Annals of Mathematics and Artificial Intelligence **36:** 153–175, 2002.

Quantum mereotopology

Barry Smith [a] and Berit Brogaard [b]

[a] *Department of Philosophy, Center for Cognitive Science and NCGIA, University at Buffalo, NY 14260, USA*
E-mail: phismith@buffalo.edu

[b] *Department of Philosophy, Southern Illinois University at Edwardsville, IL 62026, USA*
E-mail: brogaar@attglobal.net

While mereotopology – the theory of boundaries, contact and separation built up on a mereological foundation – has found fruitful applications in the realm of qualitative spatial reasoning, it faces problems when its methods are extended to deal with those varieties of spatial and non-spatial reasoning which involve a factor of granularity. This is because granularity cannot easily be represented within a mereology-based framework. We sketch how this problem can be solved by means of a theory of *granular partitions*, a theory general enough to comprehend not only the familiar sorts of spatial partitions but also a range of coarse-grained partitions of other, non-spatial sorts. We then show how these same methods can be extended to apply to finite sequences of granular partitions evolving over time, or to what we shall call coarse- and fine-grained *histories*.

Keywords: mereology, granularity, ontology, consistent histories, interpretation of quantum mechanics

1. Introduction

As a result of a series of important contributions by Bittner, Cohn, Galton, Stell, Worboys, Casati and Varzi, and others in recent years, it has become clear that many features of common-sense reasoning can be fruitfully handled within a framework of mereology conjoined with topological concepts such as *boundary*, *contact*, *separation*, *connection*, *interior* and *exterior*. The fruitfulness of this approach rests not least on the fact that mereological topology is region-based, and thus it yields a more realistic representation of the qualitative space of common sense than does standard point-set topology. Recent work (e.g., [8]) has concentrated on adding the factors of vagueness, imprecision and uncertainty to standard mereotopological approaches, and on finding ways to bridge the qualitative-quantitative divide, so that algorithms can be found for moving back and forth between the qualitative spatial representations of common sense and the discrete, digital representations used in applications [11,26,27]. Mereotopology resists a similarly realistic extension to a theory of granularity however. This is because such a theory would presuppose a means of talking about objects (at given resolutions) without at the same time talking about all the parts of those objects (at all finer resolutions). In mereology, however, if an object falls within the range over which you quantify, then so also do all the object's parts. Set theory can block this automatic recognition of an object's parts (in effect by wrapping the object in a set-theoretic coating), but only, as we shall argue, at too high a sacrifice in realism. Here we show how the problems caused for a theory of granularity within the mereological framework can be solved by adding to mereology a theory of what we shall call *granular partitions*.

The term 'partition' is here used in a way that is only distantly related to the more familiar usage which defines a partition in terms of equivalence classes. The problem with the latter notion is that it presupposes that the domain to which an equivalence relation is applied has already been divided up into units (the elements of the set with which we begin), and it is this very notion of division into units (portions, segments, items) which our present theory is designed to illuminate.

The theory is based on the one hand on the theories of discrete multi-resolution spatial knowledge advanced by Stell and Worboys in their papers listed below. It bears comparison also with the work on manipulating spatial partitions of [9] and with the conception of geographic information systems as mediators between users and the world developed in [10]. On the other hand, however, it generalizes from these to comprehend not only spatial but also certain types of non-spatial partitions. In this it exploits ideas deriving from a somewhat unusual source, namely from the theory of multi-resolution partitions put forward by [20,21], a theory which was developed as the basis of the so-called 'consistent histories' interpretation of quantum mechanics. Our title alludes to this quantum-mechanical background, and more specifically to the fact that the approach here advanced seeks to do justice to the way in which cognition induces a certain sort of quantization (or granularization) on objects in space and time.

2. Better than sets

Just as mereotopology can be seen as an extension of mereology through the addition of some topological primitive such as *connection* or *interior part*, so also set theory can be seen as an extension of mereology through the addition of the primitive set-theoretic notion of *singleton*. David Lewis [17] has shown how, with the help of this one single notion, all the standard axioms of set theory can be derived within a mereological framework. Lewis first of all defines the notion of what he calls a class, which is a set in whose content the empty set plays no role. He then shows how the theory of classes so conceived can be formally identified with the theory of mereological sums (or 'fusions') of singletons.

At the same time, however, Lewis is forced to concede that the relation between an element and its singleton is itself enveloped in mystery. As he himself puts it:

> since all classes are fusions of singletons, and nothing over and above the singletons they're made of, our utter ignorance about the nature of the singletons amounts to utter ignorance about the nature of classes generally. ... What do we know about singletons when we know only that they are atoms, and wholly distinct from the familiar individuals? What do we know about other classes, when we know only that they are composed of these atoms about which we know next to nothing [17, p. 31]?

The mystery arises, we suggest, because the relation between element and singleton (or between element and set) involves a spurious running together (and concomitant idealization) of a plurality of distinct relations each one of which is independently well understood. The relation between an object and its *location* is one such, and so also are the relations between an object and a *concept* under which it falls or between an object and a *kind* or *category* to which it belongs. Others include the various relations which an object may bear to *intervals on quantitative and qualitative scales* (for example relations between an object and its tax bracket, temperature-band, spin, quantum number, examination grade, golf handicap, Erdös number, and so on). Yet others include the relation between an object and its *role* or *function* or *office* or *niche*, or the relation between an object and the corresponding *entry in a list* or *record in a database*. For Cantor himself [6] the pertinent relation is that between an object (or what he calls a 'well-defined object of our thought') and the result of some 'collecting together into a whole'.

Set theory idealizes some of the features manifested by each of these relations, but it rides roughshod over the differences between them. Hence, if we are to do justice to the relations in question, then a more subtle framework is needed. We offer the beginnings of such a framework here: a general theory of relations of the mentioned type, each of which we shall come to recognize as involving the imposition by some cognitive agent of an appropriate sort of granular partition upon some associated portion of reality.

Because the framework we propose is so general, and is designed to comprehend not only the (comparatively well-understood) partitions of the spatial realm but also a wide range of granular partitions of a non-spatial sort, it can be developed only in outline here. We believe, however, that we have shown how it might be possible to develop on a mereotopological basis an instrument that is able to capture the relations listed, which would be comparable with set theory in the range of types of real-world objects with which it can deal.

3. What is a granular partition?

Just as sets are to all intents and purposes the mereological sums (or 'fusions') of their singletons, so granular partitions as we here conceive them are the mereological sums of their constituent *cells*. We shall accordingly develop our formal account of granular partitions in two stages:

1. In terms of a theory of the relations between cells and partitions.

2. In terms of a theory of the relations between cells and objects in reality.

(The counterpart of stage 1 in a set-theoretical context would be the study of the relations between sets and their subsets; the counterpart of stage 2 the study of relations between sets and their elements.)

We are to think of a granular partition – sometimes referred to simply as a 'partition' in what follows – as a grid of cells laid between ourselves and reality (or between ourselves and whatever is the relevant object domain) in such a way that its cells are

...	rook	bishop	pawn	knight	...
...	up	down	charm	strange	...
	John	George	Paul	Ringo	

Figure 1.

transparent. These cells allow the objects in the domain over which the partition is laid to show through in undistorted fashion. Consider the way in which in constructing weather maps we may project a grid of political divisions onto satellite images. Or consider the examples of fragments of partitions provided in figure 1, above.

A partition can be compared to a map. Indeed our present project may itself be conceived as an attempt to generalize set theory by conceiving sets as built up out of the different sorts of labeled cell-like components which we find on maps (representing counties, postal districts, census tracts) and which are themselves, in different ways, directed outwards towards objects in reality.

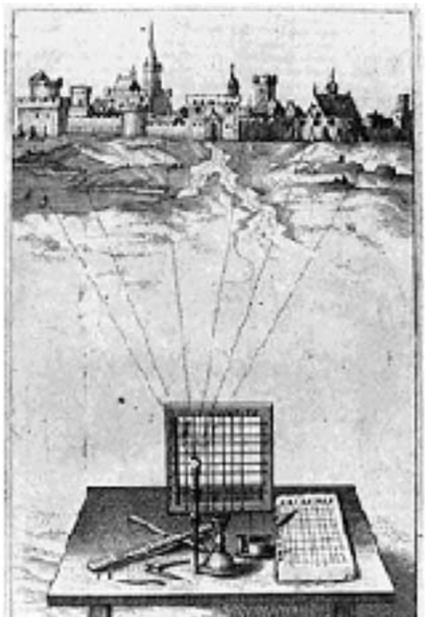

Figure 2. Alberti's grid.

A granular partition can be compared also to the drawing machines (for example Alberti's latticed grid: see figure 2) once sometimes supposed to have been used by artists as an aid to painting. (Imagine the use of such a device for tracking birds, from moment to moment, as they fly across the sky.)

Something like a grid placed over reality is involved, too, in familiar maps of brain regions or of the structure of chromosomes. Such maps are essentially spatial representations of the underlying reality. But granular partitions are involved also in representations of non-spatial knowledge – for example of the sort we find in component catalogues, or in the grid-like representations of the Periodic Table in chemical textbooks. Each of the examples listed is in our terms *transparent* to an associated portion of reality. Each picks out certain parts or features of this reality, while tracing over others.

A granular partition is a way of dividing up the world, or some portion of the world, by means of cells. Thus the verb 'to partition' is understood here as being what philosophers call a success verb. In this it is comparable with verbs like 'to know' or 'to see' or 'to find'. (If *A knows p*, then p is true. If *A sees* or *finds x*, then x exists.) This means that even though granular partitions are cognitive creations, and in this sense subjective, their relation to a reality beyond is a fully objective matter. Thus when once a given partition exists, then it is, for each cell in the partition and for each object in reality, an objective matter whether or not that object is located in that cell. (In this sense, too, granular partitions are analogous to sets.)

4. Set theory, mereology, and time

Both set theory and mereology have shared origins in explorations in the foundations of mathematics in the decades before and after the turn of the last century. The two theories were thus formulated in such a way as to relate to a static universe, a universe conceived in abstraction from time and change. A set exists outside space and time: it is determined exclusively by the (atemporally conceived) list of its elements. If the entities stipulated as the elements of a set undergo change, even change of such a sort that one or more of these entities themselves should cease to exist, then the set itself is unaffected thereby. It is for this reason that ontological commitment to sets has been characterized by philosophers as a form of Platonism.

The framework of mereology, in contrast, is non-Platonistic, which means that it is closer to the changing world of flesh-and-blood reality (and the relation of part to whole is correspondingly less mysterious). This is because, if the parts of a whole cease to exist, then so also does the whole. If you burn down the parts of a forest, then you will have burned down the whole forest, too. Yet still: mereology has resisted coherent extension of a sort which would comprehend in realistic fashion not only spatial but also temporal features of the objects in its domain. This is because mereology has been conceived in such a way as to allow unrestricted summation even in relation to objects which exist at different times. Thus it can give no satisfactory answer to the question *when* such objects (for example the fusion of Napoleon and your left foot) should properly be said to exist.

Mereo*topology* increases the resources available to mereology in countering the effects of unrestricted summation in regard to spatial wholes. For example it allows us to formulate what it means for a whole to be connected. The theory of granular partitions will enable us to push the mereotopological approach still further, by making it possible for us to take account of time and change.

5. Sets and granular partitions

Where the elements of a set exist within the set without order or location – they can be permuted at will and the set remains identical – our granular partitions typically come with a specific order and arrangement of their constituent cells. The latter are determined in part by their position within the partition, which we can think of as being fitted out with an address or coordinate system, and with a scaffolding which holds the cells together in a certain arrangement. One consequence of this is that a partition may include empty cells – indeed it may include a plurality of distinct empty cells. When you use a transparent grid for tracking birds as they move across the sky, it may well be the case that all the cells in your partition are, at any given moment, empty.

Objects as they exist in nature may stand to each other in various relations; they may have hooks of various sorts which link them together. These include common boundaries (for example as between France and Germany, or between Tibbles and her tail), and they include also relations of dependence and of functional or causal association. Mereological fusion preserves these inter-object relations, and it thus preserves the order and location of objects which fall within its grasp: if two objects are linked together in nature, then they are linked together also within their mereological fusion. In this sense mereology leaves everything as it is. If the parts of a whole are permuted in reality, then the whole, too, is thereby changed.

A set is a mereological fusion of singletons, and mereological fusion preserves order and location. How can it be, then, that the elements within a set can be permuted at will and the set be unaffected thereby? The answer is that the set is built up mereologically not out of its elements but out of its singletons, and the singleton operator has the effect of stripping away the various sorts of linkages which obtain between the objects to which it is applied as these exist in nature. It sets them apart from their surroundings and seals them off from each other and from all effects of time and change.

6. From cells to objects

The version of the theory of granular partitions presented here will mimic set theory in this respect, that if an object is located in the cell of a partition, then the whole object is located entirely within the cell, and this in completely determinate fashion. The relation between an object and its cell is modeled on the relation between an element and its singleton.

Even in the simplest version of the theory, however, granular partitions are distinguished from sets in the following respect: that where an object can be an element of a set (or singleton) in only one way, an object can be located in a cell within a partition in any number of ways. Certainly some objects are *exactly* located in the corresponding cells [7]. Exact location is illustrated for example by the relation between a concrete parcel of land and the corresponding cell in a cadastre, or (again), in set theory, by the relation of an element to its singleton. It is illustrated also by the relation between an object and its *place* (if, with Aristotle, we conceive the latter as the innermost boundary of the body of air or water by which an object is surrounded). Customers are exactly located in the cells which result when we conceive of a customer database as a partition of the corresponding population.

Here we are interested in a relation more general than that of exact location, however, a relation which holds whenever an object is entirely or completely located in a given cell even if it falls short of being exactly located therein. This relation is instantiated wherever objects are assigned (large enough) regions of space (compare an object in a cell to a guest in a hotel room). Partitions in our sense arise also when we make certain sorts of observations or experiments or whenever we sort things into categories (compare an object in a cell to a bacterium in a petri dish or to an envelope in a pigeonhole).

The requirement that an object must fit entirely within its corresponding cell can of course be generalized still further. Bittner and Stell [4] offer an approach to spatial partitions otherwise similar to the one advanced here but within which the restriction on cell-object fit is relaxed through the notion of 'rough' location. Smith and Brogaard [25] argue that the theory of granular partitions can serve, alternatively, as a framework for a new type of supervaluationist approach to the problem of vagueness. To see how this works, we note first of all that human beings and similar objects always have questionable parts (the bacteria in John's ear, the half-digested food in his stomach, the loosened molecules of skin on his back). There exists, then, a family of alternative ways of picking out a precise portion of reality which we may then choose to identify as John in a given context. Whichever such precisification we choose, however, it will still remain the case on our present theory that John is located in the *John* cell in our partition of human beings. This is because the *John* cell includes a certain amount of slack, which can be taken up in different ways by different precisifications. But only one precisification fits into a cell at any given time, just as the relation between the singleton and its element is of necessity one-to-one. In a more general theory however the relation between a cell and its occupant may be taken as one-to-many as a means of doing justice to the vagueness that is involved wherever we have a family of such alternative precisifications.

A cell is an artifact of our theoretical activity: it reflects a possible way of dividing up the world into parts and of grouping the parts together into unities of different sorts. A set is an abstract structure; its elements, in contrast (in the cases relevant to our deliberations here), are parts of concrete reality. Partitions, similarly, belong to the realm of abstracta (or better: they belong to the realm of our theoretical representations), over against the concrete realm of represented objects. Each partition brings about a demarcation of the domain upon which it is projected, a demarcation analogous to the results of drawing lines on a map. Some cells in the partition will then correspond to bona fide objects in the reality beyond, as for example in the case of the cells labeled *Guadalupe* and *Corsica* in a map of the Departments of France. Some cells will refer to fiat objects in reality, as for example in the case of cells projected onto postal districts or census tracts or other products of arbitrary legal-administrative demarcation in the spatial realm [24]. Concrete objects in the physical world are bona fide entities. A planet or tennis ball is what and where it is independently of any acts of human fiat and independently of our efforts to understand it theoretically. Granularity, we can now assert, and the associated 'discrete spaces' and 'multi-resolution spaces', are properly at home only in the fiat realm: they pertain not to the objects themselves on the side of reality, but rather to the ways we partition these objects in our activities of theorizing, classifying and mapping. If this is right, then the theory here presented can provide the basis for a mereotopological account of granularity of a truly realistic sort – where mereotopological theories of granularity have hitherto needed to rely on one or other idealizing assumption (such as is employed by Galton [11]) of discreteness on the side of the objects in the world. And if granular partitions are indeed inserted into reality by our cognitive activities, then the resultant theory – as contrasted with a theory of granularity based on a hypothesis of ultimate object-discreteness – is a theory of granularity at arbitrary resolutions.

7. Granular partitions as cognitive artifacts

Partitions as we are conceiving them here are distinct from both sets and mereological fusions in that they are not constituted out of the objects that are located in their cells at all. For partitions belong, not to the domain of objects, but rather to the domain of our theorizing and classifying and mapping activity. Partitions are grid-like *representations*. They project outwards towards those objects in reality which are located in their cells. They are thus many-rayed counterparts of concepts as conceived by Millikan [19]:

> The membership of the category "cat", like that of "Mama", is a natural unit in nature, to which the concept cat does something like pointing, and continues to point despite large changes in the properties the thinker represents the unit as having. For example, large changes can occur in the way a child identifies cats and the things it is willing to call "cat" without affecting the extension of its word "cat". The difficulty is to cash in the metaphor of "pointing" in this context.

Since some varieties of partition may exist as systems of labeled cells even independently of any objects which may at any given time be located in those cells, such partitions may remain the same – think of the transparent grid we are using to plot birds moving across the sky – even where the corresponding population of objects changes entirely. As we have stressed, however, it is at any given time a determinate matter for a given partition which objects are located in its cells.

Moreover, the assignment of objects to the cells of a partition may remain the same even though the objects towards which it is directed are subject to change. This will hold provided only that the change in question occurs beneath the threshold of what the partition recognizes. Thus the partition with just one cell labeled *Bill Clinton* picks out the same object from one day to the next even while this object gains and loses molecules. The object located in the cell labeled *cat* in your partition of biological reality is at any given time the mereological fusion of all whole, live cats. As seen through the lens of your partition, however, this total fusion is parceled out in coarse-grained fashion into individual cats (and not for example into cat-molecules or cat-organs). This parceling out is at the same time effected in such a way that the partition (or its user) does not know (or care) how many cats there are or where these cats are located. The partition traces over all the individual differences between all the different cats which fall within its scope. In this way it is able to capture all the cats in the world as forming a whole (species) and this in such a way that the latter is grasped as identical from one moment to the next even in spite of the fact that individual cats are born and die.

8. Granular partitions and relations

Some partitions are like sets in that they will apprehend the objects which are located in their respective cells independently of order or arrangement or linkage or time. Others, however, will inherit from mereology the ability to comprehend their objects in ways which map different kinds of relations that obtain among them. The cells in such partitions project their objects not in isolation, but rather in tandem with other objects located in related cells within the same partition. We can imagine, for example, two-celled partitions which capture the relations between a part and its whole or between a substance and its accident. Such partitions apply to pairs of entities in reflection of specific relations in which the latter stand to each other. John and Mary, before they wed are not, but after marriage they are, recognized by a two-celled partition of the type: *married pair*. Yet other two-celled partitions, for example the partition captured by our use of paired demonstratives such as *this* and *that*, *here* and *there*, *left* and *right*, or *first* and *second*, apply to pairs of objects only in reflection of our ways of relating to them intentionally. These are two-celled counterparts of the one-celled partitions involved in our uses of proper names or in our acts of attending cognitively to single objects. We can imagine also three-celled partitions, which might be employed for example to capture the way in which, in an action of kissing or shaking hands or congratulating, two objects become bound together by a third object – a relational event – in which the one occurs as agent, the other as patient. Partitions can manifest the feature of multi-dimensionality also in other sorts of ways. A map of the zoo, for example, might indicate not only the places where animals are located but also the sorts and sizes and proper names of the animals which are located in those places.

9. Granularity

When you think of John cooking his dinner in the kitchen, then you do not think of all the parts of John or of his surroundings. You do not think of the follicles in his arm or the freckles on his cheek. You do not think of the fly next to his ear or the neutrinos that pass through his body. Rather, you set John into relief in a highly specific way in relation to the rest of the world. You impose, in this case, a one-celled partition upon reality which induces a fiat separation between what is focused upon and what is ignored.

You effect a more complex many-celled partition when you focus on a map of France depicting its 91 *départements* or its 311 *arrondissements*. And as this last example makes clear, partitions may have different resolutions. But they must have cells of finite size. The division of the line into real or rational numbers does not define a partition, and neither does the (whole) system of lines of latitude and longitude on the surface of the globe. A partition is, intuitively, the result of applying some sort of grid to a certain portion of reality. For such a partition to do its work, its cells need to be large enough to contain the objects that are of interest in the portion of reality which concerns the user. At the same time these cells must be not too large, in the sense that they must allow the user to factor out those details which are not of concern. A granular partition is thus an instrument for focusing upon and also for ignoring things – for placing certain parts and moments of reality into the foreground of our attentions in such a way that other parts and moments are traced over in the background. (Compare Bittner [2].)

A granular partition is, we said, a labeled system of cells. The latter are then projected by the user of the partition onto the corresponding domain of reality. The cell-boundaries thereby serve to parcel out in more or less fine-grained fashion this concrete portion of the world. The cells of a partition may be purely spatial, as in a map which effects a two-dimensional partition of a certain portion of the surface of the globe. But partitions may be constructed also in such a way as to involve demarcations of a non-spatial sort. Examples are: taxonomical partitions in biology, component catalogs, customer databases. Some partitions are very simple: for example the Spinoza partition, which comprehends the whole universe in a single cell. We can analogously define for each given object x what we might call the object partition for x, consisting of one single cell in which x and x alone is located. A closely related partition has two cells, called *foreground* and *background*, one containing, precisely, x; the other containing x's complement (the mereological sum of all the objects disjoint from x). There is in addition a large family of simple partitions of reality corresponding to sequences of objects labeled by natural numbers – artifacts of that sort of cognitive act we call counting. There are partitions including empty cells (for example, the guest-list of a hotel with rooms some of which, on any given night, are unoccupied; a chessboard, some of whose squares are at any given stage in the game empty of pieces). And there are hierarchical partitions which involve cells comprehending successively more comprehensive groups of objects (species, genera, orders, classes, phyla, kingdoms, and so on) on successfully higher levels. *Dodo* is an empty cell in one standard partition of the animal kingdom.

10. Cells in granular partitions

Let variables z, z', z_1, ... range over cells (*Zellen*, in German), and A, A', A_1, ... over granular partitions (German: *Aufteilungen*). The cells in a granular partition may have subcells. Thus for example the cell *Florida* is a subcell of the cell *United States* in the standard geopolitical partition G of the surface of the globe. The cell *rabbit* is a subcell of the cell *vertebrate* in a partition of the animal kingdom, and the latter is in turn a subcell within the larger subcells *mammal*, *chordata*, and so forth. There will in general be far fewer subcells in a typical partition than there are subsets in a typical set (since the subcells are restricted, in a typical partition, to those groups of smaller cells which manifest a certain sort of naturalness or rounded-offness). This is not least because natural partitions will exclude double counting on any given level of a hierarchy of cells and subcells.

We write:

$$z \subseteq_A z'$$

as an abbreviation for: z is a subcell of the cell z' in the partition A. $\subseteq_A$ is reflexive, transitive and antisymmetric. It defines a partial order on the totality of cells in the partition A, by analogy with the usual set-theoretic subset relation. We stipulate further that it satisfies a finite chain condition to the effect that if $\cdots \subseteq_A z_i \subseteq_A z_{i-1} \subseteq_A \cdots$, then there is some n such that $z_n = z_{n+1} = \cdots$ and some m such that $z_m = z_{m-1} = \cdots$ (so that there are minimal and maximal cells at the end of each chain). An example of such a finite chain is your address (The Oval Office, The White House, 1600 Pennsylvania Avenue NW, Washington, DC 20500, USA).

We can define the property of being a minimal cell within a partition in the obvious way as follows.

$$Min_A(z) := A(z) \wedge \neg\exists z'\big(z' \subseteq_A z \wedge z' \neq z\big), \tag{1}$$

where '$A(z)$' signifies: z is a cell in the partition A. (The property of being a maximal cell can be defined similarly. In many partitions the maximal cell will coincide, in its domain of objects, with the partition considered as a whole.)

The finite chain condition tells us that every partition is in a certain sense built out of minimal cells. But consider the partition defined as follows: *Germans*, *Kant*, *Wagner*. The latter has minimal cells, but the partition as a whole can nonetheless not be identified with the mereological fusion of its minimal cells. Indeed the closest counterparts of sets within our present framework will turn out to be just those special sorts of partitions which can be identified as the sums of their minimal cells in this way. The latter then play the role played by singletons in Lewis' *Parts of Classes*. The minimal cells of the corresponding partitions represent a jointly exhaustive and pairwise disjoint tiling of the pertinent domain of objects, and every cell z in such a partition A satisfies the following:

$$\exists z_1 \ldots \exists z_n\big(Min_A(z_1) \wedge \cdots \wedge Min_A(z_n) \wedge z = z_1 \cup_A \cdots \cup_A z_n\big), \tag{2}$$

where '$\cup_A$' symbolizes the mereological fusion of cells within a partition A. We might say that the minimal cells then form a *basis* for the partition as a whole.

The possibility of decomposition into minimal cells does not hold of partitions in general. This is because partitions are artifacts of our cognition, and our cognition may be incomplete. Decomposable partitions represent a certain kind of cognitive completeness on the side of the responsible cognitive agent. Suppose you are in a crowded room and you know (who) all the people in the room (are). The maximal cell of your partition *people in this room* is then decomposable into a basis of minimal cells labeled *Jack, Jim*, *John*, *Joe*, and so on. Suppose, on the other hand, that your partition contains on the level of single persons only two cells, labeled *Jim* and *John*, since these are the only two people you recognize. Then such decomposition will not be possible. We can imagine similarly a partition of the animal kingdom containing one large cell labeled *mammal* comprehending other smaller cells labeled *rabbit*, *dog*, and so on, but where the latter are not such as to represent a complete accounting of all the species of mammal which exist. Partitions of this non-decomposable sort will be needed to capture our hierarchically organized knowledge in regard to almost all complex domains of real-world objects.

11. Union, intersection and complement of cells

We can define partition-theoretic union $z \cup_A z'$ of two cells in a partition A as a $\subseteq_A$-minimal cell satisfying the condition that it contains both z and z'. Such a union is not in general defined. (Consider our geopolitical partition G of the land surface of the globe, and take z = *Florida*, z' = *Zambia*.) And even where it is defined it is not in general unique. (As applied to *Cyprus* and *Malta*, for example, it currently yields the unique output: *British Commonwealth*; both Cyprus and Malta are however candidates for membership of the European Union.)

Partition-theoretic union is commutative, but it is not associative. That is to say $(z \cup_A z') \cup_A z''$, even where it is uniquely defined, is not in every case identical to $z \cup_A (z' \cup_A z'')$. To see why not, consider a partition P with cells exactly as follows: {a}, {b}, {c}, {d}, {e}, {a, b, d}, {b, c, e}, {a, b, c, d}, {a, b, c, d, e} (presented also in figure 3).

If we now set $z = \mathrm{a}$, $z' = \mathrm{b}$, $z'' = \mathrm{c}$, then we have $(z \cup_A z') \cup_A z'' = (\mathrm{a} \cup_A \mathrm{b}) \cup_A \mathrm{c} = \{\mathrm{a, b, d}\} \cup_A \mathrm{c} = \{\mathrm{a, b, c, d}\}$. On the other hand, however, $z \cup_A (z' \cup_A z'') = \mathrm{a} \cup_A (\mathrm{b} \cup_A \mathrm{c}) = \mathrm{a} \cup_A \{\mathrm{b, c, e}\} = \{\mathrm{a, b, c, d, e}\}$. We do however have the familiar equivalence of $z \subseteq_A z'$ and $z \cup_A z' = z'$. (See [23] for a detailed theory of such restricted union operators.)

Regarding partition-theoretic intersection, we first of all define what it is for two cells of a partition A to overlap in A, as follows:

$$z_1 \mathrm{o}_A z_2 := \exists z (z \subseteq_A z_1 \wedge z \subseteq_A z_2). \tag{3}$$

The partition-theoretic intersection of two overlapping cells in A is then defined as any $\subseteq_A$-maximal cell which is included as subcell within them both. Partition-theoretic in-

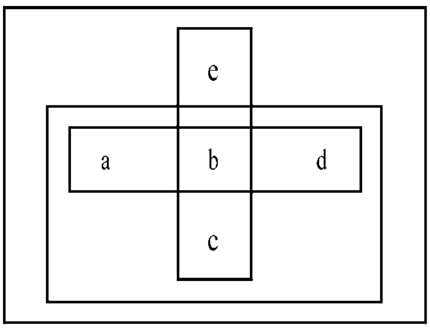

Figure 3.

tersection as thus defined is commutative, but it is not in general unique or associative. Thus for example there are two intersections of {b, c, e} and {a, b, c, d} in P, namely {b} and {c}, respectively.

Regarding partition-theoretic complement, we set $-_A z$ to be a $\subseteq_A$-maximal cell which does not overlap with z. The partition-theoretic complement of a cell, too, is not in general defined, and even where it is defined it is not in general unique. To see this, it is sufficient to consider once again our partition P, where the complements of {c} are {a, b, d} and {e}, respectively. Similarly many political entities recognized by the partition G do not have unique complements. What, for example, would be the unique $\subseteq_G$-maximal political entity which does not contain Florida as part?

That the partition-theoretic complement of a cell is not in general defined goes hand in hand with the fact that there is no analogue of the empty set in the theory of partitions. That is to say, there is no cell which is empty *per se*. This holds even for what we might call the Spinoza partition, which has a unique and maximal cell. There is no complement of this maximal cell within that partition. As we have noted, however, many partitions will contain empty cells, but these are cells which are empty *per accidens*.

Granular partitions of many specific types will of course be rather well-behaved when it comes to the taking of unions, intersections and complements. This is true of the closest approximation, within partition theory, of sets, which turn out to be those special types of partitions which have a basis in minimal cells, at most one empty cell, and which are such that every non-empty minimal cell is occupied by exactly one object.

Another important family of granular partitions is made up of those partitions which capture the sort of hierarchical knowledge of reality which is illustrated by taxonomies, partonomies, contour maps, and a range of associated systems for encoding knowledge which have in common that they can be represented in the form of trees [3,16]. In partitions of this sort there is in every case a maximal cell, which coincides with the partition as a whole and which corresponds to the root of the associated tree. This ensures that the partition-theoretic union of two cells is always defined and it is always unique. (It is the cell associated with the vertex at which the branches emanating upwards from the vertices associated with the two cells join.) If two cells intersect, then only because one is a subcell of the other. Cells on the same level within the hierarchy do not intersect at all. Here, too, partition-theoretic complements are not in general defined. A unique complement of a cell associated with a given level in the tree will exist only where the tree has exactly two vertices on that level.

We can go back and forth between trees and partitions of the given sort in virtue of a standard graph-theoretical result [18] to the effect that there is a one-to-one correspondence between families of non-intersecting, possibly nested regions in the plain and trees in which each vertex represents a region in the array and each link in the tree represents a relation of immediate containment between two nested regions.

12. The location of objects in cells

Let variables $x, x', x_1, y, \ldots$ range over objects. We write:

$$L_A(x, z)$$

for 'object x is located in cell z in partition A'. Exact location can then be defined as follows:

$$L^*_A(x, z) := L_A(x, z) \wedge \forall x' \big(L_A\big(x', z\big) \Rightarrow x' \leq x \big), \tag{4}$$

which means that an object x is exactly located in a cell z if and only if x is a maximal occupant of z. (Here and in what follows '$\leq$' abbreviates: 'is a proper or improper part of' understood according to the usual axioms of classical extensional mereology [22].)

Given the presence of empty cells, it is necessary to distinguish in our theory between partition-theoretic intersection in the general sense discussed above and that specific sort of intersection of cells which turns upon a sharing of objects. The following then appears to be an attractive axiom governing overlap for L:

$$L_A(x, z) \wedge L_A\big(x, z'\big) \Rightarrow z \operatorname{o}_A z'. \tag{5}$$

It tells us that, if an object is located in two cells (which means: fully located, located in its entirety, in both of these cells), then the latter overlap. (Here '$p \Rightarrow q$' is strict implication. It abbreviates: it is not possible for p to be true and q false.) Familiar object-based mereological intersection entails partition-theoretic intersection (but not, in general, *vice versa*).

It follows from the Overlap Axiom that if an object is in two distinct cells within a partition, then these cells are not both minimal, and they possess an intersection-cell.

13. Granular partitions and mereology

We can address the relation between granular partitions and the underlying mereology on the side of objects by isolating the special class of what we shall call *distributive* partitions. A partition is distributive if it satisfies a condition to the effect that if object x is part of object y, and if y is located in a cell z, then x is also located in the cell z:

$$dist(A) := \forall x \forall y \forall z \big(x \leq y \wedge L_A(y, z) \Rightarrow L_A(x, z) \big). \tag{6}$$

For distributive partitions, the Overlap Axiom can be proved. Spatial partitions are always distributive in the sense specified. If John is in Salzburg, then so are all his bodily parts.

Closely related to distributivity is the property of some partitions expressed by:

$$L_A(x, z) \wedge L_A(y, z) \Rightarrow L_A(x +_A y, z) \tag{7}$$

(we might call it Weak Antidistributivity) to the effect that if two objects are located in a given cell of a partition, then their sum, too, is located in that cell. Again, spatial partitions trivially satisfy this requirement: if John is in Salzburg and Mary is in Salzburg, then their sum is in Salzburg, too.

A set is a simple example of a non-distributive partition, and the same applies also to a partition generated by kinds or concepts. A partition recognizing cats does not *ipso facto* recognize parts of cats. We note also that if Bruno is a cat and Tibbles is a cat, and Tibbles and Bruno are not identical and do not overlap, then the sum of Bruno and Tibbles is not itself a cat (and nor need this sum be included in any cell of a taxonomical partition of which *cat* is a subcell).

It seems reasonable to insist, however, that all partitions satisfy a restricted version of the principle of distributivity which can be specified as follows. We first of all introduce formally the notion of recognition (a notion which has been used informally already above). To say that an object x is *recognized by* a partition A (symbolized: $x \in A$), is to say that x is located in some cell z in A, or in other words:

$$x \in A := \exists z\big(L_A(x, z)\big). \tag{8}$$

The restricted *axiom of weak distributivity* can now be formulated for all partitions A as follows:

$$L_A(x, z) \wedge y \leq x \wedge y \in A \Rightarrow L_A(y, z). \tag{9}$$

We can also define ‘minimal object’ relative to a partition A in the obvious way:

$$M_A(x) := x \in A \wedge \neg\exists y(y < x \wedge y \in A). \tag{10}$$

When you see John cooking his dinner in the kitchen, then the partition effected by your act of visual perception recognizes John, but it does not recognize all the parts of John or of his surroundings. Thus it does not recognize the cells in his lungs or the bacteria crawling in his ear or the half-digested food passing through his alimentary tract. Again, your partition induces a fiat separation of the relevant portion of reality, drawing a line between what is focused upon and what is ignored. Note, though, that this partition cannot be understood in any simple topological or geometrical terms (by analogy with the relation between Beverly Hills and the surrounding territory of Los Angeles). For while John himself is recognized by your partition, there are many interior parts of John which your partition does not recognize.

14. Relations between granular partitions

We shall say that one partition is *extended by* another partition if all of the cells in the former are also cells in the latter. We write ‘$A \leq A'$’ to signify: A is extended by A', which we define, simply enough, as follows:

$$A \leq A' := \forall z\big(A(z) \Rightarrow A'(z)\big). \tag{11}$$

A partition may be extended either by enlargement or by refinement. If a partition is *enlarged*, then more cells are added at its outer border. If a partition is *refined*, then more cells are included in its interior. This can occur either via imposition of a finer grain in the existing dimensions of the partition, or through combination of partitions. The latter can itself occur either through amalgamation along the lines set out in [28], or through a partition-theoretic analogue of Cartesian products. Consider what happens, for example, when a map of the spatial layout of the cages in your local zoo is supplemented by information as to the sorts and sizes and proper names of the animals located in those cages.

We can now assert as *axiom governing extensions*:

$$A \leq A' \Rightarrow \forall x \forall z \big(L_A(x, z) \Rightarrow L_{A'}(x, z)\big). \tag{12}$$

If A is extended by A', then all object-cell relations true in A are also true in A'.

To see why the converse does not hold, we need only consider the special role of empty cells within the theory of partitions. These ensure that the otherwise intuitively attractive definition of $z \subseteq_A z'$ in terms of $\forall x(L_A(x, z) \Rightarrow L_A(x, z'))$ here fails. Two otherwise identical partitions of the mammals, one with and one without a cell labeled *Dodo*, will at all future times have exactly the same population of objects located in exactly the same way within their respective systems of cells. The two partitions are nonetheless distinct.

We can define 'consistency' of partitions as follows:

$$A \Delta A' := \exists A'' \big(A \leq A'' \wedge A' \leq A''\big). \tag{13}$$

Two partitions are called *mutually consistent* when there is some third partition which extends them both.

All the partitions with which we have had to deal here are consistent in a trivial sense. This holds even of those partitions which cut through reality in ways that are skew to each other. One partition may, for example, divide the territory of a state into its separate counties, and a second partition may divide this same territory according to varieties of land use or soil type. We can still create a single partition which extends them both, however, again by a process of amalgamation [28]. Other partitions may relate to portions of reality which are entirely disjoint: for example the partition of the United States into *states* and of Canada into *provinces*. Here again we can create a single partition which extends them both, in this case via a process of topological gluing.

15. Histories as sequences of granular partitions

Consider a chess game. This can be conceived in terms of the theory of partitions as follows. The game determines a partition having 768 minimal cells (for the 64 squares $\times$ 6 different types of pieces $\times$ 2 different colors). At most 32 of these cells have objects located within them at any given time. The minimal objects relative to this partition are then the 32 separate pieces (and not, for example, the molecules of wood and paint from out of which these pieces are formed). Clearly, however, we need here to take into account not just one partition but rather an entire sequence of partitions, corresponding to the successive positions in the game.

We shall call such a sequence of partitions a *history*. A single partition stands to a history as an instantaneous snapshot stands to the sequence of successive frames within a film. A history may correspond, for example, to a sequence of successive observations made in the course of a physical experiment, or to a sequence of states in the execution of a computer program.

A history can be described by means of a conjunction of sentences of the form:

$$L_{A_i}(x, z)$$

to be read as: the individual x is located at time i in the cell z of partition A. Here i is an index for the successive reference times on the basis of which a given history is constructed. We have great flexibility in the choice of reference times. Thus there is no requirement that the times referred to by i be absolute smallest time-units or time-points. Reference *intervals* (of phoneme- or word-length) would be required, for example, in order to exploit the machinery of partitions and histories for purposes of linguistics. Indeed we can utilize in this respect the full generality of the approach to time granularity proposed in [1], where all that is required is (a) that the set of index times be a discrete linearly ordered set isomorphic to a subset of the integers with the usual order relation, and (b) that there be some order-preserving mapping μ from this set to the set of absolute times which is such that, if two indices are assigned a value under μ, then so also is every index which falls between them.

A partition is more or less coarse-grained according to the number of cells we use in its construction. A history may be more or less coarse-grained according to the granularity of its associated partitions but also according to the number of index-times employed in its construction (and in principle also according to the type of the mapping μ and of the choice of absolute time set).

Suppose John's travel agent issues him with a flight itinerary indicating his location at three successive (clock-) times. The rest of the world at the three times is ignored, as are all matters pertaining to the world at all other times. Suppose John's locations (cells) at these three times are successively: Kennedy, De Gaulle, and Abu Dhabi airports. The itinerary then describes John's movements in terms of a three-cell partition and three reference-times. It is not concerned with how he gets to the airport from his home, or with the other people at the airport, or with the locations during flight of the successive planes John takes, or with the food he eats on the journey. These things, whatever they are, could have varied without affecting any detail of the given history.

We can, however, create a finer-grained history by constructing partitions that contain either more details about John and the places at which he is located, or by taking more and finer-grained reference times. We use 'H' as a variable ranging over histories (finite sequences of partitions) and we write $A \in_i H$ for: A is a partition in history H at index-time i. A history H is *extended* by another history H' if and only if, at each index-time, all partitions in H are extended by partitions in H':

$$H \leq H' := \forall A \forall i \big(A \in_i H \Rightarrow \exists A' \big(A' \in_i H' \wedge A \leq A'\big)\big). \tag{14}$$

Whatever holds (eventuates) in a history H holds (eventuates) in all extensions of H.

16. Libraries as complete families of histories

There are alternatives to any given coarse-grained history H. John might fly to Abu Dhabi via London instead of via Paris. The coin, which landed on its head, might have landed on its tail. A coarse-grained history H' that is an alternative to H employs the same reference-times, but the objects are distributed differently across the cells of the underlying partitions. When we move, in this fashion, to consider not only what *is* but also what *might have been*, then the predicate L is not a location (or instantiation or occupation) predicate *simpliciter*. Rather it is a predicate affirming location with respect to a given history H.

Suppose your entire knowledge of John's trip to Abu Dhabi is encapsulated by the given coarse-grained history. There are then many finer-grained histories all of which are consistent with your knowledge (though of course not all of these need correspond to what in fact eventuates). Each coarse-grained history can be identified with a certain class of fine-grained histories, namely the class of fine-grained histories that vary in respect of the details ignored in the given coarse-grained history.

We shall say that two fine-grained histories H' and H'' are equivalent with respect to a coarse-grained history H if they satisfy:

$$H' \approx_H H'' := H \leq H' \wedge H \leq H''. \tag{15}$$

In this way we can extend to histories the tools for refinement and coarsening of graphs developed in [26]. Each relatively coarse-grained history can be associated with an equivalence class of alternative fine-grained histories, all of which agree regarding some features while other features are allowed to vary. It is such equivalence classes, which trace over insignificant details, which serve as the basis for all planning and decision-making.

The class of alternative histories, for each given level of granularity, can now be conceived as a more homely analogue of the set of possible worlds of the modal realists. To see how this works, consider a sequence of three successive tosses of a single coin. Here it is very easy to construct a complete family of alternative histories modulo a certain granularity determined, in this case, by a certain two-celled partition (with cells labeled *heads* and *tails*), and an index set consisting of three reference times. When all other features of the system in hand are traced over, then the complete family of alternative histories can be represented as in figure 4.

We could similarly, though with more difficulty, construct a complete family of alternative histories over John's behavior from the beginning to the end of his journey to

histories:	1		2		3		4		5		6		7		8	
	H	T	H	T	H	T	H	T	H	T	H	T	H	T	H	T
time 1	x		x		x		x			x		x		x		x
time 2	x		x			x		x	x		x			x		x
time 3	x			x	x			x	x			x	x			x

Figure 4. Histories (H = heads, T = tails).

Abu Dhabi, where the pertinent granular partition might be determined, for example, by the network of intervening airports. (There are only finitely many ways to fly from New York to Abu Dhabi.)

We shall call such a maximal family of consistent coarse-grained histories a *library*. A library specifies all possible ways in which, considered at a given level of granularity and in relation to a given set of reference times, a given system of objects may behave. The concept hereby defined is highly general. A library is determined by how the objects are distributed over given granular partitions associated with given reference-times, but there is no requirement that the same partitions should be employed at each of the successive times involved.

The reader will have recognized already from inspection of figure 4 above that a library is analogous to a truth-table. Omnès [20] calls a library a 'logic', and his work shows how the concept of truth-table and the framework of classical logic which it represents can be generalized in such a way that it can be applied to all coarse-grained histories of the sort described above. The resultant truth-tables are, as we might say, n-valued, where n is the number of cells in the partition associated with a given time-index. But they are nonetheless fully comparable to the truth-tables of classical logic (thus they have nothing in common with the many-valued truth-tables developed for the purposes of representing for example non-classical logics of vagueness). Partition-theoretic truth-tables allow us to define analogues of all the classical logical constants (conjunction, disjunction, inference, negation, etc.), which behave exactly as within the two-valued framework. (Omnès, op. cit.) And just as only one assignment of truth-values to a proposition will correspond to reality in the framework of classical logic, so only one of the alternative histories within any given library will, in fact, be actualized. The coarse-grained history in which John goes via Orly, and the alternative history in which he goes via Heathrow, are mutually exclusive. That is, there is no larger history that contains them both.

We write '$H \xi L$' for: H is a history in library L. We can then define an equivalence relation on fine-grained histories, relative to a given library of coarse-grained histories, as follows:

$$H' \approx_L H'' := \exists H \xi L \big(H \leq H' \wedge H \leq H'' \big). \tag{16}$$

Two histories are equivalent relative to a library L if and only if there is some history in L which both extend.

Each library is maximal relative to some given granularity of cells and reference-times and relative to a given domain of constituent partitions. However, a library can itself be *extended* by increasing the number of reference times or by imposing an extended partition for cells. We write $L \leq L'$ as an abbreviation for: library L' is an extension of library L. Two libraries L and L' are then called *mutually consistent* when there is a larger library of histories extending them both:

$$L \Delta L' := \exists L'' \big(L \leq L'' \wedge L' \leq L'' \big). \tag{17}$$

Otherwise the libraries L and L' are called *complementary*.

17. Coda on quantum mechanics

If granular partitions and histories are employed for purposes of representing classical physical phenomena, then the corresponding libraries are in every case consistent. The same applies in relation to all transparent partitions and histories constructed in relation to that medium-sized world of what happens and is the case in relation to which we humans act. When we use partitions and histories to represent quantum physical systems, however, then complementarity arises. The distinction between the quantum and the classical world (and it is a deep distinction) lies precisely in the fact that, to do justice to the evolution of even a single physical system within the quantum world, we typically need to employ not one but a plurality of libraries which are complementary in the sense defined. There can, accordingly, be no single aggregate library which would represent all phenomena in the quantum world.

The theory of consistent histories and of probability assignments to histories within libraries was originally developed by Griffiths in [14,15] and also by Gell-Mann, Hartle, and Omnès as the basis for a new interpretation of quantum mechanics which has since established itself under the label 'consistent histories'. It now forms part of the New Decoherentist Orthodoxy in interpretation theory [5]. Experiments, from this perspective, are courses of events like any other. Thus they, too, are apprehended within consistent histories (and thus within encompassing libraries) of appropriate type. There is therefore no analogue within the consistent histories approach of any 'collapse of the wave function' arising as a result of some special role of observers or of consciousness. Observations may, certainly, disturb the systems towards which they are directed; but then the total physical system compounded out of the observation event and of the system observed is itself susceptible to treatment within the consistent histories approach just like any other physical system.

Complementarity arises in virtue of the fact that in the quantum world there are sets of properties which conflict with each other in the sense that they cannot be used simultaneously without limitations. The characteristics of a particle may for example be described by giving either the position or the momentum of the particle as a function of time; a photon may be regarded either as a particle or as a wave. To represent such a state of affairs in coherent fashion, the consistent historians hold, it is necessary for physicists to embrace different and mutually incompatible libraries in relation to one and the same physical system. All reasoning about that system must then take place exclusively within some one of these selected libraries. If reasoning takes place across libraries, then inconsistency will result.

Suppose physicists A and B have each made calculations with respect to the behavior of photons within some given apparatus involving, say, a photon source, a screen with right and left slits, and a detector. They each are allowed to set up experiments to measure the location of photons in order to test the accuracy of their calculations. A, working within one library and its associated repertoire of experiments, conceives the photon as a particle and constructs appropriate types of experiments designed to detect whether the photon goes through either the right or the left slit in the apparatus. B, working within a complementary library and repertoire of experiments, conceives the photon as a wave and constructs experiments designed to measure interference effects as the wave passes through both slits. Both libraries give rise to predictions of astonishing accuracy, which are repeatedly confirmed in successive experiments. A's and B's predictions are, to be sure, inconsistent with each other. But such inconsistency can never be detected in relation to any given system of photons, since it is impossible for A and B to carry out the necessary experiments simultaneously, since each would need a quite different sort of apparatus.

Each experiment carried out by A corresponds to one library (to one family of coarse-grained histories in our terminology above), each experiment carried out by B corresponds to another library. The two libraries are inconsistent with each other, but they each give rise to equally good predictions.

Provided that a history is a member of a consistent family of histories, it can be assigned a probability [14,15], and within a given consistent family the probabilities function in the same way as do those of a classical stochastic theory: one and only one history occurs, just as, when we are tossing coins, one and only one succession of heads and tails in fact corresponds to reality. But histories can be assigned probabilities only if they are of sufficiently coarse grain [12,13].

The importance of the work of Omnès and of the other consistent historians turns on the fact that it shows how the theory of consistent histories can be used to derive in rigorous fashion a theory of the 'quasi-classical' physics governing the macroscopic phenomena of our everyday reality. This is done effectively by showing how, as more coarse-grained partitions are substituted for the finer-grained partitions employed at quantum levels, superposition phenomena 'decohere' (which means: they become negligible). This implies that we can understand why from a physical point of view macroscopic objects are so astonishingly well-behaved even in spite of the fact that the quantum phenomena from out of which they are constructed involve logically and ontologically monstrous superpositions.

Niels Bohr believed that complementarity is a concept of wide application. He held, for example, that when we describe living things as biological and as physical systems, then we are employing complementary descriptions which cannot be used simultaneously without limitations. Our remarks on the quantum-mechanical background of the theory of granular partitions suggest that it may be time to look again at such ideas. The hitherto unnoticed commonality between the approach to partitions at multiple resolutions developed in the study of macro-level spatial reasoning and the theory of granular partitions as a means of representing physical phenomena in the quantum realm may then prove to be of quite general ontological significance.

Acknowledgements

An early version of this paper was presented at the AAAI-2000 Workshop on Spatial and Temporal Granularity. Thanks are due to the participants at that meeting, and especially to Thomas Bittner, for helpful comments. Thanks are due also to Eero Byckling, Achille Varzi and to Ulrich Steinvorth, Carola Eschenbach and their colleagues in Hamburg. Support from the NCGIA (Buffalo), from the American Philosophical Society, and from the NSF (Research Grant BCS-9975557: “Geographic Categories: An Ontological Investigation”) is gratefully acknowledged.

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