lambda_1}\right)$ (Commutative law)
2. $\left(F_{\Lambda_1} \cup F_{\Lambda_2}\right) \cup F_{\Lambda_3} = F_{\Lambda_1} \cup \left(F_{\Lambda_2} \cup F_{\Lambda_3}\right)$ (Associative law)

Proof 1. In the following proof first two cases are trivial, we consider only the third case in this proposition

$\left(F_{\Lambda_1} \cup F_{\Lambda_2}\right) = \{< u, (max\ \{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, min\ \{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}) >\}$

$= \{< u, (max\{T(F_{\Lambda_2}), T(F_{\Lambda_1})\}, min\ \{I(F_{\Lambda_2}), I(F_{\Lambda_1})\}, min\{F(F_{\Lambda_2}), F(F_{\Lambda_1})\}) >\}$

$= \left(F_{\Lambda_2} \cup F_{\Lambda_1}\right)$

Proof 2: Let F_{Λ_1}, F_{Λ_2} and F_{Λ_3} are NHSSs than

$F_{\Lambda_1} \cup F_{\Lambda_2} = \{< u, (Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, Min\ \{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, Min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}) >\}$

$(F_{\Lambda_1} \cup F_{\Lambda_2}) \cup F_{\Lambda_3}$ =

$\{< u, \max\{\max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, T(F_{\Lambda_3})\}, \min\{\min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, I(F_{\Lambda_3})\}, \min\{\min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}, F(F_{\Lambda_3})\} >\}$

$= \left\{< u, \max\{T(F_{\Lambda_1}), T(F_{\Lambda_2}), T(F_{\Lambda_3})\}, \min\{\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, I(F_{\Lambda_3})\}, \min\left\{\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}, F(F_{\Lambda_3})\right\} >\right\}$

$= \left\{< u, \max\left\{T(F_{\Lambda_1}), \max\{T(F_{\Lambda_2}), T(F_{\Lambda_3})\}\right\}, \min\left\{I(F_{\Lambda_1}), \min\{I(F_{\Lambda_2}), I(F_{\Lambda_3})\}\right\}, \min\left\{F(F_{\Lambda_1}), \min\{F(F_{\Lambda_2}), F(F_{\Lambda_3})\}\right\} >\right\}$

$= F_{\Lambda_1} \cup \left(F_{\Lambda_2} \cup F_{\Lambda_3}\right)$

Definition 3.7 Extended Intersection of Two Neutrosophic Hypersoft Set

Let F_{Λ_1}, F_{Λ_2} ∈ NHSS, then their extended intersection is

$$T\ (F_{\Lambda_1} \cap F_{\Lambda_2}) = \begin{cases} T(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ T(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\left(T(F_{\Lambda_1}), T(F_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$I\ (F_{\Lambda_1} \cap F_{\Lambda_2}) = \begin{cases} I(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Max\left(I(F_{\Lambda_1}), I(F_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$F\ (F_{\Lambda_1} \cap F_{\Lambda_2}) = \begin{cases} F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Max\left(F(F_{\Lambda_1}), F(F_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

Proposition 3.8 Let F_{Λ_1}, F_{Λ_2} and F_{Λ_3} are NHSSs than

1. $F_{\Lambda_1} \cap F_{\Lambda_2} = F_{\Lambda_2} \cap F_{\Lambda_1}$ (Commutative law)
2. $\left(F_{\Lambda_1} \cap F_{\Lambda_2}\right) \cap F_{\Lambda_3} = F_{\Lambda_1} \cap \left(F_{\Lambda_2} \cap F_{\Lambda_3}\right)$ (Associative law)

Proof 1. Similar to Proposition 3.6.

Proposition 3.9 Let F_{Λ_1}, F_{Λ_2} are NHSSs then

1. $(F_{\Lambda_1} \cup F_{\Lambda_2})^c = F^c(\Lambda_1) \cap F^c(\Lambda_2)$
2. $(F_{\Lambda_1} \cap F_{\Lambda_1})^c = F^c(\Lambda_1) \cup F^c(\Lambda_2)$

Proof 1. Let F_{Λ_1} and F_{Λ_1} ∈ NHSS, such as follows

$F_{\Lambda_1} = \{< \text{u}, \{T(F_{\Lambda_1}), I(F_{\Lambda_1}), F(F_{\Lambda_1})\} >\}$ and $F_{\Lambda_2} = \{< \text{u}, \{T(F_{\Lambda_2}), I(F_{\Lambda_2}), F(F_{\Lambda_2})\} >\}$

$(F_{\Lambda_1} \cup F_{\Lambda_2})^c = \left\{< \text{u}, \left(max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, min\ \{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\right) >\right\}^c$

$= \{< \text{u}, \left(min\{1 - T(F_{\Lambda_1}), 1 - T(F_{\Lambda_2})\}, max\ \{1 - I(F_{\Lambda_1}), 1 - I(F_{\Lambda_2})\}, max\{1 - F(F_{\Lambda_1}), 1 - F(F_{\Lambda_2})\right) >\}$

$= \{< \text{u}, (min\{T(F^c(\Lambda_1)), T(F^c(\Lambda_2))\}, max\ \{I(F^c(\Lambda_1)), I(F^c(\Lambda_2))\}, max\{F(F^c(\Lambda_1)), F(F^c(\Lambda_2))\}) >\}$

$= F^c(\Lambda_1) \cap F^c(\Lambda_2)$

Proof 2. Similarly, we can prove 2.

Definition 3.10 OR-Operation of Two Neutrosophic Hypersoft Set

Let F_{Λ_1}, F_{Λ_2} ∈ NHSS. Consider $k_1, k_2, k_3, \ldots, k_n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $K_1, K_2, K_3, \ldots, K_n$ with $K_i \cap K_j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \ldots n\}$ and their relation $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \Lambda$, then $F_{\Lambda_1} \vee F_{\Lambda_2} = F_{\Lambda_1 \times \Lambda_2}$, then

$$T\ (F_{\Lambda_1 \times \Lambda_2}) = Max\left(T(F_{\Lambda_1}), T(F_{\Lambda_2})\right),$$

$$I\ (F_{\Lambda_1 \times \Lambda_2}) = Min\left(I(F_{\Lambda_1}), I(F_{\Lambda_2})\right),$$

$$F\ (F_{\Lambda_1 \times \Lambda_2}) = Min\left(F(F_{\Lambda_1}), F(F_{\Lambda_2})\right).$$

Example 3.11 Reconsider example 3.5

$F_{\Lambda_1} \vee F_{\Lambda_2} = F_{\Lambda_1 \times \Lambda_2}$

$= \{< (\text{u}_1, \text{u}_2), (\acute{\varepsilon}_1\{0.7, 0.4, 0.5\}, \acute{\varepsilon}_2\{0.8, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.7, 0.4, 0.6\}, \acute{\varepsilon}_4\{0.7, 0.6, 0.2\}) >$

$< (\text{u}_1, \text{u}_4), (\acute{\varepsilon}_1\{0.6, 0.2, 0.5\}, \acute{\varepsilon}_2\{0.8, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.6, 0.5, 0.5\}, \acute{\varepsilon}_4\{0.5, 0.6, 0.2\}) >$

$< (\text{u}_4, \text{u}_2), (\acute{\varepsilon}_1\{0.7, 0.4, 0.2\}, \acute{\varepsilon}_2\{0.6, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.8, 0.4, 0.6\}, \acute{\varepsilon}_4\{0.7, 0.4, 0.3\}) >$

$< (\text{u}_4, \text{u}_4), (\acute{\varepsilon}_1\{0.6, 0.2, 0.2\}, \acute{\varepsilon}_2\{0.6, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.8, 0.4, 0.5\}, \acute{\varepsilon}_4\{0.6, 0.4, 0.3\}) >\}$

Definition 3.12 AND-Operation of Two Neutrosophic Hypersoft Set

Let F_{Λ_1}, F_{Λ_2} ∈ NHSS. Consider $k_1, k_2, k_3, \ldots, k_n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $K_1, K_2, K_3, \ldots, K_n$ with $K_i \cap K_j = = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \ldots n\}$ and their relation $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \Lambda$ then $F_{\Lambda_1} \wedge F_{\Lambda_2} = F_{\Lambda_1 \times \Lambda_2}$, then

$$T\ (F_{\Lambda_1 \times \Lambda_2}) = Min\left(T(F_{\Lambda_1}), T(F_{\Lambda_2})\right),$$

$$I\ (F_{\Lambda_1 \times \Lambda_2}) = Max\left(I(F_{\Lambda_1}), I(F_{\Lambda_2})\right),$$

$$\boldsymbol{F}\ (\boldsymbol{F}_{\Lambda_1 \times \Lambda_2}) = \boldsymbol{Max}\left(\boldsymbol{F}(\boldsymbol{F}_{\Lambda_1}), \boldsymbol{F}(\boldsymbol{F}_{\Lambda_2})\right).$$

Proposition 3.13 Let F_{Λ_1}, F_{Λ_2} are NHSSs then

1. $(F_{\Lambda_1} \vee F_{\Lambda_2})^c = F^c(\Lambda_1) \wedge F^c(\Lambda_2)$
2. $(F_{\Lambda_1} \wedge F_{\Lambda_2})^c = F^c(\Lambda_1) \vee F^c(\Lambda_2)$

Proof 1. Let F_{Λ_1} and F_{Λ_1} ∈ NHSS, such as follows

$F_{\Lambda_1} = \{< u_i, \{T(F_{\Lambda_1}), I(F_{\Lambda_1}), F(F_{\Lambda_1})\} >: u_i \in U\}$ and $F_{\Lambda_2} = \{< u_j, \{T(F_{\Lambda_2}), I(F_{\Lambda_2}), F(F_{\Lambda_2})\} >: u_j \in U\}$

By using definition 3.10 we get

$F_{\Lambda_1} \vee F_{\Lambda_2} = \{< (u_i, u_j), [e, \max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}] >\}$

$(F_{\Lambda_1} \vee F_{\Lambda_2})^c = \{< (u_i, u_j), [e, 1-\max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, 1-\min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, 1-\min\{F(F_{\Lambda_1}), 1-F(F_{\Lambda_2})\}] >\}$

$(F_{\Lambda_1} \vee F_{\Lambda_2})^c = \{< (u_i, u_j), [e, \min\{1-T(F_{\Lambda_1}), 1-T(F_{\Lambda_2})\}, \max\{1-I(F_{\Lambda_1}), 1-I(F_{\Lambda_2})\}, \max\{1-F(F_{\Lambda_1}), 1-F(F_{\Lambda_2})\}] >\}$

$(F_{\Lambda_1} \vee F_{\Lambda_2})^c = \{< (u_i, u_j), [e, \min\{T(F^c(\Lambda_1)), T(F^c(\Lambda_2))\}, \max\{I(F^c(\Lambda_1)), I(F^c(\Lambda_2))\}, \max\{F(F^c(\Lambda_1)), F(F^c(\Lambda_2))\}] >\}$

Since

$F^c(\Lambda_1) = \{< u_i, \{T(F^c(\Lambda_1)), I(F^c(\Lambda_1)), F(F^c(\Lambda_1))\} >: u_i \in U\}$ and

$F^c(\Lambda_2) = \{< u_j, \{T(F^c(\Lambda_2)), I(F^c(\Lambda_2)), F(F^c(\Lambda_2))\} >: u_j \in U\}$

By using definition 3.12, we get

$F^c(\Lambda_1) \wedge F^c(\Lambda_2) = \{< (u_i, u_j), [e, \min\{T(F^c(\Lambda_1)), T(F^c(\Lambda_2))\}, \max\{I(F^c(\Lambda_1)), I(F^c(\Lambda_2))\}, \max\{F(F^c(\Lambda_1)), F(F^c(\Lambda_2))\}] >\}$

So

$(F_{\Lambda_1} \vee F_{\Lambda_2})^c = F^c(\Lambda_1) \wedge F^c(\Lambda_2)$.

Similarly, we can prove 2.

4. Necessity and Possibility Operations

The necessity and possibility operations on NHSS with some properties are presented in the following section.

Definition 4.1 Necessity operation

Let F_Λ ∈ NHSS, then necessity operation on NHSS represented by $\oplus\ F_\Lambda$ and defined as follows

$\oplus\ F_\Lambda = \{< \mathrm{u}, \{T(F_\Lambda), I(F_\Lambda), 1-T(F_\Lambda)\} >\}$ for all $u \in U$.

Example 4.2 Reconsider example 2.6

$$\oplus\ F_\Lambda = \{< \mathrm{u}_1, (\acute{\varepsilon}_1\{0.4, 0.7, 0.6\}, \acute{\varepsilon}_2\{0.8, 0.5, 0.2\}, \acute{\varepsilon}_3\{0.6, 0.5, 0.4\}) >$$
$$< \mathrm{u}_2, (\acute{\varepsilon}_1\{0.1, 0.5, 0.9\}, \acute{\varepsilon}_2\{0.5, 0.6, 0.5\}, \acute{\varepsilon}_3\{0.7, 0.4, 0.3\}) >\}$$

Proposition 4.3

1. $\oplus\ (F_{\Lambda_1} \cup F_{\Lambda_2}) = \oplus\ F_{\Lambda_2} \cup \oplus\ F_{\Lambda_1}$
2. $\oplus\ (F_{\Lambda_1} \cap F_{\Lambda_2}) = \oplus\ F_{\Lambda_2} \cap \oplus\ F_{\Lambda_1}$

Proof 1. Let $F_{\Lambda_1} \cup F_{\Lambda_2} = F_{\Lambda_3}$, then

$$T\ (F_{\Lambda_3}) = \begin{cases} T(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ T(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$I\ (F_{\Lambda_3}) = \begin{cases} I(F_{\Lambda_1}) & if\ u\ \in\ \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u\ \in\ \Lambda_2 - \Lambda_1 \\ Min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\} & if\ u\ \in\ \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$F\ (F_{\Lambda_3}) = \begin{cases} F(F_{\Lambda_1}) & if\ u\ \in\ \Lambda_1 - \Lambda_2 \\ F(F_{\Lambda_2}) & if\ u\ \in\ \Lambda_2 - \Lambda_1 \\ Min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\} & if\ u\ \in\ \Lambda_1 \cap \Lambda_2 \end{cases}$$

By using the definition of necessity operation

$\oplus F_{\Lambda_3} = \{< u, \{\oplus\ T(F_{\Lambda_3}), \oplus\ I(F_{\Lambda_3}), \oplus\ F(F_{\Lambda_3})\} >: u\ \in U\}$, where

$$\oplus\ T\ (F_{\Lambda_3}) = \begin{cases} T(F_{\Lambda_1}) & if\ u\ \in\ \Lambda_1 - \Lambda_2 \\ T(F_{\Lambda_2}) & if\ u\ \in\ \Lambda_2 - \Lambda_1 \\ Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\} & if\ u\ \in\ \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$\oplus\ I\ (F_{\Lambda_3}) = \begin{cases} I(F_{\Lambda_1}) & if\ u\ \in\ \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u\ \in\ \Lambda_2 - \Lambda_1 \\ Min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\} & if\ u\ \in\ \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$\oplus\ F\ (F_{\Lambda_3}) = \begin{cases} 1 - T(F_{\Lambda_1}) & if\ u\ \in\ \Lambda_1 - \Lambda_2 \\ 1 - T(F_{\Lambda_2}) & if\ u\ \in\ \Lambda_2 - \Lambda_1 \\ 1 - Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\} & if\ u\ \in\ \Lambda_1 \cap \Lambda_2 \end{cases}$$

Assume

$\oplus F_{\Lambda_1} = \{< u, \{T(F_{\Lambda_1}), I(F_{\Lambda_1}), 1 - T(F_{\Lambda_1})\} >: u\ \in U\}$

$\oplus F_{\Lambda_2} = \{< u, \{T(F_{\Lambda_2}), I(F_{\Lambda_2}), 1 - T(F_{\Lambda_2})\} >: u\ \in U\}$

$\oplus F_{\Lambda_1}\ \cup\ \oplus F_{\Lambda_2} = F_\delta$, where

$F_\delta = \{< u, \{T(F_\delta), I(F_\delta), F(F_\delta)\} >: u\ \in U\}$, such that

$$T\ (F_\delta) = \begin{cases} T(F_{\Lambda_1}) & if\ u\ \in\ \Lambda_1 - \Lambda_2 \\ T(F_{\Lambda_2}) & if\ u\ \in\ \Lambda_2 - \Lambda_1 \\ Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\} & if\ u\ \in\ \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$I\ (F_\delta) = \begin{cases} I(F_{\Lambda_1}) & if\ u\ \in\ \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u\ \in\ \Lambda_2 - \Lambda_1 \\ Min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\} & if\ u\ \in\ \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$F\ (F_\delta) = \begin{cases} 1 - T(F_{\Lambda_1}) & if\ u\ \in\ \Lambda_1 - \Lambda_2 \\ 1 - T(F_{\Lambda_2}) & if\ u\ \in\ \Lambda_2 - \Lambda_1 \\ Min\{1 - T(F_{\Lambda_1}), 1 - T(F_{\Lambda_2})\} & if\ u\ \in\ \Lambda_1 \cap \Lambda_2 \end{cases} \text{ OR}$$

$$F\ (F_\delta) = \begin{cases} 1 - T(F_{\Lambda_1}) & if\ u\ \in\ \Lambda_1 - \Lambda_2 \\ 1 - T(F_{\Lambda_2}) & if\ u\ \in\ \Lambda_2 - \Lambda_1 \\ 1 - Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\} & if\ u\ \in\ \Lambda_1 \cap \Lambda_2 \end{cases}$$

Consequently $\oplus F_{\Lambda_3}$ and F_δ are same. So

$\oplus\ (F_{\Lambda_1}\ \cup\ F_{\Lambda_2}) = \oplus\ F_{\Lambda_2}\ \cup\ \oplus\ F_{\Lambda_1}$.

Similarly, we can prove 2.

Definition 4.4 Possibility operation

Let $F_\Lambda\ \in$ NHSS, then possibility operation on NHSS represented by $\otimes\ F_\Lambda$ and defined as follows

$\otimes\ F_\Lambda = \{< \mathrm{u}, \{1 - F(F_\Lambda), I(F_\Lambda), F(F_\Lambda)\} >\}$ for all $u\ \in U$.

Example 4.5 Reconsider the example 2.6

$$\otimes\ F_\Lambda = \{< \mathrm{u}_1, (\acute{\varepsilon}_1\{0.5, 0.7, 0.5\}, \acute{\varepsilon}_2\{0.7, 0.5, 0.3\}, \acute{\varepsilon}_3\{0.1, 0.5, 0.9\}) >$$

$< u_2, (\acute{\varepsilon}_1\{0.3, 0.5, 0.7\}, \acute{\varepsilon}_2\{0.8, 0.6, 0.2\}, \acute{\varepsilon}_3\{0.4, 0.4, 0.6\}) >\}$

Proposition 4.6

1. $\otimes\ (F_{\Lambda_1} \cup F_{\Lambda_2}) = \otimes\ F_{\Lambda_2} \cup \otimes\ F_{\Lambda_1}$
2. $\otimes\ (F_{\Lambda_1} \cap F_{\Lambda_2}) = \otimes\ F_{\Lambda_2} \cap \otimes\ F_{\Lambda_1}$

Proof 1. Let $F_{\Lambda_1} \cup F_{\Lambda_2} = F_{\Lambda_3}$, then

$$T\ (F_{\Lambda_3}) = \begin{cases} T(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ T(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$I\ (F_{\Lambda_3}) = \begin{cases} I(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$F\ (F_{\Lambda_3}) = \begin{cases} F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

By using the definition of necessity operation

$\otimes F_{\Lambda_3} = \{< u, \{\otimes T(F_{\Lambda_3}), \otimes I(F_{\Lambda_3}), \otimes F(F_{\Lambda_3})\} >: u \in U\}$, where

$$\otimes\ T\ (F_{\Lambda_3}) = \begin{cases} 1 - F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ 1 - F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ 1 - Max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$= \begin{cases} 1 - F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ 1 - F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{1 - F(F_{\Lambda_1}), 1 - F(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$\otimes\ I\ (F_{\Lambda_3}) = \begin{cases} I(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$\otimes\ F\ (F_{\Lambda_3}) = \begin{cases} F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

Assume

$\otimes F_{\Lambda_1} = \{< u, \{1 - F(F_{\Lambda_1}), I(F_{\Lambda_1}), F(F_{\Lambda_1})\} >: u \in U\}$

$\otimes F_{\Lambda_2} = \{< u, \{1 - F(F_{\Lambda_2}), I(F_{\Lambda_2}), F(F_{\Lambda_2})\} >: u \in U\}$

$\otimes F_{\Lambda_1} \cup \oplus F_{\Lambda_2} = F_\delta$, where

$F_\delta = \{< u, \{T(F_\delta), I(F_\delta), F(F_\delta)\} >: u \in U\}$, such that

$$\otimes\ T\ (F_\delta) = \begin{cases} 1 - F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ 1 - F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ 1 - Max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$\otimes\ I\ (F_\delta) = \begin{cases} I(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ I(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

$$\otimes\ F\ (F_\delta) = \begin{cases} F(F_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2 \\ F(F_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1 \\ Min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\} & if\ u \in \Lambda_1 \cap \Lambda_2 \end{cases}$$

Consequently $\otimes F_{\Lambda_3}$ and F_δ are same. So

$\otimes\left(F_{\Lambda_1} \cup F_{\Lambda_2}\right) = \otimes F_{\Lambda_2} \cup \otimes F_{\Lambda_1}$

Similarly, we can prove 2.

Proposition 4.7 Let F_{Λ_1} and $F_{\Lambda_2} \in$ NHSS, than we have the following

1. $\oplus\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right) = \oplus F_{\Lambda_1} \wedge \oplus F_{\Lambda_2}$
2. $\oplus\left(F_{\Lambda_1} \vee F_{\Lambda_2}\right) = \oplus F_{\Lambda_1} \vee \oplus F_{\Lambda_2}$
3. $\otimes\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right) = \otimes F_{\Lambda_1} \wedge \otimes F_{\Lambda_2}$
4. $\otimes\left(F_{\Lambda_1} \vee F_{\Lambda_2}\right) = \otimes F_{\Lambda_1} \vee \otimes F_{\Lambda_2}$

Proof 1. Assume $F_{\Lambda_1} \wedge F_{\Lambda_2} = F_{\Lambda_1 \times \Lambda_2}$, where $(u_i, u_j) \in \Lambda_1 \times \Lambda_2$

$F_{\Lambda_1 \times \Lambda_2} = \{< (u_i, u_j), [e, \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}] >\}$

By using definition 4.1, we have

$\oplus\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right) = \{< (u_i, u_j), [e, \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, 1 - \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}] >\}$

Since

$\oplus F_{\Lambda_1} = \{< \mathrm{u}, \{T(F_{\Lambda_1}), I(F_{\Lambda_1}), 1 - T(F_{\Lambda_1})\} >\}$, and

$\oplus F_{\Lambda_2} = \{< \mathrm{u}, \{T(F_{\Lambda_2}), I(F_{\Lambda_2}), 1 - T(F_{\Lambda_2})\} >\}$, then by using AND-operation, we get

$\oplus F_{\Lambda_1} \wedge \oplus F_{\Lambda_2} =$

$\{< (u_i, u_j), [e, \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \max\{1 - T(F_{\Lambda_1}), 1 - T(F_{\Lambda_2})\}] >\}$

$= \{< (u_i, u_j), [e, \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, 1 - \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}] >\}$

$= \oplus\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right)$

Proof 2. Similar to Assertion 1.

Proof 3. Assume $F_{\Lambda_1} \wedge F_{\Lambda_2} = F_{\Lambda_1 \times \Lambda_2}$, where $(u_i, u_j) \in \Lambda_1 \times \Lambda_2$

$F_{\Lambda_1 \times \Lambda_2} = \{< (u_i, u_j), [e, \min\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}] >\}$

By using definition 4.4, we have

$\otimes\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right) = \{< (u_i, u_j), [e, 1 - \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}] >\}$

Since

$\otimes F_{\Lambda_1} = \{< \mathrm{u}, \{1 - F(F_{\Lambda_1}), I(F_{\Lambda_1}), F(F_{\Lambda_1})\} >\}$, and

$\otimes F_{\Lambda_2} = \{< \mathrm{u}, \{1 - F(F_{\Lambda_2}), I(F_{\Lambda_2}), F(F_{\Lambda_2})\} >\}$, then by using AND-operation, we get

$\otimes F_{\Lambda_1} \wedge \oplus F_{\Lambda_2} =$

$\{< (u_i, u_j), [e, \min\{1 - F(F_{\Lambda_1}), 1 - F(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}] >\}$

$= \{< (u_i, u_j), [e, 1 - \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}, \max\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \max\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}] >\}$

$= \otimes\left(F_{\Lambda_1} \wedge F_{\Lambda_2}\right)$

Proof 4. Assume $F_{\Lambda_1} \vee F_{\Lambda_2} = F_{\Lambda_1 \times \Lambda_2}$, where $(u_i, u_j) \in \Lambda_1 \times \Lambda_2$

$F_{\Lambda_1 \times \Lambda_2} = \{< (u_i, u_j), [e, \max\{T(F_{\Lambda_1}), T(F_{\Lambda_2})\}, \min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}] >\}$

By using definition 4.4, we have

$\otimes\left(F_{\Lambda_1} \vee F_{\Lambda_2}\right) = \{< (u_i, u_j), [e, 1 - \min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}, \min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}] >\}$

Since

$\otimes F_{\Lambda_1} = \{< \mathrm{u}, \{1 - F(F_{\Lambda_1}), I(F_{\Lambda_1}), F(F_{\Lambda_1})\} >\}$, and

$\otimes F_{\Lambda_2} = \{< \mathrm{u}, \{1 - F(F_{\Lambda_2}), I(F_{\Lambda_2}), F(F_{\Lambda_2})\} >\}$, then by using OR-operation, we get

$\otimes F_{\Lambda_1} \vee \oplus F_{\Lambda_2} =$

$\{< (u_i, u_j), [e, \max\{1 - F(F_{\Lambda_1}), 1 - F(F_{\Lambda_2})\}, \min\{I(F_{\Lambda_1}), I(F_{\Lambda_2})\}, \min\{F(F_{\Lambda_1}), F(F_{\Lambda_2})\}] >\}$

$= \{< (u_i, u_j), [e, 1 - \min\{F(F_{\wedge_1}), F(F_{\wedge_2})\}, \min\{I(F_{\wedge_1}), I(F_{\wedge_2})\}, \min\{F(F_{\wedge_1}), F(F_{\wedge_2})\}] >\}$

$= \otimes (\boldsymbol{F}_{\wedge_1} \wedge \boldsymbol{F}_{\wedge_2})$

5. Conclusion

In this paper, we study neutrosophic hypersoft set with some basic definition. We proposed the generalized aggregate operators on neutrosophic hypersoft sets such as complement, extended union, extended intersection, And-operation, and Or-operation with their properties and proved the commutative and associative laws on NHSS by using extended union and extended intersection. Finally, the concept of necessity and possibility operations on NHSS with suitable numerical examples and properties are presented. For future trends, we can develop the distance-based similarity measure and will be used for decision making, medical diagnoses, pattern recognition, etc. We also develop the neutrosophic hypersoft matrices with its operations and properties by using proposed operations and use for decision making.

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Composite Neutrosophic Finite Automata

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Abstract. The idea behind the neutrosophic set is we can connect the concept by dynamics of opposite interacts and its neutral that are uncertain and get common parts. Automata theory is beneficial to solve computational complexity problem and also it is an influential mathematical modeling tool in computer science. Inspired by the concepts of neutrosophic sets and automata theory, here, we are introducing and discussing the algebraic concept of neutrosophic finite automata based on the paper [10]. Generally, composite machines can be achieved by the output of the one machine that will be used as input for another machines. This paper introduced the concept of composite automata under the environment of the neutrosophic set and also examined the box function between the composite neutrosophic finite automata.

Keywords: automata theory, stable, composite, box function, neutrosophic set

1. Introduction

Smarandache [27, 28] has proposed an idea of neutrosophic sets which was extending from fuzzy sets. Neutrosophic sets have membership values lies in $]0^-, 1^+[$, the nonstandard unit interval [23] which includes the degree of truth, indeterminacy, and falsity. It is a device for handling the computational complexity of real-life and scientific problems whereas the fuzzy set has limited sources to depict it. The neutrosophic sets are different from intuitionistic fuzzy sets, it is because the neutrosophic set degree of indeterminacy can be defined independently since it is quantified explicitly. Aftermath, there are lots of research works done in various fields such as algebraic structures [5,21,29], topological structures [8,20,24], control theory [17,18,36], decision-making [2, 3, 14, 22, 34], medical [1, 25, 35] and smart product-service system [4].

Generally, computational complexity problems are solved by the automata theory. It has a wide application in computer science and discrete mathematics which is also used to study the behavior of dynamical discrete systems. Fuzzy automata emerge from the inclusion of fuzzy logic into automata theory. Fuzzy finite automata are beneficial to model uncertainties which inherent in many applications [6]. Wee [33] and Santos [26] first introduced the theory of fuzzy finite automata to deal with the notions frequently encountered in the study of natural languages such as vagueness and imprecision. Malik et al [16] introduced a considerably simpler notion of a fuzzy finite state machine that is almost identical to fuzzy finite automatons and greatly contributed to the algebraic study of the fuzzy automaton and fuzzy languages. In addition, several researchers contributed to the development of the theory of fuzzy automata ([11]). Fuzzy finite automata with output offer further inclination in providing output compare to one without outputs. For each assigning input, the machine will generate output and its value is a function of the current state and the current input. Verma and Tiwari [32] recently introduced and studied the concepts of state distinguishability, input-distinguishability, and output completeness of states of a crisp deterministic fuzzy automaton with output function based on [7].

In recent years neutrosophic sets and systems have become an area of interest for many researchers in different areas because it can provide a practical way to address real-world problems more efficiently along with indeterminacy naturally especially in the realm of decision-making. Neutrosophic automata is a newer model, which is extended from a fuzzy automata theory. The neutrosophic set idea was incorporated in automata theory by many researchers in different forms such as finite state machine and its switchboard machine was introduced by under the concept of interval neutrosophic sets [30] and single-valued neutrosophic sets [31]. Further, the finite automata theory has been extended by the concept of general fuzzy automata under the environment of neutrosophic sets, which is called as neutrosophic general finite automata [12]. In addition, the concept of distinguishability and inverse of neutrosophic finite automata was introduced by Kavikumar et al. in [10]. However, still, there are many algebraic structures of neutrosophic automata theory that haven't been studied yet especially automaton with output. Hence, it is important to study more algebraic structures on neutrosophic automata theory with outputs. Therefore, our motive is to study and introduce the concept of composite neutrosophic finite automata which we can obtain by using the outputs of one automaton as inputs to another automaton.

2. Preliminaries

Definition 2.1. Let X be a universe of discourse. The neutrosophic set is an object having the form $A = \{\prec x, \delta_1(x), \delta_2(x), \delta_3(x) \succ |\forall x \in X\}$ where the functions can be defined by $\delta_1, \delta_2, \delta_3 : X \to]0, 1[$ and δ_1 is the degree of membership or truth, δ_2 is the degree of indeterminancy and δ_3 is the degree of non-membership or false of the element $x \in X$ to the set A with the condition $\delta_1(x) + \delta_2(x) + \delta_3(x) \leq 3$.

Let X be a universe of discourse and λ is a neutrosophic subset of X. A map $\lambda : X \to L$, where L is a lattice-ordered monoid. The definition of lattice-ordered monoid is as follows:

Definition 2.2. An algebra $\mathbb{L} = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ is called a lattice-ordered monoid if

(1) $\mathbb{L} = (L, \leq, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the element element 1.
(2) $(L, \bullet, 1)$ is a monoid with 1 identity $1 \in L$ such that $a, b, c \in L$.
  (a) $a \bullet 0 = 0 \bullet a = 0$,
  (b) $a \leq b \Rightarrow a \bullet x \leq b \bullet b, \forall x \in L$,
  (c) $a \bullet (b \vee c) = (a \bullet b) \vee (b \bullet c)$ and $(b \vee c) \bullet a = (b \bullet a) \vee (c \bullet a)$.

Throughout, we work with a lattice-ordered monoid $\mathbb{L}$ so that the monoid $(L, \bullet, 1)$ satisfies the left cancellation law. A neutrosophic finite automaton with outputs (in short; neutrosophic finite automata (NFA)) has considered with neutrosophic transition function and neutrosophic output function.

Definition 2.3. A NFA is a five-tuple $\mathbb{M} = (Q, \Sigma, Z, \delta, \sigma)$, where Q is a finite non-empty set of states, Σ is a finite set of input alphabet, Z is a finite set of output alphabet, δ is a neutrosophic subset of $Q \times \Sigma \times Q$ which represents neutrosophic transition function, and σ is a neutrosophic subset of $Q \times \Sigma \times Z$ which represents neutrosophic output function.

Definition 2.4. Let $\mathbb{M} = (Q, \Sigma, Z, \delta, \sigma)$ be a NFA.

(1) $Q = \{q_1, q_2, \ldots, q_n\}$, is a finite set of states,
(2) $\Sigma = \{x_1, x_2, \ldots, x_n\}$, is a finite set of input symbols,
(3) $Z = \{y_1, y_2, \ldots, y_n\}$, is a finite set of output symbols,
(4) Let $\delta =\prec \delta_1, \delta_2, \delta_3 \succ$ is a neutrosophic subset of $Q \times \Sigma \times Q$ such that the neutrosophic transition function $\delta : A \times \Sigma \times Q \to L \times L \times L$ is defined as follows: $\forall q_i, q_j \in Q$ and $x_1, x_2 \in \Sigma$,

$$\delta_1(q_i, \Lambda, q_j) = \begin{cases} 1 & \text{if } q_i = q_j \\ 0 & \text{if } q_i \neq q_j \end{cases}$$
$$\delta_2(q_i, \Lambda, q_j) = \begin{cases} 0 & \text{if } q_i = q_j \\ 1 & \text{if } q_i \neq q_j \end{cases}$$
$$\delta_3(q_i, \Lambda, q_j) = \begin{cases} 0 & \text{if } q_i = q_j \\ 1 & \text{if } q_i \neq q_j \end{cases}$$

and

$$\delta_1(q_i, x_1x_2, q_j) = \bigvee_{r\in Q} \{\delta_1(q_i, x_1, r) \wedge \delta_1(r, x_2, q_j)\}$$
$$\delta_2(q_i, x_1x_2, q_j) = \bigwedge_{r\in Q} \{\delta_2(q_i, x_1, r) \vee \delta_2(r, x_2, q_j)\}$$
$$\delta_3(q_i, x_1x_2, q_j) = \bigwedge_{r\in Q} \{\delta_3(q_i, x_1, r) \vee \delta_3(r, x_2, q_j)\}$$

(5) Let $\sigma = \prec \sigma_1, \sigma_2, \sigma_3 \succ$ is a neutrosophic subset of $Q\times\Sigma\times Z$ such that the neutrosophic output function $\sigma : Q\times\Sigma\times Z \to L\times L\times L$ is defined as follows: $\forall q_i, q_j \in Q, x_1, x_2 \in \Sigma$ and $y_1, y_2 \in Z$,

$$\sigma_1(q_i, x_1, q_j) = \begin{cases} 1 & \text{if } x_1 = y_1 = \Lambda \\ 0 & \text{if } x_1 = \Lambda, y_1 \neq \Lambda \text{ or } x_1 \neq \Lambda, y_1 = \Lambda \end{cases}$$
$$\sigma_2(q_i, x_1, q_j) = \begin{cases} 0 & \text{if } x_1 = y_1 = \Lambda \\ 1 & \text{if } x_1 = \Lambda, y_1 \neq \Lambda \text{ or } x_1 \neq \Lambda, y_1 = \Lambda \end{cases}$$
$$\sigma_3(q_i, x_1, q_j) = \begin{cases} 0 & \text{if } x_1 = y_1 = \Lambda \\ 1 & \text{if } x_1 = \Lambda, y_1 \neq \Lambda \text{ or } x_1 \neq \Lambda, y_1 = \Lambda \end{cases}$$

and

$$\sigma_1(q_i, x_1x_2, y_1y_2) = \sigma_1(q_i, x_1, y_1) \bullet \bigvee_{r\in Q} \{\delta_1(q_i, x_1, r) \wedge \sigma_1(r, x_2, y_2)\}$$
$$\sigma_2(q_i, x_1x_2, y_1y_2) = \sigma_2(q_i, x_1, y_1) \bullet \bigwedge_{r\in Q} \{\delta_2(q_i, x_1, r) \vee \sigma_2(r, x_2, y_2)\}$$
$$\sigma_3(q_i, x_1x_2, y_1y_2) = \sigma_3(q_i, x_1, y_1) \bullet \bigwedge_{r\in Q} \{\delta_3(q_i, x_1, r) \vee \sigma_3(r, x_2, y_2)\}$$

3. Composite Neutrosophic Finite Automata

This section is interested in the concept of composite finite automata under the environment of neutrosophic sets.

Definition 3.1. For $i \leq n$, let $\mathbb{M}_i = (Q_i, \Sigma_i, Z_i, \delta^i, \sigma^i)$ be NFA's. Let $\mathbb{M}_T = \mathbb{M}_1 \to \mathbb{M}_2 \to \cdots \to \mathbb{M}_n$ be a composite NFA, where $(q_1, q_2, \ldots, q_n) = q_T \in Q_T$ and each $q_i \in Q_i$ if

(1) $Z_i \subseteq \Sigma_{i+1}$, for $i \leq n-1$.

(2) let $\{(x_T \in \Sigma_T \Rightarrow x_1 \in \Sigma_1)(y_T \in Z_T \Rightarrow y_n \in Z_n) | \sigma_1^1(q_1, x_T, y_1) > 0, \sigma_2^1(q_1, x_T, y_1) < 1, \sigma_3^1(q_1, x_T, y_1) < 1, \text{ for } i = 1\}$ then define

$$\delta_1^T\left[(q_1, q_2, \ldots, q_n), x_T, (q_1', q_2', \ldots, q_n')\right] = \begin{cases} \delta_1^1(q_1, x_1, q_1') > 0 & \text{for } i = 1, \\ \delta_1^i(q_i, (\sigma_1^i(q_i, y_{i-1}, y_i)), q_i') & \text{for } i > 1. \end{cases}$$

,

$$\delta_2^T\left[(q_1, q_2, \ldots, q_n), x_T, (q_1', q_2', \ldots, q_n')\right] = \begin{cases} \delta_2^1(q_1, x_1, q_1') < 1 & \text{for } i = 1, \\ \delta_2^i(q_i, (\sigma_2^i(q_i, y_{i-1}, y_i)), q_i') & \text{for } i > 1. \end{cases}$$

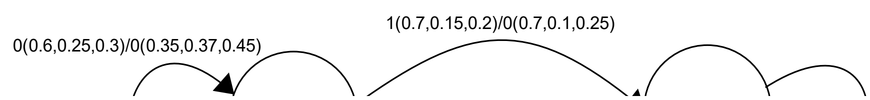
$$\delta_3^T\left[(q_1, q_2, \ldots, q_n), x_T, (q_1', q_2', \ldots, q_n')\right] = \begin{cases} \delta_3^1(q_1, x_1, q_1') < 1 & \text{for } i = 1, \\ \delta_3^i(q_i, (\sigma_3^i(q_i, y_{i-1}, y_i)), q_i') & \text{for } i > 1. \end{cases}$$

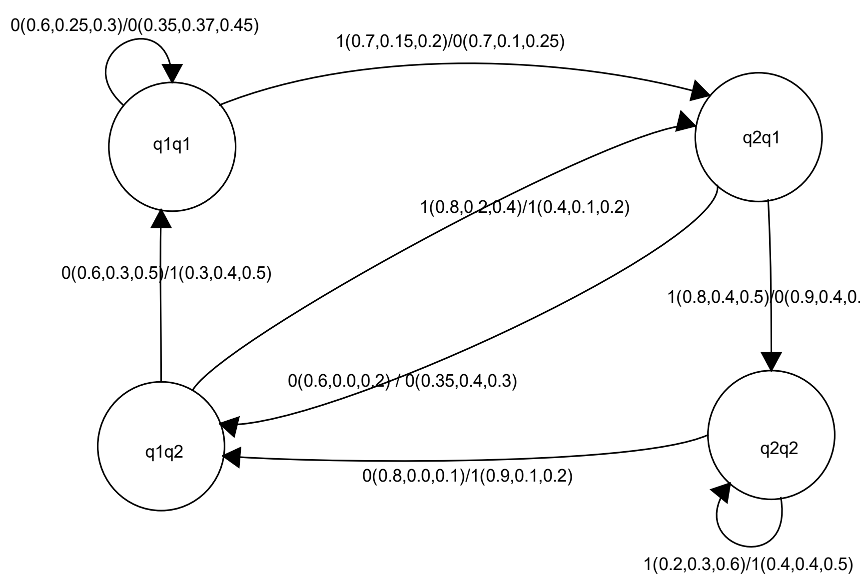
and

$$\sigma_1^T((q_1, q_2, \ldots, q_n), x_T, y_n) = \begin{cases} 1 & \text{if } x_T = y_n = \Lambda \\ 0 & \text{if either } x_T \neq \Lambda \text{ and } y_n = \Lambda \text{ or } x_T = \Lambda \text{ and } y_n \neq \Lambda \end{cases}$$

$$\sigma_2^T((q_1, q_2, \ldots, q_n), x_T, y_n) = \begin{cases} 0 & \text{if } x_T = y_n = \Lambda \\ 1 & \text{if either } x_T \neq \Lambda \text{ and } y_n = \Lambda \text{ or } x_T = \Lambda \text{ and } y_n \neq \Lambda \end{cases}$$

$$\sigma_3^T((q_1, q_2, \ldots, q_n), x_T, y_n) = \begin{cases} 0 & \text{if } x_T = y_n = \Lambda \end{cases}$$





Then the output for input $x_T = 1001$ is $y_T = 0010$.

Definition 3.3. Let $\mathbb{M} = (Q, \Sigma, Z, \delta, \sigma)$ be a NFA. A non-empty set of states $Q_A \subseteq \mathbb{M}$ is said to be stable if

$$\delta_1(q, x, p) > 0, \delta_2(q, x, p) < 1, \delta_3(q, x, p) < 1,$$

for all $q, p \in Q_A$ and $x \in \Sigma$.

Definition 3.4. Two NFA's $\mathbb{M}_1 = (Q_1, \Sigma_1, Z_1, \delta^1, \sigma^1)$ and $\mathbb{M}_2 = (Q_2, \Sigma_2, Z_2, \delta^2, \sigma^2)$ are said to be homomorphism if $\alpha\left[\delta^1(q, x, p)\right] = \delta^2(\alpha(q), \beta(x), \alpha(p))$ and $\sigma^1(q, x, y) \leq \sigma^2(\alpha(q), \beta(x), \gamma(y))$, $\forall q, p \in Q_1$, $x \in \Sigma_1$ and $y \in Z_1$, where the mapping $\alpha : Q_1 \to Q_2$, $\beta : \Sigma_1 \to \Sigma_2$ and $\gamma : Z_1 \to Z_2$ are monoid homomorphisms. Moreover, two NFA's are said to be isomorphism when the mapping α, β and γ are bijective.

Lemma 3.5. *Let $\mathbb{M}_1 = (Q_1, \Sigma_1, Z_1, \delta^1, \sigma^1)$, $\mathbb{M}_2 = (Q_2, \Sigma_2, Z_2, \delta^2, \sigma^2)$ and $\mathbb{M}_3 = (Q_3, \Sigma_3, Z_3, \delta^3, \sigma^3)$ be NFA's. Then $\mathbb{M}_1 \to (\mathbb{M}_2 \to \mathbb{M}_3)$ and $(\mathbb{M}_1 \to \mathbb{M}_2) \to \mathbb{M}_3$ are isomorphic.*

Proof. Since one neutrosophic finite automaton outputs are used as the another neutrosophic finite automaton inputs and omit the parentheses as follows $\mathbb{M}_1 \to \mathbb{M}_2 \to \mathbb{M}_3$. Now, we have an initial inputs for $\mathbb{M}_1$ and its outputs will become an input of $\mathbb{M}_2$. Then, the outputs of $\mathbb{M}_2$ will be an input of $\mathbb{M}_3$. In this manner, $\mathbb{M}_1 \to (\mathbb{M}_2 \to \mathbb{M}_3)$ and $(\mathbb{M}_1 \to \mathbb{M}_2) \to \mathbb{M}_3$ are isomorphic.

Remark 3.6. Lemma 3.5 can be easily extend to four or more NFA's.

Lemma 3.7. *Let $\mathbb{M}_i = (Q_i, \Sigma_i, Z_i, \delta^i, \sigma^i)$, where $i = 1, 2, \ldots, n$, be NFA's. If $\mathbb{M}_1 \to \mathbb{M}_2 \to \cdots \to \mathbb{M}_n$ is a composite NFA if and only if $\mathbb{M}_n$ is a NFA.*

Proof. Assume that $\mathbb{M}_1 \to \mathbb{M}_2 \to \cdots \to \mathbb{M}_n$ is a composite NFA. Then, by lemma 3.5, it is clear that $\mathbb{M}_n$ is a NFA. Conversely, since $\mathbb{M}_n$ is a NFA, the input of $\mathbb{M}_n$ is a output of the $\mathbb{M}_{n-1}$, so in this manner, $\mathbb{M}_1 \to \mathbb{M}_2 \to \cdots \to \mathbb{M}_n$ is a composite NFA.

Definition 3.8. A NFA $\mathbb{M} = (Q, \Sigma, Z, \delta, \sigma)$ is called free if $\forall q_i \in Q$, $x \in \Sigma$ $\exists$ $y \in Z$ such that

$$\sigma_1(q_i, x, y) > 0, \quad \sigma_2(q_i, x, y) < 1, \quad \text{and} \quad \sigma_3(q_i, x, y) < 1.$$

Theorem 3.9. *For each positive integer $i \leq n$, let $\mathbb{M}_i$ is a free NFA, then $\mathbb{M}_1 \to \mathbb{M}_2 \to \cdots \to \mathbb{M}_n$ is a composite NFA.*

Proof. Suppose $\mathbb{M}_i$, $i = 1, 2, \ldots, n$ is a NFA. Let $q, p \in Q_1$ and $x_1 \in \Sigma_1$ and $y_1 \in Z_1$. We prove the theorem by induction on $|i| = n$.

If $n = 1$, then $\mathbb{M}_1$ is a free NFA. Now, we have

$$\sigma_1^1(q_1, x_1, y_1) > 0, \quad \sigma_2^1(q_1, x_1, y_1) < 1, \quad \text{and} \quad \sigma_3^1(q_1 x_1, y_1) < 1,$$

since $\delta_1^1(q_1, x_1, p_1) > 0, \delta_2^1(q_1, x_1, p_1) < 1$ and $\delta_3^1(q_1, x_1, p_1) < 1$. This implies that $\mathbb{M}_1$ is a composite NFA. Hence, the theorem is true for $n = 1$.

Suppose the result is true for all $x_i \in \Sigma_i$ and $y_i \in Z_i$ such that $|i| = n-1$. Let $Z_i \subseteq \Sigma_{i+1}$ for $i \leq n-1$, $n > 1$, so that $\mathbb{M}_{n-1}$ is a free NFA. Now, we have,

$$\sigma_1^{n-1}(q_{n-1}, x_{n-1}, y_{n-1}) > 0, \quad \sigma_2^{n-1}(q_{n-1}, x_{n-1}, y_{n-1}) < 1, \quad \text{and} \quad \sigma_3^{n-1}(q_{n-1}, x_{n-1}, y_{n-1}) < 1.$$

Then by Definition 3.1, we have

$$\delta_1^n(q_n, y_{n-1}, p_n) > 0, \quad \delta_2^n(q_n, y_{n-1}, p_n) < 1 \quad \text{and} \quad \delta_3^n(q_n, y_{n-1}, p_n) < 1.$$

By the induction hypothesis and consider $y_{n-1} = x_n$, then we have

$$\delta_1^n(q_n, x_n, p_n) > 0, \quad \delta_2^n(q_n, x_n, p_n) < 1 \quad \text{and} \quad \delta_3^n(q_n, x_n, p_n) < 1.$$

This implies that, for $x_n \in \Sigma_n$ there exists $y_n \in Z_n$ such that

$$\sigma_1^n(q_n, x_n, y_n) > 0, \quad \sigma_2^n(q_n, x_n, y_n) < 1, \quad \text{and} \quad \sigma_3^n(q_n, x_n, y_n) < 1.$$

Hence, the theorem is true for induction.

Remark 3.10. The converse of Theorem 3.9 is not true since the outputs of composite NFA need not be satisfy the condition of free NFA.

Definition 3.11. Let $\mathbb{M}_1 = (Q_1, \Sigma_1, Z_1, \delta^1, \sigma^1)$ and $\mathbb{M}_2 = (Q_2, \Sigma_2, Z_2, \delta^2, \sigma^2)$ be NFA's. A box function β of $(\mathbb{M}_1, \mathbb{M}_2)$ is satisfy the following conditions, where $\beta : Q_1 \to Q_2$ such that

(1) $\Sigma_1 \subseteq Z_2$

(2) for all $q, p \in Q_1$ and $x \in \Sigma_1$ there exists $y \in Z_1$ such that

$$\beta\left[\delta^1(q, x, p)\right] = \delta^2\left[\beta(q), \sigma^1(q, x, y), \beta(p)\right].$$

Definition 3.12. Let $\mathbb{M}_i = (Q_i, \Sigma_i, Z_i, \delta^i, \sigma^i)$, i=1,2,…,n, be NFA's. To each box functions β_i of $(\mathbb{M}_i, \mathbb{M}_{i+1})$ for $1 \leq i \leq n-1$, there is a corresponding sub NFA $\mathbb{N}(\beta_1, \beta_2, \ldots, \beta_{n-1})$ of $\mathbb{M}_T = \mathbb{M}_1 \to \mathbb{M}_2 \to \cdots \to \mathbb{M}_n$.

Proposition 3.13. *Let $\mathbb{M}_T = (Q_T, \Sigma_T, Z_T, \delta^T, \sigma^T)$ be a composite NFA and $\mathbb{N} = (Q_N, \Sigma_N, Z_N, \delta^N, \sigma^N) \subseteq \mathbb{M}$, where $Q_N = \{(q_1, q_2, \ldots, q_n) | q_1 \in \mathbb{M}$ and $q_i = \beta_{i-1}(q_{i-1})$ for $i > 1\}$. If Q_T is stable, then $\mathbb{N}$ is a compositie NFA.*

Proof. Let $q = (q_1, \ldots, q_n), q' = (q_1', \ldots, q_n') \in Q_N, x_T \in \Sigma_T$ and $y_i \in Z_T$. Then, by definition 3.1 and $y_{i-1} = x_i$. Since $Q_N \subseteq Q_T$, it is enough to prove that Q_N is stable, for each $i > 1$. Then

$$\begin{aligned}
\delta_1^i(q_i, x_i, q_i') &= \delta_1^i\left[\beta_{i-1}(q_{i-1}), (\sigma_1^{i-1}(q_{i-1}, y_{i-2}, y_{i-1})), \beta_{i-1}(q_{i-1}')\right] \\
&= \beta_{i-1}\left[\delta_1^{i-1}(q_{i-1}, x_{i-1}, q_{i-1}')\right], \text{since } \beta_{i-1} \text{ is a box function of } (\mathbb{M}_{i-1}, \mathbb{M}_i), \\
&= \delta_1^{i-1}\left[\beta_{i-1}(q_{i-1}), x_{i-1}, \beta_{i-1}(q_{i-1}')\right]
\end{aligned}$$

This implies that $\delta_1^{i-1}\left[\beta_{i-1}(q_{i-1}), x_{i-1}, \beta_{i-1}(q_{i-1}')\right]$ is stable, since $\delta_1^{i-1}(q_{i-1}, x_{i-1}, q_{i-1}')$ is stable. Hence, Q_N is stable. Therefore, $\mathbb{N}$ is a composite NFA.

Theorem 3.14. *Let $\mathbb{M}_1 = (Q_1, \Sigma_1, Z_1, \delta^1, \sigma^1)$ and $\mathbb{M}_2 = (Q_2, \Sigma_2, Z_2, \delta^2, \sigma^2)$ be two NFA's and let $\mathbb{H}$ be a NFA with inputs Σ_H which generating inputs set for Σ_1. Suppose $Z_1 \subseteq \Sigma_2$ and for all $p, q \in Q_1$, $x_1 \in \Sigma_H$, the map $\beta : Q_1 \to Q_2$ such that $\beta[\delta^1(q, x_1, p)] = \delta^2[\beta(q), \sigma^1(q, x_1, y_1), \beta(p)]$. Then β is a box function of $(\mathbb{M}_1, \mathbb{M}_2)$.*

Proof. We will prove the result by mathematical induction on the generated set of inputs Σ_H. For $n = 1$, let $x_1 \in \Sigma_H$ the result follows from 3.11.

For $n = 2$, let $x_1, x_2 \in \Sigma_H$ and $q, p \in Q_1$, then

$$\begin{aligned}
\beta\left[\delta^1(q, x_1x_2, p)\right] &= \beta\left[\bigvee\nolimits_{r\in Q_1}\left\{\delta^1(q, x_1, r) \wedge \delta^1(r, x_2, p)\right\}\right]\\
&= \bigvee\nolimits_{r\in Q_1}\left\{\beta(\delta^1(q, x_1, r)) \wedge \beta(\delta^1(r, x_2, p))\right\}\\
&= \bigvee\nolimits_{\beta(r)\in Q_2}\left\{\delta^2(\beta(q), \sigma^1(q, x_1, y_1), \beta(r)) \wedge \delta^2(\beta(r), \sigma^1(q, x_2, y_2), \beta(p))\right\}\\
&= \delta^2\left[\beta(q), \sigma^1(q, x_1, y_1) \bullet \sigma^1(q, x_2, y_2), \beta(p)\right]\\
&= \delta^2\left[\beta(q), \sigma^1(q, x_1x_2, y_1y_2), \beta(p)\right]
\end{aligned}$$

If the induction continues for any finite sequence of inputs such as $n > 2$ for each $x_i \in \Sigma_H$, the results follows by induction. Hence β is a box function of $(\mathbb{M}_1, \mathbb{M}_2)$.

4. Conclusions

The main focus of this paper is to study the algebraic automata theory based on the concept of neutrosophic sets. Thus, this investigation contributes a small portion to algebraic automata theory such as composite neutrosophic finite automata which is established by outputs of one automaton as the inputs of another automaton. The future study will be concerned with similar concepts but the approaches are based on the combination of N-fuzzy structures [9,13] and type-2 fuzzy structures [15,19] under the environment of neutrosophic sets [27,28].

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Fixed Point Results for Contraction Theorems in Neutrosophic Metric Spaces

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Abstract. In this article, we present fixed and common fixed point results for Banach and Edelstein contraction theorems in neutrosophic metric spaces. Then some properties and examples are given for neutrosophic metric spaces. Thus, we added a new path in neutrosophic theory to obtain fixed point results. we investigate and prove some contraction theorems that are extended to neutrosophic metric space with the assistance of Grabiec.

Keywords: Fixed point; Neutrosophic Metric Space; Banach Contraction; Edelstein Contraction.

1. **Introduction**

Fuzzy Sets was presented by Zadeh [20] as a class of elements with a grade of membership. Kramosil and Michalek [9] defined new notion called Fuzzy Metric Space (FMS). Later, many authors have examined the concept of fuzzy metric in various aspects. In 1984 Kaleva and Seikkala [8] have characterized the FMS, where separation between any two points to be positive number. In particular, George and Veeramani [4, 5] redefined the concept of fuzzy metric space with the assistance of continuous t-norm, and continuous t-co norm. FMS has utilized in applied science fields such as fixed point theory, decision making, medical imaging and signal processing. Heilpern [7] defined fuzzy contraction for Fixed point theorem. Park [14] defined Intuitionistic Fuzzy Metric Space (IFMS) from the concept of FMS and given some fixed point results. Fixed point theorems related to FMS and IFMS given by Alaca et al [2] and nemerous researchers [13, 19].In 1998, Smarandache [16] characterized the new concept called neutrosophic logic and neutrosophic set. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. A neutrosophic value is appeared by (T, I, F). Hence, neutrosophic logic and neutrosophic set assists us to brief many uncertainties in our lives. In addition, several researchers have made significant development on this theory [26–30]. Recently, Baset et al. [22–25] explored the neutrosophic applications in different fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making and financial performance evaluation of manufacturing industries. In fact, the idea of fuzzy sets deals with only a degree of membership. In addition, the concept of intuitionistic fuzzy set established while adding degree of non - membership with degree of membership. But these degrees are characterized relatively one another. Therefore, neutrosophic set is a generalized state of fuzzy and intuitionistic fuzzy set by incorporating degree of indeterminacy. In 2019, Kirisci et al [10, 11] defined neutrosophic metric space as a generalization of IFMS and brings about fixed point theorems in complete neutrosophic metric space.

In this paper, we investigate and prove some contraction theorems that are extended to neutrosophic metric space with the assistance of Grabiec [6].

2. Preliminaries

Definition 2.1 [17] Let Σ be a non-empty fixed set. A Neutrosophic Set (NS for short) N in Σ is an object having the form $N = \{\langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle : a \in \Sigma\}$ where the functions $\xi_N(a), \varrho_N(a)$ and $\nu_N(a)$ represent the degree of membership, degree of indeterminacy and the degree of non-membership respectively of each element a $\in$ N to the set Σ.

A neutrosophic set $N = \{\langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle : a \in \Sigma\}$ is expressed as an ordered triple $N = \langle a, \xi_N(a), \varrho_N(a), \nu_N(a)\rangle$ in Σ.

In NS, there is no restriction on $(\xi_N(a), \varrho_N(a), \nu_N(a))$ other than they are subsets of $]^-0, 1^+[$

Remark 2.2 [10] Neutrosophic Set N is included in another Neutrosophic set Γ ($N \subseteq \Gamma$) if and only if

$$inf\ \xi_N(a) \leq inf \xi_\Gamma(a) \quad sup\ \xi_N(a) \leq sup\ \xi_\Gamma(a)$$

$$inf\ \varrho_N(a) \geq inf \varrho_\Gamma(a) \quad sup\ \varrho_N(a) \geq sup\ \varrho_\Gamma(a)$$

$$inf\ \nu_N(a) \geq inf \nu_\Gamma(a) \quad sup\ \nu_N(a) \geq sup\ \nu_\Gamma(a)$$

Triangular Norms (TNs) were initiated by menger. Triangular co norms(TCs) knowns as dual operations of triangular norms (TNs).

Definition 2.3 [4] A binary operation $\star : [0, 1] \times [0, 1] \to [0, 1]$ is called continuous t - norm (CTN) if it satisfies the following conditions;

For all $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0,\ 1]$

(i) $\varepsilon_1 \star 0 = \varepsilon_1$;

(ii) If $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$ then $\varepsilon_1 \star \varepsilon_2 \leq \varepsilon_3 \star \varepsilon_4$;

(iii) $\star$ is continuous;

(iv) $\star$ is commutative and associative.

Definition 2.4 [4] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is called continuous t - co norm (CTC) if it satisfies the following conditions;

For all $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0,\ 1]$

(i) $\varepsilon_1 \diamond 0 = \varepsilon_1$;

(ii) If $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$ then $\varepsilon_1 \diamond \varepsilon_2 \leq \varepsilon_3 \diamond \varepsilon_4$;

(iii) $\diamond$ is continuous;

(iv) $\diamond$ is commutative and associative.

Remark 2.5 From the definitions of CTN and CTC, we note that if we take $0 < \varepsilon_1, \varepsilon_2 < 1$ for $\varepsilon_1 < \varepsilon_2$ then there exist $0 < \varepsilon_3, \varepsilon_4 < 1$ such that $\varepsilon_1 \star \varepsilon_3 \geq \varepsilon_2$ and $\varepsilon_1 \geq \varepsilon_2 \diamond \varepsilon_4$.

Further we choose $\varepsilon_5 \in (0,1)$ then there exists $\varepsilon_6, \varepsilon_7 \in (0,1)$ such that $\varepsilon_6 \star \varepsilon_6 \geq \varepsilon_5$ and $\varepsilon_7 \diamond \varepsilon_7 \leq \varepsilon_5$.

Definition 2.6 [13] A Sequence $\{t_n\}$ is called s - non-decreasing sequence if there exists $m_0 \in \mathbb{N}$ such that $t_m \leq t_{m+1}$ for all $m > m_0$.

3. **Neutrosophic Metric Space**

In this section, we apply neutrosophic theory to generalize the Intuitionistic fuzzy metric space. we also discuss some properties and examples in it.

Definition 3.1 A 6 - tuple $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$is called Neutrosophic Metric Space(NMS), if Σ is an arbitrary non empty set, $\star$ is a neutrosophic CTN and $\diamond$ is a neutrosophic CTC and Ξ, Θ, Υ are neutrosophic sets on $\Sigma^2 \times \mathbb{R}^+$ satisfying the following conditions:

For all $\zeta, \eta, \omega \in \Sigma, \lambda \in \mathbb{R}^+$

(i) $0 \leq \Xi(\zeta,\eta,\lambda) \leq 1;\ 0 \leq \Theta(\zeta,\eta,\lambda) \leq 1;\ 0 \leq \Upsilon(\zeta,\eta,\lambda) \leq 1$;

(ii) $\Xi(\zeta,\eta,\lambda) + \Theta(\zeta,\eta,\lambda) + \Upsilon(\zeta,\eta,\lambda) \leq 3$;

(iii) $\Xi(\zeta,\eta,\lambda) = 1$ if and only if $\zeta = \eta$;

(iv) $\Xi(\zeta,\eta,\lambda) = \Xi(\eta,\zeta,\lambda)$ for $\lambda > 0$;

(v) $\Xi(\zeta,\eta,\lambda) \star \Xi(\eta,\zeta,\mu) \leq \Xi(\zeta,\omega,\lambda+\mu)$, for all $\lambda, \mu > 0$;

(vi) $\Xi(\zeta,\eta,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous ;

(vii) $lim_{\lambda\to\infty}\Xi(\zeta,\eta,\lambda) = 1$ for all $\lambda > 0$;

(viii) $\Theta(\zeta,\eta,\lambda) = 0$ if and only if $\zeta = \eta$;

(ix) $\Theta(\zeta,\eta,\lambda) = \Theta(\eta,\zeta,\lambda)$ for $\lambda > 0$;

(x) $\Theta(\zeta,\eta,\lambda) \diamond \Theta(\zeta,\omega,\mu) \geq \Theta(\zeta,\omega,\lambda+\mu)$, for all $\lambda, \mu > 0$;

(xi) $\Theta(\zeta,\eta,.) : [0,\infty) \to [0,1]$ is neutrosophic continuous ;

(xii) $lim_{\lambda\to\infty}\Theta(\zeta,\eta,\lambda) = 0$ for all $\lambda > 0$;

(xiii) $\Upsilon(\zeta,\eta,\lambda) = 0$ if and only if $\zeta = \eta$;

(xiv) $\Upsilon(\zeta, \eta, \lambda) = \Upsilon(\eta, \zeta, \lambda)$ for $\lambda > 0$;

(xv) $\Upsilon(\zeta, \eta, \lambda) \diamond \Upsilon(\zeta, \omega, \mu) \geq \Upsilon(\zeta, \omega, \lambda + \mu)$, for all $\lambda, \mu > 0$;

(xvi) $\Upsilon(\zeta, \eta, .) : [0, \infty) \to [0, 1]$ is neutrosophic continuous ;

(xvii) $lim_{\lambda \to \infty} \Upsilon(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;

(xviii) If $\lambda > 0$ then $\Xi(\zeta, \eta, \lambda) = 0, \Theta(\zeta, \eta, \lambda) = 1, \Upsilon(\zeta, \eta, \lambda) = 1$.

Then (Ξ, Θ, Υ) is called Neutrosophic Metric on Σ. The functons Ξ, Θ and Υ denote degree of closedness, neturalness and non - closedness between ζ and η with respect to λ respectively.

Example 3.2 Let (Σ, d) be a metric space. Define $\zeta \star \eta = min\{\zeta, \eta\}$ and $\zeta \diamond \eta = max\{\zeta, \eta\}$, and $\Xi, \Theta, \Upsilon : \Sigma^2 \times \mathbb{R}^+ \to [0, 1]$ defined by , we define

$$\Xi(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}; \quad \Theta(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)}; \quad \Upsilon(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda}$$

for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is called neutrosophic metric space induced by a metric d the standard neutrosophic metric.

Example 3.3 If we take $\Sigma = \mathbb{N}$, consider the CTN, CTC are $\zeta \star \eta = min\{\zeta, \eta\}$ and $\zeta \diamond \eta = max\{\zeta, \eta\}$, $\Xi, \Theta, \Upsilon : \Sigma^2 \times \mathbb{R}^+ \to [0, 1]$ defined by

$$\Xi(\zeta, \eta, \lambda) = \begin{cases} \frac{\zeta}{\eta} & if \quad \zeta \leq \eta \\ \frac{\eta}{\zeta} & if \quad \eta \leq \zeta \end{cases}$$

$$\Theta(\zeta, \eta, \lambda) = \begin{cases} \frac{\eta - \zeta}{\eta} & if \quad \zeta \leq \eta \\ \frac{\zeta - \eta}{\zeta} & if \quad \eta \leq \zeta \end{cases}$$

$$\Upsilon(\zeta, \eta, \lambda) = \begin{cases} \eta - \zeta & if \quad \zeta \leq \eta \\ \zeta - \eta & if \quad \eta \leq \zeta \end{cases}$$

for all $\zeta, \eta \in \Sigma \; and \; \lambda > 0$. Then $\Xi, \Theta, \Upsilon : \Sigma^2 \times \mathbb{R}^+ \to [0, 1]$ is a NMS.

Remark 3.4 In Neutrosophic Metric space Ξ is non - decreasing , Θ is a non - increasing , Υ is decreasing for all $\zeta, \eta \in \Sigma$.

Definition 3.5 Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ be neutrosophic metric space . Then

(a) a sequence $\{\zeta_n\}$ in Σ is converging to a point $\zeta \in \Sigma$ if for each $\lambda > 0$

$$lim_{\lambda \to \infty} \Xi(\zeta, \eta, \lambda) = 1; \quad lim_{\lambda \to \infty} \Theta(\zeta, \eta, \lambda) = 0; \quad lim_{\lambda \to \infty} \Upsilon(\zeta, \eta, \lambda) = 0.$$

(b) a sequence ζ_n in Σ is said to be Cauchy if for each $\epsilon > 0$ and $\lambda > 0$ there exist $N \in \mathbb{N}$ such that $\Xi(\zeta_n, \zeta_m, \lambda) > 1 - \epsilon$; $\Theta(\zeta_n, \zeta_m, \lambda) < \epsilon$; $\Upsilon(\zeta_n, \zeta_m, \lambda) < \epsilon$ for all n, m $\leq$ N.

(c) $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is said to be complete neutrosophic metric space if every Cauchy sequence is convergent.

(d) $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ is called compact neutrosophic metric space if every sequence contains convergent sub sequence.

4. **Main Results**

Theorem 4.1 (Neutrosophic Banach Contraction Theorem) Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ be a complete neutrosophic metric space. Let $F : \Sigma \to \Sigma$ be a function satisfying

$$\Xi(F\zeta, F\eta, \lambda) \geq \Xi(\zeta, \eta, \lambda); \quad \Theta(F\zeta, F\eta, \lambda) \leq \Theta(\zeta, \eta, \lambda); \quad \Upsilon(F\zeta, F\eta, \lambda) \leq \Upsilon(\zeta, \eta, \lambda) \qquad (4.1.1)$$

for all $\zeta, \eta \in \Sigma$. $0 < k < 1$. Then F has unique fixed point.
Proof: Let $\zeta \in \Sigma$ and $\{\zeta_n\} = F^n(a)$ $(n \in \mathbb{N})$. By Mathematical induction, we obtain

$$\Xi(\zeta_n, \zeta_{n+1}, \lambda) \geq \Xi(\zeta, \zeta_1, \frac{\lambda}{k^n}); \Theta(\zeta_n, \zeta_{n+1}, \lambda) \leq \Theta(\zeta, \zeta_1, \frac{\lambda}{k^n}); \Upsilon(\zeta_n, \zeta_{n+1}, \lambda) \leq \Upsilon(\zeta, \zeta_1, \frac{\lambda}{k^n}) \; ....(4.1.2)$$

for all $n > 0$ and $\lambda > 0$. Thus for any non-negative integer p, we have

$$\begin{aligned}
\Xi(\zeta_n, \zeta_{n+p}, \lambda) &\geq \Xi(\zeta, \zeta_{n+1}, \frac{\lambda}{p}) \star \dots^{(p-times)} \dots \star \Xi(\zeta_{n+p-1}, \zeta_{n+p}, \frac{\lambda}{p}) \\
&\geq \Xi(\zeta, \zeta_1, \frac{\lambda}{pk^n}) \star \dots^{(p-times)} \dots \star \Xi(\zeta, \zeta_1, \frac{\lambda}{pk^{n+p-1}}) \\
\Theta(\zeta_n, \zeta_{n+p}, \lambda) &\leq \Theta(\zeta, \zeta_{n+1}, \frac{\lambda}{p}) \diamond \dots^{(p-times)} \dots \diamond \Theta(\zeta_{n+p-1}, \zeta_{n+p}, \frac{\lambda}{p}) \\
&\leq \Theta(\zeta, \zeta_1, \frac{\lambda}{pk^n}) \diamond \dots^{(p-times)} \dots \diamond \Theta(\zeta, \zeta_1, \frac{\lambda}{pk^{n+p-1}}) \\
\Upsilon(\zeta_n, \zeta_{n+p}, \lambda) &\leq \Upsilon(\zeta, \zeta_{n+1}, \frac{\lambda}{p}) \diamond \dots^{(p-times)} \dots \diamond \Upsilon(\zeta_{n+p-1}, \zeta_{n+p}, \frac{\lambda}{p}) \\
&\leq \Upsilon(\zeta, \zeta_1, \frac{\lambda}{pk^n}) \diamond \dots^{(p-times)} \dots \diamond \Upsilon(\zeta, \zeta_1, \frac{\lambda}{pk^{n+p-1}})
\end{aligned}$$

by (4.1.2) and the definition of NMS conditions, we get

$$\begin{aligned}
lim_{n\to\infty}\Xi(\zeta_n, \zeta_{n+p}, \lambda) &\geq 1 \star \dots^{(p-times)} \dots \star 1 = 1 \\
lim_{n\to\infty}\Theta(\zeta_n, \zeta_{n+p}, \lambda) &\leq 0 \diamond \dots^{(p-times)} \dots \diamond 0 = 0 \\
lim_{n\to\infty}\Upsilon(\zeta_n, \zeta_{n+p}, \lambda) &\leq 0 \diamond \dots^{(p-times)} \dots \diamond 0 = 0.
\end{aligned}$$

Therefore, $\{\zeta_n\}$ is Cauchy sequence and it is convergent to a limit, let the limit point is η. Thus, we get

$$\begin{aligned}
\Xi(F\eta, \eta, t) &\geq \Xi(F\eta, F\zeta_n, \frac{\lambda}{2}) \star \Xi(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \\
&\geq \Xi(\eta, \zeta_n, \frac{\lambda}{2k}) \star \Xi(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \to 1 \star 1 = 1. \\
\Theta(F\eta, \eta, \lambda) &\leq \Theta(F\eta, F\zeta_n, \frac{\lambda}{2}) \diamond \Theta(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \\
&\leq \Theta(\eta, \zeta_n, \frac{\lambda}{2k}) \diamond \Theta(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \to 0 \diamond 0 = 0. \\
\Upsilon(F\eta, \eta, \lambda) &\leq \Upsilon(F\eta, F\zeta_n, \frac{\lambda}{2}) \diamond \Upsilon(\zeta_{n+1}, \eta, \frac{\lambda}{2})) \\
&\leq \Upsilon(\eta, \zeta_n, \frac{\lambda}{2k}) \diamond \Upsilon(\zeta_{n+1}, \eta, \frac{\lambda}{2}) \to 0 \diamond 0 = 0.
\end{aligned}$$

Since we see that

$$\Xi(\zeta,\eta,\lambda)=1 \ \ iff \ \ \zeta=\eta; \quad \Theta(\zeta,\eta,\lambda)=0 \ \ iff \ \ \zeta=\eta; \quad \Upsilon(\zeta,\eta,\lambda)=0 \ \ iff \ \ \zeta=\eta$$

we get $F\eta \ = \ \eta$, which is the fixed point of Neutrosophic metric space.
To show the uniqueness, let us assume that $F\omega \ = \omega$ for some $\omega \in \Sigma$

$$\begin{aligned}
&1 \geq \Xi(\zeta,\omega,\lambda)=\Xi(F\eta,F\omega,\lambda)\geq \Xi(\zeta,\omega,\frac{\lambda}{k})=\Xi(F\eta,F\eta,\frac{\lambda}{k})\geq \Xi(\zeta,\omega,\frac{\lambda}{k^2})\\
&\quad \geq \cdots \geq \Xi(\zeta,\omega,\frac{\lambda}{k^n})\to 1 \ as \ n\to\infty\\
&0 \leq \Theta(\zeta,\omega,\lambda)=\Theta(F\eta,F\omega F\omega,\lambda)\leq \Theta(\zeta,\omega,\frac{\lambda}{k})=\Theta(F\eta,F\omega,\frac{\lambda}{k})\leq \Theta(\zeta,\omega,\frac{\lambda}{k^2})\\
&\quad \leq \cdots \leq \Theta(\zeta,\omega,\frac{\lambda}{k^n})\to 0 \ as \ n\to\infty\\
&0 \leq \Upsilon(\zeta,\omega,\lambda)=\Upsilon(F\eta,F\omega,\lambda)\leq \Upsilon(\zeta,\omega,\frac{\lambda}{k})=\Upsilon(F\eta,F\omega,\frac{\lambda}{k})\leq \Upsilon(\zeta,\omega,\frac{\lambda}{k^2})\\
&\quad \leq \cdots \leq \Upsilon(\zeta,\omega,\frac{\lambda}{k^n})\to 0 \ as \ n\to\infty.
\end{aligned}$$

From the definition of NMS, We get $\eta=\omega$. Therefor, F has a unique fixed point.
Lemma 4.2 (a) If $lim_{n\to\infty}\zeta_n=\zeta$ and $lim_{n\to\infty}\eta_n=\eta$, then

$$\begin{aligned}
\Xi(\zeta,\eta,\lambda-\epsilon)&\leq lim_{n\to\infty}inf \ \Xi(\zeta_n,\eta_n,\lambda)\\
\Theta(\zeta,\eta,\lambda-\epsilon)&\geq lim_{n\to\infty}sup \ \Theta(\zeta_n,\eta_n,\lambda)\\
\Upsilon(\zeta,\eta,\lambda-\epsilon)&\geq lim_{n\to\infty}sup \ \Upsilon(\zeta_n,\eta_n,\lambda)
\end{aligned}$$

(b) If $lim_{n\to\infty}\zeta_n=\zeta$ and $lim_{n\to\infty}\eta_n=\eta$, then

$$\begin{aligned}
\Xi(\zeta,\eta,\lambda+\epsilon)&\geq lim_{n\to\infty}sup \ \Xi(\zeta_n,\eta_n,\lambda)\\
\Theta(\zeta,\eta,\lambda+\epsilon)&\leq lim_{n\to\infty}inf \ \Theta(\zeta_n,\eta_n,\lambda)\\
\Upsilon(\zeta,\eta,\lambda+\epsilon)&\leq lim_{n\to\infty}inf \ \Upsilon(\zeta_n,\eta_n,\lambda)
\end{aligned}$$

for all $\lambda > 0$ and $0 < \epsilon < \lambda$.

Proof for(a): By the definition of NMS, conditions (v),(x) and (xv)

$$\Xi(\zeta_n, \eta_n, \lambda) \geq \Xi(\zeta_n, \zeta, \frac{\epsilon}{2}) \star \Xi(\zeta, \eta, \lambda - \epsilon) \star \Xi(\eta, \eta_n, \frac{\epsilon}{2})$$

$$lim_{n\to\infty} inf\ \Xi(\zeta_n, \eta_n, \lambda) \geq 1 \star \Xi(\zeta, \eta, \lambda - \epsilon) \star 1$$

$$Hence, \quad lim_{n\to\infty} inf\ \Xi(\zeta_n, \eta_n, \lambda) \geq \Xi(\zeta, \eta, \lambda - \epsilon)$$

$$\Theta(\zeta_n, \eta_n, \lambda) \leq \Theta(\zeta_n, \zeta, \frac{\epsilon}{2}) \diamond \Theta(\zeta, \eta, \lambda - \epsilon) \diamond \Theta(\eta, \eta_n, \frac{\epsilon}{2})$$

$$lim_{n\to\infty} sup\ \Theta(\zeta_n, \eta_n, \lambda) \leq 0 \diamond \Theta(\zeta, \eta, \lambda - \epsilon) \diamond 0$$

$$Hence, \quad lim_{n\to\infty} sup\ \Theta(\zeta_n, \eta_n, \lambda) \leq \Theta(\zeta, \eta, \lambda - \epsilon)$$

$$\Upsilon(\zeta_n, \eta_n, \lambda) \leq \Upsilon(\zeta_n, \zeta, \frac{\epsilon}{2}) \diamond \Upsilon(\zeta, \eta, \lambda - \epsilon) \diamond \Upsilon(\eta, \eta_n, \frac{\epsilon}{2})$$

$$lim_{n\to\infty} sup\ \Upsilon(\zeta_n, \eta_n, \lambda) \leq 0 \diamond \Upsilon(\zeta, \eta, \lambda - \epsilon) \diamond 0$$

$$Hence, \quad lim_{n\to\infty} sup\ \Upsilon(\zeta_n, \eta_n, \lambda) \leq \Theta(\zeta, \eta, \lambda - \epsilon)$$

Proof for (b):By the definition of NMS, conditions (v),(x) and (xv)

$$\Xi(\zeta, \eta, \lambda + \epsilon) \geq \Xi(\zeta, \zeta_n, \frac{\epsilon}{2}) \star \Xi(\zeta_n, \eta_n, \epsilon) \star \Xi(\eta_n, \eta, \frac{\epsilon}{2})$$

$$Hence, \quad \Xi(\zeta, \eta, \lambda + \epsilon) \geq lim_{n\to\infty} sup \Xi(\zeta_n, \eta_n, \epsilon)$$

$$\Theta(\zeta, \eta, \lambda + \epsilon) \leq \Xi(\zeta, \zeta_n, \frac{\epsilon}{2}) \diamond \Theta(\zeta_n, \eta_n, \epsilon) \diamond \Theta(\eta_n, \eta, \frac{\epsilon}{2})$$

$$Hence, \quad \Theta(\zeta, \eta, \lambda + \epsilon) \leq lim_{n\to\infty} inf \Theta(\zeta_n, \eta_n, \epsilon)$$

$$\Upsilon(\zeta, \eta, \lambda + \epsilon) \leq \Upsilon(\zeta, \zeta_n, \frac{\epsilon}{2}) \diamond \Upsilon(\zeta_n, \eta_n, \epsilon) \diamond \Upsilon(\eta_n, \eta, \frac{\epsilon}{2})$$

$$Hence, \quad \Upsilon(\zeta, \eta, \lambda + \epsilon) \leq lim_{n\to\infty} inf \Upsilon(\zeta_n, \eta_n, \epsilon)$$

Corollary 4.3 If $lim_{n\to\infty}\zeta_n = a$ and $lim_{n\to\infty}\eta_n = \eta$, then

$$(a) \quad \Xi(\zeta, \eta, \lambda) \leq lim_{n\to\infty} inf\ \Xi(\zeta_n, \eta_n, \lambda);$$

$$\Theta(\zeta, \eta, \lambda) \geq lim_{n\to\infty} sup\ \Theta(\zeta_n, \eta_n, \lambda);$$

$$\Upsilon(\zeta, \eta, \lambda) \geq lim_{n\to\infty} sup\ \Upsilon(\zeta_n, \eta_n, \lambda)....(4.3.1)$$

$$(b) \quad \Xi(\zeta, \eta, \lambda) \geq lim_{n\to\infty} sup\ \Xi(\zeta_n, \eta_n, \lambda)$$

$$\Theta(\zeta, \eta, \lambda) \leq lim_{n\to\infty} inf\ \Theta(\zeta_n, \eta_n, \lambda)$$

$$\Upsilon(\zeta, \eta, \lambda) \leq lim_{n\to\infty} inf\ \Upsilon\zeta_n, \eta_n, \lambda)....(4.3.2)$$

for all $\lambda > 0$ and $0 < \epsilon < \lambda$.

Theorem 4.4 (Neutrosophic Edelstein Contraction Theorem) Let $(\Sigma, \Xi, \Theta, \Upsilon, \star, \diamond)$ be compact neutrosophic metric space. Let $F : \Sigma \to \Sigma$ be a function satisfying

$$\Xi(F\zeta, F\eta, .) > \Xi(\zeta, \eta, .); \Theta(F\zeta, F\eta, .) < \Theta(\zeta, \eta, .); \Upsilon(F\zeta, F\eta, .) < \Upsilon(\zeta, \eta, .).....(4.4.1)$$

Then F has fixed point.

Proof: Let $a \in \Sigma$ and $a_n = F^n\zeta \quad (n \in \mathbb{N})$. Assume $\zeta_n \neq \zeta_{n+1}$ for each n (If not $F\zeta_n = \zeta_n$) consequently $a_n \neq a_{n+1} \quad (n \neq m)$, For otherwise we get

$$\Xi(\zeta_n, \zeta_{n+1}, .) = \Xi(\zeta_m, \zeta_{m+1}, .) > \Xi(\zeta_{m-1}, \zeta_m, .) > \cdots > \Xi(\zeta_n, \zeta_{n+1}, .)$$
$$\Theta(\zeta_n, \zeta_{n+1}, .) = \Theta(\zeta_m, \zeta_{m+1}, .) < \Theta(\zeta_{m-1}, \zeta_m, .) < \cdots < \Theta(\zeta_n, \zeta_{n+1}, .)$$
$$\Upsilon(\zeta_n, \zeta_{n+1}, .) = \Upsilon(\zeta_m, \zeta_{m+1}, .) < \Upsilon(\zeta_{m-1}, \zeta_m, .) < \cdots < \Upsilon(\zeta_n, \zeta_{n+1}, .)$$

where $m > n$, which is a contradiction. Since Σ is compact set, $\{\zeta_n\}$ has convergent sub sequence $\{\zeta_{n_i}\}$. Let $\eta = lim_{i\to\infty}\zeta_{n_i}$, Also we assume that η such that $F\eta \in \{\zeta_{n_i}; i \in \mathbb{N}\}$. According to the above assumption, we may now write,

$$\Xi(F\zeta_{n_i}, F\eta, .) > \Xi(\zeta_{n_i}, \eta, .); \quad \Theta(F\zeta_{n_i}, F\eta, .) < \Theta(\zeta_{n_i}, \eta, .); \quad \Upsilon(F\zeta_{n_i}, F\eta, .) < \Upsilon(\zeta_{n_i}, \eta, .)$$

for all $i \in \mathbb{N}$. Then by equation (4.3.1) we obtain

$$lim\ inf \Xi(F\zeta_{n_i}, F\eta, \lambda) \geq lim\ \Xi(\zeta_{n_i}, \eta, \lambda) = \Xi(\eta, \eta, \lambda) = 1$$
$$lim\ sup \Theta(F\zeta_{n_i}, F\eta, \lambda) \leq lim\ \Theta(\zeta_{n_i}, \eta, \lambda) = \Theta(\eta, \eta, \lambda) = 0$$
$$lim\ sup \Upsilon(F\zeta_{n_i}, F\eta, \lambda) \leq lim\ \Upsilon(\zeta_{n_i}, \eta, \lambda) = \Upsilon(\eta, \eta, \lambda) = 0$$

for each $\lambda > 0$. Hence

$$lim\ F\zeta_{n_i} = F\eta....(4.4.2)$$

Similarly

$$lim\ F^2\zeta_{n_i} = lim F^2\eta...(4.4.3)$$

(we recall that $lim\ F\zeta_{n_i} = F\eta$ for all $(i \in \mathbb{N})$), Now observe that,

$$\begin{aligned}
\Xi(\zeta_{n_i}, F\zeta_{n_i}, \lambda) &\leq \Xi(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \leq \cdots \leq \Xi(\zeta_{n_i}, F\zeta_{n_i}, \lambda) \\
&\leq \Xi(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \leq \cdots \leq \Xi(F\zeta_{n_{i+1}}, F^2\zeta_{n_{i+1}}, \lambda) \\
&\leq \Xi(F\zeta_{n_{i+1}}, F^2\zeta_{n_{i+1}}, \lambda) \leq \cdots \leq 1. \\
\Theta(\zeta_{n_i}, F\zeta_{n_i}, \lambda) &\geq \Theta(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Theta(\zeta_{n_i}, F\zeta_{n_i}, \lambda) \\
&\geq \Theta(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Theta(F\zeta_{n_{i+1}}, F^2\zeta_{n_{i+1}}, \lambda) \\
&\geq \Theta(F\zeta_{n_{i+1}}, F^2\zeta_{n_{i+1}}, \lambda) \geq \cdots \geq 0. \\
\Upsilon(\zeta_{n_i}, F\zeta_{n_i}, \lambda) &\geq \Upsilon(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Upsilon(\zeta_{n_i}, F\zeta_{n_i}, \lambda) \\
&\geq \Upsilon(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \geq \cdots \geq \Upsilon(F\zeta_{n_{i+1}}, F^2\zeta_{n_{i+1}}, \lambda) \\
&\geq \Upsilon(F\zeta_{n_{i+1}}, F^2\zeta_{n_{i+1}}, \lambda) \geq \cdots \geq 0.
\end{aligned}$$

for all $\lambda > 0$. Thus $\{\Xi(\zeta_{n_i}, F\zeta_{n_i}, \lambda)\}, \{\Theta(\zeta_{n_i}, F\zeta_{n_i}, \lambda)\}, \{\Upsilon(\zeta_{n_i}, F\zeta_{n_i}, \lambda)\}$ and $\{(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda)\}$ $(\lambda > 0)$ are convergent to a common limit point . So by equations (4.3.1) , (4.3.2) and (4.4.1) and we get,

$$\begin{aligned}
\Xi(\eta, F\eta, \lambda) &\geq lim\ sup\ \Xi(\zeta_{n_i}, F\zeta_{n_i}, \lambda) = lim\ sup\ (F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \\
&\geq lim\ inf \Xi(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \\
&\geq \Xi(F\eta, F^2\eta, \lambda) \\
\Theta(\eta, F\eta, \lambda) &\leq lim\ inf\ \Theta(\zeta_{n_i}, F\zeta_{n_i}, \lambda) = lim\ inf\ \Theta(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \\
&\leq lim\ sup \Theta(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \\
&\leq \Theta(F\eta, F^2\eta, \lambda) \\
\Upsilon(\eta, F\eta, \lambda) &\leq liminf\ \Upsilon(\zeta_{n_i}, F\zeta_{n_i}, \lambda) = lim\ inf\ \Upsilon(F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \\
&\leq lim\ sup \Upsilon(F\zeta F\zeta_{n_i}, F^2\zeta_{n_i}, \lambda) \\
&\leq \Upsilon(F\eta, F^2\eta, \lambda)
\end{aligned}$$

for all $\lambda > 0$. Suppose $b \neq F\eta$, By equation (4.4.1)

$$\Xi(\eta, F\eta, .) < \Xi(F\eta, F^2\eta, .); \quad \Theta(\eta, F\eta, .) > \theta(F\eta, F^2\eta, .); \quad \Upsilon(\eta, F\eta, .) > \Upsilon(F\eta, F^2\eta, .).$$

which is a contradiction , because all the above functions are left continuous , non -decreasing and right continuous , non - increasing respectively. Hence $\eta = F\eta$ is a fixed point.
To prove the uniqueness of the fixed point, let us consider $F(\zeta) = \omega$ for some $\zeta \in \Sigma$.
Then

$$\begin{aligned}
1 &\geq \Xi(\zeta, \omega, \lambda) = \Xi(F\eta, F\omega, \lambda) \geq \Xi(\zeta, \omega, \frac{\lambda}{k}) = \Xi(F\eta, F\omega, \frac{\lambda}{k}) \geq \cdots \geq \Xi(\zeta, \omega, \frac{\lambda}{k^n}) \\
0 &\leq \Theta(\zeta, \omega, \lambda) = \Theta(F\eta, F\omega, \lambda) \leq \Theta(\zeta, \omega, \frac{\lambda}{k}) = \Theta(F\eta, F\omega, \frac{\lambda}{k}) \leq \cdots \leq \Theta(\zeta, \omega, \frac{\lambda}{k^n}) \\
0 &\leq \Upsilon(\zeta, \omega, \lambda) = \Upsilon(F\eta, F\omega, \lambda) \leq \Upsilon(\zeta, \omega, \frac{\lambda}{k}) = \Upsilon(F\omega, F\omega, \frac{\lambda}{k}) \leq \cdots \leq \Upsilon(\zeta, \omega, \frac{\lambda}{k^n})
\end{aligned}$$

Now , we easily verify that $\{\frac{\lambda}{k^n}\}$ is an s - increasing sequence, then by assumption for a given $\epsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\Xi(\zeta, \omega, \frac{\lambda}{k^n}) \geq 1 - \epsilon; \quad \Theta(\zeta, \omega, \frac{\lambda}{k^n}) \leq \epsilon; \quad \Upsilon(\zeta, \omega, \frac{\lambda}{k^n}) \leq \epsilon.$$

Clearly

$$lim_{n\to\infty}\Xi(\zeta, \omega, \frac{\lambda}{k^n}) = 1;\ lim_{n\to\infty}\Theta(\zeta, \omega, \frac{\lambda}{k^n}) = 0;\ lim_{n\to\infty}\Upsilon(\zeta, \omega, \frac{\lambda}{k^n}) = 0.$$

Hence $\Xi(\zeta, \omega, \lambda) = 1; \ \Theta(\zeta, \omega, \lambda) = 0; \ \Upsilon(\zeta, \omega, \lambda) = 0$. Thus $\eta = \omega$. Hence proved.

Conclusion: In this study, we have investigated the concept of Neutrosophic Metric Space and its properties. We have proved fixed point results for contraction theorems in the setting of neutrosophic metric Space. There is a scope to establish many fixed point results in the areas such as fuzzy metric, generalized fuzzy metric, bipolar and partial fuzzy metric spaces by using the concept of Neutrosophic Set.

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Neutrosophy for Survey Analysis in Social Sciences

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Carlos Rosero Martínez, German Acurio Hidalgo, Marvelio Alfaro Matos, Florentin Smarandache (2020). Neutrosophy for Survey Analysis in Social Sciences. *Neutrosophic Sets and Systems* 37, 409-416

Abstract. The survey is a research procedure used in sociology to determine the thoughts and feelings of a social group at a given time and context. Within the survey, the questionnaire is considered as a very useful instrument used to measure the state of opinions of social groups. Although it has been demonstrated that fuzzy responses to questionnaires are more appropriate than crisp ones, there may be indeterminacy and thus fuzzy processing does not accurately capture the thought that the respondent wants to express, due to doubts, unclear and vague thoughts, among others. Modeling such scenario by means of neutrosophic sets provides respondents a greater range of possible responses and hence it is more appropriate. In this paper, we propose a method to design single-valued neutrosophic sets from questionnaires to social groups. This method, inspired by another fuzzy one, allows us to create membership functions of truthfulness, indeterminacy and falseness through experimental data, which will let us find the essence of the thought of the human group under study to be captured with greater accuracy.

Keywords: Neutrosociology, survey, questionnaire, single-valued neutrosophic set.

1 Introduction

Sociology is the social science that studies the collective phenomena produced by the social activity of human beings, within the historical-cultural context in which they are immersed. In sociology, multiple interdisciplinary research techniques are used to analyze and interpret from different theoretical perspectives the causes, meanings and cultural influences that motivate the appearance of various behavioral trends in the human being, especially when it is in social coexistence and within a shared habitat. One of the most widely used research methods is the survey.

A survey is a research procedure, within descriptive research designs (not experimental) in which the researcher seeks to collect data through a previously designed questionnaire or an interview with someone, without modifying the environment or the phenomenon where the information is collected (just like in an experiment), [1]. The data are obtained by carrying out a set of standardized questions addressed to a representative sample or to the total set of the statistical population under study, often made up of people, companies or institutional entities, in order to know states of opinion, ideas, characteristics or facts. The researcher must select the most suitable questions, according to the nature of the investigation.

On the other hand, a questionnaire is a research instrument that consists of a set of questions and other indications to obtain information from those consulted [1,2]. Although they are often designed to allow statistical analysis of responses, this is not always the case. The questionnaire is a document formed by a set of questions that must be drafted in a coherent way. Those questions must be organized, sequenced and structured according to a certain planning, so that answers can offer us all the required information.

The survey is often carried out based on a questionnaire, which is therefore the basic document to obtain information in the vast majority of research and market studies. Questionnaires have advantages over other types of surveys in that they are inexpensive, do not require much effort on the part of the respondent, such as oral or telephone surveys, and often have standardized responses that make data tabulation simpler.

In sociology, surveys are usually designed such that the possible responses to the questionnaires are fixed values. An example of a sociological questionnaire is the opinion on the number of children that an ideal family should have, which can force respondents to answer with a number (2, 3, 4) even though the respondent wishes to answer more exactly, although imprecise as in interval form such as 2 to 4, [3]. Some authors have studied and demonstrated the fuzzy rather than crisp essence of surveys, [3-7]. Fuzzy sets have been proven more effective in dealing with measurements related to human thought than classical sets.

In this paper, we defend the thesis that neutrosophic sets are even more suitable than fuzzy sets to represent the possible responses to questionnaires. The former one allow the surveyed person to be able to express even more accurately and also with greater indeterminacy about their true thoughts and feelings, due to the indeterminacy membership function [8], which allows modeling the lack of knowledge, doubts or contradictions that may exist in the responses of any human being.

Neutrosophic Sociology or Neutrosociology is the study of sociology using neutrosophic scientific methods, [9-13], because the data of sociology can be vague, incomplete, contradictory, hybrid, biased, ignorant, redundant, superfluous, meaningless, ambiguous, and unclear, among others. In this new approach to the study of sociology, the concepts are represented in the form of <A>, which is the primary concept, <Anti A>, which is its opposite, and <Neut A>, which represents those that are neither <A> nor <Anti A>.

In this paper, we are inspired by a method in [4] for the construction of fuzzy membership functions [14,15] to construct neutrosophic sets as a result of the responses of a survey by a group of individuals under study. To design a priori fuzzy membership functions or neutrosophic sets is not sufficient and yet it is of great interest finding a more adequate application of these theories. With the use of neutrosophic sets instead of fuzzy sets, greater accuracy of the results is obtained, since the single-valued neutrosophic sets allow a greater range of expression of the thoughts and feelings of the respondents, since they cannot only express their ideas, but also what they consider false and what they consider indeterminate. This method, like its predecessor, stands out for its simplicity and applicability.

This paper is structured into the following sections: Section 2, which recalls the main concepts of Neutrosophy that will be used in the proposed method. In Section 3, we introduce the method proposed in the paper and we develop two illustrative examples. The last section contains the conclusions.

2 Basic concepts of Neutrosophy

This section describes the main concepts of Neutrosophy, such as neutrosophic sets, single-valued neutrosophic sets, and single-valued neutrosophic numbers, among others. In addition, the main definitions of neutrosophic statistics are described.

Definition 1: ([8]) Let X be a universe of discourse. A *Neutrosophic Set* (NS) is characterized by three membership functions, $u_A(x), r_A(x), v_A(x) : X \rightarrow\ ]^-0, 1^+[$, which satisfy the condition $^-0 \leq \inf u_A(x) + \inf r_A(x) + \inf v_A(x) \leq \sup u_A(x) + \sup r_A(x) + \sup v_A(x) \leq 3^+$ for all x∈X. $u_A(x), r_A(x)$ and $v_A(x)$ are the membership functions of truthfulness, indeterminacy and falseness of x in A, respectively, and their images are standard or non-standard subsets of $]^-0, 1^+[$.

Definition 2: (8]) Let X be a universe of discourse. A *Single-Valued Neutrosophic Set* (SVNS) A on X is a set of the form:

$$A = \{\langle x, u_A(x), r_A(x), v_A(x)\rangle : x \in X\}c \quad (1)$$

Where $u_A, r_A, v_A : X \rightarrow [0,1]$, satisfy the condition $0 \leq u_A(x) + r_A(x) + v_A(x) \leq 3$ for all x∈X. $u_A(x), r_A(x)$ and $v_A(x)$ denotes the membership functions of truthfulness, indeterminate and falseness of x in A, respectively. For convenience a *Single-Valued Neutrosophic Number* (SVNN) will be expressed as $A = (a, b, c)$, where $a, b, c \in [0,1]$ and satisfy $0 \leq a + b + c \leq 3$.

Definition 3: (8]) A *neutrosophic number* N is defined as a number in the following expression:

$$N = d + I \quad (2)$$

Where d is called *determinate part* and I is called *indeterminate part*.

Given $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ two neutrosophic numbers, some operations between them are defined as follows:

$N_1 + N_2 = a_1 + a_1 + (b_1 + b_2)I$ (Addition);

$N_1 - N_2 = a_1 - a_1 + (b_1 - b_2)I$ (Difference),

$N_1 \times N_2 = a_1 a_2 + (a_1 b_2 + b_1 a_2 + b_1 b_2)I$ (Multiplication),

$\frac{N_1}{N_2} = \frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} I$ (Division).

Neutrosophy studies triads, where if <A> is an item or a concept then the triad is (<A>, <neut A>, <anti A>),[9,10]. Neutrosociology is based on triads. E.g., the concept A = imperialist society, has an antiA = communist society, and neutA = neutral society.

Neutrosophic Statistics extends the classical statistics, such that we deal with set values rather than crisp values, [16-22]. Neutrosophic Statistics can be used as a quantitative research method in sociology for testing social hypotheses.

Neutrosophic Descriptive Statistics is comprised of all techniques to summarize and describe the neutrosophic numerical data characteristics.

Neutrosophic Inferential Statistics consists of methods to allow the generalization of a neutrosophic sampling to a population from which the sample was selected.

Neutrosophic Data is the piece of information that contains some indeterminacy. Similar to the classical statistics, it can be classified as:

- *Discrete neutrosophic data*, if the values are isolated points.
- *Continuous neutrosophic data*, if the values form one or more intervals.

Another classification is:

- *Quantitative (numerical) neutrosophic data*; for example: a number in the interval [2, 5] (we do not know exactly), 47, 52, 67 or 69 (we do not know exactly);
- *Qualitative (categorical) neutrosophic data*; for example: blue or red (we do not know exactly), white, black or green or yellow (not knowing exactly).

The *univariate neutrosophic data* is a neutrosophic data that consists of observations on a neutrosophic single attribute.

Multivariable neutrosophic data is neutrosophic data that consists of observations on two or more attributes.

A *Neutrosophic Statistical Number* N has the form $N = d + i$, like Equation 2.

A *Neutrosophic Frequency Distribution* is a table displaying the categories, frequencies, and relative frequencies with some indeterminacies. Most often, indeterminacies occur due to imprecise, incomplete or unknown data related to frequency. Therefore, relative frequency becomes imprecise, incomplete, or unknown too.

Neutrosophic Survey Results are survey results that contain some indeterminacy.

A *Neutrosophic Population* is a population not well determined at the level of membership (i.e. not sure if some individuals belong or not to the population).

A *simple random neutrosophic sample* of size n from a classical or neutrosophic population is a sample of n individuals such that at least one of them has some indeterminacy.

A *stratified random neutrosophic sampling* the researcher groups the (classical or neutrosophic) population by a strata according to a classification; afterwards the researcher takes a random sample (of appropriate size according to a criterion) from each group. If there is some indeterminacy, we deal with neutrosophic sampling.

3 Application of neutrosophic theory in sociological surveys

In the study carried out by Li in [4] about how to measure the people's thoughts, the author acknowledges the existence of possible responses like "1 or 2 (sorry)" with respect to the size of an small family, whereas other answer is "not an exact age" for the question about the exact age of a "young person". Thus, it is necessary to include the indeterminacy like a possible result of a survey. On the other hand, Li deals with indeterminacy when the range of responses is an interval rather than a single value.

In this section, we deal with indeterminacy based on single-valued neutrosophic sets and Neutrosociology concepts. The method consists of the following aspects:

1. Firstly, the sociologist must determine the primary concept he/she wants to measure, e.g., A = "small family". Next, he/she determines anti A, e.g. "big family", and neut A, e.g. "optimal family". In addition, he/she establishes the social group to analyse.
2. He/she asks to the group the questions he/she designed aiming to have information about the triad (<A>, <neut A>, <anti A>). Every question should have three variants, one of them related to one of the three elements of the triad.
   The ambiguous or vague answers like "I don't know", "certain number", and so on are associated with <neut A>, even though they were responses for questions of <A> or <anti A>.
   The interviewer remarks that the responses can be given in form of intervals in case it makes sense or if respondent considers it better corresponds to his/her opinions.
   Questionnaires can also include answers in form of linguistic values.
   The respondent should feel free to write what he/she thinks on the subject of the questions.

Let us denote as $X_j = \{x_i^j\}_{i=1}^{m_j}$ the set of possible responses to question q_j (j = 1, 2,…, n).

The frequency of every possible response is calculated for every element of the triad, let us call them $f_{<A>}(x_i^j)$, $f_{<neut\ A>}(x_i^j)$, and $f_{<anti\ A>}(x_i^j)$.

If N is the size of the social group to study, we calculate the following probabilities:

$$p_{<A>}(x_i^j) = \frac{f_{<A>}(x_i^j)}{N} \qquad (3)$$

$$p_{<neut\ A>}\left(x_i^j\right) = \frac{f_{<neut\ A>}\left(x_i^j\right)}{N} \tag{4}$$

$$p_{<anti\ A>}\left(x_i^j\right) = \frac{f_{<anti\ A>}\left(x_i^j\right)}{N} \tag{5}$$

The properties of $p_{<A>}\left(x_i^j\right)$, $p_{<neut\ A>}\left(x_i^j\right)$, and $p_{<anti\ A>}\left(x_i^j\right)$ are the following:

- For every X_j then $p_{<A>}\left(x_i^j\right), p_{<neut\ A>}\left(x_i^j\right), p_{<anti\ A>}\left(x_i^j\right) \in [0,1]$.
- For every X_j then $\sum_{i=1}^{m_j}\left(p_{<A>}\left(x_i^j\right) + p_{<anti\ A>}\left(x_i^j\right)\right) \leq 1$.
- For every X_j then $\sum_{i=1}^{m_j}\left(p_{<A>}\left(x_i^j\right) + p_{<neut\ A>}\left(x_i^j\right) + p_{<anti\ A>}\left(x_i^j\right)\right) \geq 1$.

Let us remark that the probabilities $p_{<A>}\left(x_i^j\right)$ and $p_{<anti\ A>}\left(x_i^j\right)$ should satisfy the property of subjective probability approach, [23], whereas, when $p_{<neut\ A>}\left(x_i^j\right)$ is included then the sum can exceed the unity. This is because of $p_{<neut\ A>}\left(x_i^j\right)$ and the others two may have common answers for some individuals.

Now, for every concept A the sociologists have a single-valued neutrosophic set defined as follows:

$$A = \left\{\langle x, \min_j\left(p_{<A>}\left(x_i^j\right)\right), \max_j\left(p_{<neut\ A>}\left(x_i^j\right)\right), \max_j\left(p_{<anti\ A>}\left(x_i^j\right)\right)\rangle : x \in \Pi_{j=1}^n X_j\right\} \tag{6}$$

Let us note that Π is the Cartesian product and the set A contains the definition of n-norm, [17]. Also, let us remark we are using neutrosophic statistics with neutrosophic data.

The single-valued neutrosophic set A can be de-neutrosophied to a crisp set where the elements of the triad are reduced to numerical values using the scoring function or a precision index.

A scoring function $s: [0,1]^3 \to [0,3]$ is defined in Formula 7, it is an adapted scoring function from the one defined in [24].

$$s(a) = 2 + T - F - I \tag{7}$$

Where a is a SVNN with values (T, I, F).

The definition of precision index is given in Equation 8.

$$a(a) = T - F \tag{8}$$

Where $a: [0,1]^3 \to [-1,\ 1]$.

Below, we illustrate the method through two examples.

Example 1

Here, we revisit the example appeared in [4]. The survey aims to investigate what people considers is an ideal family size, thus <A> = <ideal family size>, <anti A> = <non- ideal family size>, and <neut A> = <indeterminate ideal family size>. Let us note we are dealing with three variants of the same concept instead of only one of them. Here, we use only one question, which is:

1. Use any number (0, 1,2,...) or any range (1-4, 2-3,...) to indicate your perception of:
   1.1. the ideal family size.
   1.2. you cannot determinate it is neither ideal nor not an ideal family size.
   1.3. non- ideal family size.

Let us assume that the population contains 6 respondents, which answer in the following way, where $R_i = \left(R_i^{<A>},\ R_i^{<neut\ A>},\ R_i^{<anti\ A>}\right)$, correspond to the responses given by the i-th respondent for the triad $(< A >, < neut\ A >,\ < anti\ A >)$, respectively:

$$R_1 = (\{1,2,3,4\}, \{5\}, \{0,6,7,8,9,10\})$$

$$R_2 = (\{2\}, \{3,4\}, \{0,1,5,6,7,8,9,10\})$$

$$R_3 = (\{2,3\}, \{1\}, \{0,4,5,6,7,8,9,10\})$$

$$R_4 = (\{1,2\}, \{0\}, \{3,4,5,6,7\})$$

$$R_5 = (\{0\}, \emptyset, \{x: x > 0\})$$

$$R_6 = (\{2,3,4\}, \{5,6\}, \{1,7,8,9,10\}).$$

That means, e.g., the first respondent thinks the ideal family size (number of children) is from 1 to 4, whereas to have not child or more than 6 is not ideal, however, 5 children is indeterminate for him/her. Contrarily, respondent 5 is against to have any child.

Table 1 summarizes the frequency of each possible response in the example:

Responses X_1	$f_{<A>}(x_i^1)$	$f_{<neut\ A>}(x_i^1)$,	$f_{<anti\ A>}(x_i^1)$.
0	1	1	3
1	2	1	2
2	5	0	1
3	3	1	2
4	2	1	3
5	0	1	4
6	0	1	5
7	0	0	6
8	0	0	5
9	0	0	5
10	0	0	5

Table 1: Frequencies of the responses.

The probabilities are obtained dividing the frequencies by N = 6. The truthfulness, indeterminacy and falseness membership functions are depicted in Figure 1.

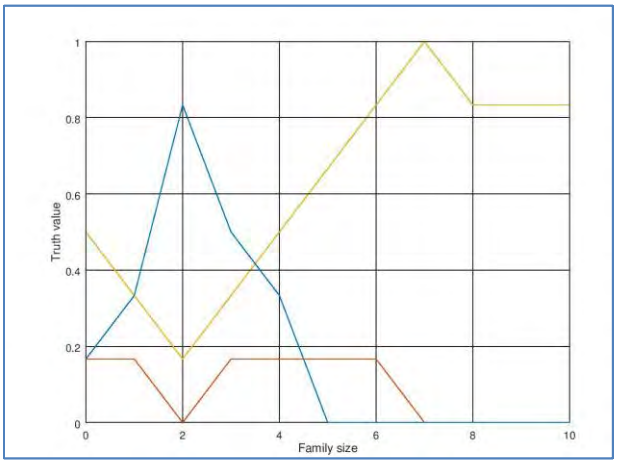

Figure 1: Truthfulness-membership function in blue lines, indeterminacy-membership function in red lines, and falseness-membership function in yellow lines, for the concept "ideal family size".

In Figure 2 shows the scoring function using Equation 7 for the possible responses about the concept "ideal family size".

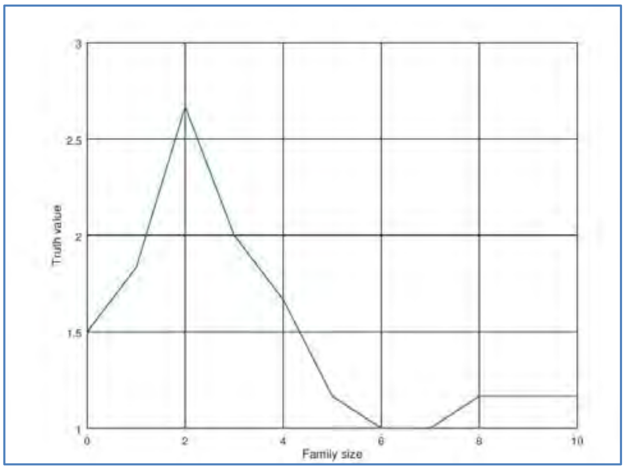

Figure 2: Scoring function of the single-valued neutrosophic set in Figure 1.

Evidently, the ideal family size can be considered equal to 2 for this social group.

Example 2

In a community, sociologists want to know how members perceive two concepts: *young* and *educated*. To do this, they design a questionnaire, one where the triad is that of (<young>, <middle-aged>, <old>), while the second triad is (<instructed>, <borderline instruction>, <unlearned>).

Questions are:

1. How old must a person be to be considered:
    1.1. young?
    1.2. middle-aged?
    1.3. old?
2. ¿What level of education must a person have to be considered:
    2.1. instructed?
    2.2. borderline educated?
    2.3. not educated?

For the first question, the possible answer is an age between 0 and 120 years old, it can also be expressed in the form of intervals, that is, $X_1 = \{G: G \subset [0, 120]\}$.

For the second question, the possible answers are: "primary level of education", "secondary level of education", "upper secondary level of education", "higher level of education", and "MSc. or PhD degrees ”, these are the elements of X_2.

Suppose the population of study consists of 180 members. The results are shown in Figure 3:

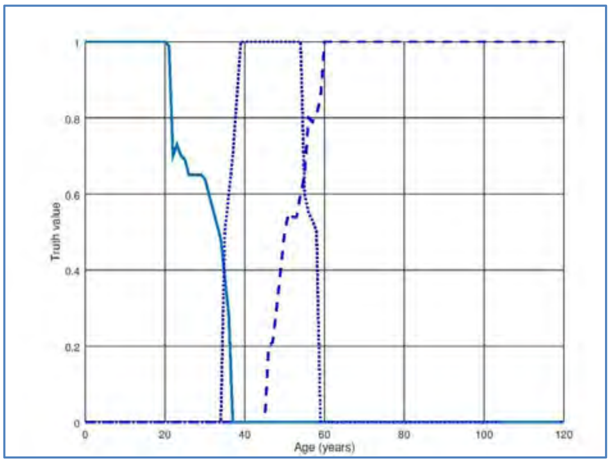

Figure 3: Truthfulness-membership function in solid lines, indeterminacy-membership function in dotted lines, and falseness-membership function in dashed lines, for the concept “young”.

In Figure 4 it is depicted the scoring function of the triad related with young people.

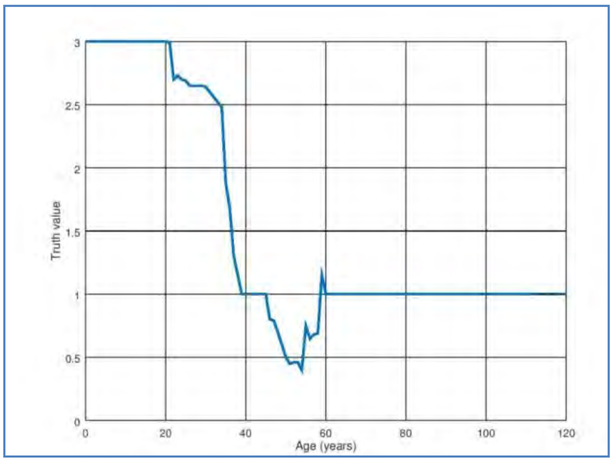

Figure 4: Scoring function for the single-valued neutrosophic set in Figure 3.

Regarding the level of instruction, let us assume that the results were the following:

- Primary level of education has the triple $(0, 0.01, 0.93)$,

- Secondary level of education has the triple $(0.1, 0.6, 0.8)$,
- Upper secondary level of education has the triple $(0.6, 0.4, 0.1)$,
- Higher level of education has the triple $(1, 0.1, 0)$, and
- MSc. or PhD degrees has the triple $(1,0,0)$.

Thus, to define the conjunction of young and instructed person, it is necessary to obtain the Cartesian product between the pair age and education level, where the n-norm of the triple of each of them is calculated. E.g., one young 20 years old person AND having a primary level of instruction has a triad value obtained since the n-norm between $(1, 0, 0)$ for young and $(0, 0.01, 0.93)$ for educated, which results in $(0, 0.01, 0.93)$ for this combination. Calculating the scoring function we have the value -0.94, thus it is very low.

Conclusion

This paper introduces a neutrosophic method for survey analysis in social sciences. The new method is inspired by another one where fuzzy sets were used. The advantage of the neutrosophic approach is that the respondents can express more accurately their thoughts and feelings, because indeterminacy is considered as well as an independent membership function of falseness. The method consists of designing a single-valued neutrosophic set from the collected data. This set serves to evaluate the satisfaction of a concept by a social group. The method is also based on the Neutrosociology theory, where the set A includes the notion of the triad of the aforementioned theory. This neutrosophic approach is applied to questionnaires where both discrete numerical and linguistic responses are possible.

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Neutrosophic Sociogram for Group Analysis

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Abstract. The sociogram is a technique of sociometry widely used in the field of sociology due to its simplicity and effectiveness. The purpose of this method is the graphical visualization of the relationships among the members of a social group. The sociogram has been extended to the fuzzy framework to include uncertainty in the so-called fuzzy sociograms. However, there could be indeterminate relationships among some members of the group, because they have not experience in performing some activities together, although potentially either future links or disagreements could be established among them. In this paper, we propose a neutrosophic sociogram, which allows representing the indeterminacies in the relationships among some members of a group. The advantage of neutrosophic sociograms over fuzzy sociograms is that the representation and calculation considering indeterminacy, allow us to achieve greater accuracy in the results, and a greater approach to the potentialities of the group in terms of the future bond among the members. A hypothetical example is proposed to illustrate the applicability of the method.

Keywords: Sociogram, neutrosophic sociogram, Neutrosociology, group analysis, sociometry analysis.

1 Introduction

The sociogram is a data analysis technique that focuses its attention on the way in which social relationships are established within any group, [1]. Jacob Levy Moreno, a Romanian psychiatrist, developed the technique in the mid-30s of the 20th century as a tool for exploratory and diagnostic purposes. Since its creation, sociometry appears as one of the most advanced and ordered strategies to describe and measure group dynamics, since it allows the quantitative study of interpersonal relationships in groups. The sociogram is an important example within sociometry.

In essence, the sociogram allows us to study the existing interpersonal preferences in a group of people. Currently it is widely used in various organizational settings, from small schools to large companies. It is also used in intelligence work in order to detect criminal networks. They can be briefly defined as graphics or tools used to determine the sociometry of a social space.

A social bond is a set of social relationships established between two or more individuals, which together, results in a group of social interaction, that is, when several members establish social bonds between them, forming a small social group. The social position is the specific place that every member occupies either in relation to the group of interaction or to the group in general.

This way, when applying a sociometric test or sociogram in a social group, the researcher may have knowledge of the way in which the group is socially related to each other, as well as the benefits and repercussions that this interaction has on each one of the members individually. This is very useful, since many times the degree of integration of an individual directly influences their performance. It is not groups dynamic but an easy-to-apply technique that can help us to better understand the world of relationships that is established in a social group.

Specifically, the sociogram starts from a questionnaire applied to the social group under investigation, where each member of the group specifies, in order of preference, with which other members they would like to carry out the activities asked in the questionnaire. This way, it starts with a matrix that is represented in the form of a graph, where the individual of the group preferred by the others and the isolated individual are determined.

In the classical sociogram, each member evaluates their preference through crisp values; however, some authors introduce the uncertainty that exists in these relationships, by using fuzzy graphs instead of crisp graphs with the so-called fuzzy sociogram, [2, 3]. Others make this type of graph even more complex with the definition of fuzzy graphs for polyfactor analysis, that is, fuzzy graphs that allow us studying more than one relationship between members of the social group. Some of these methods link this tool with classic cooperative game solutions such as Shapley value [2]. Sociograms can be applied in more than one moment to measure the change in relationships within the group.

It is not difficult to accept that the relationships between the members of the social groups may contain indeterminacies. Some members of the group may not know each other well, or may doubt on the behavior of the other in some activity. Therefore, in the classical sociogram and in the fuzzy sociogram it is not differentiated whether there is a mutual rejection between these individuals and therefore there is no possibility of a future relationship, or there is simply a potential bond that has not developed yet.

This fact has motivated the authors to propose a neutrosophic sociogram, where indeterminacy is included as part of the relationships between two individuals, because they are not well known, or there has been no possibility of creating a link between them or they have not determined the impossibility of such relationship.

Neutrosophy has served as the basis for sociology with the so-called Neutrosophic Sociology or Neutrosociology, which is defined as the study of sociology using neutrosophic scientific methods, [4, 5]. There are also neutrosophic graphs that allow us to measure concepts using graphs within the framework of Neutrosophy.

In this paper, neutrosophic sociograms are introduced, where the relationships among the members of a social group are graphically represented and quantitatively measured, including the indeterminacy of these relationships. Indeterminate relationships are considered as potential relationships in short, medium or long term, therefore it is a more accurate indicator than sociograms or fuzzy sociograms, since it guarantees a more precise measurement of group dynamics.

The paper is structured into the following sections: section 2 contains the main concepts related to sociograms and Neutrosophy. In section 3 the method proposed in this paper is introduced and a hypothetical example is used to illustrate how to apply it. The last section contains the conclusions.

2 Preliminaries

In this section, we summarize the main concepts of sociogram and Neutrosophy that will be used in this paper.

2.1 Sociogram

A sociogram is a graph that represents the relationship among the members of a social group. Firstly, the social group is identified. Then the investigator explains to the members the objective of the research. Next, the investigator designs a questionnaire for each member about the other members of the group he/she prefers to join in certain activities. E.g., in a group of students the teacher can ask every one of the members the following three questions [1]:

In order of preference, write the friends with whom
Q_1 : you want to join a quiz program.
Q_2 : you want to study in group.
Q_3 : you want to do volunteer activity.
Let us assume $S = \{s_1, s_2, \cdots, s_n\}$ denotes the set of interviewed. The results are represented in Table 1:

	Q_1	Q_2	Q_3
s_1	S_{11}	S_{12}	S_{13}
s_2	S_{21}	S_{22}	S_{23}
⋮	⋮	⋮	⋮
s_n	S_{n1}	S_{n2}	S_{n3}

Table 1: Generic table representing the relationship among the members of the social group.

The elements of Table 1 are the sets of members $S_{ij} \subset S$ $(i = 1,2,\dots,n)(j = 1,2,3)$ such that the member s_i has chosen for answering the j-th question (Q_1, Q_2, or Q_3).

The classical sociogram is formed from a square matrix where every member of S is represented in one row and one column, such that elements of the matrix contain one number from 1 to 3, which is used by every s_k to evaluate his/her preference for member s_l.

The results are depicted in a directed graph, where every node represents a member of the social group and the edges E_{kl} represent that k-th member of the group selected the l-th member. An example of sociogram is depicted in Figure 1.

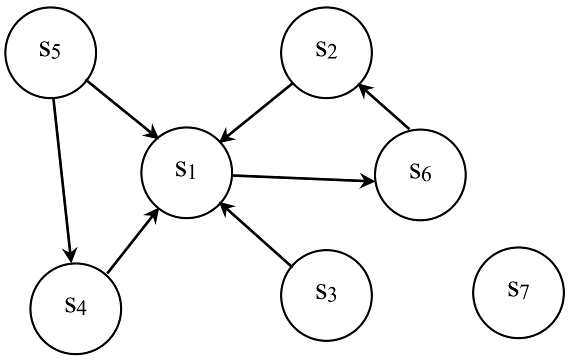

Figure 1: Example of sociogram of a group with seven members.

For example, in Figure 1 a social group of 7 members is investigated, where every node represents a member and every edge represents that one member prefers the other. Let us note in the example most of the members preferred s_1, while s_7 is isolated, he/she does not prefer anybody and nobody prefers him/her.

In the crisp sociogram, the graph is the final result, whereas in fuzzy sociogram the strength of every node (member) is measured with a function. f: $\{1, 2, \cdots, n\} \rightarrow [0, 1]$, where the closer is f(i) to 0 the more isolated member *i* is, thus it is an unpopular member possibly discriminated by the others, and the closer is f(i) to 1 the more linked member *i* is, then, *i* is a popular member or possibly the group's leader. This function can depend on fuzzy operators like t-norms or compensatory ones.

On the other hand, the preferred member can be selected using Shapley value [2]. Sometimes dendrograms are used to represent the sociogram [3].

2.2 Basic concepts on Neutrosophy

Definition 1: [6] Let X be a universe of discourse. A *Neutrosophic Set* (NS) is characterized by three membership functions, $u_A(x), r_A(x), v_A(x) : X \rightarrow]^-0, 1^+[$, which satisfy the condition $^-0 \leq \inf u_A(x) + \inf r_A(x) + \inf v_A(x) \leq \sup u_A(x) + \sup r_A(x) + \sup v_A(x) \leq 3^+$ for all x∈X. $u_A(x), r_A(x)$ and $v_A(x)$ denote the membership functions of truthfulness, indetermination and falsehood of x in A, respectively, and their images are standard or non-standard subsets of $]^-0, 1^+[$.

NS are useful only as a philosophical approach, so a *Single-Valued Neutrosophic Set* is defined to guarantee the applicability of Neutrosophy, see Definition 2.

Definition 2: ([6]) Let X be a universe of discourse. A *Single-Valued Neutrosophic Set* (SVNS) A on X is an object of the form:

$$A = \{\langle x, u_A(x), r_A(x), v_A(x)\rangle : x \in X\} \quad (1)$$

Where $u_A, r_A, v_A : X \rightarrow [0,1]$, satisfy the condition $0 \leq u_A(x) + r_A(x) + v_A(x) \leq 3$ for all x∈X. $u_A(x), r_A(x)$ and $v_A(x)$ denote the membership functions of truthfulness, indetermination and falsehood of x in A, respectively. For convenience, a *Single-Valued Neutrosophic Number* (SVNN)[7, 8] will be expressed as $A = (a, b, c)$, where $a, b, c \in [0,1]$ and satisfies $0 \leq a + b + c \leq 3$.

Neutrosophic Logic (NL) extends fuzzy logic. As stated by Florentin Smarandache, its author, a proposition P is characterized by three components; see [9-12]:

$$NL(P) = (T,I,F) \quad (2)$$

Where component T is the degree of truthfulness, F is the degree of falsehood and I is the degree of indetermination. T, I and F belong to the interval [0, 1], and they are independent from each other.

A *neutrosophic number* is formed by the algebraic structure a+bI, where I = indetermination. Below we formally describe some important concepts.

Definition 3: ([13-18]) Let R be a ring. The *neutrosophic ring* $\langle R \cup I\rangle$ is also a ring, generated by R and I under the operation of R, where I is a neutrosophic element that satisfies the property $I^2 = I$. Given an integer n, then, n+I and nI are neutrosophic elements of $\langle R \cup I\rangle$ and in addition $0 \cdot I = 0$. Also, I^{-1}, the inverse of I is not defined.

E.g., a neutrosophic ring is $\langle \mathbb{Z} \cup I\rangle$ generated by $\mathbb{Z}$, which is the set of integers.

Some operation using I is $I + I + \ldots + I = nI$.

Definition 4: ([19, 20]) A *neutrosophic number* N is also defined as a number:

$$N = d + I \quad (3)$$

Where d is the *determined part* and I is the *indeterminate part* of N.

Example 1. N = 1+I, where 1 is the determined part and I is the indeterminate part, and for I = [0, 1] we have N = [1, 2].

Let $N_1 = a_1 + b_1 I$ and $N_2 = a_2 + b_2 I$ be two neutrosophic numbers, then some operations between them are:

1. $N_1 + N_2 = a_1 + a_1 + (b_1 + b_2)I$ (Addition),

2. $N_1 - N_2 = a_1 - a_1 + (b_1 - b_2)I$ (Difference),
3. $N_1 \times N_2 = a_1a_2 + (a_1b_2 + b_1a_2 + b_1b_2)I$ (Multiplication),
4. $\frac{N_1}{N_2} = \frac{a_1+b_1I}{a_2+b_2I} = \frac{a_1}{a_2} + \frac{a_2b_1-a_1b_2}{a_2(a_2+b_2)}I$ (Division).

A *neutrosophic matrix* is a matrix whose components are elements of $\langle R \cup I \rangle$.

Thus, it is possible to generalize the operations between vectors and matrices on R to the ring $\langle R \cup I \rangle$. See Example 2.

Example 2. Given two matrices, $A = \begin{pmatrix} 3 & 9 \\ -1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 8 & I \\ 1 & 3 & 2I \end{pmatrix}$, $AB = \begin{pmatrix} 12 & 51 & 21I \\ 6 & 13 & 13I \end{pmatrix}$.

A *neutrosophic graph* is a graph with at least one neutrosophic edge linking two nodes, that is to say, there is an edge with an indetermination on its two nodes connection, [6, 21-23], see Figure 2.

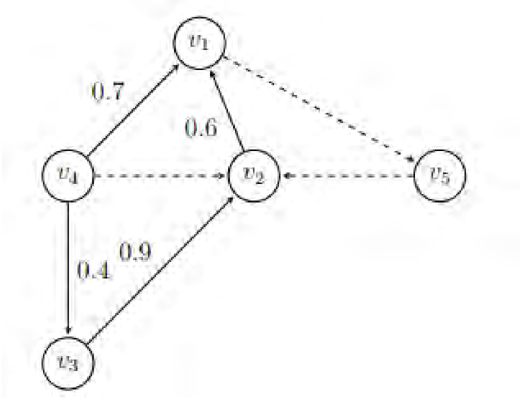

Figure 2: Example of neutrosophic graph. Source [6].

The de-neutrosophication process was introduced by Salmeron and Smarandache in [19], which converts a neutrosophic number in one numeric value. This process provides a range of numbers for centrality using as a base the maximum and minimum values of $I = [a_1, a_2] \subseteq [0, 1]$, based on Equation 4:

$$\lambda([a_1, a_2]) = \frac{a_1 + a_2}{2} \quad (4)$$

3 Neutrosophic sociogram

In this section, we introduce for the first time the concepts of neutrosophic sociograms. Firstly, the interviewers have to explain to the members of the social group the goal for applying the questionnaire and the type of possible answers required by the researchers[24, 25].

The new questionnaire is a variant of that summarized in Table 1. Now, we have $Q_1, Q_2, \ldots, Q_m$ the questions to be answered. Again, $S = \{s_1, s_2, \cdots, s_n\}$ denotes the set of interviewed.

The possible questions are the following:
In order of preference, write the friends with whom:
Q_1 : you want to join quiz program.
Q_2 : you want to study in group.
Q_3 : you want to do volunteer activity.
Apart, write the members of the group with whom:
Q_1 : you are not sure to join quiz program.
Q_2 : you are not sure to study in group.
Q_3 : you are not sure to do volunteer activity.

With this new method we maintain the elements of Table 1 like $S_{ij} \subset S$ $(i = 1,2, \ldots, n)(j = 1,2, \cdots, m)$ meaning the answers of s_i about his/her preferred members for doing activity asked in Q_j. Additionally, $O_{ij} \subset S$ $(i = 1,2, \ldots, n)(j = 1,2, \cdots, m)$ means the list of the members of the group which s_i is not sure to join in the activity asked in question Q_j, they satisfy $S_{ij} \cap O_{ij} = \emptyset$. Also, interviewer provides a weight to every question, which is denoted by $\Omega = \{\omega_1, \omega_2, \cdots, \omega_m\}$, where $\sum_{j=1}^{m} \omega_j = 1$ and $\omega_j \in [0, 1]$.

Then Table 1 converts into Table 2, where sets O_{ij} are included.

	Q_1	Q_2	$\cdots$	Q_m
s_1	$S_{11}; O_{11}$	$S_{12}; O_{12}$	$\cdots$	$S_{1m}; O_{1m}$
s_2	$S_{21}; O_{21}$	$S_{22}; O_{22}$	$\cdots$	$S_{2m}; O_{2m}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s_n	$S_{n1}; O_{n1}$	$S_{n2}; O_{n2}$	$\cdots$	$S_{nm}; O_{nm}$

Table 2: Generic table representing the relationship among the members of the social group for the neutrosophic sociogram.

According to Table 2, the interviewed has also the possibility to include those members of the group whom he/she is not sure to carry out the activity. We consider this indeterminate selected group is the potential extension of the links among the members of the group. The advantage is that we can influence those imprecise relationships to strength the group unity, instead of carrying out some external exercise, e.g. didactic activity in the group class, and later to apply another sociogram to study the dynamical changes in the social group.

Using Table 2 the *evaluation matrix* $R^j = \left(r_{kl}^j\right)$, where r_{kl}^j is the number of times (0 or 1) that s_k selects s_l in Q_j. When k = l we define $r_{kl}^j = 1$.

Thus, $F = \sum_{j=1}^{m} \omega_j R^j$, $F = (f_{kl})$ and if k = l we have $f_{kl} = 1$. f_{kl} means the degree of preference of s_l by s_k. If $f_{kl} = 1$ then s_k strongly prefers s_l and $f_{kl} = 0$ means s_k never prefers s_l.

The *fuzzy amicable degree* g_{kl} between s_k and s_l is calculated through formula 5:

$$\frac{2}{g_{kl}} = \frac{1}{f_{kl}} + \frac{1}{f_{lk}} \tag{5}$$

Where the arithmetic $1/_0 = \infty$ and $1/_\infty = 0$ is used.

Equivalently, $T^j = \left(t_{kl}^j\right)$, where t_{kl}^j is the number of times that s_k selects or hesitates about s_l in Q_j (0 or 1), $T = \sum_{j=1}^{m} \omega_j T^j$. When k = l we define $t_{kl}^j = 1$. Matrix T determines the preferences of s_l by s_k or the possibility that he/she would prefer him/her in the future. Therefore, the *neutrosophic amicable degree* u_{kl} between s_k and s_l is calculated with Equation 6:

$$\frac{2}{u_{kl}} = \frac{1}{t_{kl}} + \frac{1}{t_{lk}} \tag{6}$$

The fuzzy sociogram is represented with the elements of F, whereas the *neutrosophic sociogram* is a neutrosophic graph, such that the elements of the fuzzy sociogram are represented with continuous lines, and the other edges are represented with dashed lines. Every edge of the neutrosophic sociogram is associated with the fuzzy value g_{kl} and the other edges are associated with symbol I. Let us note that we are dealing with non-directed graphs.

The interval of indeterminacy is calculated as $I_{kl} = [g_{kl}, u_{kl}]$. $\lambda(I_{kl})$ indicates a unique value for representing the amicable relationship between s_k and s_l, according to Equation 4, whereas $I_{kl} = u_{kl} - g_{kl}$ measures the degree of indeterminacy.

The leadership of the k-th member of the group is measured with the following index [2]:

$$\mu(k) = \frac{\sum_l g_{kl}}{\sum_k \sum_l g_{kl}} \tag{7}$$

Additionally, the potential leadership of the k-th member of the group is measured with the following index:

$$\theta(k) = \frac{\sum_l u_{kl}}{\sum_k \sum_l u_{kl}} \tag{8}$$

Below, we use an example for demonstrating how to use neutrosophic sociograms in a simulated case.

Example 3.

A teacher of a group of 10 elementary school students wants to investigate the relationships between the children and the potential links among the group members. To do this, he asks three questions to analyze preferences and possible future links among students. He also uses this study to determine current and potential leaders within the group and if there is any isolated student. That is why he decides to apply the neutrosophic sociogram.

The total questionnaire consists of the following pairs of questionnaires:

Write your friends with whom:

Q_1: you want to join a quiz program.

Q_2: you want to study in group.

Q_3: you want to do volunteer activity.

Apart, write the members of the group in with whom:

Q_1^I : you are not sure to join quiz program.

Q_2^I : you are not sure to study in group.

Q_3^I : you are not sure to do volunteer activity.

We denote by $S = \{s_1, s_2, \cdots, s_{10}\}$ the set of members of the group of class. Results are shown in Table 3.

	$Q_1; Q_1^I$	$Q_2; Q_2^I$	$Q_3; Q_3^I$
s_1	$s_3, s_6, s_8; s_4$	$s_2, s_6, s_{10}; s_4$	$s_3, s_6, s_9; s_4$
s_2	$s_3, s_4, s_5; s_7$	$s_1, s_4, s_9; s_7$	$s_1, s_3, s_4; s_7$
s_3	$s_1, s_2, s_9; s_6$	$s_5, s_8, s_9; s_2$	$s_2, s_6, s_7; s_{10}$
s_4	$s_2, s_5, s_6; s_9$	$s_3, s_6, s_{10}; s_9$	$s_3, s_5, s_6; s_2$
s_5	$s_3, s_7, s_{10}; s_9$	$s_4, s_7, s_8; s_1$	$s_4, s_8, s_{10}; s_1$
s_6	$s_1, s_7, s_8; s_9$	$s_1, s_7, s_8; s_9$	$s_1, s_2, s_9; s_3$
s_7	$s_4, s_5, s_9; s_2$	$s_2, s_4, s_9; s_5$	$s_6, s_8, s_9; s_1$
s_8	$s_5, s_7, s_9; s_3$	$s_4, s_6, s_9; s_7$	$s_1, s_2, s_5; s_3$
s_9	$s_1, s_8, s_{10}; s_2$	$s_2, s_4, s_5; s_1$	$s_1, s_4, s_5; s_2$
s_{10}	$s_2, s_5, s_7; s_9$	$s_2, s_3, s_5; s_4$	$s_2, s_4, s_5; s_8$

Table 3: Preferences and potential links between the members of the group.

In Table 3, for every child in the row, before the semicolon we have the children he/she prefers for performing the activity asked in questions Q_j. After the semicolon there are the children that the student is not sure about to perform activity asked in Q_j^I. Interestingly, student denoted by s_3 prefers to join quiz program with s_2, however he/she is not sure to study in group with s_2. This shows the capacity of the neutrosophic method to model more feelings of the members than its precedents do.

Here we assumed the three questions are equally important, thus, $\omega_j = \frac{1}{3}$ $(j = 1,2,3)$.

Tables 4, 5, and 6 contains the evaluation matrices for Q_1, Q_2, and Q_3, respectively.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_1	1	0	1	0	0	1	0	1	0	0
s_2	0	1	1	1	1	0	0	0	0	0
s_3	1	1	1	0	0	0	0	0	1	0
s_4	0	1	0	1	1	1	0	0	0	0
s_5	0	0	1	0	1	0	1	0	0	1
s_6	1	0	0	0	0	1	1	1	0	0
s_7	0	0	0	1	1	0	1	0	1	0
s_8	0	0	0	0	1	0	1	1	1	0
s_9	1	0	0	0	0	0	0	1	1	1
s_{10}	0	1	0	0	1	0	1	0	0	1

Table 4: Evaluation matrix R^1 for Q_1.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_1	1	1	0	0	0	1	0	0	0	1
s_2	1	1	0	1	0	0	0	0	1	0
s_3	0	0	1	0	1	0	0	1	1	0
s_4	0	0	1	1	0	1	0	0	0	1
s_5	0	0	0	1	1	0	1	1	0	0
s_6	1	0	0	0	0	1	1	1	0	0
s_7	0	1	0	1	0	0	1	0	1	0
s_8	0	0	0	1	0	1	0	1	1	0
s_9	0	1	0	1	1	0	0	0	1	0
s_{10}	0	1	1	0	1	0	0	0	0	1

Table 5: Evaluation matrix R^2 for Q_2

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_1	1	0	1	0	0	1	0	0	1	0
s_2	1	1	1	1	0	0	0	0	0	0
s_3	0	1	1	0	0	1	1	0	0	0
s_4	0	0	1	1	1	1	0	0	0	0
s_5	0	0	0	1	1	0	0	1	0	1
s_6	1	1	0	0	0	1	0	0	1	0
s_7	0	0	0	0	0	1	1	1	0	1
s_8	1	1	0	0	1	0	0	1	0	0
s_9	1	0	0	1	1	0	0	0	1	0
s_{10}	0	1	0	1	1	0	0	0	0	1

Table 6: Evaluation matrix R^3 for Q_3.

Table 7 contains the result of F, the fuzzy matrix.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_1	1	0.33	0.66	0	0	1	0	0.33	0.33	0.33
s_2	0.66	1	0.66	1	0.33	0	0	0	0.33	0
s_3	0.33	0.66	1	0	0.33	0.33	0.33	0.33	0.66	0
s_4	0	0.33	0.66	1	0.66	1	0	0	0	0.33
s_5	0	0	0.33	0.66	1	0	0.66	0.66	0	0.66
s_6	1	0.33	0	0	0	1	0.66	0.66	0.33	0
s_7	0	0.33	0	0.66	0.33	0.33	1	0.33	0.66	0.33
s_8	0.33	0.33	0	0.33	0.66	0.33	0.33	1	0.66	0
s_9	0.66	0.33	0	0.66	0.66	0	0	0.33	1	0.33
s_{10}	0	1	0.33	0.33	1	0	0.33	0	0	1

Table 7: Fuzzy matrix.

Table 8 summarizes the matrix G of fuzzy amicable degree.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_1	1	0.44	0.44	0	0	1	0	0.33	0.44	0
s_2	0.44	1	0.66	0.5	0	0	0	0	0.33	0
s_3	0.44	0.66	1	0	0.33	0	0	0	0	0
s_4	0	0.5	0	1	0.66	0	0	0	0	0.33
s_5	0	0	0.33	0.66	1	0	0.44	0.66	0	0.8
s_6	1	0	0	0	0	1	0.44	0.44	0	0
s_7	0	0	0	0	0.44	0.44	1	0.33	0	0.33
s_8	0.33	0	0	0	0.66	0.44	0.33	1	0.44	0
s_9	0.44	0.33	0	0	0	0	0	0.44	1	0
s_{10}	0	0	0	0.33	0.8	0	0.33	0	0	1

Table 8: Matrix of fuzzy amicable degree.

Equivalently, we calculate matrices T^1, T^2, and T^3, which are summarized in Tables 9, 10, and 11, respectively.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_1	1	0	1	1	0	1	0	1	0	0
s_2	0	1	1	1	1	0	1	0	0	0
s_3	1	1	1	0	0	1	0	0	1	0
s_4	0	1	0	1	1	1	0	0	1	0
s_5	0	0	1	0	1	0	1	0	1	1
s_6	1	0	0	0	0	1	1	1	1	0
s_7	0	1	0	1	1	0	1	0	1	0
s_8	0	0	1	0	1	0	1	1	1	0
s_9	1	1	0	0	0	0	0	1	1	1
s_{10}	0	1	0	0	1	0	1	0	1	1

Table 9: Evaluation matrix T^1 for Q_1.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_1	1	1	0	1	0	1	0	0	0	1
s_2	1	1	0	1	0	0	1	0	1	0
s_3	0	1	1	0	1	0	0	1	1	0
s_4	0	0	1	1	0	1	0	0	1	1
s_5	1	0	0	1	1	0	1	1	0	0
s_6	1	0	0	0	0	1	1	1	1	0
s_7	0	1	0	1	1	0	1	0	1	0
s_8	0	0	0	1	0	1	1	1	1	0
s_9	1	1	0	1	1	0	0	0	1	0
s_{10}	0	1	1	1	1	0	0	0	0	1

Table 10: Evaluation matrix T^2 for Q_2.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_1	1	0	1	1	0	1	0	0	1	0
s_2	1	1	1	1	0	0	1	0	0	0
s_3	0	1	1	0	0	1	1	0	0	1
s_4	0	1	1	1	1	1	0	0	0	0
s_5	1	0	0	1	1	0	0	1	0	1
s_6	1	1	1	0	0	1	0	0	1	0
s_7	1	0	0	0	0	1	1	1	0	1
s_8	1	1	1	0	1	0	0	1	0	0
s_9	1	1	0	1	1	0	0	0	1	0
s_{10}	0	1	0	1	1	0	0	1	0	1

Table 11: Evaluation matrix T^3 for Q_3.

Table 12 contains the values of matrix T.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_1	1	0.33	0.66	1	0	1	0	0.33	0.33	0.33
s_2	0.66	1	0.66	1	0.33	0	1	0	0.33	0
s_3	0.33	1	1	0	0.33	0.66	0.33	0.33	0.66	0.33
s_4	0	0.66	0.66	1	0.66	1	0	0	0.66	0.33
s_5	0.66	0	0.33	0.66	1	0	0.66	0.66	0.33	0.66
s_6	1	0.33	0.33	0	0	1	0.66	0.66	1	0
s_7	0.33	0.66	0	0.66	0.66	0.33	1	0.33	0.66	0.33
s_8	0.33	0.33	0.66	0.33	0.66	0.33	0.66	1	0.66	0
s_9	1	1	0	0.66	0.66	0	0	0.33	1	0.33
s_{10}	0	1	0.33	0.66	1	0	0.33	0.33	0.33	1

Table 12: Matrix T.

Table 13 summarizes the values of the amicable degrees in matrix $U = (u_{kl})$

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
s_1	1	0.44	0.44	0	0	1	0	0.33	0.5	0
s_2	0.44	1	0.8	0.8	0	0	0.8	0	0.5	0
s_3	0.44	0.8	1	0	0.33	0.44	0	0.44	0	0.33
s_4	0	0.8	0	1	0.66	0	0	0	0.66	0.44
s_5	0	0	0.33	0.66	1	0	0.66	0.66	0.44	0.8
s_6	1	0	0.44	0	0	1	0.44	0.44	0	0
s_7	0	0.8	0	0	0.66	0.44	1	0.44	0	0.33
s_8	0.33	0	0.44	0	0.66	0.44	0.44	1	0.44	0
s_9	0.5	0.5	0	0.66	0.44	0	0	0.44	1	0.33
s_{10}	0	0	0.33	0.44	0.8	0	0.33	0	0.33	1

Table 13: Matrix U.

According to Tables 8 and 13 we have that, for example, the relationship between the students s_2 and s_3 and its potentiality is $I_{23} = I_{32} = [0.66, 0.8]$, which means that currently the amicable degree between them is 0.66, however this degree can be potentially increased up to 0.8 in the future. Thus, the teacher should work to strengthen the relationship between these two students, instead of students s_1 and s_2 with $I_{12} = I_{21} = [0.44, 0.44]$, which seems to do not have the opportunity of changing and is weaker than the relationship between s_2 and s_3.

Calculating the leadership index with Equations 7 and 8, we have the following results:

$\mu =$
$(0.127217, 0.102159, 0.084811, 0.086739, 0.135698, 0.100231, 0.088666, 0.111796, 0.077101, 0.085582)$.

Whereas, $\theta =$
$(0.098068, 0.114461, 0.100117, 0.094262, 0.120609, 0.087822, 0.097190, 0.099532, 0.102459, 0.085480)$.

It is interpreted that student 5 is the leader according to $\mu(5) = 0.135698$ that is a maximum; however, potentially his/her leadership can slightly diminish because of $\theta(5) = 0.120609$.

Finally, we depict the neutrosophic sociogram of the example. See Figure 3.

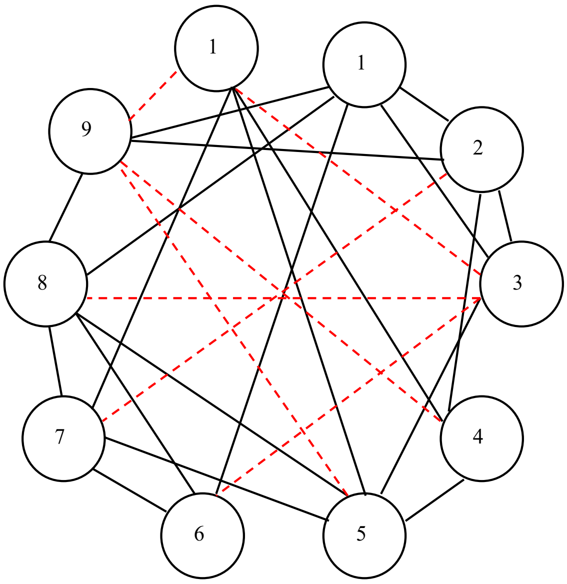

Figure 3: Neutrosophic sociogram of the example.

Let us note that the neutrosophic sociogram in Figure 3 shows in black continuous lines the relationships between the pair of students with fuzzy amicable degree bigger than 0, from Table 8. With dashed lines in red, we represent the edges with amicable degree bigger than 0 in matrix U and null value in matrix G, according to Table 13.

The results represented with continuous lines model the current preferences and the dashed lines represent the potential future links. For simplicity, we omitted in the graph the fuzzy amicable degrees values associated with the edges and the symbol I associated with the lines in red. Let us remark that this is a non-directed graph.

Conclusion

This paper introduces the neutrosophic sociograms. The crisp and fuzzy sociograms only take into account the preference relationships between individuals of the social group under investigation. However, it is possible that there are individuals in the group, especially if it is a large group, where some individuals do not know each other well and therefore are not designated as preferred ones. This type of relationship with lack of knowledge or lack of trust between two members can be consider an indeterminate relationship, where the future of the bond may be a preference relationship or a non-preference relationship, depending on group dynamics. Neutrosophic sociograms consider these indeterminacies, which are measured a possible future relationship. They are non-directed neutrosophic graphs. In this paper, we introduce a method to calculate the matrix of the graph, which is a neutrosophic matrix. The calculations include the weights or importance of each of the questions used to measure the preferred individuals to carry out the activities with. The advantage of defining a neutrosophic sociogram, instead of a crisp or fuzzy sociogram, is that it achieves greater accuracy in the representation of social relationships, and offers a better idea of what group dynamics is like and in which individuals the group cohesion can be strengthened. In future works we will study in depth the relationship of neutrosophic sociograms with Shapley value, taking into account that in the offsets [26, 27] there is an example of a solution for cooperative n-personal games using these neutrosophic sets [28].

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The Score, Accuracy, and Certainty Functions determine a Total Order on the Set of Neutrosophic Triplets (T, I, F)

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Abstract: In this paper we prove that the Single-Valued (and respectively Interval-Valued, as well as Subset-Valued) Score, Accuracy, and Certainty Functions determine a total order on the set of neutrosophic triplets (*T, I, F*). This total order is needed in the neutrosophic decision-making applications.

Keywords: single-valued neutrosophic triplet numbers; single-valued neutrosophic score function; single-valued neutrosophic accuracy function; single-valued neutrosophic certainty function.

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1. Introduction

We reveal the easiest to use single-valued neutrosophic score, accuracy, and certainty functions that exist in the literature and the algorithm how to use them all together. We present Xu and Da's Possibility Degree that an interval is greater than or equal to another interval, and we prove that this method is equivalent to the intervals' midpoints comparison. Also, Hong-yu Zhang et al.'s interval-valued neutrosophic score, accuracy, and certainty functions are listed, that we simplify these functions. Numerical examples are provided.

2. Single-Valued Neutrosophic Score, Accuracy, and Certainty Functions

We firstly present the most known and used in literature single-valued score, accuracy, and certainty functions.

Let M be the set of single-valued neutrosophic triplet numbers,

$M = \{(T, I, F), \text{where } T, I, F \in [0, 1], 0 \leq T + I + F \leq 3\}$. (1)

Let $N = (T, I, F) \in M$ be a generic single-valued neutrosophic triplet number. Then:

T = truth (or membership) represents the positive quality of N;

I = indeterminacy represents a negative quality of N,

hence $1 - I$ represents a positive quality of N;

F = falsehood (or nonmembership) represents also a negative quality of N, hence $1 - F$ represents a positive quality of N.

We present the three most used and best functions in the literature:

2.1. The Single-Valued Neutrosophic Score Function

$s: M \to [0, 1]$

$$s(T, I, F) = \frac{T + (1 - I) + (1 - F)}{3} = \frac{2 + T - I - F}{3} \quad (2)$$

that represents the average of positiveness of the single-valued neutrosophic components T, I, F.

2.2. The Single-Valued Neutrosophic Accuracy Function

$a: M \to [-1, 1]$

$a(T, I, F) = T - F$ (3)

2.3. The Single-Valued Neutrosophic Certainty Function

$c: M \to [0, 1]$

$c(T, I, F) = T$ (4)

3. Algorithm for Ranking the Single-Valued Neutrosophic Triplets

Let (T_1, I_1, F_1) and (T_2, I_2, F_2) be two single-valued neutrosophic triplets from M, i.e. $T_1, I_1, F_1, T_2, I_2, F_2 \in [0, 1]$.

Apply the *Neutrosophic Score Function*.

1. If $s(T_1, I_1, F_1) > s(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) > (T_2, I_2, F_2)$.
2. If $s(T_1, I_1, F_1) < s(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) < (T_2, I_2, F_2)$.
3. If $s(T_1, I_1, F_1) = s(T_2, I_2, F_2)$, then apply the *Neutrosophic Accuracy Function*:

3.1 If $a(T_1, I_1, F_1) > a(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) > (T_2, I_2, F_2)$.

3.2 If $a(T_1, I_1, F_1) < a(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) < (T_2, I_2, F_2)$.

3.3 If $a(T_1, I_1, F_1) = a(T_2, I_2, F_2)$, then apply the *Neutrosophic Certainty Function*.

3.3.1 If $c(T_1, I_1, F_1) > c(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) > (T_2, I_2, F_2)$.

3.3.2 If $c(T_1, I_1, F_1) < c(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) < (T_2, I_2, F_2)$.

3.3.1 If $c(T_1, I_1, F_1) = c(T_2, I_2, F_2)$, then $(T_1, I_1, F_1) \equiv (T_2, I_2, F_2)$, i.e. $T_1 = T_2,\ I_1 = I_2,\ F_1 = F_2$.

3.1. Theorem

We prove that the single-valued neutrosophic score, accuracy, and certainty functions all together form a *total order relationship* on M. Or:

for any two single-valued neutrosophic triplets (T_1, I_1, F_1) and (T_2, I_2, F_2) we have:

a) Either $(T_1, I_1, F_1) > (T_2, I_2, F_2)$

b) Or $(T_1, I_1, F_1) < (T_2, I_2, F_2)$

c) Or $(T_1, I_1, F_1) \equiv (T_2, I_2, F_2)$, which means that $T_1 = T_2,\ I_1 = I_2,\ F_1 = F_2$.

Therefore, on the set of single-valued neutrsophic triplets $M = \{(T, I, F), \text{with } T, I, F \in [0, 1], 0 \leq T + I + F \leq 3\}$, the score, accuracy, and certainty functions altogether form a *total order relationship*.

Proof.

Firstly we apply the score function.

The only problematic case is when we get equality:

$s(T_1, I_1, F_1) = s(T_2, I_2, F_2)$. (5)

That means:

$\frac{2+T_1-I_1-F_1}{3} = \frac{2+T_2-I_2-F_2}{3}$

or $T_1 - I_1 - F_1 = T_2 - I_2 - F_2$.

Secondly we apply the accuracy function.

Again the only problematic case is when we get equality:

$a(T_1, I_1, F_1) = a(T_2, I_2, F_2)$ or $T_1 - F_1 = T_2 - F_2$.

Thirdly, we apply the certainty function.

Similarly, the only problematic case may be when we get equality:

$c(T_1, I_1, F_1) = c(T_2, I_2, F_2)$ or $T_1 = T_2$.

For the most problematic case, we got the following linear algebraic system of 3 equations of 6 variables:

$$\begin{cases} T_1 - I_1 - F_1 = T_2 - I_2 - F_2 \\ T_1 - F_1 = T_2 - F_2 \\ T_1 = T_2 \end{cases}$$

Let's solve it.

Since $T_1 = T_2$, replacing this into the second equation we get $F_1 = F_2$.

Now, replacing both $T_1 = T_2$ and $F_1 = F_2$ into the first equation, we get $I_1 = I_2$.

Therefore the two neutrosophic triplets are identical: $(T_1, I_1, F_1) \equiv (T_2, I_2, F_2)$, i.e. equivalent (or equal), or $T_1 = T_2$, $I_1 = I_2$, and $F_1 = F_2$.

In conclusion, for any two single-valued neutrosophic triplets, either one is bigger than the other, or both are equal (identical).

4. Definition of Neutrosophic Negative Score Function

We have introduce in 2017 for the first time [1] the *Average Negative Quality Neutrosophic Function* of a single-valued neutrosophic triplet**,** defined as:

$$s^- : [0,1]^3 \to [0,1], s^-(t,i,f) = \frac{(1-t)+i+f}{3} = \frac{1-t+i+f}{3}. \quad (6)$$

4.1. Theorem

The average positive quality (score) neutrosophic function and the average negative quality neutrosophic function are complementary to each other, or

$$s^+(t,i,f) + s^-(t,i,f) = 1. \quad (7)$$

Proof.

$$s^+(t,i,f) + s^-(t,i,f) = \frac{2+t-i-f}{3} + \frac{1-t+i+f}{3} = 1. \quad (8)$$

The *Neutrosophic Accuracy Function* has been defined by:

$h: [0, 1]^3 \to [-1, 1], h(t, i, f) = t - f.$ (9)

We have also introduce [1] for the first time the *Extended Accuracy Neutrosophic Function*, defined as follows:

$h_e: [0, 1]^3 \to [-2, 1], h_e(t, i, f) = t - i - f,$ (10)

which varies on a range: from the worst negative quality (-2) [or minimum value], to the best positive quality (+1) [or maximum value].

4.2. Theorem

If $s(T_1, I_1, F_1) = s(T_2, I_2, F_2)$, $a(T_1, I_1, F_1) = a(T_2, I_2, F_2)$, and $c(T_1, I_1, F_1) = c(T_2, I_2, F_2)$, then $T_1 = T_2$, $I_1 = I_2$, $F_1 = F_2$, or the two neutrosophic triplets are identical:

$(T_1, I_1, F_1) \equiv (T_2, I_2, F_2)$.

Proof:

It results from the proof of Theorem 3.1.

5. Xu and Da's Possibility Degree

Xu and Da [3] have defined in 2002 the possibility degree $P(.)$ that an interval is greater than another interval:

$[a_1, a_2] \geq [b_1, b_2]$

for $a_1, a_2, b_1, b_2 \in [0,1]$ and $a_1 \leq a_2, b_1 \leq b_2$, in the following way:

$P([a_1, a_2] \geq [b_1, b_2]) = max\left\{1 - max\left(\frac{b_2 - a_1}{a_2 - a_1 + b_2 - b_1}, 0\right), 0\right\}$,

where $a_2 - a_1 + b_2 - b_1 \neq 0$ (i.e. $a_2 \neq a_1$ *or* $b_2 \neq b_1$.

They proved the following:

5.1. Properties

1) $P([a_1, a_2] \geq [b_1, b_2]) \in [0,1]$;
2) $P([a_1, a_2] \approx [b_1, b_2]) = 0.5$;
3) $P([a_1, a_2] \geq [b_1, b_2]) + P([b_1, b_2] \geq [a_1, a_2]) = 1$.

5.2. Example

Let $[0.4, 0.7]$ and $[0.3, 0.6]$ be two intervals.

Then,

$$P([0.4, 0.7] \geq [0.3, 0.6]) = max\left\{1 - max\left(\frac{0.6 - 0.4}{0.7 - 0.4 + 0.6 - 0.3}, 0\right), 0\right\} = max\left\{1 - max\left(\frac{0.2}{0.6}\right), 0\right\}$$

$$= max\left\{1 - \frac{0.2}{0.6}, 0\right\} = \frac{0.4}{0.6} \approx 0.66 > 0.50,$$

therefore $[0.4, 0.7] \geq [0.3, 0.6]$.

The opposite:

$$P\big((0.3, 0.6) \geq ([0.4, 0.7])\big) = max\left\{1 - max\left(\frac{0.7 - 0.3}{0.6 - 0.3 + 0.7 - 0.4}, 0\right), 0\right\} = max\left\{1 - max\left(\frac{0.4}{0.6}, 0\right), 0\right\}$$

$$= max\left\{1 - \frac{0.4}{0.6}, 0\right\} = \frac{0.2}{0.6} \approx 0.33 < 0.50,$$

therefore $[0.3, 0.6] \leq [0.4, 0.7]$.

We see that

$P([0.4, 0.7] \geq [0.3, 0.6]) + P([0.3, 0.6] \geq [0.4, 0.7]) = \frac{0.4}{0.6} + \frac{0.2}{0.6} = 1$.

Another method of ranking two intervals is the midpoint one.

6. Midpoint Method

Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two intervals included in or equal to [0, 1], with $m_A = (a_1 + a_2)/2$ and $m_B = (b_1 + b_2)/2$ the midpoints of A and respectively B. Then:

1) If $m_A < m_B$ then $A < B$.
2) If $m_A > m_B$ then $A > B$.
3) If $m_A = m_B$ then $A =_N B$, i.e. A is neutrosophically equal to B.

6.1. Example

1) We take the previous example,

where $A = [0.4, 0.7]$, and $m_A = \frac{0.4+0.7}{2} = 0.55$;

and $B = [0.3, 0.6]$, and $m_B = \frac{0.3+0.6}{2} = 0.45$.

Since $m_A = 0.55 > 0.45 = m_B$, we have $A > B$.
Let $C = [0.1, 0.7]$ and $D = [0.3, 0.5]$.

Then $m_C = \frac{0.1+0.7}{2} = 0.4$, and $m_D = \frac{0.3+0.5}{2} = 0.4$.

Since $m_C = m_D = 0.4$, we get $C =_N D$.

Let's verify the ranking relationship between C and D using Xu and Da's possibility degree method.

$$P([0.1, 0.7] \geq [0.3, 0.5]) = max\left\{1 - max\left(\frac{0.5 - 0.1}{0.7 - 0.1 + 0.5 - 0.3}, 0\right), 0\right\} = max\left\{1 - max\left(\frac{0.4}{0.8}, 0\right), 0\right\}$$

$$= max\left\{1 - \frac{0.4}{0.8}, 0\right\} = max\left\{\frac{0.4}{0.8}, 0\right\} = 0.5;$$

and $P([0.3, 0.5] \geq [0.1, 0.7]) = max\left\{1 - max\left(\frac{0.7-0.3}{0.5-0.3+0.7-0.1}, 0\right), 0\right\} = max\left\{1 - max\left(\frac{0.4}{0.8}, 0\right), 0\right\} = max\left\{1 - \frac{0.4}{0.8}, 0\right\} = max\left\{\frac{0.4}{0.8}, 0\right\} = 0.5;$

thus, $[0.1, 0.7] =_N [0.3, 0.5]$.

6.2. Corollary

The possibility method for two intervals having the same midpoint gives always 0.5.

For example:

p([0.3, 0.5] ≥ [0.2, 0.6]) = max{1 - max(((0.6-0.3) / (0.5-0.3 + 0.6-0.2)), 0), 0} =
= max{1 - max(((0.3) / (0.6)), 0), 0} = max{1 - max(0.5, 0), 0} = 0.5.

Similarly,

p([0.2, 0.6] ≥ [0.3, 0.5]) = max{1 - max(((0.5-0.2) / (0.6-0.2 + 0.5-0.3)), 0), 0} = 0.5.

Hence, none of the intervals [0.3, 0.5] and [0.2, 0.6]) is bigger than the other.

Therefore, we may consider that the intervals [0.3, 0.5] $=_N$ [0.2, 0.6] are neutrosophically equal (or neutrosophically equivalent).

7. Normalized Hamming Distance between Two Intervals

Let's consider the *Normalized Hamming Distance* between two intervals $[a_1, a_2]$ and $[b_1, b_2]$

$$h : int([0, 1]) \times int([0, 1]) \rightarrow [0, 1]$$

defined as follows:

$$h([a_1, b_1], [a_2, b_2]) = ½(|a_1 - b_1| + |a_2 - b_2|).$$

7.1. Theorem

7.1.1. The Normalized Hamming Distance between two intervals having the same midpoint and the negative-ideal interval *[0, 0]* is the same.

7.1.2. The Normalized Hamming Distance between two intervals having the same midpoint and the positive-ideal interval *[1, 1]* is also the same (Jun Ye [4, 5]).

Proof.

Let $A = [m - a, m + a]$ and $B = [m - b, m + b]$ be two intervals from *[0, 1]*, where $m-a$, $m+a$, $m-b$, $m+b$, $a, b, m \in [0, 1]$. A and B have the same midpoint m.

7.1.1. $h([m - a, m + a], [0, 0]) = ½(|m - a - 0| + |m + a - 0|) = ½(m - a + m + a) = m$, and

$h([m - b, m + b], [0, 0]) = ½(|m - b - 0| + |m + b - 0|) = ½(m - b + m + b) = m$,

7.1.2. $h([m - a, m + a], [1, 1]) = ½(|m - a - 1| + |m + a - 1|) = ½(1 - m + a + 1 - m - a) = 1 - m$, and

$h([m - b, m + b], [1, 1]) = ½(|m - b - 1| + |m + b - 1|) = ½(1 - m + b + 1 - m - b) = 1 - m$.

8. Xu and Da's Possibility Degree Method is equivalent to the Midpoint Method

We prove the following:

8.1. Theorem

The Xu and Da's Possibility Degree Method is equivalent to the Midpoint Method in ranking two intervals included in *[0, 1]*.

Proof.

Let A and B be two intervals included in [0,1]. Without loss of generality, we write each interval in terms of each midpoint:

$A = [m_1 - a, m_1 + a]$ and $B = [m_2 - b, m_2 + b]$,

where $m_1, m_2 \in [0,1]$ are the midpoints of A and respectively B, and $a, b \in [0,1]$, $A, B \subseteq [0,1]$.

(For example, if $A = [0.4, 0.7]$, $m_A = \frac{0.4+0.7}{2} = 0.55$, 0.55-0.4=0.15, then $A = [0.55 - 0.15, 0.55 + 0.15]$).

1) First case: $m_1 < m_2$. According to the Midpoint Method, we get $A < B$. Let's prove the same inequality results with the second method.

Let's apply Xu and Da's Possibility Degree Method:

$$P(A \geq B) = P([m_1 - a, m_1 + a] \geq [m_2 - b, m_2 + b])$$

$$= max\left\{1 - max\left(\frac{(m_2 + b) - (m_1 - a)}{(m_1 + a) - (m_1 - a) + (m_2 + b) - (m_2 - b)}, 0\right), 0\right\}$$

$$= max\left\{1 - max\left(\frac{m_2 - m_1 + a + b}{2a + 2b}, 0\right), 0\right\}$$

$$= max\left\{1 - \frac{m_2 - m_1 + a + b}{2a + 2b}, 0\right\}, \text{because } m_1 < m_2,$$

$$= max\left\{\frac{2a + 2b - m_2 + m_1 - a - b}{2a + 2b}, 0\right\} = max\left\{\frac{a + b + m_1 - m_2}{2a + 2b}, 0\right\}$$

i) If $a + b + m_1 - m_2 \leq 0$, then $p(A \geq B) = max\left\{\frac{a+b+m_1-m_2}{2a+2b}, 0\right\} = 0$, hence $A < B$.

ii) If $a + b + m_1 - m_2 > 0$,

then $p(A \geq B) = max\left\{\frac{a+b+m_1-m_2}{2a+2b}, 0\right\} = \frac{a+b+m_1-m_2}{2a+2b} > 0$.

We need to prove that $\frac{a+b+m_1-m_2}{2a+2b} < 0.5$,

or $a+b+m_1-m_2 < 0.5(2a+2b)$,

or $a+b+m_1-m_2 < a+b$,

or $m_1 - m_2 < 0$,

or $m_1 < m_2$, which is true according to the first case assumption.

2) Second case: $m_1 = m_2$. According to the Midpoint Method, A is neutrosophically equal to B (we write $A =_N B$).

Let's prove that we get the same result with Xu and Da's Method.

Then $A = [m_1 - a, m_1 + a]$, and $B = [m_1 - b, m_1 + b]$.

Let's apply Xu and Da's Method:

$$P(A \geq B) = max\left\{1 - max\left(\frac{(m_1+b)-(m_1-a)}{(m_1+a)-(m_1-a)+(m_1+b)-(m_1-b)}, 0\right), 0\right\}$$

$$= max\left\{1 - max\left(\frac{a+b}{2a+2b}, 0\right), 0\right\} = max\left\{1 - \frac{1}{2}, 0\right\} = 0.5$$

Similarly:

$$P(B \geq A) = max\left\{1 - max\left(\frac{(m_1+a)-(m_1-b)}{(m_1+b)-(m_1-b)+(m_1+a)-(m_1-a)}, 0\right), 0\right\}$$

$$= max\left\{1 - max\left(\frac{a+b}{2a+2b}, 0\right), 0\right\} = 0.5$$

Therefore, again $A =_N B$.

3) If $m_1 > m_2$, according to the Midpoint Method, we get $A > B$.

Let's prove the same inequality using Xu and Da's Method.

$$P(A \geq B) = P([m_1 - a, m_1 + a] \geq [m_2 - b, m_2 - b]) =$$

$$max\left\{1 - max\left(\frac{(m_2+b)-(m_1-a)}{(m_1+a)-(m_1-a)+(m_2+b)-(m_2-b)}, 0\right), 0\right\}$$

$$= max\left\{1 - max\left(\frac{m_2 - m_1 + a + b}{2a+2b}, 0\right), 0\right\}$$

i) If $m_2 - m_1 + a + b \leq 0$, then $P(A \geq B) = max\{1-0, 0\} = 1$, therefore $A > B$.

ii) If $m_2 - m_1 + a + b > 0$, then

$$P(A \geq B) = max\left\{1 - \frac{m_2 - m_1 + a + b}{2a+2b}, 0\right\} = \frac{2a+2b-m_2+m_1-a-b}{2a+2b} = \frac{a+b+m_1-m_2}{2a+2b}$$

We need to prove that $\frac{a+b+m_1-m_2}{2a+2b} > 0.5$,

or $a+b+m_1-m_2 > 0.5(2a+2b)$

or $a+b+m_1-m_2 > a+b$

or $m_1 - m_2 > 0$

or $m_1 > m_2$, which is true according to the third case. Thus $A > B$.

8.2. Consequence

All intervals, included in [0, 1], with the same midpoint are considered neutrosophically equal.

$C(m) = \{[m-a, m+a], \text{where all } m, a, m-a, m+a \in [0,1]\}$

represents the class of all neutrosophically equal intervals included in [0, 1] whose midpoint is m.

i) If $m = 0$ or $m = 1$, there is only one interval centered in 0, i.e. [0, 0], and only one interval centered in 1, i.e. [1, 1].

ii) If $m \notin \{0, 1\}$, there are infinitely many intervals from [0, 1], centered in m.

8.3. Consequence

Remarkably we can rank an interval $[a, b] \subseteq [0,1]$ with respect to a number $n \in [0, 1]$ since the number may be transformed into an interval $[n, n]$ as well.

For example $[0.2, 0.8] > 0.4$ since the midpoint of [0.2, 0.8] is 0.5, and the midpoint of [0.4, 0.4]= 0.4, hence $0.5 > 0.4$.

Similarly, $0.7 > (0.5, 0.8)$.

9. Interval (-Valued) Neutrosophic Score, Accuracy, and Certainty Functions

Let $T, I, F \subseteq [0,1]$ be three open, semi-open / semi-closed, or closed intervals.

Let $T^L = infT$ and $T^U = supT$; $I^L = infI$ and $I^U = supI$; $F^L = infF$ and $F^U = supF$.

Let $T^L, T^U, I^L, I^U, F^L, F^U \in [0,1]$, with $T^L \leq T^U, I^L \leq I^U, F^L \leq F^U$.

We consider all possible types of intervals: open (*a, b*), semi-open / semi-closed (*a, b*] and [*a, b*), and closed [*a, b*]. For simplicity of notations, we are using only [*a, b*], but we understand all types.

Then $A = ([T^L, T^U], [I^L, I^U], [F^L, F^U])$ is an Interval Neutrosophic Triplet.

T^L is the lower limit of the interval T,

T^U is the upper limit of the interval T,

and similarly for I^L, I^U, and F^L, F^U for the intervals I, and respectively F.

Hong-yu Zhang, Jian-qiang Wang, and Xiao-hong Chen [2] in 2014 defined the Interval Neutrosophic Score, Accuracy, and Certainty Functions as follows.

Let's consider *int([0, 1])* the set of all (open, semi-open/semi-closed, or closed) intervals included in or equal to *[0, 1]*, where the abbreviation and index *int* stand for *interval*, and *Zhang* stands for Hong-yu Zhang, Jian-qiang Wang, and Xiao-hong Chen.

9.1. Zhang Interval Neutrosophic Score Function

$$s_{\text{int}}^{Zhang} : \{\text{int}([0,1])\}^3 \to \text{int}([0,1])$$

$$S_{int}^{Zhang}(A) = [T^L + 1 - I^U + 1 - F^U,\ T^U + 1 - I^L + 1 - F^L] \tag{11}$$

9.2. Zhang Interval Neutrosophic Accuracy Function

$$a_{\text{int}}^{Zhang} : \{\text{int}([0,1])\}^3 \to \text{int}([0,1])$$

$$a_{\text{int}}^{Zhang}(A) = [min\{T^L - F^L, T^U - F^U\}, max\{T^L - F^L, T^U - F^U\}] \tag{12}$$

9.3. Zhang Interval Neutrosophic Certainty Function

$$c_{\text{int}}^{Zhang} : \{\text{int}([0,1])\}^3 \to \text{int}([0,1])$$

$$c_{int}^{Zhang}(A) = [T^L,\ T^U] \tag{13}$$

9. New Interval Neutrosophic Score, Accuracy, and Certainty Functions

Since comparing/ranking two intervals is equivalent to comparing/ranking two members (i.e. the intervals' midpoints), we simplify Zhang Interval Neutrosophic Score (S_{int}^{Zhang}), Accuracy (a_{int}^{Zhang}), Certainty (c_{int}^{Zhang}) functions, as follows:

$$s_{\text{int}}^{FS} : \{\text{int}([0,1])\}^3 \to [0,1]$$
$$a_{\text{int}}^{FS} : \{\text{int}([0,1])\}^3 \to [-1,1]$$
$$c_{\text{int}}^{FS} : \{\text{int}([0,1])\}^3 \to [0,1]$$

where the upper index *FS* stands for our name's initials, in order to distinguish these new functions from the previous ones:

10.1. New Interval Neutrosophic Score Function

$$s_{\text{int}}^{FS}\left(([T^L,T^U],[I^L,I^U],[F^L,F^U])\right) = \frac{T^L+T^U+(1-I^L)+(1-I^U)+(1-F^L)+(1-F^U)}{6} = \frac{4+T^L+T^U-I^L-I^U-F^L-F^U}{6},$$

which means the average of six positivenesses;

10.2. New Interval Neutrosophic Accuracy Function

$a_{\text{int}}^{FS}\left(([T^L,T^U],[I^L,I^U],[F^L,F^U])\right) = \frac{T^L+T^U-F^L-F^U}{2}$, which means the average of differences between positiveness and negativeness;

10.3. New Interval Neutrosophic Certainty Function

$$c_{\text{int}}^{FS}\left(([T^L,T^U],[I^L,I^U],[F^L,F^U])\right) = \frac{T^L+T^U}{2},$$

which means the average of two positivenesses.

10.4. Theorem

Let $\mathcal{M}_{int} = \{(T,I,F), \text{where } T,I,F \subseteq [0,1], \text{and } T,I,F \text{ are intervals}\}$, be the set of interval neutrosophic triplets.

The New Interval Neutrosophic Score, Accuracy, and Certainty Functions determine a total order relationship on the set $\mathcal{M}_{int}$ of Interval Neutrosophic Triplets.

Proof.

Let's assume we have two interval neutrosophic triplets:
$P_1 = ([T_1^L,T_1^U],[I_1^L,I_1^U],[F_1^L,F_1^U])$,
and $P_1 = ([T_2^L,T_2^U],[I_2^L,I_2^U],[F_2^L,F_2^U])$, both from M_{int}.
We have to prove that: either $P_1 > P_2$, or $P_1 < P_2$, or $P_1 = P_2$.

Apply the new interval neutrosophic score function (s_{int}^{FS}) to both of them:

$$s_{\text{int}}^{FS}(P_1) = \frac{4+T_1^L+T_1^U-I_1^L-I_1^U-F_1^L-F_1^U}{6}$$
$$s_{\text{int}}^{FS}(P_2) = \frac{4+T_2^L+T_2^U-I_1^L-I_1^U-F_1^L-F_1^U}{6}$$

If $s_{\text{int}}^{FS}(P_1) > s_{\text{int}}^{FS}(P_2)$, then $P_1 > P_2$.

If $s_{\text{int}}^{FS}(P_1) < s_{\text{int}}^{FS}(P_2)$, then $P_1 < P_2$.

If $s_{\text{int}}^{FS}(P_1) = s_{\text{int}}^{FS}(P_2)$, then we get from equating the above two equalities that:

$T_1^L + T_1^U - I_1^L - I_1^U - F_1^L - F_1^U = T_2^L + T_2^U - I_2^L - I_2^U - F_2^L - F_2^U$

In this problematic case, we apply the new interval neutrosophic accuracy function (a_{int}^{FS}) to both P_1 and P_2, and we get:

$$a_{\text{int}}^{FS}(P_1) = \frac{T_1^L + T_1^U - F_1^L - F_1^U}{2}$$

$$a_{\text{int}}^{FS}(P_2) = \frac{T_2^L + T_2^U - F_2^L - F_2^U}{2}$$

If $a_{\text{int}}^{FS}(P_1) > a_{\text{int}}^{FS}(P_2)$, then $P_1 > P_2$.

If $a_{\text{int}}^{FS}(P_1) < a_{\text{int}}^{FS}(P_2)$, then $P_1 < P_2$.

If $a_{\text{int}}^{FS}(P_1) = a_{\text{int}}^{FS}(P_2)$, then we get from equating the two above equalities that:

$T_1^L + T_1^U - F_1^L - F_1^U = T_2^L + T_2^U - F_2^L - F_2^U$

Again, a problematic case, so we apply the new interval neutrosophic certainty function (c_{int}^{FS}) to both P_1 and P_2, and we get:

$$c_{\text{int}}^{FS}(P_1) = T_1^L + T_1^U$$

$$c_{\text{int}}^{FS}(P_2) = T_2^L + T_2^U$$

If $c_{\text{int}}^{FS}(P_1) > c_{\text{int}}^{FS}(P_2)$, then $P_1 > P_2$.

If $c_{\text{int}}^{FS}(P_1) < c_{\text{int}}^{FS}(P_2)$, then $P_1 < P_2$.

If $c_{\text{int}}^{FS}(P_1) = c_{\text{int}}^{FS}(P_2)$, then we get:

$T_1^L + T_1^U = T_2^L + T_2^U$

We prove that in the last case we get:

$P_1 =_N P_2$ (or P_1 is neutrosophically equal to P_2).

We get the following linear algebraic system of 3 equations and 12 variables:

$$\begin{cases} T_1^L + T_1^U - I_1^L - I_1^U - F_1^L - F_1^U = T_2^L + T_2^U - I_2^L - I_2^U - F_2^L - F_2^U \\ T_1^L + T_1^U - F_1^L - F_1^U = T_2^L + T_2^U - F_2^L - F_2^U \\ T_1^L + T_1^U = T_2^L + T_2^U \end{cases}$$

Second equation minus the third equation gives us:
$-F_1^L - F_1^U = -F_2^L - F_2^U$, or $F_1^L + F_1^U = F_2^L + F_2^U$.
First equation minus the second equation gives us:
$-I_1^L - I_1^U = -I_2^L - I_2^U$, or $I_1^L + I_1^U = I_2^L + I_2^U$.
The previous system is now equivalent to the below system:

$$\begin{cases} T_1^L + T_1^U = T_2^L + T_2^U \\ I_1^L + I_1^U = I_2^L + I_2^U \\ F_1^L + F_1^U = F_2^L + F_2^U \end{cases} \Leftrightarrow \begin{cases} \frac{T_1^L + T_1^U}{2} = \frac{T_2^L + T_2^U}{2} \\ \frac{I_1^L + I_1^U}{2} = \frac{I_2^L + I_2^U}{2} \\ \frac{F_1^L + F_1^U}{2} = \frac{F_2^L + F_2^U}{2} \end{cases}$$

which means that:

i) the intervals $[T_1^L, T_1^U]$ and $[T_2^L, T_2^U]$ have the same midpoint, therefore they are neutrosophically equal.

ii) the intervals $[I_1^L, I_1^U]$ and $[I_2^L, I_2^U]$ have also the same midpoint, so they are neutrosophically equal.

iii) similarly, the intervals $[F_1^L, F_1^U]$ and $[F_2^L, F_2^U]$ have the same midpoint, and again they are neutrosophically equal.

Whence, the interval neutrosophic triplets P_1 and P_2 are neutrosophically equal, i.e. $P_1 =_N P_2$.

10.5. Theorem

Let's consider the ranking of intervals defined by Xu and Da, which is equivalent to the ranking of intervals' midpoints. Then, the algorithm by Hong-yu Zhang et al. for ranking the interval neutrosophic triplets in equivalent to our algorithm.

Proof

Let's consider two interval neutrosophic triplets, P_1 and $P_2 \in \mathcal{M}_{int}$,
$P_1 = ([T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U])$,
and $P_2 = ([T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U])$.

Let's rank them using both methods and prove we get the same results. We denote by s_{int}^{Zhang}, a_{int}^{Zhang}, c_{int}^{Zhang}, and s_{int}^{FS}, a_{int}^{FS}, c_{int}^{FS} the Interval Neutrosophic Score, Accuracy, and Certainty Functions, by Hong-yu Zhang et al. and respectively by us.

Interval Neutrosophic Score Function

$$s_{\text{int}}^{Zhang}(P_1) = [T_1^L + 1 - I_1^U + 1 - F_1^U, T_1^U + 1 - I_1^L + 1 - F_1^L]$$

$$s_{\text{int}}^{Zhang}(P_2) = [T_2^L + 1 - I_2^U + 1 - F_2^U, T_2^U + 1 - I_2^L + 1 - F_2^L]$$

a) If $s_{\text{int}}^{Zhang}(P_1) > s_{\text{int}}^{Zhang}(P_2)$, then

the midpoint of the interval $s_{\text{int}}^{Zhang}(P_1) >$ midpoint of the interval $s_{\text{int}}^{Zhang}(P_2)$, or $\frac{T_1^L+1-I_1^U+1-F_1^U+T_1^U+1-I_1^L+1-F_1^L}{2} > \frac{T_2^L+1-I_2^U+1-F_2^U+T_2^U+1-I_2^L+1-F_2^L}{2}$,

or $T_1^L + T_1^U - I_1^L - I_1^U - F_1^L - F_1^U > T_2^L + T_2^U - I_2^L - I_2^U - F_2^L - F_2^U$,

or $\frac{4+T_1^L+T_1^U-I_1^L-I_1^U-F_1^L-F_1^U}{6} > \frac{4+T_2^L+T_2^U-I_2^L-I_2^U-F_2^L-F_2^U}{6}$,

or $s_{int}^{FS}(P_1) > s_{int}^{FS}(P_2)$.

b) If $s_{\text{int}}^{Zhang}(P_1) < s_{\text{int}}^{Zhang}(P_2)$, the proof is similar, we only replace the inequality symbol $>$ by $<$ into the above proof.

c) If $s_{\text{int}}^{Zhang}(P_1) = s_{\text{int}}^{Zhang}(P_2)$, the proof again is similar with the above, we only replace $>$ by $=$ into the above proof.

Interval Neutrosophic Accuracy Function

$$a_{\text{int}}^{Zhang}(P_1) = [min\{T_1^L - F_1^L, T_1^U - F_1^U\}, max\{T_1^L - F_1^L, T_1^U - F_1^U\}],$$

$$a_{\text{int}}^{Zhang}(P_2) = [min\{T_2^L - F_2^L, T_2^U - F_2^U\}, max\{T_2^L - F_2^L, T_2^U - F_2^U\}].$$

a) If $a_{\text{int}}^{Zhang}(P_1) > a_{\text{int}}^{Zhang}(P_2)$, then

the midpoint of the interval $a_{int}^{Hong}(P_1) >$ the midpoint of the interval $a_{int}^{Hong}(P_2)$,

or $\frac{T_1^L-F_1^L+T_1^U-F_1^U}{2} > \frac{T_2^L-F_2^L+T_2^U-F_2^U}{2}$,

or $a_{int}^{FS}(P_1) > a_{int}^{FS}(P_2)$.

b) Similarly, if $a_{\text{int}}^{Zhang}(P_1) < a_{\text{int}}^{Zhang}(P_2)$, just replacing $>$ by $<$ into the above proof.

c) Again, similarly if $a_{\text{int}}^{Zhang}(P_1) = a_{\text{int}}^{Zhang}(P_2)$, only replacing $>$ by $=$ into the above proof.

Interval Neutrosophic Certainty Function

$$c_{\text{int}}^{Zhang}(P_1) = [T_1^L, T_1^U]$$

$$c_{\text{int}}^{Zhang}(P_2) = [T_2^L, T_2^U]$$

a) If $c_{\text{int}}^{Zhang}(\boldsymbol{P_1}) > c_{\text{int}}^{Zhang}(\boldsymbol{P_2})$, then

b) the midpoint of the interval $c_{\text{int}}^{Zhang}(\boldsymbol{P_1}) >$ the midpoint of the interval $c_{\text{int}}^{Zhang}(\boldsymbol{P_2})$,

or $\frac{T_1^L+T_1^U}{2} > \frac{T_2^L+T_2^U}{2}$

or $c_{int}^{FS}(P_1) > c_{int}^{FS}(P_2)$.

b) Similarly, if $c_{\text{int}}^{Zhang}(P_1) < c_{\text{int}}^{Zhang}(P_2)$, just replacing $>$ by $<$ into the above proof.

c) Again, if $c_{\text{int}}^{Zhang}(P_1) = c_{\text{int}}^{Zhang}(P_2)$, only replace $>$ by $=$ into the above proof.

Therefore, we proved that, for any interval neutrosophic triplet P,

$s_{\text{int}}^{Zhang}(P) \sim s_{int}^{FS}(P)$, where $\sim$ means equivalent;

$a_{\text{int}}^{Zhang}(P) \sim a_{int}^{FS}(P)$,

and $c_{\text{int}}^{Zhang}(P) \sim c_{int}^{FS}(P)$.

11. Subset Neutrosophic Score, Accuracy, and Certainty Functions

Let $M_{\text{subset}} = \{(T_{\text{subset}}, I_{\text{subset}}, F_{\text{subset}})$, where the subsets $T_{\text{subset}}, I_{\text{subset}}, F_{\text{subset}} \subseteq [0,1]\}$.

We approximate each subset by the smallest closed interval that includes it.

Let's denote:

$T^L = inf(T_{\text{subset}})$ and $T^U = sup(T_{\text{subset}})$; therefore $T_{\text{subset}} \subseteq [T^L, T^U]$;

$I^L = inf(I_{\text{subset}})$ and $I^U = sup(I_{\text{subset}})$; therefore $I_{\text{subset}} \subseteq [I^L, I^U]$;

$F^L = inf(F_{\text{subset}})$ and $F^U = sup(F_{\text{subset}})$; therefore $F_{\text{subset}} \subseteq [F^L, F^U]$.

Then:

$$M_{\text{subset}} \approx \left\{ \begin{matrix} ([T^L, T^U], [I^L, I^U], [F^L, F^U]), \text{ where } T^L, T^U, I^L, I^U, F^L, F^U \in [0,1], \\ \text{and } T^L \le T^U, I^L \le I^U, F^L \le F^U \end{matrix} \right\}$$

11.1. Definition of Subset Neutrosophic Score, Accuracy, and Certainty Functions

Then, the formulas for Subset Neutrosophic Score, Accuracy, and Certainty Functions will coincide with those for Interval Neutrosophic Score Accuracy, and Certainty Functions by Hong-yu Zhang, and respectively by us:

11.2. Theorem

Let N be the Interval Neutrosophic Triplet

$N = ([T^L, T^U], [I^L, I^U], [F^L, F^U])$,

where $T^L \le T^U$, $I^L \le I^U$, $F^L \le F^U$,

and all $[T^L, T^U], [I^L, I^U], [F^L, F^U] \subseteq [0,1]$.

If each interval collapses to a single point, i.e.

$T^L = T^U = T$, then $[T^L, T^U] = [T, T] \equiv T \in [0,1]$,

$I^L = I^U = I$, then $[I^L, I^U] = [I, I] \equiv I \in [0,1]$,

$F^L = F^U = F$, then $[F^L, F^U] = [F, F] \equiv T \in [0,1]$,

then $s_{int}^{FS}(N) = s(N)$, $a_{int}^{FS}(N) = a(N)$, and $c_{int}^{FS}(N) = c(N)$.

Proof

$$s_{int}^{FS}(N) = \frac{4 + T^L + T^U - I^L - I^U - F^L - F^U}{6} = \frac{4 + T + T - I - I - F - F}{6} = \frac{4 + 2T - 2I - 2F}{6}$$

$$= \frac{2 + T - I - F}{3} = s(N).$$

$$a_{int}^{FS}(N) = \frac{T^L + T^U - F^L - F^U}{2} = \frac{T + T - F - F}{2} = \frac{2(T - F)}{2} = a(N).$$

$$c_{int}^{FS}(N) = \frac{T^L + T^U}{2} = \frac{T + T}{2} = \frac{2T}{2} = c(N).$$

12. Conclusion

The most used and easy for ranking the Neutrosophic Triplets (T, I, F) are the following functions, that provide a total order:

Single-Valued Neutrosophic Score, Accuracy, and Certainty Functions:

$$s(T,I,F) = \frac{2 + T - I - F}{3}$$

$$a(T,I,F) = T - F$$

$$c(T,I,F) = T$$

Interval-Valued Neutrosophic Score, Accuracy, and Certainty Functions:

$$s_{\text{int}}^{FS}\left(([T^L,T^U],[I^L,I^U],[F^L,F^U])\right) = \frac{4 + T^L + T^U - I^L - I^U - F^L - F^U}{6}$$

$$a_{\text{int}}^{FS}\left(([T^L,T^U],[I^L,I^U],[F^L,F^U])\right) = \frac{T^L + T^U - F^L - F^U}{2}$$

$$c_{\text{int}}^{FS}\left(([T^L,T^U],[I^L,I^U],[F^L,F^U])\right) = \frac{T^L + T^U}{2}$$

All these functions are very much used in decision-making applications.

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Some Fundamental Operations on Interval Valued Neutrosophic Hypersoft Set with Their Properties

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Abstract: Multi-criteria decision-making (MCDM) focuses on coordination, choice and planning issues, including multi-criteria. the neutrosophic soft set cannot handle environments involving multiple attributes. In order to overcome these obstacles, the neutrosophic hypersoft set (NHSS) and Interval Value neutrosophic hypersoft set (IVNHSS) are defined. In this paper, we extend the concept of IVNHSS with basic properties. We also developed some basic operations on IVNHSS such as union, intersection, addition, difference, Truth-favorite, and False-favorite, etc. with their desirable properties. Finally, the necessity and possibility operations on IVNHSS with properties are presented in the following research.

Keywords: Soft set; Neutrosophic Set; Interval-valued neutrosophic set; Hypersoft set; Interval-valued neutrosophic hypersoft set.

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1. Introduction

Anxiety performs a dynamic part in lots of areas of life such as modeling, medicine, and engineering. However, people have raised a general question, that is, how can we verbalize anxiety in mathematical modeling. Several investigators all over the world have recommended and advised different methodologies to minimize uncertainty. First of all, Zadeh planned the idea of fuzzy sets [1] to resolve these complications which contain anxiety as well as ambiguity. It is seen that sometimes; fuzzy sets can't deal with scenarios. To overcome such scenarios, Turksen [2] suggested the concept of interval-valued fuzzy sets (IVFS). In some cases, we need to debate the suitable representation of the object under the circumstances of anxiety and uncertainty, and regard its unbiased membership value and non-membership value of the suitable representation of the object, that cannot be processed by these fuzzy sets or IVFS. To overcome such concerns, Atanassov projected the theory of IFS in [3]. The theory proposed by Atanassov only considers membership and non-membership values to deal with insufficient data, but the IFS theory cannot deal with incompatible and imprecise information. To deal with this incompatible and imprecise data, Smarandache proposed the idea of NS [4]. Molodtsov [5] proposed a general mathematical tool to deal with uncertain, ambiguous, and undefined substances, called soft sets (SS). Maji et al. [6] extended the work of SS and defined some operations and their attributes. In [7], they also use SS theory to make decisions. Ali et al. [8] Modified the Maji method of SS and developed some new operations with its properties. In [9], they proved De Morgan's SS theory and law by using different operators. Cagman and Enginoglu [10] proposed the concept of soft matrices with operations and discussed their properties. They also introduced a decision-making method to solve problems that contain uncertainty. In [11], they modified the actions proposed by Molodtsov's SS. In [12], the author proposed some new operations for soft matrices, such as soft difference product, soft restricted difference product, soft extended difference product, and weak extended difference product.

Maji [13] put forward the idea of NSS with necessary operations and characteristics. The idea of Possibility NSS was proposed by Karaaslan [14] and introduced a neutrosophic soft decision method to solve those uncertain problems based on And-product. Broumi [15] developed a generalized NSS with certain operations and properties and used the proposed concept for decision-making. To solve the MCDM problem with single-valued neutrosophic numbers proposed by Deli and Subas in [16], they constructed the concept of the cut set of single-valued neutrosophic numbers. Based on the correlation of IFS, the term correlation coefficient of SVNS is introduced [17]. In [18], the idea of simplifying NS introduced some algorithms and aggregation operators, such as weighted arithmetic operators and weighted geometric average operators. They constructed the MCDM method based on the proposed aggregation operator. Zulqarnain et al. [19] extended the fuzzy TOPSIS technique to the Neutrosophic TOPSIS technique and used the developed approach to solve the MCDM problem. Abdel-basset et al [20] presented the integration of TOPSIS methodology decision-making test as well as evaluation laboratory (DEMATEL) solution (TOPSIS) CIIC environment delivers a new method to pick out the proper project. Abdel-basset Mohamed [21] developed an MCDM model to discover along with display screen cancer addressing obscure, anxiety, the incompleteness of reported signs as well as handicapping apparently within cancer or replaceable ailments in the signs and symptoms. Abdel-Basset et al. [22] raised the issue of assessment of the smart emergency response techniques is interpreted as MCDM problem. they suggested a framework by combining three common MCDM strategies which are AHP, TOPSIS, and VIKOR.

All the above-mentioned studies cannot deal with the problems in which attributes of the alternates have their corresponding sub-attributes. To handle such compilations Smarandache [23] generalized the SS to HSS by converting the function to a multi-attribute function to deal with uncertainty. Saqlain et al. [24] developed the generalization of TOPSIS for the NHSS, by using accuracy function they transformed the fuzzy neutrosophic numbers to crisp form. Zulqarnain et al. [25] extended the notion of NHSSs and presented the generalized operations for NHSSs, they also developed the necessity and possibility operations and discussed their desirable features. In [26], the author's proposed the fuzzy Plithogenic hypersoft set in matrix form with some basic operations and properties. Saqlain et al. [27] proposed the aggregate operators on NHSS. In [28], the author extended the NHSS approach and introduced IVNHSS, m-polar, and m-polar IVNHSS. Zulqarnain et al. [29] presented the intuitionistic fuzzy hypersoft set, they developed the TOPSIS technique by developing a correlation coefficient to solve multi-attribute decision making problems. Many other novel researchers are done under neutrosophic environment and their applications in everyday life [30-34].

The following research is organized as follows: Some basic definitions recalled in section 2, which are used in the following research such as SS, NS, NSS, HSS, NHSS, and IVNHSS. We present different operators on IVNHSS such as union, intersection, addition, difference, extended union, extended intersection, truth-favorite, and false-favorite operations in section 3 with properties and prove the De Morgan laws by using union and intersection operators. We also proposed the necessity and possibility operators, OR, and operations with some properties in section 4.

2. Preliminaries

In this section, we recollect some basic definitions such as SS, NSS, NHSS, and IVNHSS which use in the following sequel.

Definition 2.1 [5]

The soft set is a pair (F, Λ) over $\mathbb{U}$ if and only if $\mathrm{F}: \Lambda \rightarrow P(\mathbb{U})$ is a mapping. That is the parameterized family of subsets of $\mathbb{U}$ known as a SS.

Definition 2.2 [4]

Let $\mathbb{U}$ be a universe and Λ be an NS on $\mathbb{U}$ is defined as $\Lambda = \{< u, u_A(u), v_A(u), w_A(u) > : u \in \mathbb{U}\}$, where u, v, w: $\mathbb{U} \to]0^-, 1^+[$ and $0^- \leq u_\Lambda(u) + v_\Lambda(u) + w_\Lambda(u) \leq 3^+$.

Definition 2.3 [13]

Let $\mathbb{U}$ and Ḝ are universal set and set of attributes respectively. Let P($\mathbb{U}$) be the set of Neutrosophic values of $\mathbb{U}$ and $\Lambda \subseteq$ Ḝ. A pair (F, Λ) is called an NSS over $\mathbb{U}$ and its mapping is given as

F: $\Lambda \to (\mathbb{U})$

Definition 2.4 [35]

Let $\mathbb{U}$ be a universal set, then interval valued neutrosophic set can be expressed by the set $\boldsymbol{A} = \{< \boldsymbol{u}, \boldsymbol{u_A}(\boldsymbol{u}), \boldsymbol{v_A}(\boldsymbol{u}), \boldsymbol{w_A}(\boldsymbol{u}) > : \boldsymbol{u} \in \mathbb{U}\}$, where $\boldsymbol{u_A}$, $\boldsymbol{v_A}$, and $\boldsymbol{w_A}$ are truth, indeterminacy and falsity membership functions for $\boldsymbol{A}$ respectively, $\boldsymbol{u_A}$, $\boldsymbol{v_A}$, and $\boldsymbol{w_A} \subseteq [0, 1]$ for each $\boldsymbol{u} \in \mathbb{U}$. Where

$\boldsymbol{u_A}(\boldsymbol{u}) = [\boldsymbol{u_A^L}(\boldsymbol{u}), \boldsymbol{u_A^U}(\boldsymbol{u})]$

$\boldsymbol{v_A}(\boldsymbol{u}) = [\boldsymbol{v_A^L}(\boldsymbol{u}), \boldsymbol{v_A^U}(\boldsymbol{u})]$

$\boldsymbol{w_A}(\boldsymbol{u}) = [\boldsymbol{w_A^L}(\boldsymbol{u}), \boldsymbol{w_A^U}(\boldsymbol{u})]$

For each point $\boldsymbol{u} \in \mathbb{U}$, $0 \leq \boldsymbol{u_A}(\boldsymbol{u}) + \boldsymbol{v_A}(\boldsymbol{u}) + \boldsymbol{w_A}(\boldsymbol{u}) \leq 3$ and IVN($\mathbb{U}$) represents the family of all interval valued neutrosophic sets.

Definition 2.5 [23]

Let $\mathbb{U}$ be a universal set and $P(\mathbb{U})$ be a power set of $\mathbb{U}$ and for $n \geq 1$, there are n distinct attributes such as $k_1, k_2, k_3, \ldots, k_n$ and $K_1, K_2, K_3, \ldots, K_n$ are sets for corresponding values attributes respectively with following conditions such as $K_i \cap K_j = \emptyset$ $(i \neq j)$ and $i, j \in \{1,2,3 \ldots n\}$. Then the pair (F, $K_1 \times K_2 \times K_3 \times \ldots \times K_n$) is said to be HSS over $\mathbb{U}$ where F is a mapping from $K_1 \times K_2 \times K_3 \times \ldots \times K_n$ to $P(\mathbb{U})$.

Definition 2.6 [23]

Let $\mathbb{U}$ be a universal set and $P(\mathbb{U})$ be a power set of $\mathbb{U}$ and for $n \geq 1$, there are n distinct attributes such as $k_1, k_2, k_3, \ldots, k_n$ and $K_1, K_2, K_3, \ldots, K_n$ are sets for corresponding values attributes respectively with following conditions such as $K_i \cap K_j = \emptyset$ $(i \neq j)$ and $i, j \in \{1,2,3 \ldots n\}$. Then the pair (F, Λ) is said to be NHSS over $\mathbb{U}$ if there exists a relation $K_1 \times K_2 \times K_3 \times \ldots \times K_n = \Lambda$. F is a mapping from $K_1 \times K_2 \times K_3 \times \ldots \times K_n$ to $P(\mathbb{U})$ and $\mathrm{F}(K_1 \times K_2 \times K_3 \times \ldots \times K_n) = \{< u, u_A(u), v_A(u), w_A(u) > : u \in \mathbb{U}\}$ where u, v, w are membership values for truthness, indeterminacy and falsity respectively such that u, v, w: $\mathbb{U} \to]0^-, 1^+[$ and $0^- \leq u_\Lambda(u) + v_\Lambda(u) + w_\Lambda(u) \leq 3^+$.

Definition 2.7 [28]

Let $\mathbb{U}$ be a universal set and $P(\mathbb{U})$ be a power set of $\mathbb{U}$ and for $n \geq 1$, there are n distinct attributes such as $k_1, k_2, k_3, \ldots, k_n$ and $K_1, K_2, K_3, \ldots, K_n$ are sets for corresponding values attributes respectively with following conditions such as $K_i \cap K_j = \emptyset$ $(i \neq j)$ and $i, j \in \{1,2,3 \ldots n\}$. Then the pair (F, A) is said to be IVNHSS over $\mathbb{U}$ if there exists a relation $K_1 \times K_2 \times K_3 \times \ldots \times K_n = A$. Where

F: $K_1 \times K_2 \times K_3 \times \ldots \times K_n \to (\mathbb{U})$ and

$F(K_1 \times K_2 \times K_3 \times \ldots \times K_n) = \{< u, [u_A^L(u), u_A^U(u)], [v_A^L(u), v_A^U(u)], [w_A^L(u), w_A^U(u)] > : u \in \mathbb{U}\}$, where u_A^L, v_A^L, and w_A^L are lower and u_A^U, v_A^U, and w_A^U are upper membership values for truthiness, indeterminacy, and falsity respectively for A and $[u_A^L(u), u_A^U(u)]$, $[v_A^L(u), v_A^U(u)]$, $[w_A^L(u), w_A^U(u)] \subseteq [0, 1]$ and $0 \leq sup\,u_A(u) + sup\,v_A(u) + sup\,w_A(u) \leq 3$ for each $u \in \mathbb{U}$.

Example 1 Assume $\mathbb{U} = \{u_1, u_2\}$ be a universe of discourse and $E = \{x_1, x_2, x_3, x_4\}$ be a set of attributes. Consider F_A be an IVNHSS over $\mathbb{U}$ can be expressed as follows

$$F_A = \{(x_1, \{\langle u_1, [.6, .8], [.5, 0.9], [.1, .4]\rangle, \langle u_2, [.4, .7], [.3, .9], [.2, .6]\rangle\}),$$
$$(x_2, \{\langle u_1, [.4, .7], [.3, .9], [.3, .5]\rangle, \langle u_2, [0, .3], [.6, .8], [.3, .7]\rangle\}),$$
$$(x_3, \{\langle u_1, [.2, .9], [.1, .5], [.7, .8]\rangle, \langle u_2, [.4, .9], [.1, .6], [.5, .7]\rangle\}),$$
$$(x_4, \{\langle u_1, [.6, .9], [.6, .9], [1, 1]\rangle, \langle u_2, [.5, .9], [.6, .8], [.1, .8]\rangle\})\}.$$

Tablur representation of IVNHSS F_A over $\mathbb{U}$ given as follows

Table 1: Tablur representation of IVNHSS F_A

$\mathbb{U}$	u_1	u_1
x_1	$\langle[.6, .8], [.5, .9], [.1, .4]\rangle$	$\langle[.4, .7], [.3, .9], [.2, .6]\rangle$
x_2	$\langle[.4, .7], [.3, .9], [.3, .5]\rangle$	$\langle[0, .3], [.6, .8], [.3, .7]\rangle$
x_3	$\langle[.2, .9], [.1, .5], [.7, .8]\rangle$	$\langle[.4, .9], [.1, .6], [.5, .7]\rangle$
x_4	$\langle[.6, .9], [.6, .9], [1, 1]\rangle$	$\langle[.5, .9], [.6, .8], [.1, .8]\rangle$

3. Operations on Interval Valued Neutrosophic Hypersoft Set with Properties

In this section, we extend the concept of IVNHSS and introduce some fundamental operations on IVNHSS with their properties.

Definition 3.1

Let F_A and $G_B \in$ IVNHSS over $\mathbb{U}$, then $F_A \subseteq G_B$ if

$inf\,u_A(u) \leq inf\,u_B(u),\ sup\,u_A(u) \leq sup\,u_B(u)$

$inf\,v_A(u) \geq inf\,v_B(u),\ sup\,v_A(u) \geq sup\,v_B(u)$

$inf\,w_A(u) \geq inf\,w_B(u),\ sup\,w_A(u) \geq sup\,w_B(u)$

Example 2 Assume $\mathbb{U} = \{u_1,\ u_2\}$ be a universe of discourse and $E = \{x_1,\ x_2,\ x_3,\ x_4\}$ be a set of attributes. Consider G_B be an IVNHSS over $\mathbb{U}$ can be expressed as follows and F_A given in example 1

$$G_B = \{(x_1, \{\langle u_1, [.6, .9], [.3, .7], [.1, .3]\rangle, \langle u_2, [.6, .9], [.3, .5], [.1, .4]\rangle\}),$$
$$(x_2, \{\langle u_1, [.6, .8], [.2, .5], [.2, .3]\rangle, \langle u_2, [.3, .5], [.4, .7], [.1, .4]\rangle\}),$$
$$(x_3, \{\langle u_1, [.4, .9], [.1, .3], [.4, .6]\rangle, \langle u_2, [.6, 1], [.1, .4], [.3, .4]\rangle\}),$$
$$(x_4, \{\langle u_1, [.7, .9], [.4, .6], [.6, 1]\rangle, \langle u_2, [.5, .7], [.4, .7], [.1, .4]\rangle\})\}.$$

Thus

$F_A \subseteq G_B$.

Definition 3.2

Let $F_A \in$ IVNHSS over $\mathbb{U}$, then

i. Empty IVNHSS can be represented as $F_{\breve{0}}$, and defined as follows $F_{\breve{0}} = \{< u, [0, 0],\ [1, 1],\ [1, 1] > : u \in \mathbb{U}\}$.

ii. Universal IVNHSS can be represented as $F_{\breve{E}}$, and defined as follows $F_{\breve{E}} = \{< u, [0, 0],\ [1, 1],\ [1, 1] > : u \in \mathbb{U}\}$.

iii. The complement of IVNHSS can be defined as follows $F_A^c = \{< u, [w_A^L(u), w_A^U(u)], [1 - v_A^U(u), 1 - v_A^L(u)], [u_A^L(u), u_A^U(u)] >: u \in \mathbb{U}\}$.

Example 3 Assume $\mathbb{U} = \{u_1, u_2\}$ be a universe of discourse and $E = \{x_1, x_2, x_3, x_4\}$ be a set of attributes. The tabular representation of $F_{\breve{0}}$ and $F_{\breve{E}}$ given as follows in table 2 and table 3 respectively.

Table 2:Tablur representation of IVNHSS $\boldsymbol{F_{\breve{0}}}$

$\mathbb{U}$	$\boldsymbol{u_1}$	$\boldsymbol{u_1}$
$\boldsymbol{x_1}$	$\langle[0,0],[1,1],[1,1]\rangle$	$\langle[0,0],[1,1],[1,1]\rangle$
$\boldsymbol{x_2}$	$\langle[0,0],[1,1],[1,1]\rangle$	$\langle[0,0],[1,1],[1,1]\rangle$
$\boldsymbol{x_3}$	$\langle[0,0],[1,1],[1,1]\rangle$	$\langle[0,0],[1,1],[1,1]\rangle$
$\boldsymbol{x_4}$	$\langle[0,0],[1,1],[1,1]\rangle$	$\langle[0,0],[1,1],[1,1]\rangle$

Table 3:Tablur representation of IVNHSS $\boldsymbol{F_{\breve{E}}}$

$\mathbb{U}$	$\boldsymbol{u_1}$	$\boldsymbol{u_1}$
$\boldsymbol{x_1}$	$\langle[1,1],[0,0],[0,0]\rangle$	$\langle[1,1],[0,0],[0,0]\rangle$
$\boldsymbol{x_2}$	$\langle[1,1],[0,0],[0,0]\rangle$	$\langle[1,1],[0,0],[0,0]\rangle$
$\boldsymbol{x_3}$	$\langle[1,1],[0,0],[0,0]\rangle$	$\langle[1,1],[0,0],[0,0]\rangle$
$\boldsymbol{x_4}$	$\langle[1,1],[0,0],[0,0]\rangle$	$\langle[1,1],[0,0],[0,0]\rangle$

Proposition 3.3

If $F_{\mathrm{A}} \in$ IVNHSS, then

1. $(F_A^c)^c = F_{\mathrm{A}}$
2. $(F_{\breve{0}})^c = F_{\breve{E}}$
3. $(F_{\breve{E}})^c = F_{\breve{0}}$

Proof 1 Let $F_{\mathrm{A}} = \{< u, [u_A^L(u), u_A^U(u)], [v_A^L(u), v_A^U(u)], [w_A^L(u), w_A^U(u)] >: u \in \mathbb{U}\}$ be an IVNHSS. Then by using definition 3.3(iii), we have

$F_A^c = \{< u, [w_A^L(u), w_A^U(u)], [1 - v_A^U(u), 1 - v_A^L(u)], [u_A^L(u), u_A^U(u)] >: u \in \mathbb{U}\}$

Thus

$(F_A^c)^c = \left\{< u, [u_A^L(u), u_A^U(u)], \left[1 - \left(1 - v_A^L(u)\right), 1 - (1 - v_A^U(u))\right], [w_A^L(u), w_A^U(u)] >: u \in \mathbb{U}\right\}$

$(F_A^c)^c = \{< u, [u_A^L(u), u_A^U(u)], [v_A^L(u), v_A^U(u)], [w_A^L(u), w_A^U(u)] >: u \in \mathbb{U}\}$

$(F_A^c)^c = F_{\mathrm{A}}$

Proof 2

As we know that $F_{\breve{0}} = \{< u, [0,0], [1,1], [1,1] >: u \in \mathbb{U}\}$

By using definition 3.3(iii), we get

$(F_{\breve{0}})^c = \{< u, [1,1], [0,0], [0,0] >: u \in \mathbb{U}\} = F_{\breve{E}}$.

Similarly, we can prove 3.

Definition 3.4

Let F_A and G_B $\in$ IVNHSS over $\mathbb{U}$, then

$$F_A \cup G_B = \left\{ \begin{array}{c} (< u, [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}], \\ [min\{inf\,v_A(u),\ inf\,v_B(u)\}, min\{sup\,v_A(u),\ sup\,v_B(u)\}], \\ [min\{inf\,w_A(u),\ inf\,w_B(u)\}, min\{sup\,w_A(u),\ sup\,w_B(u)\}] >/\ u \in \mathbb{U}) \end{array} \right\}. \quad (1)$$

Example 4 Assume $\mathbb{U} = \{u_1,\ u_2\}$ be a universe of discourse and $E = \{x_1,\ x_2,\ x_3,\ x_4\}$ be a set of attributes. Consider F_A and G_B are IVNHSS over $\mathbb{U}$ can be given as follows

$$\begin{array}{l} F_A = \{(x_1, \{\langle u_1, [.6,.8], [.5,.9], [.1,.4]\rangle, \langle u_2, [.4,.7], [.3,.9], [.2,.6]\rangle\}), \\ (x_2, \{\langle u_1, [.4,.7], [.3,.9], [.3,.5]\rangle, \langle u_2, [.2,.8], [.6,.8], [.3,.7]\rangle\}), \\ (x_3, \{\langle u_1, [.2,.9], [.1,.5], [.4,.7]\rangle, \langle u_2, [.4,.9], [.1,.6], [.5,.7]\rangle\}), \\ (x_4, \{\langle u_1, [.6,.9], [.6,.9], [1,1]\rangle, \langle u_2, [.5,.9], [.6,.8], [.1,.8]\rangle\})\} \end{array}$$

$$\begin{array}{l} G_B = \{(x_1, \{\langle u_1, [.5,.7], [.5,.7], [.4,.6]\rangle, \langle u_2, [.3,.9], [.3,.6], [.4,.7]\rangle\}), \\ (x_2, \{\langle u_1, [.3,.8], [.4,.5], [.4,.9]\rangle, \langle u_2, [.4,.7], [.5,.9], [.4,.6]\rangle\}), \\ (x_3, \{\langle u_1, [.3,.5], [.2,.6], [.3,.8]\rangle, \langle u_2, [.3,1], [.2,.7], [.3,.8]\rangle\}), \\ (x_4, \{\langle u_1, [.4,.6], [.7,.8], [.4,1]\rangle, \langle u_2, [.4,.8], [.3,.6], [.2,.6]\rangle\})\} \end{array}$$

Then

$$\begin{array}{l} F_A \cup G_B = \{(x_1, \{\langle u_1, [.6,.8], [.5,.7], [.1,.4]\rangle, \langle u_2, [.4,.9], [.3,.6], [.2,.6]\rangle\}), \\ (x_2, \{\langle u_1, [.4,.8], [.3,.5], [.3,.5]\rangle, \langle u_2, [.4,.8], [.5,.8], [.3,.6]\rangle\}), \\ (x_3, \{\langle u_1, [.3,.9], [.1,.5], [.3,.7]\rangle, \langle u_2, [.4,1], [.1,.6], [.3,.7]\rangle\}), \\ (x_4, \{\langle u_1, [.6,.9], [.6,.8], [.4\ 1]\rangle, \langle u_2, [.5,.9], [.3,.6], [.1,.6]\rangle\})\} \end{array}$$

Proposition 3.5

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, $\mathcal{H}_{\breve{C}}$ $\in$ IVNHSS over $\mathbb{U}$. Then

1. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{A}} = \mathcal{F}_{\breve{A}}$
2. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{0}} = \mathcal{F}_{\breve{0}}$
3. $\mathcal{F}_{\breve{A}} \cup \mathcal{F}_{\breve{E}} = \mathcal{F}_{\breve{A}}$
4. $\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cup \mathcal{F}_{\breve{A}}$
5. $(\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \cup \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \cup (\mathcal{G}_{\breve{B}} \cup \mathcal{H}_{\breve{C}})$

Proof By using definition 3.4 we can prove easily.

Definition 3.6

Let F_A and G_B $\in$ IVNHSS over $\mathbb{U}$, then

$$F_A \cap G_B = \left\{ \begin{array}{c} < u, [min\{inf\,u_A(u),\ inf\,u_B(u)\}, min\{sup\,u_A(u),\ sup\,u_B(u)\}], \\ [max\{inf\,v_A(u),\ inf\,v_B(u)\}, max\{sup\,v_A(u),\ sup\,v_B(u)\}], \\ [max\{inf\,w_A(u),\ inf\,w_B(u)\}, max\{sup\,w_A(u), sup\,w_B(u)\}] >/\ u \in \mathbb{U}) \end{array} \right\}. \quad (2)$$

Example 5 Reconsider example 4

$$\begin{array}{l} F_A = \{(x_1, \{\langle u_1, [.6,.8], [.5,.9], [.1,.4]\rangle, \langle u_2, [.4,.7], [.3,.9], [.2,.6]\rangle\}), \\ (x_2, \{\langle u_1, [.4,.7], [.3,.9], [.3,.5]\rangle, \langle u_2, [.2,.8], [.6,.8], [.3,.7]\rangle\}), \end{array}$$

$(x_3, \{\langle u_1, [.2, .9], [.1, .5], [.4, .7]\rangle, \langle u_2, [.4, .9], [.1, .6], [.5, .7]\rangle\}),$

$(x_4, \{\langle u_1, [.6, .9], [.6, .9], [1, 1]\rangle, \langle u_2, [.5, .9], [.6, .8], [.1, .8]\rangle\})\}$

$G_B = \{(x_1, \{\langle u_1, [.5, .7], [.5, .7], [.4, .6]\rangle, \langle u_2, [.3, .9], [.3, .6], [.4, .7]\rangle\}),$

$(x_2, \{\langle u_1, [.3, .8], [.4, .5], [.4, .9]\rangle, \langle u_2, [.4, .7], [.5, .9], [.4, .6]\rangle\}),$

$(x_3, \{\langle u_1, [.3, .5], [.2, .6], [.3, .8]\rangle, \langle u_2, [.3, 1], [.2, .7], [.3, .8]\rangle\}),$

$(x_4, \{\langle u_1, [.4, .6], [.7, .8], [.4, 1]\rangle, \langle u_2, [.4, .8], [.3, .6], [.2, .6]\rangle\})\}$

Then

$F_A \cap G_B = \{(x_1, \{\langle u_1, [.5, .7], [.5, .9], [.4, .6]\rangle, \langle u_2, [.3, .7], [.3, .9], [.4, .7]\rangle\}),$

$(x_2, \{\langle u_1, [.3, .7], [.4, .9], [.4, .9]\rangle, \langle u_2, [.2, .7], [.6, .9], [.4, .7]\rangle\}),$

$(x_3, \{\langle u_1, [.2, .5], [.2, .6], [.4, .8]\rangle, \langle u_2, [.3, .9], [.2, .7], [.5, .8]\rangle\}),$

$(x_4, \{\langle u_1, [.4, .6], [.7, .9], [1, 1]\rangle, \langle u_2, [.4, .8], [.6, .8], [.2, .8]\rangle\})\}$

Proposition 3.7

Let $\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}, \mathcal{H}_{\breve{C}} \in$ IVNHSS over $\mathbb{U}$. Then

1. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{A}} = \mathcal{F}_{\breve{A}}$
2. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{0}} = \mathcal{F}_{\breve{A}}$
3. $\mathcal{F}_{\breve{A}} \cap \mathcal{F}_{\breve{E}} = \mathcal{F}_{\breve{E}}$
4. $\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}} = \mathcal{G}_{\breve{B}} \cap \mathcal{F}_{\breve{A}}$
5. $(\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) \cap \mathcal{H}_{\breve{C}} = \mathcal{F}_{\breve{A}} \cap (\mathcal{G}_{\breve{B}} \cap \mathcal{H}_{\breve{C}})$

Proof By using definition 3.6 we can prove easily.

Proposition 3.8

Let F_A and $G_B \in$ IVNHSS over $\mathbb{U}$, then

1. $(F_A \cup G_B)^C = {F_A}^C \cap {G_B}^C$
2. $(F_A \cap G_B)^C = {F_A}^C \cup {G_B}^C$

Proof 1 As we know that

$F_A = \{< u, u_A(u), v_A(u), w_A(u) > : u \in \mathbb{U}\}$ and $G_B = \{< u, u_B(u), v_B(u), w_B(u) > : u \in \mathbb{U}\}$. Where

$u_A(u) = [inf u_A(u),\ sup u_A(u)]$ or $[u_A^L(u), u_A^U(u)]$, $u_A^L(u) = inf u_A(u)$ and $u_A^U(u) = sup u_A(u)$

$v_A(u) = [inf v_A(u),\ sup v_A(u)]$ or $[v_A^L(u), v_A^U(u)]$, $v_A^L(u) = inf v_A(u)$ and $v_A^U(u) = sup v_A(u)$

$w_A(u) = [\ inf w_A(u),\ sup w_A(u)\ ]$ or $[w_A^L(u), w_A^U(u)]$, $w_A^L(u) = inf w_A(u)$ and $w_A^U(u) = sup w_A(u)$

$u_B(u) = [inf u_B(u),\ sup u_B(u)]$ or $[u_B^L(u), u_B^U(u)]$, $u_B^L(u) = inf u_B(u)$ and $u_B^U(u) = sup u_B(u)$

$v_B(u) = [inf v_B(u),\ sup v_B(u)]$ or $[v_B^L(u), v_B^U(u)]$, $v_B^L(u) = inf v_B(u)$ and $v_B^U(u) = sup v_B(u)$

$w_B(u) = [\ inf w_B(u),\ sup w_B(u)\ ]$ or $[w_B^L(u), w_B^U(u)]$, $w_B^L(u) = inf w_B(u)$ and $w_B^U(u) = sup w_B(u)$

Then by using Equation 1

$$F_A \cup G_B = \left\{ \begin{array}{c} (< u, [max\{inf u_A(u),\ inf u_B(u)\}, max\{sup u_A(u),\ sup u_B(u)\}], \\ [min\{inf v_A(u),\ inf v_B(u)\}, min\{sup v_A(u),\ sup v_B(u)\}], \\ [min\{inf w_A(u),\ inf w_B(u)\}, min\{sup w_A(u),\ sup w_B(u)\}] >/ u \in \mathbb{U}) \end{array} \right\}$$

By using definition 3.3(iii), we get

$$(F_A \cup G_B)^C = \left\{ \begin{array}{c} (< u, [min\{inf\,w_A(u),\ inf\,w_B(u)\}, min\{sup\,w_A(u),\ sup\,w_B(u)\}], \\ [1 - min\{sup\,v_A(u),\ sup\,v_B(u)\}, 1 - min\{inf\,v_A(u),\ inf\,v_B(u)\}], \\ [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}] > / \ u \in \mathbb{U}) \end{array} \right\}$$

Now

$$F_A{}^C = \{< u, [inf\,w_A(u), sup\,w_A(u)], [1 - sup\,v_A(u), 1 - inf\,v_A(u)], [inf\,u_A(u), sup\,u_A(u)] > : u \in \mathbb{U}\}$$

$$G_B{}^C = \{< u, [inf\,w_B(u), sup\,w_B(u)], [1 - sup\,v_B(u), 1 - inf\,v_B(u)], [inf\,u_B(u), sup\,u_B(u)] > : u \in \mathbb{U}\}$$

$$F_A{}^C \cap G_B{}^C = \left\{ \begin{array}{c} (< u, [min\{inf\,w_A(u),\ inf\,w_B(u)\}, min\{sup\,w_A(u),\ sup\,w_B(u)\}], \\ [max\{1 - sup\,v_A(u),\ 1 - sup\,v_B(u)\}, max\{1 - inf\,v_A(u),\ 1 - inf\,v_B(u)\}], \\ [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}] > / \ u \in \mathbb{U}) \end{array} \right\}$$

$$F_A{}^C \cap G_B{}^C = \left\{ \begin{array}{c} (< u, [min\{inf\,w_A(u),\ inf\,w_B(u)\}, min\{sup\,w_A(u),\ sup\,w_B(u)\}], \\ [1 - min\{sup\,v_A(u), sup\,v_B(u)\}, 1 - min\{inf\,v_A(u),\ inf\,v_B(u)\}], \\ [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}] > / \ u \in \mathbb{U}) \end{array} \right\}$$

Hence

$(F_A \cup G_B)^C = F_A{}^C \cap G_B{}^C$

Proof 2

Similar to assertion 1.

Proposition 3.9

Let $\mathcal{F}_{\breve{A}}, \mathcal{G}_{\breve{B}}, \mathcal{H}_{\breve{C}} \in$ IVNHSS over $\mathbb{U}$. Then

1. $\mathcal{F}_{\breve{A}} \cup (\mathcal{G}_{\breve{B}} \cap \mathcal{H}_{\breve{C}}) = (\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \cap (\mathcal{F}_{\breve{A}} \cup \mathcal{H}_{\breve{C}})$
2. $\mathcal{F}_{\breve{A}} \cap (\mathcal{G}_{\breve{B}} \cup \mathcal{H}_{\breve{C}}) = (\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) \cup (\mathcal{F}_{\breve{A}} \cap \mathcal{H}_{\breve{C}})$
3. $\mathcal{F}_{\breve{A}} \cup (\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) = \mathcal{F}_{\breve{A}}$
4. $\mathcal{F}_{\breve{A}} \cap (\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) = \mathcal{F}_{\breve{A}}$

Proof 1 From Equation 2, we have

$$\mathcal{G}_{\breve{B}} \cap \mathcal{H}_{\breve{C}} = \left\{ \begin{array}{c} (< u, [min\{inf\,u_B(u),\ inf\,u_C(u)\}, min\{sup\,u_B(u),\ sup\,u_C(u)\}], \\ [max\{inf\,v_B(u),\ inf\,v_C(u)\}, max\{sup\,v_B(u),\ sup\,v_C(u)\}], \\ [max\{inf\,w_B(u),\ inf\,w_C(u)\}, max\{sup\,w_B(u),\ sup\,w_C(u)\}] > / \ u \in \mathbb{U}) \end{array} \right\}$$

$\mathcal{F}_{\breve{A}} \cup (\mathcal{G}_{\breve{B}} \cap \mathcal{H}_{\breve{C}}) =$

$$\left\{ \begin{array}{c} (< u, [\max\{inf\,u_A(u), min\{inf\,u_B(u),\ inf\,u_C(u)\}\}, \max\{sup\,u_A(u), min\{sup\,u_B(u), sup\,u_C(u)\}\}], \\ [\min\{inf\,v_A(u), max\{inf\,v_B(u),\ inf\,v_C(u)\}\}, \min\{sup\,v_A(u), max\{sup\,v_B(u), sup\,v_C(u)\}\}], \\ [min\{inf\,w_A(u), max\{inf\,w_B(u),\ inf\,w_C(u)\}\}, min\{sup\,w_A(u), max\{sup\,w_B(u),\ sup\,w_C(u)\}\}] > / \ u \in \mathbb{U}) \end{array} \right\}$$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}} = \left\{ \begin{array}{c} (< u, [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}], \\ [min\{inf\,v_A(u),\ inf\,v_B(u)\}, min\{sup\,v_A(u),\ sup\,v_B(u)\}], \\ [min\{inf\,w_A(u),\ inf\,w_B(u)\}, min\{sup\,w_A(u),\ sup\,w_B(u)\}] > / \ u \in \mathbb{U}) \end{array} \right\}.$$

$$\mathcal{F}_{\breve{A}} \cup \mathcal{H}_{\breve{C}} = \left\{ \begin{array}{c} (< u, [max\{inf\,u_A(u),\ inf\,u_C(u)\}, max\{sup\,u_A(u),\ sup\,u_C(u)\}], \\ [min\{inf\,v_A(u),\ inf\,v_C(u)\}, min\{sup\,v_A(u),\ sup\,v_C(u)\}], \\ [min\{inf\,w_A(u),\ inf\,w_C(u)\}, min\{sup\,w_A(u),\ sup\,w_C(u)\}] > / \ u \in \mathbb{U}) \end{array} \right\}.$$

$(\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \cap (\mathcal{F}_{\breve{A}} \cup \mathcal{H}_{\breve{C}}) =$

$$\left\{\begin{array}{c}(< u, [\min\{max\{inf\mathcal{u}_A(u),\ inf\mathcal{u}_B(u)\}, max\{inf\mathcal{u}_A(u),\ inf\mathcal{u}_C(u)\}\}, \min\{max\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_B(u)\}, max\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_C(u)\}\}],\\ [\max\{min\{inf\mathcal{v}_A(u),\ inf\mathcal{v}_B(u)\}, min\{inf\mathcal{v}_A(u),\ inf\mathcal{v}_C(u)\}\}, \max\{min\{sup\mathcal{v}_A(u),\ sup\mathcal{v}_B(u)\}, min\{sup\mathcal{v}_A(u),\ sup\mathcal{v}_C(u)\}\}],\\ [max\{min\{inf\mathcal{w}_A(u),\ inf\mathcal{w}_B(u)\}, min\{inf\mathcal{w}_A(u),\ inf\mathcal{w}_C(u)\}\}, max\{min\,\{sup\mathcal{w}_A(u),\ sup\mathcal{w}_B(u)\}, sup\mathcal{w}_A(u),\ sup\mathcal{w}_C(u)\}\}] >/ u \in \mathbb{U})\end{array}\right\}$$

$(\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \cap (\mathcal{F}_{\breve{A}} \cup \mathcal{H}_{\breve{C}}) =$

$$\left\{\begin{array}{c}(< u, [\max\{inf\mathcal{u}_A(u), min\{inf\mathcal{u}_B(u),\ inf\mathcal{u}_C(u)\}\}, \max\{sup\mathcal{u}_A(u), min\{sup\mathcal{u}_B(u), sup\mathcal{u}_C(u)\}\}],\\ [\min\{inf\mathcal{v}_A(u), max\{inf\mathcal{v}_B(u),\ inf\mathcal{v}_C(u)\}\}, \min\{sup\mathcal{v}_A(u), max\{sup\mathcal{v}_B(u), sup\mathcal{v}_C(u)\}\}],\\ [min\{inf\mathcal{w}_A(u), max\{inf\mathcal{w}_B(u),\ inf\mathcal{w}_C(u)\}\}, min\{sup\mathcal{w}_A(u), max\{sup\mathcal{w}_B(u),\ sup\mathcal{w}_C(u)\}\}] >/ u \in \mathbb{U})\end{array}\right\}$$

Hence

$\mathcal{F}_{\breve{A}} \cup (\mathcal{G}_{\breve{B}} \cap \mathcal{H}_{\breve{C}}) = (\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \cap (\mathcal{F}_{\breve{A}} \cup \mathcal{H}_{\breve{C}})$.

Similarly, we can prove other results.

Definition 3.10

Let $F_A,\ G_B \in$ IVNHSS, then their extended union is

$$\mathcal{u}\ (F_A \cup G_B) = \begin{cases}[inf\mathcal{u}_A(u), sup\mathcal{u}_A(u)] & if\ u \in A - B\\ [inf\mathcal{u}_B(u), sup\mathcal{u}_B(u)] & if\ u \in B - A\\ [max\{inf\mathcal{u}_A(u),\ inf\mathcal{u}_B(u)\}, max\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_B(u)\}] & if\ u \in A \cap B\end{cases}$$

$$\mathcal{v}\ (F_A \cup G_B) = \begin{cases}[inf\mathcal{v}_A(u), sup\mathcal{v}_A(u)] & if\ u \in A - B\\ [inf\mathcal{v}_B(u), sup\mathcal{v}_B(u)] & if\ u \in B - A\\ [min\{inf\mathcal{v}_A(u),\ inf\mathcal{v}_B(u)\}, min\{sup\mathcal{v}_A(u),\ sup\mathcal{v}_B(u)\}] & if\ u \in A \cap B\end{cases}$$

$$\mathcal{w}\ (F_A \cup G_B) = \begin{cases}[inf\mathcal{w}_A(u), sup\mathcal{w}_A(u)] & if\ u \in A - B\\ [inf\mathcal{w}_B(u), sup\mathcal{w}(u)] & if\ u \in B - A\\ [min\{inf\mathcal{w}_A(u),\ inf\mathcal{w}_B(u)\}, min\{sup\mathcal{w}_A(u),\ sup\mathcal{w}_B(u)\}] & if\ u \in A \cap B\end{cases}$$

Definition 3.11

Let $F_A,\ G_B \in$ IVNHSS, then their extended intersection is

$$\mathcal{u}\ (F_A \cap G_B) = \begin{cases}[inf\mathcal{u}_A(u), sup\mathcal{u}_A(u)] & if\ u \in A - B\\ [inf\mathcal{u}_B(u), sup\mathcal{u}_B(u)] & if\ u \in B - A\\ [min\{inf\mathcal{u}_A(u),\ inf\mathcal{u}_B(u)\}, min\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_B(u)\}] & if\ u \in A \cap B\end{cases}$$

$$\mathcal{v}\ (F_A \cap G_B) = \begin{cases}[inf\mathcal{v}_A(u), sup\mathcal{v}_A(u)] & if\ u \in A - B\\ [inf\mathcal{v}_B(u), sup\mathcal{v}_B(u)] & if\ u \in B - A\\ [max\{inf\mathcal{v}_A(u),\ inf\mathcal{v}_B(u)\}, max\{sup\mathcal{v}_A(u),\ sup\mathcal{v}_B(u)\}] & if\ u \in A \cap B\end{cases}$$

$$\mathcal{w}\ (F_A \cap G_B) = \begin{cases}[inf\mathcal{w}_A(u), sup\mathcal{w}_A(u)] & if\ u \in A - B\\ [inf\mathcal{w}_B(u), sup\mathcal{w}(u)] & if\ u \in B - A\\ [max\{inf\mathcal{w}_A(u),\ inf\mathcal{w}_B(u)\}, max\{sup\mathcal{w}_A(u),\ sup\mathcal{w}_B(u)\}] & if\ u \in A \cap B\end{cases}$$

Definition 3.12

Let F_A and $G_B \in$ IVNHSS over $\mathbb{U}$, then their difference defined as follows

$$F_A \setminus G_B = \left\{\begin{array}{c}(< u, [min\{inf\mathcal{u}_A(u),\ inf\mathcal{u}_B(u)\}, min\{sup\mathcal{u}_A(u),\ sup\mathcal{u}_B(u)\}],\\ [max\{inf\mathcal{v}_A(u),\ 1 - \sup\mathcal{v}_B(u)\}, max\{sup\mathcal{v}_A(u),\ 1 - \inf\mathcal{v}_B(u)\}],\\ [max\{inf\mathcal{w}_A(u),\ inf\mathcal{w}_B(u)\}, max\{sup\mathcal{w}_A(u),\ sup\mathcal{w}_B(u)\}] >/ u \in \mathbb{U})\end{array}\right\}. \quad (3)$$

Example 6 Reconsider example 4

$$F_A \setminus G_B = \{(x_1, \{\langle u_1, [.5, .7], [.5, .9], [.4, .6]\rangle, \langle u_2, [.3, .7], [.4, .9], [.4, .7]\rangle\}),$$
$$(x_2, \{\langle u_1, [.3, .7], [.5, .9], [.4, .9]\rangle, \langle u_2, [.2, .7], [.6, .8], [.4, .7]\rangle\}),$$

$$(x_3, \{\langle u_1, [.2, .5], [.4, .8], [.4, .8]\rangle, \langle u_2, [.3, .9], [.3, .8], [.5, .8]\rangle\}),$$
$$(x_4, \{\langle u_1, [.4, .6], [.6, .9], [1, 1]\rangle, \langle u_2, [.4, .8], [.6, .8], [.2, .8]\rangle\})\}$$

Definition 3.13

Let F_A and G_B ∈ IVNHSS over $\mathbb{U}$, then their addition defined as follows

$$F_A + G_B = \begin{Bmatrix} (< u, [min\{inf\,u_A(u) + inf\,u_B(u), 1\}, min\{sup\,u_A(u) + sup\,u_B(u), 1\}], \\ [min\{inf\,v_A(u) + inf\,v_B(u), 1\}, min\{sup\,v_A(u) + sup\,v_B(u), 1\}], \\ [min\{inf\,w_A(u) + inf\,w_B(u), 1\}, min\{sup\,w_A(u) + sup\,w_B(u), 1\}] > / \, u \in \mathbb{U}) \end{Bmatrix}. \quad (4)$$

Example 7 Reconsider example 4

$$F_A + G_B = \{(x_1, \{\langle u_1, [1.0, 1.0], [1.0, 1.0], [0.5, 1.0]\rangle, \langle u_2, [0.7, 1.0], [0.6, 1.0], [0.6, 1.0]\rangle\}),$$
$$(x_2, \{\langle u_1, [0.7, 1.0], [0.7, 1.0], [0.7, 1.0]\rangle, \langle u_2, [0.6, 1.0], [1.0, 1.0], [0.7, 1.0]\rangle\}),$$
$$(x_3, \{\langle u_1, [0.5, 1.0], [0.3, 1.0], [0.7, 1.0]\rangle, \langle u_2, [0.7, 1.0], [0.3, 1.0], [0.8, 1.0]\rangle\}),$$
$$(x_4, \{\langle u_1, [1.0, 1.0], [1.0, 1.0], [1.0, 1.0]\rangle, \langle u_2, [0.9, 1.0], [0.9, 1.0], [0.3, 1.0]\rangle\})\}.$$

Definition 3.14

Let F_A ∈ IVNHSS over $\mathbb{U}$, then its scalar multiplication is represented as $F_A.\breve{a}$, where $\breve{a} \in [0, 1]$ and defined as follows

$$F_A.\breve{a} = \begin{Bmatrix} (< u, [min\{inf\,u_A(u).\breve{a}, 1\}, min\{sup\,u_A(u).\breve{a}, 1\}], \\ [min\{inf\,v_A(u).\breve{a}, 1\}, min\{sup\,v_A(u).\breve{a}, 1\}], \\ [min\{inf\,w_A(u).\breve{a}, 1\}, min\{sup\,w_A(u).\breve{a}, 1\}] > / \, u \in \mathbb{U}) \end{Bmatrix}. \quad (5)$$

Definition 3.15

Let F_A ∈ IVNHSS over $\mathbb{U}$, then its scalar division is represented as $F_A/\breve{a}$, where $\breve{a} \in [0, 1]$ and defined as follows

$$F_A/\breve{a} = \begin{Bmatrix} (< u, [min\{inf\,u_A(u)/\breve{a}, 1\}, min\{sup\,u_A(u)/\breve{a}, 1\}], \\ [min\{inf\,v_A(u)/\breve{a}, 1\}, min\{sup\,v_A(u)/\breve{a}, 1\}], \\ [min\{inf\,w_A(u)/\breve{a}, 1\}, min\{sup\,w_A(u)/\breve{a}, 1\}] > / \, u \in \mathbb{U}) \end{Bmatrix}. \quad (6)$$

Definition 3.16

Let F_A ∈ IVNHSS over $\mathbb{U}$, then Truth-Favorite operator on F_A is denoted by $\widetilde{\Delta} F_A$ and defined as follows

$$\widetilde{\Delta} F_A = \begin{Bmatrix} (< u, [min\{inf\,u_A(u) + inf\,v_A(u), 1\}, min\{sup\,u_A(u) + sup\,v_A(u), 1\}], [0, 0], \\ [inf\,w_A(u), sup\,w_A(u)] > / \, u \in \mathbb{U}) \end{Bmatrix}. \quad (7)$$

Example 8 Reconsider example 1

$$\widetilde{\Delta} F_A = \{(x_1, \{\langle u_1, [1, 1], [0, 0], [.1, .4]\rangle, \langle u_2, [.7, 1], [0, 0], [.2, .6]\rangle\}),$$
$$(x_2, \{\langle u_1, [.7, 1], [0, 0], [.3, .5]\rangle, \langle u_2, [.6, 1], [0, 0], [.3, .7]\rangle\}),$$
$$(x_3, \{\langle u_1, [.3, 1], [0, 0], [.7, .8]\rangle, \langle u_2, [.5, 1], [0, 0], [.5, .7]\rangle\}),$$
$$(x_4, \{\langle u_1, [1, 1], [0, 0], [1, 1]\rangle, \langle u_2, [1, 1], [0, 0], [.1, .8]\rangle\})\}$$

Proposition 3.17

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$ ∈ IVNHSS over $\mathbb{U}$, then

1. $\widetilde{\Delta}\widetilde{\Delta}\mathcal{F}_{\breve{A}} = \widetilde{\Delta}\mathcal{F}_{\breve{A}}$
2. $\widetilde{\Delta}(\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \subseteq \widetilde{\Delta}\mathcal{F}_{\breve{A}} \cup \widetilde{\Delta}\mathcal{G}_{\breve{B}}$

3. $\widetilde{\Delta}(\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) \subseteq \widetilde{\Delta}\mathcal{F}_{\breve{A}} \cap \widetilde{\Delta}\mathcal{G}_{\breve{B}}$
4. $\widetilde{\Delta}(\mathcal{F}_{\breve{A}} + \mathcal{G}_{\breve{B}}) = \widetilde{\Delta}\mathcal{F}_{\breve{A}} + \widetilde{\Delta}\mathcal{G}_{\breve{B}}$

Proof of the above proposition is easily obtained by using definitions 3.4, 3.6, 3.13, and 3.16.

Definition 3.18

Let $F_A \in$ IVNHSS over $\mathbb{U}$, then False-Favorite operator on F_A is denoted by $\tilde{\nabla}F_A$ and defined as follows

$$\tilde{\nabla}F_A = \left\{\begin{matrix} (< u, [inf\,\mathcal{u}_A(u), sup\,\mathcal{u}_A(u)], [0,0], \\ [min\{inf\,\mathcal{w}_A(u) + inf\,\mathcal{v}_A(u), 1\}, min\{sup\,\mathcal{w}_A(u) + sup\,\mathcal{v}_A(u), 1\}] >/ u \in \mathbb{U}) \end{matrix}\right\}. \quad (8)$$

Example 9 Reconsider example 1

$$\tilde{\nabla}F_A = \{(x_1, \{\langle u_1, [.6, .8], [0,0], [.6, 1]\rangle, \langle u_2, [.4, .7], [0,0], [.5, 1]\rangle\}),$$
$$(x_2, \{\langle u_1, [.4, .7], [0,0], [.6, 1]\rangle, \langle u_2, [0, .3], [0,0], [.9, 1]\rangle\}),$$
$$(x_3, \{\langle u_1, [.2, .9], [0,0], [.8, 1]\rangle, \langle u_2, [.4, .9], [0,0], [.6, 1]\rangle\}),$$
$$(x_4, \{\langle u_1, [.6, .9], [0,0], [1, 1]\rangle, \langle u_2, [.5, .9], [0,0], [.7, 1]\rangle\})\}$$

Proposition 3.19

Let $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}} \in$ IVNHSS over $\mathbb{U}$, then

1. $\tilde{\nabla}\tilde{\nabla}\mathcal{F}_{\breve{A}} = \tilde{\nabla}\mathcal{F}_{\breve{A}}$
2. $\tilde{\nabla}(\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) \subseteq \tilde{\nabla}\mathcal{F}_{\breve{A}} \cup \tilde{\nabla}\mathcal{G}_{\breve{B}}$
3. $\tilde{\nabla}(\mathcal{F}_{\breve{A}} \cap \mathcal{G}_{\breve{B}}) \subseteq \tilde{\nabla}\mathcal{F}_{\breve{A}} \cap \tilde{\nabla}\mathcal{G}_{\breve{B}}$
4. $\tilde{\nabla}(\mathcal{F}_{\breve{A}} + \mathcal{G}_{\breve{B}}) = \tilde{\nabla}\mathcal{F}_{\breve{A}} + \tilde{\nabla}\mathcal{G}_{\breve{B}}$

Proof of the above proposition is easily obtained by using definitions 3.4, 3.6, 3.13, and 3.18.

4. Necessity and Possibility Operations on IVNHSS

In this section, some further operations on IVNHSS are developed such as OR-Operation, And-Operation, necessity, and possibility operations with some properties.

Definition 4.1

Let F_A and $G_B \in$ IVNHSS over $\mathbb{U}$, then OR-Operator is represented by $F_A \vee G_B$ and defined as follows

$$\mathcal{u}\,(F_{A\times \mathrm{B}}) = [max\{inf\,\mathcal{u}_A(u), inf\,\mathcal{u}_B(u)\}, max\{sup\,\mathcal{u}_A(u), sup\,\mathcal{u}_B(u)\}],$$
$$\mathcal{v}\,(F_{A\times \mathrm{B}}) = [min\{inf\,\mathcal{v}_A(u), inf\,\mathcal{v}_B(u)\}, min\{sup\,\mathcal{v}_A(u), sup\,\mathcal{v}_B(u)\}],$$
$$\mathcal{w}\,(F_{A\times \mathrm{B}}) = [min\{inf\,\mathcal{w}_A(u), inf\,\mathcal{w}_B(u)\}, min\{sup\,\mathcal{w}_A(u), sup\,\mathcal{w}_B(u)\}].$$

Definition 4.2

Let F_A and $G_B \in$ IVNHSS over $\mathbb{U}$, then And-Operator is represented by $F_A \wedge G_B$ and defined as follows

$$\mathcal{u}\,(F_{A\times \mathrm{B}}) = [min\{inf\,\mathcal{u}_A(u), inf\,\mathcal{u}_B(u)\}, min\{sup\,\mathcal{u}_A(u), sup\,\mathcal{u}_B(u)\}],$$
$$\mathcal{v}\,(F_{A\times \mathrm{B}}) = [max\{inf\,\mathcal{v}_A(u), inf\,\mathcal{v}_B(u)\}, max\{sup\,\mathcal{v}_A(u), sup\,\mathcal{v}_B(u)\}],$$
$$\mathcal{w}\,(F_{A\times \mathrm{B}}) = [max\{inf\,\mathcal{w}_A(u), inf\,\mathcal{w}_B(u)\}, max\{sup\,\mathcal{w}_A(u), sup\,\mathcal{w}_B(u)\}].$$

Proposition 4.3

Let $\mathcal{F}_{\breve{A}}$, $\mathcal{G}_{\breve{B}}$, $\mathcal{H}_{\breve{C}} \in$ IVNHSSs, then

1. $\mathcal{F}_{\check{A}} \vee \mathcal{G}_{\check{B}} = \mathcal{G}_{\check{B}} \vee \mathcal{F}_{\check{A}}$
2. $\mathcal{F}_{\check{A}} \wedge \mathcal{G}_{\check{B}} = \mathcal{G}_{\check{B}} \wedge \mathcal{F}_{\check{A}}$
3. $\mathcal{F}_{\check{A}} \vee (\mathcal{G}_{\check{B}} \vee \mathcal{H}_{\check{C}}) = (\mathcal{F}_{\check{A}} \vee \mathcal{G}_{\check{B}}) \vee \mathcal{H}_{\check{C}}$
4. $\mathcal{F}_{\check{A}} \wedge (\mathcal{G}_{\check{B}} \wedge \mathcal{H}_{\check{C}}) = (\mathcal{F}_{\check{A}} \wedge \mathcal{G}_{\check{B}}) \wedge \mathcal{H}_{\check{C}}$
5. $(\mathcal{F}_{\check{A}} \vee \mathcal{G}_{\check{B}})^c = \mathcal{F}^c(\check{A}) \wedge \mathcal{G}^c(\check{B})$
6. $(\mathcal{F}_{\check{A}} \wedge \mathcal{G}_{\check{B}})^c = \mathcal{F}^c(\check{A}) \vee \mathcal{G}^c(\check{B})$

Proof We can prove easily by using definitions 4.1 and 4.2.

Definition 4.4

Let $F_A \in$ IVNHSS over $\mathbb{U}$, then necessity operator IVNHSS represented as $\oplus F_A$ and defined as follows

$$\oplus F_A = \{< u, [inf u_A(u), sup u_A(u)], [inf v_A(u), sup v_A(u)], [1 - sup u_A(u), 1 - inf u_A(u)] >: u \in \mathbb{U}\}$$

Example 10 Reconsider example 1

$$\begin{aligned}\oplus F_A = \{&(x_1, \{\langle u_1, [.6, .8], [.5, 0.9], [.2, .4]\rangle, \langle u_2, [.4, .7], [.3, .9], [.3, .6]\rangle\}),\\ &(x_2, \{\langle u_1, [.4, .7], [.3, .9], [.3, .6]\rangle, \langle u_2, [0, .3], [.6, .8], [.7, 1]\rangle\}),\\ &(x_3, \{\langle u_1, [.2, .9], [.1, .5], [.1, .8]\rangle, \langle u_2, [.4, .9], [.1, .6], [.1, .6]\rangle\}),\\ &(x_4, \{\langle u_1, [.6, .9], [.6, .9], [.1, .4]\rangle, \langle u_2, [.5, .9], [.6, .8], [.1, .5]\rangle\})\}\end{aligned}$$

Definition 4.5

Let $F_A \in$ IVNHSS over $\mathbb{U}$, then possibility operator on IVNHSS represented as $\otimes F_A$ and defined as follows

$$\otimes F_A = \{(< u, [1 - sup w_A(u), 1 - inf w_A(u)],\ [inf v_A(u),\ sup v_A(u)],\ [inf w_A(u),\ sup w_A(u)] >/ u \in \mathbb{U}\})\}$$

Example 11 Reconsider example 1

$$\begin{aligned}\otimes F_A = \{&(x_1, \{\langle u_1, [.6, .9], [.5, 0.9], [.1, .4]\rangle, \langle u_2, [.4, .8], [.3, .9], [.2, .6]\rangle\}),\\ &(x_2, \{\langle u_1, [.5, .7], [.3, .9], [.3, .5]\rangle, \langle u_2, [.3, .7], [.6, .8], [.3, .7]\rangle\}),\\ &(x_3, \{\langle u_1, [.2, .3], [.1, .5], [.7, .8]\rangle, \langle u_2, [.3, .5], [.1, .6], [.5, .7]\rangle\}),\\ &(x_4, \{\langle u_1, [0, 0], [.6, .9], [1, 1]\rangle, \langle u_2, [.2, .9], [.6, .8], [.1, .8]\rangle\})\}\end{aligned}$$

Proposition 4.6

Let $\mathcal{F}_{\check{A}}$ and $\mathcal{G}_{\check{B}} \in$ IVNHSS over $\mathbb{U}$, then

1. $\oplus (\mathcal{F}_{\check{A}} \cup \mathcal{G}_{\check{B}}) = \oplus \mathcal{F}_{\check{A}} \cup \oplus \mathcal{G}_{\check{B}}$
2. $\oplus (\mathcal{F}_{\check{A}} \cap \mathcal{G}_{\check{B}}) = \oplus \mathcal{F}_{\check{A}} \cap \oplus \mathcal{G}_{\check{B}}$

Proof 1. As we know that

$\mathcal{F}_{\check{A}} = \{(< u, [inf u_A(u),\ sup u_A(u)],\ [inf v_A(u),\ sup v_A(u)],\ [inf w_A(u),\ sup w_A(u)] >/ u \in \mathbb{U}\})\}$ and

$\mathcal{G}_{\check{B}} = \{(< u, [inf u_B(u),\ sup u_B(u)],\ [inf v_B(u),\ sup v_B(u)],\ [inf w_B(u),\ sup w_B(u)] >/ u \in \mathbb{U}\})\}$

Then by using definition 3.5, we get

$$\mathcal{F}_{\check{A}} \cup \mathcal{G}_{\check{B}} = \left\{\begin{matrix} (< u, [max\{inf u_A(u),\ inf u_B(u)\}, max\{sup u_A(u),\ sup u_B(u)\}], \\ [min\{inf v_A(u),\ inf v_B(u)\}, min\{sup v_A(u),\ sup v_B(u)\}], \\ [min\{inf w_A(u),\ inf w_B(u)\}, min\{sup w_A(u),\ sup w_B(u)\}] >/ u \in \mathbb{U}) \end{matrix}\right\}.$$

By using the necessity operator, we get

$$\oplus\ (\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) = \left\{\begin{matrix} (< u, [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}], \\ [min\{inf\,v_A(u),\ inf\,v_B(u)\}, min\{sup\,v_A(u),\ sup\,v_B(u)\}], \\ [1 - max\{sup\,u_A(u),\ sup\,u_B(u)\}, 1 - max\{inf\,u_A(u),\ inf\,u_B(u)\}] >/ u \in \mathbb{U}) \end{matrix}\right\}.$$

$\oplus\ (\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) =$

$$\left\{\begin{matrix} (< u, [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}], \\ [min\{inf\,v_A(u),\ inf\,v_B(u)\}, min\{sup\,v_A(u),\ sup\,v_B(u)\}], \\ [min\{1 - sup\,u_A(u),\ 1 - sup\,u_B(u)\}, min\{1 - inf\,u_A(u),\ 1 - inf\,u_B(u)\}] >/ u \in \mathbb{U}) \end{matrix}\right\}.$$

$\oplus\ \mathcal{F}_{\breve{A}} = \{(< u, [inf\,u_A(u),\ sup\,u_A(u)],\ [inf\,v_A(u),\ sup\,v_A(u)],\ [1 - sup\,u_A(u), 1 - inf\,u_A(u)] >/ u \in \mathbb{U}\})\}$ and

$\oplus\ \mathcal{G}_{\breve{B}} = \{(< u, [inf\,u_B(u),\ sup\,u_B(u)],\ [inf\,v_B(u),\ sup\,v_B(u)],\ [1 - sup\,u_B(u), 1 - inf\,u_B(u)] >/ u \in \mathbb{U}\})\}$

Again, by using definition 3.5 we get

$\oplus\ \mathcal{F}_{\breve{A}} \cup \oplus\ \mathcal{G}_{\breve{B}} =$

$$\left\{\begin{matrix} (< u, [max\{inf\,u_A(u),\ inf\,u_B(u)\}, max\{sup\,u_A(u),\ sup\,u_B(u)\}], \\ [min\{inf\,v_A(u),\ inf\,v_B(u)\}, min\{sup\,v_A(u),\ sup\,v_B(u)\}], \\ [min\{1 - sup\,u_A(u),\ 1 - sup\,u_B(u)\}, min\{1 - inf\,u_A(u),\ 1 - inf\,u_B(u)\}] >/ u \in \mathbb{U}) \end{matrix}\right\}$$

Hence

$\oplus\ (\mathcal{F}_{\breve{A}} \cup \mathcal{G}_{\breve{B}}) = \oplus\ \mathcal{F}_{\breve{A}} \cup \oplus\ \mathcal{G}_{\breve{B}}$

Similarly, we can prove assertion 2.

Proposition 4.7

Let $\mathcal{F}_{\breve{A}}$ and $\mathcal{G}_{\breve{B}} \in$ IVNHSS, then we have the following

1. $\oplus(\mathcal{F}_{\breve{A}} \wedge \mathcal{G}_{\breve{B}}) = \oplus\mathcal{F}_{\breve{A}} \wedge \oplus\mathcal{G}_{\breve{B}}$
2. $\oplus(\mathcal{F}_{\breve{A}} \vee \mathcal{G}_{\breve{B}}) = \oplus\mathcal{F}_{\breve{A}} \vee \oplus\mathcal{G}_{\breve{B}}$
3. $\otimes\ (\mathcal{F}_{\breve{A}} \wedge \mathcal{G}_{\breve{B}}) = \otimes\ \mathcal{F}_{\breve{A}} \wedge \otimes\ \mathcal{G}_{\breve{B}}$
4. $\otimes\ (\mathcal{F}_{\breve{A}} \vee \mathcal{G}_{\breve{B}}) = \otimes\ \mathcal{F}_{\breve{A}} \vee \otimes\ \mathcal{G}_{\breve{B}}$

Proof By using definitions 4.1, 4.2, 4.4, and 4.5 the proof of the above proposition can be obtained easily.

5. Conclusion

In this paper, we study NHSS and IVNHSS with some basic definitions and examples. We extend the work on IVNHSS and proposed some fundamental operations on IVNHSS such as union, intersection, extended union, extended intersection, addition, and difference, etc. are developed with their properties and proved the De Morgan laws by using union, intersection, OR-operation, and And-Operation. We also developed the addition, difference, scalar multiplication, Truth-Favorite, and False-Favorite operators on IVNHSS. Finally, the concept of necessity and possibility operations on IVNHSS with properties are presented. For future trends, we can develop the interval-valued neutrosophic hypersoft matrices by using proposed operations and use them for decision making. Furthermore, several other operators such as weighted average, weighted geometric, interaction weighted average, interaction weighted geometric, etc. can be developed with their decision-making approaches to solve MCDM problems.

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An integrated model of Neutrosophic TOPSIS with application in Multi-Criteria Decision-Making Problem

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Abstract: Multi-criteria decision making (MCDM) is the technique of selecting the best alternative from multiple alternatives and multiple conditions. The technique for order preference by similarity to an ideal solution (TOPSIS) is a crucial practical technique for ranking and selecting different options by using a distance measure. In this article, we protract the fuzzy TOPSIS technique to neutrosophic fuzzy TOPSIS and prove the accuracy of the method by explaining the MCDM problem with single-valued neutrosophic information and use the method for supplier selection in the production industry. We hope that this article will promote future scientific research on numerous existing issues based on multi-criteria decision making.

Keywords: Neutrosophic set, Single valued Neutrosophic set, TOPSIS, MCDM

1. Introduction

We faced a lot of complications in different areas of life which contain vagueness such as engineering, economics, modeling, and medical diagnoses, etc. However, a general question is raised that in mathematical modeling how we can express and use the uncertainty. A lot of researchers in the world proposed and recommended different approaches to solve those problems that contain uncertainty. In decision-making problems, multiple attribute decision making (MADM) is the most essential part which provides us to find the most appropriate and extraordinary alternative. However, choosing the appropriate alternative is very difficult because of vague information in some cases. To overcome such situations, Zadeh developed the notion of fuzzy sets (FSs) [1] to solve those problems which contain uncertainty and vagueness. Fuzzy sets are like sets whose components have membership (Mem) degrees. In the classical set theory, the Mem degree of the elements in the set is checked in binary form according to the bivalent condition of whether the elements completely belong to the set. In contrast, the fuzzy set theory allows modern ratings of the Mem of elements in the set. This is represented by the Mem function, and the effective unit interval of the Mem function is [0, 1]. The fuzzy set is the generalization of the classical set because the indicator function of the classic set is a special case of the Mem function of the fuzzy set if the latter only takes the value 0 or 1. In the fuzzy set theory, the classical bivalent set is usually called the crisp set. Fuzzy set theory can be used in a wide range of fields with incomplete or imprecise information.

It is observed that in some cases circumstances cannot be handled by fuzzy sets, to overcome such types of situations Turksen [2] gave the idea of interval-valued fuzzy sets (IVFSs). In some cases, we must deliberate membership unbiassed as the non-membership values for the suitable representation of an object in uncertain and indeterminate conditions that could not be handled by FSs nor IVFSs. To overcome these difficulties Atanassov offered the concept of Intuitionistic fuzzy sets (IFSs) [3]. The theory which was presented by Atanassov only deals the insufficient data considering both the membership and non-membership values, but the intuitionistic fuzzy set theory cannot handle the incompatible and imprecise information. To deal with such incompatible and imprecise data Smarandache [4] extended the work of Atanassov IFSs and proposed a powerful tool comparative to FSs and IFSs to deal with indeterminate, incomplete, and inconsistent information's faced in real-life problems. Since the direct use of Neutrosophic sets (NSs) for TOPSIS is somewhat difficult. To apply the NSs, Wang et al. introduced a subclass of NSs known as single-valued Neutrosophic sets (SVNSs) in [5]. In [6] the author proposed a geometric interpretation by using NSs. Gulfam et al. [7] introduced a new distance formula for SVNSs and developed some new techniques under the Neutrosophic environment. The concept of a single-valued Neutrosophic soft expert set is proposed in [8] by combining the SVNSs and soft expert sets. To solve MCDM problems with single-valued Neutrosophic numbers (SVNNs) presented by Deli and Subas in [9], they constructed the concept of cut sets of SVNNs. On the base of the correlation of IFSs, the term correlation coefficient of SVNSs [10] introduced and proposed a decision-making method by using a weighted correlation coefficient or the weighted cosine similarity measure of SVNSs. In [11] the idea of simplified Neutrosophic sets introduced with some operational laws and aggregation operators such as real-life Neutrosophic weighted arithmetic average operator and weighted geometric average operator. They constructed an MCDM method based on proposed aggregation operators and cosine similarity measure for simplified neutrosophic sets. Sahin and Yiğider [12] extended the TOPSIS method to MCDM with a single-valued neutrosophic technique.

Hwang and Yoon [13] established TOPSIS to solve the general difficulties of DM. The TOPSIS method can effectively maintain the minimum distance from the ideal solution, thereby helping to select the finest choice. After the TOPSIS technique came out, some investigators utilized the TOPSIS technique for DM and protracted the TOPSIS technique to several other hybrid structures of FS. The most important determinant of current scientific research is to present an integrated model for neutrosophic TOPSIS to solve the MCDM problem. Chen & Hwang [14] extended the idea of the TOPSIS method and proposed a new TOPSIS model. The author uses the newly proposed decision-making method to solve uncertain data [15]. Zulqarnain et al. [16] utilized the TOPSIS method for the prediction of diabetic patients in medical diagnosis. They also utilized the TOPSIS extensions of different hybrid structures of FS [17–19] and used them for decision making. Pramanik et al. [21] established the TOPSIS to resolve the multi-attribute decision-making problem under a single-valued neutrosophic soft set expert scenario. Zulqarnain et al. [21] presented the generalized neutrosophic TOPSIS to solve the MCDM problem. Zulqarnain et al. [22] utilized fuzzy TOPSIS to solve the MCDM problem. Maji [23] proposed the concept of neutrosophic soft sets (NSSs) with some properties and operations. The authors studied NSSs and gave some new definitions on NSSs [24], they also gave the idea of neutrosophic soft matrices with some operations and proposed a decision-making method. Many researchers developed the decision-making models by using the NSSs reported in the literature [25–27]. Elhassouny and Smarandache [28] extended the work on a simplified TOPSIS method and by using single-valued Neutrosophic information they proposed Neutrosophic simplified TOPSIS method. The concept of single-valued neutrosophic cross-entropy measure introduced by Jun [29], he also constructed an MCDM method and claimed that this proposed method is more appropriate than previous methods for decision making.

Saha and Broumi [31] studied the interval-valued neutrosophic sets (IVNSs) and developed some new set-theoretic operations on IVNSs with their properties. The idea of an Interval-valued generalized single valued neutrosophic trapezoidal number (IVGSVTrN) was presented by Deli [32] with some operations and discussed their properties based on neutrosophic numbers. Hashim et al

[33], studied the vague set and interval neutrosophic set and established a new theory known as interval neutrosophic vague set (INVS), they also presented some operations for INVS with their properties and derived the properties by using numerical examples. Abdel basset et al. [34] applied TODIM and TOPSIS methods based on the best-worst method to increase the accuracy of evaluation under uncertainty according to the NSs. They also used the Plithogenic set theory to resolve the indeterminate information and evaluate the economic performance of manufacturing industries, they used the AHP method to find the weight vector of the financial ratios to achieve this goal after that they used the VIKOR and TOPSIS methods to utilize the companies ranking [35, 36]. Nabeeh et al. [37] utilized the integrating neutrosophic analytical hierarchy process (AHP) with the TOPSIS for personal selection. Nabeeh et al. [38] developed the AHP neutrosophic by merging the AHP and NS. Abdel-Basset et al. [39] merged the AHP, MCDM approach, and NS to handle the indefinite and irregularity in decision making. Abdel-Basset et al. [40] constructed the TOPSIS technique for type-2 neutrosophic numbers and utilized the presented approach for supplier selection. Abdel-Basset et al. [41] utilized the neutrosophic TOPSIS for the selection of medical instruments and many. Saqlain *et. al.* applied TOPSIS for the prediction of sports, and in MCDM problems [42-44].

The FS and IFS theories do not provide any information about the indeterminacy part of the object. Because the above work is considered to examine the environment of linear inequality between the degree of membership (MD) and the degree of non-membership (NMD) of the considered attributes. However, all existing studies only deal with the scenario by using MD and NMD of attributes. If any decision-maker considers the truthiness, falsity, and indeterminacy of any attribute of the alternatives, then clearly, we can see that it cannot be handled by the above-mentioned FS and IFS theories. To overcome the above limitations, Smarandache [4] proposed the NS to solve uncertain objects by considering the truthiness, falsity, and indeterminacy. In the following article, we explain some positive impacts of this research. The concentration of this study is to evaluate the best supplier for the production industry. This research is a very suitable illustration of Neutrosophic TOPSIS. A group of decision-makers chooses the best supplier for the production industry. The Neutrosophic TOPSIS method increases alternative performances based on the best and worst solutions. Classical TOPSIS uses clear techniques for language assessment, but due to the imprecision and ambiguity of language assessment, we propose neutrosophic TOPSIS. In this paper, we discuss the NSs and SVNSs with some operations. We presented the generalization of TOPSIS for the SVNSs and use the proposed method for supplier selection.

In Section 2, some basic definitions have been added, which will help us to design the structure of the current article. In section 3, we develop an integrated model to solve the MCDM problem under single-valued neutrosophic information. We also established the graphical and mathematical structure of the proposed TOPSIS approach. To ensure the validity of the developed methodology we presented a numerical illustration for supplier selection in the production industry in section 4.

2. Preliminaries

In this section, we remind some basic definitions such as NSs and SVNSs with some operations that will be used in the following sequel.

Neutrosophic Set (NS) [30]: Let X be a space of points and x be an arbitrary element of X. A neutrosophic set A in X is defined by a Truth-membership function $T_A(x)$, an Indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$ i.e.; $T_A(x)$, $I_A(x)$, $F_A(x)$: $X \to]0^-, 1^+[$, and $0^- \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$.

Single Valued Neutrosophic Sets [5]**:** Let E be a universe. An SVNS over E is an NS over E, but truthiness, indeterminacy, and falsity membership functions are defined

$T_A(x): X \to [0, 1]$, $I_A(x): X \to [0, 1]$, $F_A(x): X \to [0, 1]$, and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Multiplication of SVNS [11]: Let A = $\{\alpha_1, \alpha_2, \alpha_3\}$ and B = $\{\beta_1, \beta_2, \beta_3\}$ are two SVN numbers, then their multiplication is defined as follows $A \otimes B = (\alpha_1\beta_1, \alpha_2 + \beta_2 - \alpha_2\beta_2, \alpha_3 + \beta_3 - \alpha_3\beta_3)$.

3. Neutrosophic TOPSIS [11]

3. 1. Algorithm for Neutrosophic TOPSIS using SVNNs

To explain the procedure of Neutrosophic TOPSIS using SVNNs the following steps are followed. Let A = $\{A_1, A_2, A_3, \ldots, A_m\}$ be a set of alternatives and C = $\{C_1, C_2, C_3, \ldots, C_n\}$ be a set of evaluation criteria and *DM* be a set of "*l*" decision-makers as follows *DM* = $\{DM_1, DM_2, DM_3, \ldots, DM_l\}$. In the form of linguistic variables, the importance of the evaluation criteria, DMs, and alternative ratings are given in Table 1.

Step 1: Computation of weights of the DMs

Let the SVN number for rating the k^{th} DM is denoted by

$D_k = (T_k^{dm}, I_k^{dm}, F_k^{dm})$

The weight of the k^{th} DM can be found by the following formula

$$\lambda_k = \frac{1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}}{\sum_{k=1}^{l}\left(1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}\right)} \text{ ; where } \lambda_k \ge 0 \text{ and } \sum_{k=1}^{l} \lambda_k = 1 \quad (1)$$

Step 2: Computation of the Aggregated Neutrosophic Decision Matrix (ANDM)

The ANDM is given as follows

$$D = \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} = [r_{ij}]_{m\times n} \quad (2)$$

where r_{ij} can be defined as

$r_{ij} = (T_{ij}, I_{ij}, F_{ij}) = (T_{A_i}(x_j), I_{A_i}(x_j), F_{A_i}(x_j))$, where $i = 1, 2, 3, \ldots, m;\ j = 1, 2, 3, \ldots, n$

Therefore, ANDM written as follows

$$D = \begin{bmatrix} (T_{A_1}(x_1), I_{A_1}(x_1), F_{A_1}(x_1)) & (T_{A_1}(x_2), I_{A_1}(x_2), F_{A_1}(x_2)) & \cdots & (T_{A_1}(x_n), I_{A_1}(x_n), F_{A_1}(x_n)) \\ (T_{A_2}(x_1), I_{A_2}(x_1), F_{A_2}(x_1)) & (T_{A_2}(x_2), I_{A_2}(x_2), F_{A_2}(x_2)) & \cdots & (T_{A_2}(x_n), I_{A_2}(x_n), F_{A_2}(x_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{A_m}(x_1), I_{A_m}(x_1), F_{A_m}(x_1)) & (T_{A_m}(x_2), I_{A_m}(x_2), F_{A_m}(x_2)) & \cdots & (T_{A_m}(x_n), I_{A_m}(x_n), F_{A_m}(x_n)) \end{bmatrix}$$

rating for the i^{th} alternative w.r.t. the j^{th} criterion by the k^{th} *DM*

$$r_{ij}^{(k)} = (T_{ij}^{(k)}, I_{ij}^{(k)}, F_{ij}^{(k)})$$

For *DM* weights and alternative ratings r_{ij} can be calculated by using a single-valued neutrosophic weighted averaging operator (SVNWAO)

$$r_{ij} = \left[1 - \prod_{k=1}^{l}(1 - T_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(I_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(F_{ij}^{(k)})^{\lambda_k}\right] \quad (3)$$

Step 3: Computation of the weights for the criteria

Let an SVNN allocated to the criterion by X_j the k^{th} *DM* is denoted as

$$w_j^{(k)} = \left(T_j^{(k)}, I_j^{(k)}, F_j^{(k)}\right)$$

SVNWAO to compute the weights of the criteria is given as follows

$$w_j = \left[1 - \prod_{k=1}^{l}(1 - T_j^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(I_j^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(F_j^{(k)})^{\lambda_k}\right] \quad (4)$$

The aggregated weight for the criterion X_j is represented as

$$w_j = (T_j, I_j, F_j) \qquad j = 1, 2, 3, \ldots, n$$

$$W = [w_1, w_2, w_3, \ldots, w_n]^{Transpose}$$

Step 4: Computation of Aggregated Weighted Neutrosophic Decision Matrix (AWNDM)

The AWNDM is calculated as follows

$$R' = \begin{bmatrix} r'_{11} & r'_{12} & \cdots & r'_{1n} \\ r'_{21} & r'_{22} & \cdots & r'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r'_{m1} & r'_{m2} & \cdots & r'_{mn} \end{bmatrix} = \left[r'_{ij}\right]_{m\times n} \quad (5)$$

where $r'_{ij} = \left(T_{A_i.W}(x_j), I_{A_i.W}(x_j), F_{A_i.W}(x_j)\right)$ where $i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$.

Therefore, R' can be written as

$$R' = \begin{bmatrix} (T_{A_1.W}(x_1), I_{A_1.W}(x_1), F_{A_1.W}(x_1)) & (T_{A_1.W}(x_2), I_{A_1.W}(x_2), F_{A_1.W}(x_2)) & \cdots & (T_{A_1.W}(x_n), I_{A_1.W}(x_n), F_{A_1.W}(x_n)) \\ (T_{A_2.W}(x_1), I_{A_2.W}(x_1), F_{A_2.W}(x_1)) & (T_{A_2.W}(x_2), I_{A_2.W}(x_2), F_{A_2.W}(x_2)) & \cdots & (T_{A_2.W}(x_n), I_{A_2.W}(x_n), F_{A_2.W}(x_n)) \\ \vdots & \vdots & \ddots & \vdots \\ (T_{A_m.W}(x_1), I_{A_m.W}(x_1), F_{A_m.W}(x_1)) & (T_{A_m.W}(x_2), I_{A_m.W}(x_2), F_{A_m.W}(x_2)) & \cdots & (T_{A_m.W}(x_n), I_{A_m.W}(x_n), F_{A_m.W}(x_n)) \end{bmatrix}$$

To find $T_{A_i.W}(x_j)$, $I_{A_i.W}(x_j)$ and $F_{A_i.W}(x_j)$ we used

$$R \otimes W = \left\{\langle x, T_{A_i.W}(x)\rangle, \langle x, I_{A_i.W}(x)\rangle, \langle x, F_{A_i.W}(x)\rangle \,\middle|\, x \in X\right\} \quad (6)$$

The components of the product given as

$T_{A_i.W}(x) = T_{A_i}(x).\ T_j$

$I_{A_i.W}(x) = I_{A_i}(x) + I_j(x) - I_{A_i}(x) \times I_j(x)$

$F_{A_i.W}(x) = F_{A_i}(x) + F_j(x) - F_{A_i}(x) \times F_j(x)$

Step 5: Computation of Single Valued Neutrosophic Positive Ideal Solution (SVN-PIS) and Single Valued Neutrosophic Positive Ideal Solution (SVN-NIS)

Let J_1 be the benefit criteria and J_2 be the cost criteria. A^* be an SVN-PIS and A' be an SVN-NIS as follows

$A^* = (T_{A^*W}(x_j), I_{A^*W}(x_j), F_{A^*W}(x_j))$ and

$A' = (T_{A'W}(x_j), I_{A'W}(x_j), F_{A'W}(x_j))$

The components of SVN-PIS and SVN-NIS are following

$$T_{A^*W}(x_j) = \left(\left(\max_i T_{A_i.W}(x_j) \mid j \in j_1\right), \left(\min_i T_{A_i.W}(x_j) \mid j \in j_2\right)\right)$$

$$I_{A^*W}(x_j) = \left(\left(\min_i I_{A_i.W}(x_j) \mid j \in j_1\right), \left(\max_i I_{A_i.W}(x_j) \mid j \in j_2\right)\right)$$

$$F_{A^*W}(x_j) = \left(\left(\min_i F_{A_i.W}(x_j) \mid j \in j_1\right), \left(\max_i F_{A_i.W}(x_j) \mid j \in j_2\right)\right)$$

$$T_{A'W}(x_j) = \left(\left(\min_i T_{A_i.W}(x_j) \mid j \in j_1\right), \left(\max_i T_{A_i.W}(x_j) \mid j \in j_2\right)\right)$$

$$I_{A'W}(x_j) = \left(\left(\max_i I_{A_i.W}(x_j) \mid j \in j_1\right), \left(\min_i I_{A_i.W}(x_j) \mid j \in j_2\right)\right)$$

$$F_{A'W}(x_j) = \left(\left(\max_i F_{A_i.W}(x_j) \mid j \in j_1\right), \left(\min_i F_{A_i.W}(x_j) \mid j \in j_2\right)\right)$$

Step 6: Computation of Separation Measures

For the separation measures $d^{*}$and d', Normalized Euclidean Distance is used as given as

$$d_i^{*}= \left(\frac{1}{3n}\sum_{j=1}^{n}\left[\left(T_{A_i.W}(x_j)-T_{A^*W}\,(x_j)\right)^2+\left(I_{A_i.W}(x_j)-I_{A^*W}\,(x_j)\right)^2+\left(F_{A_i.W}(x_j)-F_{A^*W}\,(x_j)\right)^2\right]\right)^{0.5} \quad (7)$$

$$d_i'= \left(\frac{1}{3n}\sum_{j=1}^{n}\left[\left(T_{A_i.W}(x_j)-T_{A'W}\,(x_j)\right)^2+\left(I_{A_i.W}(x_j)-I_{A'W}\,(x_j)\right)^2+\left(F_{A_i.W}(x_j)-F_{A'W}\,(x_j)\right)^2\right]\right)^{0.5} \quad (8)$$

Step 7: Computation of Relative Closeness Coefficient (*RCC*)

The *RCC* of an alternative *Ai* w.r.t. the SVN-PIS A^* is computed as

$$RCCi = \frac{d_i'}{d_i'+d_i^*} \qquad \text{where } 0 \le RCCi \le 1 \quad (9)$$

Step 8: Ranking alternatives

After computation of *RCCi* for each alternative A_i, the rank of the alternatives presented in descending orders of *RCCi*.

The flow chart of the presented technique can be seen in Figure 1.

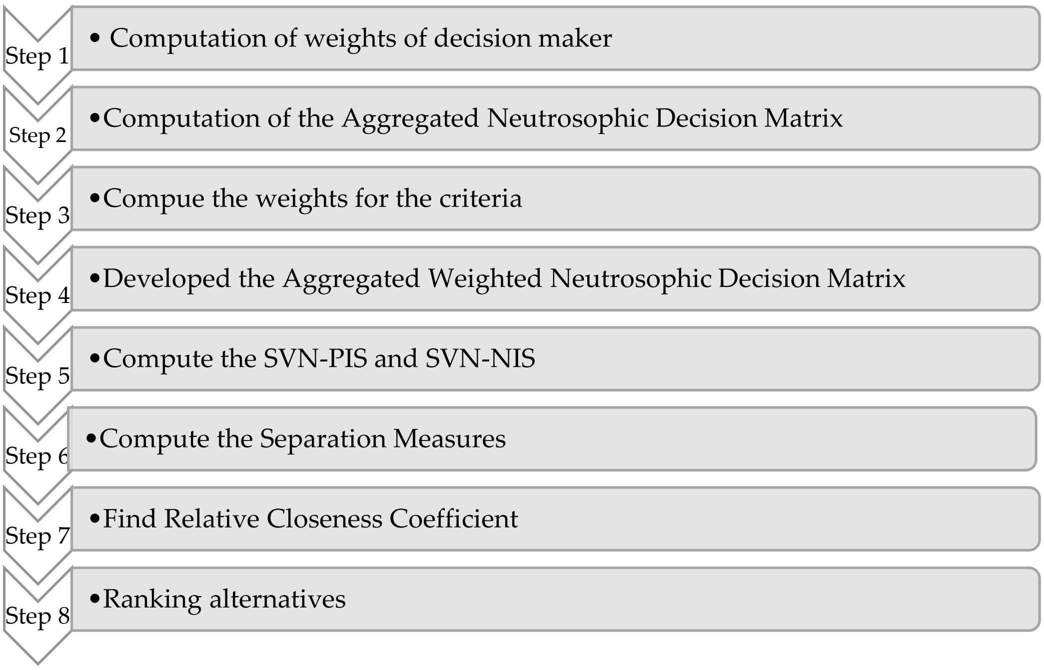

Figure 1: Flow chart of the presented approach

4. Application of Neutrosophic TOPSIS in decision making

A production industry wants to hire a supplier, for the selection of supplier managing director of the industry decides the criteria for supplier selection. The industry hires a team of decision-makers for the selection of the best supplier. Consider A = {A_i: i = 1, 2, 3, 4, 5} be a set of supplier and $DM = \{DM_1, DM_2, DM_3, DM_4\}$ be a team of decision-makers (l = 4). The evaluation criteria (n = 5) for the selection of supplier given as follows,

$$C = \begin{cases} Benifit\ Criteria \\ Cost\ Criteria \end{cases} \quad j_1 = \begin{cases} X_1: & Delivery \\ X_2: & Quality \\ X_3: & Flexibility \\ X_4: & Service \end{cases} \quad j_2 = \{X_5 : Price$$

Calculations of the problem using the proposed SVN-TOPSIS for the importance of criteria and DMs SVN rating scale is given in the following Table

Table 1. Linguistic variables LV's for rating the importance of criteria and decision-makers

LVs	SVNNs
VI	(.90, .10, .10)
I	(.75, .25, .20)
M	(.50, .50, .50)
UI	(.35, .75, .80)
VUI	(.10, .90, .90)

Where VI, I, M, UI, VUI stand for very important, important, medium, unimportant, very unimportant respectively. The alternative ratings are given in the following table

Table 2. Alternative Ratings for Linguistic Variables

LVs	SVNNs
EG	(1.0, 0.0,0.0)
VVG	(.90, .10, .10)
VG	(.80, .15, .20)
G	(.70, .25, .30)
MG	(.60, .35, .40)
M	(.50, .50, .50)
MB	(.40, .65, .60)
B	(.30, .75, .70)
VB	(.20, .85, .80)
VVB	(.10, .90, .90)
EB	(0.0,1.0,1.0)

Where EG, VVG, VG, G, MG, M, MB, B, VB, VVB, EB are representing extremely good, very very good, very good, good, medium good, medium, medium bad, bad, very bad, very very bad, extremely bad respectively.

Step 1: Determine the weights of the DMs

By using Equation 1, weights for the *DMs* are calculated as follows:

$$\lambda_k = \frac{1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}}{\sum_{k=1}^{l}\left(1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}\right)} \;;\; \lambda_k \geq 0 \text{ and } \sum_{k=1}^{l} \lambda_k = 1$$

$$\lambda_1 = \frac{1-\left[\frac{1}{3}\left\{\left(1-T_1^{dm}(x)\right)^2+\left(I_1^{dm}(x)\right)^2+\left(F_1^{dm}(x)\right)^2\right\}\right]^{0.5}}{\sum_{k=1}^{l}\left(1-\left[\frac{1}{3}\left\{\left(1-T_k^{dm}(x)\right)^2+\left(I_k^{dm}(x)\right)^2+\left(F_k^{dm}(x)\right)^2\right\}\right]^{0.5}\right)}$$

$$\lambda_1 = \frac{1-\left[\frac{1}{3}\left\{\left(1-T_1^{dm}(x)\right)^2+\left(I_1^{dm}(x)\right)^2+\left(F_1^{dm}(x)\right)^2\right\}\right]^{0.5}}{\begin{gathered}1-\left[\frac{1}{3}\left\{\left(1-T_1^{dm}(x)\right)^2+\left(I_1^{dm}(x)\right)^2+\left(F_1^{dm}(x)\right)^2\right\}\right]^{0.5}+1-\left[\frac{1}{3}\left\{\left(1-T_2^{dm}(x)\right)^2+\left(I_2^{dm}(x)\right)^2+\left(F_2^{dm}(x)\right)^2\right\}\right]^{0.5}+\\ 1-\left[\frac{1}{3}\left\{\left(1-T_3^{dm}(x)\right)^2+\left(I_3^{dm}(x)\right)^2+\left(F_3^{dm}(x)\right)^2\right\}\right]^{0.5}+1-\left[\frac{1}{3}\left\{\left(1-T_4^{dm}(x)\right)^2+\left(I_4^{dm}(x)\right)^2+\left(F_4^{dm}(x)\right)^2\right\}\right]^{0.5}\end{gathered}}$$

$$\lambda_1 = \frac{1-\left[\frac{1}{3}\{(1-0.9)^2+(0.10)^2+(0.10)^2\}\right]^{0.5}}{\begin{gathered}1-\left[\frac{1}{3}\{(1-0.9)^2+(0.10)^2+(0.10)^2\}\right]^{0.5}+1-\left[\frac{1}{3}\{(1-0.75)^2+(0.25)^2+(0.20)^2\}\right]^{0.5}+\\ 1-\left[\frac{1}{3}\{(1-0.50)^2+(0.50)^2+(0.50)^2\}\right]^{0.5}+1-\left[\frac{1}{3}\{(1-0.35)^2+(0.75)^2+(0.80)^2\}\right]^{0.5}\end{gathered}}$$

$\lambda_1 = \frac{0.9}{0.9+0.76548+0.5+0.26402}$

$\lambda_1 = \frac{0.9}{2.42950} = 0.37045$

$\lambda_1 = 0.37045$

Similarly, we get the weights for the other decision-makers as follows

$\lambda_2 = \frac{0.76548}{2.42950} = 0.31508$

$\lambda_2 = 0.31508$

$\lambda_3 = \frac{0.5}{2.42950} = 0.20580$

$\lambda_3 = 0.20580$

$\lambda_4 = \frac{0.26402}{2.42950} = 0.10867$

$\lambda_4 = 0.10867$

The weights for *DMs* are given in the following Table

Table 3. Weights of Decision Makers

Criteria	**Alternatives**	**Decision Makers**			
		DM1	**DM2**	**DM3**	**DM4**
X1	*A1*	VG (0.80,0.15,0.20)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)
		$r_{11}^{(1)} = (T_{11}^{(1)}, I_{11}^{(1)}, F_{11}^{(1)})$	$r_{11}^{(2)} = (T_{11}^{(2)}, I_{11}^{(2)}, F_{11}^{(2)})$	$r_{11}^{(3)} = (T_{11}^{(3)}, I_{11}^{(3)}, F_{11}^{(3)})$	$r_{11}^{(4)} = (T_{11}^{(4)}, I_{11}^{(4)}, F_{11}^{(4)}$
	A2	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)
		$r_{21}^{(1)} = (T_{21}^{(1)}, I_{21}^{(1)}, F_{21}^{(1)})$	$r_{21}^{(2)} = (T_{21}^{(2)}, I_{21}^{(2)}, F_{21}^{(2)})$	$r_{21}^{(3)} = (T_{21}^{(3)}, I_{21}^{(3)}, F_{21}^{(3)})$	$r_{21}^{(4)} = (T_{21}^{(4)}, I_{21}^{(4)}, F_{21}^{(4)})$
	A3	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)
		$r_{31}^{(1)} = (T_{31}^{(1)}, I_{31}^{(1)}, F_{31}^{(1)})$	$r_{31}^{(2)} = (T_{31}^{(2)}, I_{31}^{(2)}, F_{31}^{(2)})$	$r_{31}^{(3)} = (T_{31}^{(3)}, I_{31}^{(3)}, F_{31}^{(3)})$	$r_{31}^{(4)} = (T_{31}^{(4)}, I_{31}^{(4)}, F_{31}^{(4)})$
	A4	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)
		$r_{41}^{(1)} = (T_{41}^{(1)}, I_{41}^{(1)}, F_{41}^{(1)})$	$r_{41}^{(2)} = (T_{41}^{(2)}, I_{41}^{(2)}, F_{41}^{(2)})$	$r_{41}^{(3)} = (T_{41}^{(3)}, I_{41}^{(3)}, F_{41}^{(3)})$	$r_{41}^{(4)} = (T_{41}^{(4)}, I_{41}^{(4)}, F_{41}^{(4)})$
	A5	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)
		$r_{51}^{(1)} = (T_{51}^{(1)}, I_{51}^{(1)}, F_{51}^{(1)})$	$r_{51}^{(2)} = (T_{51}^{(2)}, I_{51}^{(2)}, F_{51}^{(2)})$	$r_{51}^{(3)} = (T_{51}^{(3)}, I_{51}^{(3)}, F_{51}^{(3)})$	$r_{51}^{(4)} = (T_{51}^{(4)}, I_{51}^{(4)}, F_{51}^{(4)})$
X2	*A1*	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)
		$r_{12}^{(1)} = (T_{12}^{(1)}, I_{12}^{(1)}, F_{12}^{(1)})$	$r_{12}^{(2)} = (T_{12}^{(2)}, I_{12}^{(2)}, F_{12}^{(2)})$	$r_{12}^{(3)} = (T_{12}^{(3)}, I_{12}^{(3)}, F_{12}^{(3)})$	$r_{12}^{(4)} = (T_{12}^{(4)}, I_{12}^{(4)}, F_{12}^{(4)})$
	A2	VG (0.80,0.15,0.20)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)
		$r_{22}^{(1)} = (T_{22}^{(1)}, I_{22}^{(1)}, F_{22}^{(1)})$	$r_{22}^{(2)} = (T_{22}^{(2)}, I_{22}^{(2)}, F_{22}^{(2)})$	$r_{22}^{(3)} = (T_{22}^{(3)}, I_{22}^{(3)}, F_{22}^{(3)})$	$r_{22}^{(4)} = (T_{22}^{(4)}, I_{22}^{(4)}, F_{22}^{(4)})$
	A3	M (0.50,0.50,0.50)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)
		$r_{32}^{(1)} = (T_{32}^{(1)}, I_{32}^{(1)}, F_{32}^{(1)})$	$r_{32}^{(2)} = (T_{32}^{(2)}, I_{32}^{(2)}, F_{32}^{(2)})$	$r_{32}^{(3)} = (T_{32}^{(3)}, I_{32}^{(3)}, F_{32}^{(3)})$	$r_{32}^{(4)} = (T_{32}^{(4)}, I_{32}^{(4)}, F_{32}^{(4)})$
	A4	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)
		$r_{42}^{(1)} = (T_{42}^{(1)}, I_{42}^{(1)}, F_{42}^{(1)})$	$r_{42}^{(2)} = (T_{42}^{(2)}, I_{42}^{(2)}, F_{42}^{(2)})$	$r_{42}^{(3)} = (T_{42}^{(3)}, I_{42}^{(3)}, F_{42}^{(3)})$	$r_{42}^{(4)} = (T_{42}^{(4)}, I_{42}^{(4)}, F_{42}^{(4)})$
	A5	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)
		$r_{52}^{(1)} = (T_{52}^{(1)}, I_{52}^{(1)}, F_{52}^{(1)})$	$r_{52}^{(2)} = (T_{52}^{(2)}, I_{52}^{(2)}, F_{52}^{(2)})$	$r_{52}^{(3)} = (T_{52}^{(3)}, I_{52}^{(3)}, F_{52}^{(3)})$	$r_{52}^{(4)} = (T_{52}^{(4)}, I_{52}^{(4)}, F_{52}^{(4)})$
X3	*A1*	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)

		$r_{13}^{(1)} = (T_{13}^{(1)}, I_{13}^{(1)}, F_{13}^{(1)})$	$r_{13}^{(2)} = (T_{13}^{(2)}, I_{13}^{(2)}, F_{13}^{(2)})$	$r_{13}^{(3)} = (T_{13}^{(3)}, I_{13}^{(3)}, F_{13}^{(3)})$	$r_{13}^{(4)} = (T_{13}^{(4)}, I_{13}^{(4)}, F_{13}^{(4)})$
	A_2	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)
		$r_{23}^{(1)} = (T_{23}^{(1)}, I_{23}^{(1)}, F_{23}^{(1)})$	$r_{23}^{(2)} = (T_{23}^{(2)}, I_{23}^{(2)}, F_{23}^{(2)})$	$r_{23}^{(3)} = (T_{23}^{(3)}, I_{23}^{(3)}, F_{23}^{(3)})$	$r_{23}^{(4)} = (T_{23}^{(4)}, I_{23}^{(4)}, F_{23}^{(4)})$
	A_3	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)
		$r_{33}^{(1)} = (T_{33}^{(1)}, I_{33}^{(1)}, F_{33}^{(1)})$	$r_{33}^{(2)} = (T_{33}^{(2)}, I_{33}^{(2)}, F_{33}^{(2)})$	$r_{33}^{(3)} = (T_{33}^{(3)}, I_{33}^{(3)}, F_{33}^{(3)})$	$r_{33}^{(4)} = (T_{33}^{(4)}, I_{33}^{(4)}, F_{33}^{(4)})$
	A_4	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	MG (0.60,0.35,0.40)
		$r_{43}^{(1)} = (T_{43}^{(1)}, I_{43}^{(1)}, F_{43}^{(1)})$	$r_{43}^{(2)} = (T_{43}^{(2)}, I_{43}^{(2)}, F_{43}^{(2)})$	$r_{43}^{(3)} = (T_{43}^{(3)}, I_{43}^{(3)}, F_{43}^{(3)})$	$r_{43}^{(4)} = (T_{43}^{(4)}, I_{43}^{(4)}, F_{43}^{(4)})$
	A_5	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)
		$r_{53}^{(1)} = (T_{53}^{(1)}, I_{53}^{(1)}, F_{53}^{(1)})$	$r_{53}^{(2)} = (T_{53}^{(2)}, I_{53}^{(2)}, F_{53}^{(2)})$	$r_{53}^{(3)} = (T_{53}^{(3)}, I_{53}^{(3)}, F_{53}^{(3)})$	$r_{53}^{(4)} = (T_{53}^{(4)}, I_{53}^{(4)}, F_{53}^{(4)})$
X_4	A_1	G (0.70,0.25,0.30)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)	M (0.50,0.50,0.50)
		$r_{14}^{(1)} = (T_{14}^{(1)}, I_{14}^{(1)}, F_{14}^{(1)})$	$r_{14}^{(2)} = (T_{14}^{(2)}, I_{14}^{(2)}, F_{14}^{(2)})$	$r_{14}^{(3)} = (T_{14}^{(3)}, I_{14}^{(3)}, F_{14}^{(3)})$	$r_{14}^{(4)} = (T_{14}^{(4)}, I_{14}^{(4)}, F_{14}^{(4)})$
	A_2	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)
		$r_{24}^{(1)} = (T_{24}^{(1)}, I_{24}^{(1)}, F_{24}^{(1)})$	$r_{24}^{(2)} = (T_{24}^{(2)}, I_{24}^{(2)}, F_{24}^{(2)})$	$r_{24}^{(3)} = (T_{24}^{(3)}, I_{24}^{(3)}, F_{24}^{(3)})$	$r_{24}^{(4)} = (T_{24}^{(4)}, I_{24}^{(4)}, F_{24}^{(4)})$
	A_3	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)	MG (0.60,0.35,0.40)
		$r_{34}^{(1)} = (T_{34}^{(1)}, I_{34}^{(1)}, F_{34}^{(1)})$	$r_{34}^{(2)} = (T_{34}^{(2)}, I_{34}^{(2)}, F_{34}^{(2)})$	$r_{34}^{(3)} = (T_{34}^{(3)}, I_{34}^{(3)}, F_{34}^{(3)})$	$r_{34}^{(4)} = (T_{34}^{(4)}, I_{34}^{(4)}, F_{34}^{(4)})$
	A_4	M (0.50,0.50,0.50)	MB (0.40,0.65,0.60)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)
		$r_{44}^{(1)} = (T_{44}^{(1)}, I_{44}^{(1)}, F_{44}^{(1)})$	$r_{44}^{(2)} = (T_{44}^{(2)}, I_{44}^{(2)}, F_{44}^{(2)})$	$r_{44}^{(3)} = (T_{44}^{(3)}, I_{44}^{(3)}, F_{44}^{(3)})$	$r_{44}^{(4)} = (T_{44}^{(4)}, I_{44}^{(4)}, F_{44}^{(4)})$
	A_5	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	G (0.70,0.25,0.30)
		$r_{54}^{(1)} = (T_{54}^{(1)}, I_{54}^{(1)}, F_{54}^{(1)})$	$r_{54}^{(2)} = (T_{54}^{(2)}, I_{54}^{(2)}, F_{54}^{(2)})$	$r_{54}^{(3)} = (T_{54}^{(3)}, I_{54}^{(3)}, F_{54}^{(3)})$	$r_{54}^{(4)} = (T_{54}^{(4)}, I_{54}^{(4)}, F_{54}^{(4)})$
X_5	A_1	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)
		$r_{15}^{(1)} = (T_{15}^{(1)}, I_{15}^{(1)}, F_{15}^{(1)})$	$r_{15}^{(2)} = (T_{15}^{(2)}, I_{15}^{(2)}, F_{15}^{(2)})$	$r_{15}^{(3)} = (T_{15}^{(3)}, I_{15}^{(3)}, F_{15}^{(3)})$	$r_{15}^{(4)} = (T_{15}^{(4)}, I_{15}^{(4)}, F_{15}^{(4)})$
	A_2	VG (0.80,0.15,0.20)	M (0.50,0.50,0.50)	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)
		$r_{25}^{(1)} = (T_{25}^{(1)}, I_{25}^{(1)}, F_{25}^{(1)})$	$r_{25}^{(2)} = (T_{25}^{(2)}, I_{25}^{(2)}, F_{25}^{(2)})$	$r_{25}^{(3)} = (T_{25}^{(3)}, I_{25}^{(3)}, F_{25}^{(3)})$	$r_{25}^{(4)} = (T_{25}^{(4)}, I_{25}^{(4)}, F_{25}^{(4)})$
	A_3	G (0.70,0.25,0.30)	G (0.70,0.25,0.30)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)
		$r_{35}^{(1)} = (T_{35}^{(1)}, I_{35}^{(1)}, F_{35}^{(1)})$	$r_{35}^{(2)} = (T_{35}^{(2)}, I_{35}^{(2)}, F_{35}^{(2)})$	$r_{35}^{(3)} = (T_{35}^{(3)}, I_{35}^{(3)}, F_{35}^{(3)})$	$r_{35}^{(4)} = (T_{35}^{(4)}, I_{35}^{(4)}, F_{35}^{(4)})$
	A_4	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)	MG (0.60,0.35,0.40)	G (0.70,0.25,0.30)
		$r_{45}^{(1)} = (T_{45}^{(1)}, I_{45}^{(1)}, F_{45}^{(1)})$	$r_{45}^{(2)} = (T_{45}^{(2)}, I_{45}^{(2)}, F_{45}^{(2)})$	$r_{45}^{(3)} = (T_{45}^{(3)}, I_{45}^{(3)}, F_{45}^{(3)})$	$r_{45}^{(4)} = (T_{45}^{(4)}, I_{45}^{(4)}, F_{45}^{(4)})$
	A_5	G (0.70,0.25,0.30)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)	VG (0.80,0.15,0.20)
		$r_{55}^{(1)} = (T_{55}^{(1)}, I_{55}^{(1)}, F_{55}^{(1)})$	$r_{55}^{(2)} = (T_{55}^{(2)}, I_{55}^{(2)}, F_{55}^{(2)})$	$r_{55}^{(3)} = (T_{55}^{(3)}, I_{55}^{(3)}, F_{55}^{(3)})$	$r_{55}^{(4)} = (T_{55}^{(4)}, I_{55}^{(4)}, F_{55}^{(4)})$

Table 4. Importance and Weights of Decision-Makers

	DM₁	***DM₂***	***DM₃***	***DM₄***
Linguistic	VI(0.90,0.10,0.10)	I (0.75,0.25,0.20)	M (0.50,0.50,0.50)	UI (0.35,0.75,0.80)
Variables	$(T_1^{dm}, I_1^{dm}, F_1^{dm})$	$(T_2^{dm}, I_2^{dm}, F_2^{dm})$	$(T_3^{dm}, I_3^{dm}, F_3^{dm})$	$(T_4^{dm}, I_4^{dm}, F_4^{dm})$
Weights	λ_{DM_1}= 0.37045	λ_{DM_2}= 0.31508	λ_{DM_3}= 0.20580	λ_{DM_4}= 0.10867

Step 2: Computation of Aggregated Single Valued Neutrosophic Decision Matrix (ASVNDM)

To find the ASVNDM not only the weights of the DMs, but the alternative ratings are also required. The alternative ratings, according to the *DMs* given in the following table.

Now by using Equation 3, alternative ratings $r_{ij}^{(k)}$ and the DM weights λ_k we get

$r_{ij} = \lambda_1 r_{ij}^{(1)} \oplus \lambda_2 r_{ij}^{(2)} \oplus \lambda_3 r_{ij}^{(3)} \oplus \cdots \oplus \lambda_l r_{ij}^{(l)}$

$r_{ij} = \left(1 - \prod_{k=1}^{l}(1 - T_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(I_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(F_{ij}^{(k)})^{\lambda_k}\right)$

where i = 1, 2, 3, 4, 5; j = 1, 2, 3, 4, 5 and (l = 4).

For $i = j = 1$ and $l = 4$

$r_{11} = \lambda_1 r_{11}^{(1)} \oplus \lambda_2 r_{11}^{(2)} \oplus \lambda_3 r_{11}^{(3)} \oplus \cdots \oplus \lambda_l r_{11}^{(l)}$

$r_{11} = \left(1 - \prod_{k=1}^{4}(1 - T_{11}^{(k)})^{\lambda_k}, \prod_{k=1}^{4}(I_{11}^{(k)})^{\lambda_k}, \prod_{k=1}^{4}(F_{11}^{(k)})^{\lambda_k}\right)$

r_{11} = (1- $(1 - T_{11}^{(1)})^{\lambda_1}(1 - T_{11}^{(2)})^{\lambda_2}(1 - T_{11}^{(3)})^{\lambda_3}(1 - T_{11}^{(4)})^{\lambda_4}$, $(I_{11}^{(1)})^{\lambda_1}(I_{11}^{(2)})^{\lambda_2}(I_{11}^{(3)})^{\lambda_3}(I_{11}^{(4)})^{\lambda_4}$, $(F_{11}^{(1)})^{\lambda_1}(F_{11}^{(2)})^{\lambda_2}(F_{11}^{(3)})^{\lambda_3}(F_{11}^{(4)})^{\lambda_4}$)

r_{11} = (1-($(1 - 0.8)^{0.37045}(1 - 0.6)^{0.31508}(1 - 0.8)^{0.20580}(1 - 0.7)^{0.10867}$),

($(0.15)^{0.37045}(0.35)^{0.31508}(0.15)^{0.20580}(0.25)^{0.10867}$) ,

($(0.20)^{0.37045}(0.40)^{0.31508}(0.20)^{0.20580}(0.30)^{0.10867}$))

r_{11} = (0.740, 0.207, 0.260)

Similarly, we can find other values

r_{21} = (0.711, 0.237, 0.289)

r_{31} = (0.593, 0.373, 0.407)

r_{41} = (0.661, 0.288, 0.339)

r_{51} = (0.706, 0.241, 0.294)

r_{12} = (0.682, 0.268, 0.318)

r_{22} = (0.676, 0.275, 0.324)

r_{32} = (0.681, 0.275, 0.324)

r_{42} = (0.619, 0.342, 0.381)

r_{52} = (0.695, 0.253, 0.305)

r_{13} = (0.505, 0.392, 0.429)

r_{23} = (0.773, 0.176, 0.227)

r_{33} = (0.603, 0.359, 0.397)

r_{43} = (0.661, 0.288, 0.339)

r_{53} = (0.693, 0.255, 0.307)

r_{14} = (0.605, 0.359, 0.395)

r_{24} = (0.748, 0.203, 0.252)

r_{34} = (0.600, 0.350, 0.400)

r_{44} = (0.542, 0.443, 0.458)

r_{54} = (0.693, 0.339, 0.307)

r_{15} = (0.614, 0.349, 0.386)

r_{25} = (0.697, 0.257, 0.303)

r_{35} = (0.656, 0.299, 0.344)

r_{45} = (0.548, 0.431, 0.452)

r_{55} = (0.768, 0.181, 0.232)

Table 5. Aggregated Single Valued Neutrosophic Decision Matrix D = $[r_{ij}]_{5\times4}$

	X_1	X_2	X_3	X_4	X_5
A_1	r_{11} = (0.740, 0.207, 0.260)	r_{12} = (0.682, 0.268, 0.318)	r_{13} = (0.505, 0.392, 0.429)	r_{14} = (0.605, 0.359, 0.395)	r_{15} = (0.614, 0.349, 0.386)
A_2	r_{21} = (0.711, 0.237, 0.289)	r_{22} = (0.676, 0.275, 0.324)	r_{23} = (0.773, 0.176, 0.227)	r_{24} = (0.748, 0.203, 0.252)	r_{25} = (0.697, 0.257, 0.303)

A_3	r_{31} = (0.593, 0.373, 0.407)	r_{32} = (0.681, 0.275, 0.324)	r_{33} = (0.603, 0.359, 0.397)	r_{34} = (0.600, 0.350, 0.400)	r_{35} = (0.656, 0.299, 0.344)
A_4	r_{41} = (0.661, 0.288, 0.339)	r_{42} = (0.619, 0.342, 0.381)	r_{43} = (0.661, 0.288, 0.339)	r_{43} = (0.661, 0.288, 0.339)	r_{45} = (0.548, 0.431, 0.452)
A_5	r_{51} = (0.706, 0.241, 0.294)	r_{52} = (0.695, 0.253, 0.305)	r_{53} = (0.693, 0.255, 0.307)	r_{54}= (0.693, 0.339, 0.307)	r_{55} = (0.768, 0.181, 0.232)

Step 3: Computation of the weights of the criteria

The individual weights given by each DM is given in Table 6.

Table 6. Weights of alternatives determined by the DMs $w_j^{(k)} = (T_j^{(k)}, I_j^{(k)}, F_j^{(k)})$

Criteria	*DM₁*	*DM₂*	*DM₃*	*DM₄*
X_1	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)	I (0.75,0.25,0.20)
(DELIVERY)	$w_1^{(1)} = (T_1^{(1)}, I_1^{(1)}, F_1^{(1)})$	$w_1^{(2)} = (T_1^{(2)}, I_1^{(2)}, F_1^{(2)})$	$w_1^{(3)} = (T_1^{(3)}, I_1^{(3)}, F_1^{(3)})$	$w_1^{(4)} = (T_1^{(4)}, I_1^{(4)}, F_1^{(4)})$
X_2	I (0.75,0.25,0.20)	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)	I (0.75,0.25,0.20)
(QUALITY)	$w_2^{(1)} = (T_2^{(1)}, I_2^{(1)}, F_2^{(1)})$	$w_2^{(2)} = (T_2^{(2)}, I_2^{(2)}, F_2^{(2)})$	$w_2^{(3)} = (T_2^{(3)}, I_2^{(3)}, F_2^{(3)})$	$w_2^{(4)} = (T_2^{(4)}, I_2^{(4)}, F_2^{(4)})$
X_3	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)	I (0.75,0.25,0.20)	VI (0.90,0.10,0.10)
(FLEXIBILITY)	$w_3^{(1)} = (T_3^{(1)}, I_3^{(1)}, F_3^{(1)})$	$w_3^{(2)} = (T_3^{(2)}, I_3^{(2)}, F_3^{(2)})$	$w_3^{(3)} = (T_3^{(3)}, I_3^{(3)}, F_3^{(3)})$	$w_3^{(4)} = (T_3^{(4)}, I_3^{(4)}, F_3^{(4)})$
X_4	I (0.75,0.25,0.20)	I (0.75,0.25,0.20)	M (0.50,0.50,0.50)	UI (0.35,0.75,0.80)
(SERVICE)	$w_4^{(1)} = (T_4^{(1)}, I_4^{(1)}, F_4^{(1)})$	$w_4^{(2)} = (T_4^{(2)}, I_4^{(2)}, F_4^{(2)})$	$w_4^{(3)} = (T_4^{(3)}, I_4^{(3)}, F_4^{(3)})$	$w_4^{(4)} = (T_4^{(4)}, I_4^{(4)}, F_4^{(4)})$
X_5	M (0.50,0.50,0.50)	M (0.50,0.50,0.50)	VI (0.90,0.10,0.10)	VI (0.90,0.10,0.10)
(PRICE)	$w_5^{(1)} = (T_5^{(1)}, I_5^{(1)}, F_5^{(1)})$	$w_5^{(2)} = (T_5^{(2)}, I_5^{(2)}, F_5^{(2)})$	$w_5^{(3)} = (T_5^{(3)}, I_5^{(3)}, F_5^{(3)})$	$w_5^{(4)} = (T_5^{(4)}, I_5^{(4)}, F_5^{(4)})$

By using the values from Table 6, the aggregated criteria weights are calculated as follows

$w_j = (T_j, I_j, F_j) = \lambda_1 w_j^{(1)} \oplus \lambda_2 w_j^{(2)} \oplus \lambda_3 w_j^{(3)} \oplus \cdots \oplus \lambda_l w_j^{(l)}$

$w_j = (1-\prod_{k=1}^{l}(1-T_j^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(I_j^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(F_j^{(k)})^{\lambda_k})$ where j = 1, 2, 3, 4, 5 and (l = 4).

For j = 1 and l = 4

$w_1 = \lambda_1 w_1^{(1)} \oplus \lambda_2 w_1^{(2)} \oplus \lambda_3 w_1^{(3)} \oplus \lambda_4 w_1^{(4)}$

$w_1 = (1-\prod_{k=1}^{4}(1-T_1^{(k)})^{\lambda_k}, \prod_{k=1}^{4}(I_1^{(k)})^{\lambda_k}, \prod_{k=1}^{4}(F_1^{(k)})^{\lambda_k})$

$w_1 = (1- (1-T_1^{(1)})^{\lambda_1}(1-T_1^{(2)})^{\lambda_2}(1-T_1^{(3)})^{\lambda_3}(1-T_1^{(4)})^{\lambda_4}, (I_1^{(1)})^{\lambda_1}(I_1^{(2)})^{\lambda_2}(I_1^{(3)})^{\lambda_3}(I_1^{(4)})^{\lambda_4},$
$(F_1^{(1)})^{\lambda_1}(F_1^{(2)})^{\lambda_2}(F_1^{(3)})^{\lambda_3}(F_1^{(4)})^{\lambda_4})$

$w_1 = (1 - ((1-0.9)^{0.37045}(1-0.9)^{0.31508}(1-0.9)^{0.20580}(1-0.75)^{0.10867}),$
$((0.10)^{0.37045}(0.10)^{0.31508}(0.10)^{0.20580}(0.25)^{0.10867})$,
$((0.10)^{0.37045}(0.10)^{0.31508}(0.10)^{0.20580}(0.20)^{0.10867}))$

r_{11} = (0.740, 0.207, 0.260)

$w_1 = (T_1, I_1, F_1)$ = (0.890, 0.110, 0.108)

Similarly, we can get other values

Therefore

$$W_{\{X_1,X_2,X_3,X_4\}} = \begin{bmatrix} (0.890, 0.110, 0.108) \\ (0.641, 0.359, 0.322) \\ (0.879, 0.121, 0.115) \\ (0.680, 0.325, 0.281) \\ (0.699, 0.301, 0.301) \end{bmatrix}^T$$

Step 4: Construction of Aggregated Weighted Single Valued Neutrosophic Decision Matrix (AWSVNDM)

After finding the weights of the criteria and the alternative ratings, the aggregated weighted single-valued neutrosophic ratings are calculated by using Equation 4 as follows:

$r'_{ij} = (T'_{ij},\ I'_{ij},\ rF'_{ij}) = (T_{A_i}(x).T_j,\ I_{A_i}(x) + I_j - I_{A_i}(x).I_j,\ F_{A_i}(x) + F_j - F_{A_i}(x).F_j)$

By using the above equation, we can get an aggregated weighted single-valued neutrosophic decision matrix.

Table 7. Aggregated Weighted Single Valued Neutrosophic Decision Matrix $R' = [r'_{ij}]_{5\times 5}$

	X1	X2	X3	X4	X5
A1	r'_{11} = (0.659,0.294,0.340)	r'_{12} = (0.437,0.531,0.538)	r'_{13} = (0.444,0.466,0.495)	r'_{14} = (0.411,0.567,0.565)	r'_{15} = (0.429,0.545,0.571)
A2	r'_{21} = (0.633,0.321,0.366)	r'_{22} = (0.433,0.535,0.542)	r'_{23} = (0.679,0.276,0.316)	r'_{24} = (0.509,0.462,0.462)	r'_{25} = (0.487,0.481,0.513)
A3	r'_{31} = (0.528,0.442,0.471)	r'_{32} = (0.437,0.535,0.542)	r'_{33} = (0.530,0.437,0.466)	r'_{34} = (0.408,0.561,0.569)	r'_{35} = (0.459,0.510,0.541)
A4	r'_{41} = (0.588,0.366,0.410)	r'_{42} = (0.397,0.578,0.580)	r'_{43} = (0.581,0.374,0.415)	r'_{44} = (0.037,0.624,0.610)	r'_{45} = (0.383,0.602,0.617)
A5	r'_{51} = (0.628,0.324,0.3700	r'_{52} = (0.445,0.521,0.529)	r'_{53} = (0.609,0.345,0.387)	r'_{54} = (0.471,0.554,0.502)	r'_{55} = (0.537,0.428,0.463)

Step 5: Computation of SVN-PIS and SVN-NIS

Since Delivery, Quality, Flexibility, and Services are benefit criteria that is why they are in the set

$$J_1 = \{X_1, X_2, X_3, X_4\}$$

whereas Price being the cost criteria, so it is in the set $J_2 = \{X_2\}$ SVN-PIS and SVN-NIS are calculated as,

Table 8. SVN-PIS and SVN-NIS

SVN-PIS	SVN-NIS
$\boldsymbol{T_1^+}$ = max {0.659,0.633,0.528,0.588,0.628} = 0.659	T_1^- = min {0.659,0.633,0.528,0.588,0.628} = 0.528
$\boldsymbol{I_1^+}$ = min {0.294,0.321,0.442,0.366,0.324} = 0.294	I_1^- = max {0.294,0.321,0.442,0.366,0.324} = 0.442
$\boldsymbol{F_1^+}$ = min {0.340,0.366,0.471,0.410,0.370} = 0.340	F_1^- = max {0.340,0.366,0.471,0.410,0.370} = 0.471
$\boldsymbol{T_2^+}$ = max {0.437,0.433,0.437,0.397,0.445} = 0.445	T_2^- = min {0.437,0.433,0.437,0.397,0.445} = 0.397
$\boldsymbol{I_2^+}$ = min {0.531,0.535,0.535,0.578,0.521} = 0.521	I_2^- = max {0.531,0.535,0.535,0.578,0.521} = 0.578
$\boldsymbol{F_2^+}$ = min {0.538,0.542,0.542,0.580,0.529} = 0.529	F_2^- = max {0.538,0.542,0.542,0.580,0.529} = 0.580
$\boldsymbol{T_3^+}$= max {0.444,0.679,0.530,0.581,0.609} = 0.679	T_3^- = min {0.444,0.679,0.530,0.581,0.609} = 0.444
$\boldsymbol{I_3^+}$ = min {0.466,0.276,0.437,0.374,0.345} = 0.276	I_3^- = max {0.466,0.276,0.437,0.374,0.345} = 0.466
$\boldsymbol{F_3^+}$ = min {0.495,0.316,0.466,0.415,0.387} = 0.316	F_3^- = max {0.495,0.316,0.466,0.415,0.387} = 0.495
$\boldsymbol{T_4^+}$ = max {0.411,0.509,0.408,0.037,0.471} = 0.509	T_4^- = min {0.411,0.509,0.408,0.037,0.471} = 0.037
$\boldsymbol{I_4^+}$ = min {0.567,0.462,0.561,0.624,0.554} = 0.462	I_4^- = max {0.567,0.462,0.561,0.624,0.554} = 0.624
$\boldsymbol{F_4^+}$ = min {0.565,0.462,0.569,0.610,0.502} = 0.462	F_4^- = max {0.565,0.462,0.569,0.610,0.502} = 0.610
$\boldsymbol{T_5^+}$ = min {0.429,0.487,0.459,0.383,0.537} = 0.383	T_5^- = max {0.429,0.487,0.459,0.383,0.537} = 0.537
$\boldsymbol{I_5^+}$ = max {0.545,0.481,0.510,0.602,0.428} = 0.602	I_5^- = min {0.545,0.481,0.510,0.602,0.428} = 0.428
$\boldsymbol{F_5^+}$ = max {0.571,0.513,0.541,0.617,0.463} = 0.617	F_5^- = min {0.571,0.513,0.541,0.617,0.463} = 0.463

$$A^{+} = \begin{pmatrix} (0.659, 0.294, 0.340), \\ (0.445, 0.521, 0.529), \\ (0.679, 0.276, 0.316), \\ (0.509, 0.462, 0.462), \\ (0.383, 0.602, 0.617) \end{pmatrix} \qquad A^{-} = \begin{pmatrix} (0.528, 0.442, 0.471), \\ (0.397, 0.578, 0.580), \\ (0.444, 0.466, 0.495), \\ (0.037, 0.624, 0.610), \\ (0.537, 0.428, 0.463) \end{pmatrix}$$

Step 6: Computation of Separation Measures

Normalized Euclidean Distance Measure is used to find the negative and positive separation measures $\boldsymbol{d}^{+}$ and $\boldsymbol{d}^{-}$respectively by using Equation 7, 8. Now for the SVN-PIS, we use

$$d_i^{+} = \left(\frac{1}{3n}\sum_{j=1}^{n}\left[\left(T_{A_i.W}(x_j) - T_{A^*W}\ (x_j)\right)^2 + \left(I_{A_i.W}(x_j) - I_{A^*W}\ (x_j)\right)^2 + \left(F_{A_i.W}(x_j) - F_{A^*W}\ (x_j)\right)^2\right]\right)^{0.5}$$

For i = 1and n = 5

$$d_1^{+} = \left(\frac{1}{3(5)}\sum_{j=1}^{5}\left[\left(T_{A_1.W}(x_j) - T_{A^*W}\ (x_j)\right)^2 + \left(I_{A_1.W}(x_j) - I_{A^*W}\ (x_j)\right)^2 + \left(F_{A_1.W}(x_j) - F_{A^*W}\ (x_j)\right)^2\right]\right)^{0.5}$$

$$d_1^{+} = \left(\frac{1}{15}\begin{bmatrix} \left(T_{A_1.W}(X_1) - T_{A^*W}\ (X_1)\right)^2 + \left(I_{A_1.W}(X_1) - I_{A^*W}\ (X_1)\right)^2 + \left(F_{A_1.W}(X_1) - F_{A^*W}\ (X_1)\right)^2 + \\ \left(T_{A_1.W}(X_2) - T_{A^*W}\ (X_2)\right)^2 + \left(I_{A_1.W}(X_2) - I_{A^*W}\ (X_2)\right)^2 + \left(F_{A_1.W}(X_2) - F_{A^*W}\ (X_2)\right)^2 + \\ \left(T_{A_1.W}(X_3) - T_{A^*W}\ (X_3)\right)^2 + \left(I_{A_1.W}(X_3) - I_{A^*W}\ (X_3)\right)^2 + \left(F_{A_1.W}(X_3) - F_{A^*W}\ (X_3)\right)^2 + \\ \left(T_{A_1.W}(X_4) - T_{A^*W}\ (X_4)\right)^2 + \left(I_{A_1.W}(X_4) - I_{A^*W}\ (X_4)\right)^2 + \left(F_{A_1.W}(X_4) - F_{A^*W}\ (X_4)\right)^2 + \\ \left(T_{A_1.W}(X_5) - T_{A^*W}\ (X_5)\right)^2 + \left(I_{A_1.W}(X_5) - I_{A^*W}\ (X_5)\right)^2 + \left(F_{A_1.W}(X_5) - F_{A^*W}\ (X_5)\right)^2 \end{bmatrix}\right)^{0.5}$$

$$d_1^{+} = \left(\frac{1}{15}\begin{bmatrix} (0659 - 0.659)^2 + (0.294 - 0.294)^2 + (0.340 - 0.340)^2 + \\ (0.437 - 0.445)^2 + (0.531 - 0.521)^2 + (0.538 - 0.529)^2 + \\ (0.444 - 0.679)^2 + (0.466 - 0.276)^2 + (0.495 - 0.316)^2 + \\ (0.411 - 0.509)^2 + (0.567 - 0.462)^2 + (0.565 - 0.462)^2 + \\ (0.429 - 0.383)^2 + (0.545 - 0.602)^2 + (0.571 - 0.617)^2 \end{bmatrix}\right)^{0.5}$$

$$d_1^{+} = \left[\frac{1}{15}\ (0.000245 + 0.123366 + 0.031238 + 0.007481)\right]^{0.5}$$

d_1^{+}= 0.1040

Similarly, we can find other separation measures.

Step 7: Computation of Relative Closeness Coefficient (RCC)

The RCC is calculated by using Equation 9.

RCCi = $\frac{d_i'}{d_i' + d_i^*}$; i = 1, 2, 3, 4, 5

$RCC_1 = \frac{d_1'}{d_1' + d_1^*} = \frac{0.127532}{0.127532+0.104029} = 0.551$

$RCC_2 = 0.896$

$RCC_3 = 0.505$

$RCC_4 = 0.363$

RCC5 = 0.757

The separation measure and the value of relative closeness coefficient (RCC) expressed in the following Figure 2.

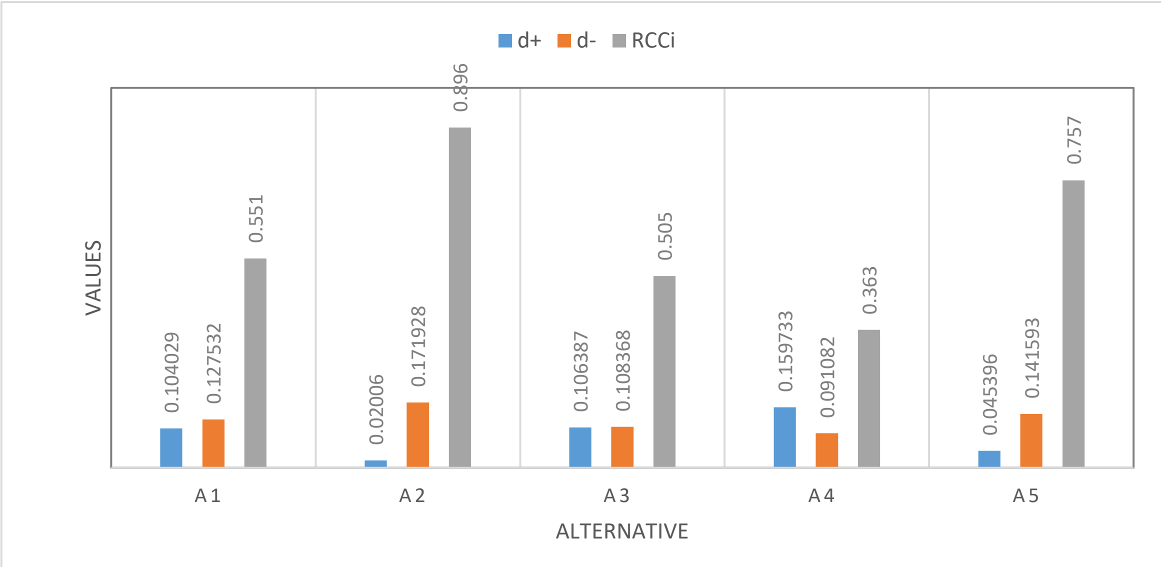

Figure 2. Separation measure and the RCC for each Alternative

Step 8: Ranking alternatives

From the above figure, we can see the RCC are ranked as follows

$RCC_2 > RCC_5 > RCC_1 > RCC_3 > RCC_4 \Rightarrow A_2 > A_5 > A_1 > A_3 > A_4$

By using the presented technique, we choose the best supplier for the production industry and observe that A_2 is the best alternative.

5. Conclusion

In this paper, we studied the neutrosophic set and SVNSs with some basic operations and developed the generalized neutrosophic TOPSIS by using single-valued neutrosophic numbers. By using crisp data, it is more difficult to solve decision-making problems in uncertain environments. Single valued neutrosophic sets can handle these limitations competently and provide the appropriate choice to decision-makers. We also developed the integrated model for neutrosophic TOPSIS. The closeness coefficient has been defined to compute the ranking of the alternatives by using an established approach under-considered environment. Moreover, for the justification of the proposed technique an illustrated example has been described for the selection of suppliers in the production industry. Consequently, relying upon the obtained results it can be confidently concluded that the proposed methodology indicates higher stability and usability for decision-makers in the DM process. Future research will surely concentrate upon presenting the TOPSIS technique based on correlation coefficient under-considered environment. The suggested approach can be applied to quite a lot of issues in real life, including the medical profession, robotics, artificial intelligence, pattern recognition, economics, etc.

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Some New Structures in Neutrosophic Metric Spaces

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Abstract: Neutrosophic sets deals with inconsistent, indeterminate and imprecise datas. The concept of Neutrosophic Metric Space (NMS) uses the idea of continuous t- norm and continuous t - conorm in intuitionistic fuzzy metric spaces. In this paper, we introduce the definition of subcompatible maps of types (J-1 and J-2). We extend the structure of weak non-Archimedian with the help of subcompatible maps of types (J-1 and J-2) in NMS. Finally, we obtain common fixed point theorems for four subcompatible maps of type (J-1) in weak non-Archimedean NMS.

Keywords: Weak non-Archimedean, NMS, Compatible map, Sub compatible, Subcompatible maps of types (J-1) and (J-2).

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1. **Introduction**

Fuzzy set was presented by Zadeh [22] as a class of elements with a grade of membership. Kramosil and Michalek [8] defined new notion called Fuzzy Metric Space (FMS). Later, many authors have examined the concept of fuzzy metric in various aspects. In 2013, Muthuraj and Pandiselvi [17] introduced the concept of compatible mappings of type (P-1) andtype (P-2) in generalized fuzzy metric spaces and obtains common fixed point theorems are obtained forcompatible maps of type (P-1) and type (P- 2). Since then, many authors have obtained fixed point results in fuzzy metric space using these compatible notions.

Atanassov [1] introduced and studied the notion of intuitionistic fuzzy set by generalizing the notion of fuzzy set. Park [9] defined the notion of intuitionistic fuzzy metric space as a generalization of fuzzy metric space. In 1998, Smarandache [14-16] characterized the new concept called neutrosophic logic and neutrosophic set and explored many results in it. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. Baset et al. [2] Explored the neutrosophic applications in dif and only iferent fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making.

In 2019, Kirisci et al [9] defined NMS as a generalization of IFMS and brings about fixed point theorems in complete NMS. Erduran et.al.[13] introduced the concept of weak non-Archimedean intuitionistic fuzzy metric space and proved a common fixed point theorem for a pair of generalized (φ, Ψ) – contractive mappings. Later Jeyaraman at el [19,20] proved Fixed point results in non-Archimedean generalized intuitionistic fuzzy metric spaces. In 2020, Sowndrarajan Jeyaraman and Florentin Smarandache [18] proved some fixed point results for contraction theorems in neutrosophic metric spaces.

In this paper, we introduce the definition of sub compatible maps and sub compatible maps of types (J-1) and (J-2) in weak non-Archimedean NMS and give some examples and relationship between these definitions. We extend the structure of weak non-Archimedian with the help of subcompatible maps of types (J-1 and J-2) in NMS. Thereafter, we prove common fixed point theorems for four subcompatible maps of type (J-1) in weak non-Archimedean NMS.

2. **Preliminaries**

Definition: 2.1

A binary operation * : [0, 1] x [0, 1] → [0, 1] is a continuous t-norm [CTN] if it satisfies the following conditions :

(i) * is commutative and associative,

(ii) * is continuous,

(iii) $\varepsilon_1 * 1 = \varepsilon_1$ for all $\varepsilon_1 \in [0, 1]$,

(iv) $\varepsilon_1 * \varepsilon_2 \leq \varepsilon_3 * \varepsilon_4$ whenever $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0, 1]$.

Definition: 2.2

A binary operation ⋄ : [0, 1] x [0, 1] → [0, 1] is a continuous t-conorm [CTC] if it satisfies the following conditions:

(i) ⋄ is commutative and associative,

(ii) ⋄ is continuous,

(iii) $\varepsilon_1 \diamond 0 = \varepsilon_1$ for all $\varepsilon_1 \in [0, 1]$,

(iv) $\varepsilon_1 \diamond \varepsilon_2 \leq \varepsilon_3 \diamond \varepsilon_4$ whenever $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$, for each ε_1, ε_2, ε_3 and $\varepsilon_4 \in [0, 1]$.

Definition: 2.3

A 6-tuple $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is said to be an NMS (shortly NMS), if Σ is an arbitrary non empty set, $*$ is a neutrosophic CTN, $\diamond$ is a neutrosophic CTC and Ξ, Θ *and* Υ are neutrosophic on $\Sigma^3 \times \mathbb{R}^+$ satisfying the following conditions:

For all $\zeta, \eta, \delta, \omega \in \Sigma$, $\lambda \in \mathbb{R}^+$.

1. $0 \leq \Xi(\zeta, \eta, \delta, \lambda) \leq 1$; $0 \leq \Theta(\zeta, \eta, \delta, \lambda) \leq 1$; $0 \leq \Upsilon(\zeta, \eta, \delta, \lambda) \leq 1$;
2. $\Xi(\zeta, \eta, \delta, \lambda) + \Theta(\zeta, \eta, \delta, \lambda) + \Upsilon(\zeta, \eta, \delta, \lambda) \leq 3$;
3. $\Xi(\zeta, \eta, \delta, \lambda) = 1$ if and only if $\zeta = \eta = \delta$;
4. $\Xi(\zeta, \eta, \delta, \lambda) = \Xi(\rho(\zeta, \eta, \delta, \lambda))$, when ρ is the permutation function;
5. $\Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, \mu) \leq \Xi(\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;

6. $\Xi(\zeta, \eta, \delta, .) : [0, \infty) \to [0, 1]$ is neutrosophic continuous ;
7. $\lim_{\lambda\to\infty} \Xi(\zeta, \eta, \delta, \lambda) = 1$ for all $\lambda > 0$;
8. $\Theta(\zeta, \eta, \delta, \lambda) = 0$ if and only if $\zeta = \eta = \delta$;
9. $\Theta(\zeta, \eta, \delta, \lambda) = \Theta(\rho(\zeta, \eta, \delta, \lambda))$, when ρ is the permutation function;
10. $\Theta(\zeta, \eta, \omega, \lambda) \diamond \Theta(\omega, \delta, \delta, \mu) \geq \Theta(\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
11. $\Theta(\zeta, \eta, \delta, .) : [0, \infty) \to [0, 1]$ is neutrosophic continuous;
12. $\lim_{\lambda\to\infty} \Theta(\zeta, \eta, \delta, \lambda) = 0$ for all $\lambda > 0$;
13. $\Upsilon(\zeta, \eta, \delta, \lambda) = 0$ if and only if $\zeta = \eta = \delta$;
14. $\Upsilon(\zeta, \eta, \delta, \lambda) = \Upsilon(\rho(\zeta, \eta, \delta, \lambda))$, when ρ is the permutation function;
15. $\Upsilon(\zeta, \eta, \omega, \lambda) \diamond \Upsilon(\omega, \delta, \delta, \mu) \geq \Upsilon(\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
16. $\Upsilon(\zeta, \eta, \delta, .) : [0, \infty) \to [0, 1]$ is neutrosophic continuous;
17. $\lim_{\lambda\to\infty} \Upsilon(\zeta, \eta, \delta, \lambda) = 0$ for all $\lambda > 0$;
18. If $\lambda > 0$ then $\Xi(\zeta, \eta, \delta, \lambda) = 0$; $\Theta(\zeta, \eta, \delta, \lambda) = 1$; $\Upsilon(\zeta, \eta, \delta, \lambda) = 1$.

Then, (Ξ, Θ, Υ) is called an NMS on Σ. The functions Ξ, Θ *and* Υ denote degree of closedness, neturalness and non-closedness between ζ, η *and* δ with respect to λ respectively.

Example: 2.4

Let (Σ, D) be a metric space. Define $\omega * \tau = \min\{\omega, \tau\}$ and $\omega \diamond \tau = \max\{\omega, \tau\}$ and $\Xi, \Theta, \Upsilon : \Sigma^3 \times \mathbb{R}^+ \to [0, 1]$ defined by, we define
$\Xi(\zeta, \eta, \delta, \lambda) = \frac{\lambda}{\lambda + D(\zeta,\eta,\delta)}$; $\Theta(\zeta, \eta, \delta, \lambda) = \frac{D(\zeta,\eta,\delta)}{\lambda + D(\zeta,\eta,\delta)}$; $\Upsilon(\zeta, \eta, \delta, \lambda) = \frac{D(\zeta,\eta,\delta)}{\lambda}$ for all $\zeta, \eta, \delta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is called NMS induced by a metric D the standard neutrosophic metric.

Remark: 2.5

In NMS$\Xi(\zeta, \eta, \delta, \lambda, .)$ is non-decreasing, $\Theta(\zeta, \eta, \delta, .)$ is non-increasing and $\Upsilon(\zeta, \eta, \delta, .)$ is decreasing for all $\zeta, \eta, \delta \in \Sigma$.

In the above definition, if the triangular inequality (v), (x) and (xv) are replaced by the following:

$$\Xi(\zeta, \eta, \delta, \max\{\lambda, \mu\}) \geq \Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, \mu),$$
$$\Theta(\zeta, \eta, \delta, \min\{\lambda, \mu\}) \leq \Theta(\zeta, \eta, \omega, \lambda) \diamond \Theta(\omega, \delta, \delta, \mu),$$
$$\Upsilon(\zeta, \eta, \delta, \min\{\lambda, \mu\}) \leq \Upsilon(\zeta, \eta, \omega, \lambda) \diamond \Upsilon(\omega, \delta, \delta, \mu)$$

or equivalently

$$\Xi(\zeta, \eta, \delta, \lambda) \geq \Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, \lambda),$$
$$\Theta(\zeta, \eta, \delta, \lambda) \leq \Theta(\zeta, \eta, \omega, \lambda) \diamond \Theta(\omega, \delta, \delta, \lambda),$$
$$\Upsilon(\zeta, \eta, \delta, \lambda) \leq \Upsilon(\zeta, \eta, \omega, \lambda) \diamond \Upsilon(\omega, \delta, \delta, \lambda).$$

Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is called non-Archimedean NMS. It is easy to check that the triangle inequality (NA) implies (5), (10) and (15), that is, every non-Archimedean NMS is itself an NMS.

Example:2.6

Let Σ be a non-empty set with at least two elements. Define $\Xi(\zeta, \eta, \delta, \lambda)$ by: If we define the neutrosophic set $(\Sigma, \Xi, \Theta, \Upsilon)$ by $\Xi(\zeta, \zeta, \zeta, \lambda) = 1, \Theta(\zeta, \zeta, \zeta, \lambda) = 0$ and $\Upsilon(\zeta, \zeta, \zeta, \lambda) = 0$ for all $\zeta \in \Sigma$ and $\lambda > 0$, and $\Xi(\zeta, \eta, \delta, \lambda) = 0$, $\Theta(\zeta, \eta, \delta, \lambda) = 1$ and $\Upsilon(\zeta, \eta, \delta, \lambda) = 1$, for $\zeta \neq \eta \neq \delta$ and $0 < \lambda \leq 1$, and $\Xi(\zeta, \eta, \delta, \lambda) = 1$, $\Theta(\zeta, \eta, \delta, \lambda) = 0$ and $\Upsilon(\zeta, \eta, \delta, \lambda) = 0$, for $\zeta \neq \eta \neq \delta$ and $\lambda > 1$. Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a non-Archimedean NMS with arbitrary $*$ is a neutrosophic CTN, $\diamond$ is a neutrosophic CTC. Clearly $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is also an NMS.

Definition:2.7

In Definition 2.3, if the triangular inequality (v), (x) and (xv) are replaced by the following:

$\Xi(\zeta, \eta, \delta, \lambda) \geq \max\{\Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, , \lambda/2), \Xi(\zeta, \eta, \omega, \lambda/2) * \Xi(\omega, \delta, \delta, \lambda)\}$,

$\Theta(\zeta, \eta, \delta, \lambda) \leq \min\{\Theta(\zeta, \eta, \omega, \lambda) \diamond \Theta(\omega, \delta, \delta, , \lambda/2), \Theta(\zeta, \eta, \omega, \lambda/2) \diamond \Theta(\omega, \delta, \delta, \lambda)\}$,

$\Upsilon(\zeta, \eta, \delta, \lambda) \leq \min\{\Upsilon(\zeta, \eta, \omega, \lambda) \diamond \Upsilon(\omega, \delta, \delta, , \lambda/2), \Upsilon(\zeta, \eta, \omega, \lambda/2) \diamond \Upsilon(\omega, \delta, \delta, \lambda)\}$,

for all $\Xi, \Theta, \Upsilon \in \Sigma$ and $\lambda > 0$, then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is said to be a Weak Non- Archimedean (WNA) NMS.

Obviously, every non-Archimedean NMS is itself a weak non-Archimedean NMS.

The inequality (WNA) does not imply that $\Xi(\zeta, \eta, \delta, \lambda, .)$ is non-decreasing , $\Theta(\zeta, \eta, \delta, .)$ is non-increasing and $\Upsilon(\zeta, \eta, \delta, .)$ is decreasing. Thus, a weak non-Archimedean NMS is not necessarily an NMS.

Example: 2.8

Let $\Sigma = [0, \infty)$ and define $\Xi(\zeta, \eta, \delta, \lambda)$; $\Theta(\zeta, \eta, \delta, \lambda)$ and $\Upsilon(\zeta, \eta, \delta, \lambda)$ by

$$\Xi(\zeta, \eta, \delta, \lambda) = \begin{cases} 1, & \zeta = \eta = \delta \\ \frac{\lambda}{\lambda + 1}, & \zeta \neq \eta \neq \delta \end{cases},$$

$$\Theta(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta \\ \frac{1}{\lambda + 1}, & \zeta \neq \eta \neq \delta \end{cases},$$

$$\Upsilon(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta \\ \lambda + 1, & \zeta \neq \eta \neq \delta \end{cases},$$

for all $\lambda > 0$. $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$.

Definition: 2.9

Let Γ and Ω be maps from an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0,$$

for all $\lambda > 0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

Definition: 2.10

Let Γ and Ω be self mappings of an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$. Then the mappings are said to be compatible of type (J-1), if

$$\lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0,$$

for all $\lambda > 0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

Definition: 2.11

Let Γ and Ω be self mappings of an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$. Then the mappings are said to be compatible of type (J-2), if

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and}$$

$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$

for all $\lambda > 0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

Definition:2.12

Let Γ and Ω be maps from an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ into itself. The maps Γ and Ω are said to be Occasionally Weakly Compatible (OWC) if and only if there is a point $\zeta \in \Sigma$ which is a coincidence point of Γ and Ωat which Γ and Ω commute i.e., there is a point $\zeta \in \Sigma$ such that $\Gamma\zeta = \Omega\zeta$ and $\Gamma\Omega\zeta = \Omega\Gamma\zeta$.

Definition:2.13

Let Γ and Ω be maps from an NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$. The maps Γ and Ω are said to be reciprocally continuous if $\lim_{n\to\infty}\Gamma\Omega\zeta_n = \Gamma\zeta$, $\lim_{n\to\infty}\Omega\Gamma\zeta_n, = \Omega\zeta$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$.

3. Types Of Subcompatible Maps In Weak Non-Archimedean NMS.

Definition:3.1

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subsequently continuous if there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfy $\lim_{n\to\infty}\Gamma\mathcal{S}\zeta_n = \Gamma\zeta$, $\lim_{n\to\infty}\Omega\Gamma\zeta_n, = \Omega\zeta$.

Clearly, if Γ and $\Omega$are continuous or reciprocally continuous, then they are subsequentially continuous, but converse is not true in general.

Example: 3.2

Let $\Sigma = [0, \infty)$ and define, for all $\lambda > 0$, $\Xi(\zeta, \eta, \delta, \lambda)$; $\Theta(\zeta, \eta, \delta, \lambda)$ and $\Upsilon(\zeta, \eta, \delta, \lambda)$ by

$$\Xi(\zeta, \eta, \delta, \lambda) = \begin{cases} 1, & \zeta = \eta = \delta, \\ \frac{\lambda}{\lambda + 1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Theta(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \frac{1}{\lambda+1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Upsilon(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \lambda + 1, & \zeta \neq \eta \neq \delta. \end{cases}$$

Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$. Define Γ and Ωas follows:

$$\Gamma\zeta = \begin{cases} 2, & \zeta < 3 \\ \zeta, & \zeta \geq 3 \end{cases}, \quad \Omega\zeta = \begin{cases} 2\zeta - 4, & \zeta \leq 3, \\ 3, & \zeta > 3. \end{cases}$$

Clearly Γ and Ω are discontinuous at $\zeta = 3$. Let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 3 - \frac{1}{n}$ for n = 1, 2 …, then $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = 2$, $2 \in \Sigma$ and $\lim_{n\to\infty}\Gamma\Omega\zeta_n = 2 = \Gamma(2)$, $\lim_{n\to\infty}\Omega\Gamma\zeta_n = 0 = \Omega(2)$. Therefore, Γ and Ω are subsequentially continuous. Now, let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 3 + \frac{1}{n}$ for n = 1,2,…, then $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = 3$, $3 \in \Sigma$ and $\lim_{n\to\infty}\Omega\Gamma\zeta_n = 3 \neq 2 = \Omega(3)$. Hence Γ and Ω are not reciprocally continuous.

Definition: 3.3

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subcompatible if and only if there exist a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta, \zeta \in \Sigma$ and satisfies

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0.$$

It is easy to see that two owc maps are subcompatible, however the converse is not true in general. It is also interesting to see the following one-way implication:

Commuting $\Rightarrow$ Weakly commuting $\Rightarrow$ Compatibility $\Rightarrow$ Weak compatibility $\Rightarrow$ OWC$\Rightarrow$ Sub compatibility.

Definition:3.4

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subcompatible of type (J-1) if there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta, \zeta \in \Sigma$ and satisfies

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda)= 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Xi\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda)= 1,$$
$$\lim_{n\to\infty} \Theta\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0, \text{ and,}$$
$$\lim_{n\to\infty} \Upsilon\,(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Clearly, if Γ and Ω are compatible of type (J-1), then they are subcompatible of type (J-1), but converse is not true in general.

Example: 3.5

Let $\Sigma = [0, \infty)$. Define $\Xi\,(\zeta, \eta, \delta, \lambda)$; $\Theta\,(\zeta, \eta, \delta, \lambda)$ and $\Upsilon\,(\zeta, \eta, \delta, \lambda)$ by $\Xi\,(\zeta, \eta, \delta, \lambda) = \frac{\lambda}{\lambda+|\zeta-\eta|+|\eta-\delta|+|\delta-\zeta|}$, $\Theta\,(\zeta, \eta, \delta, \lambda) = \frac{|\zeta-\eta|+|\eta-\delta|+|\delta-\zeta|}{\lambda+|\zeta-\eta|+|\eta-\delta|+|\delta-\zeta|}$ and $\Upsilon\,(\zeta, \eta, \delta, \lambda) = \frac{|\zeta-\eta|+|\eta-\delta|+|\delta-\zeta|}{\lambda}$ for all $\lambda > 0$. Then, $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$.

Define Γ and Ω as follows:

$$\Gamma x = \begin{cases} \zeta^2 + 1, & \zeta < 1 \\ 2\zeta - 1, & \zeta \geq 1 \end{cases}, \quad \Omega\zeta = \begin{cases} \zeta + 1, & \zeta < 1 \\ 3\zeta - 2, & \zeta \geq 1 \end{cases}.$$

Let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 1 + \frac{1}{n}$, for n = 1, 2…, then $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = 1$, $1 \in \Sigma$ and

$$\Gamma\Omega\zeta_n = \Gamma\left(1 + \frac{3}{n}\right) = 2\left(1 + \frac{3}{n}\right)\text{-}1 = 1 + \left(\frac{6}{n}\right),$$
$$\Omega\Gamma\zeta_n = \Omega\left(1 + \frac{2}{n}\right) = 3\left(1 + \frac{2}{n}\right)\text{-}2 = 1 + \left(\frac{6}{n}\right),$$

$$\Gamma\Gamma\zeta_n = \Gamma\left(1+\frac{2}{n}\right) = 2\left(1+\frac{2}{n}\right)\text{-}1 = 1+\left(\frac{4}{n}\right),$$
$$\Omega\Omega\zeta_n = \Omega\left(1+\frac{3}{n}\right) = 3\left(1+\frac{3}{n}\right)\text{-}2 = 1+\left(\frac{9}{n}\right).$$

Therefore,

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \lambda) = 0.$$

And,

$$\lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n,\ \Gamma\Gamma\zeta_n,\ \Gamma\Gamma\zeta_n,\ \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Omega\Gamma\zeta_n,\ \Gamma\Gamma\zeta_n,\ \Gamma\Gamma\zeta_n,\ \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n,\ \Gamma\Gamma\zeta_n,\ \Gamma\Gamma\zeta_n,\ \lambda) = 0.$$

That is, Γ and Ω are subcompatible of type (J-1) but if we consider a sequence $\zeta_n = 1\text{-}\frac{1}{n}$ for n = 1,2,…, then $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = 2,\ 2 \in \Sigma$ and

$$\Gamma\Omega\zeta_n = \Gamma\left(2-\frac{1}{n}\right) = 2\left(2-\frac{1}{n}\right)\text{-}1 = 3\text{-}\left(\frac{2}{n}\right),\ \Omega\Gamma\zeta_n = \Omega\left(\left(1-\frac{1}{n}\right)^2+1\right) = 3\left(\left(1-\frac{1}{n}\right)^2+1\right)\text{-}2,$$
$$\Gamma\Gamma\zeta_n = \Gamma\left(\left(1-\frac{1}{n}\right)^2+1\right) = \Gamma\left(1-\frac{2}{n}+\frac{1}{n^2}\right) = \left(1-\frac{2}{n}+\frac{1}{n^2}\right)^2+1,$$
$$\Omega\Omega\zeta_n = \Omega\left(2-\frac{1}{n}\right) = 3\left(2-\frac{1}{n}\right)\text{-}2 = 4\text{ - }\left(\frac{3}{n}\right).$$

Therefore,

$$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \lambda) \neq 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \lambda) \neq 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \lambda) \neq 0,$$
$$\lim_{n\to\infty} \Xi\,(\Omega\Gamma\zeta_n,\ \Gamma\Gamma\zeta_n,\ \Gamma\Gamma\zeta_n,\ \lambda) \neq 1,$$
$$\lim_{n\to\infty} \Theta\,(\Omega\Gamma\zeta_n,\ \Gamma\Gamma\zeta_n,\ \Gamma\Gamma\zeta_n,\ \lambda) \neq 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon\,(\Omega\Gamma\zeta_n,\ \Gamma\Gamma\zeta_n,\ \Gamma\Gamma\zeta_n,\ \lambda) \neq 0.$$

That is, Γ and Ω are not compatible of type (J-1).

Definition: 3.6

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$be a weak non-Archimedean NMS. Self- maps Γ and Ω on Σ are said to be subcompatible of type (J-1) if and only if there exist a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta,\ \zeta \in \Sigma$ and satisfies

$$\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n,\ \Omega\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n,\ \Omega\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n,\ \Omega\Omega\zeta_n,\ \Omega\Omega\zeta_n,\ \lambda) = 0.$$

Clearly, if Γ and Ω are compatible of type (J-2), then they are subcompatible of type (J-2), but converse is not true in general.

Example: 3.7

Let Σ = [0, ∞) and define $\Xi\,(\zeta, \eta, \delta, \lambda)$; $\Theta\,(\zeta, \eta, \delta, \lambda)$ and $\Upsilon\,(\zeta, \eta, \delta, \lambda)$ by

$$\Xi\,(\zeta, \eta, \delta, \lambda) = \begin{cases} 1, & \zeta = \eta = \delta, \\ \frac{\lambda}{\lambda+1}, & \zeta \neq \eta \neq \delta, \end{cases}$$

$$\Theta(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \frac{1}{\lambda+1}, & \zeta \neq \eta \neq \delta, \end{cases}$$
$$\Upsilon(\zeta, \eta, \delta, \lambda) = \begin{cases} 0, & \zeta = \eta = \delta, \\ \lambda + 1, & \zeta \neq \eta \neq \delta. \end{cases}$$

Then, $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is a weak non-Archimedean NMS with $\omega * \tau = \omega\tau$ and $\omega \diamond \tau = \{\omega + \tau - \omega\tau\}$ for every $\omega, \tau \in [0, 1]$. Define Γ and Ωas follows:

$$\Gamma\zeta = \zeta^2, \quad \Omega\zeta = \begin{cases} \zeta + 2, & \zeta \in [0,4] \cup (5, \infty) \\ \zeta + 12, & \zeta \in (4,5]\ . \end{cases}$$

Let $\{\zeta_n\}$ be a sequence in Σ defined by $\zeta_n = 2+\frac{1}{n}$ for n = 1,2…, then $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = 4$, and $\Gamma\Gamma\zeta_n = \Gamma\left(\left(2+\frac{1}{n}\right)^2\right) = \left(2+\frac{1}{n}\right)^4$, $\Omega\Omega\zeta_n = \Omega\left(4+\frac{1}{n}\right) = 4+\frac{1}{n} + 12 = 16 + \frac{1}{n}$.

Therefore,

$$\lim_{n\to\infty} \Xi(\Gamma\Gamma_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

That is, Γ and Ω are subcompatible of type (J-2) but if we consider a sequence $\zeta_n = 2-\frac{1}{n}$ for n = 1, 2,…, then $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = 4$ and $\Gamma\Gamma\zeta_n = \Gamma\left(\left(2-\frac{1}{n}\right)^2\right) = \left(2-\frac{1}{n}\right)^4$, $\Omega\Omega\zeta_n = \Omega\left(4-\frac{1}{n}\right) = 4-\frac{1}{n} + 2 = 6 - \frac{1}{n}$.

Therefore,

$$\lim_{n\to\infty} \Xi(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \neq 0.$$

That is, Γ and Ω are not compatible of type (J-2).

Preposition: 3.8

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS and $\Gamma, \Omega: \Sigma \to \Sigma$ are subsequentially continuous mappings. Γ and Ω are subcompatible maps if and only if they are not subcompatible of type (J-1).

Proof:

Suppose Γ and Ω are subcompatible, then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma_n, \Omega\Gamma_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma_n, \Omega\Gamma_n, \lambda) = 0, \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma_n, \Omega\Gamma_n, \lambda) = 0.$$

Since Γ and $\Omega$are subsequentially continuous, we have

$\lim_{n\to\infty} \Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n\to\infty} \Gamma\Gamma\zeta_n$, $\lim_{n\to\infty} \Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n\to\infty} \Omega\Omega\zeta_n$.

Thus, from the inequality (WNA), for all $\lambda > 0$,

$$\Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) * \Xi(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$
$$\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Theta(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$
$$\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Upsilon(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$

and it follows that

$$\Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,$$

$$\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \le 0 \diamond 0 = 0,$$
$$\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \le 0 \diamond 0 = 0.$$

That is, for all $\lambda > 0$,

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

By the same way,

$$\lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Consequently, Γ and Ω are subcompatible of type (J-1).

Conversely, suppose that Γ and Ω are subcompatible of type (J-1), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1, \lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0 \text{ and}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0 \text{ and } \lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n\to\infty} \Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n\to\infty} \Gamma\Gamma\zeta_n, \quad \lim_{n\to\infty} \Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n\to\infty} \Omega\Omega\zeta_n.$$

Now, from the inequality (WNA), for all $\lambda > 0$,

$$\Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \ge \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) * \Xi(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$
$$\Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \le \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \diamond \Theta(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$
$$\Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \le \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \diamond \Upsilon(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$

and, it follows that,

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \ge 1 * 1 = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \le 0 \diamond 0 = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \le 0 \diamond 0 = 0,$$

which implies that

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0.$$

Therefore, Γ and Ω are subcompatible.

Preposition: 3.9

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS and $\Gamma, \Omega: \Sigma \to \Sigma$ are subsequentially continuous mappings. Γ and Ω are subcompatible maps if and only if they are not subcompatible of type (J-2).

Proof:

Suppose Γ and Ω are subcompatible, then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \delta$, $\delta \in \Sigma$ and satisfy

$\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1$, $\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$, and $\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$.

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n\to\infty} \Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n\to\infty} \Gamma\Gamma\zeta_n,\quad \lim_{n\to\infty} \Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n\to\infty} \Omega\Omega\zeta_n.$$

Thus, from the inequality (WNA),

$$\begin{aligned}
\Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\geq \Xi\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) * \Xi\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2)\\
&\geq \Xi\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) * \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2) *\\
&\qquad \Xi(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/4),\\
\Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\leq \Theta\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Theta\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2)\\
&\leq \Theta\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2) \diamond\\
&\qquad \Theta\,(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/4) \text{ and}\\
\Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) &\leq \Upsilon\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2)\\
&\leq \Upsilon\,(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2) \diamond\\
&\qquad \Upsilon\,(\Omega\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/4),
\end{aligned}$$

for all $\lambda > 0$, and, it follows that, for all $\lambda > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,\\
&\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,\\
&\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,
\end{aligned}$$

which implies that,

$$\begin{aligned}
&\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,\\
&\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,\\
&\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.
\end{aligned}$$

Consequently, Γ and Ω are subcompatible of type (J-2). Conversely, suppose that Γ and Ω are subcompatible of type (J-2), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\begin{aligned}
&\lim_{n\to\infty} \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,\\
&\lim_{n\to\infty} \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,\\
&\lim_{n\to\infty} \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.
\end{aligned}$$

Now, from the inequality (WNA), we have

$$\begin{aligned}
\Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\geq \Xi\,(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) * \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\geq \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) * \Xi\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\qquad * \Xi\,(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/4),\\
\Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\leq \Theta\,(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Theta(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\leq \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Theta\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\qquad \diamond \Theta\,(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/4) \text{ and}\\
\Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) &\leq \Upsilon\,(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Upsilon(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\leq \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \diamond \Upsilon\,(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda/2)\\
&\qquad \diamond \Upsilon\,(\Omega\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/4),
\end{aligned}$$

and, it follows that, for all $\lambda > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \Xi\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \geq 1 * 1 * 1 = 1,\\
&\lim_{n\to\infty} \Theta\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \leq 0 \diamond 0 \diamond 0 = 0,\\
&\lim_{n\to\infty} \Upsilon\,(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) \leq 0 \diamond 0 \diamond 0 = 0,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0.$$

Therefore, Γ and Ω are subcompatible.

Preposition: 3.10

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ be a weak non-Archimedean NMS and $\Gamma, \Omega: \Sigma \to \Sigma$ are subsequentially continuous mappings. Γ and Ω are subcompatible maps of type (J-1) if and only if they are subcompatible of type (J-2).

Proof:

Suppose Γ and $\Omega$are subcompatible of type (J-1), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfy

$$\lim_{n\to\infty} \Xi(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0, \text{ and,}$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0, \text{ and,}$$
$$\lim_{n\to\infty} \Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Since Γ and Ω are subsequentially continuous, we have

$$\lim_{n\to\infty}\Gamma\Omega\zeta_n = \Gamma\zeta = \lim_{n\to\infty}\Gamma\Gamma\zeta_n,\ \lim_{n\to\infty}\Omega\Gamma\zeta_n = \Omega\zeta = \lim_{n\to\infty}\Omega\Omega\zeta_n.$$

Thus, from the inequality (WNA),

$$\Xi(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq \Xi(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) * \Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$
$$\Theta(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Theta(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$
$$\Upsilon(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Upsilon(\Gamma\Gamma\zeta_n, \Gamma\Omega\zeta_n, \Gamma\Omega\zeta_n, \lambda) \diamond \Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda/2),$$

and, it follows that

$$\lim_{n\to\infty} \Xi(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$

which implies that

$$\lim_{n\to\infty} \Xi(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

Therefore, Γ and $\Omega$are subcompatible of type (J-2).

Conversely, suppose that Γ and Ω are subcompatible of type (J-2), then there exists a sequence $\{\zeta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \zeta$, $\zeta \in \Sigma$ and satisfying

$$\lim_{n\to\infty} \Xi(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty} \Theta(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty} \Upsilon(\Gamma\Gamma\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

Now, from the inequality (WNA), we have

$$\Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq \Xi(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) * \Xi(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$
$$\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Theta(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Theta(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$
$$\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq \Upsilon(\Gamma\Omega\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) \diamond \Upsilon(\Gamma\Gamma\zeta_n, \Omega\Gamma\zeta_n, \Omega\Gamma\zeta_n, \lambda/2),$$

and, it follows that

$$\lim_{n\to\infty}(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \geq 1 * 1 = 1,$$
$$\lim_{n\to\infty}\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$
$$\lim_{n\to\infty}\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) \leq 0 \diamond 0 = 0,$$

which implies that, for all $\lambda > 0$,

$$\lim_{n\to\infty}(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty}\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty}\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = 0.$$

By the same way, we obtain that

$$\lim_{n\to\infty}(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 1,$$
$$\lim_{n\to\infty}\Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0,$$
$$\lim_{n\to\infty}\Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = 0.$$

Therefore, Γ and Ω are subcompatible of type (J-1).

4. Main Theorems

Theorem: 4.1

Let Γ, Λ, Ω and H be self-maps of a weak non-Archimedean NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ and let the pairs (Γ, Ω) and (Λ, H) are subcompatible maps of type(J-1) and subsequentially continuous.

$$\Xi(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \geq \psi(\min\{\Xi(\Omega\zeta, H\eta, H\eta, \lambda), \Xi(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Xi(\Lambda\eta, H\eta, H\eta, \lambda), \tfrac{1}{2}[\Xi(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Xi(\Gamma\zeta, H\eta, H\eta, \lambda)]\}) \quad (4.1.1)$$
$$\Theta(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \phi(\max\{\Theta(\Omega\zeta, H\eta, H\eta, \lambda), \Theta(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Theta(\Lambda\eta, H\eta, H\eta, \lambda), \tfrac{1}{2}[\Theta(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Theta(\Gamma\zeta, H\eta, H\eta, \lambda)]\}) \quad (4.1.2)$$
$$\Upsilon(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \varphi(\max\{\Upsilon(\Omega\zeta, H\eta, H\eta, \lambda), \Upsilon(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Upsilon(\Lambda\eta, H\eta, H\eta, \lambda), \tfrac{1}{2}[\Upsilon(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Upsilon(\Gamma\zeta, H\eta, H\eta, \lambda)]\}) \quad (4.1.3)$$

for all $\zeta, \eta \in \Sigma$, $\lambda > 0$, where $\psi, \phi, \varphi : [0,1] \to [0,1]$ are continuous functions such that $\psi(s) > s$, $\phi(s) < s$ and $\varphi(s) < s$ for each $s \in (0,1)$. Then Γ, Λ, Ω and H have a unique common fixed point in Σ.

Proof

Since the pairs (Γ, Ω) and (Λ, H) are subcompatible maps of type (J-1) and subsequentially continuous, then there exist two sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in Σ such that $\lim_{n\to\infty}\Gamma\zeta_n = \lim_{n\to\infty}\Omega\zeta_n = \delta$, $\delta \in \Sigma$ and satisfy

$$\lim_{n\to\infty}\Xi(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = \Xi(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda) = 1,$$
$$\lim_{n\to\infty}\Theta(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = \Theta(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda) = 0,$$
$$\lim_{n\to\infty}\Upsilon(\Gamma\Omega\zeta_n, \Omega\Omega\zeta_n, \Omega\Omega\zeta_n, \lambda) = \Upsilon(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda) = 0,$$
$$\lim_{n\to\infty}\Xi(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = \Xi(\Omega\delta, \Gamma\delta, \Gamma\delta, \lambda) = 1,$$
$$\lim_{n\to\infty}\Theta(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = \Theta(\Omega\delta, \Gamma\delta, \Gamma\delta, \lambda) = 0,$$
$$\lim_{n\to\infty}\Upsilon(\Omega\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \Gamma\Gamma\zeta_n, \lambda) = \Upsilon(\Omega\delta, \Gamma\delta, \Gamma\delta, \lambda) = 0.$$

$\lim_{n\to\infty}\Lambda\zeta_n = \lim_{n\to\infty}H\zeta_n = \omega$, $\omega \in \Sigma$, and

$$\lim_{n\to\infty}\Xi(\Lambda H\eta_n, HH\eta_n, HH\eta_n, \lambda) = \Xi(\Lambda\omega, H\omega, H\omega, \lambda) = 1,$$

$\lim_{n\to\infty} \Theta$ (ΛHη_n, HHη_n, HHη_n, λ) = Θ ($\Lambda\omega$, Hω, Hω, λ) = 0,
$\lim_{n\to\infty} \Upsilon$ (ΛHη_n, HHη_n, HHη_n, λ) = Υ($\Lambda\omega$, Hω, Hω, λ) = 0,
$\lim_{n\to\infty} \Xi$ (H$\Lambda\eta_n$, $\Lambda\Lambda\eta_n$, $\Lambda\Lambda\eta_n$, λ) = Ξ(Hω, $\Lambda\omega$, $\Lambda\omega$, λ) = 1,
$\lim_{n\to\infty} \Theta$ (H$\Lambda\eta_n$, $\Lambda\Lambda\eta_n$, $\Lambda\Lambda\eta_n$, λ) = Θ (Hω, $\Lambda\omega$, $\Lambda\omega$, λ) = 0,
$\lim_{n\to\infty} \Upsilon$ (H$\Lambda\eta_n$, $\Lambda\Lambda\eta_n$, $\Lambda\Lambda\eta_n$, λ) = Υ (Hω, $\Lambda\omega$, $\Lambda\omega$, λ) = 0.

Therefore, $\Gamma\delta = \Omega\delta$ and $\Lambda\omega$ = Hω, that is δ is a coincidence point of Γ and Ω, ω is a coincidence point of Λ and H.Now, we prove that $\delta = \omega$. By using (3.1) for $\zeta = \zeta_n$ and $\eta = \eta_n$, we get
$\Xi(\Gamma\zeta_n, \Lambda\eta_n, \Lambda\eta_n, \lambda) \geq \psi$ (min {$\Xi(\Omega\zeta_n$, Hη_n, Hη_n, λ), $\Xi(\Gamma\zeta_n, \Omega\zeta_n, \Omega\zeta_n, \lambda)$, Ξ ($\Lambda\eta_n$, Hη_n, Hη_n, λ),
$\frac{1}{2}$[Ξ ($\Lambda\eta_n, \Omega\zeta_n, \Omega\zeta_n, \lambda$) + Ξ ($\Gamma\zeta_n$, Hη_n, Hη_n, λ)]}),
Θ ($\Gamma\zeta_n, \Lambda\eta_n, \Lambda\eta_n, \lambda$) $\leq \phi$ (max {Θ ($\Omega\zeta_n$, Hη_n, Hη_n, λ) , $\Theta(\Gamma\zeta_n, \Omega\zeta_n, \Omega\zeta_n, \lambda)$, Θ ($\Lambda\eta_n$, Hη_n, Hη_n, λ),
$\frac{1}{2}$[Θ ($\Lambda\eta_n, \Omega\zeta_n, \Omega\zeta_n, \lambda$) + Θ ($\Gamma\zeta_n$, Hη_n, Hη_n, λ)]}).
$\Upsilon(\Gamma\zeta_n, \Lambda\eta_n, \Lambda\eta_n, \lambda) \leq \varphi$(max {$\Upsilon(\Omega\zeta_n$, H$\eta_n$, H$\eta_n$, λ) , $\Upsilon(\Gamma\zeta_n, \Omega\zeta_n, \Omega\zeta_n, \lambda)$, Υ ($\Lambda\eta_n$, Hη_n, Hη_n, λ),
$\frac{1}{2}$[Υ ($\Lambda\eta_n, \Omega\zeta_n, \Omega\zeta_n, \lambda$) + Υ ($\Gamma\zeta_n$, Hη_n, Hη_n, λ)]}).

Taking the limit n $\to \infty$, we have
Ξ ($\delta, \omega, \omega, \lambda$) $\geq \psi$ (min { Ξ ($\delta, \omega, \omega, \lambda$), Ξ ($\delta, \delta, \delta, \lambda$) , Ξ ($\omega, \omega, \omega, \lambda$), $\frac{1}{2}$ [Ξ ($\omega, \delta, \delta, \lambda$) + Ξ ($\delta, \omega, \omega, \lambda$)]}),
Θ ($\delta, \omega, \omega, \lambda$) $\leq \phi$ (max { $\Theta(\delta, \omega, \omega, \lambda)$, Θ ($\delta, \delta, \delta, \lambda$) , Θ ($\omega, \omega, \omega, \lambda$), $\frac{1}{2}$ [Θ ($\omega, \delta, \delta, \lambda$) + Θ ($\delta, \omega, \omega, \lambda$)]}),
$\Upsilon(\delta, \omega, \omega, \lambda) \leq \varphi$ (max { $\Upsilon(\delta, \omega, \omega, \lambda)$, $\Upsilon(\delta, \delta, \delta, \lambda)$, $\Upsilon(\omega, \omega, \omega, \lambda)$, $\frac{1}{2}$ [$\Upsilon(\omega, \delta, \delta, \lambda) + \Upsilon(\delta, \omega, \omega, \lambda)$]}),
that is,

Ξ (δ., ω, ω, λ) $\geq \psi$ (Ξ ($\delta, \omega, \omega, \lambda$)) > Ξ ($\delta, \omega, \omega, \lambda$),
Θ (δ., ω, ω, λ) $\leq \phi$ (Θ ($\delta, \omega, \omega, \lambda$)) < Θ ($\delta, \omega, \omega, \lambda$),
Υ (δ., ω, ω, λ) $\leq \varphi$ (Υ ($\delta, \omega, \omega, \lambda$)) < Υ ($\delta, \omega, \omega, \lambda$),

which yield $\delta = \omega$.

Again using (3.1) for $\zeta = \delta$ and $\eta = \eta_n$, we obtain
Ξ ($\Gamma\delta, \Lambda\eta_n, \Lambda\eta_n, \lambda$) $\geq \psi$ (min {Ξ ($\Omega\delta$, Hη_n, Hη_n, λ), Ξ ($\Gamma\delta, \Omega\delta, \Omega\delta, \lambda$), Ξ ($\Lambda\eta_n$, Hη_n, Hη_n, λ),
$\frac{1}{2}$[Ξ ($\Lambda\eta_n, \Omega\delta, \Omega\delta, \lambda$) + Ξ ($\Gamma\delta$, Hη_n, Hη_n, λ)]}),
Θ {$\Gamma\delta, \Lambda\eta_n, \Lambda\eta_n, \lambda$) $\leq \phi$ (max {Θ ($\Omega\delta$, Hη_n, Hη_n, λ), Θ ($\Gamma\delta, \Omega\delta, \Omega\delta, \lambda$), Θ ($\Lambda\eta_n$, Hη_n, Hη_n, λ),
$\frac{1}{2}$[Θ ($\Lambda\eta_n, \Omega\delta, \Omega\delta, \lambda$) + Θ ($\Gamma\delta$, Hη_n, Hη_n, λ)]}).
Υ{$\Gamma\delta, \Lambda\eta_n, \Lambda\eta_n, \lambda$) $\leq \varphi$(max {$\Upsilon(\Omega\delta$, Hη_n, Hη_n, λ), $\Upsilon(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda)$, $\Upsilon(\Lambda\eta_n$, Hη_n, Hη_n, λ),
$\frac{1}{2}$[$\Upsilon(\Lambda\eta_n, \Omega\delta, \Omega\delta, \lambda)$ + Υ ($\Gamma\delta$, Hη_n, Hη_n, λ)]}).

Taking the limit as n $\to \infty$, we have,
$\Xi(\Gamma\delta, \omega, \omega, \lambda) \geq \psi$ (min {$\Xi(\Omega\delta, \omega, \omega, \lambda)$, $\Xi(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda)$, Ξ ($\omega, \omega, \omega, \lambda$),
$\frac{1}{2}$[Ξ ($\omega, \Omega\delta, \Omega\delta, \lambda$) + Ξ ($\Gamma\delta, \omega, \omega, \lambda$)]}),
$\Theta(\Gamma\delta, \omega, \omega, \lambda) \leq \phi$ (max {$\Theta(\Omega\delta, \omega, \omega, \lambda)$, $\Theta(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda)$, Θ ($\omega, \omega, \omega, \lambda$),
$\frac{1}{2}$[Θ ($\omega, \Omega\delta, \Omega\delta, \lambda$) + Θ ($\Gamma\delta, \omega, \omega, \lambda$)]}),
$\Upsilon(\Gamma\delta, \omega, \omega, \lambda) \leq \varphi$ (max {$\Upsilon(\Omega\delta, \omega, \omega, \lambda)$, $\Upsilon(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda)$, Υ ($\omega, \omega, \omega, \lambda$),
$\frac{1}{2}$[Υ ($\omega, \Omega\delta, \Omega\delta, \lambda$) + Υ ($\Gamma\delta, \omega, \omega, \lambda$)]}).

That is,

$\Xi(\Gamma\delta, \omega, \omega, \lambda) \geq \psi$ ($\Xi(\Gamma\delta, \omega, \omega, \lambda)$) > $\Xi(\Gamma\delta, \omega, \omega, \lambda)$,
$\Theta(\Gamma\delta, \omega, \omega, \lambda) \leq \phi$ ($\Theta(\Gamma\delta, \omega, \omega, \lambda)$) < $\Theta(\Gamma\delta, \omega, \omega, \lambda)$,
$\Upsilon(\Gamma\delta, \omega, \omega, \lambda) \leq \varphi$ ($\Upsilon(\Gamma\delta, \omega, \omega, \lambda)$) < $\Upsilon(\Gamma\delta, \omega, \omega, \lambda)$.

which yield $\Gamma\delta = \omega = \delta$.

Therefore $\delta = \omega$ is a common fixed point of Γ, Λ, Ω and H.

For uniqueness, suppose that there exist another fixed point u of Γ, Λ, Ω and H.

Then from (3.1), we have

$$\begin{aligned}
\Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda) &\geq \psi(\min\{\Xi(\Omega\delta, Hu, Hu, \lambda), \Xi(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Xi(\Lambda u, Hu, Hu, \lambda), \\
&\qquad \tfrac{1}{2}[\Xi(\Lambda u, \Omega\delta, \Omega\delta, \lambda) + \Xi(\Gamma\delta, Hu, Hu, \lambda)]\}) \\
&= \psi(\min\{\Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda), 1, \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \\
&\qquad \tfrac{1}{2}[\Xi(\Lambda u, \Gamma\delta, \Gamma\delta, \lambda) + \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda)]\}) \\
&= \psi(\Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda) \\
&> \Xi(\Gamma\delta, \Lambda u, \Lambda u, \lambda),
\end{aligned}$$

$$\begin{aligned}
\Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda) &\leq \phi(\max\{\Theta(\Omega\delta, Hu, Hu, \lambda), \Theta(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Theta(\Lambda u, Hu, Hu, \lambda), \\
&\qquad \tfrac{1}{2}[\Theta(\Lambda u, \Omega\delta, \Omega\delta, \lambda) + \Theta(\Gamma\delta, Hu, Hu, \lambda)]\}) \\
&= \phi(\max\{\Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda), 0, \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \\
&\qquad \tfrac{1}{2}[\Theta(\Lambda u, \Gamma\delta, \Gamma\delta, \lambda) + \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda)]\}) \\
&= \phi(\Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda) \\
&< \Theta(\Gamma\delta, \Lambda u, \Lambda u, \lambda),
\end{aligned}$$

$$\begin{aligned}
\Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda) &\leq \varphi(\max\{\Upsilon(\Omega\delta, Hu, Hu, \lambda), \Upsilon(\Gamma\delta, \Omega\delta, \Omega\delta, \lambda), \Upsilon(\Lambda u, Hu, Hu, \lambda), \\
&\qquad \tfrac{1}{2}[\Upsilon(\Lambda u, \Omega\delta, \Omega\delta, \lambda) + \Upsilon(\Gamma\delta, Hu, Hu, \lambda)]\}) \\
&= \varphi(\max\{\Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda), 0, \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda), \\
&\qquad \tfrac{1}{2}[\Upsilon(\Lambda u, \Gamma\delta, \Gamma\delta, \lambda) + \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda)]\}) \\
&= \varphi(\Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda) \\
&< \Upsilon(\Gamma\delta, \Lambda u, \Lambda u, \lambda),
\end{aligned}$$

which yield δ = u. Therefore, uniqueness follows.

If we put Ω = H in Theorem 3.1, we get the following result.

Corollary: 4.2

Let Γ, Λ, and Ω be self-maps of a weak non-Archimedean NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ and let the pairs (Γ, Ω) and (Λ, Ω) are subcompatible maps of type (J-1) and subsequentially continuous. If

$$\Xi(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \geq \psi(\min\{\Xi(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Xi(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Xi(\Lambda\eta, \Omega\eta, \Omega\eta, \lambda), \tfrac{1}{2}[\Xi(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Xi(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}) \quad (4.2.1)$$

$$\Theta(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \phi(\max\{\Theta(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Theta(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Theta(\Lambda\eta, \Omega\eta, \Omega\eta, \lambda), \tfrac{1}{2}[\Theta(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Theta(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}) \quad (4.2.2)$$

$$\Upsilon(\Gamma\zeta, \Lambda\eta, \Lambda\eta, \lambda) \leq \varphi(\max\{\Upsilon(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Upsilon(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Upsilon(\Lambda\eta, \Omega\eta, \Omega\eta, \lambda), \tfrac{1}{2}[\Upsilon(\Lambda\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Upsilon(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}) \quad (4.2.3)$$

for all $\zeta, \eta \in \Sigma$, $\lambda > 0$, where $\psi, \phi, \varphi : [0,1] \to [0,1]$ are continuous functions such that $\psi(s) > s$, $\varphi(s) < s$ and $\varphi(s) < s$ for each $s \in (0,1)$. Then Γ, Λ and Ω have a unique common fixed point in Σ.

If we put Γ = Λ and Ω = H in Theorem 4.1, we get the following result.

Corollary: 4.3

Let Γ and Ω be self-maps of a weak non-Archimedean NMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ and let the pairs (Γ, Ω) is subcompatible maps of type (J-1) and subsequentially continuous. If

$$\Xi(\Gamma\zeta, \Gamma\eta, \Gamma\eta, \lambda) \geq \psi\,(\min\{\Xi(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Xi(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Xi(\Gamma\eta, \Omega\eta, \Omega\eta, \lambda), \tfrac{1}{2}[\Xi(\Gamma\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Xi(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}), \quad (4.3.1)$$

$$\Theta(\Gamma\zeta, \Gamma\eta, \Gamma\eta, \lambda) \leq \phi\,(\max\{\Theta(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Theta(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Theta(\Gamma\eta, \Omega\eta, \Omega\eta, \lambda), \tfrac{1}{2}[\Theta(\Gamma\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Theta(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}), \quad (4.3.2)$$

$$\Upsilon(\Gamma\zeta, \Gamma\eta, \Gamma\eta, \lambda) \leq \varphi\,(\max\{\Upsilon(\Omega\zeta, \Omega\eta, \Omega\eta, \lambda), \Upsilon(\Gamma\zeta, \Omega\zeta, \Omega\zeta, \lambda), \Upsilon(\Gamma\eta, \Omega\eta, \Omega\eta, \lambda), \tfrac{1}{2}[\Upsilon(\Gamma\eta, \Omega\zeta, \Omega\zeta, \lambda) + \Upsilon(\Gamma\zeta, \Omega\eta, \Omega\eta, \lambda)]\}), \quad (4.3.3)$$

for all $\zeta, \eta \in \Sigma$, $\lambda > 0$, where $\psi, \phi, \varphi : [0,1] \to [0,1]$ are continuous functions such that $\psi(s) > s$, $\varphi(s) < s$ and $\varphi(s) < s$ for each $s \in (0,1)$. Then Γ and Ω have a unique common fixed point in Σ.

5. Conclusion

In this work, we obtained new structure of weak non-Archimedian with the help of subcompatible maps of types (J-1) and (J-2) in NMS. Also, we proved common fixed point theorems for four subcompatible maps of type (J-1) in weak non-Archimedean NMS.

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Analysis of Neutrosophic Multiple Regression

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Abstract: The idea of Neutrosophic statistics is utilized for the analysis of the uncertainty observation data. Neutrosophic multiple regression is one of a vital roles in the analysis of the impact between the dependent and independent variables. The Neutrosophic regression equation is useful to predict the future value of the dependent variable. This paper to predict the students' performance in campus interviews is based on aptitude and personality tests, which measures conscientiousness, and predict the future trend. Neutrosophic multiple regression is to authenticate the claim and examine the null hypothesis using the F-test. This study exhibits that Neutrosophic multiple regression is the most efficient model for uncertainty rather than the classical regression models

Keywords: Neutrosophic multiple regression; Neutrosophic regression; Neutrosophic correlation

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1. Introduction

The concept of fuzzy logic was introduced by Zadeh [1], the elements in the collections are represented by the membership value in the closed interval [0,1]. Atanassov [2,3,4] introduce the intuitionistic fuzzy set that is an extension of the fuzzy set. It is useful to examine the real-life circumstances by considering membership and non-membership grades but without indeterminate membership grades. Smarandache [5, 6] extend the idea of intuitionistic fuzzy sets with the account of indeterminate membership grades, which we called Neutrosophic sets. Aftermath, Salama et al., [7] introduced the operations on Neutrosophic sets and progressed Neutrosophic sets theory in [8, 9, 10, 11, 12].

The important role of analyzing the correlation of dependent and independent variables is to estimate the strength and relation between two variables. Hanafy et al., [13] introduced the concepts of Neutrosophic correlation and its coefficients for the case of finite spaces. The Neutrosophic regression analysis is a powerful method to identify the relationships between the dependent and independent variables and also forecasting the uncertainty observation data. Some of the applications of Neutrosophic regression can be seen in literature such as Karacoska [14], Cervigon, et al., [15], Kumar & Chong [16], and Abdul et al., [17]. Smarandach [18] introduced the theory of Neutrosophic statistics that is the extension of classical statistics and also investigated Neutrosophic regression analysis. The real-time applications of Neutrosophic regression can be seen in Aslam [20], Salama et al., [21]. Prabhu et al., [22] analyzed the real-time multiple analysis. Some other contributions are in this domain have already been done by various researchers such as Tanaka & Ishibuchi [23] and Aslam [24].

Broumi & Smarandache [25] studied the weighted correlation and correlation coefficient between two interval Neutrosophic sets that were defined by Wang et al., [26]. Zhang et al., [27] explained the correlation coefficient measures and their entropy for interval Neutrosophic sets. Ye [28] proposed the two correlation coefficients between normal Neutrosophic numbers (NNSs) based on the score functions of normal Neutrosophic numbers (NNNs) and investigated their properties. He also developed a MADM method with NNSs under normal Neutrosophic numbers. Ye [29] presented a new correlation coefficient measure between dynamic single-valued Neutrosophic multisets. Karaaslan [30] studied the measures between two Neutrosophic sets; two interval-Neutrosophic sets; two Neutrosophic-refined sets and their applications of these methods are utilized in multi-criteria decision-making problems. Broumi and Smarandache [31] also proposed the correlation coefficient between interval Neutrosophic sets. Rajarajeswari and Uma [32] put forward the correlation measure for IFMS. Recently, Broumi and Smarandache [reference] defined the Hausdorff distance between Neutrosophic sets and some similarity measures based on the distance such as the set-theoretic approach and matching function to calculate the similarity degree between Neutrosophic sets. Broumi [32] explained the concept of correlation measure of Neutrosophic-refined sets that is the extension of the correlation measure of Neutrosophic sets and intuitionistic fuzzy multi-sets. Le [33] established the fuzzy decision-making method based on the weighted correlation coefficient under the intuitionistic fuzzy environment. Le [34] explained the cosine similarity measures for intuitionistic fuzzy sets and their applications. Gerstenkorn [35] studied the concept of correlation under the environment of intuitionistic fuzzy sets. Further, Hung [36] defined the correlation for intuitionistic fuzzy sets based on the centroid method. Ye [37] introduced the multicriteria decision-making method by the use of the correlation coefficient under a single-valued Neutrosophic environment. Deli [38] studied the concept of Neutrosophic-refined sets and their applications in medical diagnosis. Sahin [39] explained the correlation coefficient of single-valued Neutrosophic hesitant fuzzy sets and applied them in decision-making problems. Pramanik et al., [40] studied the multicriteria decision-making problems by applying a rough Neutrosophic correlation coefficient. Nagarajan et al., [41] explained Neutrosophic interval valued graphs. Lathamaheswari et al., [42] explained type 2 fuzzy in bio medicine. Ye [43] explained the improved correlation coefficients of single-valued Neutrosophic sets and interval Neutrosophic sets for multiple attribute decision-making problems. Liu et al., [44] established a correlation coefficient for the interval-valued Neutrosophic hesitant fuzzy sets and applied them in multiple attribute decision-making. Ye [45] studied the multi-criteria decision-making method using the correlation coefficient under a single-valued Neutrosophic environment. González-Rodríguez et al.,[46] explained ANOVA test for Fuzzy data. Jiryaei A et al.,[47] studied fuzzy random variables.

2.Preliminaries:

Regression line with dependent and one independent equation is

$$Y = a + bX + e. \qquad (1)$$

When Y is the output value on dependent, variable X is the input value of the independent variable, b is the slope, a is the intercept and e is the residual.
More than one independent variable equation as:

$$Y = a + b_1X_1 + b_2X_2 + \ldots + b_nX_n + e \qquad (2)$$

Here n number of independent variables and $b_1, b_2 \ldots b_n$ are number of slopes for each.e is the standard error . The estimation of a and b for to minimize the error of prediction equation

$$Y' = a + b_1X_1 + b_2X_2 + \ldots + b_nX_n \qquad (3)$$

The equation for a with two independent variables is:

$$a = Y - b_1X_1 - b_2X_2 \qquad (4)$$

For the two-variable case:

$$b_1 = \frac{\sum x_2^{\,2} \sum x_1 y - \sum x_1 x_2 \sum x_2 y}{\sum x_1^{\,2} \sum x_2^{\,2} - (\sum x_1 x_2)^2} \qquad (5)$$

$$b_2 = \frac{\sum x_1^{\,2} \sum x_2 y - \sum x_1 x_2 \sum x_1 y}{\sum x_1^{\,2} \sum x_2^{\,2} - (\sum x_1 x_2)^2} \qquad (6)$$

From the above equations 5 & 6 only for two variables x_1 and x_2.

Smarandache [18] is given the Neutrosophic extended for classical statistics operation. The operations are as follows.

Let's S_1 and S_2 be two sets of numbers.

$S_1+S_2 = \{x_1+x_2 \mid x_1 \in S_1 \text{ and } x_2 \in S_2\}$

$S_1-S_2 = \{x_1-x_2 \mid x_1 \in S_1 \text{ and } x_2 \in S_2\}$

$S_1 \cdot S_2 = \{x_1 \cdot x_2 \mid x_1 \in S_1 \text{ and } x_2 \in S_2\}$

$a \cdot S_1 = S_1 \cdot a = \{a \cdot x_1 \mid x_1 \in S_1\}$

$a + S_1 = S_1 + a = \{a + x_1 \mid x_1 \in S_1\}$

$a - S_1 = \{a - x_1 \mid x_1 \in S_1\}$

$S_1 - a = \{x_1 - a \mid x_1 \in S_1\}$

$S_1 S_2 = \{x_1 x_2 \mid x_1 \in S_1 , x_2 \in S_2, x_2 \neq 0 \}$

$S_1 n = \{x_1 n \mid x_1 \in S_1\}$

$S_1a = \{x_1a \mid x_1 \in S_1, a \neq 0\}$

$aS_1 = \{ax_1 \mid x_1 \in S_1, \quad x_1 \neq 0\}$

$\sqrt{S_1n} = \{\sqrt{x_1n} \mid x_1 \in S_1\}$

3.Numerical example

In table 1 shows the student performance campus interview based on aptitude and personality test, that measure the conscientiousness

Y is the dependent variable conscientiousness x_1 is the aptitude test and personality test as shown in the following table 1.

table:1 Database

Y	X_1	x_2
[1,3]	3	2
2	2	[2,1]
[2,4]	[1,2]	[3,2]
4	[2,3]	4
[1,4]	[2,1]	[4,4]
6	[2,3]	[4,5]
[2,4]	2	1
[10,13]	[5,6]	[6,7]
[14,15]	7	8
5	[7,1]	3

$$\sum x_1 y = \sum X_1 Y - \frac{\sum X_1 \sum Y}{N} \tag{7}$$

$$\sum x_2 y = \sum X_2 Y - \frac{\sum X_2 \sum Y}{N} \tag{8}$$

$$\sum x_1 x_2 = \sum X_1 X_2 - \frac{\sum X_1 \sum X_2}{N} \tag{9}$$

Using the equation 7, 8, and 9

$\sum x_1 y = [38{,}91.9]$, $\sum x_2 y = [23{,}136.1]$ $\sum x_1 x_2 = [35{,}19.9]$

Matrix form of the values is corresponding to the correlation, sum of square, and cross product of the variables as shown in the following table 2.

table 2: Matrix form of the values

	Y	X_1	X_2
Y	$\sum y^2$ =[387,532]	$\sum x_1 y$ =[218,247]	$\sum x_2 y$ =[245,310]
X_1	r_{yx1} =[0.89954, 1.31925]	$\sum x_1^2$ =[156,1236]	$\sum x_1 x_2$ =[146,142]
X_2	r_{yx2} =[1.56309, 1.17855]	r_{x1x2} =[1.172603, 0.406182]	$\sum x_2^2$ =[175,189]

Using equation 5 and 6 the value of the regression coefficient

b_1= [-2.34988, 1.650734], b_2= [0.965172, 1.093934]

from equation 4 the value of the intercept is

a= [-4.29976, 10.18347]

Therefore the Neutrosophic regression equation is

Y=[-4.29976, 10.18347] + [-2.34988, 1.650734] x_1 +[0.965172, 1.093934] x_2

The proportion of variance is in the set of independent variables is R square value. The Neutrosophic R square value is

A Neutrosophic residual sum of squares is $NRSS = \sum(y - \hat{y})^2$ (9)

$$NRSS = \sum (y - \hat{y})^2 = [183,267.7]$$

A Neutrosophic total sum of squares $NTSS = \sum(y - \bar{y})^2$ (10)

$$NTSS = \sum (y - \bar{y})^2 = [2268.2,1875]$$

A Neutrosophic coefficient of determination is $NCD = 1 - \frac{NRSS}{NTSS}$ (11)

$$NCD = 1 - \frac{NRSS}{NTSS} = [0.097,0.129]$$

The Neutrosophic mean of Y is [46,50]. The Neutrosophic r square is [0.09,0.12] from the above results shows that the variation between independent and dependent variables is 9 % and 12 %. That means the student performance campus interview variation based on aptitude and personality test is between 9 % and 12 %. Hence, it is revealed that these variables are also affected by the student performance on-campus interview.

4.Significance test of R square

Using the F test for significance of R square is

$$F = \frac{R^2 / K}{(1 - R^2)(N - K - 1)} \quad (12)$$

Which is distributed as F with K and N-K-1 degrees of freedom when the null hypothesis is true. Now R^2 represents the multiple correlations rather than the single correlation.

The null hypothesis: R square value is not zero population with degrees of freedom is N-K-1

Using (12), the Neutrosophic F value is [0.007904,0093]

Comparing the tabulated value using degrees of freedom and the calculated value. It shows that the null hypothesis is accepted.

5.Regression with beta weights

Comparison of correlation and regression equation is

$$Z_Y' = r_{xy} Z_x \quad (13)$$

But β means a b weight when X and Y are in standard scores, so for the simple regression case, r = β , and we have:

$$Z_Y' = \beta Z_x \quad (14)$$

The bottom line on this is we can estimate β weights using a correlation matrix.

$$\beta_1 = \frac{r_{yx_1} - r_{yx_2} r_{x_1x_2}}{1 - r_{x_1x_2}{}^2} \quad (15)$$

$$\beta_2 = \frac{r_{yx_2} - r_{yx_1} r_{x_1x_2}}{1 - r_{x_1x_2}{}^2} \quad (16)$$

where r_{yx1} is the correlation of y with X_1, r_{yx2} is the correlation of y with X_2, and r_{12} is the correlation of x_1 with x_2. Note that the two formulas are nearly identical and the correlation matrix shows in table:3

table :3 Correlation matrix

	Y	X_1	X_2
Y	1		
X_1	[-0.89954, -1.31925	1	
X_2	[-1.56309, -1.17855]	[1.172603, 0.406182]	1

Using the equation 15 and 16 calculate the Neutrosophic beta coefficients. That is

$$\beta_1 = [-0.50399, -1.3679], \beta_2 = [-1.2303, 0.329977]$$

Note that there is a surprisingly large difference in beta weights given the magnitude of correlations.

6.The limitations on statistics

In table 4 shows that limitation on different category statistics

table:4 Limitation on Statistics

Statistics		Limitations
Classical statistics	It is applied for the analysis to determining the sample and the parameter in the population or sample space is determined.	The analysis only for the determined parameter. Testing the analysis of variance and significance under classical statistics only for determined observation.
Fuzzy statistics	The analysis using fuzzy statistics applies to the data having uncertainty. The statistics depend on Fuzzy statistics and do not consider indeterminacy.	It will be applied for observations in Fuzzy. Under fuzzy statistics testing the analysis of variance and significance only for the observations are fuzzy and uncertain.
Intuitionistic fuzzy statistics	It is the extension of fuzzy statistics and considering membership and non-membership grades.	It will apply only intervals belongs to membership and non-membership. Under Intuitionistic statistics testing the analysis of variance and significance only for the observation are membership and non-membership that belongs to the real unit interval.
Neutrosophic statistics	It is based on Neutrosophic logic and is considered the measure of indeterminacy. It is the	It is applied to an uncertain environment. Under Neutrosophic statistics testing the analysis of

	extension of intuitionistic fuzzy sets.	variance and significance when the observations are not fuzzy in the interval and it is an extension of classical and fuzzy statistics.

7. Conclusion

In this paper, we introduce the multiple regression method under the environment of Neutrosophic sets. Moreover, we proposed a method to compute the correlation coefficient of Neutrosophic sets which is given us information about the degree of the relationships between the variables based on Neutrosophic sets. Further, the method is applied to predict the students' performance in campus interviews based on aptitude and personality tests. Based on the above method the result shows that the variation between independent and dependent variables is 9% and 12%, which means that the students' performance variations based on aptitude and personality tests are between 9% and 12%. Thus, it is revealed that aptitude and personality tests are affected students' performance in campus interviews. Future work will be focused on the concept of interval Neutrosophic multiple regression analysis.

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A New Similarity Measure Based on Falsity Value between Single Valued Neutrosophic Sets Based on the Centroid Points of Transformed Single Valued Neutrosophic Numbers with Applications to Pattern Recognition

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Abstract. In this paper, we propose some transformations based on the centroid points between single valued neutrosophic numbers. We introduce these transformations according to truth, indeterminacy and falsity value of single valued neutrosophic numbers. We propose a new similarity measure based on falsity value between single valued neutrosophic sets. Then we prove some properties on new similarity measure based on falsity value between single valued neutrosophic sets. Furthermore, we propose similarity measure based on falsity value between single valued neutrosophic sets based on the centroid points of transformed single valued neutrosophic numbers. We also apply the proposed similarity measure between single valued neutrosophic sets to deal with pattern recognition problems.

Keywords: Neutrosophic sets, Single Valued Neutrosophic Numbers, Centroid Points.

1 Introduction

In [1] Atanassov introduced a concept of intuitionistic sets based on the concepts of fuzzy sets [2]. In [3] Smarandache introduced a concept of neutrosophic sets which is characterized by truth function, indeterminacy function and falsity function, where the functions are completely independent. Neutrosophic set has been a mathematical tool for handling problems involving imprecise, indeterminant and inconsistent data; such as cluster analysis, pattern recognition, medical diagnosis and decision making.In [4] Smarandache et.al introduced a concept of single valued neutrosophic sets. Recently few researchers have been dealing with single valued neutrosophic sets [5-10].

The concept of similarity is fundamentally important in almost every scientific field. Many methods have been proposed for measuring the degree of similarity between intuitionistic fuzzy sets [11-15]. Furthermore, in [13-15] methods have been proposed for measuring the degree of similarity between intuitionistic fuzzy sets based on transformed techniques for pattern recognition. But those methods are unsuitable for dealing with the similarity measures of neutrosophic sets since intuitionistic sets are characterized by only a membership function and a non-membership function. Few researchers dealt with similarity measures for neutrosophic sets [16-22]. Recently, Jun [18] discussed similarity measures on internalneutrosophic sets, Majumdar et al.[17] discussed similarity and entropy of neutrosophic sets, Broumi et.al.[16]discussed several similarity measures of neutrosophic sets, Ye [9] discussed single-valued neutrosophic similarity measures based on cotangent function and their application in the fault diagnosis of steam turbine, Deli et.al.[10] discussed multiple criteria decision making method on single valued bipolar neutrosophic set based on correlation coefficient similarity measure, Ulucay et.al. [21] discussed Jaccard vector similarity measure of bipolar neutrosophic set based on multi-criteria decision making and Ulucay et.al.[22] discussed similarity measure of bipolar neutrosophic sets and their application to multiple criteria decision making.

In this paper, we propose methods to transform between single valued neutrosophic numbers based on centroid points. Here, as single valued neutrosophic sets are made up of three functions, to make the transformation functions be applicable to all single valued neutrosophic numbers, we divide them into four according to their truth, indeterminacy and falsity values. While grouping according to the truth values, we take into account whether the truth values are greater or smaller than the indeterminancy and falsity values. Similarly, while grouping according to the indeterminancy/falsity values, we examine the indeterminancy/falsity values and their greatness or smallness with respect to their remaining two values. We also propose a new method to measure the degree of similarity based on falsity values between single valued neutrosophic sets. Then we prove some properties of new similarity measure based on falsity value between single valued neutrosophic sets. When we take this measure with respect to truth or indeterminancy we show that it does not satisfy one of the conditions of similarity measure. We also apply the proposed new similarity measures based on falsity value between single valued neutrosophic sets to deal with pattern recognition problems. Later, we define the method based on falsity value to measure the degree of similarity between single valued neutrosophic set based on centroid points of transformed single valued neutrosophic numbers and the similarity measure based on falsity value between single valued neutrosophic sets.

In section 2, we briefly review some concepts of single valued neutrosophic sets [4] and property of similarity measure between single valued neutrosophic sets. In section 3, we define transformations between the single valued neutrosophic numbers based on centroid points. In section 4, we define the new similarity measures based on falsity value between single valued neutrosophic sets and we prove some properties of new similarity measure between single valued neutroshopic sets. We also apply the proposed method to deal with pattern recognition problems. In section 5, we define the method to measure the degree of similarity based on falsity value between single valued neutrosophicset based on the centroid point of transformed single valued neutrosophic number and we apply the measure to deal with pattern recognition problems. Also we compare the traditional and new methods in pattern recognition problems.

2 Preliminaries

Definition 2.1[3] Let U be a universe of discourse. The neutrosophic set A is an object having the farm $A = \{\langle x: T_{A(x)}, I_{A(x)}, F_{A(x)}\rangle, x \in U\}$ where the functions $T, I, F: U \to]^{-}0, 1^{+}[$ respectively the degree of membership, the degree of indeterminacy and degree of non-membership of the element $x \in U$ to the set A with the condition:

$$0^{-} \leq T_{A(x)} + I_{A(x)} + F_{A(x)} \leq 3^{+}$$

Definition 2.2 [4] Let U be a universe of discourse.The single valued neutrosophic set $A$is an object having the farm $A = \{\langle x: T_{A(x)}, I_{A(x)}, F_{A(x)}\rangle, x \in U\}$ where the functions $T, I, F: U \to [0,1]$respectively the degree of membership, the degree of indeterminacy and degree of non-membership of the element $x \in U$ to the set A with the condition:

$$0 \leq T_{A(x)} + I_{A(x)} + F_{A(x)} \leq 3$$

For convenience we can simply use x = (T,I,F) to represent an element x in SVNS, and element x can be called a single valued neutrosophic number.

Definition 2.3 [4] A single valued neutrosophic set A is equal to another single valued neutrosophic set B, $A = B$ if$\forall x \in U$,

$$T_{A(x)} = T_{B(x)},\ I_{A(x)} = I_{B(x)},\ F_{A(x)} = F_{B(x)}.$$

Definition 2.4[4] A single valued neutrosophic set A is contained in another single valued neutrosophic set B , $A \subseteq B$ if $\forall x \in U$,

$$T_{A(x)} \leq T_{B(x)},\ I_{A(x)} \leq I_{B(x)},\ F_{A(x)} \geq F_{B(x)}.$$

Definition 2.5[16] (Axiom of similarity measure)

A mapping$S(A,B): NS_{(x)} \times NS_{(x)} \to [0,1]$, where $NS_{(x)}$ denotes the set of all NS in$x = \{x_1, \ldots, x_n\}$,is said to be the degree of similarity between A and B if it satisfies the following conditions:

sP_1) $0 \leq S(A,B) \leq 1$

sp_2) $S(A,B) = 1$ if and only if $A = B$, $\forall A, B \in NS$

sP_3) $S(A,B) = S(B,A)$

sp_4) If $A \subseteq B \subseteq C$ for all $A, B, C \in NS$, then $S(A,B) \geq S(A,C)$and $S(B,C) \geq S(A,C)$.

3 The Transformation Techniques between Single Valued Neutrosophic Numbers

In this section, we propose transformation techniques between a single valued neutrosophic number $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ and a single valued neutrosophic number $C_{(x_i)}$. Here $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ denote the single valued neutrosophic numbers to represent an element x_i in the single valued neutrosophic set A, and $C_{A(x_i)}$ is the center of a triangle (SLK) which was obtained by the transformation on the three-dimensional $Z - Y - M$ plane.

First we transform single valued neutrosophic numbers according to their distinct T_A, I_A, F_A values in three parts.

3.1 Transformation According to the Truth Value

In this section, we group the single valued neutrosophic numbers after the examination of their truth values T_A's greatness or smallness against I_A and F_A values. We will shift the $T_{A(x_i)}$ and $F_{A(x_i)}$ values on the Z – axis and $T_{A(x_i)}$ and $I_{A(x_i)}$ values on the Y – axis onto each other. We take the $F_{A(x_i)}$ value on the M – axis. The shifting on the Z and Y planes are made such that we shift the smaller value to the difference of the greater value and 2, as shown in the below figures.

1. **First Group**

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$, if

$$T_{A(x_i)} \leq F_{A(x_i)}$$

and

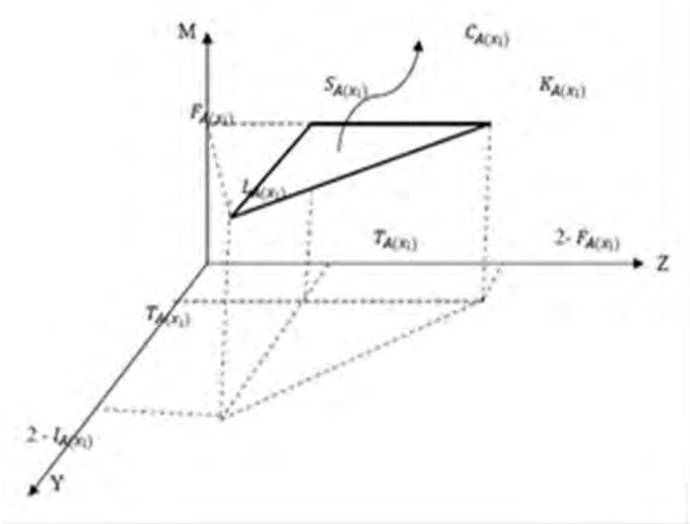
$$T_{A(x_i)} \leq I_{A(x_i)},$$

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(T_{A(x_i)}, T_{A(x_i)}, F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(2 - F_{A(x_i)},\ T_{A(x_i)},\ F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(T_{A(x_i)},\ 2 - I_{A(x_i)},\ F_{A(x_i)}\right).$$



Here, as

$$T_{C_A(x_i)} = T_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - T_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = T_{A(x_i)} + \frac{\left(2 - I_{A(x_i)} - T_{A(x_i)}\right)}{3} = \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

2. **Second Group**

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$, if

$$T_{A(x_i)} \geq F_{A(x_i)}$$

and

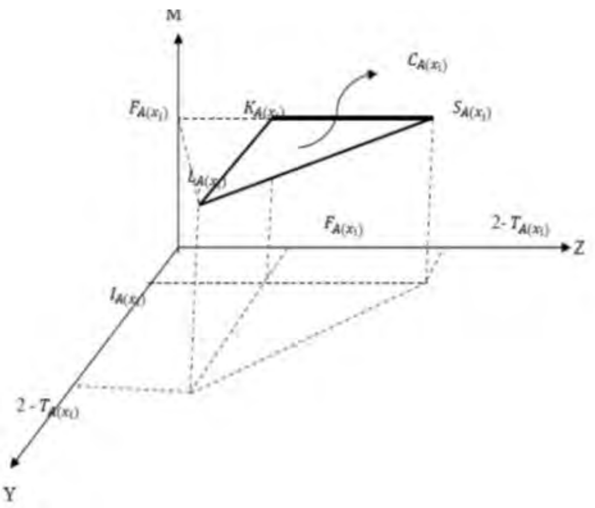
$$T_{A(x_i)} \geq I_{A(x_i)},$$

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$S_{A(x_i)} = \left(F_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\right)$

$\mathrm{L}_{A(x_i)} = \left(F_{A(x_i)}, 2 - T_{A(x_i)}, F_{A(x_i)}\right)$

$\mathrm{K}_{A(x_i)} = \left(2 - T_{A(x_i)}, I_{A(x_i)}, F_{a(x_i)}\right)$.



Here, as

$$T_{C_A(x_i)} = F_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - F_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2\,F_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = I_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2\,I_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$\mathrm{C}_{\mathrm{A}(x_i)} = \left(\frac{2 - \mathrm{T}_{\mathrm{A}(x_i)} + 2\,\mathrm{F}_{\mathrm{A}(x_i)}}{3}, \frac{2 - \mathrm{T}_{\mathrm{A}(x_i)} + 2\mathrm{I}_{\mathrm{A}(x_i)}}{3}, \mathrm{F}_{\mathrm{A}(x_i)}\right).$$

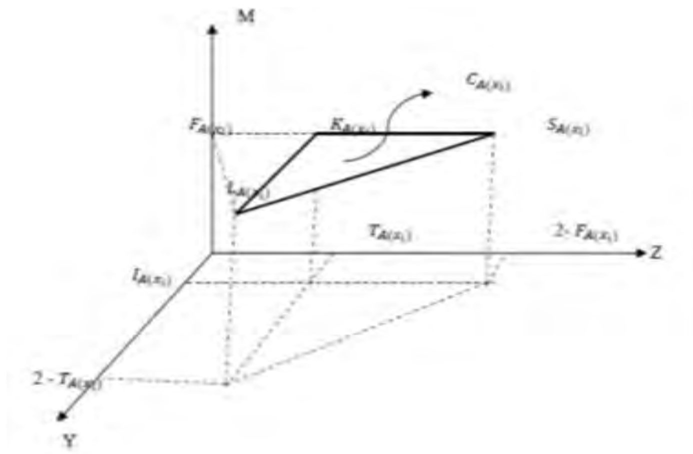
3. Third Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if$\mathrm{I}_{\mathrm{A}(x_i)} \le \mathrm{T}_{\mathrm{A}(x_i)} \le \mathrm{F}_{\mathrm{A}(x_i)}$,as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$\mathrm{S}_{\mathrm{A}(x_i)} = \left(\mathrm{T}_{\mathrm{A}(x_i)}, \mathrm{I}_{\mathrm{A}(x_i)}, \mathrm{F}_{\mathrm{A}(x_i)}\right)$

$\mathrm{L}_{\mathrm{A}(x_i)} = \left(\mathrm{T}_{\mathrm{A}(x_i)}, 2 - \mathrm{T}_{\mathrm{A}(x_i)}, \mathrm{F}_{\mathrm{A}(x_i)}\right)$

$\mathrm{K}_{\mathrm{A}(x_i)} = \left(2 - \mathrm{F}_{\mathrm{A}(x_i)}, \mathrm{I}_{\mathrm{A}(x_i)}, \mathrm{F}_{\mathrm{A}(x_i)}\right)$.



Here, as

$$T_{C_A(x_i)} = T_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - T_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = I_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2\,I_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$\mathrm{C}_{\mathrm{A}(x_i)} = \left(\frac{2 - \mathrm{F}_{\mathrm{A}(x_i)} + 2\,\mathrm{T}_{\mathrm{A}(x_i)}}{3}, \frac{2 - \mathrm{T}_{\mathrm{A}(x_i)} + 2\mathrm{I}_{\mathrm{A}(x_i)}}{3}, \mathrm{F}_{\mathrm{A}(x_i)}\right).$$

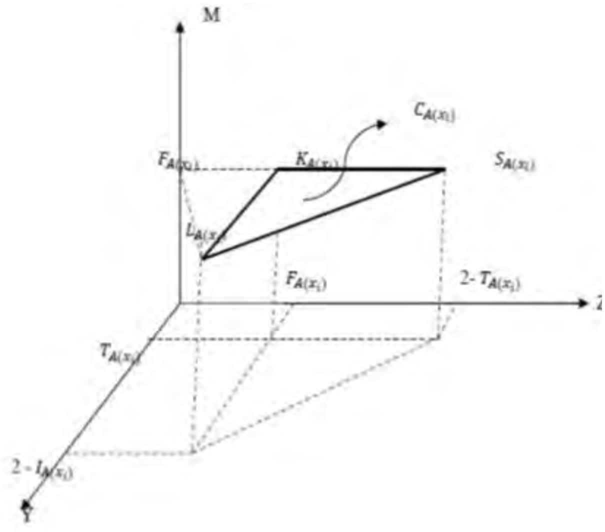
4. Fourth Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if$\mathrm{F}_{\mathrm{A}(x_i)} \le \mathrm{T}_{\mathrm{A}(x_i)} \le \mathrm{I}_{\mathrm{A}(x_i)}$,as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$$\mathrm{S}_{A(x_i)} = \left(\mathrm{F}_{A(x_i)},\ \mathrm{T}_{A(x_i)},\ \mathrm{F}_{A(x_i)}\right)$$

$$\mathrm{L}_{A(x_i)} = \left(\mathrm{F}_{A(x_i)},\ 2 - \mathrm{T}_{A(x_i)},\ \mathrm{F}_{A(x_i)}\right)$$

$$\mathrm{K}_{A(x_i)} = \left(2 - T_{A(x_i)},\ \mathrm{T}_{A(x_i)},\ \mathrm{F}_{A(x_i)}\right).$$



Example3.1.1 Transform the following single valued neutrosophic numbers according to their truth values.

$\langle 0.2,\ 0.5,\ 0.7\rangle$, $\langle 0.9,\ 0.4,\ 0.5\rangle$, $\langle 0.3,\ 0.2,\ 0.5\rangle$, $\langle 0.3,\ 0.2,\ 0.4\rangle$.

$i.$ $\langle 0.2,\ 0.5,\ 0.7\rangle$ single valued neutrosophic number belongs to the first group.

The center is calculated by the formula, $C_{A(x_i)} = \left(\frac{2- F_{A(x_i)} + 2\, T_{A(x_i)}}{3}, \frac{2- I_{A(x_i)} + 2\, T_{A(x_i)}}{3}, F_{A(x_i)}\right)$

and we have $\mathrm{C}_{A(x)} = \langle 0.566,\ 0.633,\ 0.7\rangle$.

$ii.$ $\langle 0.9,\ 0.4,\ 0.5\rangle$single valued neutrosophic number is in the second group.

The center for the values of the second group is, $\mathrm{C}_{A(x_i)} = \left(\frac{2- \mathrm{T}_{A(x_i)} + 2\, \mathrm{F}_{A(x_i)}}{3}, \frac{2- \mathrm{T}_{A(x_i)} + 2\mathrm{I}_{A(x_i)}}{3}, \mathrm{F}_{A(x_i)}\right)$

and for $\langle 0.9,\ 0.4,\ 0.5\rangle$, $\mathrm{C}_{A(x)} = \langle 0.7,\ 0.633,\ 0.5\rangle$.

$iii.$ $\langle 0.3,\ 0.2,\ 0.5\rangle$single valued neutrosophic number belongs to the third group.

The formula for the center of $\langle 0.3,\ 0.2,\ 0.5\rangle$ is $\mathrm{C}_{A(x_i)} = \left(\frac{2- \mathrm{F}_{A(x_i)} + 2\, \mathrm{T}_{A(x_i)}}{3}, \frac{2- \mathrm{T}_{A(x_i)} + 2\mathrm{I}_{A(x_i)}}{3}, \mathrm{F}_{A(x_i)}\right)$and therefore we have $\mathrm{C}_{A(x)} = \langle 0.7,\ 0.7,\ 0.5\rangle$.

$iv.$ $\langle 0.3,\ 0.2,\ 0.4\rangle$single valued neutrosophic number is in the third group and the center is calculated to be $\mathrm{C}_{A(x)} = \langle 0.733,\ 0.7,\ 0.4\rangle$.

Corollary 3.1.2 The corners of the triangles obtained using the above method need not be single valued neutrosophic number but by definition, trivially their centers are.

Note 3.1.3 As for the single valued neutrosophic number$\langle 1,$ ber$\langle 1,\ 1,\ 1\rangle$ there does not exist any transformable triangle in the above four groups, we take its transformation equal to itself.

Corollary 3.1.4 If $\mathrm{F}_{A(x_i)} = \mathrm{T}_{A(x_i)} = \mathrm{I}_{A(x_i)}$ the transformation gives the same center in all four groups. Also, if $\mathrm{T}_{A(x_i)} = \mathrm{I}_{A(x_i)} \leq \mathrm{F}_{A(x_i)}$, then the center in the first group is equal to the one in the third group and if $\mathrm{F}_{A(x_i)} \leq \mathrm{T}_{A(x_i)} = \mathrm{I}_{A(x_i)}$, the center in the second group is equal to the center in the fourth group. Similarly, if $\mathrm{T}_{A(x_i)} = \mathrm{F}_{A(x_i)} \leq \mathrm{I}_{A(x_i)}$, then the center in the first group is equal to the center in the fourth group and if $\mathrm{I}_{A(x_i)} \leq \mathrm{T}_{A(x_i)} = \mathrm{F}_{A(x_i)}$, the center in the second group is equal to the one in the third group.

3.2Transformation According to the Indeterminancy Value

In this section, we group the single valued neutrosophic numbers after the examination of their indeterminancy values I_A's greatness or smallness against T_A and$F_A$values. We will shift the $I_{A(x_i)}$ and $F_{A(x_i)}$ values on the Z – axis and $T_{A(x_i)}$ and $I_{A(x_i)}$values on the Y – axis onto each other. We take the $F_{A(x_i)}$value on the M – axis. The shifting on the Z and Y planes are made such that we shift the smaller value to the difference of the greater value and 2, as shown in the below figures.

1. **First Group**

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$, if

$$I_{A(x_i)} \leq F_{A(x_i)}$$

and

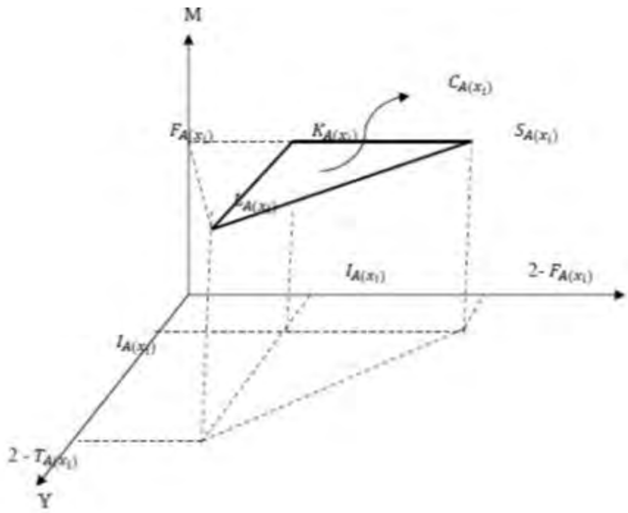
$$I_{A(x_i)} \leq F_{A(x_i)} ,$$

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(I_{A(x_i)}, \quad I_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(2 - F_{A(x_i)}, \ I_{A(x_i)}, \ F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(I_{A(x_i)}, \ 2 - T_{A(x_i)}, \ F_{A(x_i)}\right) .$$



We transformed the single valued neutrosophic number $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ into the center of the SKL triangle, namely $C_{A(x_i)}$. Here, as

$$T_{C_{A(x_i)}} = I_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2\, I_{A(x_i)}}{3}$$

$$I_{C_{A(x_i)}} = T_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2\, I_{A(x_i)}}{3}$$

and

$$F_{C_{A(x_i)}} = F_{A(x_i)} ,$$

we have

$$C_{A(x_i)} = \left(\frac{2- F_{A(x_i)}+ 2\, I_{A(x_i)}}{3}, \frac{2- T_{A(x_i)}+ 2\, I_{A(x_i)}}{3}, F_{A(x_i)}\right) .$$

2. Second Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$, if

$$I_{A(x_i)} \geq F_{A(x_i)}$$

and

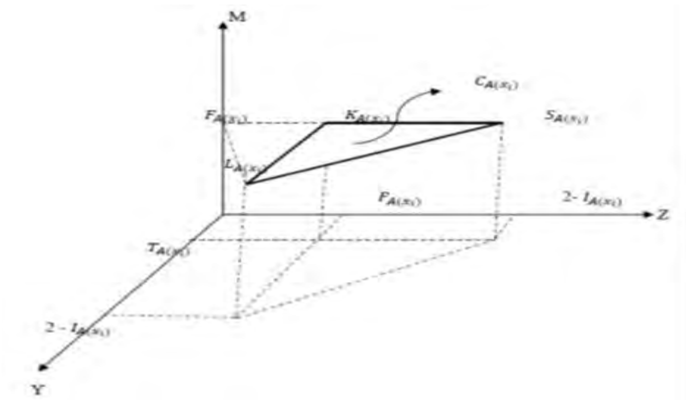
$$I_{A(x_i)} \geq F_{A(x_i)} ,$$

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(F_{A(x_i)}, \quad T_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(F_{A(x_i)}, 2 - I_{A(x_i)}, \ F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(2 - I_{A(x_i)}, \ T_{A(x_i)}, \ F_{A(x_i)}\right) .$$



Here, as

$$T_{C_{A(x_i)}} = F_{A(x_i)} + \frac{\left(2 - I_{A(x_i)} - F_{A(x_i)}\right)}{3} = \frac{2 - I_{A(x_i)} + 2\, F_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = T_{A(x_i)} + \frac{\left(2 - I_{A(x_i)} - T_{A(x_i)}\right)}{3} = \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - I_{A(x_i)} + 2\,F_{A(x_i)}}{3}, \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

3. Third Group

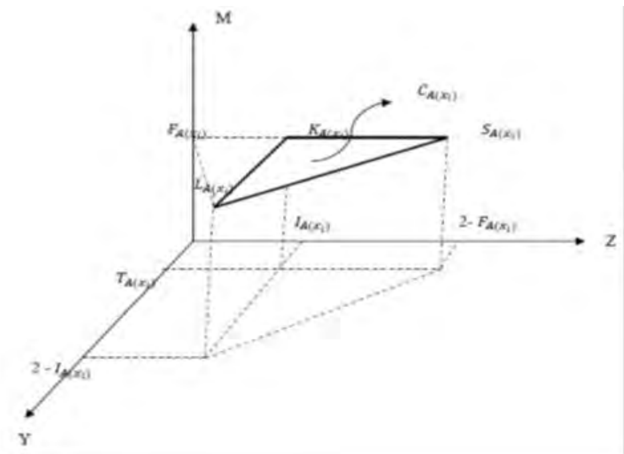
For the single valued neutrosophic number $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if $\mathrm{T}_{A(x_i)} \leq \mathrm{I}_{A(x_i)} \leq \mathrm{F}_{A(x_i)}$,

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(I_{A(x_i)}, \quad T_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(I_{A(x_i)}, 2 - I_{A(x_i)}, F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(2 - F_{A(x_i)}, T_{A(x_i)}, F_{A(x_i)}\right).$$



Here as

$$T_{C_A(x_i)} = I_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = T_{A(x_i)} + \frac{\left(2 - I_{A(x_i)} - T_{A(x_i)}\right)}{3} = \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}, \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

4. Fourth Group

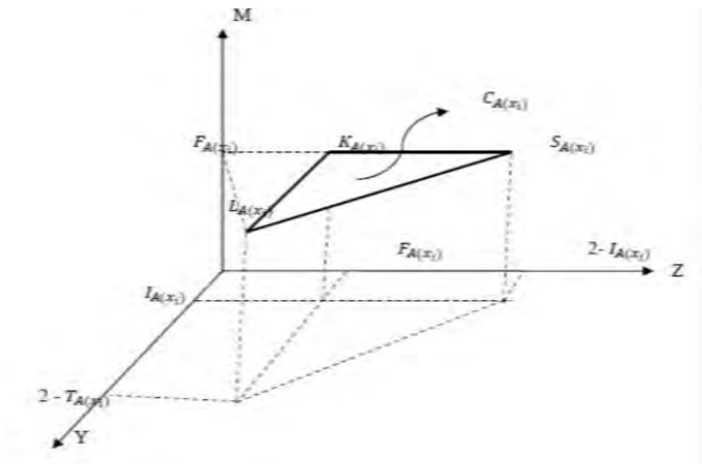
For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if $\mathrm{F}_{A(x_i)} \leq \mathrm{I}_{A(x_i)} \leq \mathrm{T}_{A(x_i)}$,

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic numbers $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(F_{A(x_i)}, \quad I_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(F_{A(x_i)}, 2 - T_{A(x_i)}, F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(2 - I_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\right).$$



Here, as

$$T_{C_A(x_i)} = F_{A(x_i)} + \frac{\left(2 - I_{A(x_i)} - F_{A(x_i)}\right)}{3} = \frac{2 - I_{A(x_i)} + 2\,F_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = I_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2\,I_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - I_{A(x_i)} + 2\,F_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2\,I_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

Example3.2.1:Transform the single neutrosophic numbers of Example 3.1.3 ,

$\langle 0.2, 0.5, 0.7\rangle$, $\langle 0.9, 0.4, 0.5\rangle$, $\langle 0.3, 0.2, 0.5\rangle$, $\langle 0.3, 0.2, 0.4\rangle$according to their indeterminancy values.

$i.$ $\langle 0.2, 0.5, 0.7\rangle$ single valued neutrosophic number is in the third group. The center is given by the formula

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}, \frac{2 - I_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

and so $C_{A(x)} = \langle 0.766, 0.633, 0.7\rangle$.

$ii.$ $\langle 0.9, 0.4, 0.5\rangle$ single valued neutrosophic number is in the first group.

By

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2\,I_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

we have $C_{A(x)} = \langle 0.733, 0.633, 0.5\rangle$.

$iii.$ $\langle 0.3, 0.2, 0.5\rangle$single valued neutrosophic number belongs to the first group and the center is

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2\,I_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

so,$C_{A(x)} = \langle 0.633, 0.9, 0.5\rangle$.

$iv.$ $\langle 0.3, 0.2, 0.4\rangle$single valued neutrosophic number is in the first group.
Using

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2\,I_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

we have $C_{A(x)} = \langle 0.666, 0.7, 0.4\rangle$.

Corollary 3.2.2 The corners of the triangles obtained using the above method need not be single valued neutrosophic numbers but by definition, trivially their centers are.

Note 3.2.3 As for the single valued neutrosophic number $\langle 1, 1, 1\rangle$ there does not exist any transformable triangle in the above four groups, we take its transformation equal to itself.

Corollary 3.2.4 If $F_{A(x_i)} = T_{A(x_i)} = I_{A(x_i)}$,the transformation gives the same center in all four groups. Also if $T_{A(x_i)} = I_{A(x_i)} \leq F_{A(x_i)}$, then the center in the first group is equal to the center in the third group, and if $F_{A(x_i)} \leq T_{A(x_i)} = I_{A(x_i)}$, then the center in the second group is the same as the one in the fiurth group. Similarly, if$F_{A(x_i)} = I_{A(x_i)} \leq T_{A(x_i)}$, then the center in the first group is equal to the one in the fourth and in the case that $T_{A(x_i)} \leq F_{A(x_i)} = I_{A(x_i)}$,the center in the second group is equal to the center in the third.

3.3 Transformation According to the Falsity Value

In this section, we group the single valued neutrosophic numbers after the examination of their indeterminancy values F_A's greatness or smallness against I_A and F_A values. We will shift the $I_{A(x_i)}$and $F_{A(x_i)}$values on the Z – axis and $T_{A(x_i)}$ and $F_{A(x_i)}$values on the Y – axis onto each other. We take the $F_{A(x_i)}$value on the M – axis. The shifting on the Z and Y planes are made such that we shift the smaller value to the difference of the greater value and 2, as shown in the below figures.

1. First Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if

$$F_{A(x_i)} \le T_{A(x_i)}$$

and

$$F_{A(x_i)} \le I_{A(x_i)},$$

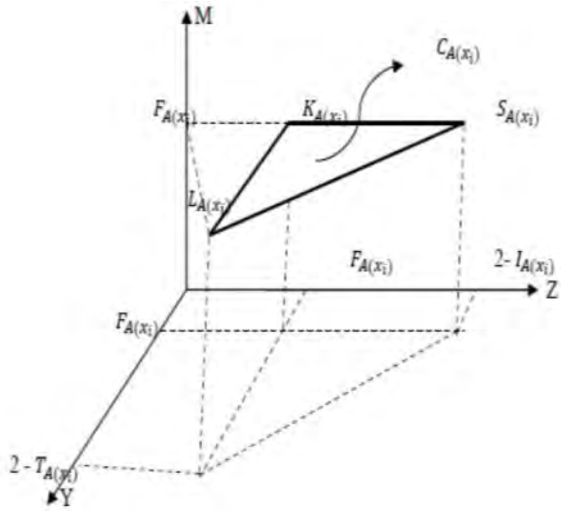
then

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic number $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(F_{A(x_i)}, \quad F_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(2 - I_{A(x_i)}, F_{A(x_i)}, F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(F_{A(x_i)}, 2 - T_{A(x_i)}, F_{A(x_i)}\right).$$



Here, as

$$T_{C_A(x_i)} = F_{A(x_i)} + \frac{\left(2 - I_{A(x_i)} - F_{A(x_i)}\right)}{3} = \frac{2 - I_{A(x_i)} + 2F_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = F_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - F_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2F_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we get

$$C_{A(x_i)} = \left(\frac{2 - I_{A(x_i)} + 2F_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2F_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

2. Second Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if

$$F_{A(x_i)} \ge T_{A(x_i)}$$

and

$$F_{A(x_i)} \ge I_{A(x_i)},$$

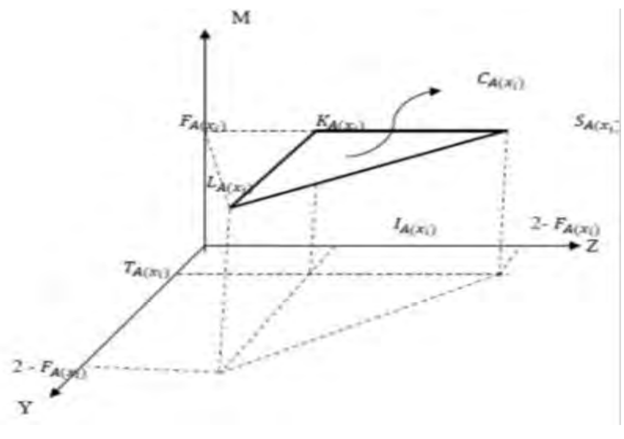
then

as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic numbers $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(I_{A(x_i)}, \quad T_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(I_{A(x_i)}, 2 - F_{A(x_i)}, F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(2 - F_{A(x_i)}, T_{A(x_i)}, F_{A(x_i)}\right).$$



Here, as

$$T_{C_A(x_i)} = I_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2I_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = T_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - T_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

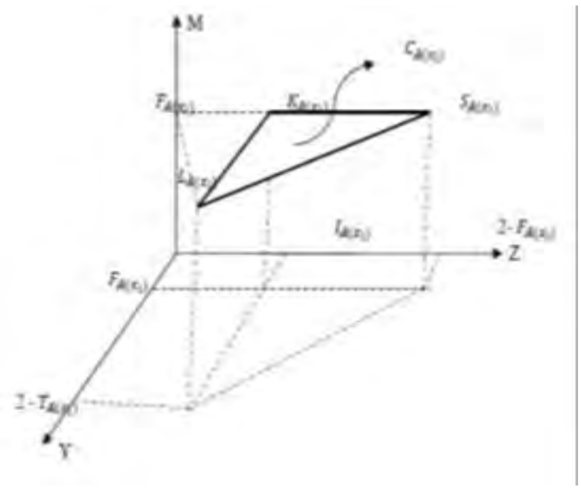
3. Third Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if $\mathrm{I}_{A(x_i)} \le \mathrm{F}_{A(x_i)} \le \mathrm{T}_{A(x_i)}$ then as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic numbers $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(I_{A(x_i)}, \quad F_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(I_{A(x_i)}, 2 - T_{A(x_i)}, \ F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(2 - F_{A(x_i)}, \ F_{A(x_i)}, \ F_{A(x_i)}\right).$$



Here, as

$$T_{C_A(x_i)} = I_{A(x_i)} + \frac{\left(2 - F_{A(x_i)} - I_{A(x_i)}\right)}{3} = \frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}$$

$$I_{C_A(x_i)} = F_{A(x_i)} + \frac{\left(2 - T_{A(x_i)} - F_{A(x_i)}\right)}{3} = \frac{2 - T_{A(x_i)} + 2\,F_{A(x_i)}}{3}$$

and

$$F_{C_A(x_i)} = F_{A(x_i)},$$

we have

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2\,F_{A(x_i)}}{3}, F_{A(x_i)}\right).$$

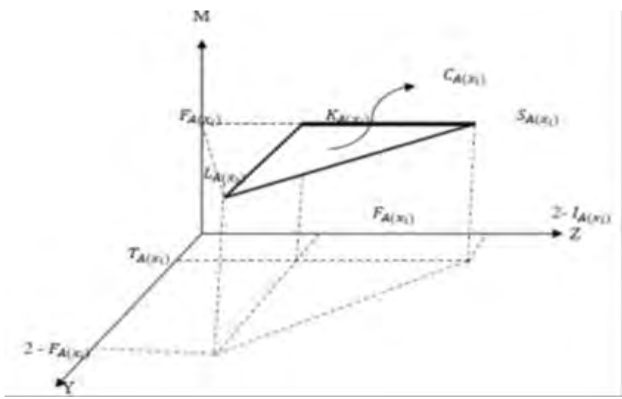
4. Fourth Group

For the single valued neutrosophic numbers $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$, if $\mathrm{T}_{A(x_i)} \le \mathrm{F}_{A(x_i)} \le \mathrm{I}_{A(x_i)}$, then as shown in the figure below, we transformed $\langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle$ into the single valued neutrosophic numbers $C_{A(x_i)}$, the center of the SKL triangle, where

$$S_{(Ax_i)} = \left(F_{A(x_i)}, \quad T_{A(x_i)}, \quad F_{A(x_i)}\right)$$

$$K_{(Ax_i)} = \left(F_{A(x_i)}, 2 - F_{A(x_i)}, \ F_{A(x_i)}\right)$$

$$L_{(Ax_i)} = \left(2 - I_{A(x_i)}, \ T_{A(x_i)}, \ F_{A(x_i)}\right).$$



Example 3.3.1: Transform the single neutrosophic numbers of Example 3.1.3.

$\langle 0.2,\ 0.5,\ 0.7\rangle$, $\langle 0.9,\ 0.4,\ 0.5\rangle$, $\langle 0.3,\ 0.2,\ 0.5\rangle$, $\langle 0.3,\ 0.2,\ 0.4\rangle$according to their falsity values.

$i.$ $\langle 0.2,\ 0.5,\ 0.7\rangle$ single valued neutrosophic number belongs to the second group. So, the center is

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

and we get $C_{A(x)} = \langle 0.766,\ 0.7,\ 0.7\rangle$.

$ii.$ $\langle 0.9,\ 0.4,\ 0.5\rangle$ single valued neutrosophic number is in the third group. Using the formula

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2\,F_{A(x_i)}}{3}, F_{A(x_i)}\right)$$

we see that$C_{A(x)} = \langle 0.766,\ 0.7,\ 0.5\rangle$.

$iii.$ $\langle 0.3,\ 0.2,\ 0.5\rangle$single valued neutrosophic number is in the second group. As

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

the center of the triangle is $C_{A(x)} = \langle 0.633,\ 0.7,\ 0.5\rangle$.

$iv.$ $\langle 0.3,\ 0.2,\ 0.4\rangle$single valued neutrosophic number belongs to the second group.

$$C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, \frac{2 - F_{A(x_i)} + 2\,T_{A(x_i)}}{3}, F_{A(x_i)}\right),$$

and so we have $C_{A(x)} = \langle 0.666,\ 0.733,\ 0.4\rangle$.

Corollary 3.3.2 The corners of the triangles obtained using the above method need not be single valued neutrosophic numbers but by definition, trivially their centers are single valued neutrosophic values.

Note 3.3.3 As for the single valued neutrosophic ber$\langle 1,\ 1,\ 1\rangle$ there does not exist any transformable triangle in the above four groups, we take its transformation equal to itself.

Corollary 3.3.4 If $F_{A(x_i)} = T_{A(x_i)} = I_{A(x_i)}$, the transformation gives the same center in all four groups. Also, if $T_{A(x_i)} = F_{A(x_i)} \leq I_{A(x_i)}$, then the center in the first group is equal to the one in the fourth group, and if $I_{A(x_i)} \leq T_{A(x_i)} = F_{A(x_i)}$, then the center in the second group is the same as the center in the third. Similarly, if $I_{A(x_i)} = F_{A(x_i)} \leq T_{A(x_i)}$, then the centers in the first and third groups are same and lastly, if $T_{A(x_i)} \leq I_{A(x_i)} = F_{A(x_i)}$, then the center in the second group is equal to the one in the fourth group.

4. A New Similarity Measure Based on Falsity Value Between Single Valued Neutrosophic Sets

In this section, we propose a new similarity measure based on falsity value between single valued neutrosophic sets.

Definition 4.1 Let A and B two single valued neutrosophic sets in $x = \{x_1, x_2, \ldots, x_n\}$.
Let $A = \{\langle x, T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle\}$

and

$B = \{\langle x, T_{B(x_i)}, I_{B(x_i)}, F_{B(x_i)}\rangle\}$.

The similarity measure based on falsity value between the neutrosophic numbers $A(x_i)$ and $B(x_i)$ is given by

$$S\left(A_{(x_i)}, B_{(x_i)}\right) = 1 - \left(\frac{\left|2\left(F_{A(x_i)} - F_{B(x_i)}\right) - \left(T_{A(x_i)} - T_{B(x_i)}\right)\right|}{9} + \frac{\left|2\left(F_{A(x_i)} - F_{B(x_i)}\right) - \left(I_{A(x_i)} - I_{B(x_i)}\right)\right|}{9} + \frac{3\left|\left(F_{A(x_i)} - F_{B(x_i)}\right)\right|}{9}\right).$$

Here, we use the values

$$2\left(F_{A(x_i)} - F_{B(x_i)}\right) - \left(T_{A(x_i)} - T_{B(x_i)}\right),$$

$$2\left(F_{A(x_i)} - F_{B(x_i)}\right) - \left(I_{A(x_i)} - I_{B(x_i)}\right),$$

$$2\left(F_{A(x_i)} - F_{B(x_i)}\right) + \left(F_{A(x_i)} - F_{B(x_i)}\right) = 3\left(F_{A(x_i)} - F_{B(x_i)}\right).$$

Since we use the falsity values $F_{A(x_i)}$ in all these three values, we name this formula as "similarity measure based on falsity value between single valued neutrosophic numbers".

Property4.2 :$0 \leq S\left(A_{(x_i)}, B_{(x_i)}\right) \leq 1$.

Proof: By the definition of Single valued neutrosophic numbers, as

$$0 \leq T_{A(x_i)}, T_{B(x_i)}, I_{A(x_i)}, I_{B(x_i)}, F_{A(x_i)}, F_{B(x_i)} \leq 1,$$

we have

$$0 \leq 2\left(F_{A(x_i)} - F_{B(x_i)}\right) - \left(T_{A(x_i)}, T_{B(x_i)}\right) \leq 3$$

$$0 \leq 2\left(F_{A(x_i)} - F_{B(x_i)}\right) - \left(I_{A(x_i)}, I_{B(x_i)}\right) \leq 3$$

and

$$0 \leq 3\left(F_{A(x_i)}, F_{B(x_i)}\right) \leq 3 .$$

So,

$$0 \leq 1 - \left(\frac{\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})\right|}{9} + \frac{\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})\right|}{9} + \frac{3\left|(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})\right|}{9}\right) \leq 1 .$$

Therefore, $0 \leq S\left(A_{(x_i)}, B_{(x_i)}\right) \leq 1$.

Property 4.3: $S\left(A_{(x_i)}, B_{(x_i)}\right) = 1 \Leftrightarrow A_{(x_i)} = B_{(x_i)}$

Proof.i) First we show $A_{(xi)} = B_{(xi)}$ when $S\left(A_{(x_i)}, B_{(x_i)}\right) = 1$.

Let $\left(A_{(x_i)}, B_{(x_i)}\right) = 1$.

$$S\left(A_{(x_i)}, B_{(x_i)}\right) = 1 - \left(\frac{\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})\right|}{9} + \frac{\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})\right|}{9} + \frac{3\left|(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})\right|}{9}\right) = 1$$

and thus,

$$\left(\frac{\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})\right|}{9} + \frac{\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})\right|}{9} + \frac{3\left|(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})\right|}{9}\right) = 0$$

So,

$$\left|(F_{A(x_i)} - F_{B(x_i)})\right| = 0,$$

$$\left|2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})\right| = 0,$$

and

$$\left|2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})\right| = 0 .$$

As $\left|(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})\right|=0$, then $\mathrm{F}_{\mathrm{A}(x_i)} = \mathrm{F}_{\mathrm{B}(x_i)}$.

If $\mathrm{F}_{\mathrm{A}(x_i)} = \mathrm{F}_{\mathrm{B}(x_i)}$,

$$\left|2\left(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}\right) - \left(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)}\right)\right| = 0$$

and

$$\mathrm{T}_{\mathrm{A}(x_i)} = \mathrm{T}_{\mathrm{B}(x_i)}.$$

When $\mathrm{F}_{\mathrm{A}(x_i)} = \mathrm{F}_{\mathrm{B}(x_i)}$,

$$\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})\right| = 0$$

and

$$\mathrm{I}_{\mathrm{A}(x_i)} = \mathrm{I}_{\mathrm{B}(x_i)}$$

Therefore, if $\left(A_{(x_i)}, B_{(x_i)}\right) = 1$, then by Definition 2.3, $\mathrm{A}_{(x_i)} = \mathrm{B}_{(x_i)}$.

ii) Now we show if $\mathrm{A}_{(x_i)} = \mathrm{B}_{(x_i)}$, then $S\left(A_{(x_i)}, B_{(x_i)}\right) = 1$. Let $\mathrm{A}_{(x_i)} = \mathrm{B}_{(x_i)}$. By Definition 2.3 ,

$$\mathrm{T}_{\mathrm{A}(x_i)} = \mathrm{T}_{\mathrm{B}(x_i)}, \ \mathrm{I}_{\mathrm{A}(x_i)} = \mathrm{I}_{\mathrm{B}(x_i)}, \ \mathrm{F}_{\mathrm{A}(x_i)} = \mathrm{F}_{\mathrm{B}(x_i)}$$

and we have

$$\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)} = 0, \ \mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)} = 0, \ \mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)} = 0 .$$

So,

$$S\left(A_{(x_i)}, B_{(x_i)}\right) = 1 - \left(\frac{\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})\right|}{9} + \frac{\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})\right|}{9} + \frac{3\left|(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})\right|}{9}\right)$$

$$= 1 - \frac{0}{9} = 1 .$$

Property4.4 :$S\left(A_{(x_i)}, B_{(x_i)}\right) = S\left(B_{(x_i)}, A_{(x_i)}\right)$.

Proof:

$$S\left(A_{(x_i)}, B_{(x_i)}\right) = 1 - \left(\frac{\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})\right|}{9} + \frac{\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})\right|}{9} + \frac{3\left|(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})\right|}{9}\right)$$

$$= 1 - \left(\frac{\left|2(-(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})) - (-(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)}))\right|}{9} + \frac{\left|2((-\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})) - (-(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)}))\right|}{9} + \frac{3\left|-(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})\right|}{9}\right)$$

$$= 1 - \left(\frac{\left|2(\mathrm{F}_{\mathrm{B}(x_i)} - \mathrm{F}_{\mathrm{A}(x_i)}) - (\mathrm{T}_{\mathrm{B}(x_i)} - \mathrm{T}_{\mathrm{A}(x_i)})\right|}{9} + \frac{\left|2(\mathrm{F}_{\mathrm{B}(x_i)} - \mathrm{F}_{\mathrm{A}(x_i)}) - (\mathrm{I}_{\mathrm{B}(x_i)} - \mathrm{I}_{\mathrm{A}(x_i)})\right|}{9} + \frac{3\left|(\mathrm{F}_{\mathrm{B}(x_i)} - \mathrm{F}_{\mathrm{A}(x_i)})\right|}{9}\right.$$

$$= S\left(B_{(x_i)}, A_{(x_i)}\right).$$

Property 4.5 : If $A \subseteq B \subseteq C$,

i) $S\left(A_{(x_i)}, B_{(x_i)}\right) \geq S\left(A_{(x_i)}, C_{(x_i)}\right)$
ii) $S\left(B_{(x_i)}, C_{(x_i)}\right) \geq S\left(A_{(x_i)}, C_{(x_i)}\right)$

Proof:

i) By the single valued neutrosophic set property, if $A \subseteq B \subseteq C$, then

$$\mathrm{T}_{\mathrm{A}(x_i)} \leq \mathrm{T}_{\mathrm{B}(x_i)} \leq \mathrm{T}_{\mathrm{C}(x_i)},$$

$$\mathrm{I}_{\mathrm{A}(x_i)} \leq \mathrm{I}_{\mathrm{B}(x_i)} \leq \mathrm{I}_{\mathrm{C}(x_i)},$$

$$\mathrm{F}_{\mathrm{A}(x_i)} \geq \mathrm{F}_{\mathrm{B}(x_i)} \geq \mathrm{F}_{\mathrm{C}(x_i)} .$$

So,

$$\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)} \leq 0,$$

$$\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)} \leq 0,$$

$$\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)} \geq 0 \qquad (1)$$

$$\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{C}(x_i)} \leq 0,$$

$$\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{C}(x_i)} \leq 0,$$

$$\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{C}(x_i)} \geq 0 \qquad (2)$$

$$\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)} \geq \mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{C}(x_i)},$$

$$\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)} \geq \mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{C}(x_i)},$$

$$\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)} \leq \mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{C}(x_i)} \qquad (3)$$

Using (1), we have

$$2\left(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}\right) - \left(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)}\right) \geq 0$$

$$2\left(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}\right) - \left(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)}\right) \geq 0$$

and

$$3\left(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)}\right) \geq 0 .$$

Thus, we get

$$S\left(A_{(x_i)}, B_{(x_i)}\right) = 1 - \left(\frac{\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})\right|}{9} + \frac{\left|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})\right|}{9} + \frac{3\left|(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})\right|}{9}\right)$$

$$= 1 - \frac{7\left(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}\right) - \left(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)}\right) - \left(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)}\right)}{9} . (4)$$

Similarly, by (2), we have

$$S(A_{(x_i)}, C_{(x_i)}) = 1 - \left(\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{C(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{C}(x_i)})|}{9} + \frac{|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{C}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{C}(x_i)})|}{9} + \frac{3|(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{C}(x_i)})|}{9}\right)$$

$$= 1 - \frac{7(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{C}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{C}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{C}(x_i)})}{9} \quad . (5)$$

Using (4) and (5) together, we get

$$S(A_{(x_i)}, B_{(x_i)}) - S(A_{(x_i)}, C_{(x_i)})$$

$$= 1 - \frac{7(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})}{9}$$

$$-1 + \frac{7(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})}{9}$$

$$= \frac{7(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})}{9} - \frac{(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})}{9} - \frac{(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})}{9}$$

$$+ \frac{7(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{C}(x_i)})}{9} - \frac{(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{C}(x_i)})}{9} - \frac{(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{C}(x_i)})}{9}$$

$$= \frac{7(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})}{9} + \frac{7(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{C}(x_i)})}{9} - \frac{(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})}{9}$$

$$- \frac{(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{C}(x_i)})}{9} - \frac{(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})}{9} - \frac{(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{C}(x_i)})}{9}$$

by (1) and (3),

$$\frac{7(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})}{9} + \frac{7(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{C}(x_i)})}{9} \geq 0,$$

$$- \frac{(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})}{9} - \frac{(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{C}(x_i)})}{9} \geq 0,$$

$$- \frac{(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})}{9} - \frac{(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{C}(x_i)})}{9} \geq 0$$

and therefore

$$S(A_{(x_i)}, B_{(x_i)}) - S(A_{(x_i)}, C_{(x_i)}) \geq 0$$

and

$$S(A_{(x_i)}, B_{(x_i)}) \geq S(A_{(x_i)}, C_{(x_i)}) \, .$$

ii. The proof of the latter part can be similarly done as the first part.

Corollary 4.6 : Suppose we make similar definitions to Definition 4.1, but this time based on truth values or indeterminancy values. If we define a truth based similarity measure, or namely,

$$S(A_{(x_i)}, B_{(x_i)}) = 1 - \left(\frac{|2(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)}) - (\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})|}{9} + \frac{|2(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})|}{9} + \frac{3|(\mathrm{T}_{\mathrm{A}(x_i)} - T_{\mathrm{B}(x_i)})|}{9}\right),$$

or if we define a measure based on indeterminancy values like

$$S(A_{(x_i)}, B_{(x_i)}) = 1 - \left(\frac{|2(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})|}{9} + \frac{|2(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)}) - (\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})|}{9} + \frac{3|(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})|}{9}\right)$$

these two definitions don't provide the conditions of Property 4.5 . For instance, for the truth value

$$S(A_{(x_i)}, B_{(x_i)}) = 1 - \left(\frac{|2(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)}) - (\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})|}{9} + \frac{|2(\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})|}{9} + \frac{3|(\mathrm{T}_{\mathrm{A}(x_i)} - T_{\mathrm{B}(x_i)})|}{9}\right),$$

when we take the single valued neutrosophic numbers $\mathrm{A}_{(x)} = \langle 0, 0.1, 0 \rangle$, $\mathrm{B}_{(x)} = \langle 1, 0.2, 0 \rangle$ and $\mathrm{C}_{(x)} = \langle 1, 0.3, 0 \rangle$, we see $S(A_{(x)}, B_{(x)}) = 0.233$ and $S(A_{(x)}, C_{(x)}) = 0.244$. This contradicts with the results of Property 4.5.

Similarly, for the indeterminancy values,

$$S(A_{(x_i)}, B_{(x_i)}) = 1 - \left(\frac{|2(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})|}{9} + \frac{|2(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)}) - (\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})|}{9} + \frac{3|(\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})|}{9}\right)$$

if we take the single valued neurosophic numbers $\mathrm{A}_{(x)} = \langle 0.1, 0, 1\rangle$, $\mathrm{B}_{(x)} = \langle 0.2, 1, 1\rangle$and $\mathrm{C}_{(x)} = \langle 0.3, 1, 1\rangle$, we have $S(A_{(x)}, B_{(x)}) = 0.233$ and $S(A_{(x)}, C_{(x)}) = 0.244$.

These results show that the definition 4.1 is only valid for the measure based on falsity values.

Defintion 4.7 As

$$S(A_{(x_i)}, B_{(x_i)}) = 1 - \left(\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})|}{9} + \frac{|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})|}{9} + \frac{3|(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})|}{9}\right),$$

The similarity measure based on the falsity value between two single valued neutrosophic sets *A* and *B* is;

$$\mathrm{S_{NS}}(A, B) = \sum_{i=1}^{n}\left(w_i \times \mathrm{S}\left(\mathrm{A}_{(x_i)}, \mathrm{B}_{(x_i)}\right)\right).$$

Here,$\mathrm{S_{NS}}(A, B) \in [0,1]$and w_i's are the weights of the x_i's with the property $\sum_{i=1}^{n} w_i = 1$. Also,

$$A = \{\langle x: T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle\},$$

$$B = \{\langle x: T_{B(x_i)}, I_{B(x_i)}, F_{B(x_i)}\rangle\}.$$

Example4.8 Let us consider three patterns$\mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3$ represerted by single valued neutrosophic sets $\widetilde{\mathrm{P}_1}$ and $\widetilde{\mathrm{P}_2}$in $X = \{x_1, x_2\}$ respectively, where $\widetilde{\mathrm{P}_1} = \{\langle \mathrm{x}_1, 0.2, 0.5, 0.7\rangle, \langle \mathrm{x}_2, 0.9, 0.4, 0.5\rangle\}$ and $\widetilde{\mathrm{P}_2} = \{\langle \mathrm{x}_1, 0.3, 0.2, 0.5\rangle, \langle \mathrm{x}_2, 0.3, 0.2, 0.4\rangle\}$. We want to classify an unknown pattern represented by a single valued neutrosophic set $\tilde{Q}$ in $X = \{x_1, x_2\}$ into one of the patterns $\widetilde{\mathrm{P}_1}$, $\widetilde{\mathrm{P}_2}$;where $\tilde{Q} = \{\langle \mathrm{x}_1, 0.4, 0.4, 0.1\rangle, \langle \mathrm{x}_2, 0.6, 0.2, 0.3\rangle\}$.

Let w_i be the weight of element w_i , where $w_i = \frac{1}{2}$ $1 \le i \le 2$,

$$\mathrm{S_{NS}}(\widetilde{P_1}, \tilde{Q}) = 0.711$$

and

$$\mathrm{S_{NS}}(\widetilde{P_1}, \tilde{Q}) = 0.772 .$$

We can see that $\mathrm{S_{NS}}(\widetilde{P_2}, \tilde{Q})$is the largest value amongthe values of $\mathrm{S_{NS}}(\widetilde{P_1}, \tilde{Q})$ and $\mathrm{S_{NS}}(\widetilde{P_2}, \tilde{Q})$.

Therefore, the unknown pattern represented by single valued neutrosophic set$\tilde{Q}$ should be classified into the pattern P_2.

5. A New Similarity Measure Based on Falsity Measure Between Neutrosophic Sets Based on the Centroid Points of Transformed Single Valued Neutrosophic Numbers

In this section, we propose a new similarity measure based on falsity value between single valued neutrosophic sets based on the centroid points of transformed single valued neutrosophic numbers.

Definition5.1:

$$S(A_{(x_i)}, B_{(x_i)}) = 1 - \left(\frac{|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{T}_{\mathrm{A}(x_i)} - \mathrm{T}_{\mathrm{B}(x_i)})|}{9} + \frac{|2(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)}) - (\mathrm{I}_{\mathrm{A}(x_i)} - \mathrm{I}_{\mathrm{B}(x_i)})|}{9} + \frac{3|(\mathrm{F}_{\mathrm{A}(x_i)} - \mathrm{F}_{\mathrm{B}(x_i)})|}{9}\right),$$

Taking the similarity measure as defined in the fourth section, and letting $\mathrm{C}_{\mathrm{A}(x_i)}$ and$\mathrm{C}_{\mathrm{B}(x_i)}$be the centers of the triangles obtained by the transformation of $\mathrm{A}_{(x_i)}$and $\mathrm{B}_{(x_i)}$in the third section respectively,the similarity measure based on falsity value between single valued neutrosophic sets A and B based on the centroid points of transformed single valued neutrosophic numbers is

$$\mathrm{S_{NSC}}(A, B) = \sum_{i=1}^{n}\left(w_i \mathrm{x} \mathrm{S}(\mathrm{C}_{\mathrm{A(xi)}}, \mathrm{C}_{\mathrm{B(xi)}})\right),$$

where

$$A = \{x: \langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}\rangle\},$$
$$B = \{x: \langle T_{B(x_i)}, I_{B(x_i)}, F_{B(x_i)}\rangle\}.$$

Here again, w_i's are the weights of the x_i's with the property $\sum_{i=1}^{n} w_i = 1$.

Example5.2: Let us consider two patterns P_1 and P_2 represented by single valued neutrosophic sets $\widetilde{P_1}$, $\widetilde{P_2}$ in $X = \{x_1, x_2\}$respectively in Example 4.8,where

$$\widetilde{P_1} = \{\langle x_1, 0.2, 0.5, 0.7\rangle, \langle x_2, 0.9, 0.4, 0.5\rangle\}$$

and

$$\widetilde{P_2} = \{\langle x_1, 0.3, 0.2, 0.5\rangle, \langle x_2, 0.3, 0.2, 0.4\rangle\}.$$

We want to classify an unknown pattern represented by single valued neutrosophic set $\tilde{Q}$ in $X = \{x_1, x_2\}$ into one of the patterns $\widetilde{P_1}$, $\widetilde{P_2}$, where

$$\tilde{Q} = \{\langle x_1, 0.4, 0.4, 0.1\rangle, \langle x_2, 0.6, 0.2, 0.3\rangle\}.$$

We make the classification using the measure in Definition 5.1, namely

$$S_{NSC}(A, B) = \sum_{i=1}^{n} \left(w_i \times S\left(C_{A(xi)}, C_{B(xi)}\right)\right).$$

Also we find the $C_{A(xi)}, C_{B(xi)}$ centers according to the truth values.

Let w_i be the weight of elementx_i, $w_i = \frac{1}{2}$; $1 \leq i \leq 2$.

$\widetilde{P_1}x_1 = \langle 0.2,\ 0.5,\ 0.7\rangle$transformed based on falsity value in Example 3.1.1

$$C_{\widetilde{P_1}x_1} = (0.566, 0.633, 0.7)$$

$\widetilde{P_1}x_2 = \langle 0.9,\ 0.4,\ 0.5\rangle$ transformed based on falsity value in Example 3.1.1

$$C_{\widetilde{P_1}x_2} = (0.7, 0.633, 0.5)$$

$\widetilde{P_2}x_1 = \langle 0.3,\ 0.2,\ 0.5\rangle$ transformed based on falsity value in Example 3.1.1

$$C_{\widetilde{P_2}x_1} = (0.7, 0.7, 0.5)$$

$\widetilde{P_2}x_2 = \langle 0.3,\ 0.2,\ 0.4\rangle$ transformed based on falsity value in Example 3.1.1

$$C_{\widetilde{P_2}x_2} = (0.733, 0.7, 0.4)$$

$\tilde{Q}_{x_1} = \langle x_1, 0.4, 0.4, 0.1\rangle$ transformed based on falsity value in Section 3.1

$C_{\tilde{Q}x_1} = \langle 0.6,\ 0.8,\ 0.1\rangle$(second group)

$\tilde{Q}_{x_2} = \langle x_2, 0.6,\ 0.2,\ 0.3\rangle$transformed based on truth falsity in Section 3.1

$C_{\tilde{Q}x_2} = \langle 0.666,\ 0.6,\ 0.3\rangle$(second group)

$$S_{NSC}(\widetilde{P_1}, \tilde{Q}) = 0.67592$$

$$S_{NSC}(\widetilde{P_2}, \tilde{Q}) = 0.80927$$

Therefore, the unknown patternQ,represented by a single valued neutrosophic set based on truth value is classified into pattern P_2.

Example5.3 : Let us consider two patterns P_1 and P_2 of example 4.8, represented by single valued neutrosophic sets $\widetilde{P_1}$, $\widetilde{P_2}$, in $X = \{x_1, x_2\}$ respectively, where

$$\widetilde{P_1} = \{\langle x_1, 0.2, 0.5, 0.7\rangle, \langle x_2, 0.9, 0.4, 0.5\rangle\}$$

and

$$\widetilde{P_2} = \{\langle x_1, 0.3, 0.2, 0.5\rangle, \langle x_2, 0.3, 0.2, 0.4\rangle\}.$$

We want to classify an unknown pattern represented by the single valued neutrosophic set $\tilde{Q}$ in $X = \{x_1, x_2\}$ into one of the patterns $\widetilde{P_1}$, $\widetilde{P_2}$,where

$$\tilde{Q} = \{\langle x_1, 0.4, 0.4, 0.1\rangle, \langle x_2, 0.6, 0.2, 0.3\rangle\}.$$

We make the classification using the measure in Definition 5.1, namely

$$S_{NSC}(A, B) = \sum_{i=1}^{n} \left(w_i x S\left(C_{A(xi)}, C_{B(xi)}\right)\right).$$

Also we find the $C_{A(xi)}, C_{B(xi)}$ centers according to the indeterminacy values.

Let w_i be the weight of elementx_i, $w_i = \frac{1}{2}$; $1 \leq i \leq 2$.

$\widetilde{P_1}x_1 = \langle 0.2,\ 0.5,\ 0.7\rangle$ transformed based on falsity value in Example 3.2.1

$$C_{\widetilde{P_1}x_1} = (0.766, 0.633, 0.7)$$

$\widetilde{P_1}x_2 = \langle 0.9,\ 0.4,\ 0.5\rangle$ transformed based on falsity value in Example 3.2.1

$$C_{\widetilde{P_1}x_2} = (0.766, 0.633, 0.5)$$

$\widetilde{P_2}x_1 = \langle 0.3,\ 0.2,\ 0.5\rangle$ transformed based on falsity value in Example 3.2.1

$$C_{\widetilde{P_2}x_1} = (0.633, 0.9, 0.5)$$

$\widetilde{P_2}x_2 = \langle 0.3,\ 0.2,\ 0.4\rangle$ transformed based on falsity value in Example 3.2.1

$$C_{\widetilde{P_2}x_2} = (0.666, 0.7, 0.4)$$

$\tilde{Q}_{x_1} = \langle x_1, 0.4, 0.4, 0.1\rangle$ transformed based on falsity value in Section 3.2

$C_{\tilde{Q}x_1} = \langle 0.6,\ 0.8,\ 0.1\rangle$(second group)

$\tilde{Q}_{x_2} = \langle x_2, 0.6,\ 0.2,\ 0.3\rangle$transformed based on truth falsity in Section 3.2

$C_{\tilde{Q}x_2} = \langle 0.7,\ 0.666,\ 0.3\rangle$ (first group)

$$S_{NSC}(\widetilde{P_1}, \tilde{Q}) = 0.67592$$

$$S_{NSC}(\widetilde{P_2}, \tilde{Q}) = 0.80927$$

Therefore, the unknown patternQ, represented by a single valued neutrosophic set based on indeterminacy value is classified into pattern P_2.

Example5.4: Let us consider in example 4.8, two patterns P_1 and P_2 represented by single valued neutrosophic sets $\widetilde{P_1}$, $\widetilde{P_2}$ in $X = \{x_1, x_2\}$ respectively ,where

$$\widetilde{P_1} = \{\langle x_1, 0.2, 0.5, 0.7\rangle, \langle x_2, 0.9, 0.4, 0.5\rangle\}$$

and

$$\widetilde{P_2} = \{\langle x_1, 0.3, 0.2, 0.5\rangle, \langle x_2, 0.3, 0.2, 0.4\rangle\}.$$

We want to classify an unknown pattern represented by single valued neutrosophic set $\tilde{Q}$ in$x = \{x_1 , x_2\}$ into one of the patterns $\widetilde{P_1}$, $\widetilde{P_2}$, where

$$\tilde{Q} = \{\langle x_1, 0.4, 0.4, 0.1\rangle, \langle x_2, 0.6, 0.2, 0.3\rangle\}.$$

We make the classification using the measure in Definition 5.1, namely

$$S_{NSC}(A, B\ ) = \sum_{i=1}^{n} \left(w_i x S\left(C_{A(xi)}\ , C_{B(xi)}\right)\right).$$

Also we find the $C_{A(xi)}$, $C_{B(xi)}$ centers according to the falsity values.

Let w_i be the weight of elementx_i, $w_i = \frac{1}{2}$; $1 \leq i \leq 2$.

$\widetilde{P_1}x_1 = \langle 0.2,\ 0.5,\ 0.7\rangle$transformed based on falsity value in Example 3.3.1

$$C_{\widetilde{P_1}x_1} = (0.766, 0.7, 0.7)$$

$\widetilde{P_1}x_2 = \langle 0.9,\ 0.4,\ 0.5\rangle$ transformed based on falsity value in Example 3.3.1

$$C_{\widetilde{P_1}x_2} = (0.766, 0.7, 0.5)$$

$\widetilde{P_2}x_1 = \langle 0.3,\ 0.2,\ 0.5\rangle$ transformed based on falsity value in Example 3.3.1

$$C_{\widetilde{P_2}x_1} = (0.633, 0.7, 0.5)$$

$\widetilde{P_2}x_2 = \langle 0.3,\ 0.2,\ 0.4\rangle$ transformed based on falsity value in Example 3.3.1

$$C_{\widetilde{P_2}x_2} = (0.666, 0.733, 0.4)$$

$\tilde{Q}_{x_1} = \langle x_1, 0.4, 0.4, 0.1\rangle$ transformed based on falsity value in Section 3.3

$C_{\tilde{Q}x_1} = \langle 0.6,\ 0.6,\ 0.1\rangle$(first group)

$\tilde{Q}_{x_2} = \langle x_2, 0.6,\ 0.2,\ 0.3\rangle$transformed based on truthfalsity in Section 3.3

$C_{\tilde{Q}x_2} = \langle 0.7,\ 0.666,\ 0.3\rangle$ (third group)

$$S_{NSC}(\widetilde{P_1}, \tilde{Q}) = 0.7091$$

$$S_{NSC}(\widetilde{P_2}, \tilde{Q}) = 0.8148$$

Therefore, the unknown pattern Q, represented by a single valued neutrosophic set based on falsity value is classified into pattern P_2.

In Example 5.2, Example 5.3 and Example 5.4, all measures according to truth, indeterminancy and falsity values give the same exact result.

Conclusion

In this study, we propose methods to transform between single valued neutrosophic numbers based on centroid points. We also propose a new method to measure the degree of similarity based on falsity values between single valued neutrosophic sets. Then we prove some properties of new similarity measure based on falsity value between single valued neutrosophic sets. When we take this measure with respect to truth or indeterminancy we show that it does not satisfy one of the conditions of similarity measure. We also apply the proposed new similarity measures based on falsity value between single valued neutrosophic sets to deal with pattern recognition problems.

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On Neutrosophic Uninorms

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Abstract. Uninorm generalizes the notion of t-norm and t-conorm in fuzzy logic theory. They are three increasing, commutative and associate operators having one neutral element. However, such specific value identifies the kind of operator it is; t-norms have the 1 as neutral element, t-conorms have the 0 and uninorms have every number lying between 0 and 1. Uninorms have been applied as aggregators in many fields of Artificial Intelligence and Decision Making. This theory has also been extended to the framework of interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and L-fuzzy sets. This paper aims to explore neutrosophic uninorms. We demonstrate that it is possible to define uninorms operators from neutrosophic logic. Additionally, we define neu-trosophic implicators induced by neutrosophic uninorms. The combination of both, Neutrosophy and uninorms, enriches the applicability of uninorms operators due to the possibility of incorporating indeterminancy as part of the Neutrosophy contribution.

Keywords: neutrosophic uninorm, uninorm, neutrosophic logic, neutrosophic implicator.

1 Introduction

Uninorms generalize the concepts of t-norm and t-conorm in fuzzy set theory, see [17]. Uninorm operators fulfill commutativity, associativity, increasing monotonicity and the existence of a neutral element *e*, in the same way that t-norm and t-conorm do, see [21]. When *e* is 1, the uninorm is a t-norm, when *e* is 0, it is a t-conorm. The generalization consists in widening to [0, 1] the range of values where the neutral element can lie.

Uninorms are not only used to extend theoretically the other aforementioned fuzzy operators, furthermore we can find in literature many fields where they are applied as aggregators, for example, in expert systems, image processing, neural networks, classifiers, among others, see [4, 10, 13, 16, 19, 22, 27]. Moreover, there exists a fuzzy implicator theory based on uninorms, [7].

G. Deschrijver and E. Kerre in [15], extend fuzzy uninorms concepts to interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and L-fuzzy sets, see [5-6, 14, 18]. They proved in [14], that these four kind of fuzzy sets are isomorphic each another, therefore, it is sufficient to prove uninorm properties in the framework of the L^*-fuzzy set theory.

On the other hand, "Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra", [23-24, 26]. The novelty of this theory is that it includes for the first time the notion of indeterminacy in fuzzy set theory, that is to say, this approach admits the membership and non membership of elements or objects to a set, akin to intuitionistic fuzzy set theory does, as well as a third function which represents indeterminacy. This theory acknowledges that ignorance, contradiction, paradox and other knowledge representation conditions, which are often considered undesirable from the classic logic viewpoint, also should be taken into account.

Neutrosophy has been applied in wide-ranging kinds of areas, e.g., image processing, decision making, clustering, among others. This is due to the nature of this theory, which allows representing and calculating with indeterminacies.

This paper is devoted to introducing neutrosophic uninorms or N-uninorms, for generalizing uninorm operators to the neutrosopic framework. It is worthily to remark that N-uninorms are used to denote neutrosophic uninorms, not n-uninorms, see [2]. To our knowledge, this seems to be the first approach to neutrosophic uninorms. In neutrosophic logic, neutrosophic norms generalize t-norms and neutrosophic conorms generalize t-conorms, hence,

N-uninorms extend fuzzy uninorms, uninorms on L^*-fuzzy sets, n-norms and n-conorms.

N-uninorms could replace fuzzy uninorms in the mathematical models where usually the latter one are employed, because this new approach keeps the advantages of uninorms as an esteemed aggregator, which is here improved with the appropriateness of neutrosophy to deal with human reasoning, knowledge representation, vagueness and uncertainty, when indeterminacy is present.

The present paper is organized as follows; the preliminary definitions and results necessaries to develop our work will be given in Section 2. Section 3 is dedicated to exposing the N-uninorm theory, including N-uninorm implicators. Finally, Section 4 draws the conclusions.

2 Preliminaries

This section is devoted to exposing the preliminary definitions and results necessaries to develop the proposed theory of N-uninorms. The first subsection is dedicated to summarizing the basic definitions and results on uninorms. In the second one we recall the definition and aspects concerning neutrosophic logic theory.

2.1 Basic notions of uninorm theory

Definition 2.1. A *uninorm* is a commutative, associative and increasing mapping $\mathrm{U}: [0,1]^2 \to [0,1]$, where there exists $e \in [0,1]$, called neutral element, such that $\forall \mathrm{x} \in [0,1], \mathrm{U}(e, \mathrm{x}) = \mathrm{x}$, [17].

If $e = 1$, U is a t-norm and if $e = 0$, U is a t-conorm.

Deschrijver and Kerre in [15] extend this definition to the framework of interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and L-fuzzy sets, which are pairwise isomorphics, therefore they restrict their theory to the set $\mathrm{L}^* = \{(\mathrm{x}_1, \mathrm{x}_2) \in [0,1]^2 \text{ and } \mathrm{x}_1 + \mathrm{x}_2 \leq 1\}$.

Let us recall two well-known algebraic definitions that we explicitly write for the sake of being self-contained. They are namely, *Partially Ordered Set* or *poset* and *Lattice*, [1, 9, 20].

Definition 2.2. A *Partially Ordered Set* or *poset* is a pair (P, ≤), where P is a set and ≤ is a binary relation over P, which satisfies for every x, y, z ∈ P, the three following conditions:

1. x≤x (Reflexive).
2. If x≤y and y≤x, then x = y (Antisymmetry).
3. If x≤y and y≤z, then x≤z (Transitivity).

An *upper bound* of X, X⊆P, is an element a∈P, such that ∀x∈X it holds x≤a. Equivalently, a *lower bound* is an element b∈P, such that ∀x∈X, b≤x. The *supremum* of X is the least upper bound and the *infimum* is the greater lower bound.

Definition 2.3. A *lattice* $(\mathrm{L}, \leq_{\mathrm{L}})$ is a poset, where every pair of elements x and y in L have an infimum or 'meet', denoted by x∧y and a supremum or 'join' denoted by x∨y.

L is a *complete lattice* if every of its subsets has an infimum and a supremum in L.

The lattice $(\mathrm{L}^*, \leq_{\mathrm{L}^*})$ is defined by the following poset:

$(\mathrm{x}_1, \mathrm{x}_2) \leq_{\mathrm{L}^*} (\mathrm{y}_1, \mathrm{y}_2) \Leftrightarrow \mathrm{x}_1 \leq \mathrm{y}_1 \text{ and } \mathrm{x}_2 \geq \mathrm{y}_2$, $\forall (\mathrm{x}_1, \mathrm{x}_2), (\mathrm{y}_1, \mathrm{y}_2) \in \mathrm{L}^*$. The units of $\mathrm{L}^*$are $0_{\mathrm{L}^*} = (0,1)$ and $1_{\mathrm{L}^*} = (1,0)$. See that $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2)$ and $\mathrm{y} = (\mathrm{y}_1, \mathrm{y}_2)$ can be incomparable with regard to $\leq_{\mathrm{L}^*}$, where either $\mathrm{x}_1 < \mathrm{y}_1$ and $\mathrm{x}_2 < \mathrm{y}_2$, or $\mathrm{x}_1 > \mathrm{y}_1$ and $\mathrm{x}_2 > \mathrm{y}_2$. It is denoted by $\mathrm{x} \parallel_{\mathrm{L}^*} \mathrm{y}$.

Evidently, $(\mathrm{x}_1, \mathrm{x}_2) \geq_{\mathrm{L}^*} (\mathrm{y}_1, \mathrm{y}_2)$ if and only if $(\mathrm{y}_1, \mathrm{y}_2) \leq_{\mathrm{L}^*} (\mathrm{x}_1, \mathrm{x}_2)$. If $(\mathrm{x}_1, \mathrm{x}_2) \leq_{\mathrm{L}^*} (\mathrm{y}_1, \mathrm{y}_2)$ and $(\mathrm{x}_1, \mathrm{x}_2) \geq_{\mathrm{L}^*} (\mathrm{y}_1, \mathrm{y}_2)$ then $(\mathrm{x}_1, \mathrm{x}_2) =_{\mathrm{L}^*} (\mathrm{y}_1, \mathrm{y}_2)$.

Formally, the uninorm on L^* is defined as follows:

Definition 2.4. A *uninorm on* L^* is a commutative, associative and increasing mapping $\mathbf{U}: \mathrm{L}^{*2} \to \mathrm{L}^*$, where there exists $e \in \mathrm{L}^*$, called neutral element, such that $\forall \mathrm{x} \in \mathrm{L}^*, \mathbf{U}(e, \mathrm{x}) = \mathrm{x}$, [15].

Here, if $e = 1_{\mathrm{L}^*}$, **U** defines a t-norm on $\mathrm{L}^*$and if $e = 0_{\mathrm{L}^*}$, it is a t-conorm on L^*. Nevertheless, the most interesting cases of uninorms are those where e satisfies $0_{\mathrm{L}^*} <_{\mathrm{L}^*} e <_{\mathrm{L}^*} 1_{\mathrm{L}^*}$.

In [15] we can find properties and their demonstrations concerning uninorms on L^* that generalize the properties of fuzzy uninorms, including those of the uninorm-based R-implicators and S-implicators. Further, we shall guide the exposition of N-uninorms theory through the theory developed in that paper. Our goal is to prove that N-uninorms extend uninorms on L^*.

2.2 Basic notions of neutrosophic logic

Definition 2.5. Given X, a universe of discourse containing elements or objects. A is a *neutrosophic set* ([25-26]) if it has the form: $\mathrm{A} = \{(\mathrm{x}: \mathrm{T_A}(\mathrm{x}), \mathrm{I_A}(\mathrm{x}), \mathrm{F_A}(\mathrm{x})),\ \mathrm{x} \in \mathrm{X}\}$, where $\mathrm{T_A}(\mathrm{x}), \mathrm{I_A}(\mathrm{x}), \mathrm{F_A}(\mathrm{x}) \subseteq]^-0, 1^+[$, i.e., they are three functions over either the standard or nonstandard subsets of $]^-0, 1^+[$. $\mathrm{T_A}(\mathrm{x})$ represents the degree of membership of x to A, $\mathrm{I_A}(\mathrm{x})$ represents its degree of indeterminacy and $\mathrm{F_A}(\mathrm{x})$ its degree of non-membership. They do not satisfy any restriction, i.e., $\forall \mathrm{x} \in \mathrm{X},\ {}^-0 \leq \inf \mathrm{T_A}(\mathrm{x}) + \inf \mathrm{I_A}(\mathrm{x}) + \inf \mathrm{F_A}(\mathrm{x}) \leq \sup \mathrm{T_A}(\mathrm{x}) + \sup \mathrm{I_A}(\mathrm{x}) + \sup \mathrm{F_A}(\mathrm{x}) \leq 3^+$.

Another particular definition is that of *Single-valued Neutrosophic set*, which is formally defined as follows:

Definition 2.6. Given X, a universe of discourse which contains elements or objects. A is a *single-valued neutrosophic set* (*SVNS*) [25] if it has the form: $A = \{(x: T_A(x), I_A(x), F_A(x)), x \in X\}$, where $T_A(x), I_A(x), F_A(x) \in [0,1]$. $T_A(x)$ represents the degree of membership of x to A, $I_A(x)$ represents its degree of indeterminacy and $F_A(x)$ its degree of non-membership. $\forall x \in X, 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

See that SVNS is derived from the definition of neutrosophic sets. In the present paper we prefer to use the former one.

In neutrosophic set theory a lattice can be defined as follows:

Given the universe of discourse X and $x(T_x, I_x, F_x)$, $y(T_y, I_y, F_y)$ two SVNS, we say that $x \leq_N y$ if and only if $T_x \leq T_y$, $I_x \geq I_y$ and $F_x \geq F_y$, $(X, \leq_N)$ is a poset. Whereas, $(L, \wedge, \vee)$ is a lattice, because it is a triple direct product of lattices, see [9]. $x \wedge y = (\min\{T_x, T_y\}, \max\{I_x, I_y\}, \max\{F_x, F_y\})$ and $x \vee y = (\max\{T_x, T_y\}, \min\{I_x, I_y\}, \min\{F_x, F_y\})$. Moreover, it is easy to prove that it is complete.

Let us remark that this definition is valid for interval-valued neutrosophic sets, when we substitute their operators by interval-valued operators.

See also that there exist two special elements, viz., $0_N = (0,1,1)$ and $1_N = (1,0,0)$, which are the infimum and the supremum respectively, of every SVNS with regard to $\leq_N$.

Given two neutrosophic sets, A and B, three basic operations over them are the following [25]:

1. $A \cap B = A \wedge B$ (Conjunction).

2. $A \cup B = A \vee B$ (Disjunction).

3. $\bar{A} = (F_A, 1 - I_A, T_A)$ (Complement).

Definition 2.7. A *neutrosophic norm* or *n-norm* N_n [25], is a mapping N_n: $(]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[)^2 \to]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, such that $N_n\left(x(T_x, I_x, F_x), y(T_y, I_y, F_y)\right) = (N_nT(x,y), N_nI(x,y), N_nF(x,y))$, where N_nT means the degree of membership, N_nI the degree of indeterminacy and N_nF the degree of non-membership of the conjunction of both, x and y.

For every x, y and z belonging to the universe of discourse, N_n must satisfy the following axioms:

1. $N_n(x, 0_N) = 0_N$ and $N_n(x, 1_N) = x$ (Boundary conditions).
2. $N_n(x,y) = N_n(y,x)$ (Commutativity).
3. If $x \leq_N y$, then $N_n(x,z) \leq_N N_n(y,z)$ (Monotonicity).
4. $N_n(N_n(x,y), z) = N_n(x, N_n(y,z))$ (Associativity).

Definition 2.8. A *neutrosophic conorm* or *n-conorm* N_c [25], is a mapping N_c: $(]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[)^2 \to]^-0, 1^+[\times]^-0, 1^+[\times]^-0, 1^+[$, such that $N_c\left(x(T_x, I_x, F_x), y(T_y, I_y, F_y)\right) = (N_cT(x,y), N_cI(x,y), N_cF(x,y))$, where N_cT means the degree of membership, N_cI the degree of indeterminacy and N_cF the degree of non-membership of the disjunction of x with y.

For every x, y and z belonging to the universe of discourse, N_c must satisfy the following axioms:

1. $N_c(x, 0_N) = x$ and $N_c(x, 1_N) = 1_N$ (Boundary conditions).
2. $N_c(x,y) = N_c(y,x)$ (Commutativity).
3. If $x \leq_N y$, then $N_c(x,z) \leq_N N_c(y,z)$ (Monotonicity).
4. $N_c(N_c(x,y),z) = N_c(x, N_c(y,z))$ (Associativity).

According to [8] a Singled-valued neutrosophic negator is defined as follows:

Definition 2.9. a *singled-valued neutrosophic negator* is a decreasing unary neutrosophic operator $N_N: [0,1]^3 \longrightarrow [0,1]^3$, satisfying the following boundary conditions:

1. $N_N(0_N) = 1_N$.
2. $N_N(1_N) = 0_N$.

It is called *involutive* if and only if $N_N(N_N(x)) = x$ for every $x \in [0,1]^3$.

In the following, we show the neutrosophic negators that we shall consider hereunder, extracted from the literature, see [25]. Given a SVNS $A(T_A, I_A, F_A)$, we have:

1. $N_N((T_A, I_A, F_A)) = (1 - T_A, 1 - I_A, 1 - F_A)$, $N_N((T_A, I_A, F_A)) = (1 - T_A, I_A, 1 - F_A)$, $N_N((T_A, I_A, F_A)) = (F_A, I_A, T_A)$and $N_N((T_A, I_A, F_A)) = (F_A, 1 - I_A, T_A)$ (Involutive negators).
2. $N_N((T_A, I_A, F_A)) = \left(F_A, \frac{F_A + I_A + T_A}{3}, T_A\right)$ and $N_N((T_A, I_A, F_A)) = \left(1 - T_A, \frac{F_A + I_A + T_A}{3}, 1 - F_A\right)$ (Non-involutive negators).

In literature, we found neutrosophic implicators, which extend only the notion of S-implications [11]. Moreover, we did not find a general definition on neutrosophic implications except in [8]. In the following, we conclude this section with such definition and properties.

Definition 2.10. A *singled-valued neutrosophic implicator* is an operator $\mathrm{I_N}: [0,1]^3 \times [0,1]^3 \to [0,1]^3$ which satisfies the following conditions, for all $\mathrm{x, x', y, y'} \in [0,1]^3$:

1. If $\mathrm{x'} \leq_{\mathrm{N}} \mathrm{x}$, then $\mathrm{I_N(x,y)} \leq_{\mathrm{N}} \mathrm{I_N(x',y)}$.
2. If $\mathrm{y} \leq_{\mathrm{N}} \mathrm{y'}$, then $\mathrm{I_N(x,y)} \leq_{\mathrm{N}} \mathrm{I_N(x,y')}$.
3. $\mathrm{I_N(0_N, 0_N) = I_N(0_N, 1_N) = I_N(1_N, 1_N) = 1_N}$.
4. $\mathrm{I_N(1_N, 0_N) = 0_N}$.

Herein we use the term neutrosophic implicator or n-implicator to mean singled-valued neutrosophic implicator.

It can satisfy the following properties for every $\mathrm{x, y, z} \in [0,1]^3$:

1. $\mathrm{I_N(1_N, x) = x}$ (Neutrality principle)
2. $\mathrm{I_N(x,y) = I_N\big(N_{IN}(y), N_{IN}(x)\big)}$, where $\mathrm{N_{IN}(x) = I_N(x, 0_N)}$ is an n-negator (Contrapositivity).
3. $\mathrm{I_N\big(x, I_N(y,z)\big) = I_N\big(y, I_N(x,z)\big)}$ (Interchangeability principle).
4. $\mathrm{x} \leq_{\mathrm{N}} \mathrm{y}$ if and only if $\mathrm{I_N(x,y) = 1_N}$ (Confinement principle).
5. $\mathrm{I_N}$ is a continuous mapping (Continuity).

3 Neutrosophic uninorms

This section is the core of the present paper, because here we explain the neutrosophic uninorm theory. We start defining this concept formally.

3.1 N-uninorms

Definition 3.1. A *neutrosophic uninorm* or *N-uninorm* $\mathbf{U}_{\mathrm{N}}$, is a commutative, increasing and associative mapping, $\mathbf{U}_{\mathrm{N}}: (]^{-}0, 1^{+}[\times]^{-}0, 1^{+}[\times]^{-}0, 1^{+}[)^2 \to]^{-}0, 1^{+}[\times]^{-}0, 1^{+}[\times]^{-}0, 1^{+}[$, such that:
$\mathbf{U}_{\mathrm{N}}\left(\mathrm{x(T_x, I_x, F_x), y\big(T_y, I_y, F_y\big)}\right) = (\mathbf{U}_{\mathrm{N}}\mathrm{T(x,y)}, \mathbf{U}_{\mathrm{N}}\mathrm{I(x,y)}, \mathbf{U}_{\mathrm{N}}\mathrm{F(x,y)})$, where $\mathbf{U}_{\mathrm{N}}\mathrm{T}$ means the degree of membership, $\mathbf{U}_{\mathrm{N}}\mathrm{I}$ the degree of indeterminacy and $\mathbf{U}_{\mathrm{N}}\mathrm{F}$ the degree of non-membership of both, x and y. Additionally, there exists a neutral element $e \in]^{-}0, 1^{+}[\times]^{-}0, 1^{+}[\times]^{-}0, 1^{+}[$, where $\forall \mathrm{x} \in]^{-}0, 1^{+}[\times]^{-}0, 1^{+}[\times]^{-}0, 1^{+}[$, $\mathbf{U}_{\mathrm{N}}(e, \mathrm{x}) = \mathrm{x}$.

Remark 3.1. See that Def. 3.1, extends Def. 2.4 in two ways, according to the differences between L* fuzzy sets and neutrosophic sets. First, $\mathbf{U}_{\mathrm{N}}$ includes the third function representing indeterminacy and secondly, there not exists constraints in the relationship among T, I and F. In addition, Def. 3.1 extends Def. 2.7 when $e = 1_{\mathrm{N}}$ and Def 2.8., when $e = 0_{\mathrm{N}}$.

Remark 3.2. For the sake of simplicity, we shall develop the theory only for singled-valued neutrosophic uninorms.

A trivial consequence of Def. 3.1 is that the neutral element is unique, which is a uninorm property in Def. 2.1 and Def. 2.4.

In the following, we explore the formulas of N-uninorms related to those corresponding to n-norms and n-conorms. For this end, first we need to describe two kinds of sets, namely, $\mathrm{E_1} = \{\mathrm{x} \in [0,1]^3 : \mathrm{x} \leq_{\mathrm{N}} e\}$ and $\mathrm{E_2} = \{\mathrm{x} \in [0,1]^3 : \mathrm{x} \geq_{\mathrm{N}} e\}$.

Lemma 3.1. Let $e \in]0,1] \times [0,1[\times [0,1[$. The mapping $\phi_{\mathrm{e}}: [0,1]^3 \to [0,1]^3$, defined by:

$$\phi_{\mathrm{e}}(\mathrm{x}) = \big(e_1\mathrm{x_1}, \mathrm{x_2} + e_2(1 - \mathrm{x_2}), \mathrm{x_3} + e_3(1 - \mathrm{x_3})\big) \quad (1)$$

for every $\mathrm{x} \in [0,1]^3$ is an increasing bijection from $[0,1]^3$ to $\mathrm{E_1}$ and ϕ_{e}^{-1} is increasing as well.

Proof. To prove ϕ_{e} is injective, let $\mathrm{x, y} \in [0,1]^3$ and suppose $\phi_{\mathrm{e}}(\mathrm{x}) = \phi_{\mathrm{e}}(\mathrm{y})$.Then, clearly the equation $\big(e_1\mathrm{x_1}, \mathrm{x_2} + e_2(1 - \mathrm{x_2}), \mathrm{x_3} + e_3(1 - \mathrm{x_3})\big) = \big(e_1\mathrm{y_1}, \mathrm{y_2} + e_2(1 - \mathrm{y_2}), \mathrm{y_3} + e_3(1 - \mathrm{y_3})\big)$ is fulfilled only if x = y, and the injection is proved, also taking into account that we excluded the cases $e_1 = 0$, $e_2 = 1$ and $e_3 = 1$.

Let us take any $\mathrm{y} \in \mathrm{E_1}$ and define $\mathrm{x = (x_1, x_2, x_3)}$, such that $\mathrm{x_1} = \frac{y_1}{e_1}$, $\mathrm{x_2} = \frac{y_2 - e_2}{1 - e_2}$ and $\mathrm{x_3} = \frac{y_3 - e_3}{1 - e_3}$. Then, $\phi_{\mathrm{e}}(\mathrm{x}) = \mathrm{y}$ and $\mathrm{x_1, x_2, x_3} \in [0,1]$, which can be proved applying $\mathrm{y} \leq_{\mathrm{N}} e$. Therefore, ϕ_{e} is surjective and evidently it is increasing. The equation of the inverse is the following:

$$\phi_{\mathrm{e}}^{-1}(\mathrm{x}) = \left(\frac{\mathrm{x_1}}{e_1}, \frac{\mathrm{x_2} - e_2}{1 - e_2}, \frac{\mathrm{x_3} - e_3}{1 - e_3}\right) \quad (2)$$

□

Lemma 3.2. Let $e \in [0,1[\times]0,1]\times]0,1]$. The mapping $\psi_e: [0,1]^3 \to [0,1]^3$, defined by:

$$\psi_e(x) = (e_1 + x_1 - e_1x_1, e_2x_2, e_3x_3) \quad (3)$$

for every $x \in [0,1]^3$ is an increasing bijection from $[0,1]^3$ to E_2 as well as ψ_e^{-1} is increasing.

Proof. This lemma can be proved similarly to the proof carried out in the Lemma 3.1. The equation of the inverse is as follows:

$$\psi_e^{-1}(x) = \left(\frac{x_1 - e_1}{1 - e_1}, \frac{x_2}{e_2}, \frac{x_3}{e_3}\right) \quad (4)$$

□

Theorem 3.3. Given $\mathbf{U}_N$ an N-uninorm with neutral element $e \in]0,\ 1[^3$. Then the following two conditions are satisfied:

i. The mapping $N_{n,\mathbf{U}_N}: [0,1]^3 \times [0,1]^3 \to [0,1]^3$ defined for all $x, y \in [0,1]^3$ by the equation:

$$N_{n,\mathbf{U}_N}(x,y) = \phi_e^{-1}\left(\mathbf{U}_N(\phi_e(x), \phi_e(y))\right) \quad (5)$$

is an n-norm.

ii. The mapping $N_{c,\mathbf{U}_N}: [0,1]^3 \times [0,1]^3 \to [0,1]^3$ defined for all $x, y \in [0,1]^3$ by the equation:

$$N_{c,\mathbf{U}_N}(x,y) = \psi_e^{-1}\left(\mathbf{U}_N(\psi_e(x), \psi_e(y))\right) \quad (6)$$

is an n-conorm.

Proof. This theorem is a consequence of Lemmas 3.1 and 3.2.□

Remark 3.3. Some cases of *e* were excluded in Lemmas 3.1, 3.2 and Theorem 3.3, for instance, $e = (0, \beta, \gamma)$, where $0 \le \beta, \gamma \le 1$ in Lemma 3.1. It is easy to prove that when *e* is one of them, there not exist any increasing bijection from $[0,1]^3$ to E_1 or E_2, because E_1 or E_2 have one constant component, and therefore they only depend on at most two components, however, $[0,1]^3$ depends on three, and that contradicts the injection. For example, if $e = (0, \beta, \gamma)$, then $E_1 = \{0\} \times [\beta, 1] \times [\gamma, 1]$, and there not exists a bijective mapping from $[0,1]^3$ to E_1.

Corollary 3.4. Given $\mathbf{U}_N$ an N-uninorm with neutral element $e \in]0,\ 1[^3$. Then the following two conditions are satisfied:

i. For every $x, y \in E_1$, $\mathbf{U}_N(x,y) = \phi_e\left(N_{n,\mathbf{U}_N}(\phi_e^{-1}(x), \phi_e^{-1}(y))\right)$.

ii. For every $x, y \in E_2$, $\mathbf{U}_N(x,y) = \psi_e\left(N_{c,\mathbf{U}_N}(\psi_e^{-1}(x), \psi_e^{-1}(y))\right)$.

Proof. The proof is obtained immediately from Theorem 3.3.□

Remark 3.4. See that Theorem 3.3 and Corollary 3.4 mean that we can define N-uninorms from n-norms and n-conorms, and vice versa.

Remark 3.5. Comparing the precedent issues with their similar ones appeared in [15], we can find few differences and numerous similarities. Indeed, so far we have proved that N-uninorms extend the approach to structures of uninorms on L^* fuzzy sets, which is valid to interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets and Goguen's L-fuzzy sets.

Definition 3.2. We say that $N_n(x,y)$ is an *Archimedean n-norm* respect to $<_N$ if for every $x \in [0,1]^3$ it satisfies: $N_n(x,x) <_N x$.

Definition 3.3. We say that $N_c(x,y)$ is an *Archimedean n-conorm* respect to $<_N$ if for every $x \in [0,1]^3$ it satisfies: $N_c(x,x) >_N x$.

Definition 3.4. $\mathbf{U}_N(x,y)$ is an *Archimedean N-uninorm* respect to $<_N$ if it satisfies the following conditions:

1. $\mathbf{U}_N(x,x) <_N x$ for every $0 <_N x <_N e$.
2. $\mathbf{U}_N(x,x) >_N x$ for every $e <_N x <_N 1_N$.

Proposition 3.5. Given $\mathbf{U}_N$ an N-uninorm with neutral element $e \in]0,\ 1[^3$. It is Archimedean if and only if the n-norm and n-conorm defined in Eq. 5 and 6, respectively, are Archimedean.

Proof Let $0 <_N x <_N e$, and $\mathbf{U}_N(x,y)$ an Archimedean N-uninorm, i.e., $\mathbf{U}_N(x,x) <_N x$, then taking into account that ϕ_e and ϕ_e^{-1} are increasing bijections, we have $N_{n,\mathbf{U}_N}(x,x) = \phi_e^{-1}\left(\mathbf{U}_N(\phi_e(x), \phi_e(x))\right) <_N \phi_e^{-1}(\phi_e(x)) = x$. Equivalently, it is easy to prove that $N_{n,\mathbf{U}_N}(x,x) <_N x$ implies $\mathbf{U}_N(x,x) <_N x$. The proof for the n-conorm is similar.□

Proposition 3.6. Given $\mathbf{U}_N$ an N-uninorm with neutral element e, and $x, y \in [0,1]^3$ are two elements such that either $x \leq_N e \leq_N y$ or $y \leq_N e \leq_N x$, then the following two inequalities hold:

$\min(x, y) \leq_N \mathbf{U}_N(x, y) \leq_N \max(x, y)$.

Proof. Without loss of generality, suppose $x \leq_N e \leq_N y$, then because of the monotonicity of the N-uninorms $\mathbf{U}_N(x, y) \leq_N \mathbf{U}_N(e, y) = y = \max(x, y)$ and $\mathbf{U}_N(x, y) \geq_N \mathbf{U}_N(x, e) = x = \min(x, y)$.□

The proposition above means that there exists a domain where $\mathbf{U}_N$ is compensatory with regard to $\leq_N$.

Let us note that there exists other sets where $x \parallel_{\leq_N} y$ or $x \parallel_{\leq_N} e$.

Example 3.1. Two examples of N-uninorms are the following:

Recalling the well-known weakest and strongest fuzzy uninorms, respectively, defined as follows:

$$\underline{U}_{e_1}(x_1, y_1) := \begin{cases} 0 & \text{if } 0 \leq x_1, y_1 < e_1 \\ \max\{x_1, y_1\} & \text{if } e_1 \leq x_1, y_1 \leq 1 \\ \min\{x_1, y_1\} & \text{otherwise} \end{cases} \text{ and } \overline{U}_{e_1}(x_1, y_1) := \begin{cases} \min\{x_1, y_1\} & \text{if } 0 \leq x_1, y_1 \leq e_1 \\ 1 & \text{if } e_1 < x_1, y_1 \leq 1 \\ \max\{x_1, y_1\} & \text{otherwise} \end{cases}$$

For every $x_1, y_1 \in [0,1]$ and $e_1 \in]0,1[$.

Let us define two N-uninorms as follows: for every $x, y \in [0,1]^3$ and $e \in [0,1]^3$ is the neutral element:

$$\underline{\mathbf{U}}_e(x, y) := \left(\underline{U}_{e_1}(x_1, y_1), \overline{U}_{e_2}(x_2, y_2), \overline{U}_{e_3}(x_3, y_3)\right) \quad (7)$$

and

$$\overline{\mathbf{U}}_e(x, y) := \left(\overline{U}_{e_1}(x_1, y_1), \underline{U}_{e_2}(x_2, y_2), \underline{U}_{e_3}(x_3, y_3)\right) \quad (8)$$

Both $\underline{\mathbf{U}}_e(x, y)$ and $\overline{\mathbf{U}}_e(x, y)$, are N-uninorms, because every one of the components are uninorms, thus, they are commutative, associative and increasing. The neutral element components are formed by the neutral elements of every individual uninorm.

Moreover, $\underline{\mathbf{U}}_e(x, y)$ is a conjunctive N-uninorm and $\overline{\mathbf{U}}_e(x, y)$ is a disjunctive N-uninorm, i.e., $\underline{\mathbf{U}}_e(0_N, 1_N) = 0_N$ and $\overline{\mathbf{U}}_e(0_N, 1_N) = 1_N$.

See that $\mathbf{U}_e(x, y) = \left(\underline{U}_{e_1}(x_1, y_1), \underline{U}_{e_2}(x_2, y_2), \underline{U}_{e_3}(x_3, y_3)\right)$ is also an N-uninorm, nevertheless, it is neither conjunctive nor disjunctive, $\mathbf{U}_e(0_N, 1_N) = (0,0,0)$.

Definition 3.5. An N-uninorm $\mathbf{U}_N$ is said to be *t-representable* if there exist three fuzzy uninorms, $U_{e_1}(x_1, y_1)$, $U_{e_2}(x_2, y_2)$ and $U_{e_3}(x_3, y_3)$, such that for all $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ it has the form $\mathbf{U}_N(x, y) = \left(U_{e_1}(x_1, y_1), U_{e_2}(x_2, y_2), U_{e_3}(x_3, y_3)\right)$.

Proposition 3.7. Let $\mathbf{U}_N$ be an N-uninorm with neutral element e and $x \in [0,1]^3$, then the following properties hold:

i. $\mathbf{U}_N(0_N, 0_N) = 0_N$ and $\mathbf{U}_N(1_N, 1_N) = 1_N$.
ii. If $e \in [0,1]^3 \setminus \{0_N, 1_N\}$, we have $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x)$, for every $x \in [0,1]^3$.
iii. If $e \in [0,1]^3 \setminus \{0_N, 1_N\}$, then either $\mathbf{U}_N(0_N, 1_N) = 0_N$ or $\mathbf{U}_N(0_N, 1_N) = 1_N$ or $\mathbf{U}_N(0_N, 1_N) \parallel_{\leq_N} e$.

Proof.

i. See that $\mathbf{U}_N(e, 0_N) = 0_N$, $\mathbf{U}_N(e, 1_N) = 1_N$ and apply the increasing axiom of N-uninorm.
ii. If $x \leq_N e$ then because $\mathbf{U}_N$ is increasing, we have $\mathbf{U}_N(0_N, x) \leq_N \mathbf{U}_N(0_N, e) = 0_N$, thus, $\mathbf{U}_N(0_N, x) = 0_N$ and $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, x), 1_N)$. Because of the commutativity and the associativity, $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x)$.
If $x \geq_N e$ then $\mathbf{U}_N(1_N, x) \geq_N \mathbf{U}_N(1_N, e) = 1_N$ and therefore, $\mathbf{U}_N(1_N, x) = 1_N$. $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N\left(0_N, \mathbf{U}_N(1_N, x)\right)$, and finally due to the commutativity and associativity, we obtain $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x)$.
If $x \parallel_{\leq_N} e$ then $x \wedge e \leq_N x \leq_N x \vee e$. We have $x \wedge e \leq_N e$ and $e \leq_N x \vee e$, thus according to the precedent results $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x \wedge e) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x \vee e)$. Applying the increasing axiom of N-uninorms we obtain $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), x)$.
iii. Suppose $\mathbf{U}_N(0_N, 1_N) \nparallel_{\leq_N} e$, that implies either $\mathbf{U}_N(0_N, 1_N) \leq_N e$ or $e \leq_N \mathbf{U}_N(0_N, 1_N)$.
If $\mathbf{U}_N(0_N, 1_N) \leq_N e$, then $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), 0_N) = 0_N$, according to ii.
If $\mathbf{U}_N(0_N, 1_N) \geq_N e$, then $\mathbf{U}_N(0_N, 1_N) = \mathbf{U}_N(\mathbf{U}_N(0_N, 1_N), 1_N) = 1_N$, according to ii.□

Let us note that the precedent issues are similar to the ones obtained in [15].

3.2 Implicators induced by N-uninorms

This subsection is dedicated to explore the notion of n-implicators induced by N-uninorms. First of all we define the concept of neutrosophic R-implicator, which is new in this framework, at least in the scope of our knowledge.

Definition 3.6. A *neutrosophic R- implicator* or *n-R-implicator* is an n-implicator defined as follows:
Given N_n an n-norm, for every $x, y \in [0,1]^3$, $RI_N(x,y) = \sup\{t \in [0,1]^3 : N_n(x,t) \leq_N y\}$.

Let us note that this definition extends both, the definition of fuzzy R-implicator, see [7], and that of L* fuzzy implicator, [15]. As well as others appeared in [3, 12].

Indeed, it is an actual n-implicator. Taking into account the properties of $\leq_N$, and the increasing property of n-norms with regard to $\leq_N$, we have that $RI_N(x,\cdot)$ is decreasing and $RI_N(\cdot,y)$ is increasing. Additionally, the satisfaction of the boundary conditions by RI_N can be verified straightforwardly.

Example 3.2. Let $a = (0.6,\ 0.2,\ 0.4)$, $b = (0.7,\ 0.1,\ 0.3)$ and $c = (0.5,\ 0.3,\ 0.5)$ be three SVNS. Observe that $c \leq_N a \leq_N b$. Consider the n-norm, $N_{n-\min}(x,y) = (\min\{T_x, T_y\}, \max\{I_x, I_y\}, \max\{F_x, F_y\})$.

Then, $RI_N(a,b) = 1_N$, $RI_N(a,c) = (0.5,\ 0.3,\ 0.5)$, $RI_N(b,a) = (0.6, 0.2, 0.4)$ and $RI_N(c,a) = 1_N$. See that $RI_N(a,c) \leq_N RI_N(a,b)$ and $RI_N(b,a) \leq_N RI_N(c,a)$.

Proposition 3.8. Let RI_N be an n-R-implicator induced by the n-norm N_n, then the two following properties hold:

i. $RI_N(1_N, y) = y$ for every $y \in [0,1]^3$ (Neutrality principle).
ii. $RI_N(x,x) = 1_N$ for every $x \in [0,1]^3$ (Identity principle).
iii. $x, y \in [0,1]^3$ and $x \leq_N y$ if and only if $RI_N(x,y) = 1_N$ (Confinement principle).

Proof.

i. For $y \in [0,1]^3$, we have $RI_N(1_N, y) = \sup\{t \in [0,1]^3 : N_n(1_N, t) = t \leq_N y\} = y$.
ii. For $x \in [0,1]^3$, we have $RI_N(x,x) = \sup\{t \in [0,1]^3 : N_n(x,t) \leq_N x\} = 1_N$, because N_n is increasing and $N_n(x, 1_N) = x$.
iii. For $x, y \in [0,1]^3$ and $x \leq_N y$, taking into account the inequalities $N_n(x,t) \leq_N N_n(x, 1_N) = x \leq_N y$ for every $t \in [0,1]^3$, we have $RI_N(x,y) = 1_N$. On the other hand, $RI_N(x,y) = 1_N$ evidently implies $x \leq_N y$, from the definition.

Theorem 3.9. Let $\mathbf{U}_N$ be an N-uninorm with neutral element $e \in]0,\ 1[^3$. Let us establish the mapping $RI_{\mathbf{U}_N} : [0,1]^3 \times [0,1]^3 \to [0,1]^3$ defined as follows:

$RI_{\mathbf{U}_N}(x,y) = \sup\{t \in [0,1]^3 : \mathbf{U}_N(x,t) \leq_N y\}$ for every $x, y \in [0,1]^3$.

It is an n-implicator if and only if there exists $\tilde{x} >_N 0_N$ such that every $x \geq_N \tilde{x}$ satisfies $\mathbf{U}_N(0_N, x) = 0_N$.

Proof. It is easy to verify that $RI_{\mathbf{U}_N}(x,\cdot)$ is decreasing and $RI_{\mathbf{U}_N}(\cdot,y)$ is increasing.

On the other hand, $RI_{\mathbf{U}_N}(0_N, 1_N) = RI_{\mathbf{U}_N}(1_N, 1_N) = 1_N$, because $\mathbf{U}_N$ is increasing and 1_N is the supremum.

See that for every $t \in [0,1]^3$, $\mathbf{U}_N(1_N, t) \geq_N \mathbf{U}_N(e,t) = t$, then $\mathbf{U}_N(1_N, t) >_N 0_N$ if and only if $t >_N 0_N$, therefore $RI_{\mathbf{U}_N}(1_N, 0_N) = 0_N$.

Additionally, if there exists $\tilde{x} >_N 0_N$ such that every $x \geq_N \tilde{x}$ satisfies $\mathbf{U}_N(0_N, x) = 0_N$, then because $\mathbf{U}_N$ is increasing and 1_N is the supremum of that set, $\mathbf{U}_N(0_N, 1_N) = 0_N$ and $RI_{\mathbf{U}_N}(0_N, 0_N) = 1_N$.□

Remark 3.6. The Theorem 3.9 is valid when $\mathbf{U}_N$ is a conjuctive N-uninorm.

Example 3.3. Given again $a = (0.6,\ 0.2,\ 0.4)$, $b = (0.7,\ 0.1,\ 0.3)$ and $c = (0.5,\ 0.3,\ 0.5)$, three SVNS, as in Example 3.2. Let us consider $\underline{\mathbf{U}}_e$ of the Example 3.1, where $e = (0.5, 0.5, 0.5)$. Recall that $\underline{\mathbf{U}}_e(0_N, 1_N) = 0_N$. Then, $RI_{\underline{\mathbf{U}}_e}(a,b) = (0.7, 0.1, 0.3)$, $RI_{\underline{\mathbf{U}}_e}(a,c) = (0.5, 0.5, 0.5)$, $RI_{\underline{\mathbf{U}}_e}(b,a) = (0.5, 0.5, 0.5)$ and $RI_{\underline{\mathbf{U}}_e}(c,a) = (0.6, 0.2, 0.4)$.

Proposition 3.10. Given $\mathbf{U}_N$ an N-uninorm with $e \in [0,1]^3 \setminus \{0_N, 1_N\}$. Then, $RI_{\mathbf{U}_N}(e,x) = x$, for every $x \in [0,1]^3$.

Proof. Let us fix $x \in [0,1]^3$, $RI_{\mathbf{U}_N}(e,x) = \sup\{t \in [0,1]^3 : \mathbf{U}_N(e,t) = t \leq_N x\} = x$.□

Proposition 3.11. Given $\mathbf{U}_N$ an N-uninorm with $e \in [0,1]^3 \setminus \{0_N, 1_N\}$. $RI_{\mathbf{U}_N}(x, 1_N) = 1_N$, for every $x \in [0,1]^3$ (Right boundary condition).

Proof. Taking into account $\mathbf{U}_N$ is increasing and 1_N is the supremum of the elements of the lattice, then, $RI_{\mathbf{U}_N}(x, 1_N) = \sup\{t \in [0,1]^3 : \mathbf{U}_N(x,t) \leq_N 1_N\} = 1_N$.□

Proposition 3.12. Given $\mathbf{U}_N$ an N-uninorm with $e \in [0,1]^3 \setminus \{0_N, 1_N\}$. If it is contrapositive respect to a negator N_N, which satisfies $N_N(e) = e$, then $N_N(x) = N_{NI_{\mathbf{U}_N}}(x) = RI_{\mathbf{U}_N}(x, e)$ for every $x \in [0,1]^3$ and $N_{NI_{\mathbf{U}_N}}$ is involutive.

Proof. Reproduce the similar proof in [15] adapted to N-uninorms.□

Proposition 3.13. Given $\mathbf{U}_N$ an N-uninorm and N_N an n-negator. The mapping $SI_{\mathbf{U}_N}(x,y) = \mathbf{U}_N(N_N(x), y)$ is an n-implicator if and only if $\mathbf{U}_N$ is disjunctive.

Proof. Reproduce the similar proof in [15] adapted to N-uninorms.□

Example 3.4. Revisiting Examples 3.2 and 3.3, where $a = (0.6,\ 0.2,\ 0.4)$, $b = (0.7,\ 0.1,\ 0.3)$ and $c = (0.5,\ 0.3,\ 0.5)$. Now we consider the n-negator $N_N\big((T_x, I_x, F_x)\big) = (F_x, I_x, T_x)$ and from the Example 3.1, $\overline{\mathbf{U}}_e(x,y)$ with $e = (0.5, 0.5, 0.5)$. There, we proved it is disjunctive.

Then, we have $SI_{\overline{\mathbf{U}}_e}(a,b) = (0.7, 0, 0.3)$, $SI_{\overline{\mathbf{U}}_e}(a,c) = (0.4,\ 0, 0.6)$, $SI_{\overline{\mathbf{U}}_e}(b,a) = (0.6,\ 0, 0.4)$ and $SI_{\overline{\mathbf{U}}_e}(c,a) = (0.6, 0, 0.4)$.

Proposition 3.14. Given $\mathbf{U}_N$ an N-uninorm and N_N an n-negator. The mapping $SI_{\mathbf{U}_N}$ satisfies the Interchangeability Principle:

$SI_{\mathbf{U}_N}\left(x, SI_{\mathbf{U}_N}(y,z)\right) = SI_{\mathbf{U}_N}\left(y, SI_{\mathbf{U}_N}(x,z)\right)$ for every $x, y, z \in [0,1]^3$.

Proof. It is proved by using the commutativity and associativity of N-uninorms.□

Conclusion

The proposed paper was devoted to define and study a new operator called neutrosophic uninorm or N-uninorm. We demonstrated that it is possible to extend the notion of uninorm to the framework of neutrosophy logic theory. In addition, we defined new neutrosophic implicators induced by N-uninorms. Moreover, we introduced a new neutrosophic implicator which generalizes the fuzzy notion of R-implicator. The importance of this new theory is that the appreciated quality of fuzzy uninorms as aggregators is enriched with the capacity of neutrosophy to deal with indeterminacy.

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H-Max Distance Measure of Bipolar Neutrosophic Sets and an Application to Medical Diagnosis

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Abstract: A single-valued neutrosophic set is one of the advanced fuzzy sets that is capable of handling complex real-world information satisfactorily. A development of single-valued neutrosophic set and fuzzy bipolar set, called a bipolar neutrosophic set, was introduced. Distance measures between fuzzy sets and advanced fuzzy sets are important tools in diagnostics and prediction problems. Sometimes they are defined without considering the condition of the inclusion relation on sets. In decision-making applications, this condition should be required (here it is called the inference of the measure). Moreover, in many cases, a distance measure capable of discriminating between two nearly identical objects is considered an effective measure. Motivated by these observations, in this paper, a new distance measure is proposed in a bipolar neutrosophic environment. Furthermore, an entropy measure is also developed by the similarity between two sets of mutual negation. Finally, an application to medical diagnosis is presented to illustrate the effective applicability of the proposed distance measure, where entropy values are used to characterize noises of different attributes.

Keywords: neutrosophic distance; similarity measure; bipolar neutrosophic sets; entropy measure; medical diagnosis

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1. Introduction

In 1965, the concept of a fuzzy set (FS) was introduced by Zadeh [1] to handle uncertainty of information in real-world inference systems. According to him, the degree of membership (positivity) of an element u to a FS on a universe U is one value $\mu(u)$, where $\mu(u) \in \lfloor 0,1 \rfloor$. The theory of FSs has reached a huge amount of achievements in a variety of application areas. However, in many real-life problems, the presence of negativity cannot be ignored. In 1983, Atanassov [2] proposed the concept of an intuitionistic fuzzy set (IFS) by considering the membership degree $\mu(u)$ as well as the non-membership degree $\nu(u)$ with the condition on their sum which is $\mu(u)+\nu(u)\le 1$. The theory and applications of IFSs have been strongly developed such as studies on logical operators [3-5] and applications in decision making [6-10].

From a philosophical perspective on the existence of the field of neutrosophy, Smarandache considers that using IFSs to treat indeterminate and inconsistent is not satisfactory enough. In 1999, Smarandache [11] introduced the concept of neutrosophic set (NS). He named its three characteristic functions the truth membership function, the indeterminacy-membership function, and falsity-membership function, denoted by $T(u)$, $I(u)$, and $F(u)$, respectively. Their outputs are real standard or nonstandard subsets of $]^-0,1^+[$. From the requirement of practical applications about representing the featured degrees by real values, Wang et al [12] provided the definition of single-valued neutrosophic sets (SVNSs). Cuong [13] also proposed the concept of picture fuzzy set (PFS) as a particular case of NSs. Some results on PFSs can be found in [14-19]. Because of the independent existence between the considered property and its corresponding implicit antagonist, Deli et al. [20] introduced the concept of bipolar neutrosophic sets (BNSs). This is a generalization of SVNSs and bipolar fuzzy sets [21]. In a BNS X, $T^{\square}(u), I^{\square}(u), F^{\square}(u)$ represent the characteristic degrees of an element $u \in U$ corresponding to X and $T^{\approx}(u), I^{\approx}(u), F^{\approx}(u)$ represent characteristic degrees of u to some implicit counter-property corresponding to X. Some research on NSs and BNSs and their applications can be found in [22-36].

The advanced fuzzy distance measures are known as effective tools for solving decision-making problems [6-10, 13, 37]. Some of distance measures of SVNSs were proposed such as Hausdorff distance [38], Cosine similarity measures [39], and the distance measures of Ye [40], Aydoğdu [41], Huang [26], and Ngan et al. [42]. In 2018, Vakkas [43] et al. introduced similarity measures of BNSs and their application to decision-making problems. Vakkas's measure was defined without considering the condition of the inclusion relation on sets. In decision-making applications, this condition (in this paper, it is called the inference of the measure) should be required. Moreover, Vakkas's proposal does not imply cross-evaluation, which is necessary to distinguish the differences and was discussed in intuitionistic fuzzy and single-value neutrosophic environments [7,10,42]. Motivated by these observations, in this paper, a new distance measure set that includes cross-evaluation and the inference of the measure is first proposed in a bipolar neutrosophic environment. Furthermore, an entropy measure is also developed by the similarity between two sets of mutual negation. Finally, an application to medical diagnosis on the UCI dataset is presented to illustrate the effective applicability of the proposed distance measure, where entropy values of different attribute sets are used to characterize their noises.

The next sections of the paper are distributed content as follows. Some basic concepts and the related measure formulas are presented in Section 2. In Section 3, the proposals on the distance measure, the similarity measure, and the entropy measure on BNSs are introduced. In Section 4, an application to medical diagnosis given to show the effectiveness of the proposed distance measure. Finally, Section 5 shows the conclusions of the study.

2. Preliminaries

Definition 1. [25] A NS X on a universe set U is characterized by three feature functions including a truth-membership function, $T_X : U \to]^-0,1^+[$, an indeterminacy-membership function, $I_X : U \to]^-0,1^+[$, and a falsity-membership function, $F_X : U \to]^-0,1^+[$, where

$$^-0 \le \sup_U T_X(z) + \sup_U I_X(z) + \sup_U F_X(z) \le 3^+,\ z \in U. \tag{1}$$

Definition 2. [20] A BNS X on U is defined by the form as follows:

$$X = \left\{ < z, T_X^{\square}(z), I_X^{\square}(z), F_X^{\square}(z), T_X^{\approx}(z), I_X^{\approx}(z), F_X^{\approx}(z) > \mid z \in U \right\} \text{ or}$$

$$X = < T_X^{\square}, I_X^{\square}, F_X^{\square}, T_X^{\approx}, I_X^{\approx}, F_X^{\approx} >, \tag{2}$$

where $T_X^{\square}, I_X^{\square}, F_X^{\square} : U \to \left[0,1\right]$, and $T_X^{\approx}, I_X^{\approx}, F_X^{\approx} : U \to \left\lfloor -1,0 \right\rfloor$.

Denoted by $BNS(U)$ the set of all BNSs on U.

Definition 3. [20] Let X_1 and X_2 be two BNSs on U, then

- $X_1 \subseteq X_2$ if and only if $T_1^{\square}(z) \le T_2^{\square}(z), I_1^{\square}(z) \ge I_2^{\square}(z), F_1^{\square}(z) \ge F_2^{\square}(z),\ T_1^{\approx}(z) \ge T_2^{\approx}(z),\ I_1^{\approx}(z) \le I_2^{\approx}(z),$ and $F_1^{\approx}(z) \le F_2^{\approx}(z)$.
- $X_1 = X_2$ if and only if $T_1^{\square}(z) = T_2^{\square}(z), I_1^{\square}(z) = I_2^{\square}(z), F_1^{\square}(z) = F_2^{\square}(z),\ T_1^{\approx}(z) = T_2^{\approx}(z), I_1^{\approx}(z) = I_2^{\approx}(z),$ and $F_1^{\approx}(z) = F_2^{\approx}(z)$.
- $X^c = \left\{ < z, F^{\square}(z), 1 - I^{\square}(z), T^{\square}(z), F^{\approx}(z), -1 - I^{\approx}(z), T^{\approx}(z) > \mid z \in U \right\}.$

Definition 4. [43] A similarity measure of BNSs is a $S : \left(BNS\left(U\right)\right)^2 \to \left[0,1\right]$ mapping satisfying

1. $0 \le S\left(X_1, X_2\right) \le 1$,
2. $S\left(X_1, X_2\right) = S\left(X_2, X_1\right)$,
3. $S\left(X_1, X_2\right) = 1$ for $X_1 = X_2$, where $X_1, X_2 \in BNS\left(U\right)$.

In 2018, Vakkas et al. [43] proposed a similarity measure of BNSs as follows:

$$S_V\left(X_1, X_2\right) = \alpha S_{V1}\left(X_1, X_2\right) + \left(1-\alpha\right) S_{V2}\left(X_1, X_2\right), \tag{3}$$

where $\alpha \in \left\lfloor 0,1 \right\rfloor$,

$$S_{V1}\left(X_1, X_2\right) = \sum_{i=1}^{n} \omega_i \left(\frac{\begin{bmatrix} \left(T_{X_1}^{\square}(z_i) T_{X_2}^{\square}(z_i) + I_{X_1}^{\square}(z_i) I_{X_2}^{\square}(z_i) + F_{X_1}^{\square}(z_i) F_{X_2}^{\square}(z_i)\right) \\ -\left(T_{X_1}^{\approx}(z_i) T_{X_2}^{\approx}(z_i) + I_{X_1}^{\approx}(z_i) I_{X_2}^{\approx}(z_i) + F_{X_1}^{\approx}(z_i) F_{X_2}^{\approx}(z_i)\right) \end{bmatrix}}{\begin{matrix} 2\Big[\left(T_{X_1}^{\square\,2}(z_i) + I_{X_1}^{\square\,2}(z_i) + F_{X_1}^{\square\,2}(z_i)\right) + \left(T_{X_2}^{\square\,2}(z_i) + I_{X_2}^{\square\,2}(z_i) + F_{X_2}^{\square\,2}(z_i)\right) \\ -\left(T_{X_1}^{\approx\,2}(z_i) + I_{X_1}^{\approx\,2}(z_i) + F_{X_1}^{\approx\,2}(z_i)\right) - \left(T_{X_2}^{\approx\,2}(z_i) + I_{X_2}^{\approx\,2}(z_i) + F_{X_2}^{\approx\,2}(z_i)\right)\Big] \end{matrix}} \right),$$

and

$$S_{V2}\left(X_1, X_2\right) = \sum_{i=1}^{n} \omega_i \left(\frac{\begin{bmatrix} \left(T_{X_1}^{\square}(x_i) T_{X_2}^{\square}(x_i) + I_{X_1}^{\square}(x_i) I_{X_2}^{\square}(x_i) + F_{X_1}^{\square}(x_i) F_{X_2}^{\square}(x_i)\right) \\ -\left(T_{X_1}^{\approx}(x_i) T_{X_2}^{\approx}(x_i) + I_{X_1}^{\approx}(x_i) I_{X_2}^{\approx}(x_i) + F_{X_1}^{\approx}(x_i) F_{X_2}^{\approx}(x_i)\right) \end{bmatrix}}{\begin{matrix} 2\Big[\sqrt{T_{X_1}^{\square\,2}(x_i) + I_{X_1}^{\square\,2}(x_i) + F_{X_1}^{\square\,2}(x_i)} \times \sqrt{T_{X_2}^{\square\,2}(x_i) + I_{X_2}^{\square\,2}(x_i) + F_{X_2}^{\square\,2}(x_i)} \\ -\sqrt{T_{X_1}^{\approx\,2}(x_i) + I_{X_1}^{\approx\,2}(x_i) + F_{X_1}^{\approx\,2}(x_i)} \times \sqrt{T_{X_2}^{\approx\,2}(x_i) + I_{X_2}^{\approx\,2}(x_i) + F_{X_2}^{\approx\,2}(x_i)}\Big] \end{matrix}} \right).$$

Note that: Vakkas's proposal is without considering the condition related to the inclusion relation on sets. Some other measures are built based on the triangle inequality condition instead of the condition related to the inclusion relation on sets, such as the Hamming distance and the Euclidean distance [44, 45].

In 2021, by reasoning about the need for the cross-evaluation, Ngan et al. [42] defined the H-max distance measure on SVNSs by

$$\begin{aligned} d_{HN}\left(X_1, X_2\right) = \sum_{i=1}^{n} \chi_i \Big(& \alpha_1 \left|T_{X_1}(z_i) - T_{X_2}(z_i)\right| + \alpha_2 \left|I_{X_1}(z_i) - I_{X_2}(z_i)\right| + \alpha_3 \left|F_{X_1}(z_i) - F_{X_2}(z_i)\right| \\ & + \alpha_4 \left|\max\left\{T_{X_1}(z_i), I_{X_2}(z_i)\right\} - \max\left\{I_{X_1}(z_i), T_{X_2}(z_i)\right\}\right| \\ & + \alpha_5 \left|\max\left\{T_{X_1}(z_i), F_{X_2}(z_i)\right\} - \max\left\{F_{X_1}(z_i), T_{X_2}(z_i)\right\}\right| \Big) \end{aligned} \tag{4}$$

where $\alpha_k \in \left(0,1\right)$, $\sum_{k=1}^{5} \alpha_k = 1,\ \ \chi_i \in \left\lfloor 0,1 \right\rfloor,\ \sum_{i=1}^{n} \chi_i = 1.$

3. H-max bipolar neutrosophic weighted measure

Now, the provided definition of distance measures of BNSs includes the inference condition. Furthermore, a specific distance measure, called H-max bipolar neutrosophic weighted measure, is introduced based on the formula of d_{HN} proposed by Ngan et al. [42].

Definition 5. For all $X_1, X_2, X_3 \in BNS(U)$ where $U = \{z_1, ..., z_n\}$, then a distance measure of BNSs is $d : (BNS(U))^2 \to [0,1]$ mapping satisfying

1. $d(X_1, X_2) = d(X_2, X_1)$,
2. $d(X_1, X_2) = 0$ if and only if $X_1 = X_2$,
3. If $X_1 \subseteq X_2 \subseteq X_3$, then $d(X_1, X_2) \le d(X_1, X_3)$ and $d(X_2, X_3) \le d(X_1, X_3)$.

Definition 6. Let $X_1, X_2 \in BNS(U)$ where $U = \{z_1, ..., z_n\}$ and

$$X_1 = \left\{ < z, T^{\square}_{X_1}(z), I^{\square}_{X_1}(z), F^{\square}_{X_1}(z), T^{\approx}_{X_1}(z), I^{\approx}_{X_1}(z), F^{\approx}_{X_1}(z) > \mid z \in U \right\},$$

$$X_2 = \left\{ < z, T^{\square}_{X_2}(z), I^{\square}_{X_2}(z), F^{\square}_{X_2}(z), T^{\approx}_{X_2}(z), I^{\approx}_{X_2}(z), F^{\approx}_{X_2}(z) > \mid z \in U \right\}.$$

Then, the formula of H-max bipolar neutrosophic weighted distance measure between X_1 and X_2 is as follows

$$d_{H-BN}(X_1, X_2) = \lambda d_{H-BN1}(X_1, X_2) + (1-\lambda) d_{H-BN2}(X_1, X_2), \tag{5}$$

where

$$\begin{aligned} d_{H-BN1}(X_1, X_2) = \sum_{i=1}^{n} \chi_i^{\square} \Big(& \alpha_1^{\square} \left| T^{\square}_{X_1}(z_i) - T^{\square}_{X_2}(z_i) \right| + \alpha_2^{\square} \left| I^{\square}_{X_1}(z_i) - I^{\square}_{X_2}(z_i) \right| + \alpha_3^{\square} \left| F^{\square}_{X_1}(z_i) - F^{\square}_{X_2}(z_i) \right| \\ & + \alpha_4^{\square} \left| \max\left\{ T^{\square}_{X_1}(z_i), I^{\square}_{X_2}(z_i) \right\} - \max\left\{ I^{\square}_{X_1}(z_i), T^{\square}_{X_2}(z_i) \right\} \right| \\ & + \alpha_5^{\square} \left| \max\left\{ T^{\square}_{X_1}(z_i), F^{\square}_{X_2}(z_i) \right\} - \max\left\{ F^{\square}_{X_1}(z_i), T^{\square}_{X_2}(z_i) \right\} \right| \Big), \end{aligned}$$

$$\begin{aligned} d_{H-BN2}(X_1, X_2) = \sum_{i=1}^{n} \chi_i^{\approx} \Big(& \alpha_1^{\approx} \left| T^{\approx}_{X_1}(z_i) - T^{\approx}_{X_2}(z_i) \right| + \alpha_2^{\approx} \left| I^{\approx}_{X_1}(z_i) - I^{\approx}_{X_2}(z_i) \right| + \alpha_3^{\approx} \left| F^{\approx}_{X_1}(z_i) - F^{\approx}_{X_2}(z_i) \right| \\ & + \alpha_4^{\approx} \left| \min\left\{ T^{\approx}_{X_1}(z_i), I^{\approx}_{X_2}(z_i) \right\} - \min\left\{ I^{\approx}_{X_1}(z_i), T^{\approx}_{X_2}(z_i) \right\} \right| \\ & + \alpha_5^{\approx} \left| \min\left\{ T^{\approx}_{X_1}(z_i), F^{\approx}_{X_2}(z_i) \right\} - \min\left\{ F^{\approx}_{X_1}(z_i), T^{\approx}_{X_2}(z_i) \right\} \right| \Big), \end{aligned}$$

where $\alpha_k^{\square}, \alpha_k^{\approx} \in (0,1)$, $\sum_{k=1}^{5} \alpha_k^{\square} = 1$, $\sum_{k=1}^{5} \alpha_k^{\square} = 1$, $\chi_i^{\square}, \chi_i^{\approx} \in [0,1]$, $\sum_{i=1}^{n} \chi_i^{\approx} = 1$, and $\lambda \in (0,1)$.

Proposition 1. d_{H-BN} satisfies the following properties for all $X_1, X_2, X_3 \in BNS(U)$.

1. $0 \le d_{H-BN}(X_1, X_2) \le 1$,
2. $d_{H-BN}(X_1, X_2) = 0$ if and only if $X_1 = X_2$,
3. $d_{H-BN}(X_1, X_2) = d_{H-BN}(X_2, X_1)$,
4. $d_{H-BN}(X_1, X_2) \le d_{H-BN}(X_1, X_3)$ and $d_{H-BN}(X_2, X_3) \le d_{H-BN}(X_1, X_3)$ if $X_1 \subseteq X_2 \subseteq X_3$.

Proof

1. Apparently, for all $i = 1, ..., n$,

$$\left| T^{\square}_{X_1}(z_i) - T^{\square}_{X_2}(z_i) \right|, \ \left| I^{\square}_{X_1}(z_i) - I^{\square}_{X_2}(z_i) \right|, \ \left| F^{\square}_{X_1}(z_i) - F^{\square}_{X_2}(z_i) \right| \in \lfloor 0,1 \rfloor,$$

$$\left|\max\left\{T^{\square}_{X_1}(z_i), I^{\square}_{X_2}(z_i)\right\} - \max\left\{I^{\square}_{X_1}(z_i), T^{\square}_{X_2}(z_i)\right\}\right| \in [0,1],$$
$$\left|\max\left\{T^{\square}_{X_1}(z_i), F^{\square}_{X_2}(z_i)\right\} - \max\left\{F^{\square}_{X_1}(z_i), T^{\square}_{X_2}(z_i)\right\}\right| \in \lfloor 0,1 \rfloor,$$

and

$$\left|T^{\approx}_{X_1}(z_i) - T^{\approx}_{X_2}(z_i)\right|, \left|I^{\approx}_{X_1}(z_i) - I^{\approx}_{X_2}(z_i)\right|, \left|F^{\approx}_{X_1}(z_i) - F^{\approx}_{X_2}(z_i)\right| \in [0,1],$$
$$\left|\min\left\{T^{\approx}_{X_1}(z_i), I^{\approx}_{X_2}(z_i)\right\} - \min\left\{I^{\approx}_{X_1}(z_i), T^{\approx}_{X_2}(z_i)\right\}\right| \in [0,1],$$
$$\left|\min\left\{T^{\approx}_{X_1}(z_i), F^{\approx}_{X_2}(z_i)\right\} - \min\left\{F^{\approx}_{X_1}(z_i), T^{\approx}_{X_2}(z_i)\right\}\right| \in [0,1].$$

Hence, $0 \le d_{H-BN}(X_1, X_2) \le 1$.

2. Clearly, $d_{H-BN}(X_1, X_2) = 0 \Leftrightarrow \begin{cases} T^{\square}_{X_1} = T^{\square}_{X_2}, T^{\approx}_{X_1} = T^{\approx}_{X_2} \\ I^{\square}_{X_1} = I^{\square}_{X_2}, I^{\approx}_{X_1} = I^{\approx}_{X_2} \\ F^{\square}_{X_1} = F^{\square}_{X_2}, F^{\approx}_{X_1} = F^{\approx}_{X_2} \end{cases} \Leftrightarrow X_1 = X_2$.

3. It can be seen that d_{H-BN} has the symmetry property.

4. Let $X_1 \subseteq X_2 \subseteq X_3$ then for all $i = 1, \ldots, n$,

$$T^{\square}_{X_1}(z_i) \le T^{\square}_{X_2}(z_i) \le T^{\square}_{X_3}(z_i), I^{\square}_{X_1}(z_i) \ge I^{\square}_{X_2}(z_i) \ge I^{\square}_{X_3}(z_i),$$
$$F^{\square}_{X_1}(z_i) \ge F^{\square}_{X_2}(z_i) \ge F^{\square}_{X_3}(z_i), T^{\approx}_{X_1}(z_i) \ge T^{\approx}_{X_2}(z_i) \ge T^{\approx}_{X_3}(z_i),$$
$$I^{\approx}_{X_1}(z_i) \le I^{\approx}_{X_2}(z_i) \le I^{\approx}_{X_3}(z_i), \text{ and } F^{\approx}_{X_1}(z_i) \le F^{\approx}_{X_2}(z_i) \le F^{\approx}_{X_3}(z_i).$$

These lead to

$$\left|T^{\square}_{X_1} - T^{\square}_{X_2}\right| \le \left|T^{\square}_{X_1} - T^{\square}_{X_3}\right|, \left|I^{\square}_{X_1} - I^{\square}_{X_2}\right| \le \left|I^{\square}_{X_1} - I^{\square}_{X_3}\right|, \left|F^{\square}_{X_1} - F^{\square}_{X_2}\right| \le \left|F^{\square}_{X_1} - F^{\square}_{X_3}\right|,$$
$$\left|T^{\approx}_{X_1} - T^{\approx}_{X_2}\right| \le \left|T^{\approx}_{X_1} - T^{\approx}_{X_3}\right|, \left|I^{\approx}_{X_1} - I^{\approx}_{X_2}\right| \le \left|I^{\approx}_{X_1} - I^{\approx}_{X_3}\right|, \left|F^{\approx}_{X_1} - F^{\approx}_{X_2}\right| \le \left|F^{\approx}_{X_1} - F^{\approx}_{X_3}\right|.$$

Moreover,

$$\max\left\{T^{\square}_{X_3}, I^{\square}_{X_1}\right\} \ge \max\left\{T^{\square}_{X_2}, I^{\square}_{X_1}\right\} \ge \max\left\{T^{\square}_{X_1}, I^{\square}_{X_2}\right\} \ge \max\left\{T^{\square}_{X_1}, I^{\square}_{X_3}\right\},$$
$$\min\left\{T^{\approx}_{X_3}, I^{\approx}_{X_1}\right\} \le \min\left\{T^{\approx}_{X_2}, I^{\approx}_{X_1}\right\} \le \min\left\{T^{\approx}_{X_1}, I^{\approx}_{X_2}\right\} \le \min\left\{T^{\approx}_{X_1}, I^{\approx}_{X_3}\right\},$$

$$\max\left\{T^{\square}_{X_3}, F^{\square}_{X_1}\right\} \ge \max\left\{T^{\square}_{X_2}, F^{\square}_{X_1}\right\} \ge \max\left\{T^{\square}_{X_1}, F^{\square}_{X_2}\right\} \ge \max\left\{T^{\square}_{X_1}, F^{\square}_{X_3}\right\},$$
$$\min\left\{T^{\approx}_{X_3}, F^{\approx}_{X_1}\right\} \le \min\left\{T^{\approx}_{X_2}, F^{\approx}_{X_1}\right\} \le \min\left\{T^{\approx}_{X_1}, F^{\approx}_{X_2}\right\} \le \min\left\{T^{\approx}_{X_1}, F^{\approx}_{X_3}\right\}.$$

Hence,

$$\left|\max\left\{T^{\square}_{X_2}, I^{\square}_{X_1}\right\} - \max\left\{T^{\square}_{X_1}, I^{\square}_{X_2}\right\}\right| \le \left|\max\left\{T^{\square}_{X_3}, I^{\square}_{X_1}\right\} - \max\left\{T^{\square}_{X_1}, I^{\square}_{X_3}\right\}\right|,$$
$$\left|\min\left\{T^{\approx}_{X_2}, I^{\approx}_{X_1}\right\} - \min\left\{T^{\approx}_{X_1}, I^{\approx}_{X_2}\right\}\right| \le \left|\min\left\{T^{\approx}_{X_3}, I^{\approx}_{X_1}\right\} - \min\left\{T^{\approx}_{X_1}, I^{\approx}_{X_3}\right\}\right|,$$

$$\left|\max\left\{T^{\square}_{X_2}, F^{\square}_{X_1}\right\} - \max\left\{T^{\square}_{X_1}, F^{\square}_{X_2}\right\}\right| \le \left|\max\left\{T^{\square}_{X_3}, F^{\square}_{X_1}\right\} - \max\left\{T^{\square}_{X_1}, F^{\square}_{X_3}\right\}\right|,$$
$$\left|\min\left\{T^{\approx}_{X_2}, F^{\approx}_{X_1}\right\} - \min\left\{T^{\approx}_{X_1}, F^{\approx}_{X_2}\right\}\right| \le \left|\min\left\{T^{\approx}_{X_3}, F^{\approx}_{X_1}\right\} - \min\left\{T^{\approx}_{X_1}, F^{\approx}_{X_3}\right\}\right|.$$

Thus, $d_{H-BN}(X_1, X_2) \le d_{H-BN}(X_1, X_3)$. Similarly, $d_{H-BN}(X_2, X_3) \le d_{H-BN}(X_1, X_3)$ is proven. □

Definition 7. Let $X_1, X_2 \in BNS(U)$ where $U = \{z_1, ..., z_n\}$. Then, the formula of H-max bipolar neutrosophic weighted similarity measure between X_1 and X_2 is as follows

$$s_{H-BN}(X_1, X_2) = 1 - d_{H-BN}(X_1, X_2). \tag{6}$$

Proposition 2. s_{H-BN} satisfies the following properties, for all $X_1, X_2, X_3 \in BNS(U)$:

1. $0 \le s_{H-BN}(X_1, X_2) \le 1$,
2. $s_{H-BN}(X_1, X_2) = 1$ if and only if $X_1 = X_2$,
3. $s_{H-BN}(X_1, X_2) = s_{H-BN}(X_2, X_1)$,
4. $s_{H-BN}(X_1, X_2) \ge s_{H-BN}(X_1, X_3)$ and $s_{H-BN}(X_2, X_3) \ge s_{H-BN}(X_1, X_3)$ if $X_1 \subseteq X_2 \subseteq X_3$.

Remark 1. The proposed distance measure overcomes the limitations of the Hamming distance, the Euclidean distance [44, 45], and Vakkas's proposal [43]. Specifically,

- The proposed measure d_{H-BN} includes cross-evaluations:

$$\left|\max\left\{T^{\square}_{X_1}(z_i), I^{\square}_{X_2}(z_i)\right\} - \max\left\{I^{\square}_{X_1}(z_i), T^{\square}_{X_2}(z_i)\right\}\right|,$$
$$\left|\max\left\{T^{\square}_{X_1}(z_i), F^{\square}_{X_2}(z_i)\right\} - \max\left\{F^{\square}_{X_1}(z_i), T^{\square}_{X_2}(z_i)\right\}\right|,$$
$$\left|\min\left\{T^{\approx}_{X_1}(z_i), I^{\approx}_{X_2}(z_i)\right\} - \min\left\{I^{\approx}_{X_1}(z_i), T^{\approx}_{X_2}(z_i)\right\}\right|,$$
$$\left|\min\left\{T^{\approx}_{X_1}(z_i), F^{\approx}_{X_2}(z_i)\right\} - \min\left\{F^{\approx}_{X_1}(z_i), T^{\approx}_{X_2}(z_i)\right\}\right|.$$

- The proposed measure satisfies the property related to the inclusion relation, i.e., the property 4 in Proposition 1.

Example 1. Let $U = \{z_1, ..., z_n\}$. Put

$$X_1 = \langle 0_U, 0.01_U, 1_U, -0.15_U, 0_U, -0.8_U \rangle,$$
$$X_2 = \langle 0.79_U, 0.01_U, 0.61_U, -0.79_U, 0_U, -0.61_U \rangle,$$
$$X_3 = \langle 0.8_U, 0_U, 0.6_U, -0.8_U, 0_U, -0.6_U \rangle.$$

Then, $X_1, X_2, X_3 \in BNS(U)$ and $X_1 \subset X_2 \subset X_3$. By the similarity measure of Vakkas et al. [43] and choosing specific values for the parameters, we have

$$S_V(X_1, X_3) = \frac{1}{2} S_{V1}(X_1, X_3) + \frac{1}{2} S_{V2}(X_1, X_3),$$
$$S_V(X_2, X_3) = \frac{1}{2} S_{V1}(X_2, X_3) + \frac{1}{2} S_{V2}(X_2, X_3),$$

where,

$$S_{V1}(X_1, X_3) = \frac{(0 \times 0.8 + 0.01 \times 0 + 1 \times 0.6) - ((-0.15)(-0.8) + 0 + (-0.8) \times (-0.6))}{2\left[(0^2 + 0.01^2 + 1^2) + (0.8^2 + 0^2 + 0.6^2) - ((-0.15)^2 + 0^2 + (-0.8)^2) - ((-0.8)^2 + 0^2 + (-0.6)^2)\right]} = 0,$$

$$S_{V2}(X_1, X_3) = \frac{(0 \times 0.8 + 0.01 \times 0 + 1 \times 0.6) - ((-0.15)(-0.8) + 0 + (-0.8) \times (-0.6))}{2\left[\sqrt{0^2 + 0.01^2 + 1^2} \times \sqrt{0.8^2 + 0^2 + 0.6^2} - \sqrt{(-0.15)^2 + 0^2 + (-0.8)^2} \times \sqrt{(-0.8)^2 + 0^2 + (-0.6)^2}\right]} = 0,$$

$$S_{V1}\left(X_2,X_3\right)=\frac{\left(0.79\times 0.8+0.01\times 0+0.61\times 0.6\right)-\left(\left(-0.79\right)\times\left(-0.8\right)+0+\left(-0.61\right)\left(-0.6\right)\right)}{2\left[\left(0.79^2+0.01^2+0.61^2\right)+\left(0.8^2+0^2+0.6^2\right)-\left(\left(-0.79\right)^2+0^2+\left(-0.61\right)^2\right)-\left(\left(-0.8\right)^2+0^2+\left(-0.6\right)^2\right)\right]}=0,$$

$$S_{V2}\left(X_2,X_3\right)=\frac{\left(0.79\times 0.8+0.01\times 0+0.61\times 0.6\right)-\left(\left(-0.79\right)\times\left(-0.8\right)+0+\left(-0.61\right)\left(-0.6\right)\right)}{2\left[\sqrt{0.79^2+0.01^2+0.61^2}\times\sqrt{0.8^2+0^2+0.6^2}-\sqrt{\left(-0.79\right)^2+0^2+\left(-0.61\right)^2}\times\sqrt{\left(-0.8\right)^2+0^2+\left(-0.6\right)^2}\right]}=0.$$

The obtained calculation results are $S_V\left(X_1,X_3\right)=0$ and $S_V\left(X_2,X_3\right)=0$.

Now, from Definition 6 and choosing specific values for the parameters, we have

$$d_{H-BN}\left(X_1,X_3\right)=\frac{1}{2}d_{H-BN1}\left(X_1,X_3\right)+\frac{1}{2}d_{H-BN2}\left(X_1,X_3\right),$$

$$d_{H-BN}\left(X_2,X_3\right)=\frac{1}{2}d_{H-BN1}\left(X_2,X_3\right)+\frac{1}{2}d_{H-BN2}\left(X_2,X_3\right),$$

where

$$d_{H-BN1}\left(X_1,X_3\right)=\frac{1}{5}\Big(\left|0-0.8\right|+\left|0.01-0\right|+\left|1-0.6\right| +\left|\max\left\{0,0\right\}-\max\left\{0.01,0.8\right\}\right|+\left|\max\left\{0,0.6\right\}-\max\left\{1,0.8\right\}\right|\Big)=0.482,$$

$$d_{H-BN2}\left(X_1,X_3\right)=\frac{1}{5}\Big(\left|0.15-0.8\right|+\left|0-0\right|+\left|0.8-0.6\right| +\left|\min\left\{-0.15,0\right\}-\min\left\{0,-0.8\right\}\right|+\left|\min\left\{-0.15,-0.6\right\}-\min\left\{-0.8,-0.8\right\}\right|\Big)=0.34,$$

$$d_{H-BN1}\left(X_2,X_3\right)=\frac{1}{5}\Big(\left|0.79-0.8\right|+\left|0.01-0\right|+\left|0.61-0.6\right| +\left|\max\left\{0.79,0\right\}-\max\left\{0.01,0.8\right\}\right|+\left|\max\left\{0.79,0.6\right\}-\max\left\{0.61,0.8\right\}\right|\Big)=0.01,$$

$$d_{H-BN2}\left(X_2,X_3\right)=\frac{1}{5}\Big(\left|0.79-0.8\right|+\left|0-0\right|+\left|0.61-0.6\right| +\left|\min\left\{-0.79,0\right\}-\min\left\{0,-0.8\right\}\right|+\left|\min\left\{-0.79,-0.6\right\}-\min\left\{-0.61,-0.8\right\}\right|\Big)=0.008.$$

Hence,

$$d_{H-BN}\left(X_1,X_3\right)=0.411>d_{H-BN}\left(X_2,X_3\right)=0.009$$
$$\left(s_{H-BN}\left(X_1,X_3\right)=0.589<s_{H-BN}\left(X_2,X_3\right)=0.991\right).$$

In this case, by observation we can also see that X_2 and X_3 are almost the same. In addition, since $X_1\subset X_2\subset X_3$, it can be deduced that the difference between X_1 and X_3 is greater than the that between X_2 and X_3. The proposed distance measure is likely to properly represent this assessment and inference and overcomes the limitation of the proposal of Vakkas et al. [43].

Example 2. Let $U=\left\{z_1,...,z_n\right\}$. Put

$$X_1=<0.4_U,0_U,0.4_U,-0.8_U,0_U,-0.8_U>,$$
$$X_2=<0.5_U,0_U,0.5_U,-0.7_U,0_U,-0.7_U>,$$
$$X_3=<0.4_U,0_U,0.6_U,-0.6_U,0_U,-0.8_U>.$$

Then, $X_1,X_2,X_3\in BNS\left(U\right)$, $X_1\not\subseteq X_2,X_2\not\subseteq X_1$, and $X_3\subset X_2$.

The Hamming distance [44, 45] on $BNS\left(U\right)$ can be defined as follows:

$$d_{Ham}\left(X_1,X_2\right)=\frac{1}{6}\sum_{i=1}^{n}\Big(\left|T_{X_1}^{+}(z_i)-T_{X_2}^{+}(z_i)\right|+\left|I_{X_1}^{+}(z_i)-I_{X_2}^{+}(z_i)\right|+\left|F_{X_1}^{+}(z_i)-F_{X_2}^{+}(z_i)\right|$$
$$+\left|T_{X_1}^{-}(z_i)-T_{X_2}^{-}(z_i)\right|+\left|I_{X_1}^{-}(z_i)-I_{X_2}^{-}(z_i)\right|+\left|F_{X_1}^{-}(z_i)-F_{X_2}^{-}(z_i)\right|\Big).$$

The Euclidean distance [44, 45] on $BNS(U)$ can be defined as follows:

$$d_{Eucl}\left(X_1,X_2\right)=\sum_{i=1}^{n}\Bigg(\frac{1}{6}\Big(\left|T_{X_1}^{+}(z_i)-T_{X_2}^{+}(z_i)\right|^2+\left|I_{X_1}^{+}(z_i)-I_{X_2}^{+}(z_i)\right|^2+\left|F_{X_1}^{+}(z_i)-F_{X_2}^{+}(z_i)\right|^2$$
$$+\left|T_{X_1}^{-}(z_i)-T_{X_2}^{-}(z_i)\right|^2+\left|I_{X_1}^{-}(z_i)-I_{X_2}^{-}(z_i)\right|^2+\left|F_{X_1}^{-}(z_i)-F_{X_2}^{-}(z_i)\right|^2\Big)\Bigg)^{\frac{1}{2}}.$$

Some of the calculation results obtained are as follows:

$$d_{Ham}\left(X_1,X_2\right)=d_{Ham}\left(X_3,X_2\right)=\frac{4}{6},$$

$$d_{Eucl}\left(X_1,X_2\right)=d_{Eucl}\left(X_3,X_2\right)=\frac{\sqrt{6}}{30},$$

$$d_{H-BN}\left(X_1,X_2\right)=0.06<d_{H-BN}\left(X_3,X_2\right)=0.08$$

Clearly, in this case, because of cross-evaluations, the proposed measure can distinguish the difference better than two related measures.

Definition 8. For $E:BNS(U)\to[0,1]$ mapping, if the following conditions are satisfied then E is an entropy measure of BNSs.

1. $E(X)=0$ if and only if X or X^c is a crisp set,
2. $E(X)=E(X^c)$; $E(X)=1$ if and only if $X=X^c$,
3. $E(X_1)\le E(X_2)$ if $X_1 \subseteq X_2$, i.e., if $T_{X_1}^{+}\le T_{X_2}^{+}$, $F_{X_1}^{+}\ge F_{X_2}^{+}$, $T_{X_1}^{-}\ge T_{X_2}^{-}$, $F_{X_1}^{-}\le F_{X_2}^{-}$ for $T_{X_2}^{+}\le F_{X_2}^{+}$, $T_{X_2}^{-}\ge F_{X_2}^{-}$, $I_{X_1}^{+}=I_{X_2}^{+}=0.5_U$, $I_{X_1}^{-}=I_{X_2}^{-}=-0.5_U$; and $T_{X_1}^{+}\ge T_{X_2}^{+}$, $F_{X_1}^{+}\le F_{X_2}^{+}$, $T_{X_1}^{-}\le T_{X_2}^{-}$, $F_{X_1}^{-}\ge F_{X_2}^{-}$ for $T_{X_2}^{+}\ge F_{X_2}^{+}$, $T_{X_2}^{-}\le F_{X_2}^{-}$, $I_{X_1}^{+}=I_{X_2}^{+}=0.5_U$, $I_{X_1}^{-}=I_{X_2}^{-}=-0.5_U$.

Proposition 3. Let $X\in BNS(U)$, where $U=\{z_1,...,z_n\}$, then $s_{H-BN}(X,X^c)$ is an entropy measure of X.

Proof.

1. If X be a crisp set, i.e., $T_X^{+}=1_U, I_X^{+}=F_X^{+}=0_U, T_X^{-}=I_X^{-}=0_U, F_X^{-}=-1_U$, or $T_X^{+}=I_X^{+}=0_U, F_X^{+}=1_U, T_X^{-}=-1_U, I_X^{-}=F_X^{-}=0_U$, then, $s_{H-BN}(X,X^c)=0.$ Similarly, if X^c is a crisp set, then $s_{H-BN}(X,X^c)=0.$ If $s_{H-BN}(X,X^c)=0,$ then it's not hard to show that X or X^c is a crisp set.
2. From Proposition 2, we obtain that $E(X)=E(X^c)$; $s_{H-BN}(X,X^c)=1$ if and only if $X=X^c$.
3. Let $X_1 \subseteq X_2$, assume that $T_{X_1}^{+}\le T_{X_2}^{+}$, $F_{X_1}^{+}\ge F_{X_2}^{+}$, $T_{X_1}^{-}\ge T_{X_2}^{-}$, $F_{X_1}^{-}\le F_{X_2}^{-}$ for $T_{X_2}^{+}\le F_{X_2}^{+}$, $T_{X_2}^{-}\ge F_{X_2}^{-}$, $I_{X_1}^{+}=I_{X_2}^{+}=0.5_U$, $I_{X_1}^{-}=I_{X_2}^{-}=-0.5_U$, then

$$T_{X_1}^{+}\le T_{X_2}^{+}\le F_{X_2}^{+}\le F_{X_1}^{+},$$
$$T_{X_1}^{-}\ge T_{X_2}^{-}\ge F_{X_2}^{-}\ge F_{X_1}^{-},$$

$$\max\left\{T_{X_1}^{+},0.5_U\right\}\le\max\left\{T_{X_2}^{+},0.5_U\right\}\le\max\left\{F_{X_2}^{+},0.5_U\right\}\le\max\left\{F_{X_1}^{+},0.5_U\right\},$$

$$\min\left\{T_{X_1}^{-},-0.5_U\right\}\ge\min\left\{T_{X_2}^{-},-0.5_U\right\}\ge\min\left\{F_{X_2}^{-},-0.5_U\right\}\ge\min\left\{F_{X_1}^{-},-0.5_U\right\},$$

$$d_{H-BN}\left(X_t, X_t^{c}\right) = \lambda \sum_{i=1}^{n} \chi_i^{\Box} \left(\left(\omega_1^{\Box} + \omega_3^{\Box} + \omega_5^{\Box}\right)\left|T_{X_t}^{\Box}(z_i) - F_{X_t}^{\Box}(z_i)\right| + \omega_4^{\Box} \left| \begin{matrix} \max\left\{T_{X_t}^{\Box}(z_i), 0.5\right\} \\ -\max\left\{0.5, F_{X_t}^{\Box}(z_i)\right\} \end{matrix} \right| \right)$$

$$+\left(1-\lambda\right) \sum_{i=1}^{n} \chi_i^{\approx} \left(\left(\omega_1^{\approx} + \omega_3^{\approx} + \omega_5^{\approx}\right)\left|T_{X_t}^{\approx}(z_i) - F_{X_t}^{\approx}(z_i)\right| + \omega_4^{\approx} \left| \begin{matrix} \min\left\{T_{X_t}^{\approx}(z_i), -0.5\right\} \\ -\min\left\{-0.5, F_{X_t}^{\approx}(z_i)\right\} \end{matrix} \right| \right), \ t = 1,2.$$

Therefore, $d_{H-BN}\left(X_1, X_1^{c}\right) \ge d_{H-BN}\left(X_2, X_2^{c}\right)$ and then $s_{H-BN}\left(X_1, X_1^{c}\right) \le s_{H-BN}\left(X_2, X_2^{c}\right)$.

Similarly, the remaining case is proved. $\Box$

4. An application of the H-Max Bipolar Neutrosophic Distance Measure to medical diagnosis

4.1. The H-BN method

A diagnostic problem is stated as follows:

- A medical dataset includes
  - m records of m corresponding patients P_i, $i = 1,2,...,m$,
  - n attributes (symptoms) A_j, $j = 1,2...,n$, of a disease D,
  - k disease classes labeled C_t, $t = 1,2,...,k$, of D.
- The problem is to build a diagnostic system with
  - the inputs are the symptoms of a patient,
  - the output is a disease label.

The proposed method:

Inspired by the diagnostic method introduced in [42] by Ngan et al, the proposed method (H-BN) includes four steps as follows.

- **Step 1.** Built two relation matrices in the bipolar neutrosophic environment:
  - Matrix 1 (M1) presents the relations between the symptoms and patients (P_i and A_j are the i[th] row and the j[th] column of M1, respectively, $i = 1,...,m; j = 1,...,n$),
  - Matrix 2 (M2) shows the relations between the symptoms and the disease or the classification results. Specifically, M2 is a $k \times n$ matrix (C_t is the t[th] row of M2, $t = 1,...,k$).
- **Step 2.** Find the entropies $E\left(A_j\right)$ of the symptoms A_j.
- **Step 3.** Calculate the similarity $s_{H-BN}\left(P_i, C_t\right)$ between the symptoms of P_i and the disease classes C_t, where $E\left(A_j\right)$ is put in the weight of A_j.
- **Step 4.** Diagnose the i[th] patient by finding the highest similarity value $\hat{s}_{H-BN}\left(P_i, C_t\right) = s_{H-BN}\left(P_i, C_{t_0}\right)$, $t_0 \in \lfloor 1, k \rfloor$. The output is t_0.

4.2. Numeric example

In this section, we use the data in the numerical example in [42] on 5 male patients (aged about 30) of Indian Liver Patient Dataset (ILPD) taken from UCI. In the dataset described in Table 1, there are 2 diagnosis labels which are La-I (liver patient) and La-II (non-liver patient). In Table 1, the considered attributes (A_1 - A_7) are Alkaline Phosphotase, Alamine Aminotransferase, Aspartate Aminotransferase, Total Bilirubin, Direct Bilirubin, Albumin, and Albumin and Globulin Ratio.

Table 1. Data of 5 male patients of the ILPD dataset.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	**Class**
P_1	1.3	0.4	482	102	80	3.3	0.9	La-I
P_2	0.8	0.2	198	26	23	4	1	La-II
P_3	0.9	0.2	518	189	17	2.3	0.7	La-I
P_4	3.8	1.5	298	102	630	3.3	0.8	La-I
P_5	0.8	0.2	156	12	15	3.7	1.1	La-II

The steps of the proposed algorithm are implemented as follows:

- **Step 1:** Input data is fuzzified by the following fuzzification functions selected by experts.

$$R^{\prec}_{\alpha,\beta}(z)=\begin{cases}0 & z\le\alpha\\ \dfrac{z-\alpha}{\beta-\alpha} & \alpha<z\le\beta,\\ 1 & \beta<z\end{cases}\qquad L^{\prec}_{\alpha',\beta'}(z)=\begin{cases}1 & z\le\alpha'\\ \dfrac{z-\beta'}{\alpha'-\beta'} & \alpha'<z\le\beta'\\ 0 & \beta'<z\end{cases}$$

$$R^{\succ}_{\rho,\ell}(z)=\begin{cases}-1 & z\le\rho\\ \dfrac{z-\ell}{\ell-\rho} & \rho<z\le\ell,\\ 0 & \ell<z\end{cases}\qquad L^{\succ}_{\rho',\ell'}(z)=\begin{cases}0 & z\le\rho'\\ \dfrac{z-\rho'}{\rho'-\ell'} & \rho'<z\le\ell'\\ -1 & \ell'<z\end{cases}$$

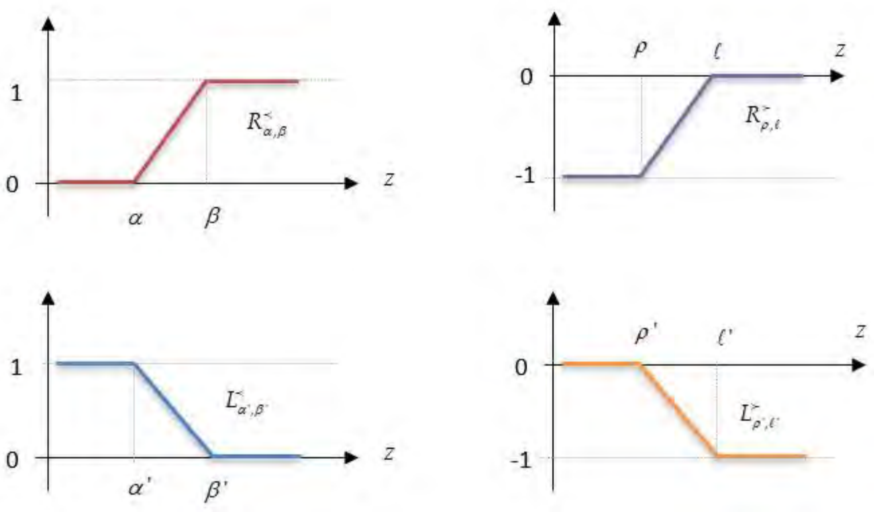

Figure 1. The fuzzification functions are illustrated by graphs.

Specifically, the symptoms on patients are represented as the following BNSs.

$$A_1=<T_1^{\square}(z),I_1^{\square}(z),F_1^{\square}(z),T_1^{\approx}(z),I_1^{\approx}(z),F_1^{\approx}(z)>=$$
$$=<R^{\prec}_{1.2,5.3}(z),L^{\prec}_{0.2,3}(z),L^{\prec}_{0.6,4}(z),R^{\succ}_{0.9,5}(z),L^{\succ}_{0.5,3.5}(z),L^{\succ}_{0.3,4.5}(z)>$$

$$A_2=<T_2^{\square}(z),I_2^{\square}(z),F_2^{\square}(z),T_2^{\approx}(z),I_2^{\approx}(z),F_2^{\approx}(z)>=$$
$$=<R^{\prec}_{0.4,2.3}(z),L^{\prec}_{0.1,1}(z),L^{\prec}_{0.15,1.5}(z),R^{\succ}_{0.2,2}(z),L^{\succ}_{0.2,1.2}(z),L^{\succ}_{0.3,2}(z)>$$

$$A_3 = < T_3^{\square}(z), I_3^{\square}(z), F_3^{\square}(z), T_3^{\approx}(z), I_3^{\approx}(z), F_3^{\approx}(z) >=$$
$$= < R_{140,486}^{\prec}(z), L_{80,250}^{\prec}(z), L_{100,400}^{\prec}(z), R_{110,450}^{\succ}(z), L_{90,300}^{\succ}(z), L_{110,420}^{\succ}(z) >$$

$$A_4 = < T_4^{\square}(z), I_4^{\square}(z), F_4^{\square}(z), T_4^{\approx}(z), I_4^{\approx}(z), F_4^{\approx}(z) > =$$
$$= < R_{33,119}^{\prec}(z), L_{5,60}^{\prec}(z), L_{30,100}^{\prec}(z), R_{25,90}^{\succ}(z), L_{10,70}^{\succ}(z), L_{40,95}^{\succ}(z) >$$

$$A_5 = < T_5^{\square}(z), I_5^{\square}(z), F_5^{\square}(z), T_5^{\approx}(z), I_5^{\approx}(z), F_5^{\approx}(z) > =$$
$$= < R_{33,100}^{\prec}(z), L_{10,90}^{\prec}(z), L_{23,95}^{\prec}(z), R_{33,100}^{\succ}(z), L_{10,90}^{\succ}(z), L_{23,95}^{\succ}(z) >$$

$$A_6 = < T_6^{\square}(z), I_6^{\square}(z), F_6^{\square}(z), T_6^{\approx}(z), I_6^{\approx}(z), F_6^{\approx}(z) > =$$
$$= < L_{2.2,3.5}^{\prec}(z), R_{2,4}^{\prec}(z), R_{3,5}^{\prec}(z), L_{2.3,3.3}^{\succ}(z), R_{2.2,4.1}^{\succ}(z), R_{2.8,5.2}^{\succ}(z) >$$

$$A_7 = < T_7^{\square}(z), I_7^{\square}(z), F_7^{\square}(z), T_7^{\approx}(z), I_7^{\approx}(z), F_7^{\approx}(z) > =$$
$$= < L_{0.5,1}^{\prec}(z), R_{0.3,1.5}^{\prec}(z), R_{0.8,2.5}^{\prec}(z), L_{0.6,1.1}^{\succ}(z), R_{0.2,1}^{\succ}(z), R_{0.7,2.8}^{\succ}(z) >$$

Two bipolar neutrosophic relation matrices M1 and M2 are placed in Tables 2 and 3.

Table 2. The relations between the symptoms and patients are presented.

(M1)	A_1	A_2	A_3	A_4	A_5	A_6	A_7
P_1	<0.02,0.6, 0.8,-0.9, -0.3,-0.2>	<0,0.6, 0.8,-0.9, -0.2,-0.06>	<0.9,0, 0,0, -1,-1>	<0.8,0, 0,0, -1,-1>	<0.7,0.1, 0.2,-0.3, -0.9,-0.8>	<0.1,0.6, 0.1,-1, -0.4,-0.8>	<0.2,0.5, 0.08,-0.6, -0.1,-0.9>
P_2	<0,0.7, 0.9,-1, -0.1,-0.1>	<0,0.8, 0.9,-1, 0,0>	<0.1,0.3, 0.6,-0.7, -0.5,-0.3>	<0,0.6, 1,-1, -0.3,0>	<0,0.8, 1,-1, -0.2,0>	<0,1, 0.5,-1, -0.05,-0.5>	<0,0.5, 0.1,-0.8, 0,-0.85>
P_3	<0,0.7, 0.9,-1, -0.1,-0.1>	<0,0.8, 0.9,-1, 0,0>	<1,0, 0,0, -1,-1>	<1,0, 0,0, -1,-1>	<0,0.9, 1,-1, -0.09,0>	<0.9,0.1, 0,0, -0.9,-1>	<0.6,0.3, 0,-0.2, -0.4,-1>
P_4	<0.6,0, 0.05,-0.3, -1,-0.8>	<0.5,0, 0,-0.3, -1,-0.7>	<0.4,0, 0.3,-0.4, -1,-0.6>	<0.8,0, 0,0, -1,-1>	<1,0, 0,0, -1,-1>	<0.1,0.6, 0.1,-1, -0.4,-0.8>	<0.4,0.4, 0,-0.4, -0.25,-0.95>
P_5	<0,0.7, 0.9,-1, -0.1,-0.1>	<0,0.8, 0.9,-1, 0,0>	<0.04,0.5, 0.8,-0.9, -0.3,-0.1>	<0,0.8, 1,-1, -0.03,0>	<0,0.9, 1,-1, -0.06,0>	<0,0.8, 0.3,-1, -0.2,-0.6>	<0,0.6, 0.2,-1, 0,-0.8>

Table 3. The relations between the symptoms and the classification results are shown.

(M2)	A_1	A_2	A_3	A_4	A_5	A_6	A_7
La-I	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>	<1,0,0, 0,-1,-1>
La-II	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>	<0,1,1, -1,0,0>

- **Step 2:** Finding the entropies $E\left(A_j\right)=s_{H-BN}(A_j,A_j^c)=1-d_{H-BN}(A_j,A_j^c)$ with $\chi_i^{\square}=\chi_i^{\approx}=\omega_j^{\square}=\omega_j^{\approx}=\dfrac{1}{5}$ $(i,j=1,\ldots,5)$ and $\lambda=\dfrac{1}{2}$:

$$E\left(A_1\right)=0.27,\quad E\left(A_2\right)=0.2,\quad E\left(A_3\right)=0.33,\quad E\left(A_4\right)=0.08,$$

$$E\left(A_5\right)=0.13,\quad E\left(A_6\right)=0.55,\quad E\left(A_7\right)=0.68.$$

- **Step 3:** Calculating the similarities $S(i-I)=s_{H-BN}\left(P_i,(\text{La - I})\right)$ and $S\left(i-II\right)=s_{H-BN}\left(P_i,(\text{La - II})\right)$ with $\omega_j^{\square}=\omega_j^{\approx}=\dfrac{1}{5}$ $(i,j=1,\ldots,5)$, $\lambda=\dfrac{1}{2}$, and $\chi_j^{\square}=\chi_j^{\approx}=\dfrac{E\left(A_j\right)}{\sum_{j=1}^{5}E\left(A_j\right)}$. The obtained results include: $\chi_1^{\square}=0.12, \chi_2^{\square}=0.09, \chi_3^{\square}=0.15, \chi_4^{\square}=0.035, \chi_5^{\square}=0.06, \chi_6^{\square}=0.245, \chi_7^{\square}=0.3,$

$$S(1-I)=0.49>S(1-II)=0.475,\quad S(2-I)=0.2<S(2-II)=0.75,$$
$$S(3-I)=0.642>S(3-II)=0.327,\quad S(4-I)=0.63>S(4-II)=0.33,$$
$$S(5-I)=0.186<S(5-II)=0.788.$$

- **Step 4.** The outputs are decided as follows: The outputs of P_1,P_2,P_3,P_4, and P_5 are La-I, La-II, La-I, La-I, and La-II, respectively. These decisions and the last column of Table 1 are the same.

4.3. Experiment

In this part, we test the proposed method on the ILPD dataset on Matlab programming with the evaluation criteria on accuracy is Mean Absolute Error (MAE) and the speed of the algorithms is measured in seconds (sec). Also on this data, Ngan et al. [8] tested 14 other diagnostic methods, denoted by M_{SK1-1}, M_{SK1-2}, M_{SK1-3}, M_{SK1-4}, M_{SK2}, M_{WX}, M_{VS}, M_{ZJ}, M_W, M_J, M_{SA}, $M_{H-\max}$, M_{C-QDM}, and M_{P-QDM}, based on the considered intuitionistic fuzzy distance measures (see Table 4).

Table 4. MAEs and Sec of the considered methods on the ILPD dataset.

Methods	MAE	Sec
M_{SK1-1}	0.3195	0.6177
M_{SK1-2}	0.3158	0.4427
M_{SK1-3}	0.3316	0.4827
M_{SK1-4}	0.2918	0.4602
M_{SK2}	0.2902	0.6527
M_{WX}	0.3227	0.4427
M_{VS}	0.2893	0.5552
M_{ZJ}	0.3096	0.5602
M_W	0.2915	0.8452
M_J	0.289	1.2077
M_{SA},	0.3031	0.8102
$M_{H-\max}$	0.2848	0.51
M_{C-QDM}	0.2836	0.155
M_{P-QDM}	0.2831	0.469

H-BN	0.2729	0.559770

In Table 4, it can be observed that the MAE value of the proposed method (H-BN), which is 0.2729, is less (better) than those of the other algorithms on the ILPD datasets. Figure 2 shows the MAE values of the considered methods on the ILPD dataset, where the heights of the vertical bars present the MAE values of the corresponding algorithms. The heights of the H-BN method (green bars) are lower than those of the remaining bars, that means, it is the best algorithm in terms of accuracy of the considered algorithms on the ILPD dataset. We note that the computation time of our algorithms is very close to the computation time of the other methods.

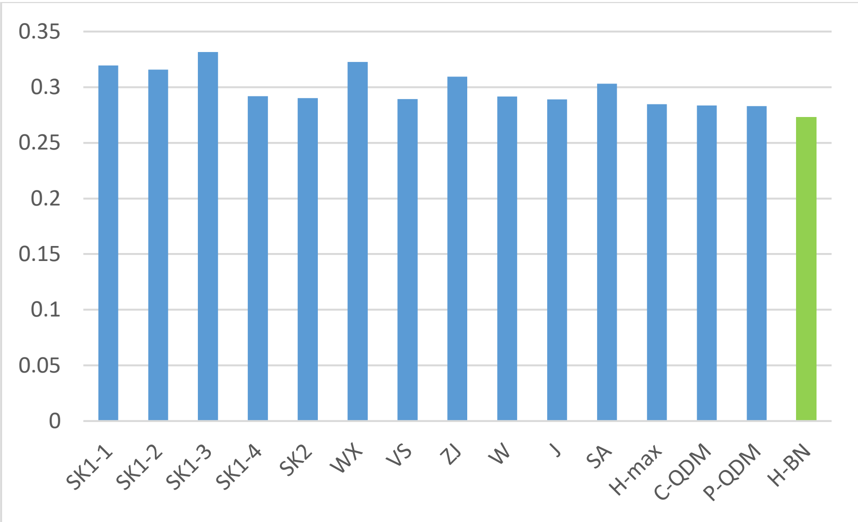

Figure 2. MAEs of the considered methods on the ILPD dataset.

5. Conclusions

In this paper, based on the H-max distance measure on IFSs and SVNSs, a new distance measure on BNSs is introduced to overcome the limitations of the related measures by including cross-evaluations and satisfying the condition of inference of a distance measure. Furthermore, a bipolar neutrosophic entropy measure and its basic properties are presented and proven. In addition, an application to medical diagnosis is shown to illustrate the effective applicability of the measures. There, the proposed diagnostic method called H-BN, a numerical example and real experiment are clarified in detail. In the future, we will test the proposed diagnostic method on other real datasets taken from UCI. Furthermore, we will develop the distance measure for interval-valued bipolar neutrosophic sets.

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Appendix

Source code and dataset of this paper can be found at this link:

https://sourceforge.net/projects/hbn-datasets-code/.

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Neutrosophic Quadratic Residues and Non-Residues

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Abstract: In this paper, we present the Neutrosophic quadratic residues and nonresidues with their basic interpretation as graphs in an algebraic manner and analog to the algebraic graphs. We establish the Neutrosophic, number-theoretic, and graph-theoretic properties of the set of Neutrosophic quadratic residues and nonresidues, many of which mirror those of the classical quadratic residues and nonresidues of modulo an odd prime. These properties, especially the algebraic ones, are connected to algebraic graphs, and thus we conclude the paper by studying the structural properties of Neutrosophic quadratic residue and quadratic nonresidue graphs.

Keywords: Quadratic residues; Quadratic nonresidues; Neutrosophic quadratic residues; Neutrosophic quadratic nonresidues; Neutrosophic quadratic residue graph; Neutrosophic quadratic nonresidue graph.

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1. Introduction

For any positive integer $n \geq 1$, the set $Z_n = \{0, 1, 2, \dots, n-1\}$ is a ring under the usual addition and multiplication modulo n. Moreover, for any prime p, the ring Z_p is a field of order p and hence $Z_p^* = \{1, 2, 3, \dots, p-1\}$ is a group under multiplication modulo p, see [1-2].

For $a \in Z_p^*$, a is a quadratic residue modulo p if and only if $a = x^2$ for some $x \in Z_p^*$. Now suppose Q_p denote the set of all quadratic residues modulo p. Then Q_p is a nonempty subset of Z_p^*, given by $Q_p = \{x^2 \in Z_p^* : x = 1, 2, \dots, \frac{p-1}{2}\}$. It is clear that for any $a, b \in Q_p$, there exists x and y in Z_p^* such that $a^{-1}b = (x^{-1}y)^2 \in Q_p$. Therefore, Q_p is a subgroup of Z_p^* and also the index $[Z_p^* : Q_p] = 2$. This implies that $xy \in Q_p$ if and only if x and y are both in Q_p or neither of them is in Q_p. This specifies that an element in Z_p^* as a residue or nonresidue according to whether or not it is a quadratic residue modulo p. In particular, the set of all quadratic nonresidues modulo p in Z_p^* is denoted by $\overline{Q_p}$. Hence $|Q_p| = |\overline{Q_p}| = \frac{p-1}{2}$. So, Q_p is the normal subgroup and $\overline{Q_p}$ is the only nonempty subset of Z_p^* whose orders are equal. For more information about Q_p and $\overline{Q_p}$, reader refer [3].

Much of the specific power and utility of modern mathematics arises from its abstraction of important features similar to various circumstances and illustrations. But many sets and systems we encounter have a usual addition and multiplication defined on their elements. These operations often satisfy a few common properties that we want to isolate and study. Besides the obvious illustrations in different number systems and algebraic systems, we can operate polynomials, functions, matrices, etc. Studying the algebraic structure of groups, rings, and fields based on number theoretic and combinatorial properties has caught the interest of many researchers order the last decades. Recently, algebraic systems associated with neutrosophic elements and sets [4] seem to be more interesting and active area compare to those associated with classical algebraic structures. For instance, the Neutrosophic set $N(Z_p^*, I)$ is generated by the multiplicative group Z_p^* and the neutrosophic unit element I ($I^2 = I$ and I^{-1} does not exist), that is, $N(Z_p^*, I)$ or equivalently $N(Z_p^*) = < Z_p^*, I >= Z_p^* \cup Z_p^* I$, where p is prime. This is a Neutrosophic group [5] concerning Neutrosophic multiplication $(aI)(bI) = abI$ for every $aI, bI \in N(Z_p^*, I)$.

The concept of the Neutrosophic graph of Neutrosophic structures was first introduced by Vasanth Kandasamy and Smarandache [6], but this work was mostly concerned with the basic properties of Neutrosophic algebraic structures. Recently, the authors Chalapathi and Kiran studied the Neutrosophic graphs [5] of finite groups. The Neutrosophic graph of a finite group G, which is denoted by $Ne(G, I)$, is an undirected simple graph whose vertices are elements of the neutrosophic group $N(G)$ with two distinct vertices x and y which are adjacent if and only if either $xy = x$ or $xy = y$.

In 1879, author Cayley considered the Cayley graph for finite groups. After that, a lot of research has been done on various families of Cayley graphs. For instance, Quadratic residue Cayley graphs [7], Quartic residue Cayley graphs [8]. Many researchers exist in the literature on Cayley graphs quadratic residues on odd prime and prime power modules. The authors studied quadratic residues modulo 2^n Cayley graphs in [9]. In this paper, we will focus on Neutrosophic quadratic residues and their corresponding algebraic graphs, which are not Cayley graphs.

2. Neutrosophic Quadratic and Non Quadratic Residues

In this section, for convenience and also for later use, we define some definitions and notations concerning integers modulo an odd prime p , and Neutrosophic quadratic and nonquadratic residue modulo p.

First, we recall some results about neutrosophic groups from [5].

Theorem 2.1:

1. $Z_p^* I = \{aI : a \in Z_p^*\}$
2. $N(Z_p^*) = Z_p^* \cup Z_p^* I$, where $Z_p^* \cap Z_p^* I = \emptyset$

Theorem 2.2: Let Z_p^* be a finite group with respect to multiplication modulo n. Then

1. $|Z_p^*| = p - 1$ and $|Z_p^* I| = p - 1$
2. $|N(Z_p^*)| = 2(p - 1)$

Let $aI \in N(Q_p)$. Then aI is a neutrosophic quadratic residue modulo p if and only if $aI = (xI)^2$ for some $xI \in Z_p^* I$. Now suppose $N(Q_p)$ denote the set of all neutrosophic quadratic residues modulo p . Then $Q_p I$ is a nonempty subset of $N(Z_p^*)$ given by $Q_p I = \{(xI)^2 \in N(Z_p^*) : x = 1, 2, \dots, \frac{p-1}{2}\}$.

Further, if for any $aI, bI \in Q_p I$, then $aI = (xI)^2$ and $bI = (yI)^2$ for some $xI, yI \in Z_p^* I$, so $(aI)(bI) = (xyI)^2 \in Q_p I$

Hence $Q_p I$ is a neutrosophic subgroup of $N(Z_p^*) = Z_p^* \cup Z_p^* I$ with neutrosophic index, by the Theorem 2.1.

$$[N(Z_p^*) : Q_p I] = \frac{|N(Z_p^*)|}{|Q_p I|} = \frac{2(p-1)}{\left(\frac{p-1}{2}\right)} = 4.$$

Similarly, the set of all neutrosophic quadratic non-residues modulo p in $Z_p^* I$ is denoted by $\overline{Q_p I}$ with $|Q_p I| = |\overline{Q_p I}| = \frac{p-1}{2}$.

Example 2.3: The following shortlist shows that the Neutrosophic quadratic and nonquadratic residues modulo 3, 5, 7, respectively.

$N(Q_3,\ I) = \{1,\ I\}$, $N(\overline{Q_3},\ I) = \{2,\ 2I\}$,

$N(Q_5,\ I) = \{1,\ 4,\ I,\ 4I\}$, $N(\overline{Q_5},\ I) = \{2,\ 3,\ 2I,\ 3I\}$,

$N(Q_7,\ I) = \{1, 3, 4, 5, 9,\ I, 3I, 4I, 5I, 9I\}$, $N(\overline{Q_7},\ I) = \{2, 6, 7, 8, 10, 2I, 6I, 7I, 8I, 10I\}$.

From the above example, we observe the following:

$N(Q_p,\ I) = Q_p \cup Q_p I$ and $N(\overline{Q_p},\ I) = \overline{Q_p} \cup \overline{Q_p I}$. In particular,

$|N(Q_p,\ I)| = |Q_p| + |Q_p I| = \frac{p-1}{2} + \frac{p-1}{2} = p - 1$ and

$|N(\overline{Q_p},\ I)| = |\overline{Q_p}| + |\overline{Q_p I}| = \frac{p-1}{2} + \frac{p-1}{2} = p - 1$.

Theorem 2.4: Given $p > 2$, $N(W_p^*, I) = W_p^* \cup W_p^* I$, is the neutrosophic prime subgroup of $N(Z_p^*, I)$, where $W_p^* = \{1, p-1\}$.

Proof: It is clear from the well-known result that W_p^* is a subgroup of the group Z_p^*, because $(p-1)^2 \equiv 1\ (mod\ p)$.

Theorem 2.5: Fundamental Theorem of Neutrosophic Quadratic Residues Modulo p

For each $p > 2$, we have the neutrosophic quotient group $\frac{N(Z_p^*,\ I)}{N(W_p^*,\ I)}$ is isomorphic to the neutrosophic group $N(Q_p, I)$.

Proof: For any $p > 2$, we have $(p-1)^2 \equiv 1\ (mod\ p)$ and $((p-1)I)^2 \equiv I\ (mod\ p)$. Therefore, $N(W_p^*,\ I) = \{1,\ p-1,\ I,\ (p-1)I\}$ is a neutrosophic subgroup of $N(Z_p^*,\ I)$. So, there exists a Neutrosophic quotient group $\frac{N(Z_p^*,\ I)}{N(W_p^*,\ I)}$. Now we claim that $\frac{N(Z_p^*,\ I)}{N(W_p^*,\ I)} \cong N(Q_p, I)$. For this, we define a map $\Psi: N(Z_p^*,\ I) \to N(Q_p, I)$ by the relation

$$\Psi(x) = \begin{cases} x^2, & if\ x \in Z_p^* \\ (xI)^2, & if\ xI \in Z_p^* I \end{cases}$$

Clearly, Ψ is a well-defined group and Neutrosophic group homomorphism, because $(ab)^2 = a^2 b^2,\ \forall\ a,\ b \in Z_p^*$ and $((aI)(bI))^2 = (aI)^2 (bI)^2,\ \forall\ aI,\ bI \in Z_p^* I$.

Now to find a kernel of Ψ. If $x \in Z_p^*$, then by the definition of kernel of group (classical) homomorphism,

$$K = \{x \in Z_p^* : x^2 = 1\}$$

$$= \{1, -1\}$$

$$= \{1,\ p-1\}.$$

Similarly, if $xI \in Z_p^* I$, then by the definition of a kernel of a Neutrosophic group homomorphism,

$$K' = \{xI \in Z_p^* I : (xI)^2 = I\}$$

$$= \{I, -I\}$$

$$= \{I, (p-1)I\}.$$

Hence, $Ker\ \Psi = K \cup K'$

$$= \{1,\ p-1,\ I, (p-1)I\}$$

$$= N(W_p^*,\ I).$$

Finally, to find image of Ψ.

$$Im\,(\Psi) = \{\Psi(x) \in N(Z_p^*,\ I) : x \in N(Z_p^*,\ I)\}$$

$$= \{x^2 \in Z_p^* : x \in Z_p^*\} \cup \{(xI)^2 \in Z_p^* I : xI \in Z_p^* I\}$$

$$= Q_p \cup Q_p I$$

$$= N(Q_p,\ I).$$

By the fundamental theorem of a Neutrosophic group homomorphism, $\frac{N(Z_p^*, I)}{Ker\ \Psi} \cong Im(\Psi)$. This shows that $\frac{N(Z_p^*,\ I)}{N(W_p^*,\ I)} \cong N(Q_p,\ I)$.

Remark 2.6: $x \in N(Z_p^*,\ I)$ is a Neutrosophic quadratic residue if and only if $x \in Im\,(\Psi)$, otherwise, it is called neutrosophic quadratic residue modulo p.

Example 2.7: For the prime $p = 5$, we have $Z_5^* = \{1, 2, 3, 4\}$, $N(Z_5^*, I) = \{1, 2, 3, 4, I, 2I, 3I, 4I\}$, $W_5^* = \{1, 4\}$, $N(W_5^*, I) = \{1, 4, I, 4I\}$, $\frac{N(Z_5^*, I)}{N(W_5^*, I)} = \{N(W_5^*, I), 2N(W_5^*, I), 3N(W_5^*, I), 4N(W_5^*, I), I\,N(W_5^*, I), 2I\,N(W_5^*, I), 3I\,N(W_5^*, I), 4I\,N(W_5^*, I)\}$.

Theorem 2.8: The neutrosophic product of two neutrosophic quadratic residues is again a neutrosophic a quadratic residue modulo p. Similarly, the neutrosophic product of two Neutrosophic quadratic nonresidues is a Neutrosophic quadratic residue modulo p.

Proof: Since $N(Q_p, I)$ is a Neutrosophic normal subgroup of the Neutrosophic group $N(Z_p^*, I)$ whose index is 4. So there exists a Neutrosophic quotient group $\frac{N(Z_p^*, I)}{N(Q_p, I)}$ such that $\left|\frac{N(Z_p^*, I)}{N(Q_p, I)}\right| = 4$, that is $\frac{N(Z_p^*, I)}{N(Q_p, I)} = \{x\,N(Q_p, I) : x \in N(Z_p^*, I)\}$.

Let $x \in Z_p^*$ such that $x \in Q_p$. Then $(x\,Q_p)^2 = {Q_p}^2 = Q_p$, since $hH = Hh = H$. Let $a \in Z_p^*$ such that $a \notin Q_p$. Then $(a\,Q_p)^2 \neq Q_p$.

Let $x \in Z_p^* I$ such that $x \in Q_p\,I$. Then $(x\,Q_p I)^2 = (Q_p I)^2 = {Q_p}^2\,I^2 = Q_p I$. Let $a \in Z_p^* I$ such that $a \notin Q_p I$. Then $(a\,Q_p I)^2 \neq Q_p I$.

Because $N(Z_p^*, I) = Z_p^* \cup Z_p^* I$ and $N(Q_p, I) = Q_p \cup Q_p I$, we know that the neutrosophic quotient group defined as $\frac{N(Z_p^*, I)}{N(Q_p, I)} = \{Q_p, aQ_p, IQ_p, a\,IQ_p\}$.

(1) If $x, y \in Q_p I$, then

$$xy\,Q_p I = (x\,Q_p I)(y\,Q_p I)$$

$$= (Q_p I)(Q_p I)$$

$$= (Q_p I)^2$$

$$= {Q_p}^2 I^2$$

$= Q_p I$, since $Q_p{}^2 = Q_p$,

and thus $xy \in Q_p I$.

(2) If $x, y \notin Q_p I$, then $x,\ y \in \overline{Q_p}$. So there exists $\bar{a}, \bar{b} \in \overline{Q_p}$ such that $x = \bar{a}\, I$ and $y = \bar{b}\, I$. Then

$$xy\, Q_p I = (\bar{a}\, I)(\bar{b}\, I)\, Q_p$$

$$= (\bar{a}\, \bar{b}) I\, Q_p$$

$$= I\left((\bar{a}\, \bar{b})\, Q_p\right)$$

$$= I\, Q_p, \text{ since } \bar{a}, \bar{b} \in \overline{Q_p} \Rightarrow \bar{a}\bar{b} \in Q_p \text{ and } \bar{a}\bar{b} Q_p = Q_p.$$

Hence $xy \in Q_p I$.

(3) If $x \in Q_p I$ and $y \notin Q_p I$, then

$$xy\, Q_p I = (x\, Q_p I)(y\, Q_p I)$$

$$= (Q_p I)(y\, Q_p I), \text{ since } x \in Q_p I \Leftrightarrow x\, Q_p I = Q_p I$$

$$= y(Q_p I)^2$$

$$= y\, Q_p{}^2 I^2$$

$$= y\, Q_p I$$

$$\neq Q_p I, \text{ since } y \notin Q_p I \text{ iff } y\, Q_p I \neq Q_p I.$$

Hence $xy \notin Q_p I$. This proves the theorem.

Now, let us start with simple undirected graphs of neutrosophic quadratic residue and Neutrosophic quadratic Nonresidue graphs of the Neutrosophic graph $N(Z_p^*, I)$ whose vertices are members in the Neutrosophic graph $N(Z_p^*, I)$ where p is an odd prime.

3. Neutrosophic Quadratic Residue Graphs

Structurally, many real-world concepts, aspects, and situations can be described by using and applying diagrams of a set of vertices with edges joining pairs of these vertices. So, a mathematical abstraction of this type of diagram gives rise to the concept of a graph. A graph G and is denoted by

$G = (V, E)$, where $V = V(G)$ and $E = E(G)$ vertex and edge sets of G, respectively. A graph G is said to be connected if there is at least one path between every two vertices in G and disconnected if G has at least one pair of vertices between which there is no path. Every graph G consists of one or more connected graphs as subgraphs, and each such connected subgraph of G is called a component of G , and each component of G is denoted by $Comp(G)$. It is clear that every connected graph contains only one component and every disconnected graph of more than one vertex contains two or more components. Now a graph G is said to be complete if every vertex in G is connected to another vertex in G.

A complete graph of order n is denoted by K_n and it has exactly $\frac{n(n-1)}{2} = n_{C_2}$ edges, and it is called the size of K_n. If u is a vertex of G, then the number of edges incident on a vertex u is called the degree of u and it is denoted by $\deg(u)$. In particular, if $\deg(u) = k$ for every vertex u in G, then G is called a $k-$regular graph. A graph G is said to be bipartite if its vertex set V can be partitioned into two non-empty disjoint subsets V_1 and V_2 such that each edge of G connects a vertex of V_1 to a vertex of V_2, and the pair (V_1, V_2) is called bipartite of G. Similarly, G is called a complete bipartite graph if each vertex of V_1 is adjacent to each vertex of V_2. Now, consider two graphs $G = (V, E)$ and $G' = (V', E')$, then G and G' are isomorphic to each other and it is denoted by $G \cong G'$ if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved, see [10].

Definition 3.1: An undirected simple graph $G(Z_p^*, Q_p, I)$ is called a Neutrosophic quadratic residue graph of the Neutrosophic group $N(Z_p^*, I)$ whose vertex set is $N(Z_p^*, I)$ and two distinct vertices x and y are adjacent in $G(Z_p^*, Q_p, I)$ if and only if $xy \in N(Q_p, I)$.

Before studying the properties of neutrosophic quadratic residue graphs, we give two examples to illustrate their usefulness.

Example 3.2: Since $N(Z_5^*, I) = \{1,\ 2,\ 3,\ 4,\ I,\ 2I,\ 3I,\ 4I\}$ is the vertex set of the graph $G(Z_5^*,\ Q_5,\ I)$, where $N(Q_5,\ I) = \{1, 4,\ I,\ 4I\}$.

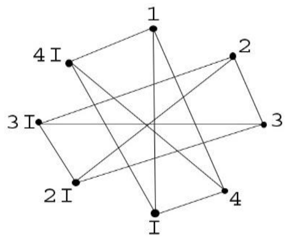


Figure 1. Neutrosophic Quadratic Residue Graph $G(Z_5^*,\ Q_5,\ I)$ of modulo 5.

Example 3.3: For $p = 7$, we have $N(Z_7^*, I) = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ I,\ 2I,\ 3I,\ 4I,\ 5I,\ 6I\}$ and $N(Q_7,\ I) = \{1,\ 2,\ 4,\ I,\ 2I,\ 4I\}$. Then the graph $G(Z_7^*,\ Q_7,\ I)$ is represented as follows.

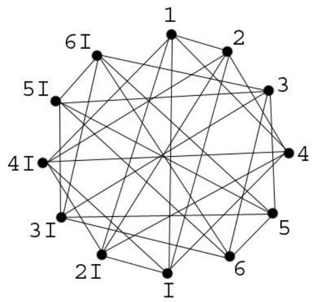


Figure 2. Neutrosophic Quadratic Residue Graph $G(Z_7^*,\ Q_7,\ I)$ of modulo 7.

In this section, the basic properties of $G(Z_p^*,\ Q_p,\ I)$ being studied. We begin with the disconnectedness of the graph $G(Z_p^*,\ Q_p,\ I)$.

Theorem 3.4: For $p > 2$, the graph $G(Z_p^*,\ Q_p,\ I)$ is disconnected. In particular, graph $G(Z_p^*,\ Q_p,\ I)$ is the disjoint union of two complete components.

Proof: Let $p > 2$ be an odd prime. Then the vertex set of neutrosophic quadratic residue graph $G(Z_p^*,\ Q_p,\ I)$ is $N(Z_p^*,\ I)$. But

$$N(Z_p^*,\ I) = N(Q_p,\ I) \cup N(\overline{Q_p},\ I)$$

$$= (Q_p \cup Q_p I) \cup (\overline{Q_p} \cup \overline{Q_p} I)$$

where $(Q_p \cup Q_p I) \cap (\overline{Q_p} \cup \overline{Q_p} I) = \emptyset$. This gives us that the vertex set $N(Z_p^*,\ I)$ is partitioned into two disjoint unions of $(Q_p \cup Q_p I)$ and $(\overline{Q_p} \cup \overline{Q_p} I)$. So, because of Theorem 2.8, we clear that $G(Z_p^*, Q_p, I)$ is disconnected. Now consider the following three cases.

Case 1: Suppose $x, y \in N(Q_p, I)$. Then obviously $xy \in N(Q_p, I)$. This implies that there exists an edge between any two vertices x and y in the graph $G(Z_p^*, Q_p, I)$. Thus, $G(Z_p^*, Q_p, I)$ has a complete subgraph, say $Comp(Z_p^*, Q_p, I)$ whose vertex set is $N(Q_p, I)$.

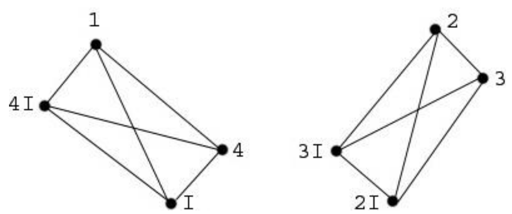
Case 2: Suppose $x, y \in N(\overline{Q_p}, I)$. Then again by Theorem 2.8, $xy \in N(\overline{Q_p}, I)$. So, in this case also there exists an edge between every two vertices x and y in the graph $G(Z_p^*, Q_p, I)$. Thus, the graph, $G(Z_p^*, Q_p, I)$ has another complete subgraph, say $Comp(Z_p^*, \overline{Q_p}, I)$ whose vertex set is $N(\overline{Q_p}, I)$.

Case 3: Suppose $x \in N(Q_p, I)$ and $y \in N(\overline{Q_p}, I)$. Then $xy \notin N(Q_p, I)$. It gives us that there is no edge between x and y when $x \in N(Q_p, I)$ and $y \in N(\overline{Q_p}, I)$.

From the above three cases, we conclude that $Comp(Z_p^*, Q_p, I)$ and $Comp(Z_p^*, \overline{Q_p}, I)$ are two disjoint complete components of the graph $G(Z_p^*, Q_p, I)$ such that

$$G(Z_p^*, Q_p, I) = Comp(Z_p^*, Q_p, I) \cup Comp(Z_p^*, \overline{Q_p}, I).$$

Example 3.5: Two components of the graph $G(Z_5^*, Q_5, I)$ as shown below.



$Comp(Z_5^*, Q_5, I)$ $Comp(Z_5^*, \overline{Q_5}, I)$

Figure 3. Components of the graph $G(Z_5^*, Q_5, I)$.

For each odd prime p, the structure of $G(Z_p^*, Q_p, I)$ is easy to describe, because it contains the following properties:

1. $G(Z_p^*, Q_p, I)$ contains two disjoint connected components for each $p > 2$.

2. Each component of $G(Z_p^*, Q_p, I)$ contains even and odd cycles for $p \geq 5$.

3. Each component of $G(Z_p^*, Q_p, I)$ is not a bipartite graph for $p \geq 3$.

The next result gives useful and important properties of the components of the graph $G(Z_p^*, Q_p, I)$ when $p > 2$.

Theorem 3.6: For each prime $p > 2$, $Comp\ (Z_p^*, Q_p, I) \cong Comp\ (Z_p^*, \overline{Q_p}, I)$.

Proof: For each prime $p > 2$, the Neutrosophic quadratic residue and non-residue sets of $N(Z_p^*, I)$ are given by $N(Q_p, I) = Q_p \cup Q_p I$ and $N(\overline{Q_p}, I) = \overline{Q_p} \cup \overline{Q_p} I$.

These are the vertex sets of the components $Comp\ (Z_p^*, Q_p, I)$ and $Comp\ (Z_p^*, \overline{Q_p}, I)$, respectively. Also, we have $|N(Q_p, I)| = \frac{p-1}{2} + \frac{p-1}{2} = p - 1 = |N(\overline{Q_p}, I)|$. Now to prove that $Comp\ (Z_p^*, Q_p, I)$ and $Comp\ (Z_p^*, \overline{Q_p}, I)$ are isomorphic as groups. For this, we define a function $f: N(Q_p, I) \to N(\overline{Q_p}, I)$ by the relation $f(u) = v$ for every $u \in N(Q_p, I)$ and $v \in N(\overline{Q_p}, I)$. Because of $|N(Q_p, I)| = p - 1$ and $|N(\overline{Q_p}, I)| = p - 1$, the map f is a one-to-one correspondence.

Now, suppose $\bar{e}$ be an edge with end vertices v and v' in the component $Comp\ (Z_p^*, \overline{Q_p}, I)$. Then $\bar{e} = (v, v') \Leftrightarrow \bar{e} = (f(u), f(u'))$

$$\Leftrightarrow \bar{e} = f(u, u')$$

$$\Leftrightarrow \bar{e} = f(e),$$

where $e = (u, u')$ be an edge with end vertices u and u' in $Comp\ (Z_p^*, Q_p, I)$. This shows that there is a one-to-one correspondence between their vertices and their edges such that the incidence relationship is preserved. Hence, $Comp\ (Z_p^*, Q_p, I) \cong Comp\ (Z_p^*, \overline{Q_p}, I)$.

The following example illustrates the procedure of the above theorem 3.6 clearly.

Example 3.7: Since $N(Q_5, I) = \{1, 4, I, 4I\}$ and $N(\overline{Q_5}, I) = \{2, 3, 2I, 3I\}$. Using the map $f: N(Q_5, I) \to N(\overline{Q_5}, I)$ as above, write the equations $f(1) = 2$, $f(4) = 3$, $f(I) = f(2I)$ and

$f(4I) = f(3I)$. These equations show that f is a one-to-one correspondence between the graph components $Comp\ (Z_5^*, Q_5, I)$ and $Comp\ (Z_5^*, \overline{Q_5}, I)$, and thus which are isomorphic as graphs.

This special case of the above theorem when $p > 2$ occurs frequently and so we isolate it as a corollary.

Corollary 3.8: Each component of the neutrosophic quadratic residue graph is isomorphic to the complete graph K_{p-1}.

Proof: Due to Theorem 3.6, the only possibility of the graph $Comp\ (Z_p^*, Q_p, I)$ is $Comp\ (Z_p^*, Q_p, I) \cong Comp\ (Z_p^*, \overline{Q_p}, I)$. Therefore, the order and size of each component are $p-1$ and $\binom{p-1}{2}$, respectively, and thus each component of the graph $G(Z_p^*, Q_p, I)$ is isomorphic to the complete graph K_{p-1}.

Example 3.9: $Comp\ (Z_5^*, Q_5, I) \cong K_4$ and $Comp\ (Z_7^*, Q_7, I) \cong K_6$.

The integer p is prime if and only if $p = 2$ or $p \equiv 3(mod\ 4)$ $p \equiv 1(mod\ 4)$. But, this paper p will denote odd prime integer such that either $p \equiv 1(mod\ 4)$ or $p \equiv 3(mod\ 4)$. These prime integers are weapons for verifying two components of the graph $G(Z_p^*, Q_p, I)$ are Eulerian or not. It is now the time for determining the cases in which the components of the graph $G(Z_p^*, Q_p, I)$ are Eulerian, but first, we recall the following well-known result.

Theorem 3.10 [10]: A connected graph G is Eulerian if and only if the degree of each vertex of G is even.

For $p \equiv 1\ (mod\ 4)$ or $p \equiv 3\ (mod\ 4)$, the following theorems show that $G(Z_p^*, Q_p, I)$ could not be Eulerian.

Theorem 3.11: If $p \equiv 1\ (mod\ 4)$ or $p \equiv 3\ (mod\ 4)$, then each component of $G(Z_p^*, Q_p, I)$ is not Eulerian.

Proof: Suppose on contrary that each component of $G(Z_p^*, Q_p, I)$ is Eulerian, which implies that the degree of each vertex is even. By Theorem 3.6, it is clear that

$$Comp\ (Z_p^*, Q_p, I) \cong Comp\ (Z_p^*, \overline{Q_p}, I) \cong K_{p-1}.$$

So, for every vertex x in $G(Z_p^*, Q_p, I)$, we have

$$\deg(x) = (p-1) - 1 = p - 2.$$

$\deg(x) = (4q+1) - 2 = 4q - 1$, which is odd. Similarly, we can show that $\deg(x) = (4q+3) - 2 = 4q + 1$, which is also odd. Hence, we found that the degree of each vertex in the graph $G(Z_p^*, Q_p, I)$ can not be even. This contraposition shows that each component of $G(Z_p^*, Q_p, I)$ is never Eulerian when $p \equiv 1(mod\ 4)$ or $p \equiv 3(mod\ 4)$.

4. Neutrosophic Quadratic Nonresidue Graphs

In this section, we establish a complement graph of the neutrosophic quadratic residue graph $G(Z_p^*, Q_p, I)$, which is denoted by $\bar{G}(Z_p^*, \overline{Q_p}, I)$ and it is called a Neutrosophic quadratic nonresidue graph whose vertex set is the Neutrosophic group $N(Z_p^*, I)$ and edge set is $E(\bar{G}(Z_p^*, \overline{Q_p}, I)) = \{(x, y) : x, y \in N(Z_p^*, I)\ and\ xy \in N(\overline{Q_p}, I)\}$.

Example 4.1: Since $N(Z_3^*, I) = \{1, 2, I, 2I\}$ and $N(\overline{Q_3}, I) = \{2, 2I\}$. The Neutrosophic quadratic nonresidue graph of $N(Z_3^*, I)$ is shown below.

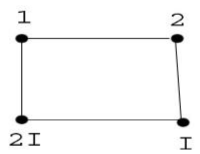


Figure 4. The graph $\bar{G}(Z_3^*, \overline{Q_3}, I)$.

Now several interesting properties of these graphs on Neutrosophic quadratic nonresidues of modulo p have been obtained.

We begin with the basic properties of $\bar{G}(Z_p^*, \overline{Q_p}, I)$.

Theorem 4.2: The Neutrosophic quadratic nonresidue graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is connected.

Proof: By the Theorem 2.8, $xy \in N(\overline{Q_p}, I)$ whenever $x \in N(Q_p, I)$ and $x \in N(\overline{Q_p}, I)$. This relates, for each $1 \le i \le \frac{p-1}{2}$, we have

$$Q_p = \left\{x_1,\ x_2,\ \ldots,\ x_{\frac{p-1}{2}}\right\},$$

$$Q_p I = \left\{x_1 I,\ x_2 I,\ \ldots,\ x_{\frac{p-1}{2}} I\right\},$$

$$\overline{Q_p} = \left\{y_1,\ y_2,\ \ldots,\ y_{\frac{p-1}{2}}\right\} \text{ and}$$

$$\overline{Q_p} I = \left\{y_1 I,\ y_2 I,\ \ldots,\ y_{\frac{p-1}{2}} I\right\}.$$

These sets determine the elements

$x_1 y_1,\ x_1 y_2, \ldots, x_1 y_i$;

$x_2 y_1,\ x_2 y_2, \ldots, x_2 y_i$;

… … …

$x_i y_1,\ x_i y_2, \ldots, x_i y_i$;

… … …

$(x_1 I)(y_1 I),\ (x_1 I)(y_2 I), \ldots, (x_1 I)(y_i I)$;

$(x_2 I)(y_1 I),\ (x_2 I)(y_2 I), \ldots, (x_2 I)(y_i I)$;

… … …

$(x_i I)(y_1 I),\ (x_i I)(y_2 I), \ldots, (x_i I)(y_i I)$;

… … …

are elements in $N(\overline{Q_p},\ I)$ and which are the edges in the graph $\bar{G}(Z_p^*,\ \overline{Q_p},\ I)$. Consequently, there is a path between any two distinct vertices in $\bar{G}(Z_p^*,\ \overline{Q_p},\ I)$ and hence $\bar{G}(Z_p^*,\ \overline{Q_p},\ I)$ is connected.

Theorem 4.3: The Neutrosophic quadratic nonresidue graph $\bar{G}(Z_p^*,\ \overline{Q_p},\ I)$ is $(p-1)$- regular.

Proof: If x is any vertex of the Neutrosophic quadratic nonresidue graph $\bar{G}(Z_p^*,\ \overline{Q_p},\ I)$, then x must be an element of the Neutrosophic group $N(Z_p^*,\ I)$. So there exist Neutrosophic quadratic residues $N(Q_p,\ I)$ and nonresidues $N(\overline{Q_p},\ I)$ such that

$$N(Z_p^*,\ I) = N(Q_p,\ I) \cup N(\overline{Q_p},\ I).$$

This partition of the vertex set of the graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ implies that either $x \in N(Q_p, I)$ or $x \in N(\overline{Q_p}, I)$.

Now $x \in N(Q_p, I)$, and if $N(\overline{Q_p}, I) = \left\{y_1, y_2, \dots, y_{\frac{p-1}{2}}, y_1 I, y_2 I, \dots, y_{\frac{p-1}{2}} I\right\}$ then by Theorem 2.8 $xN(\overline{Q_p}, I) = \left\{xy_1, xy_2, \dots, xy_{\frac{p-1}{2}}, xy_1 I, xy_2 I, \dots, xy_{\frac{p-1}{2}} I\right\} = N(\overline{Q_p}, I)$.

It gives that the vertex x is adjacent to every element in $N(\overline{Q_p}, I)$. This means that

$$\begin{aligned}\deg(x) &= |N(\overline{Q_p}, I)| \\ &= |\overline{Q_p} \cup \overline{Q_p} I| \\ &= |\overline{Q_p}| + |\overline{Q_p} I| \\ &= \tfrac{p-1}{2} + \tfrac{p-1}{2} \\ &= p-1.\end{aligned}$$

Next $x \in N(\overline{Q_p}, I)$ and if $N(Q_p, I) = \left\{x_1, x_2, \dots, x_{\frac{p-1}{2}}, x_1 I, x_2 I, \dots, x_{\frac{p-1}{2}} I\right\}$. Then, again by the Theorem 2.8,

$$x\,N(Q_p, I) = N(\overline{Q_p}, I).$$

It yields that $\deg(x) = p-1$, proving that the Neutrosophic Quadratic nonresidue Graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is $(p-1)-$ regular.

Finally looking at another basic property of the Neutrosophic quadratic nonresidue graph, we state the following fundamental theorem of graph theory.

Theorem 4.4 [10]: If G is a simple undirected graph of the size $|E|$. Then $\sum_{x \in V(G)} \deg(x) = 2|E|$.

Theorem 4.5: The size of the graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is $(p-1)^2$.

Proof: By the Theorem 4.3 and theorem 2.5, the size of the graph $\bar{G}(Z_p^*, \overline{Q_p}, I)$ is denoted by $|E(\bar{G})|$ and defined as

$$|E(\bar{G})| = \tfrac{1}{2}\sum_{x \in N(Z_p^*, I)} \deg(x)$$

$$= \frac{1}{2}\sum_{x \in N(Z_p^*,\ I)}(p-1)$$

$$= \frac{1}{2}(p-1)\left|N(Z_p^*,\ I)\right|$$

$$= \frac{1}{2}(p-1)(2p-2)$$

$$= (p-1)^2.$$

5. Conclusions

In this paper, we have studied two Neutrosophic graphical representations for determining the Neutrosophic Quadratic residues and nonresidues of the Neutrosophic group of modulo prime by using Neutrosophic algebraic theory, number theory, and classical algebraic theory. In addition to these, the Neutrosophic algebraic system can find Neutrosophic properties of Quadratic residues and nonresidues. Also, this algebraic-based application produces the complement neutrosophic graphs of each disjoint union of Neutrosophic Quadratic residue and nonresidue sets.

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Practical Applications of the Independent Neutrosophic Components and of the Neutrosophic Offset Components

Florentin Smarandache

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Abstract: The newly introduced theories, proposed as extensions of the fuzzy theory, such as the Neutrosophic, Pythagorean, Spherical, Picture, Cubic theories, and their numerous hybrid forms, are criticized by the authors of [1]. In this paper we respond to their critics with respect to the neutrosophic theories and show that the DST, that they want to replace the A-IFS with, has many flaws.

Their misunderstanding, with respect to the partial and total independence of the neutrosophic components, is that in the framework of the neutrosophic theories we deal with a *MultiVariate Truth-Value* (truth upon many independent random variables) as in our real-life world, not with a *UniVariate Truth-Value* (truth upon only one random variable) as they believe.

About the membership degrees outside of the interval [0, 1], which are now in the arXiv and HAL mainstream, it is normal that somebody who over-works (works overtime) to have an over-membership (i.e., membership degree above 1) to be distinguished from those who do not work overtime (whose membership degree is between 0 and 1). And, similarly, a negative employee (that who does only damages to the company) to have a negative membership (i.e., membership degree below 0) in order to distinguish him from the positive employees (those whose membership degree is above 0). There are elementary practical applications in this paper that allow us to think out of box (in this case the box is the interval [0, 1]).

Keywords: Neutrosophy; Neutrosophic Components; Neutrosophic Offset Components; TriVariate Truth-Value; MultiVariate Truth-Value; UniVariate Truth-Value.

1. Independence and Dependence of the Neutrosophic Components

The introduction should briefly place the study in a broad context and highlight why it is important. It should define the purpose of the work and its significance. The current state of the research field should be reviewed carefully and key publications cited. Please highlight controversial and diverging hypotheses when necessary. Finally, briefly mention the main aim of the work and highlight the principal conclusions. As far as possible, please keep the introduction comprehensible to scientists outside your particular field of research. References should be numbered in order of appearance and indicated by a numeral or numerals in square brackets, e.g., [1] or [2,3], or [4–6]. See the end of the document for further details on references.

1.1 TriVariate Truth-Value

Neutrosophy [15], as new branch of philosophy, started from the practical principle that everything (E) should be evaluated from *three independent points of view (or sources of information, or* *random variables)*: two opposite ones (positive and negative), and a third one the neutral in between them, for a fear evaluation. Thus, a neutrosophic triplet has been constructed, <positive, neutral, negative>, for studying especially contrary philosophical concepts, ideas, and schools. Therefore, one deals with a TriVariate Truth-Value because one uses three independent random variables (sources of information): one that presents the degree of positive side of E, another that presents the degree of negative side of E, and a third one that presents the degree of neutral (indeterminate) side of E.

That's what happens in our everyday life, and the most known one is in the court of law (defender, persecutor, jury). Also, everything has good, bad, and common features.

{Surely, more generally, everything may be evaluated from n points of view (n random variables, or n sources of information), for any integer $2 \le n \le \infty$, as such dealing with a *MultiVariate Truth-Value*, where the random variables may have degrees of positiveness, or negativeness, or neutrality (indeterminacy), but this case falls under the Refined Neutrosophic Logic [13], or under the Plithogenic Logic as generalization of MultiValued Logic [14], or under the Plithogenic Probability & Statistics as generalizations of MultiVariate Probability & Statistics [30], which are different stories.}

For example, in general you are evaluated by a friend in a positive way, by an enemy in a negative way, and by a neutral person in a neutral way.

Surely, in the Refined Neutrosophic Set and Logic and Probability , you may be evaluated by many friends in positive ways, and by many enemies in negative ways, and by many neutral persons in neutral ways. That's life, as in neutrosophy.

This ThreeVariate way of thinking (neutrosophic evaluation) was transferred to the scientific disciplines that resulted from neutrosophy:

Neutrosophic Set (degree of membership, degree of indeterminate-membership, degree of nonmembership);

Neutrosophic Logic (degree of truth (T), degree of indeterminate-truth (I), degree of falsehood (F));

Neutrosophic Probability (chance of an event to occur, indeterminate-chance of the event to occur or not, chance of the event not to occur); etc.

For simplicity, we preferred to use the descriptive notation (T, I, F) for all neutrosophic triplets.

Let's consider the single-valued neutrosophic components, where all $T, I, F \in [0,1]$.

Depending on each application, in the neutrosophic theories one may encounter three (or more) possibilities:

a. UniVariate Truth-Value, when only one source assigns values to the neutrosophic components, and thus the neutrosophic components are totally dependent as in the other fuzzy theories, whence $0 \le T + I + F \le 1$.
b. BiVariate Truth-Value, when two independent sources assign values to the neutrosophic components, for example one source assigns values to two neutrosophic components (let's assume to T and F, thus $0 \le T + F \le 1$) and the second one to the other neutrosophic component (which is I, thus $0 \le I \le 1$), and therefore the neutrosophic components are partially dependent and partially independent {or T and F are totally dependent of each other, while I is totally independent from both of them}, whence $0 \le T + I + F \le 2$.
c. TriVariate Truth-Value, when three independent sources assign values to the neutrosophic components, each source to one distinct neutrosophic component, thus $0 \le T + I + F \le 3$) and all three neutrosophic components are totally independent.
d. TriVariate Truth-Value, when the three sources are partially dependent and partially independent. For example, John's work is evaluated by three sources: a friend, an enemy, and a neutral person, which communicate with each other and arrive to some agreement about John's work that is interpreted as *degree of dependence* (d) between these three sources,

and to some disagreement about John's work that is interpreted as *degree of independence* (*i*) between the three sources, where $d, i \in [0,1], d + i = 1$.

e. <u>MultiVariate Truth-Value</u>, in general, for Refined Neutrosophic Set/Logic/Probability [13], and for Plithogenic Logic/Probability/Statistics [14, 30].

1.2 "Unfortunately, this fact [*independence of components – our note*] is not usually taken into account in the works, where NST was applied."

Their assertion is untrue, the independence of components was used in most of the neutrosophic applications.

The independence of the neutrosophic components comes from the unrestricted summation T + I + F that can get any value between 0 and 3. The independence comes from the fact that if a neutrosophic component gets a value, it does not affect in no way the other two neutrosophic components' values. Not restricting the value of the sum T + I + F means from the start the existence of degrees of independence and dependence between the components.

In many neutrosophic applications that presented numerical examples, looking at the neutrosophic triplets (T, I, F), you would see: some whose sum is < 1, others whose sum is > 1, and others whose sum is = 1. For example (0.1, 0.3, 0.5), or (0.9, 0.8, 0.6), or (0.7, 0.1, 0.2), etc.

Also, in all neutrosophic papers the neutrosophic operators were employed, which means that the Indeterminacy (I) was used independently from T and F into the operators' formulas, which is not the case for the previous classical, fuzzy (especially A-IFS) set and logic, and probability theories.

Unlike in other previous theories (for example in DST), no normalization is done in the neutrosophic theories, therefore, after aggregation, the resulted neutrosophic components sum may be any number between 0 and 3.

Yet, the situation is more complex, since the neutrosophic theories comprises all possibilities of the neutrosophic components, i.e.: to be totally independent, partially independent and partially dependent, and totally dependent. Not only the case of the totally independent components - as they have written in their equation (6).

1.3. In their paper [1], their equation (6):

"0 ≤ *T* + *I* + *F* ≤ 3 for the completely independent components"

is partially wrong.

The correct one is only:

" $0 \le T + I + F \le 3$ "

which means that the summation T + I + F can be any number in [0, 3], with $T, I, F \in [0,1]$,

and consequently, it comprises all possibilities, i.e. the components may be:

either totally independent, or partially independent and partially dependent, or totally dependent.

The independence and dependence of the components depend on each application and on the experts. Practical examples will follow below.

It is obvious that if $T, I, F \in [0,1]$, then of course $0 \le T + I + F \le 3$, but we emphasized this double inequality to make sure the readers would not take for granted that $0 \le T + I + F \le 1$ as in the previous classical, fuzzy set and logic, and probability theories. Therefore $0 \le T + I + F \le 3$ is no restriction at all!

1.4. "We have deep doubts about the validity of this hypothesis of the components mutual independence from its practical applicability point of view" (p. 3).

Ironically, just the practical applications have inspired us to consider the independence of the components, and very simple ones, as these authors will see below.

Their misunderstanding is that these authors are considering only the UniVariate Truth-Value {truth that depends on a single parameter (or point of view, or random variable), which enforces the sum of the neutrosophic components to be up to 1, and they are totally dependent}. But, in our everyday life, we almost always deal with a MultiVariate Truth-Value {truth that depends on many independent parameters (or random variables, or sources of information), and the neutrosophic components may be: partially dependent and partially independent, or they may be totally independent}.

Practical Examples will follow below.

In general,

UniVariate Truth-Value $\neq$ *MultiVariate Truth-Value*.

Complete Independence of the neutrosophic components means that there are different (and independent) sources of information that provide estimations on each of T, I, and F respectively.

This happens in our everyday life: an item (person, object, event, action, proposition, theory, etc.) is evaluated from many points of view (or many random variables).

1.5 "According to the independence hypothesis, the event $T = 1$, $F = 1$ and $I = 1$ is allowed in the NST and in this case, the constraint (6) is fulfilled. Suppose T, F and I are the degrees of truth, false and indeterminacy, respectively (this is the notation used in the NST). Thus, if we deal with a complete truth ($T = 1$), then in compliance with the formal logic and common sense, the measure of false is 0 ($F = 0$) without any indeterminacy ($I = 0$)."

{We used the notations T, I, F because they are more descriptive for the Truth, Indeterminacy (or Neutrality), and Falsehood respectively, instead of the Greek letters μ, π, ν that are not descriptive and were used in their paper [1].}

Here it is their confusion, these authors consider only the UniVariate Truth-Value of a proposition.

As we showed before, from a point of view a proposition may be true, from other point of view it may be false, or may be neutral (or indeterminate).

When these authors talk about "common sense" they are automatically / stereotypically referring to a single source of information that provides information about all three neutrosophic components of a proposition (therefore the components are all totally dependent). When a single source provides information about an event, it knows and adjusts the sum of the components to be 1. See the below practical examples.

1.6 "It is interesting that the events $T = 1$, $F = 1$, $I = 1$ and $T = 0$, $F = 0$, $I = 0$ are interpreted in [9] as a paradox, and its definition is treated as a merit of the NST. In our opinion, generally, it seems to be more reliable to use theories, which have no paradoxes" (p. 4).

We agree to these authors with the fact that the theories that have paradoxes are not reliable, but the Neutrosophic Logic was not designed for the theories with paradoxes.

We only proved that a proposition (not theory) P, which is a paradox (totally true and false in the same time, and totally uncertain as well), can be represented in the Neutrosophic Logic as P(1, 1, 1), while in other classical or fuzzy and fuzzy extension set, logic or probability theories the proposition P cannot be represented, since the sum of the components is not allowed to be greater than 1.

2. Practical Examples of Independent or Dependent Neutrosophic Components

Let's see several practical examples, as these authors have required:

2.1 Practical Example 1

The following event E takes place:

E = {There is a street protest in Minneapolis}.

a. From the point of view of the Human Rights Activists the protest is positive, because people have the right to express their view, and consequently the CNN television station (reflecting the *left politics*) joys it. Let's say T(positiveness) = 0.8.
b. But, from the point view of the Police, the protest is negative, since the protesters are violent and destroy and burn houses and injure people; then the Fox News television station (reflecting the *right politics*) presents the negative side of the protests: violence, destruction, arson, chaos. Let's say F(negativeness) = 0.9.
c. Let's consider an unbiased (neutral) Media that reports on the event. This is the neutral source, it evaluates the event in general as, for example, I(positiveness and negativeness) = 0.4.

As seen, T + I + F > 1, and the three neutrosophic components T, I, and F are totally independently assessed, since the Human Right Activists, the Police, and Media are three different and independent entities.

The authors wrote: "Therefore, we can say, e.g., that the high degree of truth is obligatory accompanied by the low degrees of false and indeterminacy." (p. 3).

This is true ONLY for the UniVariate Truth-Value of the Classical and Fuzzy Logic. This is false for the MultiVariate Truth-Value of the Neutrosophic Logic as we previously proved with several elementary practical examples.

To contradict these authors, let's assume, in this practical example, that the Human Rights Activists reassess their evaluation of the event, and they reassign T(positiveness) = 0.7. But this has nothing to do with the Police or Media to reassess their evaluations of F(negativeness) and I(positiveness and negativeness) respectively. Since all three sources, and thus the T, I, F, are totally independent. If a neutrosophic component increases or decreases, it may have no effect on the other neutrosophic components.

This is a TriVariate Truth-Value, since the event E is evaluated by three independent parameters (from three different points of view): Human Rights Activists, Police, and Media.

As seen, it's not fair to analyze something from only one point of view (from only one parameter).

This is a TriVariate Truth-Value.

2.2 Practical Example 2

A murderer John Doe is being tried in the court of law for having committed a crime. There are three player parts in the court:

the Persecutor team, which presents the suspect in a negative way, for example F(Doe) = 0.9;

the Defense team, that presents the suspect in a positive way, for example T(Doe) = 0.4;

and the Jury, that is neutral, where $I(Doe) \in [0,1]$.

Herein, the Persecutor and the Defense are totally independent sources (since they are opposite). Therefore, T and F are totally independent.

But the Jury is dependent on the evidences provided by both the Persecutor and the Defense.

Therefore, the neutrosophic component I is totally dependent on both T and F.

Let's assume I = 0 means not guilty, I = 1 means guilty, while $I \in (0,1)$ means a hung-jury (i.e. some jurors say he is guilty, while others say he is not guilty) or unable to reach a verdict.

This is a TriVariate Truth-Value.

2.3 Practical Example 3 that refutes their assertation

Proposition: G = George is a good student.

George is evaluated by three different independent professors.

The math prof: George is excellent in mathematics and he gets only A's. Hence T(G) = 1.

The sport prof: George is the worst athlete in the team since he cannot run, cannot play baseball. Hence F(G) = 1.

The literature prof: I am totally uncertain about George's ability to write a literary composition since he never turned in any of them. Hence I(G) = 1.

Therefore we got G(1, 1, 1).

This is a TriVariate Truth-Value.

2.4 Example 4 that refutes their assertation

A paradox is a proposition that is true and false at the same time (hence T = F = 1), and completely unclear/indeterminate (hence I = 1).

2.5 Example 5 from mathematics that refutes their assertation

Assume the proposition M is "1 + 1 = 10".

If the base of numeration is 2, then proposition M is true: T(M) = 1.

If the base of numeration is 10, the proposition M is false: F(M) = 1.

This is a proposition that is totally true and totally false, without being a paradox.

Herein one has a BiVariate Truth-Value (i.e. with respect to two parameters: Base 2, and respectively Base 10).

If the base of numeration is unknown (let's denote it by *b*), then the truth-value of M is also unknown (indeterminate): I(M) = 1.

Now one has a TriVariate Truth-Value (i.e. with respect to three parameters: Base 2, Base 5, and unknown Base *b*).

2.6 Example 6 of independent and dependent neutrosophic components

There will be a football match between Poland and Belarus. For each country there are three possibilities: to win, to draw, or to lose. Therefore, as in neutrosophic theories.

a) Totally independent neutrosophic components

Asking a Polish person what is Poland's chance to win, he may say T(Poland) = 0.8.

But a Belarusian person may say that Belarus will win, let's say F(Poland) = 0.7.

Another person, from another country (Romania), may answer that it is a chance of a tie game: I(Poland) = 0.4.

It is supposed that the three sources, the Polish, Belarusian, and Romanian persons do not communicate nor know the evaluations of the others. They are totally independent and consequently are the components T, I, F.

Herein there is a TriVariate Truth-Value.

b) Totally Dependent Neutrosophic Components

Let's assume that a Polish mathematician evaluates all three possibilities of Poland. Being a mathematician, he knows that the sum of the component has to be 1, as in the classical and fuzzy set theories, logic, or probability.

He then may say: T = 0.7, F = 0.1, I = 0.2.

The neutrosophic components are totally dependent, since all three depend on a single source.

Herein there is a UniVariate Truth-Value.

c) Partially Dependent and Partially Independent Neutrosophic Components

Another situation. Assume that a scientist George has to evaluate both chances of Poland, to win or to lose.

If he choses T = 0.6, for example, he knows that 0 ≤ F ≤ 1 – 0.6 = 0.4. Suppose he takes F = 0.3.

A second source Marcel has to evaluate the possibility of tie-game, without nothing anything about George's. Let's suppose that he says: I = 0.8.

In this case, T and F are totally dependent of each other, while I is totally independent from both T and F. Herein $0 \leq T + I + F \leq 2$.

Herein there is a BiVariate Truth-Value.

3. Neutrosophic Overset/Underset/Offset

"In our opinion, the most daring theory was proposed in [*18]. This theory allows negative and greater than 1 values of membership degrees. There are some basic definitions introduced in [*18], but here, we analyze only the most general one:

Definition 4. For T(x), I(x) and F(x) being the degrees of truth, indeterminacy and false, respectively, a Single-Valued Neutrosophic Offset A is defined as follows:

A = {(x, <T(x), I(x), F(x)>), x U},

such that there exist some elements in A that have at least one neutrosophic component that is > 1, and at least another neutrosophic component that is < 0.

For example: A = {(x1, <1.3, 0.3, 0.2>), (x2, <0.1, 0.4, -0.8>)}, since T(x1) = 1.3 > 1 and F(x2) = -0.8 < 0." (p. 6)

{We took the liberty of updating the reference citation to be adjusted to our paper. Instead of [16] as in these authors' reference, we wrote [*18]. See more papers on Neutrosophic Overset/Underset/Offset: [27-29].}

These neutrosophic overset (degree > 1), neutrosophic underset (degree < 0), and neutrosophic offset (some degree > 1 and other degree < 0) were well understood by the prestigious Cornell University arXiv (New York City) mainstream Archives that approved our book:

https://arxiv.org/ftp/arxiv/papers/1607/1607.00234.pdf

and by the mainstream French Hal Archives as well:

https://hal.archives-ouvertes.fr/hal-01340830 .

These concepts were inspired from our real life [*18, 27, 28, 29].

The authors continue with the below citation from our book:

"There is a crucial example in [*18], which clarifies the author's reasoning that we critically analyze: "In a given company a full-time employer works 40 h per week. Let's consider the last week period. Helen worked part-time, only 30 h, and the other 10 h she was absent without payment; hence, her membership degree was 30/40 = 0.75 < 1.

John worked full-time, 40 h, so he had the membership degree 40/40 = 1, with respect to this company. But George worked overtime 5 h, so his membership degree was (40+5)/40 = 45/40 = 1.125 > 1. Thus, we need to make distinction between employees who work overtime, and those who work full-time or part-time. That's why we need to associate a degree of membership strictly greater than 1 to the overtime workers." (p. 6)

The above was our practical example.

The authors reject it:

"The crucial drawback of this reasoning is the lack of the clear definition of fuzzy classes, which memberships are estimated. We can see here two distinct fuzzy classes: the class of employees working at least no more than 40 h a week and the class of employees that works more than 40 h. The first class is presented by the membership function rising from 0 to 1 in the interval [0, 40] of worked hours and equal to 0 if the sum of worked hours is greater than 40. The second class is defined by the membership function increasing from 0 to 1 in the interval of worked hours from 40 to Hmax, where Hmax is the maximal allowed by government (and trade unions) value of worked hours. We can see that such an obvious reasoning does not allow membership degrees greater than 1. The incorrect reasoning of the author of [*18] is also based on the implicit mechanical conjunction of two different classes with not intersected supports. Of course, such a conjunction can be made, but the resulting fuzzy class and the corresponding membership function should have a new sense reflecting a synthetic nature of a new class. In the considered case, we can introduce the class of "hard working employees" with the membership function rising from 0 to 1 in the interval [0, Hmax]."

There are people who invent theories and then try to squeeze the reality into them.

But, we did the opposite, we started from the real-world problems (over-work, negative work) and tried to make the theories that model / approximate the reality as accurate as possible. Late on, we improved our models little by little.

First, we do not work with fuzzy classes, but with a neutrosophic approach.

Also, we see no reason to make two classes where the membership, in both of them, starts from 0 and ends to 1. What about if one gets the same value, for example the membership degree $T = 0.3$ in both classes [or in the three classes, as they added one more similar class for the negative membership]? It's a confusion. On the other hand, these two classes cannot catch the employees with negative membership (those who produce damages to the company, $T < 0$).

These authors belong to the category of people that try to squeeze the reality (the membership degree of overtime workers which overpasses 1, or $T > 1$) to the narrow classical set theory, where the membership degree has to be $T \leq 1$. The classical set theory is not written in stone, so we may enlarge it if the reality requires it.

When Zadeh founded in 1965 the Fuzzy Set and allowed the membership degree to be any number between 0 and 1 (not only 0 or 1 as in classical Set Theory) he was criticized at that time by several scientists (as he told me in 2003 at an international conference at the University of Berkeley, California, where we met). But he prevailed, because in the real world there exist many partial memberships.

About the membership degrees that are outside of the interval [0, 1], it is normal that somebody who over-works (works overtime) to have an over-membership (i.e., membership degree above 1) to be distinguished from those who do not work overtime (whose membership degree is between 0 and 1).

Our example of negative employee who deserves a negative membership ($T < 0$), is cited by these authors:

"Let us turn to the example: "Yet, Richard, who was also hired as a full-time, not only did not come to work last week at all (0 worked hours), but he produced, by accidentally starting a devastating fire, much damage to the company, which was estimated at a value half of his salary (i.e., as he would have gotten for working 20 h that week). Therefore, his membership degree has to be less that Jane's (since Jane produced no damage). Whence, Richard's degree of membership, with respect to this company, was $-20/40 = -0.50 < 0$." " (p. 6)

The authors continue:

"As we are analyzing only the last week, we can see that Richard does not belong to any of the classes described above. It is a member of a practically unlimited class of those who do not work for a given company. We can significantly narrow this class by considering only those people who, by their actions or inaction, cause damage to the company (the most harmful are the top managers of competing firms). This way we can estimate the maximum damage Dmax (it does not matter in money or equivalent worked hours), which can be inflicted on the company by an external detractor. Thus, the class of external (nonworking for the company) people who bring company damages can be presented by the membership function varying from 0 to 1 in the interval of damages [0, Dmax]. There is no place for any negative membership degree."

These authors did not read/understand exactly: Richard is indeed a full-time employee, he works for the company, as we have written into our book: "Richard, who was also hired as a full-time" it is certainly an employee. The authors make a false statement for Richard as "nonworking for the company".

Even so, it is not clear, why did they make a third class varying from 0 to 1 for the negative employees? As such, we'd like to return the ancient Occam's wisdom back to themselves: "Entities should not be multiplied unnecessarily."

If you have a negative person in your group, for example, which creates only problems to the group, you cannot assign him a membership degree equals to zero (as for people that do neither positive nor negative things to the group), but you should assign him a negative membership degree. It is very logical this way.

A negative employee (that who does only damages to the company) has to have a negative membership (i.e., membership degree below 0) in order to distinguish him from the positive employees (those whose membership degree is above 0).

We see no reason to complicate the problem by creating three classes of membership degrees in order to avoid membership degree values greater than 1 or less than 0, instead of keeping a single class, but enlarging it to the left-hand side of 0 and respectively to the right-hand side of 1.

Because neutrosophic set has 3 components, they would need 9 classes, not talking of the refined neutrosophic set, that may have any number $2 \le n \le \infty$ of refined neutrosophic components, therefore they would need 3n classes! Better they should think out of box (in this case the box is the interval [0, 1]).

4. Applicability

The authors wrote: "there is no need for such somewhat artificial and heuristic theories as the Neutrosophic, Pythagorean and Spherical sets and their derivatives" (p. 5).

We disagree. The neutrosophic theories are not artificial, they started from our real-world practicability, where there are so many neutrosophic triplets (<A>, <neutA>, <antiA), where <A> is an item (concept, proposition, idea, etc.), formed by two opposites <A> and <antiA>, together with their neutrality (indeterminacy) <neutA>.

For examples: (friend, neutral, enemy), (positive particle, neutral particle, negative particle), (masculine, transgender, feminine), (true, indeterminate, false), (win, tie-game, defeat), (yes, uncertain, no), (take a decision, pending, not taking a decision), etc.

The neutrosophic theories have many applications [25] in various fields such as: Artificial Intelligence, Information Systems, Computer Science, Cybernetics, Theory Methods, Mathematical Algebraic Structures, Applied Mathematics, Automation, Control Systems, Big Data, Engineering, Electrical, Electronic, Philosophy, Social Science, Psychology, Biology, Genetics, Biomedical, Engineering, Medical Informatics, Operational Research, Management Science, Imaging Science, Photographic Technology, Instruments, Instrumentation, Physics, Chemistry, Optics, Economics, Mechanics, Neurosciences, Radiology Nuclear, Medicine, Medical Imaging, Interdisciplinary Applications, Multidisciplinary Sciences, etc. and there were published over 2,000 papers, books, conference presentations, MSc and PhD theses by researchers from 82 countries around the world.

With respect to what the neutrosophic theories brought new, we invite these authors to read our 2019 paper, so we do not repeat the things [26], whose weblink is provided.

Rather, these authors' transformation/substitution of the Atanassov-Intuitionistic Fuzzy Set (A-IFS) into the Dempster-Shafer Theory (DST) framework is artificial, since their transformation is not quite equivalent with the A-IFS, while practically their transformation is useless because of the very large intervals they use that supposed to catch the solution.

5. Publications

They say that the "caution of editors and reviewers of solid old journals is not caused by their conservatism at all, but by the desire to see, in addition to formal definitions of these theories and numerous theorems, the solution of real methodological and practical problems" (p. 1).

In general, in any field of knowledge, when a Theory1 is generalized by the Theory2, the proponents of Theory1 are reluctant to publish and even to admit Theory2, and the first reason is the rivalry between theories, the conservatism is only an excuse. But each theory has its flavor.

The authors are less informed, since in the last years there have been books published by prestigious publishing houses such as Springer [19, 21], Elsevier [20], IGI Global [22-24] (we cite the last ones), etc. and many high rank journals by Springer, Elsevier, IOS Press, Tayler & Francis, MDPI, Hindawi, Emerald Publishing, IGI Global, World Academy of Science Engineering and Technology, IEEE, Wiley, etc. have published papers on the neutrosophic environment, such as: Complex & Intelligent Systems, Cognitive Computation, Artificial Intelligence Review, International Journal of Fuzzy Systems, Evolving Systems, Complex & Intelligent Systems, Soft Computing, Journal of Machine Learning & Cybernetics, Multiple-Valued Logic, Design Automation for Embedded Systems, Granular Computing, Neural Computing and Applications, Journal of Systems Architecture, Applied Soft Computing, Measurement, Symmetry, Mathematics, Information, Axioms, Entropy, Computational and Applied Mathematics, BMC Medical Research Methodology, International Journal of Aerospace and Mechanical, Cognitive Systems Research, Theoretical and Applied Climatology, Journal of Metrology Society of India, Journal of King Saud University – Science, Journal of Intelligent & Fuzzy Systems, IEEE Access, Expert Systems, etc.

Further on, they will see in this paper many solutions using the neutrosophic theories to practical problems.

6. Critics of the DST

These authors [1] want to destroy the fuzzy extension theories just to promote the Dempster-Shafer Theory (DST) that they support, but from the beginning they are going on an uncertain way, since DST is a flawed theory which gives many counter-intuitive results [2-8; weblinks provided; download the papers and respond to the DST problems], as we'll show below.

They assert that all fuzzy extension theories can be substituted by the DST, which is not true.

6.1 The DST fails in the Zadeh's Counter-Example

Zadeh's Counter-Example [2], as know by all fusion community, is the following:

Two doctors examine a patient and agree that he suffers from either meningitis (M), contusion (C), or brain tumor (T). Thus $\Theta = \{M, C, T\}$ is the frame of discernment. Assume that the doctors agree in their low expectation of a tumor, but disagree in likely cause and provide the following diagnosis:

$m_1(M) = 0.99$, $m_1(T) = 0.01$, and $m_2(C) = 0.99$ $m_2(T) = 0.01$, where $m_1(.)$ represents the diagnoses provided by the first doctor, while $m_2(.)$ the diagnoses by the second doctor. If we combine the two basic belief functions using the DST (first doing the conjunctive rule, then the Dempster's rule of combination), one gets the unexpected conclusion:

$$m(T) = \frac{0.0001}{1 - 0.0099 - 0.0099 - 0.9801} = 1$$

which means that the patient suffers with certainty from brain tumor, which is wrong.

Zadeh [2] has clearly written down: "there is a serious flaw in Dempster's rule which restricts its use in many applications".

Similarly, P. M. Williams questioned the validity of Dempster's Rule [31].

6.2 The A-IFS gives a better solution to Zadeh's Counter-Example than DST

After criticizing Atanassov's Intuitionistic Fuzzy Set (A-IFS), the authors proposed "redefining the A-IFS in the framework of the more general Dempster–Shafer theory of evidence (DST)" (p. 2).

Okay, then let's set and analyze the Zadeh's Counter-Example in the frame of the A-IFS, and we show that A-IFS gives better result than DST.

Let:

D_1 = {M(0.99, 0), C(0, 0), T(0.01, 0)},

D_2 = {M(0, 0), C(0.99, 0), T(0.01, 0)},

where D_1 represents the diagnoses provided by the first doctor, i.e.

M(0.99, 0) means that the degree of membership (truth) of the patient to have meningitis is 0.99, and the degree of nonmembership (falsehood) of the patient not to have meningitis is 0;

And similarly for the other diseases.

And where D_2 represents the diagnoses provided by the second doctor.

Let's use the A-IFS min/max intersection operator ($\wedge_{A-IFS}$) for the two doctors' diagnoses:

$$D_1 \wedge_{A-IFS} D_2 = \{(0.99,\ 0) \wedge_{A-IFS} (0,\ 0), (0,\ 0) \wedge_{A-IFS} (0.99,\ 0), (0.01,\ 0) \wedge_{A-IFS} (0.01,\ 0)\} =$$
$$\{(\min\{0.99, 0\}, \max\{0, 0\}), (\min\{0, 0.99\}, \max\{0, 0\}), (\min\{0.01, 0.01\}, \max\{0, 0\})\} =$$
$$\{(0,0), (0,0), (0.01,0)\} \equiv \{M(0,0), C(0,0), T(0.01,0)\}.$$

A-IFS shows that the chance of the patient of having tumor is 0.01, which is more realistic with respect to the chance of tumor of the patient, than DST's.

More counter-examples to the Dempster's rule have been published in the literature [3-8].

After these failures of the DST, new theories have been proposed, such as TBM, DSmT [9], etc. and many quantitative and qualitative fusion rules [10-12] in order to overcome the Dempster's rule counter-intuitive results.

7. Conversion from A-IFS to DST

The authors [1] propose the conversion from the framework of the A-IFS to the DST in the following way (pp. 7-8).

Let *U* be a universe of discourse, and:

$B_{A-IFS} = \{(x, < T(x), F(x) >), T(x), F(x) \in [0,1], 0 \le T(x) + F(x) \le 1, x \in U\}$ be a non-empty subset of it, that is called an Atanassov-Intuitionistic Fuzzy Set (A-IFS).

Let's $x(T(x), F(x))$ be a generic element that belongs to B_{A-IFS} , with $T(x), F(x) \in [0,1], 0 \le T(x) + F(x) \le 1,$ whence the indeterminacy (hesitancy) is $I(x) = 1 - T(x) - F(x) \in [0,1].$

From the fusion theory, and especially from Dempster-Shafer Theory, the Basic Believe Assignment (*bba*), denoted by *m(.)*, is defined as:

$m: 2^{B_{A-IFS}} \to [0,1]$, such that $m(\phi) = 0,$ where ϕ is the empty-set, and $\sum_{x \in 2^{B_{A-IFS}}} m(x) = 1.$

And the Believe Function *Bel* and the Plausible Function *Pl* are defined as follows:

$$Bel: 2^{B_{A-IFS}} \to [0,1], Bel(x) = \sum_{y \in 2^{B_{A-IFS}}, y \subseteq x} m(y),$$

$$Pl: 2^{B_{A-IFS}} \to [0,1], Pl(x) = \sum_{y \in 2^{B_{A-IFS}}, y \cap x \ne \phi} m(y).$$

Afterwards, they approximate the above B_{A-IFS} to an interval-valued fuzzy set (IVFS), denoted as $C_{IVFS} = \{(x, [Bel(x), Pl(x)]), x \in 2^{B_{A-IFS}}\} = \{(x, [T(x), T(x) + I(x)]); T(x), I(x) \in [0,1], T(x) + I(x) \le 1; x \in 2^{B_{A-IFS}}\}$ which is not equal to B_{A-IFS} . Their approach is similar to that of a Vague Set.

The interval $[Bel(x), Pl(x)] \triangleq BI(x)$ was called Believe Interval (BI).

Mathematically, this is beautiful, but practically it is useless. When converting from an approach to another one, it is supposed to diminish the indeterminacy (hesitancy) and get better results. But, it is not the case. The higher is indeterminacy (I) the larger is the believe interval that suppose to catch the solution.

As counter-examples, let's consider the following A-IFS triplets (their components' sums are equal to 1):

(T, I, F) = (0.2, 0.5, 0.3) produces the BI = [0.2, 0.7];

(T, I, F) = (0.3, 0.6, 0.1) produces the BI = [0.3, 0.9];

(T, I, F) = (0.2, 0.8, 0.0) produces the BI = [0.2, 1], etc.

There are pretty large intervals to deal with, that make the result vaguer. To say that the solution lies inside of the interval, for example [0.2, 1], means almost nothing towards solving the problem whose solution is always between 0 and 1.

Another drawback is the fact that computing with intervals is more complicated than computing with crisp numbers.

8. Differences between A-IFS and NST

"The conceptual difference between the NST and the A - IFS is the introduction of the hypothesis of complete independence of the components" (p. 3).

By NST they meant Neutrosophic Set Theory.

This is not the only difference, another big distinction is with respect to the construction of the neutrosophic operators (negation, intersection, union, implication, equivalence, etc.), since within the frame of neutrosophic environment the Indeterminacy (I) is getting full consideration and "I" is involved in the neutrosophic operators' formulas, while in the A-IFS operators the indeterminacy (called hesitancy) is completely ignored and does not appear in none of their operators' formulas.

Even for the case when the sum of the neutrosophic components is equal to 1, as occurs for the A-IFS components, the results after applying the neutrosophic operators are different from those obtained by the A-IFS operators.

A simple example is below, for the neutrosophic conjunction ($\wedge_{NS}$) vs. A-IFS conjunction ($\wedge_{A-IFS}$).

Let's denote by $\wedge_{FS}$ the fuzzy set t-norm, and by $\vee_{FS}$ the fuzzy set co-norm.

Let (T_1, I_1, F_1) and (T_2, I_2, F_2) be two neutrosophic triplets, where $T_1, I_1, F_1, T_2, I_2, F_2 \in [0,1]$, and there is no restriction on the sums of the two neutrosophic triplets.

Then, the neutrosophic conjunction is:

$$(T_1, I_1, F_1) \wedge_{NS} (T_2, I_2, F_2) = (T_1 \wedge_{FS} T_2, I_1 \vee_{FS} I_2, F_1 \vee_{FS} F_2),$$

where we clearly see that the indeterminacy/hesitancy (I) is involved in the above formula on the right-hand side: $I_1 \vee_{FS} I_2$.

But, for the A-IFS conjunction formula the indeterminacy/hesitancy is completely ignored, which makes the operator less accurate. If $T_1 + F_1 \leq 1$, $T_2 + F_2 \leq 1$, and $T_1 + I_1 + F_1 = 1, T_2 + I_2 + F_2 = 1$, in order to comply with the A-IFS constrains, one gets:

$$(T_1, F_1) \wedge_{A-IFS} (T_2, F_2) = (T_1 \wedge_{FS} T_2, F_1 \vee_{FS} F_2),$$

unfortunately, no indeterminacy/hesitancy (*I*) is involved into the formula.

Even when the sum pf the neutrosophic components is 1, as in A-IFS, the results of the neutrosophic and respectively A-IFS operators are different. Let's see this numerical example:

$$(0.6, 0.1, 0.3) \wedge_{NS} (0.5, 0.4, 0.1) = (\min\{0.6, 0.5\}, \max\{0.1, 0.4\}, \max\{0.3, 0.1\}) = (0.5, 0.4, 0.3)$$

while

$$(0.6, 0.3) \wedge_{A-IFS} (0.5, 0.1) = (\min\{0.6, 0.5\}, \max\{0.3, 0.1\}) = (0.5, 0.3)$$

whence the indeterminacy/hesitancy = 1 – 0.5 – 0.3 = 0.2 $\neq$ 0.4.

In this case these authors agree with us:

"In the case of mutually dependent components, the main constraint $0 \leq T + F + I \leq 1$ in the NST seems to be more fruitful than that in the A - IFS ($T + F + I = 1$). This was quickly discovered and the so-called Picture fuzzy sets theory (PFS) was proposed" (p. 4).

Thanks to the indeterminacy (I), that plays an important role in the neutrosophic environment and in the real world that is full of indeterminate (vague, unclear, conflicting, incomplete, etc.) data, more fields were developed within the field of neutrosophy, such as: Neutrosophic Algebraic Structures (based on neutrosophic numbers of the form $a + bI$, where I = literal indeterminacy, and a, *b* are real or complex numbers), Neutrosophic Statistics (using classical statistical procedures and inference methods but on indeterminate data), Neutrosophic Probability (chance of an event to occur, indeterminate-chance of the event to occurring or not, and chance of the event not to occur), etc.

Therefore, there are many distinctions between the neutrosophic theories and the A-IFS.

9. Conclusions (authors also should add some future directions points related to her/his research)

Many practical applications have been given in this paper about the independence and dependence of the neutrosophic components in our every-day life.

The misunderstanding of some authors, with respect to the partial and total independence of the neutrosophic components, is that in the framework of the neutrosophic theories we deal with a *MultiVariate Truth-Value* (truth upon many independent random variables) as in our real-life world, not with a *UniVariate Truth-Value* (truth upon only one random variable) as they believe.

Similarly with respect to the degrees of memberships greater than 1 or less than 0, which are now mainstream subjects. The neutrosophic theories were inspired from the practical applications..

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